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NORMAL FORMS, HIGH-GAIN AND FUNNEL CONTROL FOR LINEAR DIFFERENTIAL-ALGEBRAIC SYSTEMS

THOMAS BERGER*, ACHIM ILCHMANN*, AND TIMO REIS†

Abstract. We consider linear differential-algebraic m -input m -output systems with positive strict relative degree or proper inverse transfer function; in the single-input single-output case these two disjoint classes make the whole of all linear DAEs without feedthrough term. Structural properties - such as normal forms (i.e. the counterpart to the Byrnes-Isidori form for ODE systems), zero dynamics, and high-gain stabilizability - are analyzed for two purposes: first, to gain insight into the system classes and secondly, to solve the output regulation problem by funnel control. The funnel controller achieves tracking of a class of reference signals within a pre-specified funnel; this means in particular, the transient behaviour of the output error can be specified and the funnel controller does neither incorporate any internal model for the reference signals nor any identification mechanism, it is simple in its design. The results are illuminated by position and velocity control of a mechanical system encompassing springs, masses, and dampers.

Key words. differential-algebraic equations, strict relative degree, zero dynamics, minimum phase, stabilization, high-gain output feedback, funnel control

AMS subject classifications. 34A09, 93D15, 93D21, 93B52, 34H15

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Nomenclature

\mathbb{N}, \mathbb{N}_0	set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, set of all integers, resp.
$\mathbb{R}_{\geq 0}$	$= [0, \infty)$
$\mathbb{C}_+, \mathbb{C}_-$	the open set of complex numbers with positive, negative real part, resp.
$\mathbf{GL}_n(\mathbb{R})$	the group of invertible real $n \times n$ matrices
$\mathbb{R}[s]$	the ring of polynomials with coefficients in \mathbb{R}
$\mathbb{R}(s)$	the quotient field of $\mathbb{R}[s]$
$R^{n,m}$	the set of $n \times m$ matrices with entries in a ring R
$\ x\ $	$= \sqrt{x^\top x}$, the Euclidean norm of $x \in \mathbb{R}^n$
$\ M\ $	$= \max_{x \in \mathbb{R}^m, \ x\ =1} \ Mx\ $, induced matrix norm of $M \in \mathbb{R}^{n,m}$
$\mathcal{C}^\ell(\mathcal{T}; \mathbb{R}^n)$	the set of ℓ -times continuously differentiable functions $f : \mathcal{T} \rightarrow \mathbb{R}^n$
$\mathcal{B}^\ell(\mathcal{T}; \mathbb{R}^n)$	$= \{f \in \mathcal{C}^\ell(\mathcal{T}; \mathbb{R}^n) \mid \frac{d^i}{dt^i} f \text{ is bounded for } i = 0, \dots, \ell\}$

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1. Introduction. We consider linear differential-algebraic systems of the form

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t), \end{aligned} \tag{1.1}$$

where $E, A \in \mathbb{R}^{n,n}$, $B, C^\top \in \mathbb{R}^{n,m}$ are such that the pencil $sE - A \in \mathbb{R}[s]^{n,n}$ is *regular*, i.e. $\det(sE - A) \in \mathbb{R}[s] \setminus \{0\}$; the set of these square (i.e., same number of inputs and outputs) systems is denoted by

$$\Sigma_{n,m} \quad \text{and we write} \quad [E, A, B, C] \in \Sigma_{n,m}.$$

The functions $u, y : \mathbb{R} \rightarrow \mathbb{R}^m$ are called *input* and *output* of the system, resp. A trajectory $(x, u, y) : \mathbb{R} \rightarrow \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$ is said to be a *solution* of (1.1) if, and only if, it belongs to the *behaviour* of (1.1):

$$\mathfrak{B}_{(1.1)} := \left\{ (x, u, y) \in \mathcal{C}^1(\mathbb{R}; \mathbb{R}^n) \times \mathcal{C}(\mathbb{R}; \mathbb{R}^m) \times \mathcal{C}(\mathbb{R}; \mathbb{R}^m) \mid \begin{array}{l} (x, u, y) \text{ solves (1.1)} \\ \text{for all } t \in \mathbb{R} \end{array} \right\}.$$

Regularity of the pencil guarantees that for each consistent initial value $x(0) = x^0 \in \mathbb{R}^n$ there exists a unique and global solution of $E\dot{x} = Ax$, see for example [16, Sec. 2.1]. More smoothness for u and y is required for some results.

We also derive frequency domain results for $[E, A, B, C] \in \Sigma_{n,m}$ and its *transfer function*, defined by

$$G(s) = C(sE - A)^{-1}B \in \mathbb{R}(s)^{m,m}.$$

Roughly speaking, and in Section 1.3 explained in detail, we divide, in the single-input single-output case, the system class $\Sigma_{n,1}$ into the disjoint sets of systems with strictly proper transfer function and with non strictly proper transfer function $g(s) = C(sE - A)^{-1}B \in \mathbb{R}(s)$, resp. Our generalization for multi-input, multi-output systems $\Sigma_{n,m}$ will treat systems with positive strict relative degree and systems with proper inverse transfer function, resp. However, these two disjoint sets do not unify to whole $\Sigma_{n,m}$ as, for example, systems with strictly proper transfer function with a non-constant vector relative degree are not encompassed.

1.1. System equivalence. We will derive normal forms of systems $[E, A, B, C] \in \Sigma_{n,m}$ belonging to certain classes specified in Section 1.3. To this end, recall the following:

Two systems $[E_i, A_i, B_i, C_i] \in \Sigma_{n,m}$, $i = 1, 2$, are called *system equivalent* if, and only if,

$$\exists W, T \in \mathbf{GL}_n(\mathbb{R}) : \begin{bmatrix} sE_1 - A_1 & B_1 \\ C_1 & 0 \end{bmatrix} = \begin{bmatrix} W & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} sE_2 - A_2 & B_2 \\ C_2 & 0 \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & I_m \end{bmatrix};$$

we write

$$[E_1, A_1, B_1, C_1] \stackrel{W,T}{\sim} [E_2, A_2, B_2, C_2]. \tag{1.2}$$

It is easy to see that system equivalence is an equivalence relation on $\Sigma_{n,m}$.

It is also easy to see that the transfer function is invariant under system equivalence; more precisely: if (1.2) holds, then

$$C_1(sE_1 - A_1)^{-1}B_1 = C_2(sE_2 - A_2)^{-1}B_2. \tag{1.3}$$

We quote the well known result by Weierstraß for regular pencils $sE - A$.

PROPOSITION 1.1 (Weierstraß form [7, Th. XII.3]).

For any regular matrix pencil $sE - A \in \mathbb{R}[s]^{n,n}$, there exist $W, T \in \mathbf{GL}_n(\mathbb{R})$ such that

$$sE - A = W \begin{bmatrix} sI_{n_s} - A_s & 0 \\ 0 & sN - I_{n_f} \end{bmatrix} T, \quad (1.4)$$

for some $A_s \in \mathbb{R}^{n_s, n_s}$ and nilpotent $N \in \mathbb{R}^{n_f, n_f}$. The dimensions $n_s, n_f \in \mathbb{N}_0$ are unique, the matrices A_s and N are unique up to similarity. \diamond

The *index of nilpotency* of a nilpotent matrix $N \in \mathbb{R}^{k,k}$ is defined to be the smallest $\nu \in \mathbb{N}$ such that $N^\nu = 0$. It can be shown (see e.g. [16, Lem. 2.10]) that the index of nilpotency ν of N in (1.4) is uniquely defined by the regular pencil $sE - A$; ν is therefore called the *index* of the pencil $sE - A$ if the nilpotent block is present and the index is $\nu = 0$ if the nilpotent block is absent (i.e., $n_f = 0$).

An immediate result from Proposition 1.1 for systems $[E, A, B, C]$ is the following.

COROLLARY 1.2 (Decoupled DAE).

Let $[E, A, B, C] \in \Sigma_{n,m}$. Then there exist $W, T \in \mathbf{GL}_n(\mathbb{R})$ such that

$$[E, A, B, C] \stackrel{W,T}{\sim} \left[\begin{bmatrix} I_{n_s} & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} A_s & 0 \\ 0 & I_{n_f} \end{bmatrix}, \begin{bmatrix} B_s \\ B_f \end{bmatrix}, [C_s \quad C_f] \right], \quad (1.5)$$

for some $B_s \in \mathbb{R}^{n_f, m}$, $B_f \in \mathbb{R}^{n_f, m}$, $C_s \in \mathbb{R}^{m, n_s}$, $C_f \in \mathbb{R}^{m, n_s}$, $A_s \in \mathbb{R}^{n_s, n_s}$ and nilpotent $N \in \mathbb{R}^{n_f, n_f}$. \diamond

The form (1.5) is interpreted, in terms of the DAE (1.1), as follows:

$$(x, u, y) \in \mathfrak{B}_{(1.1)} \quad \text{if, and only if,} \quad \begin{pmatrix} x_s(\cdot) \\ x_f(\cdot) \end{pmatrix} := Tx(\cdot)$$

solves the decoupled DAEs

$$\begin{aligned} \dot{x}_s(t) &= A_s x_s(t) + B_s u(t) \\ y_s(t) &= C_s x_s(t), \end{aligned} \quad (1.6a)$$

$$\begin{aligned} N\dot{x}_f(t) &= x_f(t) + B_f u(t) \\ y_f(t) &= C_f x_f(t), \end{aligned} \quad (1.6b)$$

$$y(t) = y_s(t) + y_f(t). \quad (1.6c)$$

If $(x, u, y) \in \mathfrak{B}_{(1.1)}$ and in addition $u \in \mathcal{C}^{\nu-1}(\mathbb{R}; \mathbb{R}^m)$, then by repeated multiplication of (1.6b) by N from the left, differentiation, and using the identity

$$(sN - I_{n_f})^{-1} = -I_{n_f} - sN - s^2N^2 - \dots - s^{\nu-1}N^{\nu-1}, \quad (1.7)$$

where ν is the index of nilpotency of N , it is easy to see that the solution of (1.6b) satisfies

$$x_f(\cdot) = - \sum_{k=0}^{\nu-1} N^k B_f u^{(k)}(\cdot). \quad (1.8)$$

1.2. System properties.

1.2.1. Zero dynamics. An essential concept in the present article is the *zero dynamics* of system (1.1):

$$\mathcal{ZD}_{(1.1)} := \{ (x, u, y) \in \mathfrak{B}_{(1.1)} \mid y = 0 \}.$$

By linearity of (1.1), $\mathcal{ZD}_{(1.1)}$ is a real vector space.

The zero dynamics of system (1.1) is called *asymptotically stable* if, and only if, $\forall \varepsilon > 0 \exists \delta > 0 \forall (x, u, y) \in \mathcal{ZD}_{(1.1)}$ s.t. $\|(x(0), u(0))\| < \delta \forall t \geq 0 : \|(x(t), u(t))\| < \varepsilon$ and

$$\forall (x, u, y) \in \mathcal{ZD}_{(1.1)} : \lim_{t \rightarrow \infty} (x(t), u(t)) = 0.$$

(Asymptotically stable) zero dynamics is the vector space of those trajectories of the system which are, loosely spoken, not visible at the output (and tend to zero).

1.2.2. Relative degree. Another fundamental concept is the *relative degree* of the transfer function $G(s) = C(sE - A)^{-1}B$ of (1.1).

A rational matrix function $G(s) \in \mathbb{R}(s)^{m,m}$ is called

$$\begin{aligned} \text{proper} & : \iff \lim_{s \rightarrow \infty} G(s) = D \text{ for some } D \in \mathbb{R}^{m,m} \\ \text{strictly proper} & : \iff \lim_{s \rightarrow \infty} G(s) = 0, \end{aligned}$$

and we say that the square rational matrix $G(s) \in \mathbb{R}(s)^{m,m}$ has *strict relative degree* $\rho \in \mathbb{Z}$ if, and only if,

$$\rho = \text{sr deg } G(s) := \sup \left\{ k \in \mathbb{Z} \mid \lim_{s \rightarrow \infty} s^k G(s) \in \mathbf{Gl}_m(\mathbb{R}) \text{ and } \lim_{s \rightarrow \infty} s^{k-1} G(s) = 0 \right\}$$

exists. Note that for any $G(s) \in \mathbb{R}(s)^{m,m}$ we have (consider the entries)

$$\lim_{s \rightarrow \infty} s^{k-1} G(s) = 0 \text{ for some } k \in \mathbb{Z} \implies \lim_{s \rightarrow \infty} s^{k-i} G(s) = 0 \text{ for all } i \in \mathbb{N}.$$

For convenience, we also say that $[E, A, B, C] \in \Sigma_{n,m}$ has strict relative degree ρ if $G(s) = C(sE - A)^{-1}B$ has strict relative degree ρ . Then the matrix

$$\Gamma := \lim_{s \rightarrow \infty} s^\rho G(s) \in \mathbf{Gl}_m(\mathbb{R})$$

is called *high frequency gain matrix*.

If $g(s) = p(s)/q(s)$, for $p(s) \in \mathbb{R}[s]$ and $q(s) \in \mathbb{R}[s] \setminus \{0\}$, is a scalar rational function, then the strict relative degree always exists and coincides with the well-known definition of relative degree:

$$\text{sr deg } g(s) = \text{deg } q(s) - \text{deg } p(s).$$

1.2.3. Controllability and observability. We recall different concepts of controllability and observability for DAEs (1.1). For brevity, we do not define the concepts in system theoretic terms but recall algebraic characterizations in Proposition 1.3; the

latter will be used in our proofs. Different notions of controllability and observability at infinity are used in the literature: [5] compares the algebraically formulated controllability/observability concepts of [17] and [19]; we go along with those in [17]. For system theoretic notions of the concepts see [6, Secs. 2 & 3].

PROPOSITION 1.3 (Controllability and observability).

A system (1.1) is

- (i) R -controllable $\iff \text{rk}[sE - A, B] = n$ for all $s \in \mathbb{C}$
- (ii) stabilizable $\iff \text{rk}[sE - A, B] = n$ for all $s \in \overline{\mathbb{C}}_+$
- (iii) controllable at infinity $\iff \text{rk}[E, B] = n$
- (iv) controllable \iff it is R -controllable and controllable at infinity
- (v) R -observable $\iff \text{rk}[sE^\top - A^\top, C^\top] = n$ for all $s \in \mathbb{C}$
- (vi) detectable $\iff \text{rk}[sE^\top - A^\top, C^\top] = n$ for all $s \in \overline{\mathbb{C}}_+$
- (vii) observable at infinity $\iff \text{rk}[E^\top, C^\top] = n$
- (viii) observable \iff it is R -observable and observable at infinity.

The properties (i)-(viii) are invariant under system equivalence.

1.2.4. Poles and zeros. Finally, we recall the definition of transmission zeros and poles of a transfer function; see for example [15, Sec. 6.5].

