Besov spaces on fractals and tempered Radon measures

Dissertation

zur Erlangung des akademischen Grades

doctor rerum naturalium (Dr. rer. nat.)

vorgelegt dem Rat der

Fakultät für Mathematik und Informatik

der Friedrich-Schiller Universität Jena

von M.Sc. Maryia Kabanava
geboren am 16. Oktober 1984 in Minsk

Gutachter

1. Prof. Dr. Hans Triebel – *summa cum laude*
2. Prof. Dr. Martina Zähle – *summa cum laude*
3. Prof. Dr. Fernando Cobos Díaz – *summa cum laude*

Tag der öffentlichen Verteidigung: 26. Juli 2011

Acknowledgements

I would like to express my deepest gratitude to my supervisors Professor Hans Triebel and Professor Hans-Jürgen Schmeisser for many fruitful discussions, suggestions and remarks. I would like to thank the whole group "Function spaces" for inspiring and friendly atmosphere. Besides, I would like to thank Professor Krotov for helpful conversations and comments.

Contents

Zusammenfassung

Viele physikalische Phänomene können durch partielle Differenzialgleichungen beschrieben werden. Dies führt häufig auf Euler-Gleichungen für bestimmte Variationsprobleme der folgenden Art: Gesucht ist das Extremum eines Funktionals der Form

$$\int_\Omega F\left(u, \frac{\partial u}{\partial x_j}, x_j\right) d\Omega + \int_\Gamma \Phi\left(u, \frac{\partial u}{\partial x_j}, x_j\right) d\gamma$$

in einer Klasse von Funktionen, wobei Γ der Rand des Gebiets $\Omega \subset \mathbb{R}^n$ ist.

In den 1930er Jahren beschäftigte sich Sobolev mit dem Variationsproblem in einer Klasse von Funktionenräumen W_p^k, die nach ihm benannt wurden [37]. Die Elemente von $W_p^k(\mathbb{R}^n)$ sind Funktionen $f \in L_p(\mathbb{R}^n)$ mit der endlichen Norm

$$\|f|W_p^k(\mathbb{R}^n)\| = \sum_{|\alpha|\le k} \|D^\alpha f|L_p(\mathbb{R}^n)\|,$$

wobei

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha^n}}$$

schwache Ableitungen von f der Ordnung $|\alpha| = \alpha_1 + \ldots + \alpha_n$ sind und $D^0 f = f$ gesetzt wird. Der Sobolevsche Einbettungssatz besagt, dass alle Funktionen aus W_p^k mit $kp > n$ stetig sind (nach Modifikation auf einer Menge vom Maß Null). Ist $kp < n$, dann gehören die Funktionen aus W_p^k zu L_q für bestimmte $q > p$.

Satz 0.1 (Sobolevscher Einbettungssatz)**.** *Es sei $k \in \mathbb{N}_0$ und $1 < p < \infty$.*

(a) Wenn $kp > n$ gilt, dann ist $W_p^k(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n)$.

(b) Wenn $kp = n$ gilt, dann ist $W_p^k(\mathbb{R}^n) \hookrightarrow L_q(\mathbb{R}^n)$ für $p \le q < \infty$.

(c) Wenn $kp < n$ gilt, dann ist $W_p^k(\mathbb{R}^n) \hookrightarrow L_q(\mathbb{R}^n)$ für $p \le q \le \frac{np}{n-kp}$.

Die Funktionenräume $W_p^k(\mathbb{R}^n)$ bilden eine diskrete Familie in Bezug auf den Parameter k, $(k = 0, 1, 2 \ldots)$. Die Verallgemeinerung von Sobolev Räumen sind Bessel-Potential-Räume $H_p^s(\mathbb{R}^n)$, wobei der Glattheitsparameter s eine beliebige positive Zahl ist. Eine Funktion f ist in $H_p^s(\mathbb{R}^n)$, $s > 0$, $1 \le p \le \infty$, falls eine Funktion $g \in L_p(\mathbb{R}^n)$ existiert, so dass $f = G_s * g$ f.ü. in $\mathbb{R}^n$ ist. Hierbei ist G_s der Besselkern in $\mathbb{R}^n$ und $*$ bezeichnet

die Faltung. Die Norm von $f = G_s * g$ ist durch $\|f|H^s_p(\mathbb{R}^n)\| = \|g|L_p(\mathbb{R}^n)\|$ gegeben. Die Räume H^s_p heißen "Sobolevsche Räume gebrochener Ordnung", wobei $H^k_p(\mathbb{R}^n) = W^k_p(\mathbb{R}^n)$ für $1 < p < \infty$ und $k \in \mathbb{N}$ ist.

Wir können den Sobolevsche Einbettungssatz verallgemeinern, indem wir die Spur von $W^k_p(\mathbb{R}^n)$ auf $\mathbb{R}^m$ mit $1 \le m < n$ betrachten.

Satz 0.2 (Sobolevscher Einbettungssatz)**.** *Es sei $k \in \mathbb{N}_0$, $1 < p < \infty$ und $m \in \mathbb{N}$ mit $n - kp < m \le n$.*

(a) Wenn $kp = n$ gilt, dann ist $W^k_p(\mathbb{R}^n) \hookrightarrow L_q(\mathbb{R}^m)$ für $p \le q < \infty$.

(b) Wenn $kp < n$ gilt, dann ist $W^k_p(\mathbb{R}^n) \hookrightarrow L_q(\mathbb{R}^m)$ für $p \le q \le \frac{mp}{n-kp}$.

Hierbei stimmt der Spurraum aus Satz 0.2 nicht mit dem gesamten Raum $L_q(\mathbb{R}^m)$ überein. Um die Spurräume von $W^k_p(\mathbb{R}^n)$ zu bestimmen, benötigt man neue Funktionenräume, $n > m$. Hierzu wurden Besovräume $B^s_{pq}(\mathbb{R}^n)$ eingeführt. Eine Funktion f gehört zu $B^s_{pq}(\mathbb{R}^n)$ mit $0 \le k < s \le k+1$, $1 \le p, q \le \infty$, falls

$$\|f|B^s_{pq}(\mathbb{R}^n)\| = \sum_{|\alpha|\le k} \|D^\alpha f|L_p(\mathbb{R}^n)\| + \sum_{|j|=k} \left(\int_{\mathbb{R}^n} \frac{\|\Delta^2_h D^j f|L_p(\mathbb{R}^n)\|^q}{|h|^{n+(s-k)q}}\, dh \right)^{1/q} < \infty.$$

Die Spursätze können nunmehr folgendermaßen formuliert werden.

Satz 0.3. *Es sei $t = s - \frac{n-m}{p} > 0$, $1 \le p, q \le \infty$ und $1 \le m < n$. Dann ist*

$$B^s_{pq}(\mathbb{R}^n)|_{\mathbb{R}^m} = B^t_{pq}(\mathbb{R}^m). \tag{1}$$

Die Interpretation der Gleichung (1) ist wie folgt: Ist $f \in B^s_{pq}(\mathbb{R}^n)$, dann existiert die Spur von f auf $\mathbb{R}^m$, bezeichnet als $\operatorname{tr} f$ oder $f|_{\mathbb{R}^m}$ f.ü. auf $\mathbb{R}^m$ und sie gehört zu $B^t_{pq}(\mathbb{R}^m)$. Der Spuroperator ist stetig und für $f \in B^t_{pq}(\mathbb{R}^m)$ gibt es eine Funktion $\operatorname{ext} f \in B^s_{pq}(\mathbb{R}^n)$, so dass $\operatorname{ext} f|_{\mathbb{R}^m} = f$ ist. Der Fortsetzungsoperator ist ein linearer, stetiger Operator.

Satz 0.4. *Es sei $t = s - \frac{n-m}{p} > 0$, $1 < p < \infty$ und $1 \le m < n$. Dann ist*

$$H^s_p(\mathbb{R}^n)|_{\mathbb{R}^m} = B^t_{pp}(\mathbb{R}^m). \tag{2}$$

Da $H^k_p(\mathbb{R}^n) = W^k_p(\mathbb{R}^n)$ für $k \in \mathbb{N}_0$ und $1 < p < \infty$ ist, erhalten wir als Folgerung von Satz 0.4 die Antwort für die von Sobolev diskutierte Frage über die Spur von $W^k_p(\mathbb{R}^n)$ auf einer glatten m-dimensionalen Mannigfaltigkeit.

Folgerung 0.5. *Es sei $t = k - \frac{n-m}{p} > 0$, $k \in \mathbb{N}$, $1 < p < \infty$ und $1 \le m < n$. Dann ist*

$$W^k_p(\mathbb{R}^n)|_{\mathbb{R}^m} = B^t_{pp}(\mathbb{R}^m). \tag{3}$$

Die Gleichungen (2) und (3) haben die gleiche Interpretation wie (1).

In der vorliegenden Arbeit sind wir an Spuren von Besovräumen auf so-genannten d-Mengen interessiert. Unter einer d-Menge, $0 < d < n$, verstehen wir eine kompakte Menge $\Gamma \subset \mathbb{R}^n$, so dass eine Radon Maß μ in $\mathbb{R}^n$ mit

$$\operatorname{supp} \mu = \Gamma \ \text{ and } \ \mu(B(\gamma, r)) \sim r^d, \ \ \gamma \in \Gamma, \ \ 0 < r \leq 1$$

existiert, wobei $B(\gamma, r)$ eine Kugel im $\mathbb{R}^n$ mit dem Mittelpunkt γ und dem Radius r ist. Übliche Beispiele von fraktalen d-Mengen sind die Cantor-Menge, die Kochsche Schneeflocke und das Sierpinski-Dreieck.

Der Besovraum $B^s_{pq}(\Gamma, \mu)$, $s > 0$, $1 < p < \infty$, $0 < q < \infty$ auf der d-Menge Γ lässt sich als Spurraum von $B^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n)$,

$$B^s_{pq}(\Gamma, \mu) = \operatorname{tr}_\mu B^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n),$$

definieren. Funktionen in Funktionenräumen sind normalerweise nur fast überall definiert. Im Kapitel 3 erklären wir, was die Spur einer Funktion auf einer d-Menge bedeutet, $0 < d < n$, wobei die Menge Γ das n-dimensionale Lebesgue-Mass Null hat. Wir erhalten auch eine Charakterisierung von $B^s_{pq}(\Gamma, \mu)$ durch einen neuen Typ von (s, p, σ)-Atomen.

Einige Besovräume auf dem Einheitsintervall können mittels Faber-Schauder-Basen beschrieben werden [48]. Im Kapitel 4 suchen wir nach ihrer Entsprechung für d-Mengen Γ. Deshalb müssen wir eine Darstellung der Funktionen in Faber-Schauder-Basen finden, die auf andere Mengen übertragen werden kann. Zu diesem Zweck benutzen wir Dirichlet-Formen [23, 39]. Man kann harmonische Funktionen auf Γ mit gegebenen Randwerten als eindeutig bestimmte Funktionen definieren, die $\mathcal{E}(f)$ minimieren. Analog können wir stückweise harmonische Funktionen definieren. Diese Funktionen können als Analogon der Faber-Schauder-Basen aufgefasst werden. Sie erlauben $f \in B^s_{pq}(\Gamma, \mu)$ in Bezug auf die Entwicklungskoeffizienten einer stückweisen harmonischen Basis zu charakterisieren. Unser Beweis basiert auf der atomaren Zerlegung von Besovräumen. Ein ähnliches Ergebnis wird auch in [32] präsentiert, wobei die durch Strichartz eingeführte harmonische Darstellung von Lipschitz Räumen $(\Lambda^{p,q}_\alpha)^{(1)}(\Gamma)$ benutzt wird. Es wurde in [3] gezeigt, dass $(\Lambda^{p,q}_\alpha)^{(1)}(\Gamma)$ mit $\operatorname{Lip}(\alpha/\alpha_0, p, q, \Gamma)$ übereinstimmt, falls Γ ein nested Fraktal ist. Auch auf diese Weise kann man die harmonische Darstellung von Besovräumen beweisen.

Manche Fraktale, die wir später selbstähnliche Kurven nennen, sind homöomorph zum Einheitsintervall $\mathrm{I} = [0, 1]$. Das erlaubt Besov-Typ-Räume $\mathbb{B}^s_{pq}(K, \mu)$ als

$$\mathbb{B}^s_{pq}(K, \mu) = \left\{ f \circ H^{-1} : f \in B^s_{pq}(\mathrm{I}) \right\} = B^s_{pq}(\mathrm{I}) \circ H^{-1}$$

zu definieren. Dabei ist

$$H : \mathrm{I} \to K$$

eine homöomorphe Abbildung. Im Kapitel 5 studieren wir folgende Probleme:

- Wie hängen die Funktionenräume $\mathbb{B}^s_{pq}(K, \mu)$ und $B^s_{pq}(K, \mu)$ zusammen?

- Wie kann man $\mathbb{B}^s_{pq}(K,\mu)$ durch Wavelets charakterisieren?
- Welcher Isomorphismus wird durch die Wavelet Darstellung erzeugt?

Die Grundidee ist, die Abbildung H zu verwenden, um orthogonale Wavelet Basen von $\mathbb{R}^n$ auf K zu übertragen. In unserer Argumentation ist die Koch-Kurve ein grundlegendes Beispiel, obwohl alle unsere Schlussfolgerungen für jede selbstähnliche Kurve richtig bleiben.

Kapitel 6 bezieht sich auf Besovräume auf der geschlossenen Schneeflocke. Die Haupteigenschaft dieser Menge ist, dass sie nicht selbstähnlich ist, aber aus drei selbstähnlichen Koch-Kurven zusammengesetzt werden kann.

Es gibt d-Mengen, die zum Einheitswürfel $[0,1]^n$ homöomorph sind, und die man als ein kartesisches Produkt von bizarren fraktalen Kurven erhalten kann. In Kapitel 7 beschreiben wir die Verbindung zwischen isotropen Besovräumen auf solchen Mengen und anisotropen Besovräumen auf $[0,1]^n$.

Introduction

Many physical phenomena can be described by partial differential equations. These equations often appear as the Euler equations for certain variational problems. Then the boundary value problem can be reduced to the problem of finding the extremum in some class of functions of a functional of the form

$$\int_\Omega F\left(u, \frac{\partial u}{\partial x_j}, x_j\right) d\Omega + \int_\Gamma \Phi\left(u, \frac{\partial u}{\partial x_j}, x_j\right) d\gamma,$$

where Γ is the boundary of the domain $\Omega \subset \mathbb{R}^n$.

In 1930s Sobolev considered the variational problem on the class of functions W_p^k, which is named after him, see [37]. The elements of $W_p^k(\mathbb{R}^n)$ are functions $f \in L_p(\mathbb{R}^n)$ with the finite norm

$$\|f|W_p^k(\mathbb{R}^n)\| = \sum_{|\alpha|\le k} \|D^\alpha f|L_p(\mathbb{R}^n)\|,$$

where

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha^n}}$$

are distributional derivatives of f of order $|\alpha| = \alpha_1 + \ldots + \alpha_n$, and $D^0 f = f$. Sobolev's embedding theorem states that if $kp > n$, then W_p^k consists of continuous functions (after modification on a set of measure zero), and if $kp < n$, then the functions in W_p^k belong to L_q for a certain $q > p$.

Theorem 0.1 (Sobolev's embedding theorem)**.** *Let $k \in \mathbb{N}_0$ and $1 < p < \infty$.*

(a) If $kp > n$, then $W_p^k(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n)$.

(b) If $kp = n$, then $W_p^k(\mathbb{R}^n) \hookrightarrow L_q(\mathbb{R}^n)$, for $p \le q < \infty$.

(c) If $kp < n$, then $W_p^k(\mathbb{R}^n) \hookrightarrow L_q(\mathbb{R}^n)$, for $p \le q \le \frac{np}{n-kp}$.

Function spaces $W_p^k(\mathbb{R}^n)$ form the discrete family with respect to parameter k, ($k = 0, 1, 2 \ldots$). The generalization of Sobolev spaces are Bessel potential spaces $H_p^s(\mathbb{R}^n)$, where the smoothness parameter s is any positive real number. A function $f \in H_p^s(\mathbb{R}^n)$, $s > 0$, $1 \le p \le \infty$, if there is a function $g \in L_p(\mathbb{R}^n)$ such that $f = G_s * g$ a.e. in $\mathbb{R}^n$, where G_s stands for the Bessel kernel in $\mathbb{R}^n$ and $*$ denotes convolution. The norm of

$f = G_s * g$ is given by $\|f|H_p^s(\mathbb{R}^n)\| = \|g|L_p(\mathbb{R}^n)\|$. H_p^s are called "fractional Sobolev spaces", since $H_p^k(\mathbb{R}^n) = W_p^k(\mathbb{R}^n)$ for $1 < p < \infty$ and $k \in \mathbb{N}$.

We may generalize the Sobolev's embedding theorem by considering the trace of $W_p^k(\mathbb{R}^n)$ to $\mathbb{R}^m$ with $1 \leq m < n$.

Theorem 0.2 (Sobolev's embedding theorem, *cont.*)**.** *Let $k \in \mathbb{N}_0$, $1 < p < \infty$ and $m \in \mathbb{N}$ satisfying $n - kp < m \leq n$.*

(a) If $kp = n$, then $W_p^k(\mathbb{R}^n) \hookrightarrow L_q(\mathbb{R}^m)$, for $p \leq q < \infty$.

(b) If $kp < n$, then $W_p^k(\mathbb{R}^n) \hookrightarrow L_q(\mathbb{R}^m)$, for $p \leq q \leq \frac{mp}{n-kp}$.

We do not get the whole spaces $L_q(\mathbb{R}^m)$ as the trace space in Theorem 0.2. This reveals the need of new function spaces which are the trace spaces to $\mathbb{R}^m$ of $W_p^k(\mathbb{R}^n)$, $n > m$. This resulted in the definition of the Besov spaces $B_{pq}^s(\mathbb{R}^n)$. A function $f \in B_{pq}^s(\mathbb{R}^n)$ with $0 \leq k < s \leq k+1$, $1 \leq p, q \leq \infty$, if

$$\|f|B_{pq}^s(\mathbb{R}^n)\| = \sum_{|\alpha| \leq k} \|D^\alpha f|L_p(\mathbb{R}^n)\| + \sum_{|j|=k} \left(\int_{\mathbb{R}^n} \frac{\|\Delta_h^2 D^j f|L_p(\mathbb{R}^n)\|^q}{|h|^{n+(s-k)q}} \, dh \right)^{1/q} < \infty.$$

Then the trace theorems can be stated in the following way.

Theorem 0.3. *Let $t = s - \frac{n-m}{p} > 0$, $1 \leq p, q \leq \infty$ and $1 \leq m < n$. Then*

$$B_{pq}^s(\mathbb{R}^n)|_{\mathbb{R}^m} = B_{pq}^t(\mathbb{R}^m). \tag{4}$$

The interpretation of the equality (4) is as follows: if $f \in B_{pq}^s(\mathbb{R}^n)$, then the restriction of f to $\mathbb{R}^m$, denoted by $\operatorname{tr} f$ or $f|_{\mathbb{R}^m}$, exists a.e. on $\mathbb{R}^m$ and belongs to $B_{pq}^t(\mathbb{R}^m)$ and the restriction operator is continuous, and if $f \in B_{pq}^t(\mathbb{R}^m)$, then there is a function $\operatorname{ext} f \in B_{pq}^s(\mathbb{R}^n)$ such that $\operatorname{ext} f|_{\mathbb{R}^m} = f$, and the extension is given by a linear and continuous operator. We will discuss later how to define the restriction of f.

Theorem 0.4. *Let $t = s - \frac{n-m}{p} > 0$, $1 < p < \infty$ and $1 \leq m < n$. Then*

$$H_p^s(\mathbb{R}^n)|_{\mathbb{R}^m} = B_{pp}^t(\mathbb{R}^m). \tag{5}$$

Since $H_p^k(\mathbb{R}^n) = W_p^k(\mathbb{R}^n)$ for $k \in \mathbb{N}_0$ and $1 < p < \infty$, as the corollary of Theorem 0.4 we get the answer to the question raised by Sobolev about the trace of $W_p^k(\mathbb{R}^n)$ on a smooth m-dimensional manifold.

Corollary 0.5. *Let $t = k - \frac{n-m}{p} > 0$, $k \in \mathbb{N}$, $1 < p < \infty$ and $1 \leq m < n$. Then*

$$W_p^k(\mathbb{R}^n)|_{\mathbb{R}^m} = B_{pp}^t(\mathbb{R}^m). \tag{6}$$

The equalities (5) and (6) have the same interpretation as (4).

In the present work we are interested in the traces of Besov spaces on the so-called d-sets. By a d-set, $0 < d < n$, we mean a compact set $\Gamma \subset \mathbb{R}^n$ such that there is a Radon measure μ in $\mathbb{R}^n$ with

$$\operatorname{supp} \mu = \Gamma \quad \text{and} \quad \mu(B(\gamma, r)) \sim r^d, \quad \gamma \in \Gamma, \quad 0 < r \leq 1,$$

where $B(\gamma, r)$ is a ball in $\mathbb{R}^n$ centered at γ and of radius r. Standard examples of fractal d-sets are Cantor set, the von Koch's snowflake and the Sierpinski gasket.

Besov spaces $B^s_{pq}(\Gamma, \mu)$, $s > 0$, $1 < p < \infty$, $0 < q < \infty$ on the d-set Γ can be defined as the trace of the Besov space $B^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n)$:

$$B^s_{pq}(\Gamma, \mu) = \operatorname{tr}_\mu B^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n).$$

Functions in function spaces are usually defined only almost everywhere. In Chapter 3 we explain what is meant with a restriction of a function to a d-set, $0 < d < n$, that has an n-dimensional Lebesgue measure zero. We also provide the characterization of $B^s_{pq}(\Gamma, \mu)$ by new type of (s, p, σ)-atoms.

Some Besov spaces on the most trivial example of a d-set, the unit interval, can be described by means of Faber-Schauder basis, see [48]. In Chapter 4 we are looking for its counterpart for the d-set Γ. So we need to find the description of functions in Faber-Schauder basis in such a way that it can be transferred to other sets. Our approach is to start with a Dirichlet form $(\mathcal{E}, \mathcal{D})$, see e.g. [23, 39]. Then the harmonic function on Γ with given boundary values can be defined as the unique function that minimizes $\mathcal{E}(f)$. Similarly we can define piecewise harmonic functions. These functions may serve as the counterpart of Faber-Schauder basis and they allow to characterize $f \in B^s_{pq}(\Gamma, \mu)$ in terms of the coefficients of its expansion in a piecewise harmonic basis. Our proof is based on the atomic characterization of Besov spaces. A similar result is also presented in the paper [32], where the harmonic representation of Lipschitz spaces $(\Lambda^{p,q}_\alpha)^{(1)}(\Gamma)$ introduced by Strichartz is stated. It was shown in [3] that $(\Lambda^{p,q}_\alpha)^{(1)}(\Gamma)$ coincide with $\operatorname{Lip}(\alpha/\alpha_0, p, q, \Gamma)$, when Γ is a nested fractal. Thus the harmonic representation of Besov spaces might be also proved by using the discrete characterizations of Besov spaces.

Certain fractals, which we call later on self-similar curves, are homeomorphic to the unit interval $\mathrm{I} = [0, 1]$. This allows to define Besov-type spaces $\mathbb{B}^s_{pq}(K, \mu)$ by

$$\mathbb{B}^s_{pq}(K, \mu) = \left\{ f \circ H^{-1} : f \in B^s_{pq}(\mathrm{I}) \right\} = B^s_{pq}(\mathrm{I}) \circ H^{-1},$$

where

$$H : \mathrm{I} \to K$$

is a homeomorphic map. In Chapter 5 we study the following problems:

- How the function spaces $\mathbb{B}^s_{pq}(K, \mu)$ and $B^s_{pq}(K, \mu)$ are interrelated.
- How $\mathbb{B}^s_{pq}(K, \mu)$ can be characterized by wavelets.

- What kind of isomorphisms are induced by wavelet representation.

The basic idea is to use transform H to transfer orthogonal wavelet bases from $\mathbb{R}^n$ to K. In our reasoning the Koch curve serves as a basic example, though all our conclusions remain true for any self-similar curve.

Chapter 6 deals with Besov spaces on the closed snowflake. The main feature of this set is that it is not self-similar but consists of three self-similar Koch curves clipped together.

There are d-sets homeomorphic to the unit cube $[0,1]^n$, which are obtained as a cartesian product of bizarre fractal curves. In Chapter 7 we follow the connection between isotropic Besov spaces on such sets and anisotropic Besov spaces on $[0,1]^n$.

CHAPTER 1

Preliminaries

1.1 Function spaces

1.1.1 Basic notation and classical Besov spaces

Let $\mathbb{N}$ be the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. $\mathbb{Z}$ is the set of all integers, $\mathbb{C}$ is the complex plane. We denote by $\mathbb{R}^n$ the Euclidean n-space, where $n \in \mathbb{N}$. Let $|\cdot - \cdot|_n$ denote the Euclidean distance in $\mathbb{R}^n$. Put $\mathbb{R} = \mathbb{R}^1$. The scalar product of $x, y \in \mathbb{R}^n$ is given by $xy = \sum_{i=1}^{n} x_i y_i$. $\mathbb{Z}^n$ denotes the lattice of all points in $\mathbb{R}^n$ with integer-valued components. Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{N}_0^n$ be a multi-index. Then $|\beta| = \beta_1 + \beta_2 + \ldots + \beta_n$,

$$x^\beta = x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}$$

and

$$D^\beta f = \frac{\partial^{|\beta|} f}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} \ldots \partial x_n^{\beta_n}}.$$

We will write $a \sim b$ to denote that there are constants $c, c' > 0$ such that

$$c' a \leq b \leq ca$$

for all admited a, b.

$S(\mathbb{R}^n)$ stands for the Schwartz space of all complex-valued, rapidly decreasing, infinitely differentiable functions on $\mathbb{R}^n$. By $S'(\mathbb{R}^n)$ we denote its topological dual, the space of all tempered distributions on $\mathbb{R}^n$. $L_p(\mathbb{R}^n)$ with $0 < p \leq \infty$, is the standard quasi-Banach space with respect to Lebesgue measure, quasi-normed by

$$\|f|L_p(\mathbb{R}^n)\| = \left(\int\limits_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{\frac{1}{p}}, \quad 0 < p < \infty,$$

$$\|f|L_\infty(\mathbb{R}^n)\| = \operatorname*{ess-sup}_{x \in \mathbb{R}^n} |f(x)|.$$

If $\varphi \in S(\mathbb{R}^n)$, then

$$\widehat{\varphi}(\xi) = \mathcal{F}\varphi(\xi) = (2\pi)^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} \varphi(x) e^{-ix\xi} \, dx, \quad \xi \in \mathbb{R}^n,$$

denotes the Fourier transform of φ. The inverse Fourier transform is given by

$$\varphi^\vee(x) = \mathcal{F}^{-1}\varphi(x) = (2\pi)^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} \varphi(\xi) e^{ix\xi} \, d\xi, \quad x \in \mathbb{R}^n.$$

We extend $\mathcal{F}$ and $\mathcal{F}^{-1}$ in the usual way from S to S'. For $f \in S'(\mathbb{R}^n)$,

$$\mathcal{F}f(\varphi) = f(\mathcal{F}\varphi), \quad \varphi \in S(\mathbb{R}^n).$$

We define $\varphi_0 \in S(\mathbb{R}^n)$ by

$$\varphi_0(x) = 1, \quad |x| \leq 1 \;\text{ and }\; \varphi_0(x) = 0, \quad |x| \geq \frac{3}{2}, \tag{1.1}$$

and let

$$\varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{-k+1}x), \quad x \in \mathbb{R}^n, \quad k \in \mathbb{N}. \tag{1.2}$$

Then, since

$$1 = \sum_{j=0}^{\infty} \varphi_j(x) \quad \text{for all } \; x \in \mathbb{R}^n, \tag{1.3}$$

the $\{\varphi_j\}$ form a dyadic resolution of unity in $\mathbb{R}^n$. According to the Paley-Wiener-Schwartz theorem $\left(\varphi_k \widehat{f}\right)^\vee$ is an entire analytic function on $\mathbb{R}^n$ for any $f \in S'(\mathbb{R}^n)$. In particular, $\left(\varphi_k \widehat{f}\right)^\vee(x)$ makes sense pointwise.

Definition 1.1. Let $\varphi = \{\varphi_j\}_{j=0}^{\infty}$ be the dyadic resolution of unity according to (1.1)-(1.3), $s \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q \leq \infty$ and

$$\|f | B^s_{pq}(\mathbb{R}^n)\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \| \left(\varphi_k \widehat{f}\right)^\vee | L_p(\mathbb{R}^n)\|^q \right)^{\frac{1}{q}}$$

(with the usual modification if $q = \infty$). Then the Besov space $B^s_{pq}(\mathbb{R}^n)$ consists of all $f \in S'(\mathbb{R}^n)$ such that $\|f | B^s_{pq}(\mathbb{R}^n)\| < \infty$.

1.1.2 Characterization of $B^s_{pq}(\mathbb{R}^n)$ by local means and atoms

Local means

Let k and k_0 be C^∞ functions with

$$\operatorname{supp} k, \; \operatorname{supp} k_0 \subset \{x \in \mathbb{R}^n : |x| \leq 1\}.$$

We assume $k_0^\vee(0) \neq 0$ and for given $s \in \mathbb{R}$

$$\begin{aligned} &k^\vee(\xi) \neq 0, && 0 < |\xi| \le \varepsilon \ \text{ for some } \varepsilon > 0,\\ &\left(D^\alpha k^\vee\right)(0) = 0, && |\alpha| \le s. \end{aligned}$$

Define

$$k(t,f)(x) = \int\limits_{\mathbb{R}^n} k(y) f(x+ty)\, dy = t^{-n} \int\limits_{\mathbb{R}^n} k\left(\frac{y-x}{t}\right) f(y)\, dy.$$

Theorem 1.2. *Let $s \in \mathbb{R}$, $0 < p \le \infty$ and $0 < q \le \infty$. Then $B^s_{pq}(\mathbb{R}^n)$ is the collection of $f \in S'(\mathbb{R}^n)$ such that*

$$\|f|B^s_{pq}(\mathbb{R}^n)\|^{k_0,k} = \|k_0(1,f)|L_p(\mathbb{R}^n)\| + \left(\sum_{j=1}^\infty 2^{jsq} \|k(2^{-j},f)|L_p(\mathbb{R}^n)\|^q\right)^{\frac{1}{q}} < \infty.$$

We refer to [46, Section 1.4].

Smooth atoms

Let $Q_{\nu m}$ denote the closed cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes, centered at $2^{-\nu} m$ and with side length $2^{-\nu+1}$, where $m \in \mathbb{Z}^n$, $\nu \in \mathbb{N}_0$. The notation cQ stands for the cube concentric with Q and with side length c times the side length of Q.

Definition 1.3.

(i) Let $K \in \mathbb{N}_0$ and $c \ge 1$. A continuous function $a : \mathbb{R}^n \to \mathbb{C}$ for which there exist all derivatives $D^\alpha a$ if $|\alpha| \le K$ is called a 1_K-atom if

$$\begin{aligned} &\operatorname{supp} a \subset cQ_{0m} && \text{for some } \ m \in \mathbb{Z}^n,\\ &|D^\alpha a(x)| \le 1 && \text{for } \ |\alpha| \le K. \end{aligned}$$

(ii) Let $s \in \mathbb{R}$, $0 < p \le \infty$, $K \in \mathbb{N}_0$, $L \ge 0$ and $c \ge 1$. A continuous function $a : \mathbb{R}^n \to \mathbb{C}$ for which there exist all derivatives $D^\alpha a$ if $|\alpha| \le K$ is called an $(s,p)_{K,L}$-atom if

$$\begin{aligned} &\operatorname{supp} a \subset cQ_{\nu m} && \text{for some } \ \nu \in \mathbb{N},\ m \in \mathbb{Z}^n,\\ &|D^\alpha a(x)| \le 2^{-\nu\left(s-\frac{n}{p}\right)+|\alpha|\nu} && \text{for } \ |\alpha| \le K,\\ &\int\limits_{\mathbb{R}^n} x^\beta a(x)\, dx = 0 && \text{for } \ |\beta| < L. \end{aligned}$$

Definition 1.4. Let $0 < p \le \infty$, $0 < q \le \infty$. Then b_{pq} is the collection of sequences $\lambda = \{\lambda_{\nu m} \in \mathbb{C} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$ such that

$$\|\lambda|b_{pq}\| = \left(\sum_{\nu=0}^\infty \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} < \infty.$$

We put

$$\sigma_p = \max\left\{n\left(\frac{1}{p}-1\right), 0\right\}.$$

We write $a_{\nu m}$ instead of a in Definition 1.3 to indicate the location and size of an atom.

Theorem 1.5. *Let $s \in \mathbb{R}$, $0 < p \leq \infty$ and $0 < q \leq \infty$. Let $K \in \mathbb{N}_0$, $L \geq 0$, $c \geq 1$ with*

$$K > s \quad \text{and} \quad L > \sigma_p - s.$$

Then $f \in S'(\mathbb{R}^n)$ belongs to $B^s_{pq}(\mathbb{R}^n)$ if and only if it can be represented as

$$f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}, \tag{1.4}$$

unconditional convergence in $S'(\mathbb{R}^n)$, where $a_{\nu m}$ are 1_K-atoms for $\nu = 0$, $(s,p)_{K,L}$-atoms for $\nu \in \mathbb{N}$, and $\lambda \in b_{pq}$. Furthermore,

$$\|f|B^s_{pq}(\mathbb{R}^n)\| \sim \inf \|\lambda|b_{pq}\|$$

are equivalent quasi-norms, where the infimum is taken over all admissible representations (1.4).

For the history of atomic representation in function spaces we refer to Section 1.9 in [42]. The decomposition of functions in Besov spaces into atoms goes back to [7, 8].

1.1.3 Weighted Besov spaces

We denote by $L_p(\mathbb{R}^n, \langle x\rangle^\alpha)$, where

$$\langle x\rangle^\alpha = \left(1+|x|^2\right)^{\frac{\alpha}{2}},$$

the weighted L_p-space quasi-normed by

$$\|f|L_p(\mathbb{R}^n, \langle x\rangle^\alpha)\| = \|\langle\cdot\rangle^\alpha f|L_p(\mathbb{R}^n)\|.$$

Definition 1.6. Let $\varphi = \{\varphi_j\}_{j=0}^\infty$ be the dyadic resolution of unity according to (1.1)-(1.3), $s \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q \leq \infty$. Then the weighted Besov space $B^s_{pq}(\mathbb{R}^n, \langle x\rangle^\alpha)$ is a collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f|B^s_{pq}(\mathbb{R}^n, \langle x\rangle^\alpha)\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \|\left(\varphi_k \widehat{f}\right)^\vee |L_p(\mathbb{R}^n, \langle x\rangle^\alpha)\|^q\right)^{\frac{1}{q}}$$

(with the usual modification if $q = \infty$) is finite.

Remark 1.7. If $\alpha = 0$ then we have the space $B^s_{pq}(\mathbb{R}^n)$ as introduced in Definition 1.1. It is also known from [5, Ch. 4.2.2] that the operator $f \mapsto \langle x\rangle^\alpha f$ is an isomorphic mapping from $B^s_{pq}(\mathbb{R}^n, \langle x\rangle^\alpha)$ onto $B^s_{pq}(\mathbb{R}^n)$. In particular,

$$\|\langle\cdot\rangle^\alpha f|B^s_{pq}(\mathbb{R}^n)\| \sim \|f|B^s_{pq}(\mathbb{R}^n, \langle x\rangle^\alpha)\|.$$

1.1.4 Anisotropic Besov spaces

Anisotropic function spaces in $\mathbb{R}^n$ characterize the smoothness properties of functions depending on the direction of the j-th coordinate.

Let $1 < p < \infty$, $1 \leq q \leq \infty$ and

$$\bar{s} = (s_1, \dots, s_n), \quad 0 < s_j < M_j \in \mathbb{N}, \quad j = 1, \dots, n.$$

Let

$$(\Delta_h f)(x) = f(x+h) - f(x),$$

where $x \in \mathbb{R}^n$ and $h \in \mathbb{R}^n$. The iterated differences are defined by

$$(\Delta_h^{m+1} f)(x) = \Delta_h(\Delta_h^m f)(x).$$

Then the iterated differences in the direction of the j-th coordinate can be written as

$$(\Delta_{t,j}^m f)(x) = (\Delta_h^m f)(x), \text{ with } h = te_j, \quad t \in \mathbb{R},$$

where $e_j = (0, \dots, 0, 1, 0, \dots, 0)$ is the unit vector with 1 at place j. A function $f \in L_p(\mathbb{R}^n)$ belongs to the classical anisotropic Besov space $B_{pq}^{\bar{s}}(\mathbb{R}^n)$ if

$$\|f|B_{pq}^{\bar{s}}(\mathbb{R}^n)\| = \|f|L_p(\mathbb{R}^n)\| + \sum_{j=1}^{n} \left(\int_0^1 t^{-s_j p} \|\Delta_{t,j}^{M_j} f|L_p(\mathbb{R}^n)\|^q \frac{dt}{t} \right)^{1/q} < \infty.$$

If

$$s_1 = s_2 = \ldots = s_n = s > 0,$$

then $B_{pq}^{\bar{s}}(\mathbb{R}^n) = B_{pq}^{s}(\mathbb{R}^n)$.

When the smoothness parameter $\bar{s}$ is small, namely $0 < s_j < 1$, and $p = q$, $B_{pp}^{\bar{s}}(\mathbb{R}^n)$ can be equivalently normed by

$$\|f|B_{pp}^{\bar{s}}(\mathbb{R}^n)\|_* = \|f|L_p(\mathbb{R}^n)\| + \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{\left(\sum_{k=1}^{n} |x_k - y_k|^{s_k/s} \right)^{n+sp}} \, dx \, dy \right)^{1/p}, \tag{1.5}$$

where s with

$$\frac{1}{s} = \frac{1}{n} \left(\frac{1}{s_1} + \cdots + \frac{1}{s_n} \right)$$

stands for the mean smoothness.

It will be convenient for us to use slightly different notation. We rely on formula (1.5) for the expression of the norm in the anisotropic Besov space.

Definition 1.8. The n-tuple

$$\alpha = (\alpha_1, \dots, \alpha_n) \ \text{ with } \ 0 < \alpha_1 \le \dots \alpha_n < \infty, \quad \sum_{j=1}^{n} \alpha_j = n$$

is called an anisotropy in $\mathbb{R}^n$.

We define an anisotropic distance in $\mathbb{R}^n$ by

$$\varrho_{\alpha,n}(x,y) = |x-y|_\alpha, \quad \text{where} \quad |x|_\alpha = \max_{k=1,\dots,n} |x_k|^{\frac{1}{\alpha_k}}. \tag{1.6}$$

Definition 1.9. Let α be an anisotropy according to Definition 1.8. Let $1 < p < \infty$ and $0 < s < \alpha_1$. Then the anisotropic Besov space $B^{s,\alpha}_{pp}(\mathbb{R}^n)$ is the collection of all $f \in L_p(\mathbb{R}^n)$ such that the norm

$$\|f|B^{s,\alpha}_{pp}(\mathbb{R}^n)\| = \|f|L_p(\mathbb{R}^n)\| + \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{\varrho_{\alpha,n}(x,y)^{n+sp}}\, dx\, dy \right)^{1/p} \tag{1.7}$$

is finite.

Remark 1.10. The parameter $\bar{s}$ in (1.5) and s, α in Definition 1.9 are related by

$$\frac{1}{s} = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{s_k} \quad \text{and} \quad \alpha_k = \frac{s}{s_k}, \quad k = 1, \dots, n.$$

Let $Q = (0,1)^n$ be the open unit cube equipped with the Lebesgue measure. Then $B^{s,\alpha}_{pp}(Q)$ is the restriction of $B^{s,\alpha}_{pp}(\mathbb{R}^n)$ to Q, equipped in the usual way with the norm

$$\|f|B^{s,\alpha}_{pp}(Q)\| = \inf\{\|g|B^{s,\alpha}_{pp}(\mathbb{R}^n)\| : g \in B^{s,\alpha}_{pp}(\mathbb{R}^n),\ f(x) = g(x)\, \text{a.e.}\}.$$

An equivalent norm in $B^{s,\alpha}_{pp}(Q)$ can be obtained by replacing $\mathbb{R}^n$ in (1.7) by Q, [27]. We will use the following definition of $B^{s,\alpha}_{pp}(Q)$.

Definition 1.11. Let α be an anisotropy according to Definition 1.8. Let $1 < p < \infty$ and $0 < s < \alpha_1$. Then the anisotropic Besov space $B^{s,\alpha}_{pp}(Q)$ is the collection of all $f \in L_p(Q)$ such that the norm

$$\|f|B^{s,\alpha}_{pp}(Q)\| = \|f|L_p(Q)\| + \left(\int_{Q} \int_{Q} \frac{|f(x)-f(y)|^p}{\varrho_{\alpha,n}(x,y)^{n+sp}}\, dx\, dy \right)^{1/p} \tag{1.8}$$

is finite.

1.1.5 Periodic Besov spaces on $\mathbb{T}^n$

Let

$$\mathbb{T}^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : 0 \le x_i \le 1, i = 1, \dots, n\}$$

be the n-dimensional torus. $x \in \mathbb{T}^n$ and $y \in \mathbb{T}^n$ are identified if and only if $x - y = k$, $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$.

By $D(\mathbb{T}^n)$ we denote the collection of all complex-valued infinitely differentiable functions on $\mathbb{T}^n$. The topology in $D(\mathbb{T}^n)$ is generated by the family of semi-norms

$$\|\varphi\|_\alpha = \sup_{x \in \mathbb{T}^n} |D^\alpha \varphi(x)|,$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is an arbitrary multi-index. $D'(\mathbb{T}^n)$ is the class of all continuous linear functionals on $D(\mathbb{T}^n)$. The continuity of a linear functional f on $D(\mathbb{T}^n)$ means that there exist $N \in \mathbb{N}$ and $c_N > 0$ such that

$$|f(\varphi)| \le c_N \sum_{|\alpha| \le N} \|\varphi\|_\alpha,$$

for all $\varphi \in D(\mathbb{T}^n)$.

Let $0 < p \le \infty$. $L_p(\mathbb{T}^n)$ is the standard quasi-Banach space with respect to Lebesgue measure, quasi-normed by

$$\|f|L_p(\mathbb{T}^n)\| = \left(\int_{\mathbb{T}^n} |f(x)|^p \, dx \right)^{\frac{1}{p}},$$

with the usual modification if $p = \infty$. If $1 \le p \le \infty$ then $f \in L_p(\mathbb{T}^n)$ can be interpreted in a unique way as an element of $D'(\mathbb{T}^n)$ by

$$f(\varphi) = \int_{\mathbb{T}^n} f(x)\varphi(x) \, dx, \quad \varphi \in D(\mathbb{T}^n). \tag{1.9}$$

Consequently, for $1 \le p \le \infty$ we have

$$D(\mathbb{T}^n) \subset L_p(\mathbb{T}^n) \subset D'(\mathbb{T}^n), \tag{1.10}$$

where " $\subset$ " here and further on means the topological embedding.

Let $f \in D'(\mathbb{T}^n)$. Then the numbers

$$\widehat{f}(k) = f(e^{-2\pi ikx}), \quad k \in \mathbb{Z}^n,$$

are said to be the Fourier coefficients of f. If $f \in L_p(\mathbb{T}^n)$, $1 \le p \le \infty$, then (1.9), (1.10) imply that

$$\widehat{f}(k) = \int_{\mathbb{T}^n} f(x) e^{-2\pi ikx}, \quad k \in \mathbb{Z}^n.$$

It is well-known that any $f \in D'(\mathbb{T}^n)$ can be represented as

$$f = \sum_{k\in\mathbb{Z}^n} a_k e^{2\pi ikx}, \quad x \in \mathbb{T}^n, \quad (\text{convergence in } D'(\mathbb{T}^n)) \tag{1.11}$$

where the Fourier coefficients $\{a_k\} \subset \mathbb{C}$ are of at most polynomial growth,

$$|a_k| \le c\,(1+|k|)^\kappa, \quad \text{for some } c > 0, \quad \kappa > 0 \text{ and all } k \in \mathbb{Z}^n.$$

Definition 1.12. Let $\varphi = \{\varphi_j\}_{j=0}^\infty$ be a dyadic resolution of unity in $\mathbb{R}^n$ according to (1.1)-(1.3), $s \in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$ and

$$\|f|B^s_{pq}(\mathbb{T}^n)\| = \left(\sum_{j=0}^\infty 2^{jsq} \| \sum_{k\in\mathbb{Z}^n} \varphi_j(2\pi k) a_k e^{2\pi ikx} | L_p(\mathbb{T}^n)\|^q\right)^{\frac{1}{q}}$$

(with the usual modification if $q = \infty$). Then the Besov space $B^s_{pq}(\mathbb{T}^n)$ consists of all $f \in D'(\mathbb{T}^n)$ such that $\|f|B^s_{pq}(\mathbb{T}^n)\| < \infty$, [34, Chapter 3].

1.1.6 Wavelets on $\mathbb{R}$ and $\mathbb{T}$

First we recall some basic definitions. Let F be a separable complex Banach space.

Definition 1.13.

(i) A sequence $\{e_i\}_{i=1}^\infty$ is called a basis in F, if every $f \in F$ can be uniquely represented by

$$f = \sum_{i=1}^\infty a_i e_i, \quad a_i \in \mathbb{C}, \tag{1.12}$$

with convergence in F.

(ii) A basis $\{e_i\}_{i=1}^\infty$ is called an unconditional basis, if for any rearrangement σ of $\mathbb{N}$ the new sequence $\{e_{\sigma(i)}\}_{i=1}^\infty$ is again a basis and

$$f = \sum_{i=1}^\infty a_{\sigma(i)} e_{\sigma(i)} \quad (\text{convergence in } F)$$

for any $f \in F$ with (1.12).

Wavelet decomposition has proved to be a useful tool in studying function spaces. For the general theory concerning wavelet bases we refer to [4, 30, 49].

Let $C^u(\mathbb{R})$, $u \in \mathbb{N}$ denote the collection of all complex-valued continuous functions on $\mathbb{R}$ having continuous bounded derivatives up to order u inclusively. Let $\psi_F \in C^u(\mathbb{R})$ and $\psi_M \in C^u(\mathbb{R})$ be a father and a mother Daubechies wavelet on $\mathbb{R}$ respectively. We recall that ψ_F and ψ_M are real and have compact support. Moreover,

$$\int_{\mathbb{R}} \psi_F(x)\, dx = 1,$$

$$\int_{\mathbb{R}} x^l \psi_M(x)\, dx = 0, \quad \text{for } \; l = 0, \dots, u-1.$$

Define ψ_j^k by

$$\psi_j^k(x) = \begin{cases} \psi_F(x-k), & j=0,\ k \in \mathbb{Z}, \\ 2^{\frac{j-1}{2}} \psi_M(2^{j-1}x-k), & j \in \mathbb{N},\ k \in \mathbb{Z}. \end{cases} \tag{1.13}$$

Then $\{\psi_j^k\}_{j\in\mathbb{N}_0, k\in\mathbb{Z}}$ is an orthonormal basis in $L_2(\mathbb{R})$. We transform the wavelet basis of $L_2(\mathbb{R})$ into a wavelet basis of $L_2(\mathbb{T})$ by periodizing each member of the basis. But first we need to mofify it a little.

Let $L \in \mathbb{N}$. One can replace ψ_F and ψ_M by

$$\psi_F^L(\cdot) = \psi_F(2^L \cdot), \quad \psi_M^L(\cdot) = \psi_M(2^L \cdot),$$

ψ_j^k by

$$\psi_j^{L,k}(\cdot) = 2^{\frac{L}{2}} \psi_j^k(2^L \cdot). \tag{1.14}$$

We choose and fix L such that

$$\operatorname{supp} \psi_F^L \subset \left\{ x : |x| < \frac{1}{2} \right\}, \ \operatorname{supp} \psi_M^L \subset \left\{ x : |x| < \frac{1}{2} \right\}. \tag{1.15}$$

Let

$$\begin{aligned} \mathbb{P}_0 &= \left\{ k \in \mathbb{Z} : 0 \le k \le 2^L - 1 \right\} \\ \mathbb{P}_j &= \left\{ k \in \mathbb{Z} : 0 \le k \le 2^{j+L-1} - 1 \right\}, \quad j \in \mathbb{N}. \end{aligned} \tag{1.16}$$

Given the functions $\psi_j^{L,k}$ on the real line we can construct their 1-periodic counterparts by the procedure

$$\psi_{j,per}^{L,k}(x) = \sum_{l=-\infty}^{\infty} \psi_j^{L,k}(x+l). \tag{1.17}$$

Define $\psi_j^{L,k,per}$ on the 1-torus $\mathbb{T}$ by

$$\psi_j^{L,k,per}(x) = \psi_{j,per}^{L,k}(x), \quad x \in \mathbb{T}.$$

Then according to the Proposition 1.34 in [47]

$$\left\{ \psi_j^{L,k,per}, j \in \mathbb{N}_0, k \in \mathbb{P}_j \right\}$$

is an orthonomal basis in $L_2(\mathbb{T})$. We simplify the notation and omit L in $\psi_j^{L,k,per}$ and $\psi_{j,per}^{L,k}$.

Before we give the description of function spaces in terms of wavelets, we introduce some sequence spaces.

Definition 1.14. Let $0 < p \le \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. Then $b_{pq}^{s,per}$ is the collection of all sequences

$$\mu = \left\{ \mu_j^k \in \mathbb{C} : j \in \mathbb{N}_0, k \in \mathbb{P}_j \right\}$$

such that

$$\|\mu | b_{pq}^{s,per}\| = \left(\sum_{j=0}^{\infty} 2^{j\left(s - \frac{1}{p}\right) q} \left(\sum_{k \in \mathbb{P}_j} |\mu_j^k|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} < \infty.$$

We use the following notation. Let

$$\sigma_p = n \left(\frac{1}{p} - 1 \right)_+ ,$$

where $b_+ = \max(b, 0)$ if $b \in \mathbb{R}$. The scalar product on the torus is defined by

$$(f, \psi_j^{k,per})_{\mathbb{T}} = \int_{\mathbb{T}} f(x) \psi_j^{k,per}(x)\, dx.$$

Theorem 1.15. *Let $\{\psi_j^{k,per}\}$ be the orthonormal basis in $L_2(\mathbb{T})$. Let $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ and*

$$u > \max(s, \sigma_p - s).$$

Let $f \in D'(\mathbb{T})$. Then $f \in B_{pq}^s(\mathbb{T})$ if, and only if, it can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{k \in \mathbb{P}_j} \mu_j^k 2^{-\frac{j+L}{2}} \psi_j^{k,per}, \quad \mu \in b_{pq}^{s,per},$$

unconditional convergence being in $D'(\mathbb{T})$ and in any space $B_{pq}^{\sigma}(\mathbb{T})$ with $\sigma < s$. Furthermore, this representation is unique,

$$\mu_j^k = 2^{\frac{j+L}{2}} (f, \psi_j^{k,per})_{\mathbb{T}},$$

and

$$I : f \to \left\{ \mu_j^k, j \in \mathbb{N}_0, k \in \mathbb{P}_j \right\}$$

is an isomorphic map of $B_{pq}^s(\mathbb{T})$ onto the sequence space $b_{pq}^{s,per}$. If, in addition, $p < \infty$, $q < \infty$, then $\{\psi_j^{k,per}\}$ is an unconditional basis in $B_{pq}^s(\mathbb{T})$, [47, Theorem 1.37].

Later on we will have the following restriction on the parameteres

$$s > 0, \quad 1 < p < \infty, \quad 0 < q < \infty. \tag{1.18}$$

Since

$$B_{pq}^s(\mathbb{T}) \hookrightarrow L_p(\mathbb{T})$$

with s, p and q satisfying (1.18), we reformulate Theorem 1.15.

Theorem 1.16. *Let $\{\psi_j^{k,per}\}$ be the orthonormal basis in $L_2(\mathbb{T})$. Let $1 < p < \infty$, $0 < q < \infty$, $s > 0$ and $u > s$. Let $f \in L_p(\mathbb{T})$. Then $f \in B^s_{pq}(\mathbb{T})$ if, and only if, it can be represented as*

$$f = \sum_{j=0}^{\infty} \sum_{k \in \mathbb{P}_j} \mu_j^k 2^{-\frac{j+L}{2}} \psi_j^{k,per}, \quad \mu \in b^{s,per}_{pq},$$

unconditional convergence being in $L_p(\mathbb{T})$. Furthermore this representation is unique,

$$\mu_j^k = 2^{\frac{j+L}{2}} (f, \psi_j^{k,per})_{\mathbb{T}},$$

and

$$I : f \to \left\{\mu_j^k, j \in \mathbb{N}_0, k \in \mathbb{P}_j\right\}$$

is an isomorphic map of $B^s_{pq}(\mathbb{T})$ onto the sequence space $b^{s,per}_{pq}$.