Let $G(s) \in \mathbb{R}(s)^{m,m}$ with *Smith-McMillan form*

$$U^{-1}(s)G(s)V^{-1}(s) = \text{diag} \left(\frac{\varepsilon_1(s)}{\psi_1(s)}, \dots, \frac{\varepsilon_r(s)}{\psi_r(s)}, 0, \dots, 0 \right) \in \mathbb{R}(s)^{m,m},$$

where $U(s), V(s) \in \mathbb{R}[s]^{m,m}$ are unimodular (i.e. invertible over $\mathbb{R}[s]^{m,m}$), $\text{rk } G(s) = r$, $\varepsilon_i(s), \psi_i(s) \in \mathbb{R}[s]$ are monic, coprime and satisfy $\varepsilon_i(s) \mid \varepsilon_{i+1}(s)$, $\psi_{i+1}(s) \mid \psi_i(s)$ for all $i = 1, \dots, r-1$. $s_0 \in \mathbb{C}$ is called *transmission zero* of $G(s)$ if $\varepsilon_r(s_0) = 0$ and a *pole* of $G(s)$ if $\psi_1(s_0) = 0$.

Note that by (1.3) the transmission zeros and poles are invariant under system equivalence.

1.3. System classes. We introduce the two main system classes investigated in the present article. In both of them, an explicit feedthrough term is not allowed. However, this assumption is only made for technical reasons. We also explain how systems with feedthrough are related to the two system classes under consideration.

1.3.1. Systems with positive strict relative degree. Consider the class of DAE systems (1.1) such that $[E, A, B, C] \in \Sigma_{n,m}$ has positive strict relative degree.

If $[E, A, B, C]$ is single-input, single-output, i.e. $m = 1$, then its transfer function $g(s) = C(sE - A)^{-1}B = p(s)/q(s)$ has positive strict relative degree if, and only if, $\deg q(s) > \deg p(s)$ or, equivalently, $g(s)$ is strictly proper.

Note that for any $G(s) \in \mathbb{R}(s)^{m,m}$ we have

$$G(s) \text{ has positive strict relative degree} \begin{array}{c} \implies \\ \not\Leftarrow \\ \text{i.g.} \end{array} G(s) \text{ is strictly proper.}$$

An ODE system $[E, A, B, C] = [I, A, B, C]$ has transfer function

$$G(s) = C(sI - A)^{-1}B = CBs^{-1} + CABs^{-2} + CA^2Bs^{-3} + \dots$$

and therefore strict relative degree $\rho \in \mathbb{N}$, if, and only if,

$$\det(CA^{\rho-1}B) \neq 0 \quad \text{and, if } \rho > 1, \quad \forall k = 0, \dots, \rho - 2 : CA^k B = 0.$$

1.3.2. Systems with proper inverse transfer function. The second class of DAE systems (1.1) considered in the present article are those DAEs $[E, A, B, C] \in \Sigma_{n,m}$ whose transfer function $G(s) = C(sE - A)^{-1}B \in \mathbb{R}(s)^{m,m}$ has *proper inverse* over $\mathbb{R}(s)^{m,m}$, i.e. $G(s)$ is invertible over $\mathbb{R}(s)$ and $\lim_{s \rightarrow \infty} G^{-1}(s) \in \mathbb{R}^{m,m}$.

In [2, Prop. 1.2] we have shown that for any $G(s) \in \mathbb{R}(s)^{m,m}$ the properties of proper inverse transfer function and strict relative degree are related as follows:

$$\text{sr deg } G(s) \leq 0 \quad \begin{array}{c} \implies \\ \not\Leftarrow \\ \text{i.g.} \end{array} \quad G(s) \text{ has proper inverse.}$$

A zero dynamics form for systems $[E, A, B, C] \in \Sigma_{n,m}$ with transfer function $G(s) \in \mathbb{R}(s)^{m,m}$ whose inverse $G^{-1}(s)$ is proper, is studied in Theorem 2.3. Conversely, if a proper transfer function $G^{-1}(s) \in \mathbb{R}(s)^{m,m}$ is given, then this may be realized as an ODE $[I, A, B, C, D]$ with feedthrough term D and it is easily verified that the regular DAE

$$\left[\begin{array}{cc} 0 & 0 \\ 0 & I \end{array} \right], \left[\begin{array}{cc} -D & -C \\ B & A \end{array} \right], \left[\begin{array}{c} I \\ 0 \end{array} \right], [I, 0] \quad (1.9)$$

without feedthrough term is a realization of $G(s)$. Furthermore, the Hautus criterion yields that (1.9) is minimal if, and only if, $[I, A, B, C, D]$ is minimal.

If $[E, A, B, C]$ is single-input, single-output, then its transfer function $g(s) = C(sE - A)^{-1}B = p(s)/q(s)$ has proper inverse if, and only if, $\deg q(s) \leq \deg p(s)$ or, equivalently, $g(s)$ is not strictly proper. In this case, there does not exist a realization of $g(s)$ as an ODE system $[I, A, B, C]$ without feedthrough term. The class of single-input single-output DAE systems $\Sigma_{n,1}$ can be decomposed into the disjoint sets of systems with strictly proper transfer function and with non strictly proper transfer function $g(s) = C(sE - A)^{-1}B \in \mathbb{R}(s)$, resp.:

$$\begin{aligned} \Sigma_{n,m} &= \{ [E, A, B, C] \in \Sigma_{n,1} \mid g(s) = \frac{p(s)}{q(s)}, \deg q(s) > \deg p(s) \} \\ &\dot{\cup} \{ [E, A, B, C] \in \Sigma_{n,1} \mid g(s) = \frac{p(s)}{q(s)}, \deg q(s) \leq \deg p(s) \} \end{aligned}$$

For multi-input, multi-output systems $\Sigma_{n,m}$, generalization to systems with positive strict relative degree and systems with proper inverse transfer function does not allow for such a disjoint union. This is the topic of the following section.

1.3.3. Systems which have neither positive strict relative degree nor proper inverse transfer function. The class of multi-input, multi-output systems $[E, A, B, C] \in \Sigma_{n,m}$ with transfer function $G(s) = C(sE - A)^{-1}B \in \mathbb{R}(s)^{m,m}$ can be

decomposed in the disjoint union of the following three sets:

$$\begin{aligned} \Sigma_{n,m} &= \{ [E, A, B, C] \in \Sigma_{n,m} \mid G(s) \text{ has positive strict relative degree} \} \\ &\dot{\cup} \{ [E, A, B, C] \in \Sigma_{n,m} \mid G(s) \text{ has proper inverse transfer function} \} \\ &\dot{\cup} \left\{ [E, A, B, C] \in \Sigma_{n,m} \mid \begin{array}{l} G(s) \text{ has neither positive strict relative degree} \\ \text{nor proper inverse transfer function} \end{array} \right\}. \end{aligned}$$

Note that the latter set is, for multi-input, multi-output systems, not empty. It contains in particular systems with a non-constant vector relative degree and it contains, for example for $m = 2$, the system

$$[E, A, B, C] := \left[\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right]$$

with transfer function $G(s) = \begin{bmatrix} s & 0 \\ 0 & s^{-1} \end{bmatrix} \in \mathbb{R}(s)^{2,2}$. We stress that $G(s)$ has neither a positive strict relative degree nor a proper inverse.

1.3.4. Systems with feedthrough. So far we have not considered systems with non-zero feedthrough term. In this section we will show that such systems can, by augmenting the state by the feedthrough term, be rewritten as a system of form (1.1); the transfer function of the new system coincides with the transfer function of the original system. Furthermore, it is shown that a system with proper transfer function can be rewritten as an ODE with feedthrough.

REMARK 1.4 (DAE with feedthrough \longrightarrow DAE without feedthrough).
Consider a DAE system with feedthrough of the form

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1.10}$$

where $E, A \in \mathbb{R}^{n,n}$, $B, C^\top \in \mathbb{R}^{n,m}$, $D \in \mathbb{R}^{m,m}$.

Then the behaviour $\mathfrak{B}_{(1.10)}$, the zero dynamics $\mathcal{ZD}_{(1.10)}$ and its asymptotic stability can be defined completely analogously as for (1.1); this is omitted.

By introducing an ‘augmented state’ of $x(t)$ and $w(t) = Du(t)$, system (1.10) can be rewritten as a DAE system without feedthrough term, namely

$$\begin{aligned} \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{pmatrix} x(t) \\ w(t) \end{pmatrix} &= \begin{bmatrix} A & 0 \\ 0 & -I \end{bmatrix} \begin{pmatrix} x(t) \\ w(t) \end{pmatrix} + \begin{bmatrix} B \\ D \end{bmatrix} u(t) \\ y(t) &= [C \quad I] \begin{pmatrix} x(t) \\ w(t) \end{pmatrix}. \end{aligned} \tag{1.11}$$

Clearly, the behaviours and zero dynamics are related in the following way

$$\begin{aligned} \mathfrak{B}_{(1.11)} &= \{ ((x^\top, w^\top)^\top, u, y) \mid (x, u, y) \in \mathfrak{B}_{(1.10)}, w = Du \}, \\ \mathcal{ZD}_{(1.11)} &= \{ ((x^\top, w^\top)^\top, u, y) \mid (x, u, y) \in \mathcal{ZD}_{(1.10)}, w = Du \}. \end{aligned}$$

The following observations can now be made:

a) The transfer functions of (1.10) and (1.11) coincide:

$$C(sE - A)^{-1}B + D = [C \quad I] \left(s \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A & 0 \\ 0 & -I \end{bmatrix} \right)^{-1} \begin{bmatrix} B \\ D \end{bmatrix}.$$

b) The zero dynamics $\mathcal{ZD}_{(1.10)}$ is asymptotically stable if, and only if, the zero dynamics $\mathcal{ZD}_{(1.11)}$ is asymptotically stable.

c) The pencil $s \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A & 0 \\ 0 & -I \end{bmatrix}$ is regular if, and only if, $sE - A$ is regular.

d) We have

$$\begin{aligned} \det \left[\begin{array}{cc|c} sE - A & 0 & B \\ 0 & I & D \\ \hline C & I & 0 \end{array} \right] &= \det \begin{bmatrix} sE - A & 0 & B \\ 0 & I & D \\ C & 0 & -D \end{bmatrix} = \det \begin{bmatrix} sE - A & 0 & B \\ 0 & I & 0 \\ C & 0 & -D \end{bmatrix} \\ &= \det \begin{bmatrix} sE - A & B \\ C & -D \end{bmatrix}. \end{aligned}$$

e) Since, for all $s \in \mathbb{C}$,

$$\operatorname{rk} \begin{bmatrix} sE - A & 0 & B \\ 0 & I & D \end{bmatrix} = \operatorname{rk} [sE - A \quad B] + m, \quad \operatorname{rk} \begin{bmatrix} sE - A & 0 \\ 0 & I \\ C & I \end{bmatrix} = \operatorname{rk} \begin{bmatrix} sE - A \\ C \end{bmatrix} + m,$$

it is easily seen that system (1.10) is stabilizable (detectable) if, and only if, system (1.11) is stabilizable (detectable).

f) If $E = I$, then (1.10) is an ODE with feedthrough and can be rewritten as a DAE with regular matrix pencil and transfer function $G(s) = C(sI - A)^{-1}B + D$, which coincides with the transfer function of the original ODE system. \diamond

REMARK 1.5 (DAE with proper transfer function \longrightarrow ODE with feedthrough).
For any DAE (1.1) with regular pencil $sE - A \in \mathbb{R}[s]^{n,n}$ and $\lim_{s \rightarrow \infty} C(sE - A)^{-1}B \in \mathbb{R}^{m,m}$ there exists an ODE system with feedthrough

$$\begin{aligned} \dot{z}(t) &= A_s z(t) + B_s u(t) \\ y(t) &= C_s z(t) - C_f B_f u(t), \end{aligned} \tag{1.12}$$

for some $B_s \in \mathbb{R}^{n_f, m}$, $B_f \in \mathbb{R}^{n_f, m}$, $C_s \in \mathbb{R}^{p, n_s}$, $C_f \in \mathbb{R}^{p, n_s}$, $A_s \in \mathbb{R}^{n_s, n_s}$ such that the transfer functions of (1.1) and (1.12) coincide:

$$C(sE - A)^{-1}B = C_s(sI - A_s)^{-1}B_s + C_f B_f.$$

This can be seen as follows: Without restriction of generality, one may assume that (1.1) is in Weierstraß form (see Proposition 1.1) and given by equations (1.6a)-(1.6c). Then its transfer function satisfies

$$\begin{aligned} C(sE - A)^{-1}B &= C_f (sN - I_{n_f})^{-1} B_f + C_s (sI_{n_s} - A_s)^{-1} B_s \\ &= - \sum_{i=0}^{\nu-1} C_f N^i B_f s^i + \sum_{i \geq 1} C_s A_s^{i-1} B_s s^{-i} \end{aligned}$$

and the assumption $\lim_{s \rightarrow \infty} C(sE - A)^{-1}B \in \mathbb{R}^{m,m}$ yields $C_f N^i B_f = 0$ for all $i \geq 1$. Now invoking (1.8) and setting $z = x_s$ in (1.12) proves the claim. \diamond

1.4. Control objectives. We consider two control strategies with respect to the two system classes: for DAE systems with positive strict relative degree, a constant proportional derivative output feedback is suggested; for DAE systems with proper inverse transfer function, a constant proportional output feedback is considered. These controllers achieve high-gain stabilization and exploit the underlying system properties. However, since no information on the system entries will be required it is unknown how “high” the feedback gain has to be chosen. To resolve this, we allow for funnel control, that is a proportional time-varying high-gain error feedback in combination with a filter (the filter “adjusts” the higher relative degree) for DAE systems with positive strict relative degree, and a proportional time-varying high-gain error feedback for DAE systems with proper inverse transfer function.

1.4.1. Constant high-gain control. First, constant high-gain proportional output feedback is given by

$$u(t) = -k \cdot p\left(\frac{d}{dt}\right) y(t) \quad \begin{array}{l} \text{for } [E, A, B, C] \in \Sigma_{n,m} \\ \text{with positive strict relative degree} \end{array} \quad (1.13)$$

$$u(t) = k \cdot y(t) \quad \begin{array}{l} \text{for } [E, A, B, C] \in \Sigma_{n,m} \\ \text{with proper inverse transfer function} \end{array} \quad (1.14)$$

where $k > 0$ and $p(s) \in \mathbb{R}[s]$ Hurwitz in (1.13), and $k \in \mathbb{R}$ in (1.14). However, these results are of more theoretical nature to show the consequences of the underlying systems properties such as asymptotically stable zero dynamics.

The control objective is output feedback regulation in the sense that the *high-gain controller*, i.e. the time-invariant proportional output-derivative feedback (1.13) or (1.14) resp., yields an asymptotically stable closed-loop system. Here k is the high-gain parameter and, in general, k has to be large to achieve that the state $x(t)$ decays to zero.