1.1.7 Haar and Faber-Schauder bases on the unit interval

The Haar system

$$\left\{h_0, h_{jm} : j \in \mathbb{N}_0, m = 0, \dots, 2^j - 1\right\} \tag{1.19}$$

is defined as follows. Let h_0 be the characteristic function of the unit interval I and

$$h_{jm}(x) = \begin{cases} 1, & 2^{-j}m \le x < 2^{-j}m + 2^{-j-1}, \\ -1, & 2^{-j}m + 2^{-j-1} \le x \le 2^{-j}(m+1), \\ 0, & \text{otherwise.} \end{cases}$$

Note that the support of function h_{jm} is the interval $[2^{-j}m, 2^{-j}(m+1)]$.

Theorem 1.17. *(i) The Haar system is an orthogonal basis in $L_2(\mathrm{I})$.*

(ii) The Haar system is an unconditional basis in $L_p(\mathrm{I})$ with $1 < p < \infty$.

(iii) The Haar system is a conditional basis in $L_1(\mathrm{I})$.

(iv) Let $f \in L_1(\mathrm{I})$ and

$$f_n(x) = \int_{\mathrm{I}} f(y)\,dy + \sum_{j=0}^{n} \sum_{m=0}^{2^j-1} 2^j \left(\int_{\mathrm{I}} f(y) h_{jm}(y)\,dy \right) h_{jm}(x), \quad x \in \mathrm{I},\ n \in \mathbb{N}_0.$$

Then

$$f_n(x) \to f(x) \quad a.e. \text{ if } n \to \infty.$$

If $f \in C(\mathrm{I})$ then

$$f_n(x) \rightrightarrows f(x), \quad n \to \infty \ \text{(uniform convergence)}.$$

Remark 1.18. System (1.19) was first introduced by Haar in [10] and now is named after him. Part (iv) of the above theorem is essentially covered by him as well. The fact that the Haar system is a basis in all spaces $L_p(\mathrm{I})$, $1 \le p < \infty$, was proved by Schauder in [33].

We wish to extend the assertions for Haar system from $L_p(\mathrm{I})$ to suitable spaces $B^s_{pq}(\mathrm{I})$. Recall that $B^s_{pq}(\mathrm{I})$, $1 < p < \infty$, $1 \le q < \infty$, $s > 0$ is defined by restriction of $B^s_{pq}(\mathbb{R}^n)$ to I:

$$B^s_{pq}(\mathrm{I}) = \left\{ f \in L_p(\mathrm{I}) : f = g|_{\mathrm{I}} \text{ for some } g \in B^s_{pq}(\mathbb{R}^n) \right\},$$
$$\|f|B^s_{pq}(\mathrm{I})\| = \inf \left\{ \|g|B^s_{pq}(\mathbb{R}^n)\| : g|_{\mathrm{I}} = f \right\}.$$

The following assertions are covered by [48, Section 2.2.4]. Since we have the additional restrictions on parameteres we present the simpler version of theorems given there. First we define sequence spaces.

Definition 1.19. Let $1 < p < \infty$, $1 \le q < \infty$ and $0 < s < 1$. Let $b^s_{pq}(\mathrm{I})$ be the set of all sequences

$$\mu = \left\{ \mu_0, \mu_{jm} : j \in \mathbb{N}_0, m = 0, \dots, 2^j - 1 \right\} \subset \mathbb{C}$$

such that

$$\|\mu|b^s_{pq}(\mathrm{I})\| = |\mu_0| + \left(\sum_{j=0}^{\infty} 2^{jsq} \left(\sum_{m=0}^{2^j-1} |\mu_{jm}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} < \infty.$$

Theorem 1.20. *Let*

$$1 < p < \infty, \quad 1 \le q < \infty \quad \text{and} \quad 0 < s < \frac{1}{p}.$$

Let $f \in L_p(\mathrm{I})$. Then $f \in B^s_{pq}(\mathrm{I})$ if, and only if, it can be represented as

$$f = \mu_0 h_0 + \sum_{j=0}^{\infty} \sum_{m=0}^{2^j-1} \mu_{jm} h_{jm}, \tag{1.20}$$

unconditional convergence being in $B^s_{pq}(\mathrm{I})$. The representation (1.20) is unique with

$$\mu_0 = \int_{\mathrm{I}} f(y)\, dy, \quad \mu_{jm} = 2^j \int_{\mathrm{I}} f(y) h_{jm}(y)\, dy, \quad j \in \mathbb{N}_0, \; m = 0, \dots, 2^j - 1,$$

and

$$J : f \mapsto \left\{ \mu_0, 2^{-\frac{j}{p}} \mu_{jm} : j \in \mathbb{N}_0, m = 0, \dots, 2^j - 1 \right\}$$

is an isomorphic map of $B^s_{pq}(\mathrm{I})$ onto $b^s_{pq}(\mathrm{I})$. In addition, (1.19) is an unconditional basis in $B^s_{pq}(\mathrm{I})$.

The Faber-Schauder system

$$\left\{v_0, v_1, v_{jm} : j \in \mathbb{N}_0, m = 0, \dots, 2^j - 1\right\} \tag{1.21}$$

is the collection of functions

$$v_0(x) = 1 - x, \;\; v_1(x) = x, \quad x \in \mathrm{I},$$

and

$$v_{jm}(x) = \begin{cases} 2^{j+1}(x - \frac{m}{2^j}), & \frac{m}{2^j} \le x < \frac{m}{2^j} + \frac{1}{2^{j+1}}, \\ 2^{j+1}(\frac{m+1}{2^j} - x), & \frac{m}{2^j} + \frac{1}{2^{j+1}} \le x < \frac{m+1}{2^j}, \\ 0, & \text{otherwise.} \end{cases}$$

Define the first and higher order differences by

$$\Delta_h^1 f(x) = f(x+h) - f(x), \quad \Delta_h^M f = \Delta_h^1 \left(\Delta_h^{M-1} f\right).$$

In particular,

$$\Delta_{2^{-j-1}}^2 f\left(\frac{m}{2^j}\right) = f\left(\frac{m+1}{2^j}\right) - 2f\left(\frac{m}{2^j} + \frac{1}{2^{j+1}}\right) + f\left(\frac{m}{2^j}\right).$$

Theorem 1.21. *The Faber-Schauder system (1.21) is a conditional basis in $C(\mathrm{I})$ and*

$$f(x) = f(0)v_0(x) + f(1)v_1(x) - \frac{1}{2}\sum_{j=0}^{\infty}\sum_{m=0}^{2^j-1} \Delta_{2^{-j-1}}^2 f\left(\frac{m}{2^j}\right) v_{jm}(x), \quad x \in \mathrm{I},$$

for any $f \in C(\mathrm{I})$.

To characterize Besov spaces $B_{pq}^s(\mathrm{I})$ by Faber-Schauder system we adapt sequence spaces $b_{pq}^s(\mathrm{I})$.

Definition 1.22. Let $1 < p < \infty$, $1 \le q < \infty$ and $\frac{1}{p} < s < 1 + \frac{1}{p}$. Let $\bar{b}_{pq}^s(\mathrm{I})$ be the set of all sequences

$$\mu = \left\{\mu_0, \mu_1, \mu_{jm} : j \in \mathbb{N}_0, m = 0, \dots, 2^j - 1\right\} \subset \mathbb{C}$$

such that

$$\|\mu|\bar{b}_{pq}^s(\mathrm{I})\| = |\mu_0| + |\mu_1| + \left(\sum_{j=0}^{\infty} 2^{jsq}\left(\sum_{m=0}^{2^j-1} |\mu_{jm}|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} < \infty.$$

Theorem 1.23. *Let*

$$1 < p < \infty, \quad 1 \le q < \infty \quad \text{and} \quad \frac{1}{p} < s < 1 + \frac{1}{p}.$$

Let $f \in L_p(\mathrm{I})$. Then $f \in B^s_{pq}(\mathrm{I})$ if, and only if, it can be represented as

$$f = \mu_0 v_0 + \mu_1 v_1 + \sum_{j=0}^{\infty} \sum_{m=0}^{2^j-1} \mu_{jm} v_{jm}, \tag{1.22}$$

unconditional convergence being in $B^s_{pq}(\mathrm{I})$ and in $C(\mathrm{I})$. The representation (1.22) is unique with

$$\mu_0 = f(0), \quad \mu_1 = f(1), \quad \mu_{jm} = -\frac{1}{2}\Delta^2_{2^{-j-1}} f\left(\frac{m}{2^j}\right), \quad j \in \mathbb{N}_0, \ m = 0, \ldots, 2^j - 1,$$

and

$$J : f \mapsto \left\{\mu_0, \mu_1, 2^{-\frac{j}{p}} \mu_{jm} : j \in \mathbb{N}_0, m = 0, \ldots, 2^j - 1\right\}$$

is an isomorphic map of $B^s_{pq}(\mathrm{I})$ onto $\bar{b}^s_{pq}(\mathrm{I})$. In addition, (1.21) is an unconditional basis in $B^s_{pq}(\mathrm{I})$.

Remark 1.24. The above theorem but in more general version is given in [48, Section 3.1.2].

1.2 d-sets

1.2.1 Basic definitions

Definition 1.25. A measure μ in $\mathbb{R}^n$ is called Radon if all Borel sets are μ-measurable and

- $\mu(K) < \infty$ for compact sets $K \subset \mathbb{R}^n$,
- $\mu(V) = \sup\{\mu(K) : \ K \subset V \text{ is compact}\}$ for open sets $V \subset \mathbb{R}^n$,
- $\mu(A) = \inf\{\mu(V) : \ A \subset V, \ V \text{is open}\}$ for $A \subset \mathbb{R}^n$.

The Radon measure μ with $\mu(\mathbb{R}^n) < \infty$ is called finite. It is called σ-finite if $\mathbb{R}^n$ is the countable union of sets of finite measure.

Let μ be a positive Radon measure in $\mathbb{R}^n$. Let T_μ,

$$T_\mu : \ \varphi \mapsto \int_{\mathbb{R}^n} \varphi(x)\, \mu(dx), \quad \varphi \in S(\mathbb{R}^n),$$

be the linear functional generated by μ.

Definition 1.26. A positive Radon measure μ is said to be tempered if $T_\mu \in S'(\mathbb{R}^n)$.

Proposition 1.27. *Let μ^1 and μ^2 be two tempered Radon measures. Then*

$$T_{\mu^1} = T_{\mu^2} \text{ in } S'(\mathbb{R}^n) \quad \text{if, and only if, } \mu^1 = \mu^2.$$

Proof. The Proposition is valid by the arguments in [46, p. 80]. □

This justifies the identification of μ and correspondent tempered disribution T_μ and we may write $\mu \in S'(\mathbb{R}^n)$.

Definition 1.28. $f \in S'(\mathbb{R}^n)$ is called a positive distribution if

$$f(\varphi) \geq 0 \quad \text{for any} \quad \varphi \in S(\mathbb{R}^n) \text{ with } \varphi \geq 0.$$

If $f \in L_1^{loc}(\mathbb{R}^n)$ then $f \geq 0$ means $f(x) \geq 0$ almost everywhere.

Remark 1.29. If f is a positive distribution, then $f \in C_0(\mathbb{R}^n)'$ and it follows from the Radon–Riesz theorem that there is a tempered Radon measure μ such that

$$f(\varphi) = \int_{\mathbb{R}^n} \varphi(x)\, \mu(dx)$$

[28, p. 61/62, 71, 75].

Definition 1.30. A compact set Γ in $\mathbb{R}^n$ is called a d-set with $0 < d < n$ if there is a Radon measure μ in $\mathbb{R}^n$ with support Γ such that for some positive constants c_1 and c_2, holds

$$c_1 r^d \leq \mu(B(\gamma, r)) \leq c_2 r^d, \quad \gamma \in \Gamma,\ 0 < r < 1,\ 0 < d < n. \tag{1.23}$$

where $B(x, r)$ is a ball in $\mathbb{R}^n$ centred at $x \in \mathbb{R}^n$ and of radius $r > 0$.

If Γ is a d-set, then the restriction to Γ of the d-dimensional Hausdorff measure H^d satisfies (1.23) and any measure μ satisfying (1.23) is equivalent to $\mathrm{H}^d|_\Gamma$. A consequence of this is that the Hausdorff dimension of Γ is d. $\mathrm{H}^d|_\Gamma$ serves as a "canonical measure" on the d-set in the same way as the Lebesgue measure on $\mathbb{R}^n$.

1.2.2 Construction of self-similar sets

Typical examples of d-sets are self-similar sets with invariant measure μ. Broadly speaking, a self-similar set is a set that is made up of parts which are in some way similar to the whole. The mathematical definition was given by Hutchinson, [13].

Definition 1.31. A mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is called a similarity (similitude), if there is a constant $0 < \rho < 1$ such that for all $x, y \in \mathbb{R}^n$ holds

$$|F(x) - F(y)| = \rho\, |x - y|\,.$$

The constant ρ is called the contraction ratio of F and is denoted by $\mathrm{Lip}(F)$.

Theorem 1.32. *Let $\{F_i\}_{i=1}^N$ be similarities in $\mathbb{R}^n$. Then there exists a unique non-empty compact set $\Gamma \subset \mathbb{R}^n$ that satisfies*

$$\Gamma = \bigcup_{i=1}^{N} F_i(\Gamma). \tag{1.24}$$

Γ is called a self-similar set with respect to $\{F_i\}_{i=1}^N$.

The idea of the proof is to show that the mapping F defined by

$$F(A) = \bigcup_{i=1}^{N} F_i(A), \quad A \text{ non-empty compact set in } \mathbb{R}^n,$$

is a contraction in the complete metric space of all non-empty compact sets in $\mathbb{R}^n$ equipped with the Hausdorff metric. Then by Schauder's fixed point theorem Γ is the unique fixed point of F, see for details [6, 13, 23].

Example 1.33. The unit interval $\mathrm{I} = [0, 1]$ is a self-similar set with respect to the similarities $F_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$,

$$F_1(x) = \frac{1}{2}x, \quad F_2(x) = \frac{1}{2}x + \frac{1}{2}.$$

Example 1.34. The Koch curve K is a self-similar set with respect to the similarities $F_i : \mathbb{R}^2 \to \mathbb{R}^2$, $i = 1, 2$,

$$F_1(x, y) = \left(\frac{1}{2}x + \frac{1}{2\sqrt{3}}y, \frac{1}{2\sqrt{3}}x - \frac{1}{2}y\right),$$
$$F_2(x, y) = \left(\frac{1}{2}x - \frac{1}{2\sqrt{3}}y + \frac{1}{2}, -\frac{1}{2\sqrt{3}}x - \frac{1}{2}y + \frac{1}{2\sqrt{3}}\right),$$

see [23], where mappings F_1, F_2 are given in a complex form.

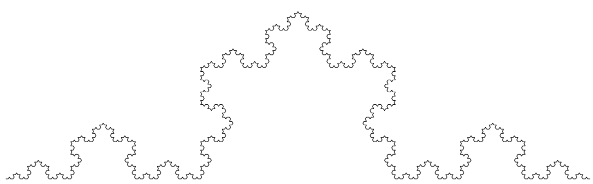


Figure 1.1: *The Koch curve*

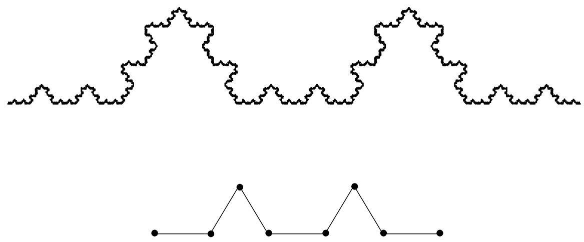


Figure 1.2: *The modified Koch curve. The Hausdorff dimension is given by* $7\left(\frac{1}{5}\right)^d = 1$ *and is equal to* $\frac{\log 7}{\log 5}$.

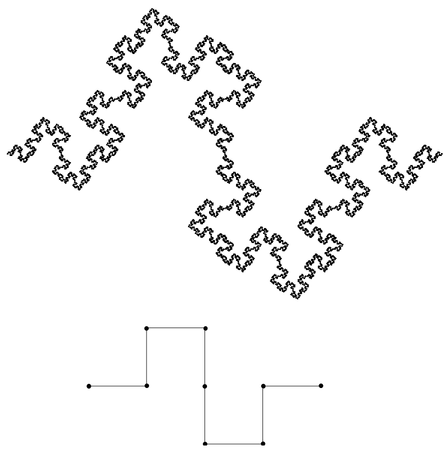


Figure 1.3: *A self-similar curve and its generator. The Hausdorff dimension is given by* $8\left(\frac{1}{4}\right)^d = 1$ *and is equal to* $\frac{3}{2}$.

Example 1.35. A certain class of self-similar sets, in particular self-similar curves, can be described by indicating an initial curve and a generator. The generator specifies the rule used to build new curve from the old one. We start with the unit interval. The generator consists of N straight line segments of equal length r. With each line segment we associate the similarity that maps the initial unit interval onto the given line segment. A self-similar curve is a set obtained by iterating the process of replacement each line segment by the generator. Some examples are shown on the Figures 1.2, 1.3.

1.2.3 Shift space

A very efficient way of representing a self-similar set Γ in (1.24) is by giving the "address" of each point in terms of iterations of the mappings F_i, $i = 1, \ldots, N$.

A set Γ_w with $w = (w_1, w_2, \ldots, w_j)$, $w_i \in \{1, \ldots, N\}$ defined by

$$\Gamma_w = F_w(\Gamma) = F_{w_1} \circ F_{w_2} \circ \ldots \circ F_{w_j}(\Gamma),$$

is called j-simplex. We call w a word of length $j = |w|$ and denote the collection of all words of length j by W_j. Then holds

$$\Gamma = \bigcup_{w \in W_j} F_w(\Gamma). \tag{1.25}$$

Let Σ be a set of all infinite sequences

$$\Sigma = \{(\omega_1, \omega_2, \ldots) : \omega_i \in \{1, 2, \ldots, N\}\}.$$

In literature Σ is called sometimes a code space or generalized Cantor set. For any $\omega = (\omega_1, \omega_2, \ldots) \in \Sigma$ define a continuous surjective map $\pi : \Sigma \to \Gamma$ by

$$\pi(\omega) = \bigcap_{m=1}^{\infty} \Gamma_{\omega_1 \omega_2 \ldots \omega_m}.$$

Let

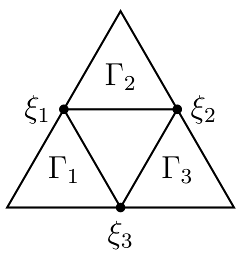
$$C = \bigcup_{i \neq j} (\Gamma_i \cap \Gamma_j),$$

$$\mathcal{C} = \pi^{-1}(C) \quad \text{and} \quad \mathcal{P} = \bigcup_{n \geq 1} \sigma^n(\mathcal{C}),$$

where $\sigma : \Sigma \to \Sigma$ is the shift map defined by

$$\sigma(\omega_1, \omega_2, \ldots) = (\omega_2, \omega_3, \ldots),$$

see Figure 1.4. If $\mathcal{P}$ is finite, then Γ is referred to as post-critically finite self-similar set.



$$C = \{\xi_1, \xi_2, \xi_3\}$$
$$\xi_1 = \pi(1\dot{2}) = \pi(2\dot{1})$$
$$\xi_2 = \pi(2\dot{3}) = \pi(3\dot{2})$$
$$\xi_3 = \pi(1\dot{3}) = \pi(3\dot{1})$$
$$\mathcal{P} = \{\dot{1}, \dot{2}, \dot{3}\}$$

Figure 1.4: *The post-critical set*

Let

$$V_0 = \pi(\mathcal{P}) \ \text{ and } \ V_j = \bigcup_{i=1}^{N} F_i(V_{j-1}),$$

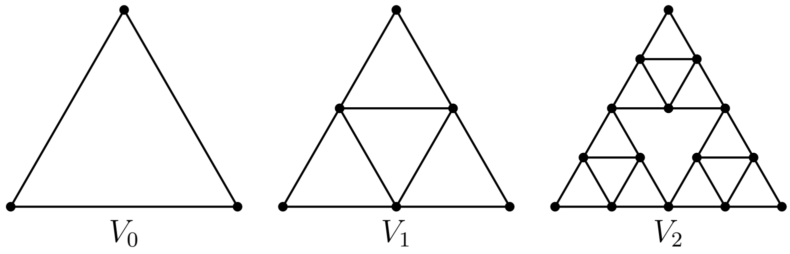


Figure 1.5: *Sets V_j*

or equivalently

$$V_j = \bigcup_{w \in W_j} F_w(V_0),$$

see Figure 1.5. Then V_j describes the set of boundary points of simplexes of fixed level j. It is clear that

$$V_j \subset V_{j+1}.$$

Let $V_* = \bigcup_{j=0}^{\infty} V_j$, then $\Gamma = \overline{V_*}$ in the Euclidean topology. We followed [23, Sections 1.2-1.3]. We form a graph G_j with vertices V_j and edge relation $\xi \sim_j \eta$ holding if and only if there exists a j-simplex containing both ξ and η as boundary points.

The shift space Σ supports various measures.

Theorem 1.36. *Let $\{p_1, p_2, \dots, p_N\}$ be numbers such that*

$$0 < p_i < 1 \text{ for all } i = 1, \dots, N \quad \text{and} \quad \sum_{i=1}^{N} p_i = 1.$$

Then there exists a unique Radon measure ν on Σ that satisfies

$$\nu(\Sigma_w) = p_{w_1} p_{w_2} \dots p_{w_m}, \text{ for any } w = (w_1, w_2, \dots, w_m) \in \bigcup_{j=0}^{\infty} W_j.$$

This measure ν is called the Bernoulli measure on Σ with weight $(p_1, p_2, \dots, p_N)$.

1.2.4 The snowflaked transform

An interesting fact about self-similar curves discussed in the Section 1.2.2 is that they are homeomorphic to the unit interval. If we treat the unit interval I as a self-similar curve that consists of N segments, each of length $\frac{1}{N}$, and denote a mapping π that corresponds to I by π_{I} and the one that corresponds to the self-similar curve K by π_K, i.e.

$$\pi_{\mathrm{I}}(\omega) = \bigcap_{m=1}^{\infty} \mathrm{I}_{\omega_1 \omega_2 \dots \omega_m}, \quad \pi_K(\omega) = \bigcap_{m=1}^{\infty} K_{\omega_1 \omega_2 \dots \omega_m},$$

then

$$H = \pi_K \circ \pi_{\mathrm{I}}^{-1} \tag{1.26}$$

is a homeomorphic map, which we call the snowflaked transform, see Figure 1.6. We refer to [23, Example 1.2.7], some information can be also find in [46, Section 8.2.2].

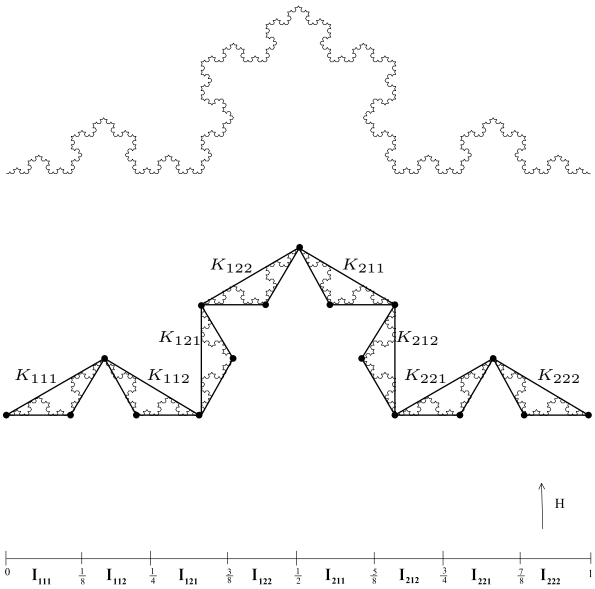


Figure 1.6: *The snowflaked transform*

1.2.5 Nested fractals

In the present work we consider sets Γ which are self-similar with respect to the similarities with the same contraction ratio $0 < \rho < 1$, that is

$$|F_i(x) - F_i(y)| = \rho\,|x - y|\,. \tag{1.27}$$

There is a special kind of sets that are self-similar with respect to similarities (1.27), satisfying some additional properties, known as nested fractals. They were first introduced by Lindstrøm [26], and afterwards were studied by many authors, e.g. [25, 31]. Nested fractals should satisfy following conditions:

C0. $\#V_0 \geq 2$.

C1. Open set condition
The family of similarities $\{F_i\}_{i=1}^N$ satisfies the open set condition if there exists an open, bounded, nonempty set $O \subset \mathbb{R}^n$ such that

$$F_i(O) \cap F_j(O) = \emptyset \ \text{ for } \ i \neq j$$

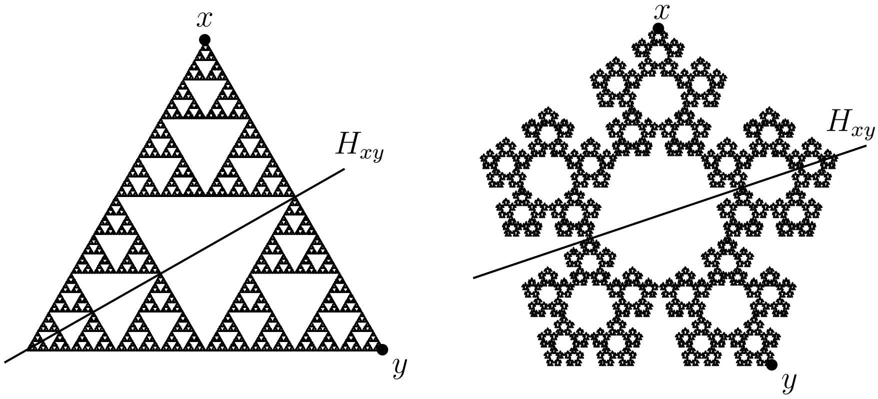


Figure 1.7: *Symmetry of nested fractals*

and

$$\bigcup_{i=1}^{N} F_i(O) \subset O.$$

When the open set condition is satisfied, the Hausdorff dimension d of Γ is

$$d = \frac{\log N}{\log \frac{1}{\rho}},$$

we refer to [6, 13].

C2. Nesting

If $j \geq 1$ and $w = (w_1, w_2, \ldots, w_j)$ and $w' = (w'_1, w'_2, \ldots, w'_j)$ are distinct elements of W_j, then

$$\Gamma_w \cap \Gamma_{w'} = F_w(V_0) \cap F_{w'}(V_0).$$

C3. Connectivity

The graph (V_1, G_1) is connected.

C4. Symmetry

For any $x, y \in \mathbb{R}^n$ with $x \neq y$, let H_{xy} denote the hyperplane given by

$$H_{xy} = \{z \in \mathbb{R}^n : |z - x| = |z - y|\}$$

and let R_{xy} denote the reflection with respect to H_{xy}. Then for any $x, y \in V_0$ with $x \neq y$, R_{xy} maps j-cells to j-cells, and maps any j-cell which contains elements in both sides of H_{xy} to itself for each $j \geq 0$, see Figure 1.7.

The simplest example of the nested fractal is the Sierpinski gasket SG, see the left part of Figure 1.7, which is generated by three similarities in the plane $F_i : \mathbb{R}^2 \to \mathbb{R}^2$, $i = 1, 2, 3$, defined by

$$F_i(x) = \frac{1}{2}(x - \xi_i) + \xi_i, \tag{1.28}$$

where ξ_i are the vertices of an equilateral triangle, see [39, Section 1.1].

Further on in this chapter we assume that the diameter of Γ is 1. Then the diameter of each j-simplex $\Gamma_{w_1 \dots w_j}$ is ρ^j, where ρ is from (1.27). In case of I and SG we get $\rho = \frac{1}{2}$, in case of K we have $\rho = \frac{1}{\sqrt{3}}$.

1.2.6 Measures on the self-similar set

Let Γ be a self-similar with respect to the similarities with the same contraction ratio $0 < \rho < 1$, that is

$$|F_i(x) - F_i(y)| = \rho \, |x - y| \, .$$

Γ can be a unit interval I, a self-similar curve K or a nested fractal. Then the Hausdorff dimension of Γ is equal to

$$d = \frac{\log N}{\log \frac{1}{\rho}}.$$

See for details [6, 13, 29] and references given there.

According to [13] there exists a unique normalized Radon measure μ with the support Γ such that

$$\mu(A) = \sum_{i=1}^{N} \rho^d \mu(F_i^{-1}(A)) = \frac{1}{N} \sum_{i=1}^{N} \mu(F_i^{-1}(A)), \tag{1.29}$$

for all Borel sets $A \subset \Gamma$. In particular,

$$\mu(\Gamma_{w_1 w_2 \dots w_m}) = (\rho^m)^d = \left(\frac{1}{N}\right)^m. \tag{1.30}$$

In [13] it was also shown that μ is a multiple of $\mathrm{H}^d|_\Gamma$ and it is clear that Γ with μ defined by (1.30) is a d-set.

Remark 1.37. In fact, μ is the image of the Bernoulli measure ν on Σ with the weight $(\frac{1}{N}, \dots, \frac{1}{N})$ under the transform π, i.e.

$$\mu(A) = \nu(\pi^{-1}(A)),$$

for any Borel set $A \subset \Gamma$, [13].

Taking into account the Remark 1.37 and the homeomorphism H in (1.26), we get

$$\mu(A) = \nu(\pi_K^{-1}(A)) = \nu \left[\pi_{\mathrm{I}}^{-1}(H^{-1}(A))\right] = \mu_L(H^{-1}(A)), \tag{1.31}$$

for any Borel set $A \subset K$. This means that μ is the image of the Lebesgue measure μ_L under the mapping H and for a function $\tilde{f}$ defined on K we get

$$\int\limits_K \tilde{f}(\gamma) \, \mu(d\gamma) = \int\limits_0^1 (\tilde{f} \circ H)(x) \, \nu(dx) = \int\limits_0^1 (\tilde{f} \circ H)(x) \, dx, \tag{1.32}$$

[29, Theorem 1.19].

Since K with measure μ is a d-set, then for $\gamma = H(x), \delta = H(y) \in \Gamma$, $x, y \in \mathrm{I}$, from (1.23) together with (1.31) follows

$$|H(x) - H(y)|_2 = |\gamma - \delta|_2 \sim \mu\left(B(\gamma, |\gamma - \delta|_2)\right)^{\frac{1}{d}} =$$
$$= \mu_L\left(B(x, |x - y|_1)\right)^{\frac{1}{d}} \sim |x - y|_1^{\frac{1}{d}}. \tag{1.33}$$

Recall that $|\cdot - \cdot|_n$ denotes the Euclidian distance in $\mathbb{R}^n$.

1.2.7 Dirichlet forms and piecewise harmonic functions

Suppose a real-valued function u is given on the vertices V_j. Then there is a natural Dirichlet form

$$E_j(u) = \sum_{\xi \sim_j \eta} \left(u(\xi) - u(\eta)\right)^2.$$

We need to multiply E_j by the renormalization factor α^j in order to have the following consistency property:

Lemma 1.38. *For every function u on V_j there exists a unique extension $\tilde{u}$ to V_{j+1} minimizing E_{j+1}, i.e.*

$$E_{j+1}(\tilde{u}) = \min\left\{E_{j+1}(u') : u'|_{V_j} = u\right\},$$

and

$$\alpha^j E_j(u) = \alpha^{j+1} E_{j+1}(\tilde{u}). \tag{1.34}$$

For I and K the renormalization factor α is equal to 2, for SG we have $\alpha = \frac{5}{3}$, [39, Section 1.3]. The number $d_w = \frac{\log N\alpha}{\log \frac{1}{\rho}}$ is called the walk dimension of Γ. The renormalized graph energies are defined by

$$\mathcal{E}_j(u) = \alpha^j E_j(u).$$

Then (1.34) can be reformulated as

$$\mathcal{E}_j(u) = \mathcal{E}_{j+1}(\tilde{u}).$$

The function $\tilde{u}$ is called a harmonic extension of u.

Definition 1.39. A continuous function $h : V_* \to \mathbb{R}$ is called harmonic if it minimizes $\mathcal{E}_j$ at all levels for given boundary values on V_0:

$$\mathcal{E}_j(h) = \min\left\{\mathcal{E}_j(u) : u|_{V_0} = \rho\right\}.$$

According to Theorem 3.2.4 in [23] for any harmonic function u there exists a unique extension $\tilde{u} \in C(\Gamma)$ such that

$$\tilde{u}|_{V_*} = u|_{V_*}.$$

Thus, we identify u with its extension $\tilde{u}$ and think of a harmonic function as a continuous function on Γ. The maximum and the minimum of the harmonic function are attained at the boundary V_0. This assertion is known as the maximum principle [23].

Definition 1.40. A continuous function $\psi : V_* \to \mathbb{R}$ is called piecewise harmonic of level j if $\psi \circ F_w$ is harmonic for all $|w| = j$.

We denote the set of piecewise harmonic functions of level j by H_j. These functions minimize $\mathcal{E}_m$ at all levels $m \geq j$ for given boundary values on V_j. Note that $H_{j-1} \subset H_j$.

For $u : V_* \to \mathbb{R}$ define

$$\mathcal{E}(u) = \lim_{j\to\infty} \mathcal{E}_j(u),$$
$$\tilde{\mathcal{D}} = \{u : V_* \to \mathbb{R}, \mathcal{E}(u) < \infty\}.$$

If $u \in \tilde{\mathcal{D}}$, then it is uniformly continuous on V_*, hence it has a unique continuous extension to Γ. Let

$$\mathcal{D} = \{u \in C(\Gamma) : \mathcal{E}(u) < \infty\}.$$

Then $(\mathcal{E}, \mathcal{D})$ is regular Dirichlet form on $L_2(\Gamma, \mu)$.

By effective resistance metric on the set Γ we mean a function $R : \Gamma \times \Gamma \to [0, \infty]$ defined by $R(x, x) = 0$ for $x \in \Gamma$ and

$$R(x, y)^{-1} = \inf\{\mathcal{E}(u) : u(x) = 0, u(y) = 1\}.$$

Let ψ^j_ξ, $\xi \in V_j$, be a piecewise harmonic function of level j which equals 1 at ξ and 0 at any other vertex of V_j:

$$\psi^j_\xi(x) = \delta_{\xi x} = \begin{cases} 1, & x = \xi \\ 0, & x \in V_j \setminus \{\xi\}. \end{cases}$$

Note that

$$\operatorname{supp} \psi^j_\xi \subset B(\xi, \rho^j).$$

In the case of the unit interval I piecewise harmonic functions are just piecewise linear functions. In fact, for $x = \frac{m}{2^{j-1}} + \frac{1}{2^j} \in V_j \setminus V_{j-1}$

$$\psi^j_x(t) = \begin{cases} 2^j(t - \frac{m}{2^{j-1}}), & \frac{m}{2^{j-1}} \leq t < \frac{m}{2^{j-1}} + \frac{1}{2^j}, \\[2ex] 2^j(\frac{m+1}{2^{j-1}} - t), & \frac{m}{2^{j-1}} + \frac{1}{2^j} \leq t < \frac{m+1}{2^{j-1}}, \\[2ex] 0, & \text{otherwise}, \end{cases}$$

and it holds

$$\left|\psi^j_x(t) - \psi^j_x(s)\right| \leq c\,|t - s| \quad \text{for all } t, s \in \mathrm{I}. \tag{1.35}$$

We have mentioned in Section 1.2.4 that the unit interval I can be regarded as a self-similar set with respect to N contractions with contraction ratio $\frac{1}{N}$. Piecewise harmonic functions on I in this case are piecewise linear functions as well and it holds

$$\left|\psi^j_x(t) - \psi^j_x(s)\right| \leq c\,|t - s| \quad \text{for all } t, s \in \mathrm{I}. \tag{1.36}$$

In general, it was shown in [24] that harmonic functions on Γ are uniformly Lipschitz continuous with respect to the resistance metric $R(x,y)$. From [12] follows that for a certain class of nested fractals there exist constants $c, c' > 0$ such that for all $x, y \in \Gamma$

$$c' \left|x-y\right|^{\frac{\log \frac{1}{\alpha}}{\log \rho}} \leq R(x,y) \leq c \left|x-y\right|^{\frac{\log \frac{1}{\alpha}}{\log \rho}},$$

note that $\frac{\log \frac{1}{\alpha}}{\log \rho} = d_w - d$. Thus piecewise harmonic functions on certain nested fractals satisfy

$$\left|\psi_\xi^j(x) - \psi_\xi^j(y)\right| \leq c \left|x-y\right|^{\sigma}, \tag{1.37}$$

with $\sigma = d_w - d$. In particular, piecewise harmonic functions on the Sierpinski gasket satisfy

$$\left|\psi_\xi^j(x) - \psi_\xi^j(y)\right| \leq c \left|x-y\right|^{\beta}, \ \text{ for all } \ x, y \in \mathrm{SG},$$

where $\beta = \frac{\ln(5/3)}{\ln 2}$.

Thus the family of piecewise harmonic functions may be regarded as the counterpart of Faber-Schauder basis.

CHAPTER 2

Tempered Radon measures

A substantial part of fractal geometry and fractal analysis deals with Radon measures in $\mathbb{R}^n$ (also called fractal measures) with compact support. One may consult [46] and the references given there. In the present chapter we clarify the relation between arbitrary σ-finite Radon measure in $\mathbb{R}^n$, tempered distributions and weighted Besov spaces. It comes out that a σ-finite Radon measure μ in $\mathbb{R}^n$ can be identified with a tempered distribution $\mu \in S'(\mathbb{R}^n)$ if and only if there is a real number β such that

$$\mu_\beta(\mathbb{R}^n) < \infty, \quad \text{where} \quad \mu_\beta = \left(1 + |x|^2\right)^{\frac{\beta}{2}} \mu.$$

Finite Radon measures can be identified with the positive cone $\overset{+}{B}{}^0_{1\infty}(\mathbb{R}^n)$ of the distinguished Besov space $B^0_{1\infty}(\mathbb{R}^n)$ and

$$\|\mu | B^0_{1\infty}(\mathbb{R}^n)\| \sim \mu(\mathbb{R}^n)$$

(equivalent norms).

2.1 Properties of weighted Besov spaces

Proposition 2.1. *For fixed* $0 < p, q \le \infty$

$$S(\mathbb{R}^n) = \bigcap_{\alpha, s \in \mathbb{R}} B^s_{pq}(\mathbb{R}^n, \langle x \rangle^\alpha) \tag{2.1}$$

and

$$S'(\mathbb{R}^n) = \bigcup_{\alpha, s \in \mathbb{R}} B^s_{pq}(\mathbb{R}^n, \langle x \rangle^\alpha).$$

Proof. Step 1. The inclusion

$$S(\mathbb{R}^n) \subset \bigcap_{\alpha, s \in \mathbb{R}} B^s_{pq}(\mathbb{R}^n, \langle x \rangle^\alpha)$$

is clear.

To prove that any $f \in \bigcap\limits_{\alpha,s\in\mathbb{R}} B^s_{pq}(\mathbb{R}^n, \langle x\rangle^\alpha)$ belongs to $S(\mathbb{R}^n)$, it is sufficient to show that for any fixed $N \in \mathbb{N}$ there are $\alpha(N) \in \mathbb{R}$ and $s(N) \in \mathbb{R}$ such that

$$\sup_{|\beta|\le N} \sup_{x\in\mathbb{R}^n} \langle x\rangle^{2N} \left|D^\beta f(x)\right| \le c\|f|B^s_{pq}(\mathbb{R}^n, \langle x\rangle^\alpha)\|.$$

For any multiindex β there are polynomials P^β_γ, $\deg P^\beta_\gamma \le 2N$ such that

$$\langle x\rangle^{2N} D^\beta f(x) = \sum_{\gamma\le\beta} D^\gamma \left[(P^\beta_\gamma f)(x)\right].$$

Hence

$$\sup_{|\beta|\le N} \sup_{x\in\mathbb{R}^n} \langle x\rangle^{2N} \left|D^\beta f(x)\right| = \sup_{|\beta|\le N} \sup_{x\in\mathbb{R}^n} \left|\sum_{\gamma\le\beta} D^\gamma \left[(P^\beta_\gamma f)(x)\right]\right| \le$$

$$\le \sup_{|\beta|\le N} \sum_{|\gamma|\le N} \sup_{x\in\mathbb{R}^n} \left|D^\gamma \left[(P^\beta_\gamma f)(x)\right]\right| \le \sup_{|\beta|\le N} \sum_{|\gamma|\le N} \|P^\beta_\gamma f|C^N(\mathbb{R}^n)\|. \tag{2.2}$$

Due to the embedding theorems [41, Ch. 2.7.1]

$$\|P^\beta_\gamma f|C^N(\mathbb{R}^n)\| \le c\|P^\beta_\gamma f|B^{N+\frac{n}{p}+\varepsilon}_{pq}(\mathbb{R}^n)\| = c\|\frac{P^\beta_\gamma}{\langle x\rangle^{2N}}\langle x\rangle^{2N} f|B^{N+\frac{n}{p}+\varepsilon}_{pq}(\mathbb{R}^n)\| \tag{2.3}$$

for any $\varepsilon > 0$. $\frac{P^\beta_\gamma}{\langle x\rangle^{2N}}$ is a pointwise multiplier for $B^{N+\frac{n}{p}+\varepsilon}_{pq}(\mathbb{R}^n)$, [41, Ch. 2.8.2], therefore

$$\|\frac{P^\beta_\gamma}{\langle x\rangle^{2N}}\langle x\rangle^{2N} f|B^{N+\frac{n}{p}+\varepsilon}_{pq}(\mathbb{R}^n)\| \le$$

$$\le c\|\frac{P^\beta_\gamma}{\langle x\rangle^{2N}}|\mathcal{C}^{N+\frac{n}{p}+\varepsilon}(\mathbb{R}^n)\| \cdot \|\langle x\rangle^{2N} f|B^{N+\frac{n}{p}+\varepsilon}_{pq}(\mathbb{R}^n)\|. \tag{2.4}$$

According to Remark 1.7

$$\|\langle x\rangle^{2N} f|B^{N+\frac{n}{p}+\varepsilon}_{pq}(\mathbb{R}^n)\| \sim \|f|B^{N+\frac{n}{p}+\varepsilon}_{pq}(\mathbb{R}^n, \langle x\rangle^{2N})\|. \tag{2.5}$$

Combining (2.2), (2.3), (2.4), (2.5), one gets

$$\sup_{|\beta|\le N} \sup_{x\in\mathbb{R}^n} \langle x\rangle^{2N} \left|D^\beta f(x)\right| \le c \sum_{|\gamma|\le N} \|\langle x\rangle^{2N} f|B^{N+\frac{n}{p}+\varepsilon}_{pq}(\mathbb{R}^n)\| \le$$

$$\le c\|f|B^{N+\frac{n}{p}+\varepsilon}_{pq}(\mathbb{R}^n, \langle x\rangle^{2N})\| \tag{2.6}$$

and it follows (2.1).

Step 2. Let $1 < p \leq \infty$, $1 < q \leq \infty$ and let p' and q' be defined in the standard way by

$$\frac{1}{p} + \frac{1}{p'} = 1, \quad \frac{1}{q} + \frac{1}{q'} = 1.$$

The inclusion

$$\bigcup_{\alpha, s \in \mathbb{R}} B^s_{pq}\left(\mathbb{R}^n, \langle x \rangle^\alpha\right) \subset S'(\mathbb{R}^n)$$

is evident.

As far as the opposite inclusion is concerned, we recall that $f \in S'(\mathbb{R}^n)$ if and only if there are $l \in \mathbb{N}$ and $m \in \mathbb{N}$ such that

$$|f(\varphi)| \leq c \sup_{|\alpha| \leq m} \sup_{x \in \mathbb{R}^n} \langle x \rangle^l \left| D^\alpha \varphi(x) \right|,$$

for all $\varphi \in S(\mathbb{R}^n)$. By (2.6)

$$\sup_{|\alpha| \leq m} \sup_{x \in \mathbb{R}^n} \langle x \rangle^l \left| D^\alpha \varphi(x) \right| \leq c \| \varphi | B^{m + \frac{n}{p} + \varepsilon}_{p'q'}\left(\mathbb{R}^n, \langle x \rangle^l\right) \|.$$

According to our choice of p and q, it follows that $1 \leq p' < \infty$ and $1 \leq q' < \infty$. Thus by [41, Ch. 2.11.2]

$$f \in \left(B^{m + \frac{n}{p} + \varepsilon}_{p'q'}\left(\mathbb{R}^n, \langle x \rangle^l\right) \right)' = B^{-(m + \frac{n}{p} + \varepsilon)}_{pq}\left(\mathbb{R}^n, \langle x \rangle^{-l}\right).$$

This means

$$S'(\mathbb{R}^n) \subset \bigcup_{\alpha, s \in \mathbb{R}} B^s_{pq}(\mathbb{R}^n, \langle x \rangle^\alpha).$$

Step 3. Let $0 < p \leq 1$, $1 < q \leq \infty$. By the arguments above, for $f \in S'(\mathbb{R}^n)$ there are $\alpha \in \mathbb{R}$ and $s \in \mathbb{R}$ such that

$$f \in B^s_{\infty q}(\mathbb{R}^n, \langle x \rangle^\alpha).$$

We want to show that

$$f \in B^s_{pq}(\mathbb{R}^n, \langle x \rangle^{\alpha - \gamma}), \quad \gamma > \frac{n}{p}.$$

Indeed,

$$\|f | B^s_{pq}(\mathbb{R}^n, \langle x \rangle^{\alpha - \gamma})\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \| \langle x \rangle^{\alpha - \gamma} \left(\varphi_j \widehat{f} \right)^\vee | L_p(\mathbb{R}^n) \|^q \right)^{\frac{1}{q}} \leq$$

$$\leq \left(\sum_{j=0}^{\infty} 2^{jsq} \sup_{x \in \mathbb{R}^n} \left[\langle x \rangle^\alpha \left| \left(\varphi_j \widehat{f} \right)^\vee (x) \right| \right]^q \left(\int_{\mathbb{R}^n} \langle x \rangle^{-\gamma p} \, dx \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \leq$$

$$\leq c \|f | B^s_{\infty q}(\mathbb{R}^n, \langle x \rangle^\alpha)\|.$$

Step 4. When $0 < q \le 1$, first we may find $\alpha \in \mathbb{R}$ and $s \in \mathbb{R}$ such that

$$f \in B^s_{pq^*}(\mathbb{R}^n, \langle x \rangle^\alpha),$$

$q^* > 1$, and then use the fact that

$$B^s_{pq*}(\mathbb{R}^n, \langle x \rangle^\alpha) \subset B^{s-\varepsilon}_{pq}(\mathbb{R}^n, \langle x \rangle^\alpha), \quad \varepsilon > 0.$$

□

2.2 Main assertions

Our next result refers to tempered measures.

Theorem 2.2. *(i) A Radon measure μ in $\mathbb{R}^n$ is tempered if, and only if, there is a real number β such that $\langle x \rangle^\beta \mu$ is finite.*

(ii) Let μ be a tempered Radon measure in $\mathbb{R}^n$. Let $j \in \mathbb{N}$,

$$A_j = \left\{ x : \; 2^{j-1} \le |x| \le 2^{j+1} \right\}, \quad A_0 = \{ x : \; |x| \le 2 \}.$$

Then for some $c > 0$, $\alpha \ge 0$,

$$\mu(A_k) \le c 2^{k\alpha} \text{ for all } k \in \mathbb{N}_0.$$

Proof. Step 1. First we prove part (ii). Suppose that the assertion does not hold. Then for $c = 1$ and $l \in \mathbb{N}$ there is $k_l \in \mathbb{N}_0$ such that

$$\mu\left(A_{k_l}\right) > 2^{k_l l}. \tag{2.7}$$

As soon as it is found one k_l with (2.7), it follows that there are infinitely many k_l^m, $m \in \mathbb{N}$ that satisfy (2.7).

With $j \in \mathbb{N}$,

$$A^*_j = \left\{ x : \; 2^{j-2} \le |x| \le 2^{j+2} \right\}, \quad A^*_0 = \{ x : \; |x| \le 4 \}.$$

For $l = 1$ take any of k_1^m, let it be k_1. For $l = 2$ choose $k_2 \gg k_1$ in such a way that $A^*_{k_1}$ and $A^*_{k_2}$ have an empty intersection. For arbitrary $l \in \mathbb{N}$ take

$$k_l \gg k_{l-1} \text{ and } A^*_{k_{l-1}} \cap A^*_{k_l} = \emptyset.$$

Let φ_0 be a C^∞ function on $\mathbb{R}^n$ with

$$\varphi_0(x) = 1, \;\; |x| \le 2 \text{ and } \;\; \varphi_0(x) = 0, \;\; |x| \ge 4.$$

Let $k \in \mathbb{N}$ and

$$\varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{-k+3}x), \;\; x \in \mathbb{R}^n.$$

Then we have
$$\operatorname{supp}\varphi_k \subset A_k^*$$
and
$$\varphi_k(x) = 1, \quad x \in A_k.$$
Let
$$\varphi(x) = \sum_{l=1}^{\infty} 2^{-lk_l}\varphi_{k_l}(x).$$
For any fixed $N \in \mathbb{N}_0$
$$\sup_{|\alpha|\le N}\sup_{x\in\mathbb{R}^n}\left(1+|x|^2\right)^N |D^\alpha\varphi(x)| =$$
$$= \sup_{|\alpha|\le N}\sup_{x\in\mathbb{R}^n}\left(1+|x|^2\right)^N \left|D^\alpha\left(\sum_{l=1}^{\infty} 2^{-lk_l}\varphi_{k_l}(x)\right)\right| \le$$
$$\le \sup_{l\in\mathbb{N}}\sup_{|\alpha|\le N}\sup_{x\in\mathbb{R}^n} 2^{-lk_l}2^{-|\alpha|k_l}2^{|\alpha|}\left(1+|x|^2\right)^N \left|\left(D^\alpha\varphi_1\right)\left(2^{-k_l+1}x\right)\right|.$$
The last inequality holds, since the functions φ_{k_l} have disjoint supports. With the change of variables
$$x' = 2^{-k_l+1}x$$
one gets
$$\sup_{|\alpha|\le N}\sup_{x\in\mathbb{R}^n}\left(1+|x|^2\right)^N |D^\alpha\varphi(x)| \le$$
$$\le \sup_{l\in\mathbb{N}}\sup_{|\alpha|\le N} 2^{-lk_l}2^{-|\alpha|k_l}2^{|\alpha|}2^{2(k_l-1)N}\sup_{x\in\mathbb{R}^n}\left(1+|x|^2\right)^N |D^\alpha\varphi_1(x)| \le$$
$$\le c\sup_{l\in\mathbb{N}}\sup_{|\alpha|\le N} 2^{-k_l(l+|\alpha|-2N)} \le c\sup_{l\in\mathbb{N}} 2^{-k_l(l-2N)}.$$
Since N is fixed and l is tending to infinity, $2^{-k_l(l-2N)}$ is bounded. Thus $\varphi \in S(\mathbb{R}^n)$.

According to the definition of tempered Radon measures
$$\int_{\mathbb{R}^n}\psi(x)\,\mu(dx) < +\infty$$
for any $\psi \in S(\mathbb{R}^n)$, but
$$\int_{\mathbb{R}^n}\varphi(x)\,\mu(dx) \ge \sum_{l=1}^{\infty}\int_{A_{k_l}}\varphi(x)\,\mu(dx) \ge \sum_{l=1}^{\infty} 2^{-lk_l}2^{lk_l} = +\infty.$$
This means that our assertion (2.7) is false.