In Section 4 we will show that the assumption of asymptotically stable zero dynamics of a system (1.1) which has either positive strict relative degree or proper inverse transfer function implies high-gain stabilizability in the following sense: If the system has positive strict relative degree, $k \gg 1$, and $p(s)$ is Hurwitz with positive leading coefficient, then the high-gain feedback (1.13) or (1.14) resp., applied to (1.1) forces the output “very quickly” to zero. If the system has proper inverse transfer function, then it is sufficient to assume $|k| \gg 1$ and $p(s) \equiv 1$ to achieve this goal.

1.4.2. Funnel control. The *control objective* is output feedback regulation in the sense that the *funnel controller*, applied to any system $[E, A, B, C] \in \Sigma_{n,m}$ achieves tracking of the output of any reference signal $y_{\text{ref}} \in \mathcal{B}^{\nu+1}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$, where ν is the index of the $sE - A$, with pre-specified transient behaviour:

$$\forall t > 0: \|e(t)\| < 1/\varphi(t).$$

The transient behaviour is pre-specified by a funnel boundary $1/\varphi(\cdot)$ given by

$$\varphi \in \Phi^\mu := \left\{ \varphi \in \mathcal{B}^\mu(\mathbb{R}_{\geq 0}, \mathbb{R}) \mid \varphi(0) = 0, \varphi(s) > 0 \text{ for all } s > 0 \text{ and } \liminf_{s \rightarrow \infty} \varphi(s) > 0 \right\},$$

with which we associate, see Fig. 1.1, the *performance funnel*

$$\mathcal{F}_\varphi := \left\{ (t, e) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \mid \varphi(t)\|e\| < 1 \right\}, \quad (1.15)$$

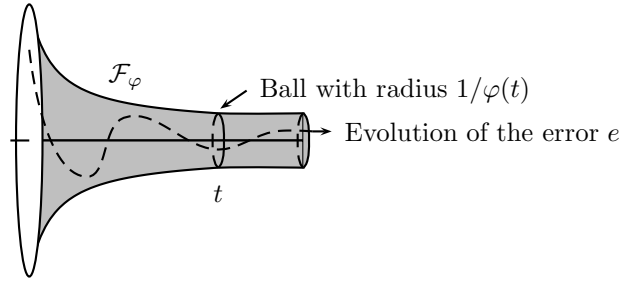


Fig. 1.1: Error evolution in the funnel \mathcal{F}_φ with “width ∞ ” at $t = 0$, i.e. $\varphi(0) = 0$

Note that there is no need that the funnel is monotone. However, the funnel is not allowed to “shrink” to zero as $t \rightarrow \infty$. But the minimal distance (at infinity) between the funnel boundary and zero, i.e. $\lambda = \liminf_{s \rightarrow \infty} \varphi(s)^{-1}$, must be positive albeit can be arbitrarily small; from a practical point of view, this is not a limitation.

The funnel controller takes two forms. For DAE systems $[E, A, B, C] \in \Sigma_{n,m}$ with positive strict relative degree the higher degree is an obstacle; in (1.13) we have used derivative feedback while now we will incorporate a filter. This idea goes back to ODEs, it is shown in [12] that funnel control is feasible if a filter is incorporated in the feedback. This filter is constructed as follows

$$\dot{z}(t) = \underbrace{\begin{bmatrix} -I_m & I_m & 0 & \cdots & 0 & 0 \\ 0 & -I_m & I_m & \cdots & 0 & 0 \\ 0 & 0 & -I_m & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -I_m & I_m \\ 0 & 0 & 0 & \cdots & 0 & -I_m \end{bmatrix}}_{=:F_\rho} z(t) + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ I_m \end{bmatrix}}_{=:G_\rho} u(t), \quad z(0) = z^0 \quad (1.16)$$

with initial data $z^0 \in \mathbb{R}^{(\rho-1)m}$. The feedback law is defined recursively by the C^∞ -functions

$$\begin{aligned} \gamma_1 : \quad & \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}^m, \\ & (k, e) \mapsto k e, \\ \gamma_2 : \quad & \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}^m, \\ & (k, e, z_1) \mapsto \gamma_1(k, e) + \|D\gamma_1(k, e)\|^2 k^4 (1 + \|z_1\|^2) (z_1 + \gamma_1(k, e)) \end{aligned}$$

and, for $i = 3, \dots, \rho$,

$$\begin{aligned} \gamma_i : \quad & \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^{(i-1)m} \rightarrow \mathbb{R}^m, \quad (k, e, (z_1, \dots, z_{i-1})) \mapsto \\ & \gamma_{i-1}(k, e, (z_1, \dots, z_{i-2})) + \|D\gamma_{i-1}(k, e, (z_1, \dots, z_{i-2}))\|^2 k^4 (1 + \|(k, e, (z_1, \dots, z_{i-1}))\|^2) \\ & \quad \times (z_{i-1} + \gamma_{i-1}(k, e, (z_1, \dots, z_{i-2}))), \end{aligned}$$

where D denotes the derivative (Jacobian matrix). For a lengthy discussion of the intuition for the filter see [12]. Now the funnel controller (with filter (1.16)) for systems $[E, A, B, C] \in \Sigma_{n,m}$ with positive strict relative degree $\rho \in \mathbb{N}$ takes the form

$$\boxed{\begin{aligned} u(t) &= -\gamma_\rho(k(t), e(t), z(t)), & e(t) &= y(t) - y_{\text{ref}}(t), \\ \dot{z}(t) &= F_\rho z(t) + G_\rho u(t), & k(t) &= \frac{1}{1 - \varphi(t)^2 \|e(t)\|^2}. \end{aligned}} \quad (1.17)$$

The funnel controller for systems $[E, A, B, C] \in \Sigma_{n,m}$ with proper inverse transfer function is much simpler: For appropriate chosen gain parameter $\hat{k} \in \mathbb{R} \setminus \{0\}$ it is

$$\boxed{\begin{aligned} u(t) &= -k(t) e(t), & e(t) &= y(t) - y_{\text{ref}}(t), \\ k(t) &= \frac{\hat{k}}{1 - \varphi(t)^2 \|e(t)\|^2}. \end{aligned}} \quad (1.18)$$

In Section 5 we will show that the assumption of asymptotically stable zero dynamics of a system (1.1) which has either positive strict relative degree or proper inverse transfer function implies feasibility of the funnel controller. In view of the fact that such systems are high-gain stabilizable (see Section 4), intuitively we may believe that if $\|e(t)\|$ is close to the funnel boundary $\varphi(t)^{-1}$, then the high-gain $k(t)$ forces $\|e(t)\|$ away from the funnel boundary. This is the essential property to allow for funnel control of these systems: $k(t)$ is designed in such a way that it is large if the the error $\|e(t)\|$ is close to the funnel boundary $\varphi(t)^{-1}$, hence avoiding contact.

The funnel controller is a high-gain adaptive controller which is not high-gain and not adaptive: Certainly, the gain $k(t)$ in (1.17) and (1.18) takes high values if $k(t)$ is “close” to the funnel boundary; but it is by no means monotone and exploits the high-gain property of the system class (i.e. asymptotically stable zero dynamics) only if necessary. The gain $k(t)$ is adapted by the output, more precisely by the distance of the norm of the output error to the funnel boundary.

1.5. Literature and outline. The present paper is based on the following precursors: The Byrnes-Isidori form for ODE systems with strictly proper transfer function is derived in [13], see also [14, Sec. 5.1]. The zero dynamics form for DAE systems with proper inverse transfer function is derived in [2]. The concept of funnel control is introduced in [11]; for a further discussion of funnel control see the survey article [10].

In this article we consider two classes of linear multi-input, multi-output DAE systems: systems with positive strict relative degree and systems with proper inverse transfer function; in the single-input single-output case, these two disjoint classes are the whole of all linear DAEs without feedthrough term. In Section 2, zero dynamics forms for these system classes are derived, they are the counterpart to the Byrnes-Isidori form for ODE systems. The normal forms may be interesting in their own right and are here essential for later proofs of funnel control. In Section 3, parametrization of the zero dynamics and characterization of asymptotically zero dynamics in terms of a matrix condition, the zero dynamics form, and the transfer function are derived. In Section 4, it is shown that a system with asymptotically stable zero dynamics is high-gain stabilizable with constant gain (in case of proper inverse transfer function) and constant gain in combination with a filter (in case of positive strict relative degree of the transfer function). In Section 5 we are ready to present two funnel controllers for the two system classes. These funnel controllers achieve tracking of a reference signal (belonging to a rather large class) within a pre-specified funnel; this means in particular that transient behaviour of the output error can be pre-specified and the funnel controller does not incorporate any internal model but is simple in its design.

These results are illustrated in Section 6 by the position control and velocity control of a mechanical system consisting of springs, masses, and dampers. The proofs are delegated to Section 7.

2. Normal forms. The notion ‘normal form’ means that it is weaker than ‘canonical form’: Recall that if a set (of systems) may be divided into equivalence classes and if each orbit (i.e. equivalence class) has a unique representative, then the mapping from the set to each representative within the orbit is called a *canonical form*. The Byrnes-Isidori form for ODE state space systems with strictly proper transfer function (see [10] and also [14, Sec. 5.1]) is not a canonical form. However, the freedom left within the non-zero entries of the Byrnes-Isidori form can be specified and is not significant; this justifies the notion ‘normal form’.

In the present section, certain normal forms, the so called zero dynamics forms, are studied for two classes of DAE systems with positive strict relative degree and with proper inverse transfer function. They are interesting in their own right since they give insight in the properties of the systems class, we will use them to derive results on zero dynamics and on high-gain output feedback and funnel control.

We first present a zero dynamics form (2.1) for descriptor systems with positive strict relative degree, see Section 1.3.1.

THEOREM 2.1 (Zero dynamics form for systems with positive strict relative degree).

Suppose $[E, A, B, C] \in \Sigma_{n,m}$ has strict relative degree $\rho \in \mathbb{N}$. Then there exist $W, T \in \mathbf{GL}_n(\mathbb{R})$ such that

$$[E, A, B, C] \stackrel{W,T}{\sim} [\hat{E}, \hat{A}, \hat{B}, \hat{C}]$$

with

$$\hat{A} = \left[\begin{array}{cccccc|cc} 0 & I_m & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & I_m & 0 & & 0 & 0 & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & I_m & 0 & 0 & 0 \\ R_1 & R_2 & \cdots & R_{\rho-1} & R_\rho & S & 0 & 0 \\ P & 0 & \cdots & 0 & 0 & Q & 0 & 0 \\ \hline 0 & 0 & \cdots & 0 & 0 & 0 & I_{n_c} & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & I_{n_{\bar{c}}} \end{array} \right], \quad \hat{B} = \left[\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \\ \Gamma \\ 0 \\ \hline B_c \\ 0 \end{array} \right], \quad (2.1)$$

$$\hat{E} = \left[\begin{array}{cccccc|cc} I_m & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 \\ 0 & I_m & 0 & & & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots \\ 0 & & 0 & I_m & 0 & 0 & 0 & 0 \\ 0 & & & 0 & I_m & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & I_\mu & 0 & 0 \\ \hline 0 & 0 & \cdots & 0 & 0 & 0 & N_c & N_{c\bar{c}} \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & N_{\bar{c}} \end{array} \right], \quad \hat{C} = \left[\begin{array}{c} I_m \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ C_{\bar{c}}^\top \end{array} \right]^\top,$$

where, for some $n_c, n_{\bar{c}} \in \mathbb{N}_0$, and $\mu = n - n_c - n_{\bar{c}} - \rho m$,

$$\begin{aligned} \Gamma &= \lim_{s \rightarrow \infty} s^\rho C(sE - A)^{-1}B \in \mathbf{GI}_m(\mathbb{R}) \text{ is the high frequency gain matrix,} \\ S &\in \mathbb{R}^{m \times \mu}, P \in \mathbb{R}^{\mu \times m}, Q \in \mathbb{R}^{\mu \times \mu}, [R_1, \dots, R_\rho] \in \mathbb{R}^{m \times (\rho m)}, \\ B_c &\in \mathbb{R}^{n_c \times m}, C_{\bar{c}} \in \mathbb{R}^{m \times n_{\bar{c}}}, N_{c\bar{c}} \in \mathbb{R}^{n_c \times n_{\bar{c}}} \\ N_c &\in \mathbb{R}^{n_c \times n_c}, N_{\bar{c}} \in \mathbb{R}^{n_{\bar{c}} \times n_{\bar{c}}} \text{ are nilpotent, and } \text{rk}[N_c, B_c] = n_c. \end{aligned} \quad (2.2)$$

The entries R_1, \dots, R_ρ are unique, system $[I, Q, P, S]$ is unique up to system equivalence $\hat{T}^{-1} \sim \hat{T}$, and the matrices N_c and $N_{\bar{c}}$ are unique up to similarity.

If E is invertible, then $n_c = n_{\bar{c}} = 0$, this means only the upper left block in \hat{E} is present.

The form (2.1) is called zero dynamics form of (1.1). The transfer function satisfies, where $R_{\rho+1} := -I$,

$$C(sE - A)^{-1}B = - \left[\sum_{i=1}^{\rho+1} R_i s^{i-1} + S(sI_{n-n_c-n_{\bar{c}}-\rho m} - Q)^{-1}P \right]^{-1} \Gamma. \quad (2.3)$$

The proof is in Appendix 7.

REMARK 2.2 (Zero dynamics form for DAEs).

An immediate consequence of Theorem 2.1 is the simplified representation of system $[E, A, B, C] \in \Sigma_{n,m}$: If ν denotes the index of $sE - A$, then a trajectory satisfies

$$(x, u, y) \in \mathfrak{B}_{(1.1)} \cap (\mathcal{C}^1(\mathbb{R}; \mathbb{R}^n) \times \mathcal{C}^{\nu-1}(\mathbb{R}; \mathbb{R}^m) \times \mathcal{C}^\rho(\mathbb{R}; \mathbb{R}^m))$$

if, and only if, $Tx = \left(y^\top, \dot{y}^\top, \dots, y^{(\rho-1)\top}, \eta^\top, x_c^\top, x_{\bar{c}}^\top \right)^\top \in \mathcal{C}^1(\mathbb{R}; \mathbb{R}^n)$ fulfills

$$\boxed{\begin{aligned} y^{(\rho)}(t) &= \sum_{i=1}^{\rho} R_i y^{(i-1)}(t) + S\eta(t) + \Gamma u(t) \\ \dot{\eta}(t) &= P y(t) + Q \eta(t) \\ x_c(t) &= - \sum_{i=0}^{\nu-1} N_c^i B_c u^{(i)}(t) \\ x_{\bar{c}}(t) &= 0. \end{aligned}} \quad (2.4)$$

See Figure 2.1. ◇

The second zero dynamics form is for systems $[E, A, B, C] \in \Sigma_{n,m}$ where the transfer function $C(sE - A)^{-1}B$ has proper inverse, see Section 1.3.2. This is the counterpart to (2.1).

THEOREM 2.3 (Zero dynamics form for systems with proper inverse transfer function [2, Thm. 2.3]).

Suppose $[E, A, B, C] \in \Sigma_{n,m}$ is such that $C(sE - A)^{-1}B$ has proper inverse. Then there exist $W, T \in \mathbf{GI}_n(\mathbb{R})$ such that

$$[E, A, B, C] \stackrel{W, T}{\sim} [\hat{E}, \hat{A}, \hat{B}, \hat{C}]$$

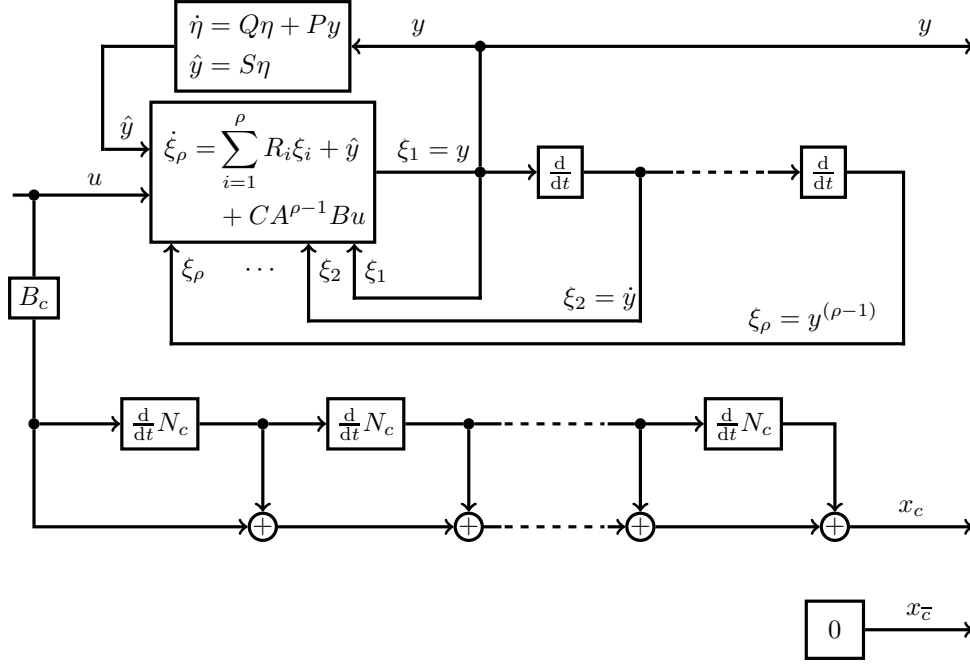


Fig. 2.1: Zero dynamics form for systems with positive strict relative degree

with

$$\hat{A} = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ A_{21} & Q & 0 & 0 \\ 0 & 0 & I_{n_c} & 0 \\ 0 & 0 & 0 & I_{n_{\bar{c}}} \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} I_m \\ 0_{n_2, m} \\ 0_{n_c, m} \\ 0_{n_{\bar{c}}, m} \end{bmatrix}, \quad (2.5)$$

$$\hat{E} = \begin{bmatrix} 0 & 0 & 0 & E_{\bar{c}} \\ 0 & I_{n_2} & 0 & 0 \\ E_c & 0 & N_c & N_{c\bar{c}} \\ 0 & 0 & 0 & N_{\bar{c}} \end{bmatrix}, \quad \hat{C} = [I_m, 0_{m, n_2}, 0_{m, n_c}, 0_{m, n_{\bar{c}}}],$$

where, for some $n_2, n_c, n_{\bar{c}} \in \mathbb{N}_0$,

$$\begin{aligned} A_{11} &\in \mathbb{R}^{m, m}, \quad A_{12} \in \mathbb{R}^{m, n_2}, \quad A_{21} \in \mathbb{R}^{n_2, m}, \quad Q \in \mathbb{R}^{n_2, n_2}, \\ E_c &\in \mathbb{R}^{n_c, m}, \quad E_{\bar{c}} \in \mathbb{R}^{m, n_{\bar{c}}}, \quad N_{c\bar{c}} \in \mathbb{R}^{n_c, n_{\bar{c}}}, \\ N_c &\in \mathbb{R}^{n_c, n_c}, \quad N_{\bar{c}} \in \mathbb{R}^{n_{\bar{c}} \times n_{\bar{c}}} \text{ are nilpotent, and } \text{rk}[N_c, E_c] = n_c. \end{aligned} \quad (2.6)$$

The form (2.5) is called zero dynamics form of (1.1). Furthermore, the following holds:

- (i) $N_c^\nu = 0$ and $N_{\bar{c}}^\nu = 0$, where ν denotes the index of the pencil $sE - A$.
- (ii) the transfer function satisfies

$$C(sE - A)^{-1}B = -(A_{11} + A_{12}(sI_{n_2} - Q)^{-1}A_{21})^{-1}.$$

- (iii) $\text{sr deg}(C(sE - A)^{-1}B) = 0$, if, and only if, $A_{11} \in \mathbf{GL}_m(\mathbb{R})$.

- (iv) $\text{sr deg } (C(sE - A)^{-1}B) = \rho < 0$, if, and only if, $A_{11} = 0$ and $\text{sr deg } (A_{12}(sI_{n_2} - Q)^{-1}A_{21}) = -\rho$.

For uniqueness we have:

- (v) If $[E, A, B, C]$, $[\hat{E}, \hat{A}, \hat{B}, \hat{C}] \in \Sigma_{n,m,m}$ are both in zero dynamics form (2.5) and

$$[\hat{E}, \hat{A}, \hat{B}, \hat{C}] \stackrel{W,T}{\sim} [E, A, B, C] \quad \text{for some } W, T \in \mathbf{GL}_n(\mathbb{R}), \quad (2.7)$$

then there exist $W_{ii} \in \mathbf{GL}_{n_i}(\mathbb{R})$, $i \in \{2, c, \bar{c}\}$, $W_{c\bar{c}} \in \mathbb{R}^{n_c, n_{\bar{c}}}$ such that

$$W = T^{-1} = \begin{bmatrix} I_m & 0 & 0 & 0 \\ 0 & W_{22} & 0 & 0 \\ 0 & 0 & W_{cc} & W_{c\bar{c}} \\ 0 & 0 & 0 & W_{\bar{c}\bar{c}} \end{bmatrix}. \quad (2.8)$$

- (vi) The dimensions $n_2, n_c, n_{\bar{c}}$ are unique; the matrices $N_c, N_{\bar{c}}, Q$ are unique up to similarity, so in particular the spectrum of Q is unique; and $A_{11} = -\lim_{s \rightarrow \infty} (C(sE - A)^{-1}B)^{-1}$.

The proof is in [2, Theorem 2.3].

REMARK 2.4 (Zero dynamics form for DAE).

An immediate consequence of Theorem 2.3 is the representation as a DAE: If ν is the index of $sE - A$, then a trajectory satisfies

$$(x, u, y) \in \mathfrak{X}_{(1.1)} \cap (\mathcal{C}^1(\mathbb{R}; \mathbb{R}^n) \times \mathcal{C}^1(\mathbb{R}; \mathbb{R}^m) \times \mathcal{C}^\nu(\mathbb{R}; \mathbb{R}^m))$$

if, and only if, $Tx = (y, x_2^\top, x_c^\top, x_{\bar{c}}^\top)^\top \in \mathcal{C}^1(\mathbb{R}; \mathbb{R}^n)$ fulfills

$$\boxed{\begin{aligned} 0 &= A_{11} y(t) + A_{12} x_2(t) + u(t) \\ \dot{x}_2(t) &= A_{21} y(t) + Q x_2(t) \\ x_c(t) &= \sum_{i=0}^{\nu-1} N_c^i E_c y^{(i+1)}(t) \\ x_{\bar{c}}(t) &= 0. \end{aligned}} \quad (2.9)$$

See Figure 2.2. ◇

REMARK 2.5 (ODE systems with feedthrough).

For ODE systems with feedthrough, i.e. (1.10) with $E = I$, and strict relative degree zero we have $D \in \mathbf{GL}_m(\mathbb{R})$. In this case, by using the transcription of (1.10) as a DAE system (1.11), the zero dynamics form (2.9) can be achieved without changing the coordinates of the state, namely

$$\begin{aligned} 0 &= -D^{-1} y(t) + D^{-1} C x(t) + u(t) \\ \dot{x}(t) &= BD^{-1} y(t) + (A - BD^{-1} C) x(t). \end{aligned} \quad (2.10)$$

In the notation of (2.9), the quantities n_c and $n_{\bar{c}}$ are then trivial, while $n_2 = n$ and $A_{11} = -D^{-1}$, $A_{12} = D^{-1} C$, $A_{21} = A - BD^{-1} C$, and $Q = A - BD^{-1} C$. ◇

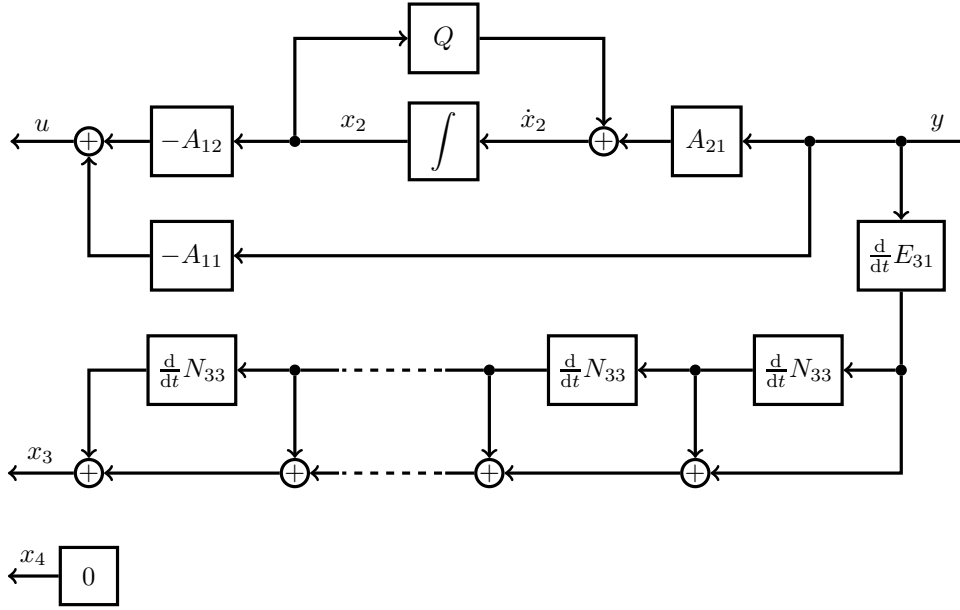


Fig. 2.2: Zero dynamics form for systems with proper inverse transfer function

3. Zero dynamics. Before we characterize asymptotically stable zero dynamics, we show that both zero dynamics forms allow to parameterize the zero dynamics.

REMARK 3.1 (Parametrization of zero dynamics).

- (i) Suppose that $[E, A, B, C] \in \Sigma_{n,m}$ has positive strict relative degree and let ν be the index of $sE - A$. Then the zero dynamics are given, in terms of the matrices in (2.1), by

$$\mathcal{ZD}_{(1.1)} = \left\{ \left(T^{-1} \begin{pmatrix} 0_{\rho m} \\ e^{Q \cdot} \eta^0 \\ \sum_{i=0}^{\nu-1} N_c^i B_c \Gamma^{-1} S Q^i e^{Q \cdot} \eta^0 \\ 0_{n_{\bar{c}}} \end{pmatrix}, -\Gamma^{-1} S e^{Q \cdot} \eta^0, 0 \right) \middle| \eta^0 \in \mathbb{R}^{\mu} \right\}. \quad (3.1)$$

- (ii) Suppose that $[E, A, B, C] \in \Sigma_{n,m}$ has proper inverse transfer function. Then the zero dynamics are given, in terms of (2.5), by

$$\mathcal{ZD}_{(1.1)} = \left\{ \left(T^{-1} \begin{pmatrix} 0_m \\ e^{Q \cdot} x_2^0 \\ 0_{n_c} \\ 0_{n_{\bar{c}}} \end{pmatrix}, -A_{12} e^{Q \cdot} x_2^0, 0 \right) \middle| x_2^0 \in \mathbb{R}^{n_2} \right\}. \quad (3.2)$$

◇

We are now in a position to characterize asymptotic stable zero dynamics of systems $[E, A, B, C] \in \Sigma_{n,m}$ with positive strict relative degree or proper inverse transfer function.

THEOREM 3.2 (Asymptotically stable zero dynamics).

Consider a system $[E, A, B, C] \in \Sigma_{n,m}$ and suppose it has positive strict relative degree or proper inverse transfer function. Then the following statements are, in terms of the system properties defined in Section 1.2, equivalent:

- (i) The zero dynamics of system (1.1) are asymptotically stable;
- (ii) $\forall s \in \overline{\mathbb{C}}_+ : \det \begin{bmatrix} sE - A & B \\ C & 0 \end{bmatrix} \neq 0$;
- (iii) system (1.1) is minimum phase, i.e.
 - (a) (1.1) is stabilizable,
 - (b) (1.1) is detectable,
 - (c) $C(sE - A)^{-1}B$ does not have any transmission zeros in $\overline{\mathbb{C}}_+$;
- (iv) $\sigma(Q) \subseteq \mathbb{C}_-$, where Q is in the zero dynamics form (2.1) or (2.5), respectively.

The proof is in Appendix 7.

Condition (ii) in Theorem 3.2 is a convenient mathematical frequency domain characterization which leads to condition (iv); the latter is the important property to show constant high-gain stabilizability in Section 4. Condition (iii) is the traditional definition of a minimum phase system. (ii) and (iii) are very similar to what is known for asymptotically stable zero dynamics of ODE systems, see [14, Sec. 4.3].

The following two remarks, although not exploited in the next section, may be worth knowing in its own right.

REMARK 3.3 (Zero dynamics and behaviour).