Step 2. We prove part (i). Since $\langle x\rangle^{\beta}\mu$ is finite it tempered. Then μ is also tempered. To prove the other direction we take $\beta = -(\alpha+1)$. Then we get

$$\langle\cdot\rangle^{\beta}\mu(\mathbb{R}^n) = \int_{\mathbb{R}^n} \langle x\rangle^{-(\alpha+1)}\,\mu(dx) \le \sum_{k=0}^{\infty}\int_{A_k} \langle x\rangle^{-(\alpha+1)}\,\mu(dx) \le$$

$$\le c\sum_{k=0}^{\infty} 2^{-k(\alpha+1)} \int_{A_k} \mu(dx) \le c\sum_{k=0}^{\infty} 2^{-k(\alpha+1)}2^{k\alpha} < \infty.$$

$\square$

In order to characterize finite Radon measures we define the positive cone $\overset{+}{B}{}^{s}_{pq}(\mathbb{R}^n)$ as the collection of all positive $f \in B^s_{pq}(\mathbb{R}^n)$.

Theorem 2.3. *Let $M(\mathbb{R}^n)$ be the collection of all finite Radon measures. Then*

$$M(\mathbb{R}^n) = \overset{+}{B}{}^{0}_{1\infty}(\mathbb{R}^n)$$

and

$$\mu(\mathbb{R}^n) \sim \|\mu|B^0_{1\infty}(\mathbb{R}^n)\|, \quad \mu \in M(\mathbb{R}^n). \tag{2.8}$$

Proof. By the proof in [46, p.82/83], Proposition 1.127,

$$\|\mu|B^0_{1\infty}(\mathbb{R}^n)\| \le \mu(\mathbb{R}^n) \ \text{ if } \ \mu \in M(\mathbb{R}^n).$$

In order to prove the converse inequality, one use the characterisation of Besov spaces via local means. Let k_0 be a C^∞ non-negative function with

$$\operatorname{supp} k_0 \subset \{x : |x| \le 1\} \ \text{ and } \ k_0^{\vee}(0) \ne 0.$$

If $f \in \overset{+}{B}{}^{0}_{1\infty}(\mathbb{R}^n)$, then $f = \mu$ is a tempered measure. By Theorem 1.2

$$\|\mu|B^0_{1\infty}(\mathbb{R}^n)\| \ge c\|k_0(1,\mu)|L_1(\mathbb{R}^n)\| = c\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} k_0(x-y)\,d\mu(y)\,dx.$$

Applying Fubini's theorem, one gets

$$\|\mu|B^0_{1\infty}(\mathbb{R}^n)\| \ge c\mu(\mathbb{R}^n).$$

$\square$

Corollary 2.4. *Let $f \in L_1(\mathbb{R}^n)$ and $f(x) \ge 0$ almost everywhere. Then*

$$\|f|L_1(\mathbb{R}^n)\| \sim \|f|B^0_{1\infty}(\mathbb{R}^n)\|.$$

Proof. Let $\mu = f\mu_L$, where μ_L is the Lebesgue measure. Then

$$\mu(\mathbb{R}^n) = \int_{\mathbb{R}^n} f(x)\,\mu_L(dx) = \|f|L_1(\mathbb{R}^n)\|$$

and

$$\|\mu|B^0_{1\infty}(\mathbb{R}^n)\| = \|f|B^0_{1\infty}(\mathbb{R}^n)\|.$$

From (2.8) follows the statement in the Corollary. □

The question arises whether Corollary 2.4 can be extended to all $f \in L_1(\mathbb{R}^n)$. We have

$$L_1(\mathbb{R}^n) \hookrightarrow B^0_{1\infty}(\mathbb{R}^n), \ \text{ hence } \ \|f|B^0_{1\infty}(\mathbb{R}^n)\| \le c\|f|L_1(\mathbb{R}^n)\|$$

for all $f \in L_1(\mathbb{R}^n)$. But the converse is not true even for functions $f \in L_1(\mathbb{R}^n)$ with compact support in the unit ball.

Proposition 2.5. *There are functions $f_j \in L_1(\mathbb{R}^n)$ with*

$$\operatorname{supp} f_j \subset \{y : |y| \le 1\}\,, \quad j \in \mathbb{N},$$

such that $\{f_j\}$ is a bounded set in $B^0_{1\infty}(\mathbb{R}^n)$, but

$$\|f_j|L_1(\mathbb{R}^n)\| \to \infty \ \text{ if } \ j \to \infty.$$

Proof. We may assume $n = 1$.

Let $a \in C^1(\mathbb{R})$ be an odd function with

$$\operatorname{supp} a \subset \{x : \ |x| \le 2\}, \quad a(x) \ge 0, \ x \ge 0$$

and

$$\max_{-2 \le x \le 2} |a(x)| = |a(-1)| = a(1) = 1.$$

If $c = \max_{-2 \le x \le 2} |a'(x)|$, then $c \ge 1$. Define $a_0 \in C^1(\mathbb{R})$ by

$$a_0(x) = c^{-1} a(x).$$

Then one has for any $x \in \mathbb{R}$,

$$|a_0(x)| \le c^{-1} \le 1, \ \ |a_0'(x)| \le 1 \ \text{ and } \ \int_{\mathbb{R}} a_0(x)\,dx = 0.$$

Define a function a_ν, $\nu \in \mathbb{N}$, by

$$a_\nu(x) = 2^\nu a_0(2^\nu x).$$

Then

$$\operatorname{supp} a_\nu \subset \left[-2^{-\nu+1}, 2^{-\nu+1}\right]$$

and

$$|a_\nu(x)| \le c^{-1}2^\nu, \quad |a'_\nu(x)| \le 2^{2\nu}, \quad \int\limits_{\mathbb{R}} a_\nu(x)\,dx = 0.$$

According to Definition 1.3, a_0 is 1_1-atom and a_ν are $(0,1)_{1,1}$-atoms. It follows from Theorem 1.5 that $\sum\limits_{\nu=1}^{\infty} a_\nu(x)$ converges in $S'(\mathbb{R}^n)$ and represents an element of $B^0_{1\infty}(\mathbb{R}^n)$. Let $f \overset{S'}{=} \sum\limits_{\nu=1}^{\infty} a_\nu$.

Let

$$f_j(x) = \sum_{\nu=1}^{j} a_\nu(x).$$

Then $\operatorname{supp} f_j \subset [-1,1]$,

$$\|f_j|L_1(\mathbb{R}^n)\| \ge \int\limits_0^{+\infty} f_j(x)\,dx = \int\limits_0^{+\infty} \sum_{\nu=1}^{j} a_\nu(x)\,dx =$$

$$= j\int\limits_0^{+\infty} a_0(x)\,dx \to \infty, \quad j \to \infty.$$

On the other hand one has by the above atomic argument

$$\|f_j|B^0_{1\infty}(\mathbb{R})\| \le 1 \ \text{ for } \ j \in \mathbb{N}.$$

□

Corollary 2.6. *Not any characteristic function of a measurable subset of $\mathbb{R}^n$ is a pointwise multiplier in $B^0_{1\infty}(\mathbb{R}^n)$.*

Proof. Let $f \in L_1(\mathbb{R}^n)$ real. Let M_+ be a set of points x such that $f(x) \ge 0$ and $M_- = \{x : f(x) < 0\}$. Then

$$\|f|L_1(\mathbb{R}^n)\| = \|\chi_{M_+}f|L_1(\mathbb{R}^n)\| + \|\chi_{M_-}f|L_1(\mathbb{R}^n)\|,$$

where χ_{M_+}, χ_{M_-} are characteristic functions of sets M_+ and M_- respectively. One may apply Corollary 2.4 to the functions $\chi_{M_+}f$ and $\chi_{M_-}f$ and get

$$\|f|L_1(\mathbb{R}^n)\| \le c\|\chi_{M_+}f|B^0_{1\infty}(\mathbb{R}^n)\| + c\|\chi_{M_-}f|B^0_{1\infty}(\mathbb{R}^n)\|.$$

If any characteristic function of a set in $\mathbb{R}^n$ would be a pointwise multiplier in $B^0_{1\infty}(\mathbb{R}^n)$, then

$$\|\chi_{M_+}f|B^0_{1\infty}(\mathbb{R}^n)\| \le c\|f|B^0_{1\infty}(\mathbb{R}^n)\|, \quad \|\chi_{M_-}f|B^0_{1\infty}(\mathbb{R}^n)\| \le c\|f|B^0_{1\infty}(\mathbb{R}^n)\|,$$

hence

$$\|f|L_1(\mathbb{R}^n)\| \le c\|f|B^0_{1\infty}(\mathbb{R}^n)\|.$$

Since for any function $f \in L_1(\mathbb{R}^n)$ holds

$$\|f|B^0_{1\infty}(\mathbb{R}^n)\| \leq c\|f|L_1(\mathbb{R}^n)\|,$$

one gets

$$\|f|L_1(\mathbb{R}^n)\| \sim \|f|B^0_{1\infty}(\mathbb{R}^n)\|, \quad \text{for real } \ f \in L_1(\mathbb{R}^n).$$

This can be also extended to complex functions $f \in L_1(\mathbb{R}^n)$. But acoording to the Proposition 2.5 this is not true. $\square$

CHAPTER 3

Trace spaces

Functions in function spaces are usually defined only almost everywhere. Then we need to explain what is meant with a restriction of a function to a d-set, $0 < d < n$, that has an n-dimensional Lebesgue measure zero.

3.1 Basic definitions

Let $\Gamma \subset \mathbb{R}^n$ be a d-set. If the function f is continuous in $\mathbb{R}^n$ or has a continuous representative, then the restriction or trace of f to Γ is defined pointwise. In other cases we have two approaches to define the trace of functions. One is by Triebel, by using inequalities, approximation and completion. Another one is due to Jonsson and Wallin, by applying strictly defined functions.

Approach by Triebel

Definition 3.1. Let Γ be a d-set and

$$s > 0, \quad 1 < p < \infty, \quad 0 < q < \infty. \tag{3.1}$$

Let for some $c > 0$,

$$\int_\Gamma |\varphi(\gamma)| \ \mu(d\gamma) \le c \|\varphi | B^s_{pq}(\mathbb{R}^n)\|, \ \text{ for all } \varphi \in S(\mathbb{R}^n).$$

Then the trace operator

$$\operatorname{tr}_\mu : B^s_{pq}(\mathbb{R}^n) \hookrightarrow L_1(\Gamma, \mu),$$

is the completion of the pointwise trace $(\operatorname{tr}_\mu \varphi)(\gamma) = \varphi(\gamma)$ with $\varphi \in S(\mathbb{R}^n)$. $g \in \operatorname{tr}_\mu B^s_{pq}(\mathbb{R}^n) \subset L_1(\Gamma, \mu)$ is quasi-normed by

$$\|g| \operatorname{tr}_\mu B^s_{pq}(\mathbb{R}^n)\| = \inf \left\{ \|f|B^s_{pq}(\mathbb{R}^n)\| : \operatorname{tr}_\mu f = g \right\}.$$

The definition above is justified since $S(\mathbb{R}^n)$ is dense in $B^s_{pq}(\mathbb{R}^n)$ with the restrictions on the parameters (3.1).

Proposition 3.2. *Let* $1 < p < \infty$ *and* $0 < q \leq 1$. *Then*

$$\operatorname{tr}_\mu B^{\frac{n-d}{p}}_{pq}(\mathbb{R}^n) = L_p(\Gamma, \mu).$$

Remark 3.3. For the proof we refer to Theorem 18.6 in [43]. From the proposition above follows that for any $\varphi \in S(\mathbb{R}^n)$

$$\left(\int_\Gamma |\varphi(\gamma)|^p \, \mu(d\gamma) \right)^{1/p} \leq c \|\varphi | B^{\frac{n-d}{p}}_{pq}(\mathbb{R}^n)\|. \tag{3.2}$$

For d-sets the trace operator exists whenever $s > \frac{n-d}{p}$. Then we may formulate the following definition.

Definition 3.4. Let $1 < p < \infty$, $0 < q < \infty$ and $s > 0$. Then

$$B^s_{pq}(\Gamma, \mu) = \operatorname{tr}_\mu B^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n)$$

and

$$B^0_{p1}(\Gamma, \mu) = \operatorname{tr}_\mu B^{\frac{n-d}{p}}_{p1}(\mathbb{R}^n) = L_p(\Gamma, \mu).$$

This definition is in good agreement with the well-known trace theorems from $\mathbb{R}^n$ onto m-dimensional hyper-planes,

$$\operatorname{tr} B^{s+\frac{n-m}{p}}_{pq}(\mathbb{R}^n) = B^s_{pq}(\mathbb{R}^m), \;\; 1 \leq m < n.$$

Now let us consider the case when $q = \infty$. Let $f \in B^s_{p\infty}(\mathbb{R}^n)$, $s > \frac{n-d}{p}$. From the properties of Besov spaces follows $B^s_{p\infty}(\mathbb{R}^n) \hookrightarrow B^{s+\varepsilon}_{pv}(\mathbb{R}^n)$ for $\varepsilon > 0$ and $0 < v < \infty$. Then we can consider f as an element of $B^{s+\varepsilon}_{pv}(\mathbb{R}^n)$ and define its trace on Γ.

When we write that $f \in B^s_{pq}(\Gamma, \mu)$, this means that there is a function $g \in B^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n)$ such that $\operatorname{tr}_\mu g = f$ and in general we can not recover g by f. The situation essentially improves, when the smoothness parameter is small.

Theorem 3.5. *Let*

$$0 < s < 1, \;\; 1 < p < \infty, \;\; 1 \leq q \leq \infty, \;\; t = s + (n-d)/p.$$

Then there is a linear and bounded extension operator ext_μ *with*

$$\operatorname{ext}_\mu : B^s_{pq}(\Gamma, \mu) \hookrightarrow B^t_{pq}(\mathbb{R}^n) \tag{3.3}$$

and

$$\operatorname{tr}_\mu \circ \operatorname{ext}_\mu = \mathrm{id} \quad (\textit{identity in } B^s_{pq}(\Gamma, \mu)). \tag{3.4}$$

Approach by Jonsson and Wallin

For $f \in L_1^{loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, define

$$\bar{f}(x) = \lim_{r \to 0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} f(y)\,dy \tag{3.5}$$

if the limit exists (here $|B(x,r)|$ stands for Lebesgue measure of a ball $B(x,r)$). Those points, where the limit in (3.5) exists, are called Lebesgue points. Lebesgue's differentiation theorem implies that $f = \bar{f}$ a.e. in $\mathbb{R}^n$. Then by restriction of f to Γ we mean the pointwise restriction of $\bar{f}$ to Γ and we denote it by $f|_\Gamma$.

Definition 3.6. Let Γ be a d-set. Let $0 < s < 1$ and $1 < p < \infty$. Then $\mathbf{B}^s_{pp}(\Gamma,\mu)$ is the collection of all $f \in L_p(\Gamma,\mu)$ such that

$$\|f|\mathbf{B}^s_{pp}(\Gamma,\mu)\| = \|f|L_p(\Gamma,\mu)\| + \left(\int\limits_\Gamma \int\limits_\Gamma \frac{|f(\gamma)-f(\delta)|^p}{|\gamma-\delta|^{d+sp}}\,\mu(d\delta)\mu(d\gamma) \right)^{\frac{1}{p}}. \tag{3.6}$$

The integration in the inner integral can be reduced to the ball $B(\gamma,1)$, since

$$\left(\int\limits_\Gamma \int\limits_{\Gamma \setminus B(\gamma,1)} \frac{|f(\gamma)-f(\delta)|^p}{|\gamma-\delta|^{d+sp}}\,\mu(d\delta)\mu(d\gamma) \right)^{\frac{1}{p}} \le c\|f|L_p(\Gamma,\mu)\|.$$

Theorem 3.7. *Let Γ be a d-set and $0 < s < 1$. Then*

$$\mathbf{B}^s_{pp}(\Gamma,\mu) = B^{s+\frac{n-d}{p}}_{pp}(\mathbb{R}^n)|_\Gamma.$$

Theorem 3.7 consists of two parts. First, it states that for every function $f \in B^{s+\frac{n-d}{p}}_{pp}(\mathbb{R}^n)$ its trace defined by $f|_\Gamma$ belongs to $\mathbf{B}^s_{pp}(\Gamma,\mu)$. Second, there is a linear and bounded operator

$$\mathrm{ext} : \mathbf{B}^s_{pp}(\Gamma,\mu) \hookrightarrow B^{s+\frac{n-d}{p}}_{pp}(\mathbb{R}^n)$$

such that

$$(\mathrm{ext}\, f)|_\Gamma = f$$

for every $f \in \mathbf{B}^s_{pp}(\Gamma,\mu)$.

Comparison

Let $0 < s < 1$ and $1 < p < \infty$. We want to show that in this case Besov spaces defined by both approaches coincide.

The comparison between these approaches relies on the notion of capacity. But before we recall some properties of fractional Sobolev spaces.

As it was mentioned in the Introduction the generalization of classical Sobolev spaces are so called Bessel potential spaces $H^s_p(\mathbb{R}^n)$.

Definition 3.8. Let $s > 0$ and $1 < p < \infty$. A function $f \in H^s_p(\mathbb{R}^n)$ if

$$\|f|H^s_p(\mathbb{R}^n)\| = \| \left(\left(1+|\xi|^2\right)^{s/2} \widehat{f} \right)^{\vee} |L_p(\mathbb{R}^n)\| < \infty.$$

It is known that

$$H^s_p(\mathbb{R}^n) = F^s_{p2}(\mathbb{R}^n), \quad s > 0, \; 1 < p < \infty, \tag{3.7}$$

where $F^s_{p2}(\mathbb{R}^n)$ are Triebel-Lizorkin spaces, [41, Section 2.3.5]. The relation between Besov spaces and Triebel-Lizorkin spaces is given by the following embedding

$$B^s_{p\min(p,q)}(\mathbb{R}^n) \hookrightarrow F^s_{pq}(\mathbb{R}^n) \hookrightarrow B^s_{p\max(p,q)}(\mathbb{R}^n). \tag{3.8}$$

In particular,

$$B^s_{pp}(\mathbb{R}^n) = F^s_{pp}(\mathbb{R}^n) \quad \text{(equivalent norms)}. \tag{3.9}$$

Then combining (3.2), (3.7) and (3.8), we get

$$\|\varphi|L_p(\Gamma,\mu)\| \le c\|\varphi|H_p^{\frac{n-d}{p}+\varepsilon}(\mathbb{R}^n)\|, \quad \varepsilon > 0, \quad \text{for all} \quad \varphi \in S(\mathbb{R}^n).$$

Let K be a compact set in $\mathbb{R}^n$. Let $1 < p < \infty$ and $\alpha > 0$. Then

$$C_{\alpha,p}(K) = \inf\left\{\|\varphi|H^\alpha_p(\mathbb{R}^n)\|^p : \varphi \in S(\mathbb{R}^n), \varphi \ge 1 \text{ on } K\right\}$$

is called the (α,p)-capacity of K. A property is said to hold (α,p)-quasi-everywhere, if it is true for all $x \in \mathbb{R}^n$ except the set of $C_{\alpha,p}$- capacity zero.
Let Γ be a d-set. Then for $\alpha = s + \frac{n-d}{p}$, $1 < p < \infty$, $0 < s < 1$ and all $\varphi \in S(\mathbb{R}^n)$ we have

$$\left(\int_\Gamma |\varphi(\gamma)|^p \, \mu(d\gamma) \right)^{1/p} \le c\|\varphi|H^\alpha_p(\mathbb{R}^n)\|. \tag{3.10}$$

Take any compact $K \subset \mathbb{R}^n$. Let φ be real and $\varphi \ge 1$ on K. Then from (3.10) follows

$$\mu(K) \le \int_\Gamma |\varphi(\gamma)|^p \, \mu(d\gamma) \le c\|\varphi|H^\alpha_p(\mathbb{R}^n)\|^p,$$

that implies

$$\mu(K) \le c\, C_{\alpha,p}(K). \tag{3.11}$$

The relation (3.9) implies that

$$B^s_{pp}(\Gamma,\mu) = \operatorname{tr}_\mu F_{pp}^{s+\frac{n-d}{p}}(\mathbb{R}^n).$$

According to Theorem 9.21 in [45] the trace of Triebel-Lizorkin spaces is independent of parameter q. Hence

$$B^s_{pp}(\Gamma,\mu) = \operatorname{tr}_\mu H_p^{s+\frac{n-d}{p}}(\mathbb{R}^n).$$

Moreover, it was shown in [18, Chapter VII] that the trace of the Sobolev space $H_p^{s+\frac{n-d}{p}}(\mathbb{R}^n)$ is the Besov space $\mathbf{B}_{pp}^s(\Gamma,\mu)$

$$\mathbf{B}_{pp}^s(\Gamma,\mu) = H_p^{s+\frac{n-d}{p}}(\mathbb{R}^n)|_\Gamma.$$

Let $f \in H_p^\alpha(\mathbb{R}^n)$. We may assume that f has a compact support in $\mathbb{R}^n$. As before define

$$\bar{f}(x) = \lim_{r\to 0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} f(y)\,dy,$$

if the limit exists. Then according to [1]

$$f = \bar{f} \ \ (\alpha,p) - \text{q.e. in } \mathbb{R}^n.$$

Define Sobolev mollifiers for $\bar{f}$ by

$$f_h(x) = h^{-n} \int_{\mathbb{R}^n} \omega\left(\frac{x-y}{h}\right) f(y)\,dy, \quad 0 < h < 1,$$

where $0 \le \omega \in D(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} \omega(x)\,dx = 1.$$

Recall that

$$f_h \to f \ \text{ in } \ H_p^\alpha(\mathbb{R}^n) \ \text{ if } \ h \to 0.$$

Moreover, from Theorem 1.25 in [38] follows

$$\lim_{h\to 0} f_h(x) = f(x)$$

for every Lebesgue point x. This means

$$f_h(x) \to f(x) \ \text{ pointwise } \ (\alpha,p) - \text{q.e.}$$

and together with (3.11) it implies

$$f_h(x) \to f(x) \ \text{ pointwise } \ \mu - \text{a.e.}$$

Set $\varphi_j = f_{2^{-j}}$ and take $\{\varphi_j\}$ as an approximating sequence for f. Then

$$B_{pp}^s(\Gamma,\mu) \ni \operatorname{tr}_\mu f = \lim_{j\to\infty} \varphi_j|_\Gamma = f|_\Gamma \in \mathbf{B}_{pp}^s(\Gamma,\mu) \quad \mu - \text{a.e.}$$

Remark 3.9. The comparison is based on the proof of Proposition 3.1. in [44].

Thus we no longer distinguish between $B^s_{pp}(\Gamma,\mu)$ and $\mathbf{B}^s_{pp}(\Gamma,\mu)$ and write $B^s_p(\Gamma)$ for short. Besov spaces $B^s_p(\Gamma)$, $1<p<\infty$, $0<s<1$, are normed by

$$\|f|B^s_{pp}(\Gamma,\mu)\| = \|f|L_p(\Gamma,\mu)\| + \left(\int_\Gamma \int_\Gamma \frac{|f(\gamma)-f(\delta)|^p}{|\gamma-\delta|^{d+sp}}\,\mu(d\delta)\mu(d\gamma)\right)^{\frac{1}{p}} \tag{3.12}$$

or equivalently

$$\|f|B^s_{pp}(\Gamma,\mu)\| = \|f|L_p(\Gamma,\mu)\| + \left(\int_\Gamma \int_{B(\gamma,1)} \frac{|f(\gamma)-f(\delta)|^p}{|\gamma-\delta|^{d+sp}}\,\mu(d\delta)\mu(d\gamma)\right)^{\frac{1}{p}}. \tag{3.13}$$

Most of our results will be first stated for the spaces $B^s_p(\Gamma)$ and then they will be extended to $B^s_{pq}(\Gamma,\mu)$ by real interpolation.

3.2 Real interpolation for the spaces $B^s_{pq}(\Gamma,\mu)$

Theorem 3.10. *Let Γ be a d-set in $\mathbb{R}^n$. Let $0<\theta<1$, $1<p<\infty$, $1\le q<\infty$, $0<s_0<1$, $0<s_1<1$, $s_0\neq s_1$ and $s=(1-\theta)s_0+\theta s_1$. Then*

$$\left(B^{s_0}_{pq_0}(\Gamma,\mu), B^{s_1}_{pq_1}(\Gamma,\mu)\right)_{\theta,q} = B^s_{pq}(\Gamma,\mu). \tag{3.14}$$

Proof. We put

$$P = \mathrm{ext}_\mu \circ \mathrm{tr}_\mu : B^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n) \hookrightarrow B^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n).$$

Then P is a linear and bounded map. From (3.4) it follows that

$$P^2 = \mathrm{ext}_\mu \circ \mathrm{tr}_\mu \circ \mathrm{ext}_\mu \circ \mathrm{tr}_\mu = P.$$

Hence P is a projection of $B^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n)$ onto $PB^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n)$. By $P\circ \mathrm{ext}_\mu = \mathrm{ext}_\mu$, one gets that ext_μ maps $B^s_{pq}(\Gamma,\mu)$ into $PB^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n)$. On the other hand, if $f\in PB^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n)$, then $f=\mathrm{ext}_\mu(\mathrm{tr}_\mu(f))$, $\mathrm{tr}_\mu f\in B^s_{pq}(\Gamma)$. Hence ext_μ maps $B^s_{pq}(\Gamma,\mu)$ onto $PB^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n)$. Since tr_μ and ext_μ are linear bounded operators, one has

$$\|f|B^s_{pq}(\Gamma,\mu)\| \sim \|\,\mathrm{ext}_\mu f|B^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n)\| \tag{3.15}$$

and it follows that

$$\mathrm{ext}_\mu : B^s_{pq}(\Gamma,\mu) \Leftrightarrow PB^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n)$$

is an isomorphic map.

Let

$$\left(B^{s_0}_{pq_0}(\Gamma,\mu), B^{s_1}_{pq_1}(\Gamma,\mu)\right)_{\theta,q} = B_\theta(\Gamma).$$

It is known that

$$\left(B^{s_0+\frac{n-d}{p}}_{pq_0}(\mathbb{R}^n), B^{s_1+\frac{n-d}{p}}_{pq_1}(\mathbb{R}^n)\right)_{\theta,q} = B^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n). \tag{3.16}$$

We denote the right-hand side of (3.16) by $B_\theta(\mathbb{R}^n)$.
By the interpolation property for the spaces on $\mathbb{R}^n$ and Γ

$$\|f|B_\theta(\Gamma)\| = \|\operatorname{tr}_\mu \circ \operatorname{ext}_\mu f|B_\theta(\Gamma)\| \le c\|\operatorname{ext}_\mu f|B_\theta(\mathbb{R}^n)\| \le c'\|f|B_\theta(\Gamma)\|. \tag{3.17}$$

Hence

$$\|f|B_\theta(\Gamma)\| \sim \|\operatorname{ext}_\mu f|B^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n)\|. \tag{3.18}$$

Together with (3.15) this leads to

$$\|f|B_\theta(\Gamma)\| \sim \|f|B^s_{pq}(\Gamma,\mu)\|.$$

This completes the proof. □

Remark 3.11. The proof essentially uses the way of reasoning in [46, Ch. 1.11.8].

3.3 Intrinsic atomic characterization of $B^s_p(\Gamma)$

Besov spaces $B^s_p(\Gamma)$ with $0 < s < 1$ and $1 < p < \infty$ can be characterized in terms of intrinsic building blocks, namely atoms.
Let for $\delta > 0$

$$\Gamma_\delta = \bigcup_{\gamma\in\Gamma} B(\gamma,\delta),$$

where

$$B(\gamma,\delta) = \{x \in \mathbb{R}^n : |x-\gamma| < \delta\}, \tag{3.19}$$

be a δ-neighbourhood of Γ. Let $0 < r < 1$ be fixed. Let for $j \in \mathbb{N}_0$,

$$\{\gamma_{j,m}\}_{m=1}^{M_j} \subset \Gamma \tag{3.20}$$

be the lattice of points with the following properties:

- For some $c_1 > 0$

$$|\gamma_{j,m_1} - \gamma_{j,m_2}| \ge c_1 r^j, \quad j \in \mathbb{N}_0, \quad m_1 \ne m_2. \tag{3.21}$$

- For some some $c_2 > 0$

$$\Gamma_{c_2 r^j} \subset \bigcup_{m=1}^{M_j} B(\gamma_{j,m}, r^j), \quad j \in \mathbb{N}_0, \tag{3.22}$$

  where $B(\gamma_{j,m}, r^j)$ are given by (3.19).

Let

$$B^{\Gamma}_{j,m} = \{\gamma \in \Gamma : |\gamma - \gamma_{j,m}| < r^j\}, \quad j \in \mathbb{N}_0, \ m = 1, \dots, M_j, \tag{3.23}$$

be the intersection of balls $B(\gamma_{j,m}, r^j)$ with Γ.

Definition 3.12. Let Γ be a d-set in $\mathbb{R}^n$. Let $1 < p < \infty$ and $0 < s < 1$. Then a continuous function a_{jm} on Γ is called an $(s,p)^*$-atom, if for $j \in \mathbb{N}_0$ and $m = 1, \dots, M_j$,

$$\operatorname{supp} a_{jm} \subset B^{\Gamma}_{j,m}, \tag{3.24}$$

$$|a_{jm}(\gamma)| \le \mathrm{H}^d \left(B^{\Gamma}_{j,m}\right)^{\frac{s}{d} - \frac{1}{p}}, \quad \gamma \in \Gamma, \tag{3.25}$$

and

$$|a_{jm}(\gamma) - a_{jm}(\delta)| \le \mathrm{H}^d \left(B^{\Gamma}_{j,m}\right)^{\frac{s-1}{d} - \frac{1}{p}} |\gamma - \delta| \tag{3.26}$$

with $\gamma, \delta \in \Gamma$, [46, Section 8.1.3].

Since Γ is a d-set, we can reformulate (3.25) and (3.26) as

$$|a_{jm}(\gamma)| \le c r^{j\left(s - \frac{d}{p}\right)},$$
$$|a_{jm}(\gamma) - a_{jm}(\delta)| \le c r^{j\left(s - 1 - \frac{d}{p}\right)} |\gamma - \delta| .$$

For our further purposes we need the following assertion which is covered by the Proposition 8.10 in [46].

Lemma 3.13. *Let Γ be a d-set. Let $r \ge 0$ and*

$$B^{\Gamma}(r) = \{\gamma \in \Gamma : |\gamma - \gamma_0| < r\} \quad \text{for some} \ \ \gamma_0 \in \Gamma,$$

and

$$B(2r) = \{x \in \mathbb{R}^n : |x - \gamma_0| < 2r\} .$$

Let

$$f \in B^s_p(\Gamma) \ \ \text{with} \ \ \operatorname{supp} f \subset B^{\Gamma}(r).$$

Then

$$\|f|B^s_p(\Gamma)\| = \inf \|g|B^t_p(\mathbb{R}^n)\|, \ \ t = s + (n-d)/p,$$

where the infimum is taken over all

$$g \in B^t_p(\mathbb{R}^n), \ \ g|_{\Gamma} = f, \ \ \operatorname{supp} g \subset B(2r).$$

Now we can formulate an intrinsic atomic decomposition of the trace spaces $B^s_p(\Gamma)$.

Theorem 3.14. *Let $1 < p < \infty$ and $0 < s < 1$. Then $B^s_p(\Gamma)$ is the collection of all $f \in L_1(\Gamma, \mu)$ which can be represented as*

$$f(\gamma) = \sum_{j=0}^{\infty} \sum_{m=1}^{M_j} \lambda^j_m a_{jm}(\gamma), \ \ \gamma \in \Gamma, \tag{3.27}$$

where

$$\|\lambda\| = \left(\sum_{j=0}^{\infty} \sum_{m=1}^{M_j} \left|\lambda_m^j\right|^p \right)^{\frac{1}{p}} < \infty,$$

a_{jm} are $(s,p)^*$-atoms according to Definition 3.12 and (3.27) converges absolutely in $L_1(\Gamma,\mu)$. Furthermore,

$$\|f|B_p^s(\Gamma)\| \sim \inf \|\lambda\| \tag{3.28}$$

where infimum is taken over all admissible representations (3.27), [46, Chapter 8.1.3].

We introduce new type of atoms, that we call (s,p,σ)-atoms.

Definition 3.15. Let $1 < p < \infty$, $0 < \sigma < 1$ and $0 < s < \sigma$. Then a continuous function a_{jm} on Γ is called an (s,p,σ)-atom, if for $j \in \mathbb{N}_0$ and $m = 1,\dots,M_j$,

$$\operatorname{supp} a_{jm} \subset B_{j,m}^{\Gamma}, \tag{3.29}$$

$$|a_{jm}(\gamma)| \le c r^{j\left(s-\frac{d}{p}\right)}, \quad \gamma \in \Gamma, \tag{3.30}$$

and

$$|a_{jm}(\gamma) - a_{jm}(\delta)| \le c r^{j\left(s-\sigma-\frac{d}{p}\right)} |\gamma - \delta|^{\sigma} \tag{3.31}$$

with $\gamma,\delta \in \Gamma$.

Let a_{jm} be an $(s,p)^*$-atom and $0 < s < \sigma$. Then

$$\begin{aligned} |a_{jm}(\gamma) - a_{jm}(\delta)| &\le c r^{j\left(s-1-\frac{d}{p}\right)} |\gamma - \delta| \\ = c r^{j\left(s-1-\frac{d}{p}\right)} |\gamma-\delta|^{1-\sigma} |\gamma-\delta|^{\sigma} &\le c r^{j\left(s-1-\frac{d}{p}\right)} r^{j(1-\sigma)} |\gamma-\delta|^{\sigma} \\ &= c r^{j\left(s-\sigma-\frac{d}{p}\right)} |\gamma-\delta|^{\sigma}, \end{aligned}$$

which shows that any $(s,p)^*$-atom is an (s,p,σ)-atom.

Theorem 3.16. *Let $1 < p < \infty$, $0 < \sigma < 1$ and $0 < s < \sigma$. Then $B_p^s(\Gamma)$ is the collection of all $f \in L_1(\Gamma,\mu)$ which can be represented as*

$$f(\gamma) = \sum_{j=0}^{\infty} \sum_{m=1}^{M_j} \lambda_m^j a_{jm}(\gamma), \quad \gamma \in \Gamma, \tag{3.32}$$

where

$$\|\lambda\| = \left(\sum_{j=0}^{\infty} \sum_{m=1}^{M_j} \left|\lambda_m^j\right|^p \right)^{\frac{1}{p}} < \infty,$$

a_{jm} are (s,p,σ)-atoms according to Definition 3.15 and (3.32) converges absolutely in $L_1(\Gamma,\mu)$. Furthermore,

$$\|f|B_p^s(\Gamma)\| \sim \inf \|\lambda\|$$

where infimum is taken over all admissible representations (3.32).

Proof. The proof is the adaption of reasoning in [46, Section 8.1.3]. The representation (3.27) with $(s,p)^*$-atoms is a special case of the representation (3.32) and it holds (3.28). Hence it remains to show that from the representation (3.32) follows that

$$f \in B_p^s(\Gamma) \quad \text{and} \quad \|f|B_p^s(\Gamma)\| \leq c\|\lambda\|.$$

First we estimate the norm of (s,p,σ)-atoms in $B_p^s(\Gamma)$. Let L be a number such that $\operatorname{diam}\Gamma \leq 2^L$. Then

$$\int\limits_\Gamma \int\limits_\Gamma \frac{|a_{jm}(\gamma) - a_{jm}(\delta)|^p}{|\gamma-\delta|^{d+sp}} \mu(d\delta)\mu(d\gamma) \leq c \int\limits_\Gamma \int\limits_\Gamma \frac{1}{|\gamma-\delta|^{d+(s-\sigma)p}} \mu(d\delta)\mu(d\gamma)$$

$$= c \int\limits_\Gamma \sum_{i=-\infty}^{L} \int\limits_{B(\gamma,2^i)\setminus B(\gamma,2^{i-1})} \frac{1}{|\gamma-\delta|^{d+(s-\sigma)p}} \mu(d\delta)\mu(d\gamma)$$

$$\leq c \int\limits_\Gamma \sum_{i=-\infty}^{L} \int\limits_{B(\gamma,2^i)\setminus B(\gamma,2^{i-1})} \frac{1}{2^{i(d+(s-\sigma)p)}} \mu(d\delta)\mu(d\gamma)$$

$$\leq c\mu(\Gamma) \sum_{i=-\infty}^{L} \frac{2^{id}}{2^{i(d+(s-\sigma)p)}} = c\mu(\Gamma) \frac{2^{L(s-\sigma)p}}{1-2^{(s-\sigma)p}} \leq C.$$

Moreover,

$$\int\limits_\Gamma |a_{jm}(\gamma)|^p \, \mu(d\gamma) \leq \int\limits_{B_{jm}} \mu(B_{jm})^{\frac{sp}{d}-1} \mu(d\gamma) \leq \mu(\Gamma)^{\frac{sp}{d}} = C.$$

This means that there is a constant $C > 0$ such that

$$\|a_{jm}|B_p^s(\Gamma)\| \leq C$$

for all (s,p,σ)-atoms. Furthermore, for $0 < s \leq \bar{s} < \sigma$ we can write

$$a_{jm}(\gamma) = r^{j(s-\bar{s})} b_{jm}(\gamma),$$

where

$$b_{jm}(\gamma) = r^{j(\bar{s}-s)} a_{jm}(\gamma).$$

For each $j \in \mathbb{N}_0$ and $m = 1, \ldots, M_j$ we have

$$\operatorname{supp} b_{jm} = \operatorname{supp} a_{jm} \subset B_{jm}^\Gamma,$$

$$|b_{jm}(\gamma)| \leq c r^{j\left(\bar{s}-\frac{d}{p}\right)}$$

and

$$|b_{jm}(\gamma) - b_{jm}(\delta)| \leq c r^{j\left(\bar{s}-\sigma-\frac{d}{p}\right)} |\gamma-\delta|^\sigma.$$

This shows that b_{jm} are $(\bar{s}, p, \sigma)$- atoms and

$$\|b_{jm}|B_p^{\bar{s}}(\Gamma)\| \leq C.$$

Hence

$$\|a_{jm}|B_p^{\bar{s}}(\Gamma)\| \leq Cr^{j(s-\bar{s})}.$$

We apply Lemma 3.13 to a_{jm}. Then it follows that there are functions

$$A_{jm} \in B_{pp}^{\bar{t}}(\mathbb{R}^n), \quad \text{where} \quad \bar{t} = \bar{s} + \frac{n-d}{p},$$

such that

$$\operatorname{tr}_\mu A_{jm} = a_{jm}, \quad \operatorname{supp} A_{jm} \subset \left\{x \in \mathbb{R}^n : |x - \gamma_{jm}| \leq c_1 r^j\right\}$$

and

$$\|A_{jm}|B_{pp}^{\bar{t}}(\mathbb{R}^n)\| \leq c_2 r^{j(t-\bar{t})}, \quad t = s + \frac{n-d}{p}.$$

Then according to Definition 2.7 in [46] A_{jm} are non-smooth atoms for $B_{pp}^t(\mathbb{R}^n)$ and from Theorem 2.3 in [46] follows that

$$F = \sum_{j=0}^{\infty} \sum_{m=1}^{M_j} \lambda_m^j A_{jm} \quad \text{with} \quad \|\lambda\| < \infty$$

belongs to $B_{pp}^t(\mathbb{R}^n)$ and

$$\|F|B_{pp}^t(\mathbb{R}^n)\| \leq c\|\lambda\|.$$

Taking into account that $f = \operatorname{tr}_\mu F$, we may conclude

$$\|f|B_p^s(\Gamma)\| \leq c\|\lambda\|.$$

□

CHAPTER 4

Besov spaces on nested fractals

In this chapter we assume that $\Gamma \subset \mathbb{R}^n$ is a nested fractal such that

$$R(x,y) \sim |x-y|^{d_w - d}, \tag{4.1}$$

where $R(x,y)$ stands for the effective resistance metric and $|x-y|$ is the Euclidean distance in $\mathbb{R}^n$, for details we refer to Section 1.2.7. In particular, Γ can be the Sierpinski gasket or pentakun. Then relation (4.1) implies that piecewise harmonic functions belong to the Hölder class with exponent $\sigma = d_w - d$, i.e.

$$\left|\psi_\xi^j(x) - \psi_\xi^j(y)\right| \le c\,|x-y|^\sigma .$$

This enables us to treat piecewise harmonic functions as (s,p,σ)-atoms. Thus functions from $B^s_{pq}(\Gamma,\mu)$ can be characterized in terms of the coefficients of its expansion in a piecewise harmonic basis.

4.1 Representation of a function by piecewise harmonic basis

We start with the following observation. Let h be the function which equals identically 1 on Γ. Then h is harmonic and $h|_{V_0} = 1$. On the other hand, function

$$g = \sum_{\xi \in V_0} \psi_\xi^0$$

is also harmonic and $g|_{V_0} = 1$. Due to the uniqueness of the harmonic function with given boundary values we get that $h \equiv g$ or equivalently

$$\sum_{\xi \in V_0} \psi_\xi^0 \equiv 1 \ \text{ on } \ \Gamma.$$

This statement has following counterpart for $(j-1)$-harmonic functions. Let $\xi \in V_j \backslash V_{j-1}$, $j \ge 1$ be fixed. There is an $\omega \in \Sigma$ such that

$$\xi = \pi(\omega). \tag{4.2}$$

We define $\Delta(\xi)$ by
$$\Delta(\xi) = \left\{\eta \in V_{j-1} : \eta \in F_{\omega_1\omega_2\ldots\omega_{j-1}}(\Gamma)\right\},$$
where ω is chosen according to (4.2). $\Delta(\xi)$ consists of vertices of $(j-1)$-simplex that ξ belongs to. It is the same as the one defined in [16, Section 4.1]. Note that
$$\psi_\zeta^{j-1}(\xi) = 0, \quad V_{j-1} \ni \zeta \notin \Delta(\xi) \tag{4.3}$$
and
$$\sum_{\eta\in\Delta(\xi)} \psi_\eta^{j-1}(\xi) = 1. \tag{4.4}$$

Let $f \in C(\Gamma)$. There exists the unique harmonic function $P_0 f$ that interpolates f at all points of V_0
$$\begin{aligned} &P_0 f|_{V_0} = f|_{V_0},\\ &P_0 f - \text{ harmonic.}\end{aligned}$$
It is clear that
$$P_0 f = \sum_{\xi\in V_0} f(\xi)\psi_\xi^0.$$
Let $P_n f$, $n \geq 1$, be the unique piecewise harmonic function in H_n which interpolates f at all points in V_n
$$P_n f = \sum_{\xi\in V_n} f(\xi)\psi_\xi^n.$$
Let us take the approximation of f by 1-harmonic functions
$$P_1 f = \sum_{\xi\in V_1} f(\xi)\psi_\xi^1 = \sum_{\xi\in V_0} f(\xi)\psi_\xi^1 + \sum_{\xi\in V_1\setminus V_0} f(\xi)\psi_\xi^1 = F + G. \tag{4.5}$$

Function F is a 1-harmonic function that coincides with f on V_0 and is 0 at all points of $V_1 \setminus V_0$
$$\begin{aligned} &F|_{V_0} = f,\\ &F|_{V_1\setminus V_0} = 0.\end{aligned}$$
Since every harmonic function is also 1-harmonic and there is unique 1-harmonic function for given boundary values, we may conclude
$$F = P_0 f - \sum_{\xi\in V_1\setminus V_0} (P_0 f)(\xi)\psi_\xi^1. \tag{4.6}$$

Replacing F in (4.5) by (4.6) we get
$$P_1 f = P_0 f + \sum_{\xi\in V_1\setminus V_0} \left[f(\xi) - P_0 f(\xi)\right]\psi_\xi^1. \tag{4.7}$$

For $\xi \in V_j \setminus V_{j-1}$, $j \geq 1$, define
$$c_\xi(f) = P_j f(\xi) - P_{j-1} f(\xi) = f(\xi) - P_{j-1} f(\xi), \tag{4.8}$$

complemented by

$$c_\xi(f) = f(\xi), \quad \xi \in V_0.$$

Then (4.7) can be written in the following way

$$P_1 f = P_0 f + \sum_{\xi \in V_1 \setminus V_0} c_\xi(f)\psi_\xi^1.$$

By method of mathematical induction we may prove that

$$P_n f = \sum_{\xi \in V_0} f(\xi)\psi_\xi^0 + \sum_{j=1}^{n} \sum_{\xi \in V_j \setminus V_{j-1}} c_\xi(f)\psi_\xi^j, \quad n \geq 1.$$

Indeed,

$$\begin{aligned} P_n f = \sum_{\xi \in V_n} f(\xi)\psi_\xi^n &= \sum_{\xi \in V_n \setminus V_{n-1}} f(\xi)\psi_\xi^n + \sum_{\xi \in V_{n-1}} f(\xi)\psi_\xi^n \\ &= \sum_{\xi \in V_n \setminus V_{n-1}} f(\xi)\psi_\xi^n + P_{n-1} f - \sum_{\xi \in V_n \setminus V_{n-1}} P_{n-1} f(\xi)\psi_\xi^n \\ &= P_0 f + \sum_{j=1}^{n-1} \sum_{\xi \in V_j \setminus V_{j-1}} c_\xi(f)\psi_\xi^j + \sum_{\xi \in V_n \setminus V_{n-1}} c_\xi(f)\psi_\xi^j. \end{aligned}$$

Let $V_{-1} = \emptyset$. Then

$$P_n f = \sum_{j=0}^{n} \sum_{\xi \in V_j \setminus V_{j-1}} c_\xi(f)\psi_\xi^j. \tag{4.9}$$

For the coefficients $c_\xi(f)$, $\xi \in V_j \setminus V_{j-1}$, $j \geq 1$, we may write

$$c_\xi(f) = f(\xi) - \sum_{\eta \in V_{j-1}} f(\eta)\psi_\eta^{j-1}(\xi).$$

Taking into account (4.3) and (4.4), we get

$$c_\xi(f) = f(\xi) - \sum_{\eta \in \Delta(\xi)} \alpha_{\xi\eta} f(\eta) \tag{4.10}$$

with

$$\sum_{\eta \in \Delta(\xi)} \alpha_{\xi\eta} = 1, \tag{4.11}$$

where $\alpha_{\xi\eta} = \psi_\eta^{j-1}(\xi)$.

Lemma 4.1. *Let $f \in C(\Gamma)$ and $P_n f$ be given by (4.9). Then $P_n f$ tends to f uniformly on Γ as $n \to \infty$.*

Proof. Let $\varepsilon > 0$. Then there is $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon, \quad \text{if} \ \ |x - y| < \delta.$$

Choose $N \in \mathbb{N}$ in such a way that for $n \geq N$

$$|x - y| < \delta \ \text{ if } \ x, y \in F_w(\Gamma), \ w \in W_n.$$

Take any $x \in \Gamma$. From the maximum principle for harmonic functions follows

$$|f(x) - P_n f(x)| \leq |f(x) - f(x_n)|,$$

where x_n is one of the vertices of the simplex $F_w(\Gamma)$, $w \in W_n$, that x belongs to. Since $|x - x_n| < \delta$, we get $|f(x) - P_n f(x)| < \varepsilon$. □

Proposition 1.3.2 in [35] implies that $\{\psi_\xi^j, \xi \in V_j \setminus V_{j-1}, j \geq 0\}$ is an interpolating basis. This means that $f \in C(\Gamma)$ has the unique representation

$$f = \sum_{j=0}^{\infty} \sum_{\xi \in V_j \setminus V_{j-1}} c_\xi(f) \psi_\xi^j.$$

4.2 Characterization of $B^s_{pq}(\Gamma, \mu)$ by piecewise harmonic basis

The question arises whether Besov spaces with a certain range of parameters can be characterized by coefficients $c_\xi(f)$. We give an affirmative answer in the following theorem.

Theorem 4.2. *Let Γ be the above d-set with ρ as in (1.27) and σ as in (1.37). Let*

$$1 < p < \infty \ \ \textit{and} \ \ \frac{d}{p} < s < \min\{1, \sigma\}. \tag{4.12}$$

Then $f \in C(\Gamma)$ belongs to $B^s_p(\Gamma)$ if and only if it can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{\xi \in V_j \setminus V_{j-1}} c_\xi(f) \psi_\xi^j, \tag{4.13}$$

where

$$C^s_p(f) = \left(\sum_{j=0}^{\infty} \rho^{j\left(\frac{d}{p} - s\right)p} \sum_{\xi \in V_j \setminus V_{j-1}} |c_\xi(f)|^p \right)^{\frac{1}{p}} < \infty,$$

unconditional convergence being in $C(\Gamma)$. Furthermore,

$$\|f|B^s_p(\Gamma)\| \sim C^s_p(f).$$

Proof. The idea of the proof is the same as in [16, Theorem 5.1]. Let

$$a_{j\xi}(x) = \rho^{j\left(s-\frac{d}{p}\right)}\psi_\xi^j(x), \quad j \in \mathbb{N}_0, \ \xi \in V_j \setminus V_{j-1}.$$

Then $a_{j\xi}$ satisfy (3.29)-(3.31). Taking into account that $C(\Gamma) \subset L_1(\Gamma)$ we get that (4.13) is an atomic representation of f and from Theorem 3.16 it follows that

$$\|f|B_p^s(\Gamma)\| \le c\, C_p^s(f).$$

To prove the converse, let $f \in B_p^s(\Gamma)$ and let

$$f = \sum_{j=0}^{\infty}\sum_{m=1}^{M_j} \lambda_m^j a_{jm}$$

be an atomic decomposition of f into (s,p,σ)-atoms with $r = \rho$ in (3.23), (3.30) and (3.31) such that

$$\|\lambda\| \le c\|f|B_p^s(\Gamma)\|. \tag{4.14}$$

Then taking into account (3.30) and that $s > \frac{d}{p}$ we get

$$\left|\sum_{m=1}^{M_j} \lambda_m^j a_{jm}\right| \le \sup_m \left|\lambda_m^j\right| \sum_{m=1}^{M_j} \rho^{j\left(s-\frac{d}{p}\right)} \le c\rho^{\left(s-\frac{d}{p}\right)} \sup_m \left|\lambda_m^j\right|$$

$$\le c\rho^{\left(s-\frac{d}{p}\right)} \left(\sum_{m=1}^{M_j} \left|\lambda_m^j\right|^p\right)^{\frac{1}{p}}.$$

The Weierstrass test together with the estimate (4.14) imply that the series $\sum_{j=0}^{\infty}\sum_{m=1}^{M_j} \lambda_m^j a_{jm}$ converges uniformly and it follows

$$c_\xi(f) = \sum_{j=0}^{\infty}\sum_{m=1}^{M_j} \lambda_m^j c_\xi(a_{jm}).$$

From the formula (4.10) together with (4.11) and the property (3.30) of (s,p,σ)-atoms follows

$$|c_\xi(a_{jm})| \le 2\rho^{j\left(s-\frac{d}{p}\right)}. \tag{4.15}$$

Moreover, for $i > 0$ the property (3.31) implies

$$|c_\xi(a_{jm})| = \left|a_{jm}(\xi) - \sum_{\eta\in\Delta(\xi)} \alpha_{\xi\eta} a_{jm}(\eta)\right| = \left|\sum_{\eta\in\Delta(\xi)} \alpha_{\xi\eta}(a_{jm}(\xi) - a_{jm}(\eta))\right|$$

$$\le \rho^{i\sigma}\rho^{j\left(s-\sigma-\frac{d}{p}\right)}, \quad \xi \in V_i \setminus V_{i-1}. \tag{4.16}$$

Let us split $c_\xi(f)$ into two parts

$$c_\xi(f) = \sum_{j=0}^{i} \sum_{m=1}^{M_j} \lambda_m^j c_\xi(a_{jm}) + \sum_{j=i+1}^{\infty} \sum_{m=1}^{M_j} \lambda_m^j c_\xi(a_{jm}) = x_\xi(f) + y_\xi(f).$$

Taking into account the support condition for atoms (3.29), we get that for all ξ and j the number of atoms such that $c_\xi(a_{jm}) \neq 0$ is finite.