- (i) Suppose that $[E, A, B, C] \in \Sigma_{n,m}$ has positive strict relative degree. Then the behaviour $\mathfrak{B}_{(1.1)}$ can be decomposed, in terms of the transformation matrix T from Theorem 2.1, into a direct sum of the zero dynamics and a summand as

$$\mathfrak{B}_{(1.1)} = \mathcal{ZD}_{(1.1)} \oplus \mathcal{R},$$

where

$$\mathcal{R} := \left\{ \begin{array}{l} (x, u, y) \in \mathcal{C}^1(\mathbb{R}; \mathbb{R}^n) \\ \times \mathcal{C}(\mathbb{R}; \mathbb{R}^m) \times \mathcal{C}(\mathbb{R}; \mathbb{R}^m) \end{array} \left| \begin{array}{l} (x, u, y) \text{ solves (1.1) and} \\ [0_{\mu, \rho m}, I_\mu, 0_{\mu, n_c}, 0_{\mu, n_{\bar{c}}}] T x(0) = 0 \end{array} \right. \right\}.$$

In terms of (3.1), the representation is immediate from

$$\begin{aligned} (x(\cdot), u(\cdot), y(\cdot)) &= \left(T^{-1} \begin{pmatrix} 0_{\rho m \times \rho m} \\ e^{Q \cdot} \eta^0 \\ \sum_{i=0}^{\nu-1} N_c^i B_c \Gamma^{-1} S Q^i e^{Q \cdot} \eta^0 \\ 0_{n_{\bar{c}} \times n_{\bar{c}}} \end{pmatrix}, -\Gamma^{-1} S e^{Q \cdot} \eta^0, 0 \right) \\ &+ \left(x(\cdot) - T^{-1} \begin{pmatrix} 0_{\rho m \times \rho m} \\ e^{Q \cdot} \eta^0 \\ \sum_{i=0}^{\nu-1} N_c^i B_c \Gamma^{-1} S Q^i e^{Q \cdot} \eta^0 \\ 0_{n_{\bar{c}} \times n_{\bar{c}}} \end{pmatrix}, u(\cdot) + \Gamma^{-1} S e^{Q \cdot} \eta^0, y(\cdot) \right), \end{aligned}$$

- for any $(x, u, y) \in \mathfrak{B}_{(1.1)}$, where $\eta^0 = [0_{\mu, \rho m}, I_\mu, 0_{\mu, n_c}, 0_{\mu, n_\tau}] Tx(0)$.
- (ii) Suppose that $[E, A, B, C] \in \Sigma_{n, m}$ has proper inverse transfer function. Then the behaviour $\mathfrak{B}_{(1.1)}$ can be decomposed, in terms of the transformation matrix T from Theorem 2.3, into a direct sum of the zero dynamics and a summand as

$$\mathfrak{B}_{(1.1)} = \mathcal{ZD}_{(1.1)} \oplus \mathcal{R},$$

where

$$\mathcal{R} := \left\{ \begin{array}{l} (x, u, y) \in \mathcal{C}^1(\mathbb{R}; \mathbb{R}^n) \\ \times \mathcal{C}(\mathbb{R}; \mathbb{R}^m) \times \mathcal{C}(\mathbb{R}; \mathbb{R}^m) \end{array} \left| \begin{array}{l} (x, u, y) \text{ solves (1.1) and} \\ [0_{n_2, m}, I_{n_2}, 0_{n_2, n_c}, 0_{n_2, n_\tau}] Tx(0) = 0 \end{array} \right. \right\}.$$

This fact is shown in [2, Remark 3.3]. \diamond

Finally, we show that the zero dynamics carries, in a certain sense, the structure of a dynamical system.

REMARK 3.4 (Zero dynamics are a dynamical system).

Let ν be the index of the pencil $sE - A$. The transition map of system (1.1) is defined, in terms of Proposition 1.1, as

$$\begin{aligned} \varphi : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{C}^{\nu-1}(\mathbb{R}; \mathbb{R}^m) &\rightarrow \mathbb{R}^n \\ (t, t_0, x^0, u(\cdot)) &\mapsto T^{-1} \begin{bmatrix} e^{As(t-t_0)} & 0 \\ 0 & 0 \end{bmatrix} Tx^0 \\ &\quad + \int_{t_0}^t T^{-1} \begin{bmatrix} e^{As(t-\tau)} & 0 \\ 0 & 0 \end{bmatrix} W^{-1} Bu(\tau) d\tau \\ &\quad - \sum_{k=0}^{\nu-1} T^{-1} \begin{bmatrix} 0 & 0 \\ 0 & N^k \end{bmatrix} W^{-1} Bu^{(k)}(t). \end{aligned}$$

We have shown in [3, Prop. 2.20] that for any $(t_0, x^0, u) \in \mathbb{R} \times \mathbb{R}^n \times \mathcal{C}^{\nu-1}(\mathbb{R}; \mathbb{R}^m)$ the map $t \mapsto x(t) := \varphi(t, t_0, x^0, u(\cdot))$ solves the initial value problem

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad x(t_0) = x^0 \quad (3.3)$$

if, and only if,

$$x^0 \in \mathcal{V}_{t_0, u(\cdot)} := \left\{ x \in \mathbb{R}^n \left| x + \sum_{i=0}^{\nu-1} T^{-1} \begin{bmatrix} 0 & 0 \\ 0 & N^i \end{bmatrix} W^{-1} Bu^{(i)}(t_0) \in \text{im } T^{-1} \begin{bmatrix} I_{n_s} \\ 0 \end{bmatrix} \right. \right\}.$$

Therefore, consistency of the initial value x^0 depends on the initial time t_0 and the input $u(\cdot)$. The output map of system (1.1) is defined by

$$\eta : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m, \quad (t, x, u) \mapsto Cx.$$

It is readily verified that the structure $(\mathbb{R}, \mathbb{R}^m, \mathcal{C}^{\nu-1}(\mathbb{R}; \mathbb{R}^m), \mathbb{R}^n, \mathbb{R}^m, \varphi, \eta)$, where $\varphi : \mathcal{D}_\varphi \rightarrow \mathbb{R}^n$ is the restriction of the transition map (by abuse of notation we write the same symbol) on

$$\mathcal{D}_\varphi := \left\{ \begin{array}{l} (t, t_0, x^0, u) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{C}^{\nu-1}(\mathbb{R}; \mathbb{R}^m) \\ \left| \begin{array}{l} x^0 \in \mathcal{V}_{t_0, u(\cdot)}, \\ C\varphi(\cdot; t_0, x^0, u(\cdot)) = 0 \end{array} \right. \right\},$$

is an \mathbb{R} -linear time-invariant dynamical system as defined in [8, Defs. 2.1.1, 2.1.24, 2.1.26].

Next let $u \in C^{\nu-1}(\mathbb{R}; \mathbb{R}^m)$. As a consequence of uniqueness and global existence of the solution of the initial value problem (3.3) for $x^0 \in \mathcal{V}_{0,u(\cdot)}$ (see again [3, Prop. 2.20]), the map

$$\Psi: \mathcal{D}_{\varphi,0} \rightarrow \mathcal{ZD}_{(1.1)}, \quad (0, 0, x^0, u(\cdot)) \mapsto (\varphi(\cdot; 0, x^0, u(\cdot)), u(\cdot), C\varphi(\cdot; 0, x^0, u(\cdot)))$$

is well-defined, where

$$\mathcal{D}_{\varphi,0} := \{(0, 0, x^0, u) \in \mathcal{D}_{\varphi}\} \subset \mathcal{D}_{\varphi}.$$

Most importantly, if $[E, A, B, C]$ has positive strict relative degree or proper inverse transfer function, then Ψ is an isomorphism: it is surjective since a pre-image of $(x, u, 0) \in \mathcal{ZD}_{(1.1)}$ is $(0, 0, x(0), u(\cdot)) \in \mathcal{D}_{\varphi,0}$ (note that $u \in C^{\nu-1}(\mathbb{R}; \mathbb{R}^m)$ by Remark 3.1), it is injective by uniqueness of the solution of the initial value problem (3.3). In this sense, we may say that $\mathcal{ZD}_{(1.1)}$ is a dynamical system. \diamond

4. Stabilization by constant high-gain output feedback. Next we investigate whether asymptotically stable zero dynamics is a sufficient (or even necessary) condition for high-gain stabilizability in the sense that the feedback (1.13) or (1.14) resp., applied to (1.1) yields an asymptotically stable closed-loop system.

THEOREM 4.1 (High-gain stabilizability).

Consider a system $[E, A, B, C] \in \Sigma_{n,m}$. Then the following statements hold:

- (i) Suppose $[E, A, B, C]$ has strict relative degree $\rho \in \mathbb{N}$ and positive definite high frequency gain matrix $\Gamma \in \mathbb{R}^{m,m}$ (cf. (2.2)). Let $p(s) = \sum_{i=0}^{\rho-1} p_i s^i \in \mathbb{R}[s]$ be Hurwitz and $p_{\rho-1} > 0$. Then

$$\mathcal{ZD}_{(1.1)} \text{ is asympt. stable} \begin{array}{l} \implies \\ \not\Leftarrow \\ \text{i.g.} \end{array} \left\{ \begin{array}{l} \exists k^* \geq 0 \forall k \geq k^* : \\ \text{'(1.1) \& (1.13)'} \text{ is asympt. stable.} \end{array} \right.$$

- (ii) Suppose $[E, A, B, C]$ has proper inverse transfer function. Then

$$\mathcal{ZD}_{(1.1)} \text{ is asympt. stable} \begin{array}{l} \implies \\ \not\Leftarrow \\ \text{i.g.} \end{array} \left\{ \begin{array}{l} \exists k^* \geq 0 \forall k \in \mathbb{R}, |k| \geq k^* \forall \text{sln. } x(\cdot) \text{ of} \\ \text{'(1.1) \& (1.14)'}: \lim_{t \rightarrow \infty} x(t) = 0. \end{array} \right.$$

The proof is in Appendix 7.

REMARK 4.2 (High-gain stabilizability).

- (i) In case of strict relative degree one, the feedback law (1.13) reduces to the proportional output feedback $u(t) = -k y(t)$. If the system has higher relative degree, (1.13) incorporates a compensator $p(s)$ (and thus derivative feedback) to achieve a relative degree one system.

For ODE systems, the result is proven in [4] for relative degree one. By using the form (2.1), this result can be generalized to differential-algebraic systems with positive strict relative degree by using the same techniques.

- (ii) Consider $[E, A, B, C]$ with proper inverse transfer function. It may be surprising that the sign of k in (1.14) is irrelevant and, furthermore, no compensator is required.

In view of Remark 2.4, the closed-loop system ' $u(t) = ky(t)$ & (1.1)' is equivalent to

$$\begin{aligned} -(A_{11} + kI_m)y(t) &= A_{12}x_2(t) \\ \dot{x}_2(t) &= Qx_2(t) + A_{21}y(t) \\ x_c(t) &= \sum_{i=0}^{\nu-1} N_c^i E_c y^{(i+1)}(t) \end{aligned}$$

and, if $k > \|A_{11}\|$, equivalent to

$$\begin{aligned} y(t) &= -(A_{11} + kI_m)^{-1} A_{12}x_2(t) \\ \dot{x}_2(t) &= [Q - A_{21}(A_{11} + kI_m)^{-1} A_{12}] x_2(t) \\ x_c(t) &= \sum_{i=0}^{\nu-1} N_c^i E_c y^{(i+1)}(t). \end{aligned} \tag{4.1}$$

Note that $x_2 \in C^\infty(\mathbb{R}; \mathbb{R}^{n_2})$ yields $y \in C^\infty(\mathbb{R}; \mathbb{R}^m)$ and so the algebraic equation for x_3 is well defined. Note also that asymptotic stability of the zero dynamics is equivalent to $\sigma(Q) \subseteq \mathbb{C}_-$ by Theorem 3.2. Since

$$\lim_{k \rightarrow \pm\infty} \sigma(Q - A_{21}(A_{11} + kI_m)^{-1} A_{12}) = \sigma(Q),$$

the assumptions ' $|k|$ sufficiently large' and '(1.1) has asymptotically stable zero dynamics' yield exponential decay of $x_2(\cdot)$, and therefore $x_3(\cdot)$ and $y(\cdot)$ decay exponentially, too. \diamond

5. Funnel control. In this section we will show that the two funnel controllers (1.17) and (1.18) achieve output tracking of a rather general reference signal within a pre-specified funnel.

THEOREM 5.1 (Funnel control).

Suppose that $[E, A, B, C] \in \Sigma_{n,m}$ has asymptotically stable zero dynamics, and let ν be the index of $sE - A$. Let $\varphi \in \Phi^{\nu+1}$ define a performance funnel \mathcal{F}_φ . The initial data are an arbitrary consistent initial value $x^0 \in \mathbb{R}^n$ and a reference signal $y_{\text{ref}} \in \mathcal{B}^{\nu+1}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$.

- (i) If $[E, A, B, C]$ has positive strict relative degree $\rho \in \mathbb{N}$ and positive definite high frequency gain matrix $\Gamma = \lim_{s \rightarrow \infty} s^\rho C(sE - A)^{-1} B$, then the application of the funnel controller (1.17) to (1.1) yields a closed-loop initial value problem with precisely one maximal continuously differentiable solution $x: [0, \omega) \rightarrow \mathbb{R}^n$ and this solution is global (i.e. $\omega = \infty$), and all functions x, z, k, u are bounded.
- (ii) If $[E, A, B, C]$ has proper inverse transfer function and the initial gain $k(0) = \hat{k} \in \mathbb{R}$ satisfies $|\hat{k}| > \lim_{s \rightarrow \infty} \|G^{-1}(s)\|$, then the application of the funnel controller (1.18) to (1.1) yields a closed-loop initial value problem with precisely one maximal continuously differentiable solution $x: [0, \omega) \rightarrow \mathbb{R}^n$ and this solution is global (i.e. $\omega = \infty$), and all functions x, k, u are bounded.

Most importantly, in both cases (i) and (ii), the tracking error $e(\cdot) = Cx(\cdot) - y_{\text{ref}}(\cdot)$ satisfies

$$\exists \varepsilon > 0 \forall t \geq 0 : \|e(t)\| \leq \varphi(t)^{-1} - \varepsilon, \tag{5.1}$$

(that means $e(\cdot)$ evolves within the performance funnel \mathcal{F}_φ and is uniformly bounded away from the boundary) and for the same ε the gain is bounded by

$$\forall t \geq 0 : \|k\|_\infty \leq \frac{|\hat{k}|}{1 - (1 - \varepsilon\|\varphi\|_\infty)^2}, \quad (5.2)$$

where $\hat{k} = 1$ in case of (i).

The proof is in Appendix 7.

REMARK 5.2.