First we deal with

$$X_{i,p} = \left(\sum_{\xi \in V_i \setminus V_{i-1}} |x_\xi(f)|^p \right)^{1/p}.$$

Note that

$$\{\xi \in V_i \setminus V_{i-1} : c_\xi(a_{jm}) \neq 0\} \subset \left\{\xi \in V_i \cap B^\Gamma(\gamma_{jm}, \rho^j)\right\}.$$

The balls $B^\Gamma(\xi, \frac{\rho^i}{2})$ corresponding to different $\xi \in V_i \cap B^\Gamma(\gamma_{jm}, \rho^j)$ are disjoint and for $j < i$ they are contained in $B^\Gamma(\gamma_{jm}, 2\rho^j)$. Thus

$$\sum_{\xi \in V_i \cap B^\Gamma(\gamma_{jm}, \rho^j)} \mu\left(B^\Gamma\left(\xi, \rho^i/2\right)\right) \leq \mu\left(B^\Gamma\left(\gamma_{j,m}, 2\rho^j\right)\right).$$

Since μ is a d-measure this implies that $\left\{\xi \in V_i \cap B^\Gamma(\gamma_{jm}, \rho^j)\right\}$ can have at most $c\left(\frac{\rho^j}{\rho^i}\right)^d$ elements. Hence

$$\#\left\{\xi \in V_i \setminus V_{i-1} : c_\xi(a_{jm}) \neq 0\right\} \leq c\rho^{(j-i)d}, \quad j < i.$$

By Minskowski's and Hölder's inequalities together with (4.16) follows

$$X_{i,p} = \left(\sum_{\xi \in V_i \setminus V_{i-1}} |x_\xi(f)|^p \right)^{1/p} \leq \sum_{j=0}^{i} \left(\sum_{\xi \in V_i \setminus V_{i-1}} \left(\sum_{m=1}^{M_j} \left|\lambda_m^j\right| |c_\xi(a_{jm})| \right)^p \right)^{1/p}$$

$$\leq \sum_{j=0}^{i} \sum_{m=1}^{M_j} \left(\sum_{\xi \in V_i \setminus V_{i-1}} \left|\lambda_m^j\right|^p |c_\xi(a_{jm})|^p \right)^{1/p} \leq \sum_{j=0}^{i} \sum_{m=1}^{M_j} \left|\lambda_m^j\right| \rho^{i\sigma} \rho^{j\left(s-\sigma-\frac{d}{p}\right)} \rho^{(j-i)\frac{d}{p}}$$

$$\leq c\rho^{i\left(\sigma-\frac{d}{p}\right)} \sum_{j=0}^{i} \rho^{j(s-\sigma)} \left(\sum_{m=1}^{M_j} \left|\lambda_m^j\right|^p \right)^{1/p}$$

and it follows

$$X_{i,p,s} = \rho^{i\left(\frac{d}{p}-s\right)} X_{i,p} \leq c\rho^{i(\sigma-s)} \sum_{j=0}^{i} \rho^{j(s-\sigma)} \left(\sum_{m=1}^{M_j} \left|\lambda_m^j\right|^p \right)^{1/p}.$$

Jensen's inequality implies

$$X_{i,p,s}^p \le c\rho^{i(\sigma-s)p}\rho^{i(s-\sigma)(p-1)}\sum_{j=0}^{i}\rho^{j(s-\sigma)}\sum_{m=1}^{M_j}\left|\lambda_m^j\right|^p$$

$$= c\rho^{i(\sigma-s)}\sum_{j=0}^{i}\rho^{j(s-\sigma)}\sum_{m=1}^{M_j}\left|\lambda_m^j\right|^p.$$

Then

$$\left(\sum_{i=0}^{\infty}X_{i,p,s}^p\right)^{1/p} \le c\left(\sum_{i=0}^{\infty}\rho^{i(\sigma-s)}\sum_{j=0}^{i}\rho^{j(s-\sigma)}\sum_{m=1}^{M_j}\left|\lambda_m^j\right|^p\right)^{1/p}$$

$$= c\left(\sum_{j=0}^{\infty}\left(\sum_{i=j}^{\infty}\rho^{i(\sigma-s)}\right)\rho^{j(s-\sigma)}\sum_{m=1}^{M_j}\left|\lambda_m^j\right|^p\right)^{1/p} \le c\left(\sum_{j=0}^{\infty}\sum_{m=1}^{M_j}\left|\lambda_m^j\right|^p\right)^{1/p} = c\|\lambda\|.$$

To estimate

$$Y_{i,p} = \left(\sum_{\xi\in V_i\setminus V_{i-1}}|y_\xi(f)|^p\right)^{1/p}$$

we use Minkowski's and Hölder's inequalities together with the property (4.15). Then we get

$$Y_{i,p} \le \sum_{j=i}^{\infty}\left(\sum_{\xi\in V_i\setminus V_{i-1}}\left(\sum_{m=1}^{M_j}\left|\lambda_m^j\right|\left|c_\xi(a_{jm})\right|\right)^p\right)^{1/p}$$

$$\le \sum_{j=i}^{\infty}\sum_{m=1}^{M_j}\left(\sum_{\xi\in V_i\setminus V_{i-1}}\left|\lambda_m^j\right|^p\left|c_\xi(a_{jm})\right|^p\right)^{1/p} \le c\sum_{j=i}^{\infty}\sum_{m=1}^{M_j}\left|\lambda_m^j\right|\rho^{j\left(s-\frac{d}{p}\right)}$$

$$\le c\sum_{j=i}^{\infty}\rho^{j\left(s-\frac{d}{p}\right)}\left(\sum_{m=1}^{M_j}\left|\lambda_m^j\right|^p\right)^{1/p}.$$

Hence we have

$$Y_{i,p,s} = \rho^{i\left(\frac{d}{p}-s\right)}Y_{i,p} \le c\rho^{i\left(\frac{d}{p}-s\right)}\sum_{j=i}^{\infty}\rho^{j\left(s-\frac{d}{p}\right)}\left(\sum_{m=1}^{M_j}\left|\lambda_m^j\right|^p\right)^{1/p}.$$

Applying Jensen's inequality we get

$$Y_{i,p,s}^p \le c\rho^{i\left(\frac{d}{p}-s\right)p}\rho^{i\left(\frac{d}{p}-s\right)(p-1)}\sum_{j=i}^{\infty}\rho^{j\left(s-\frac{d}{p}\right)}\sum_{m=1}^{M_j}\left|\lambda_m^j\right|^p$$

$$\leq c\rho^{i\left(\frac{d}{p}-s\right)}\sum_{j=i}^{\infty}\rho^{j\left(s-\frac{d}{p}\right)}\sum_{m=1}^{M_j}\left|\lambda_m^j\right|^p.$$

Then

$$\left(\sum_{i=0}^{\infty}Y_{i,p,s}^p\right)^{1/p}\leq c\left(\sum_{i=0}^{\infty}\sum_{j=i}^{\infty}\rho^{i\left(\frac{d}{p}-s\right)}\rho^{j\left(s-\frac{d}{p}\right)}\sum_{m=1}^{M_j}\left|\lambda_m^j\right|^p\right)^{1/p}$$

$$\leq c\left(\sum_{j=0}^{\infty}\left(\sum_{i=0}^{j}\rho^{i\left(\frac{d}{p}-s\right)}\right)\rho^{j\left(s-\frac{d}{p}\right)}\sum_{m=1}^{M_j}\left|\lambda_m^j\right|^p\right)^{1/p}\leq c\left(\sum_{j=0}^{\infty}\sum_{m=1}^{M_j}\left|\lambda_m^j\right|^p\right)^{1/p}=c\|\lambda\|.$$

Thus

$$C_p^s(f)=\left(\sum_{i=0}^{\infty}\rho^{i\left(\frac{d}{p}-s\right)p}\sum_{\xi\in V_i\setminus V_{i-1}}|c_\xi(f)|^p\right)^{\frac{1}{p}}$$

$$\leq\left(\sum_{i=0}^{\infty}X_{i,p,s}^p\right)^{\frac{1}{p}}+\left(\sum_{i=0}^{\infty}Y_{i,p,s}^p\right)^{\frac{1}{p}}\leq c\|\lambda\|\leq c\|f|B_p^s(\Gamma)\|.$$

□

Corollary 4.3. *Let*

$$1<p<\infty \quad \text{and} \quad \frac{d}{p}<s<\min\{1,\sigma\}.$$

The system of functions $\{\psi_\xi^j, j\in\mathbb{N}_0, \xi\in V_j\setminus V_{j-1}\}$ *is an unconditional basis in* $B_p^s(\Gamma)$.

Proof. Let $f\in B_p^s(\Gamma)$. Then f has the unique representation

$$f=\sum_{j=0}^{\infty}\sum_{\xi\in V_j\setminus V_{j-1}}c_\xi(f)\psi_\xi^j \tag{4.17}$$

with the convergence first being in $C(\Gamma)$. It is left to show that (4.17) converges in $B_p^s(\Gamma)$.

Let us show that the sequence of partial sums

$$S_n=\sum_{j=0}^{n}\sum_{\xi\in V_j\setminus V_{j-1}}c_\xi(f)\psi_\xi^j$$

is a Cauchy sequence in $B_p^s(\Gamma)$. For $n>m$

$$\|S_n-S_m|B_p^s(\Gamma)\|=\|\sum_{j=m+1}^{n}\sum_{\xi\in V_j\setminus V_{j-1}}c_\xi(f)\psi_\xi^j|B_p^s(\Gamma)\|$$

$$\sim \left(\sum_{j=m+1}^{n} \rho^{j(\frac{d}{p}-s)p} \sum_{\xi \in V_j \setminus V_{j-1}} |c_\xi(f)|^p \right)^{\frac{1}{p}} \to 0, \quad n, m \to \infty.$$

Since $B^s_p(\Gamma)$ is complete, the series (4.17) converges to f in $B^s_p(\Gamma)$. □

Theorem 4.2 establishes isomorphism between function spaces $B^s_p(\Gamma)$ and certain sequence spaces.

Definition 4.4. Let $0 < s < 1$, $1 < p < \infty$ and $1 \le q < \infty$. Let b^{*s}_{pq} be the space of all sequences $a = \{a_{jm}, j \in \mathbb{N}_0, m = 1, \dots, M_j\}$ such that

$$\|a|b^{*s}_{pq}\| = \left(\sum_{j=0}^{\infty} \rho^{-jsq} \left(\sum_{m=1}^{M_j} |a_{jm}|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} < \infty.$$

We can reformulate the above result in the following manner.

Theorem 4.5. *Let Γ be a d-set with ρ as in (1.27) and σ as in (1.37). Let*

$$1 < p < \infty \quad \text{and} \quad \frac{d}{p} < s < \sigma.$$

Then $f \in C(\Gamma)$ belongs to $B^s_p(\Gamma)$ if and only if it can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{\xi \in V_j \setminus V_{j-1}} c_\xi(f) \psi^j_\xi,$$

unconditional convergence being in $C(\Gamma)$. Furthermore this representation is unique and

$$I : f \to \left\{ \rho^{\frac{jd}{p}} c_\xi(f), j \in \mathbb{N}_0, \xi \in V_j \setminus V_{j-1} \right\} \tag{4.18}$$

is an isomorphic map of $B^s_p(\Gamma)$ onto the sequence space b^{*s}_{pp}.

To extend Theorem 4.5 to spaces $B^s_{pq}(\Gamma, \mu)$ we apply real interpolation.

Let $0 < \theta < 1$, $1 < p < \infty$, $1 \le q < \infty$, $0 < s_0, s_1 < 1$, $s_0 \ne s_1$ and $s = (1-\theta)s_0 + \theta s_1$. Theorem 3.10 implies

$$\left(B^{s_0}_p(\Gamma), B^{s_1}_p(\Gamma) \right)_{\theta,q} = B^s_{pq}(\Gamma, \mu). \tag{4.19}$$

As for the interpolation of the sequence spaces b^{*s}_{pq}, we have the following statement.

Theorem 4.6. *Let $0 < \theta < 1$, $1 < p < \infty$, $1 \le q < \infty$, $0 < s_0, s_1 < 1$, $s_0 \ne s_1$ and $s = (1-\theta)s_0 + \theta s_1$. Then*

$$\left(b^{*s_0}_{pp}, b^{*s_1}_{pp} \right)_{\theta,q} = b^{*s}_{pq}. \tag{4.20}$$

The proof follows the same lines of Theorem in [40, Chapter 1.18.2].

Combining (4.19), (4.20) and (4.18) we deduce following corollary.

Corollary 4.7. *Let*

$$\frac{d}{p} < s < \sigma, \;\; 1 < p < \infty \;\; 1 \le q < \infty.$$

Then $f \in C(\Gamma)$ belongs to $B^s_{pq}(\Gamma, \mu)$ if and only if it can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{\xi \in V_j \setminus V_{j-1}} c_\xi(f) \psi^j_\xi,$$

unconditional convergence being in $C(\Gamma)$. Furthermore this representation is unique and

$$I : f \to \left\{ \rho^{\frac{jd}{p}} c_\xi(f), j \in \mathbb{N}_0, \xi \in V_j \setminus V_{j-1} \right\}$$

is an isomorphic map of $B^s_{pq}(\Gamma, \mu)$ onto the sequence space b^{*s}_{pq}.

CHAPTER 5

Besov spaces on the Koch curve

As we discussed before, any self-similar set, in particular any self-similar curve K from Section 1.2.2, is a d-set, with d being the Hausdorff dimension of the set. One of the ways to define Besov spaces $B^s_{pq}(K,\mu)$ on K is by traces, see Chapter 3. The second way is to use the snowflaked transform H, defined by (1.26), to introduce Besov spaces $\mathbb{B}^s_{pq}(K,\mu)$. The question arises how the function spaces $B^s_{pq}(K,\mu)$ and $\mathbb{B}^s_{pq}(K,\mu)$ are interrelated. In particular, we shift the characterization in terms of Daubechies wavelets from $(\mathbb{T}, \rho = |x-y|^{1/d}, \mu_L)$, [46, p. 360], to K.

In our further reasoning the Koch curve serves as an example, though all our conclusions remain true for any self-similar curve from Section 1.2.2. From now on we also identify the unit interval I and the 1-torus $\mathbb{T}$.

5.1 Besov spaces $\mathbb{B}^s_{pq}(K,\mu)$

Let K be the Koch curve in $\mathbb{R}^2$, discussed in the Example 1.34. It is a d-set with $d = \frac{\log 4}{\log 3}$. We endow K with measure μ defined by (1.29), which is a multiple of the Hausdorff measure.

Let

$$\mathbb{B}^s_{pq}(K,\mu) = \left\{ f \circ H^{-1} : f \in B^s_{pq}(\mathbb{T}) \right\} = B^s_{pq}(\mathbb{T}) \circ H^{-1}$$

with

$$\|f \circ H^{-1} | \mathbb{B}^s_{pq}(K,\mu)\| = \|f | B^s_{pq}(\mathbb{T})\|.$$

We are interested in wavelet expansions for the spaces $\mathbb{B}^s_{pq}(K,\mu)$. Define $\widetilde{\psi}^k_j$ by

$$\widetilde{\psi}^k_j(\gamma) = \psi^{k,per}_j \circ H^{-1}(\gamma).$$

From (1.32) follows that the system $\left\{ \widetilde{\psi}^k_j, j \in \mathbb{N}_0, k \in \mathbb{P}_j \right\}$ is orthonormal in $L_2(K,\mu)$.

The counterpart of Theorem 1.16 for the spaces $\mathbb{B}^s_{pq}(K,\mu)$ reads as follows.

Theorem 5.1. *Let $1 < p < \infty$, $0 < q < \infty$ and $s > 0$. Let $\widetilde{f} \in L_p(K,\mu)$. Then $\widetilde{f} \in \mathbb{B}^s_{pq}(K,\mu)$ if, and only if, it can be represented as*

$$\widetilde{f} = \sum_{j=0}^{\infty} \sum_{k \in \mathbb{P}_j} \mu_j^k 2^{-\frac{j+L}{2}} \widetilde{\psi}_j^k, \tag{5.1}$$

unconditional convergence being in $L_p(K,\mu)$. Furthermore this representation is unique,

$$\mu_j^k = 2^{\frac{j+L}{2}} (\widetilde{f}, \widetilde{\psi}_j^k)_K = 2^{\frac{j+L}{2}} \int_K \widetilde{f}(\gamma) \widetilde{\psi}_j^k(\gamma)\, \mu(d\gamma),$$

and

$$I : \widetilde{f} \to \left\{ \mu_j^k, j \in \mathbb{N}_0, k \in \mathbb{P}_j \right\} \tag{5.2}$$

is an isomorphic map of $\mathbb{B}^s_{pq}(K,\mu)$ onto the sequence space $b^{s,per}_{pq}$.

5.2 Comparison of $B^s_{pq}(K,\mu)$ and $\mathbb{B}^s_{pq}(K,\mu)$

We first deal with the case $1 < p = q < \infty$, $0 < s < 1$.

We recall that the spaces $B^s_{pp}(\mathbb{T})$ can be normed by

$$\|f|B^s_{pp}(\mathbb{T})\|_* = \left(\int_0^1 |f(x)|^p \, dx \right)^{\frac{1}{p}} + \left(\int_0^1 \int_0^1 \frac{|f(x) - f(y)|^p}{|x-y|^{1+sp}} \, dx\, dy \right)^{\frac{1}{p}}. \tag{5.3}$$

According to (1.33), (5.3) is equivalent to

$$\left(\int_K \left| \tilde{f}(\gamma) \right|^p \mu(d\gamma) \right)^{\frac{1}{p}} + \left(\int_K \int_K \frac{\left| \tilde{f}(\gamma) - \tilde{f}(\delta) \right|^p}{|\gamma - \delta|^{d+sdp}} \mu(d\gamma)\mu(d\delta) \right)^{\frac{1}{p}},$$

where $\tilde{f} = f \circ H^{-1}$. We endow the spaces $\mathbb{B}^s_{pp}(K,\mu)$ with the equivalent norm

$$\|\tilde{f}|\mathbb{B}^s_{pp}(K,\mu)\|_* = \|f|L_p(K,\mu)\| + \left(\int_K \int_K \frac{\left| \tilde{f}(\gamma) - \tilde{f}(\delta) \right|^p}{|\gamma - \delta|^{d+sdp}} \mu(d\gamma)\mu(d\delta) \right)^{\frac{1}{p}}. \tag{5.4}$$

Together with (3.12) this leads to

$$B^s_{pp}(K,\mu) = \mathbb{B}^{\frac{s}{d}}_{pp}(K,\mu). \tag{5.5}$$

The analogue of Theorem 5.1 for the spaces $B^s_{pp}(K,\mu)$ reads as follows.

Theorem 5.2. *Let $1 < p < \infty$ and $0 < s < 1$. Let $\widetilde{f} \in L_p(K,\mu)$. Then $\widetilde{f} \in B^s_{pp}(K,\mu)$ if, and only if, it can be represented as*

$$\widetilde{f} = \sum_{j=0}^{\infty} \sum_{k \in \mathbb{P}_j} \mu_j^k 2^{-\frac{j+L}{2}} \widetilde{\psi}_j^k,$$

unconditional convergence being in $L_p(K,\mu)$. Furthermore this representation is unique,

$$\mu_j^k = 2^{\frac{j+L}{2}} (\widetilde{f}, \widetilde{\psi}_j^k)_K,$$

and

$$I : \widetilde{f} \to \{\mu_j^k, j \in \mathbb{N}_0, k \in \mathbb{P}_j\}$$

is an isomorphic map of $B^s_{pp}(K,\mu)$ onto the sequence space $b_{pp}^{\frac{s}{d},per}$.

Proof. This follows from the observation (5.5). □

To compare $B^s_{pq}(K,\mu)$ and $\mathbb{B}^s_{pq}(K,\mu)$ with $1 < p < \infty$, $0 < q < \infty$ and $0 < s < 1$ we use the real interpolation.

Let $0 < \theta < 1$, $1 < p < \infty$, $0 < q < \infty$, $0 < s_0 < 1$, $0 < s_1 < 1$, $s_0 \neq s_1$ and $s = (1-\theta)s_0 + \theta s_1$. Then from Theorem 1 in [34, Ch. 3.6.1] follows

$$\left(B^{s_0}_{pp}(\mathbb{T}), B^{s_1}_{pp}(\mathbb{T})\right)_{\theta,q} = B^s_{pq}(\mathbb{T}).$$

Since spaces $B^s_{pq}(\mathbb{T})$ are isomorphic to sequence spaces $b^{s,per}_{pq}$, it follows that

$$\left(b^{s_0,per}_{pp}, b^{s_1,per}_{pp}\right)_{\theta,q} = b^{s,per}_{pq}.$$

Using the isomorphic map in (5.1) one gets

$$\left(\mathbb{B}^{s_0}_{pp}(K,\mu), \mathbb{B}^{s_1}_{pp}(K,\mu)\right)_{\theta,q} = \mathbb{B}^s_{pq}(K,\mu). \tag{5.6}$$

Using (5.5), (5.6) and (3.14) one gets that for $0 < s < 1$, $1 < p < \infty$, $1 \le q < \infty$

$$B^s_{pq}(K,\mu) = \mathbb{B}^{\frac{s}{d}}_{pq}(K,\mu).$$

Thus we may conclude that the following theorem holds.

Theorem 5.3. *Let $1 < p < \infty$, $1 \le q < \infty$ and $0 < s < 1$. Let $\widetilde{f} \in L_p(K,\mu)$. Then $\widetilde{f} \in B^s_{pq}(K,\mu)$ if, and only if, it can be represented as*

$$\widetilde{f} = \sum_{j=0}^{\infty} \sum_{k \in \mathbb{P}_j} \mu_j^k 2^{-\frac{j+L}{2}} \widetilde{\psi}_j^k,$$

unconditional convergence being in $L_p(K,\mu)$. Furthermore this representation is unique,

$$\mu_j^k = 2^{\frac{j+L}{2}} (\widetilde{f}, \widetilde{\psi}_j^k)_K,$$

and

$$I : \widetilde{f} \to \{\mu_j^k, j \in \mathbb{N}_0, k \in \mathbb{P}_j\}$$

is an isomorphic map of $B^s_{pq}(K,\mu)$ onto the sequence space $b_{pq}^{\frac{s}{d},per}$.

5.3 Faber-Schauder basis on the Koch curve

In Section 1.2.7 we defined piecewise harmonic functions on a self-similar set, which are regarded as analogue of Faber-Schauder functions.

For the Koch curve K and any other self-similar curve from Example 1.35 a piecewise harmonic function ψ^j_ξ with $\xi = H(x)$, is the composition of ψ^j_x with the transform H^{-1} from (1.26),

$$\psi^j_\xi = \psi^j_x \circ H^{-1}.$$

Taking into account (1.35), (1.36) and (1.33) we get

$$\left|\psi^j_\xi(\gamma) - \psi^j_\xi(\delta)\right| \leq c\left|\gamma - \delta\right|^d, \tag{5.7}$$

where d is the Hausdorff dimension of the curve.

Similarly to the reasoning in Chapter 4 we may expand a function $f \in C(K)$ in a piecewise harmonic basis. Define

$$c_\xi(f) = \begin{cases} f(\xi), & \xi \in V_0, \\ f(\xi) - \sum\limits_{\eta \sim_j \xi} \alpha_{\xi\eta} f(\eta), & \xi \in V_j \setminus V_{j-1},\ j \geq 1. \end{cases}$$

Recall that $\alpha_{\xi\eta} = \psi^{j-1}_\eta(\xi)$. Hence for the Koch curve K

$$\alpha_{\xi\eta} = \frac{1}{2}, \quad \xi \in V_j \setminus V_{j-1},\ \eta \sim_j \xi,\ j \geq 1,$$

and

$$c_\xi(f) = f(\xi) - \frac{1}{2}\sum_{\eta \sim_j \xi} f(\eta)$$

is the counterpart of the second difference. Define

$$P_n f = \sum_{j=0}^{n} \sum_{\xi \in V_j \setminus V_{j-1}} c_\xi(f)\psi^j_\xi.$$

Then $P_n f$ tends to f uniformly on K as $n \to \infty$ and $f \in C(K)$ has the unique representation

$$f = \sum_{j=0}^{\infty} \sum_{\xi \in V_j \setminus V_{j-1}} c_\xi(f)\psi^j_\xi.$$

We can say whether $f \in B^s_{pq}(K,\mu)$ judging by the coefficients in the expansion to the piecewise harmonic basis.

Theorem 5.4. *Let K be the Koch curve. Let*

$$\frac{d}{p} < s < 1, \quad 1 < p < \infty \quad \text{and} \quad 1 \leq q < \infty.$$

Then $f \in C(K)$ *belongs to* $B^s_{pq}(K,\mu)$ *if and only if it can be represented as*

$$f = \sum_{j=0}^{\infty} \sum_{\xi \in V_j \setminus V_{j-1}} c_\xi(f)\psi^j_\xi, \tag{5.8}$$

where

$$C^s_{pq}(f) = \left(\sum_{j=0}^{\infty} \sqrt{3}^{jsq} \left(\frac{1}{2^j} \sum_{\xi \in V_j \setminus V_{j-1}} |c_\xi(f)|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} < \infty,$$

unconditional convergence being in $C(K)$*. Furthermore,*

$$\|f|B^s_{pq}(K,\mu)\| \sim C^s_{pq}(f).$$

The proof for the case $p = q$ is the same as for Theorem 4.2. To extend the result to $p \neq q$ we apply real interpolation.

5.4 Haar wavelets on the Koch curve

The characterization of Besov spaces $B^s_{pq}(K,\mu)$ by Faber-Schauder basis involves pointwise evaluation of $f \in B^s_{pq}(K,\mu)$. This means that we consider only those spaces $B^s_{pq}(K,\mu)$ where $s > \frac{d}{p}$. In order to consider function spaces with $0 < s < \frac{d}{p}$ we introduce Haar wavelets.

Wavelets of Haar type on self-similar fractals were introduced in [14, 15]. These functions differ from ordinary Haar wavelets even on the unit interval, since they are piecewise polynomials instead of piecewise constants.

Our approach to defining Haar wavelets on the self-similar curves is the same as in Section 5.1. We apply mapping H from (1.26) to transfer ordinary Haar wavelets $\{h_0, h_{jm} : j \in \mathbb{N}_0, m = 0, \dots, 2^j - 1\}$ from I to K. Let

$$\widetilde{h}_0(\gamma) = h_0 \circ H^{-1}(\gamma), \quad \widetilde{h}_{jm}(\gamma) = h_{jm} \circ H^{-1}(\gamma). \tag{5.9}$$

Observe that the support of $\widetilde{h}_{jm}$, $m = 0, \dots, 2^j - 1$, are sets

$$\operatorname{supp} \widetilde{h}_{jm} = K_{w_1 w_2 \dots w_j}, \quad (w_1, w_2, \dots, w_j) \in W_j.$$

Then (1.32) implies that the system $\left\{\widetilde{h}_0, \widetilde{h}_{jm} : j \in \mathbb{N}_0, m = 0, \dots, 2^j - 1\right\}$ is orthonormal in $L_2(K,\mu)$.

Taking into acount considerations in Section 1.1.7, Section 5.1 and Section 5.2 we get the followong theorem.