- (i) *Minimum-phase systems (1.10) with transfer function $G(s) = C(sE - A)^{-1}B + D$ that has either a proper inverse or has positive strict relative degree, do also allow funnel control in the sense of Theorem 5.1. This follows from the artificial extension (1.11).*
- (ii) *Consistency of the initial value $x^0 \in \mathbb{R}^n$ means that the closed-loop system (1.1), (1.17) or (1.1), (1.18) resp., with $x(0) = x^0$ has a solution $x : [0, \omega) \rightarrow \mathbb{R}^n$ for some $\omega \in (0, \infty]$. In practice, consistency of the initial state of the “unknown” system should be satisfied as far as the DAE $[E, A, B, C]$ is the correct model.*
- (iii) *In case of Theorem 5.1(ii), the assumption on the initial gain $k(0) = \hat{k}$ reduces to $\hat{k} > 0$ if $\lim_{s \rightarrow \infty} G^{-1}(s) = 0$; the latter means that the transfer function cannot be realized by an ODE system with feedthrough, see [2, Rem. 6.3].*
- (iv) *In case of ODE systems (1.10), $E = I$, with invertible feedthrough term, the transfer function $G(s) = C(sI - A)^{-1}B + D$ has proper inverse and the assumption on the initial gain $k(0) = \hat{k}$ is equivalent to $|\hat{k}| > \|D^{-1}\|$.*
- (v) *Theorem 5.1 is on system class $\Sigma_{n,m}$. A careful inspection of the proof of the theorem shows that one actually only needs that the nominal system can be brought into the form (2.4) or (2.9), resp. Thus in particular, the pencil $sE - A$ does not have to be regular; important is $\sigma(Q) \subseteq \mathbb{C}_-$. \diamond*

6. Application: Position and velocity control of a mechanical system with springs, masses and dampers.

6.1. The mechanical model. For purposes of illustration, we consider a mechanical system, see Fig. 6.1, with springs, masses and dampers with single-input spatial distance between the two masses and single-output position of one mass. We are indebted to Professor P.C. Müller (BU Wuppertal) for suggesting this example to us. The masses m_1, m_2 , damping constants d_1, d_2 and spring constants c_1, c_2 are all assumed to be positive. As output $y(t) = z_2(t)$ we take the position of the mass m_2 . In the first example, the input $u(t) = z_2(t) - z_1(t)$ is the spatial distance between the masses m_1 and m_2 , whereas, in the second example, the input is the relative velocity between the masses m_1 and m_2 , i.e. $u(t) = \dot{z}_2(t) - \dot{z}_1(t)$.

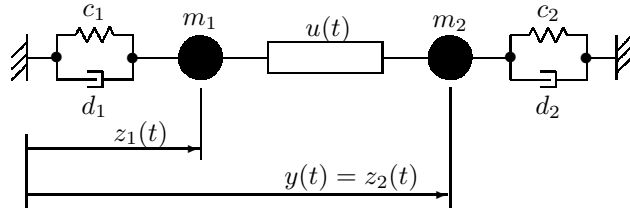


Fig. 6.1: Mass-spring-damper system

6.1.1. Position control. With input $u(t) = z_2(t) - z_1(t)$, the mechanical system in Fig. 6.1 may be modelled by the second-order differential-algebraic equation

$$\begin{aligned} m_1 \ddot{z}_1(t) + d_1 \dot{z}_1(t) + c_1 z_1(t) - \lambda(t) &= 0 \\ m_2 \ddot{z}_2(t) + d_2 \dot{z}_2(t) + c_2 z_2(t) + \lambda(t) &= 0 \\ z_2(t) - z_1(t) &= u(t) \\ y(t) &= z_2(t) \end{aligned} \quad (6.1)$$

where $\lambda(\cdot)$ is a constraint force viewed as a variable. Defining the state

$$x(t) = (z_1(t), \dot{z}_1(t), z_2(t), \dot{z}_2(t), \lambda(t))^\top, \quad (6.2)$$

model (6.1) may be rewritten as the linear differential-algebraic input-output system (1.1) for

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & m_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -c_1 & -d_1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -c_2 & -d_2 & -1 \\ 1 & 0 & -1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}^\top. \quad (6.3)$$

We may immediately see that the pencil $sE - A$ is regular and has index $\nu = 3$; The transfer function

$$G(s) = C(sE - A)^{-1}B = \frac{m_1 s^2 + d_1 s + c_1}{(m_1 + m_2)s^2 + (d_1 + d_2)s + (c_1 + c_2)}$$

has proper inverse: $\lim_{s \rightarrow \infty} G^{-1}(s) = (m_1 + m_2)/m_1$. The zero dynamics of (6.3) is asymptotically stable: setting $y(\cdot) = 0$ in (6.1) yields $z_2(\cdot) = 0$, $\lambda(\cdot) = 0$, $z_1(\cdot) = -u(\cdot)$ and $m_1 z_1(t) + d_1 \dot{z}_1(t) + c_1 z_1(t) = 0$ for all $t \geq 0$; positivity of m_1 , d_1 and c_1 then gives $\lim_{t \rightarrow \infty} \dot{z}_1(t) = \lim_{t \rightarrow \infty} z_1(t) = 0$. Summarizing, system (6.3) satisfies the assumptions of Theorem 5.1(ii).

6.1.2. Velocity control. With input $u(t) = \dot{z}_2(t) - \dot{z}_1(t)$, the mechanical system in Fig. 6.1 may, analogous to position control, be modelled by the second-order differential-algebraic equation

$$\begin{aligned} m_1 \ddot{z}_1(t) + d_1 \dot{z}_1(t) + c_1 z_1(t) - \lambda(t) &= 0 \\ m_2 \ddot{z}_2(t) + d_2 \dot{z}_2(t) + c_2 z_2(t) + \lambda(t) &= 0 \\ \dot{z}_2(t) - \dot{z}_1(t) &= u(t) \\ y(t) &= z_2(t). \end{aligned} \quad (6.4)$$

Defining the state as in (6.2), the model (6.4) may be rewritten as the linear differential-algebraic input-output system (1.1) for

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -c_1 & -d_1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -c_2 & -d_2 & -1 \\ 0 & 1 & 0 & -1 & 0 \end{bmatrix} \quad \text{and } E, B, C \text{ as in (6.3)}. \quad (6.5)$$

We may immediately see that the pencil $sE - A$ is regular and has index $\nu = 2$; The transfer function

$$G(s) = C(sE - A)^{-1}B = \frac{m_1 s^2 + d_1 s + c_1}{(m_1 + m_2)s^3 + (d_1 + d_2)s^2 + (c_1 + c_2)s},$$

has strict relative degree 1: $\lim_{s \rightarrow \infty} s \cdot G(s) = m_1 / (m_1 + m_2)$. Similar to Section 6.1.1, we may see that the zero dynamics of (6.5) are asymptotically stable, whence we are in the situation of Theorem 5.1(i).

6.2. Simulations. In both examples, as reference signal $y_{\text{ref}} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$, we take the first component of the solution of the following initial-value problem for the Lorenz system

$$\begin{aligned} \dot{\xi}_1(t) &= 10(\xi_2(t) - \xi_1(t)), & \xi_1(0) &= 5 \\ \dot{\xi}_2(t) &= 28\xi_1(t) - \xi_1(t)\xi_3(t) - \xi_2(t), & \xi_2(0) &= 5 \\ \dot{\xi}_3(t) &= \xi_1(t)\xi_2(t) - \frac{8}{3}\xi_3(t), & \xi_3(0) &= 5. \end{aligned} \quad (6.6)$$

This may be viewed as a rather academic choice, however it is well known (see for example [18, App. C]) that the Lorenz system is chaotic (and thus the reference signal is rather “wild”), the unique global solution of (6.6) is bounded with bounded derivative on the positive real axis (and thus our assumptions on the class of reference signals are satisfied). The solution of (6.6) is depicted in Fig. 6.2. The funnel \mathcal{F}_φ is determined by the function

$$\varphi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}, \quad t \mapsto 0.5 te^{-t} + 2 \arctan t. \quad (6.7)$$

Note that this prescribes an exponentially (exponent 1) decaying funnel in the transient phase $[0, T]$, where $T \approx 3$, and a tracking accuracy quantified by $\lambda = 1/\pi$ thereafter, see e.g. Fig. 6.3d.

Spring and damping constants, masses and their initial positions are chosen, for the simulations, as

$$\begin{aligned} m_1 &= 1, & m_2 &= 3, & c_1 &= 2, & c_2 &= 1, & d_1 &= 3, & d_2 &= 5, \\ z_1(0) &= 101, & z_2(0) &= 21 & \text{and} & \hat{k} &= 5. \end{aligned} \quad (6.8)$$

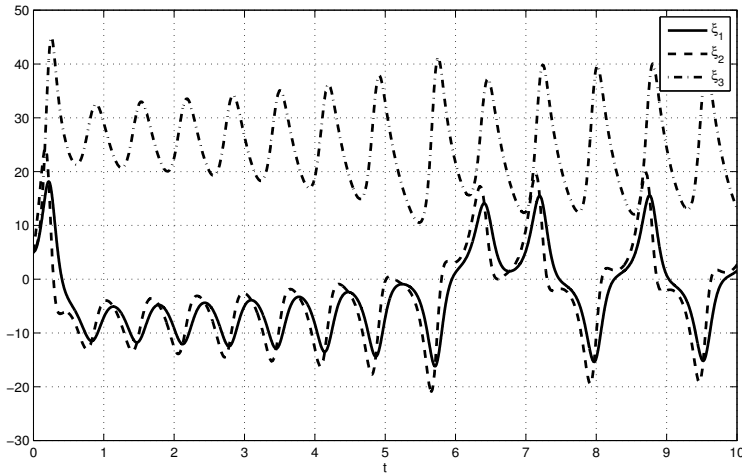


Fig. 6.2: Solution of the Lorenz system (6.6)

For position control, straightforward calculations show that the closed-loop system (1.18), (6.1) has uniquely determined initial velocities $\dot{z}_1(0)$, $\dot{z}_2(0)$ as well as initial constraint force $\lambda(0)$ and that the initialization is consistent.

In the case of velocity control we further choose the initial velocities

$$\dot{z}_1(0) = -11, \quad \dot{z}_2(0) = -3 \quad (6.9)$$

and clearly there is a unique initial constraint force $\lambda(0)$ and the initialization of (1.17), (6.5) is consistent.

All numerical simulations are performed by MATLAB.

6.2.1. Position control. Since

$$\hat{k} = 5 > 4 = \lim_{s \rightarrow \infty} G^{-1}(s),$$

all assumptions of Theorem 5.1(ii) are satisfied and we may apply the funnel controller (1.18) with funnel boundary specified in (6.7) and reference signal $y_{\text{ref}}(\cdot) = \xi_1(\cdot)$ given in (6.6). The simulations over the time interval $[0, 10]$ are depicted in Fig. 6.3: Fig. 6.3a shows the output $y(\cdot)$ tracking the rather “vivid” reference signal $y_{\text{ref}}(\cdot)$ within the funnel shown in Fig. 6.3d. Note that the input $u(\cdot)$ in Fig. 6.3c as well as the gain function $k(\cdot)$ in Fig. 6.3b have spikes at those times t when the norm of the error $\|e(t)\|$ is “close” to the funnel boundary $\varphi(t)^{-1}$; this is due to rapid change of the reference signal. We stress that the gain function $k(\cdot)$ is non-monotone.

6.2.2. Velocity control. Since the system has relative degree one with positive high frequency gain $\Gamma = m_1/(m_1 + m_2) = 1/4$, all assumptions of Theorem 5.1 (i) are satisfied and we may apply the funnel controller (1.17) with funnel boundary specified in (6.7) and reference signal $y_{\text{ref}}(\cdot) = \xi_1(\cdot)$ given in (6.6).

The simulations over the time interval $[0, 10]$ are depicted in Fig. 6.4: Fig. 6.4a shows the output $y(\cdot)$ tracking the reference signal $y_{\text{ref}}(\cdot)$; the error within the funnel

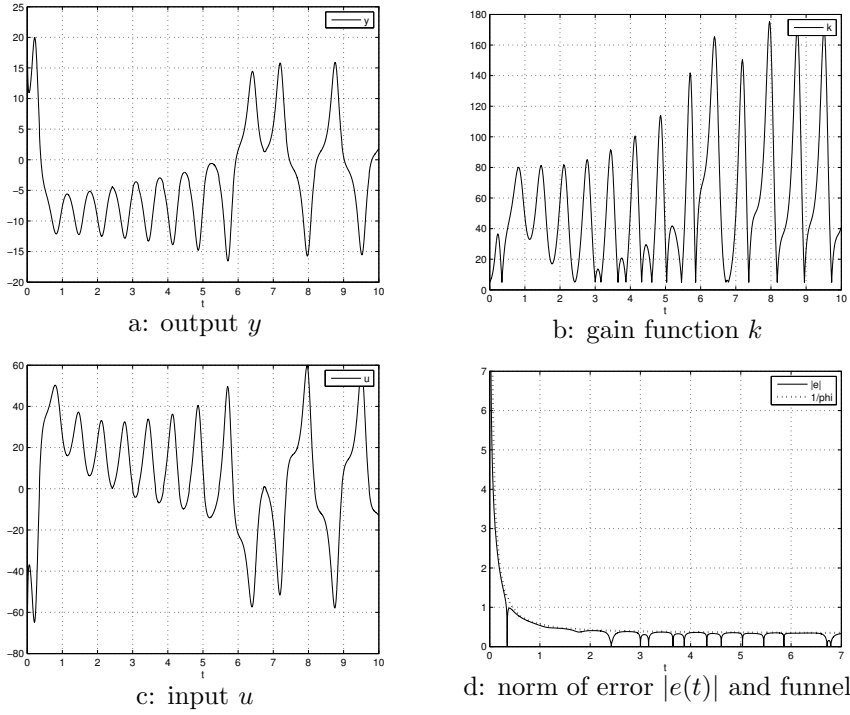


Fig. 6.3: Position control: Simulation of the funnel controller (1.18) with funnel boundary specified in (6.7) and reference signal $y_{\text{ref}}(\cdot) = \xi_1(\cdot)$ given in (6.6) applied to the mechanical model (6.1) with data (6.8).

is depicted in Fig. 6.4d. Note that, due to the rather “academic choice” of the example, the input $u(\cdot)$ (in Fig. 6.4c) and the gain function $k(\cdot)$ (in Fig. 6.4b) both take considerable larger values than for position control. Another reason for this behaviour is that we have kept the funnel as tight as for position control simulated in Fig. 6.3, and the velocity exhibits a very “vivid” behaviour which causes the error to approach the funnel boundary faster, resulting in the high values of the gain function.

7. Appendix: Proofs. This section contains the proofs of the results presented in Sections 1-5. First we prove a preliminary lemma.

LEMMA 7.1. *Suppose $[E, A, B, C] \in \Sigma_{n,m}$ has strictly proper transfer function $G(s) = C(sE - A)^{-1}B$. Then there exists $W, T \in \mathbf{GL}_n(\mathbb{R})$ such that*

$$[E, A, B, C] \stackrel{W, T}{\sim} \left[\begin{bmatrix} I_{n_s} & 0 & 0 \\ 0 & N_c & N_{c\bar{c}} \\ 0 & 0 & N_{\bar{c}} \end{bmatrix}, \begin{bmatrix} A_s & 0 & 0 \\ 0 & I_{n_{f_c}} & 0 \\ 0 & 0 & I_{n_{f_{\bar{c}}}} \end{bmatrix}, \begin{bmatrix} B_s \\ B_{f_c} \\ 0 \end{bmatrix}, [C_s \ 0 \ C_{f_{\bar{c}}}] \right] \quad (7.1)$$

for some $A_s \in \mathbb{R}^{n_s, n_s}$, $B_s \in \mathbb{R}^{n_s, m}$, $C_s \in \mathbb{R}^{m, n_s}$, $N_c \in \mathbb{R}^{n_{f_c}, n_{f_c}}$, $N_{c\bar{c}} \in \mathbb{R}^{n_{f_c}, n_{f_{\bar{c}}}}$, $N_{\bar{c}} \in \mathbb{R}^{n_{f_{\bar{c}}}, n_{f_{\bar{c}}}}$, $B_{f_c} \in \mathbb{R}^{n_{f_c}, m}$ and $C_{f_{\bar{c}}} \in \mathbb{R}^{m, n_{f_{\bar{c}}}}$, where $N_c, N_{\bar{c}}$ are nilpotent and $\text{rk}[N_c, B_{f_c}] = n_{f_c}$. The dimensions $n_s, n_{f_c}, n_{f_{\bar{c}}} \in \mathbb{N}_0$ are unique, the matrices $A_s,$

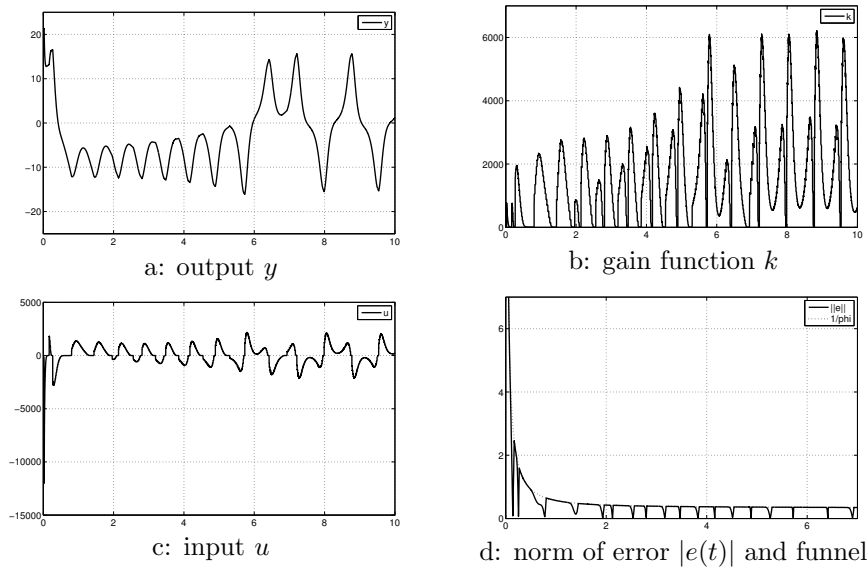


Fig. 6.4: Velocity control: Simulation of the funnel controller (1.17) with funnel boundary specified in (6.7) and reference signal $y_{\text{ref}}(\cdot) = \xi_1(\cdot)$ given in (6.6) applied to the mechanical model (6.4) with data (6.8), (6.9).

N_c and $N_{\bar{c}}$ are unique up to similarity. Furthermore, system $[E, A, B, C]$ has strict relative degree $\rho > 0$ if, and only if,

$$\det(C_s A_s^{\rho-1} B_s) \neq 0 \quad \text{and, if } \rho > 1, \quad \forall k = 0, \dots, \rho - 2: C_s A_s^k B_s = 0. \quad (7.2)$$

Proof. We proceed in several steps.

Step 1: We show that there exist $W, T \in \mathbf{GL}_n(\mathbb{R})$ such that (7.1) holds true.

Corollary 1.2 yields (1.5) for some $W_1, T_1 \in \mathbf{GL}_n(\mathbb{R})$. It follows from [6, Sec. 2-1.] that system $[N, I_{n_f}, B_f, C_f]$ may be decomposed into controllability form so that, for some $T_2 \in \mathbf{GL}_{n_f}(\mathbb{R})$,

$$[N, I_{n_f}, B_f, C_f] \stackrel{T_2^{-1}, T_2}{\sim} \left[\begin{array}{cc} N_c & N_{c\bar{c}} \\ 0 & N_{\bar{c}} \end{array} \right], \left[\begin{array}{cc} I_{n_{fc}} & 0 \\ 0 & I_{n_{f\bar{c}}} \end{array} \right], \left[\begin{array}{c} B_{fc} \\ 0 \end{array} \right], [C_{fc}, C_{f\bar{c}}],$$

where $N_c \in \mathbb{R}^{n_{fc}, n_{fc}}$, $N_{\bar{c}} \in \mathbb{R}^{n_{f\bar{c}}, n_{f\bar{c}}}$, $N_{12} \in \mathbb{R}^{n_{fc}, n_{f\bar{c}}}$, $B_{fc} \in \mathbb{R}^{n_{fc}, m}$, $C_{fc} \in \mathbb{R}^{m, n_{fc}}$, and $C_{f\bar{c}} \in \mathbb{R}^{m, n_{f\bar{c}}}$, such that $N_c, N_{\bar{c}}$ are nilpotent and $\text{rk}[N_c, B_{fc}] = n_{fc}$.

We show that $C_{fc} = 0$: Since the transfer function is invariant under system equivalence we have, using (1.7),

$$G(s) = C(sE - A)^{-1}B = C_s(sI_{n_s} - A_s)^{-1}B_s - \sum_{k=0}^{\nu-1} s^k C_{fc} N_c^k B_{fc},$$

and since $G(s)$ is strictly proper, it follows that $C_{fc} N_c^i B_{fc} = 0$ for $i = 1, \dots, \nu - 1$. The nilpotency of N_c gives $C_{fc} N_c^{\nu-1} [N_c, B_{fc}] = 0$, whence $C_{fc} N_c^{\nu-1} = 0$. Repeating this argumentation $\nu - 1$ times, we obtain $C_{fc} = 0$.

Setting $W := W_1 \begin{bmatrix} I & 0 \\ 0 & T_2^{-1} \end{bmatrix}$ and $T := \begin{bmatrix} I & 0 \\ 0 & T_2 \end{bmatrix} T_1$, we obtain (7.1).

Step 2: We show that the dimensions $n_s, n_{fc}, n_{f\bar{c}} \in \mathbb{N}_0$ are unique and that the matrices A_s, N_c and $N_{\bar{c}}$ are unique up to similarity: Assume that

$$\begin{aligned} & \left[\begin{bmatrix} I_{n_{s1}} & 0 & 0 \\ 0 & N_{c1} & N_{c\bar{c}1} \\ 0 & 0 & N_{\bar{c}1} \end{bmatrix}, \begin{bmatrix} A_{s1} & 0 & 0 \\ 0 & I_{n_{fc1}} & 0 \\ 0 & 0 & I_{n_{f\bar{c}1}} \end{bmatrix}, \begin{bmatrix} B_{s1} \\ B_{fc1} \\ 0 \end{bmatrix}, [C_{s1} \ 0 \ C_{f\bar{c}1}] \right] \\ & \stackrel{W,T}{\sim} \left[\begin{bmatrix} I_{n_{s2}} & 0 & 0 \\ 0 & N_{c2} & N_{c\bar{c}2} \\ 0 & 0 & N_{\bar{c}2} \end{bmatrix}, \begin{bmatrix} A_{s2} & 0 & 0 \\ 0 & I_{n_{fc2}} & 0 \\ 0 & 0 & I_{n_{f\bar{c}2}} \end{bmatrix}, \begin{bmatrix} B_{s2} \\ B_{fc2} \\ 0 \end{bmatrix}, [C_{s2} \ 0 \ C_{f\bar{c}2}] \right]. \end{aligned}$$

Proposition 1.1 implies that $n_{s1} = n_{s2}$ as well as the similarity of A_{s1} and A_{s2} . Proposition 1.1 also yields the existence of $T_{11} \in \mathbb{R}^{n_{fc1}, n_{fc2}}$, $T_{12} \in \mathbb{R}^{n_{fc1}, n_{f\bar{c}2}}$, $T_{21} \in \mathbb{R}^{n_{f\bar{c}1}, n_{fc2}}$, and $T_{22} \in \mathbb{R}^{n_{f\bar{c}1}, n_{f\bar{c}2}}$ such that

$$\begin{aligned} T &= \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \in \mathbf{G}\mathbf{l}_{n_f}(\mathbb{R}), \text{ and} \\ \begin{bmatrix} N_{c1} & N_{c\bar{c}1} \\ 0 & N_{\bar{c}1} \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} &= \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} N_{c2} & N_{c\bar{c}2} \\ 0 & N_{\bar{c}2} \end{bmatrix}, \begin{bmatrix} B_{fc1} \\ 0 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} B_{fc2} \\ 0 \end{bmatrix}. \end{aligned}$$

Therefore, $0 = T_{21}B_{fc2}$ and $T_{21}N_{c2} = N_{\bar{c}1}T_{21}$.

Hence, for $k = 1, \dots, \nu - 1$, we have $T_{21}N_{c2}^k B_{fc2} = N_{\bar{c}1}^k T_{21}B_{fc2} = 0$, and so $T_{21}N_{c2}^{\nu-1} [N_{c2} \ B_{fc2}] = 0$, whence $T_{21}N_{c2}^{\nu-1} = 0$. Repeating this argumentation $\nu - 1$ times, we obtain $T_{21} = 0$. Then $T \in \mathbf{G}\mathbf{l}_{n_f}(\mathbb{R})$ yields $n_{fc2} \leq n_{fc1}$. By reversing the roles of the above matrices, we analogously obtain $n_{fc1} \leq n_{fc2}$ and thus $n_{fc1} = n_{fc2}$, $n_{f\bar{c}1} = n_{f\bar{c}2}$. This shows that T_{11} and T_{22} are square. Together with $T \in \mathbf{G}\mathbf{l}_{n_f}$, we obtain $T_{11} \in \mathbf{G}\mathbf{l}_{n_{fc}}(\mathbb{R})$ and $T_{22} \in \mathbf{G}\mathbf{l}_{n_{f\bar{c}}}(\mathbb{R})$. Hence N_{c1}, N_{c2} and $N_{\bar{c}1}, N_{\bar{c}2}$ are similar, resp.

Step 3: We show that $[E, A, B, C]$ has strict relative degree $\rho > 0$ if, and only if, (7.2) holds.

This is an immediate consequence of the fact that, due to Step 1, the transfer function has the representation

$$G(s) = C(sE - A)^{-1}B = C_s(sI_{n_s} - A_s)^{-1}B_s.$$

This completes the proof of the lemma. \square

Proof of Theorem 2.1. We proceed in several steps.

Step 1: We show that there exist $W, T \in \mathbf{G}\mathbf{l}_n(\mathbb{R})$ such that $[E, A, B, C] \stackrel{W,T}{\sim} [\hat{E}, \hat{A}, \hat{B}, \hat{C}]$ for $[\hat{E}, \hat{A}, \hat{B}, \hat{C}]$ as in (2.1).

Since a positive strict relative degree implies that $G(s)$ is strictly proper, we may apply Lemma 7.1 to obtain (7.1) for some $W_1, T_1 \in \mathbf{G}\mathbf{l}_n(\mathbb{R})$. Furthermore, (7.2) holds and hence we may transform $[I, A_s, B_s, C_s]$ into Byrnes-Isidori form (see [13, Lemma

Since the transfer function is invariant under system equivalence we have

$$\begin{aligned}
 C(sE - A)^{-1}B &= \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B} \\
 &= [I_m \ 0 \ \cdots \ 0] \left(sI_{n-n_{fc}-n_{f\bar{c}}} - \begin{bmatrix} 0 & I_m & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_m & 0 & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & I_m & 0 \\ R_1 & R_2 & \cdots & R_{\rho-1} & R_\rho & S \\ P & 0 & \cdots & 0 & 0 & Q \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \Gamma \\ 0 \end{bmatrix} \\
 &\quad + [0 \ C_{f\bar{c}}] \begin{bmatrix} sN_c - I_{n_{fc}} & sN_{c\bar{c}} \\ 0 & sN_{\bar{c}} - I_{n_{f\bar{c}}} \end{bmatrix}^{-1} \begin{bmatrix} B_{fc} \\ 0 \end{bmatrix}. \\
 &= [I_m \ 0 \ \cdots \ 0] \left(sI_{n-n_{fc}-n_{f\bar{c}}} - \begin{bmatrix} 0 & I_m & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_m & 0 & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & I_m & 0 \\ R_1 & R_2 & \cdots & R_{\rho-1} & R_\rho & S \\ P & 0 & \cdots & 0 & 0 & Q \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \Gamma \\ 0 \end{bmatrix} \\
 &= X_1(s) = - \left[\sum_{i=1}^{\rho+1} R_i s^{i-1} + S(sI_{n-n_{fc}-n_{f\bar{c}}-\rho m} - Q)^{-1}P \right]^{-1} \Gamma.
 \end{aligned}$$

This proves (2.3). Finally,

$$\begin{aligned}
 \Gamma &= \lim_{s \rightarrow \infty} - \left[\sum_{i=1}^{\rho+1} R_i s^{i-1} + S(sI_{n-n_{fc}-n_{f\bar{c}}-\rho m} - Q)^{-1}P \right] G(s) \\
 &= - \sum_{i=1}^{\rho} R_i \lim_{s \rightarrow \infty} s^{i-1} G(s) + \lim_{s \rightarrow \infty} s^\rho G(s) - \lim_{s \rightarrow \infty} S(sI_{n-n_{fc}-n_{f\bar{c}}-\rho m} - Q)^{-1}P G(s) \\
 &= \lim_{s \rightarrow \infty} s^\rho G(s)
 \end{aligned}$$

and the proof of the theorem is complete. \square

Proof of Theorem 3.2.

(i) \Leftrightarrow (iv): This is immediate from Remark 3.1.

For the remainder of the proof we only consider systems with positive strict relative degree; the proof for systems which have a proper inverse transfer function is in [2, Theorem 5.4]. In view of invariance of the transfer function under system equivalence we may assume, without loss of generality, that $[E, A, B, C]$ is in form (2.1).

(ii) \Leftrightarrow (iv): First note that Schur's complement formula (see for example [8, Lemma A.1.17]) yields, for all $X \in \mathbb{R}^{k \times m}$, $Y \in \mathbb{R}^{k \times k}$, $Z \in \mathbb{R}^{m \times m}$,

$$\det \begin{bmatrix} X & Y \\ Z & 0 \end{bmatrix} = \det \begin{bmatrix} 0 & I_k \\ I_m & 0 \end{bmatrix} \cdot \det \begin{bmatrix} Z & 0 \\ X & Y \end{bmatrix} = (-1)^{km} \det Z \cdot \det Y.$$

This and the fact that $\det(sN - I_k) = (-1)^k$ for any nilpotent $N \in \mathbb{R}^{k,k}$ yields

$$\begin{aligned}
& \det \begin{bmatrix} sE - A & B \\ C & 0 \end{bmatrix} \\
&= \det \left[\begin{array}{cccccc|cc} sI_m & -I_m & & & & & 0 & \\ & \ddots & \ddots & & & & \vdots & \\ & & sI_m & -I_m & & & 0 & \\ -R_1 & \dots & -R_{\rho-1} & sI_m - R_\rho & -S & & \Gamma & \\ -P & 0 & \dots & 0 & sI_\mu - Q & & 0 & \\ 0 & 0 & & \dots & 0 & sN_c - I_{n_c} & sN_{c\bar{c}} & B_c \\ 0 & 0 & & \dots & 0 & 0 & sN_{\bar{c}} - I_{n_{\bar{c}}} & 0 \\ \hline I_m & 0 & \dots & 0 & 0 & 0 & C_{\bar{c}} & 0 \end{array} \right] \\
&= (-1)^m \det \left[\begin{array}{cccccc|cc} 0 & -I_m & & & & & sI_m & \\ \vdots & \ddots & \ddots & & & & & \\ 0 & & sI_m & -I_m & & & & \\ \Gamma & \dots & -R_{\rho-1} & sI_m - R_\rho & -S & & -R_1 & \\ 0 & 0 & \dots & 0 & sI_\mu - Q & & -P & \\ B_c & 0 & 0 & \dots & 0 & sN_c - I_{n_c} & sN_{c\bar{c}} & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & sN_{\bar{c}} - I_{n_{\bar{c}}} & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & C_{\bar{c}} & I_m \end{array} \right] \\
&= (-1)^m \det \begin{bmatrix} sN_{\bar{c}} - I_{n_{\bar{c}}} & 0 \\ C_{\bar{c}} & I_m \end{bmatrix} \cdot \det(sN_c - I_{n_c}) \cdot \det(sI_\mu - Q) \\
&\quad \cdot \det \begin{bmatrix} 0 & -I_m & & & & \\ 0 & sI_m & -I_m & & & \\ \vdots & & \ddots & \ddots & & \\ 0 & & & sI_m & -I_m & \\ \Gamma & -R_2 & \dots & -R_{\rho-1} & sI_m - R_\rho & \end{bmatrix} \\
&= (-1)^{m+n_c+n_{\bar{c}}} \det(sI_\mu - Q) \det \Gamma.
\end{aligned}$$

This proves the claim.

(iii) \Leftrightarrow (iv): Set

$$[A_s, B_s, C_s] = \left[\begin{bmatrix} 0 & I_m & 0 & \dots & 0 & 0 \\ 0 & 0 & I_m & 0 & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & I_m & 0 \\ R_1 & R_2 & \dots & R_{\rho-1} & R_\rho & S \\ P & 0 & \dots & 0 & 0 & Q \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \Gamma \\ 0 \end{bmatrix}, [I_m \ 0 \ \dots \ 0] \right].$$

We immediately see that

- $\forall s \in \mathbb{C} : \text{rk}[sE - A, B] = \text{rk}[sI_{n_s} - A_s, B_s] + n_c + n_{\bar{c}},$
- $\forall s \in \mathbb{C} : \text{rk}[sE^\top - A^\top, C^\top] = \text{rk}[sI_{n_s} - A_s^\top, C_s^\top] + n_c + n_{\bar{c}},$
- $G(s) = C(sE - A)^{-1}B = C_s(sI_{n_s} - A_s)^{-1}B_s,$

where c) is proved similar to Step 3 in the proof of Theorem 2.1. Therefore, the equivalence of (iii) and (iv) has been reduced to the ODE system $[I_{n_s}, A_s, B_s, C_s]$. Now the claim follows from [9, Proposition 2.1.2]. This completes the proof of the theorem. \square

For the proof of Theorem 4.1 a lemma is required.

LEMMA 7.2 (High-gain lemma). *Consider, for $\Gamma \in \mathbb{C}^{m \times m}$, $\tilde{R} \in \mathbb{C}^{m \times m}$, \tilde{S}^\top , $\tilde{P} \in \mathbb{C}^{(n-m) \times m}$, $\tilde{Q} \in \mathbb{C}^{(n-m) \times (n-m)}$, the parameterized matrix*

$$A_\kappa := \begin{bmatrix} \tilde{R} - \kappa\Gamma & \tilde{S} \\ \tilde{P} & \tilde{Q} \end{bmatrix}, \quad \kappa \geq 0.$$

Suppose that the spectrum $\sigma(\Gamma) = \{\gamma_1, \dots, \gamma_m\} \subseteq \mathbb{C} \setminus \{0\}$ consists of semi-simple non-zero eigenvalues only, and denote the spectrum of \tilde{Q} by $\sigma(\tilde{Q}) = \{q_{m+1}, \dots, q_n\}$. Then there exist $r_1, \dots, r_m \in \mathbb{C}$ only depending on \tilde{R} and Γ with the following property: For all $\varepsilon > 0$ there exist $\rho_1, \dots, \rho_m \geq 0$ and $\kappa^ \geq 1$ such that, for all $\kappa \geq \kappa^*$ and a suitable numeration of the eigenvalues $\lambda_1(A_\kappa), \dots, \lambda_n(A_\kappa)$ of A_κ , we have*

$$\lambda_i(A_\kappa) \in \bigcup_{j=1}^m B(r_j - \kappa\gamma_j, \rho_j) \quad \forall i = 1, \dots, m$$

and $\lambda_i(A_\kappa) \in \bigcup_{j=m+1}^n B(q_j, \varepsilon) \quad \forall i = m+1, \dots, n,$

where $B(z, \varepsilon) = \{w \in \mathbb{C} \mid |z - w| < \varepsilon\}$ denotes the ball of radius ε around z in \mathbb{C} .

Proof. Choose $U_1 \in \mathbf{GL}_m(\mathbb{C})$, $U_2 \in \mathbf{GL}_{n-m}(\mathbb{C})$ such that, for appropriately chosen $\delta_{m+1}, \dots, \delta_{n-1} \in \{0, 1\}$ we have

$$U_1 \Gamma U_1^{-1} = \begin{bmatrix} \gamma_1 & & \\ & \ddots & \\ & & \gamma_m \end{bmatrix}, \quad U_2 \tilde{Q} U_2^{-1} = \begin{bmatrix} q_{m+1} & \delta_{m+1} & & \\ & \ddots & \ddots & \\ & & q_{n-1} & \delta_{n-1} \\ & & & q_n \end{bmatrix},$$

and let

$$T_\alpha := \text{diag}(\alpha^1, \alpha^2, \dots, \alpha^{n-m}), \quad \text{where } \alpha > 0.$$

Then we define

$$\begin{aligned} M(\alpha, \kappa) &:= \begin{bmatrix} U_1^{-1} & 0 \\ 0 & T_\alpha^{-1} U_2^{-1} \end{bmatrix} \begin{bmatrix} \tilde{R} - \kappa\Gamma & \tilde{S} \\ \tilde{P} & \tilde{Q} \end{bmatrix} \begin{bmatrix} U_1 & 0 \\ 0 & U_2 T_\alpha \end{bmatrix} \\ &= \begin{bmatrix} U_1^{-1} \tilde{R} U_1 - \kappa \begin{bmatrix} \gamma_1 & & \\ & \ddots & \\ & & \gamma_m \end{bmatrix} & U_1^{-1} \tilde{S} U_2 T_\alpha \\ T_\alpha^{-1} U_2^{-1} \tilde{P} U_1 & \begin{bmatrix} q_{m+1} & \delta_{m+1}/\alpha & & \\ & \ddots & \ddots & \\ & & q_{n-1} & \delta_{n-1}/\alpha \\ & & & q_n \end{bmatrix} \end{bmatrix}. \end{aligned}$$

Define $r_i := (U_1^{-1} \tilde{R} U_1)_{ii}$, $i = 1, \dots, m$ and the off-diagonal row sums

$$\rho_i(\alpha) := \sum_{\substack{j=1 \\ j \neq i}}^n |M(\alpha, \kappa)_{ij}|, \quad i = 1, \dots, n.$$

Note that the off-diagonal row sums are independent of κ and the γ_i . Fix $\varepsilon > 0$. We may now choose $\alpha > 0$ sufficiently large so that the effect of the scaling matrix T_α is that

$$\forall i \in \{m+1, \dots, n\} : \rho_i := \rho_i(\alpha) \in [0, \varepsilon].$$

Since $\gamma_i \neq 0$, $i = 1, \dots, m$ we may now choose $\kappa^* \geq 1$ sufficiently large so that

$$\forall \kappa \geq \kappa^* : \bigcup_{i=1}^m B(r_i - \kappa \gamma_i, \rho_i) \cap \bigcup_{j=m+1}^n B(q_j, \varepsilon) = \emptyset.$$

We are now in a position to apply Gershgorin's disks, see [8, Theorem 4.2.19], to deduce the claim. This completes the proof of the lemma. \square

Proof of Theorem 4.1.

(i): We prove “ \Rightarrow ”. By Theorem 2.1, $[E, A, B, C]$ is equivalent to a system in the form (2.4). We introduce the “new states”

$$\xi := \frac{1}{p_{\rho-1}} \cdot p\left(\frac{d}{dt}\right)y, \quad \chi := \left(y^\top, \dot{y}^\top, \dots, \left(y^{(\rho-2)} \right)^\top, \eta^\top \right)^\top,$$

and observe that

$$\begin{aligned} \dot{\xi}(t) &= \tilde{R} \xi(t) + \tilde{S} \chi(t) + \Gamma u(t) \\ \dot{\chi}(t) &= \tilde{P} \xi(t) + \tilde{Q} \chi(t), \end{aligned} \tag{7.4}$$

where \tilde{R}, \tilde{S} are matrices of appropriate size and

$$\tilde{P} = [0, \dots, 0, I_m, 0]^\top, \quad \tilde{Q} = \begin{bmatrix} \hat{A} & 0 \\ \hat{P} & Q \end{bmatrix},$$

with $\hat{A} = \begin{bmatrix} 0 & I & & \\ & \ddots & \ddots & \\ & & 0 & I \\ -\frac{p_0}{p_{\rho-1}} I & \dots & -\frac{p_{\rho-3}}{p_{\rho-1}} I & -\frac{p_{\rho-2}}{p_{\rho-1}} I \end{bmatrix}, \quad \hat{P} = [P, 0, \dots, 0].$

Note that \hat{A} is Hurwitz since $p(s)$ is Hurwitz. The feedback (1.17) reads in the new coordinates

$$u(t) = -k p\left(\frac{d}{dt}\right)y(t) = -k p_{\rho-1} \xi(t), \tag{7.5}$$

and therefore the application of (1.17) to $[E, A, B, C]$ results, in terms of the new system (7.4), in the closed-loop system

$$\frac{d}{dt} \begin{pmatrix} \xi(t) \\ \chi(t) \end{pmatrix} = \underbrace{\begin{bmatrix} \tilde{R} - k p_{\rho-1} \Gamma & \tilde{S} \\ \tilde{P} & \tilde{Q} \end{bmatrix}}_{=: A_k} \begin{pmatrix} \xi(t) \\ \chi(t) \end{pmatrix}. \tag{7.6}$$

Note that the closed-loop system (1.1), (1.13) is asymptotically stable if (7.6) is asymptotically stable. This is due to the fact that $p(s)$ is Hurwitz and if y decays exponentially so do all derivatives of y and, by (2.4), also u and all derivatives of u , which finally gives that x_c decays exponentially.

We show that (7.6) is asymptotically stable. Note that \tilde{Q} is Hurwitz since Q is Hurwitz. Therefore, by $p_{\rho-1} > 0$, Γ positive definite and $\sigma(\tilde{Q}) \subseteq \mathbb{C}_-$, we may apply Lemma 7.2 to conclude that

$$\exists k^* \geq 0 \forall k \geq k^* : \sigma(A_k) \subseteq \mathbb{C}_-.$$

This proves the claim.

To see that “ \Leftarrow ” does not hold true, consider system (1.1) for

$$E = I, \quad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = [1, 0], \quad (7.7)$$

which is in zero dynamics form (2.1) and has strict relative degree 1. We may observe that $Q = 0$ and therefore the zero dynamics of (7.7) are not asymptotically stable. However, the closed-loop system (7.7), (1.13) where $p(s) \equiv p_0 > 0$ takes the form

$$\dot{x}(t) = A_k x(t) = \begin{bmatrix} -kp_0 & 1 \\ -1 & 0 \end{bmatrix} x(t),$$

which is, since

$$\sigma(A_k) = \left\{ -\frac{p_0 k}{2} \pm \sqrt{\frac{(p_0 k)^2}{4} - 1} \right\},$$

asymptotically stable for all $k > 0$. This shows that $[E, A, B, C]$ is high-gain stabilizable.

(ii): The proof of “ \Rightarrow ” has been carried out in Remark 4.2(ii). It remains to show that “ \Leftarrow ” does not hold true. Consider system (1.1) for

$$E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad (7.8)$$

which is in zero dynamics form (2.5) with $n_1 = n_2 = 2$, $n_c = n_{\bar{c}} = 0$ and

$$A_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

Since Q has an eigenvalue at the origin, Theorem 3.2 yields that the zero dynamics are not asymptotically stable. The asymptotic stability of the closed-loop system ‘(7.8) & (1.14)’ is determined by the spectrum of the matrix

$$Q - A_{21}(kI_2 + A_{11})^{-1}A_{12} = \begin{bmatrix} 0 & \frac{1}{k} \\ -\frac{1}{k} & -1 \end{bmatrix},$$

and this satisfies, for all $k \in \mathbb{R} \setminus \{0\}$,

$$\sigma(Q - A_{21}(kI_2 + A_{11})^{-1}A_{12}) = \left\{ -\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{k^2}}, -\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{k^2}} \right\} \subseteq \mathbb{C}_-.$$

This shows that $[E, A, B, C]$ is high-gain stabilizable and completes the proof of the theorem. \square

Proof of Theorem 5.1.

(i): Without restriction of generality, one may consider $[E, A, B, C]$ in the form (2.4). Ignoring the bottom two algebraic equations in (2.4), the claim in (i) and the bound on e follow from [12, Theorem 2]. Since γ_ρ is a C^∞ -function, it is easy to see that u is $(\nu - 1)$ -times continuously differentiable and all of these derivatives are bounded functions. Therefore, x_c and \bar{x}_c in (2.4) are bounded functions. It remains to show the bound on k in (5.2): We clearly have that

$$\forall t > 0 : \|k\|_\infty \leq \frac{1}{1 - \|\varphi\|_\infty^2 \|e(t)\|^2} \stackrel{(5.1)}{\leq} \frac{1}{1 - (\varphi(t)^{-1} \|\varphi\|_\infty - \varepsilon \|\varphi\|_\infty)^2}.$$

Taking the infimum over all $t > 0$ and observing that $\inf_{t>0} \varphi(t)^{-1} = (\sup_{t>0} \varphi(t))^{-1} = \|\varphi\|_\infty^{-1}$, yields (5.2).

(ii): This result is shown in [2, Thm. 6.2]. \square

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