Theorem 5.5. *Let $1 < p < \infty$, $1 \leq q < \infty$ and $0 < s < \frac{d}{p}$. Let $\widetilde{f} \in L_p(K,\mu)$. Then $\widetilde{f} \in B^s_{pq}(K,\mu)$ if, and only if, it can be represented as*

$$\widetilde{f} = \sum_{j=0}^{\infty} \sum_{k \in \mathbb{P}_j} \mu_j^k 2^{-\frac{j+L}{2}} \widetilde{\psi}_j^k,$$

unconditional convergence being in $L_p(K,\mu)$. Furthermore this representation is unique,

$$\mu_j^k = 2^{\frac{j+L}{2}} (\widetilde{f}, \widetilde{\psi}_j^k)_K,$$

and

$$I : \widetilde{f} \to \left\{\mu_j^k, j \in \mathbb{N}_0, k \in \mathbb{P}_j\right\}$$

is an isomorphic map of $B^s_{pq}(K,\mu)$ onto the sequence space $b_{pq}^{\frac{s}{d},per}$.

Theorem 5.6. *Let*

$$1 < p < \infty, \;\; 1 \leq q < \infty \;\; \text{and} \;\; 0 < s < \frac{d}{p}.$$

Let $\widetilde{f} \in L_p(K,\mu)$. Then $\widetilde{f} \in B^s_{pq}(K,\mu)$ if, and only if, it can be represented as

$$\widetilde{f} = \mu_0 \widetilde{h}_0 + \sum_{j=0}^{\infty} \sum_{m=0}^{2^j-1} \mu_{jm} \widetilde{h}_{jm}, \tag{5.10}$$

unconditional convergence being in $B^\sigma_{pq}(K,\mu)$, $\sigma < s$. The representation (5.10) is unique with

$$\mu_0 = \int_K \widetilde{f}(\gamma)\,\mu(d\gamma), \quad \mu_{jm} = 2^j \int_K \widetilde{f}(\gamma)\widetilde{h}_{jm}(\gamma)\,\mu(d\gamma), \;\; j \in \mathbb{N}_0, \; m = 0, \ldots, 2^j - 1,$$

and

$$J : \widetilde{f} \mapsto \left\{\mu_0, 2^{-\frac{j}{p}} \mu_{jm} : j \in \mathbb{N}_0, m = 0, \ldots, 2^j - 1\right\}$$

is an isomorphic map of $B^s_{pq}(K,\mu)$ onto $b_{pq}^{\frac{s}{d}}(\mathrm{I})$. In addition, (5.9) is an unconditional basis in $B^s_{pq}(K,\mu)$.

CHAPTER 6

Besov spaces on the snowflake

Three Koch curves clipped together form the snowflake curve SF, see Figure 6.1. Due to the isomorphism between $[0,1]$ and the Koch curve K we may establish isomorphism $\widetilde{H}$ between $[0,3]$ and SF. The snowflake is not a self-similar set, but it is a d-set with $d = \frac{\log 4}{\log 3}$ being its Hausdorff dimension and a measure μ being equivalent to the Hausdorff measure $\mathrm{H}^d|_{\mathrm{SF}}$. Let μ be chosen in such a way that it is the image of the Lebesgue measure under $\widetilde{H}$.

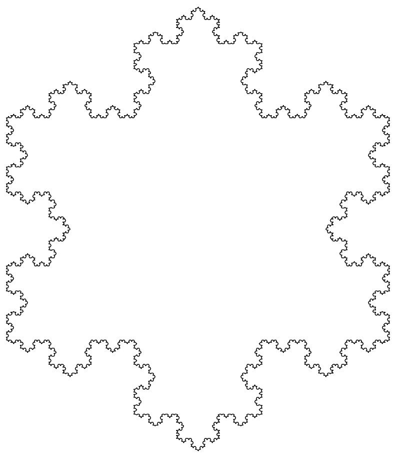


Figure 6.1: *The snowflake*

6.1 New periodic wavelets on $\mathbb{T}$ and $\mathbb{R}$

In Section 1.1.5 we have considered the theory of periodic Besov spaces. We slightly modify the definitions and theorems given there to consider 3-periodic functions.

Let

$$\mathbb{T} = \{x \in \mathbb{R} : 0 \leq x \leq 1\},$$

where the points 0 and 1 are identified. Let

$$3\mathbb{T} = \{x \in \mathbb{R} : 0 \leq x \leq 3\}$$

with the points 0 and 3 being identified. We can interpret $3\mathbb{T}$ as a circle of radius $\frac{3}{2\pi}$ with the centre at the origin. We define the distance $\rho(x,y)$ between two points $x, y \in 3\mathbb{T}$ as the length of the shortest arc on the circle connecting them, i.e.

$$\rho(x,y) = \min\{|x-y|, 3-|x-y|\}. \tag{6.1}$$

By $D(3\mathbb{T})$ we denote the collection of all complex-valued infinitely differentiable functions on $3\mathbb{T}$. The topology in $D(3\mathbb{T})$ is generated by the family of semi-norms

$$\|\varphi\|_\alpha = \sup_{x\in 3\mathbb{T}} |D^\alpha \varphi(x)|, \quad \alpha \in \mathbb{N}_0.$$

$D'(3\mathbb{T})$ is the class of all continuous linear functionals on $D(3\mathbb{T})$. The continuity of a linear functional f on $D(3\mathbb{T})$ means that there exist $N \in \mathbb{N}$ and $c_N > 0$ such that

$$|f(\varphi)| \leq c_N \sum_{\alpha \leq N} \|\varphi\|_\alpha,$$

for all $\varphi \in D(3\mathbb{T})$.

Let $0 < p \leq \infty$. $L_p(3\mathbb{T})$ is the standard quasi-Banach space with respect to Lebesgue measure, quasi-normed by

$$\|f|L_p(3\mathbb{T})\| = \left(\int_0^3 |f(x)|^p \, dx \right)^{\frac{1}{p}},$$

with the usual modification if $p = \infty$. If $1 \leq p \leq \infty$ then $f \in L_p(3\mathbb{T})$ can be interpreted in a unique way as an element of $D'(3\mathbb{T})$ by

$$f(\varphi) = \int_0^3 f(x)\varphi(x)\, dx, \quad \varphi \in D(3\mathbb{T}). \tag{6.2}$$

Consequently, for $1 \leq p \leq \infty$ we have

$$D(3\mathbb{T}) \subset L_p(3\mathbb{T}) \subset D'(3\mathbb{T}), \tag{6.3}$$

where " $\subset$ " here and further on means the topological embedding.

Let $f \in D'(3\mathbb{T})$. Then the numbers

$$\widehat{f}(k) = \frac{1}{3} f(e^{-\frac{2\pi}{3}ikx}), \quad k \in \mathbb{Z},$$

are said to be the Fourier coefficients of f. If $f \in L_p(3\mathbb{T})$, $1 \le p \le \infty$, then (6.2), (6.3) imply that

$$\widehat{f}(k) = \frac{1}{3} \int_0^3 f(x) e^{-\frac{2\pi}{3} ikx}\, dx, \quad k \in \mathbb{Z}.$$

Any $f \in D'(3\mathbb{T})$ can be represented as

$$f = \sum_{k\in\mathbb{Z}} a_k e^{\frac{2\pi}{3} ikx}, \quad x \in 3\mathbb{T}, \quad (\text{convergence in } D'(3\mathbb{T})) \tag{6.4}$$

where the Fourier coefficients $\{a_k\} \subset \mathbb{C}$ are of at most polynomial growth,

$$|a_k| \le c\,(1 + |k|)^{\kappa}, \quad \text{for some } c > 0, \quad \kappa > 0 \text{ and all } k \in \mathbb{Z}.$$

Definition 6.1. Let $\varphi = \{\varphi_j\}_{j=0}^{\infty}$ be a dyadic resolution of unity in $\mathbb{R}$ according to (1.1)-(1.3), $s \in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$ and

$$\|f|B^s_{pq}(3\mathbb{T})\| = \left(\sum_{j=0}^{\infty} 2^{jsq} \| \sum_{k\in\mathbb{Z}} \varphi_j \left(\frac{2\pi k}{3} \right) a_k e^{\frac{2\pi}{3} ikx} | L_p(3\mathbb{T})\|^q \right)^{\frac{1}{q}}$$

(with the usual modification if $q = \infty$). Then the Besov space $B^s_{pq}(3\mathbb{T})$ consists of all $f \in D'(3\mathbb{T})$ such that $\|f|B^s_{pq}(3\mathbb{T})\| < \infty$, [34, Chapter 3].

Our approach to defining Besov spaces on the snowflake is the same as in Section 5.1. We start with the same restrictions on the parameters

$$0 < s < 1, \quad 1 < p = q < \infty$$

and then extend our result to the case when $p \ne q$. Since $0 < s < 1$ it is enough to consider $\psi_F \in C^1(\mathbb{R})$ and $\psi_M \in C^1(\mathbb{R})$ in (1.13). Now we slightly modify the construction of periodic wavelets on $\mathbb{R}$ in order to introduce wavelets on the closed snowflake.

Let

$$N = \sup_{x\in\mathbb{R}} |\psi_F'(x)|, \quad M = \sup_{x\in\mathbb{R}} |\psi_M'(x)|.$$

ψ_F and ψ_M are Lipschitz-continuous functions. For the functions $\psi_j^{L,k}$ defined by (1.13) and (1.14) holds

$$\left| \psi_0^{L,k}(x) - \psi_0^{L,k}(y) \right| \le 2^{\frac{3L}{2}} N\, |x - y|, \quad x, y \in \mathbb{R},$$

$$\left| \psi_j^{L,k}(x) - \psi_j^{L,k}(y) \right| \le 2^{\frac{3}{2}(j+L-1)} M\, |x - y|, \quad j \in \mathbb{N}, \quad x, y \in \mathbb{R}.$$

We construct 3-periodic counterparts of $\psi_j^{L,k}$ by the procedure

$$\psi_{j,3per}^{L,k}(x) = \sum_{l=-\infty}^{\infty} \psi_j^{L,k}(x + 3l). \tag{6.5}$$

Define $\psi_j^{L,k,3per}$ on $3\mathbb{T}$ by

$$\psi_j^{L,k,3per}(x) = \psi_{j,3per}^{L,k}(x), \quad x \in 3\mathbb{T}.$$

Let

$$\mathbb{P}_0^3 = \left\{k \in \mathbb{Z} : 0 \le k \le 3 \cdot 2^L - 1\right\}$$
$$\mathbb{P}_j^3 = \left\{k \in \mathbb{Z} : 0 \le k \le 3 \cdot 2^{j+L-1} - 1\right\}, \quad j \in \mathbb{N}.$$

Then for $j \in \mathbb{N}_0$ there exists the set of points $\{x_{j,k}\}_{k\in\mathbb{P}_j^3} \subset 3\mathbb{T}$ such that

$$\operatorname{supp} \psi_0^{L,k,3per} \subset \{x \in 3\mathbb{T} : \rho(x, x_{0,k}) < \frac{1}{2}\} = B_{0,k}^{3\mathbb{T}},$$
$$\operatorname{supp} \psi_j^{L,k,3per} \subset \{x \in 3\mathbb{T} : \rho(x, x_{j,k}) < 2^{-j}\} = B_{j,k}^{3\mathbb{T}}.$$

Recall that $\rho(\cdot,\cdot)$ is the metric on $3\mathbb{T}$ given by (6.1). For the points $x, y \in B_{j,k}^{3\mathbb{T}}$, $j \in \mathbb{N}_0$, $k \in \mathbb{P}_j^3$ holds

$$\left|\widetilde{H}(x) - \widetilde{H}(y)\right| \sim |x-y|^{\frac{1}{d}}.$$

We recall that $\widetilde{H}$ is the adaption of mapping H from (1.26) with property (1.33).

Similarly to the Proposition 1.34 in [47] one gets that

$$\left\{\psi_j^{L,k,3per}, j \in \mathbb{N}_0, k \in \mathbb{P}_j^3\right\}$$

is an orthornomal basis in $L_2(3\mathbb{T})$. We simplify the notation and omit L in $\psi_j^{L,k,3per}$.

To characterize periodic Besov spaces in terms of wavelets we first introduce the corresponding sequence spaces.

Definition 6.2. Let $0 < p \le \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. Then $b_{pq}^{s,3per}$ is the collection of all sequences

$$\mu = \left\{\mu_j^k \in \mathbb{C} : j \in \mathbb{N}_0, k \in \mathbb{P}_j^3\right\}$$

such that

$$\|\mu|b_{pq}^{s,3per}\| = \left(\sum_{j=0}^{\infty} 2^{j\left(s-\frac{1}{p}\right)q} \left(\sum_{k\in\mathbb{P}_j^3} |\mu_j^k|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} < \infty.$$

Theorem 6.3. *Let $\{\psi_j^{k,3per}\}$ be the orthonormal basis in $L_2(3\mathbb{T})$. Let $0 < p \le \infty$, $0 < q \le \infty$ and $0 < s < 1$. Let $f \in D'(3\mathbb{T})$. Then $f \in B_{pq}^s(3\mathbb{T})$ if, and only if, it can be represented as*

$$f = \sum_{j=0}^{\infty} \sum_{k\in\mathbb{P}_j^3} \mu_j^k 2^{-\frac{j+L}{2}} \psi_j^{k,3per}, \quad \mu \in b_{pq}^{s,3per},$$

unconditional convergence being in $D'(3\mathbb{T})$ and in any space $B_{pq}^{\sigma}(3\mathbb{T})$ with $\sigma < s$. Furthermore, this representation is unique,

$$\mu_j^k = 2^{\frac{j+L}{2}} \int_0^3 f(x)\psi_j^{k,3per}(x)\,dx,$$

and

$$I : f \to \{\mu_j^k, j \in \mathbb{N}_0, k \in \mathbb{P}_j^3\}$$

is an isomorphic map of $B^s_{pq}(3\mathbb{T})$ onto the sequence space $b^{s,3per}_{pq}$. If, in addition, $p < \infty$, $q < \infty$, then $\{\psi_j^{k,per}\}$ is an unconditional basis in $B^s_{pq}(3\mathbb{T})$.

Remark 6.4. This assertion is the counterpart of Theorem 1.37 in [47] for $B^s_{pq}(3\mathbb{T})$.

Since

$$B^s_{pq}(3\mathbb{T}) \hookrightarrow L_p(3\mathbb{T})$$

with $s > 0$, $1 < p < \infty$, $0 < q < \infty$, (see [34, Chapter 3.5.1]), we reformulate Theorem 6.3 with additional restrictions on the parameteres.

Theorem 6.5. *Let $\{\psi_j^{k,3per}\}$ be the above orthonormal basis in $L_2(3\mathbb{T})$. Let $1 < p < \infty$, $0 < q < \infty$ and $0 < s < 1$. Let $f \in L_p(3\mathbb{T})$. Then $f \in B^s_{pq}(3\mathbb{T})$ if, and only if, it can be represented as*

$$f = \sum_{j=0}^{\infty} \sum_{k \in \mathbb{P}_j^3} \mu_j^k 2^{-\frac{j+L}{2}} \psi_j^{k,3per}, \quad \mu \in b^{s,3per}_{pq},$$

unconditional convergence being in $L_p(3\mathbb{T})$. Furthermore this representation is unique,

$$\mu_j^k = 2^{\frac{j+L}{2}} \int_0^3 f(x) \psi_j^{k,3per}(x)\, dx,$$

and

$$I : f \to \{\mu_j^k, j \in \mathbb{N}_0, k \in \mathbb{P}_j^3\}$$

is an isomorphic map of $B^s_{pq}(3\mathbb{T})$ onto the sequence space $b^{s,3per}_{pq}$.

6.2 Besov spaces $\mathbb{B}^s_{pq}(\mathrm{SF}, \mu)$

Let

$$\mathbb{B}^s_{pq}(\mathrm{SF}, \mu) = \left\{ f \circ \widetilde{H}^{-1} : f \in B^s_{pq}(3\mathbb{T}) \right\} = B^s_{pq}(3\mathbb{T}) \circ \widetilde{H}^{-1}$$

with

$$\| f \circ \widetilde{H}^{-1} | \mathbb{B}^s_{pq}(\mathrm{SF}, \mu) \| = \| f | B^s_{pq}(3\mathbb{T}) \|.$$

Define $\widetilde{\psi}_{jk}$ by

$$\widetilde{\psi}_{jk}(\gamma) = \psi_j^{k,3per} \circ \widetilde{H}^{-1}(\gamma).$$

From the corresponding properties of functions $\psi_j^{k,3per}$ and transform $\widetilde{H}$ follow the properties of $\widetilde{\psi}_{jk}$, namely:

- The system $\left\{ \widetilde{\psi}_{jk}, j \in \mathbb{N}_0, k \in \mathbb{P}_j^3 \right\}$ is an orthonormal basis in $L_2(\mathrm{SF}, \mu)$.

- For $j \in \mathbb{N}_0$ there is the set of points $\{\gamma_{j,k}\}_{k\in\mathbb{P}^3_j} \subset \mathrm{SF}$ such that

$$\operatorname{supp} \widetilde{\psi}_{0k} \subset \{\gamma \in \mathrm{SF} : |\gamma - \gamma_{0,k}| \le c2^{-\frac{1}{d}}\} = B^{\mathrm{SF}}_{0,k}, \quad k \in \mathbb{P}^3_0,$$

$$\operatorname{supp} \widetilde{\psi}_{jk} \subset \{\gamma \in \mathrm{SF} : |\gamma - \gamma_{j,k}| \le c2^{-\frac{j}{d}}\} = B^{\mathrm{SF}}_{j,k}, \quad k \in \mathbb{P}^3_j.$$

- For $\gamma, \delta \in \operatorname{supp} \widetilde{\psi}_{jk}$ holds

$$\left|\widetilde{\psi}_{jk}(\gamma) - \widetilde{\psi}_{jk}(\delta)\right| \le c2^{\frac{3j}{2}} |\gamma - \delta|^d =$$

$$= c2^{\frac{3j}{2}} |\gamma - \delta|^{d-1} |\gamma - \delta| \le c2^{-j\left(-\frac{1}{d}-\frac{1}{2}\right)} |\gamma - \delta| .$$

The last inequality is due to the fact that for $\gamma, \delta \in B^{\mathrm{SF}}_{j,k}$

$$|\gamma - \delta| \le |\gamma - \gamma_{j,k}| + |\gamma_{j,k} - \delta| \le c2^{-\frac{j}{d}}.$$

Define $\widetilde{a}_{jk}$ by

$$\widetilde{a}_{jk} = \begin{cases} 2^{-\frac{L}{2}} \widetilde{\psi}_{jk}, & j = 0,\ k \in \mathbb{P}^3_j, \\ 2^{-j(s-\frac{1}{p})} 2^{-\frac{j+L-1}{2}} \widetilde{\psi}_{jk}, & j \in \mathbb{N},\ k \in \mathbb{P}^3_j. \end{cases}$$

Then

$$\operatorname{supp} \widetilde{a}_{jk} \subset B^{\mathrm{SF}}_{j,k},$$

$$|\widetilde{a}_{jk}(\gamma)| \le c2^{-j(s-\frac{1}{p})} \le c\,\mathrm{H}^d \left(B^{\mathrm{SF}}_{j,k}\right)^{s-\frac{1}{p}}, \quad \text{for any } \gamma \in \mathrm{SF},$$

and for any $\gamma, \delta \in \operatorname{supp} \widetilde{a}_{jk}$

$$|\widetilde{a}_{jk}(\gamma) - \widetilde{a}_{jk}(\delta)| \le c2^{-j(s-\frac{1}{d}-\frac{1}{p})} |\gamma - \delta| \le c\,\mathrm{H}^d \left(B^{\mathrm{SF}}_{j,k}\right)^{s-\frac{1}{d}-\frac{1}{p}} |\gamma - \delta| .$$

According to Definition 3.12 $\widetilde{a}_{jk}$ are (sd, p)-atoms.

Theorem 6.6. *Let $1 < p < \infty$, $0 < q < \infty$ and $0 < s < 1$. Let $\widetilde{f} \in L_p(\mathrm{SF}, \mu)$. Then $\widetilde{f} \in \mathbb{B}^s_{pq}(\mathrm{SF}, \mu)$ if, and only if, it can be represented as*

$$\widetilde{f} = \sum_{j=0}^{\infty} \sum_{k\in\mathbb{P}^3_j} \mu^k_j 2^{-\frac{j+L}{2}} \widetilde{\psi}_{jk}, \tag{6.6}$$

unconditional convergence being in $L_p(\mathrm{SF}, \mu)$. Furthermore this representation is unique,

$$\mu^k_j = 2^{\frac{j+L}{2}} (\widetilde{f}, \widetilde{\psi}_{jk})_{\mathrm{SF}} = 2^{\frac{j+L}{2}} \int\limits_{\mathrm{SF}} \widetilde{f}(\gamma) \widetilde{\psi}_{jk}(\gamma)\, \mu(d\gamma),$$

and

$$I : \widetilde{f} \to \left\{\mu^k_j, j \in \mathbb{N}_0, k \in \mathbb{P}^3_j\right\} \tag{6.7}$$

is an isomorphic map of $\mathbb{B}^s_{pq}(\mathrm{SF}, \mu)$ onto the sequence space $b^{s,3per}_{pq}$.

6.3 Comparison of $\mathbb{B}^s_{pq}(\mathrm{SF},\mu)$ and $B^s_{pq}(\mathrm{SF},\mu)$

We have

$$\mathbb{B}^{\frac{s}{d}}_{pp}(\mathrm{SF},\mu) = B^s_{pp}(\mathrm{SF},\mu). \tag{6.8}$$

The inclusion from left to the right follows from Theorem 3.14 and Theorem 6.6. To get the opposite one, we need the characterization of periodic Besov spaces in terms of first differences, we refer to [34, Section 3.5]. The idea is the same as in Chapter 5.

To compare $\mathbb{B}^s_{pq}(\mathrm{SF},\mu)$ and $B^s_{pq}(\mathrm{SF},\mu)$ with $0 < s < 1$ and $p \neq q$ we use the real interpolation.

Let $0 < \theta < 1$, $1 < p < \infty$, $0 < q < \infty$, $0 < s_0 < 1$, $0 < s_1 < 1$, $s_0 \neq s_1$ and $s = (1-\theta)s_0 + \theta s_1$. Then from Theorem 1 in [34, Ch. 3.6.1] follows

$$\left(B^{s_0}_{pp}(3\mathbb{T}), B^{s_1}_{pp}(3\mathbb{T})\right)_{\theta,q} = B^s_{pq}(3\mathbb{T}).$$

Since spaces $B^s_{pq}(3\mathbb{T})$ are isomorphic to sequence spaces $b^{s,3per}_{pq}$, it follows that

$$\left(b^{s_0,3per}_{pp}, b^{s_1,3per}_{pp}\right)_{\theta,q} = b^{s,3per}_{pq}.$$

Using the isomorphic map in (6.7) one gets

$$\left(\mathbb{B}^{s_0}_{pp}(\mathrm{SF},\mu), \mathbb{B}^{s_1}_{pp}(\mathrm{SF},\mu)\right)_{\theta,q} = \mathbb{B}^s_{pq}(\mathrm{SF},\mu). \tag{6.9}$$

Using (6.8), (6.9) and (3.14) one gets that for $0 < s < 1$, $1 < p < \infty$, $1 \le q < \infty$

$$B^s_{pq}(\mathrm{SF},\mu) = \mathbb{B}^{\frac{s}{d}}_{pq}(\mathrm{SF},\mu).$$

Thus we may conclude that the following theorem holds.

Theorem 6.7. *Let $1 < p < \infty$, $1 \le q < \infty$ and $0 < s < 1$. Let $\widetilde{f} \in L_p(\mathrm{SF},\mu)$. Then $\widetilde{f} \in B^s_{pq}(\mathrm{SF},\mu)$ if, and only if, it can be represented as*

$$\widetilde{f} = \sum_{j=0}^{\infty} \sum_{k\in\mathbb{P}_j} \mu^k_j 2^{-\frac{j+L}{2}} \widetilde{\psi}^k_j,$$

unconditional convergence being in $L_p(\mathrm{SF},\mu)$. Furthermore this representation is unique,

$$\mu^k_j = 2^{\frac{j+L}{2}} (\widetilde{f}, \widetilde{\psi}^k_j)_{\mathrm{SF}},$$

and

$$I : \widetilde{f} \to \{\mu^k_j, j \in \mathbb{N}_0, k \in \mathbb{P}_j\} \tag{6.10}$$

is an isomorphic map of $B^s_{pq}(\mathrm{SF},\mu)$ onto the sequence space $b^{\frac{s}{d},3per}_{pq}$.

6.4 Faber-Schauder basis on the snowflake

There are two different decompositions of the Koch snowflake into three Koch curves

$$\mathrm{SF} = \bigcup_{i=1}^{3} K_i = \bigcup_{i=4}^{6} K_i,$$

see Fig. 6.2.

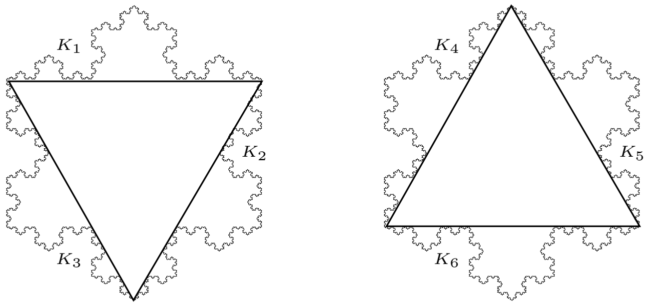


Figure 6.2: *Decomposition of the snowflake*

In [9] it was shown that the energy on the snowflake SF can be obtained as the sum of energies correspondent to three Koch curves comprising SF. Moreover, it is independent of the decomposition.

The procedure is as follows. Let $\{V_j^{(i)}\}$ be the sequence of finite sets of points approximating K_i according to the way described in Section 1.2.3, i.e.

$$K_i = \overline{\bigcup_{j=0}^{\infty} V_j^{(i)}}.$$

We denote by $G_j^{(i)}$ the graphs with vertices $V_j^{(i)}$ and edge relation $\xi \sim_j \eta$ defined in Section 1.2.3. Let

$$V_j = \bigcup_{i=1}^{3} V_j^{(i)} = \bigcup_{i=4}^{6} V_j^{(i)}, \quad j \geq 1,$$

$$V_* = \bigcup_{j=1}^{\infty} V_j.$$

Then

$$\mathrm{SF} = \overline{V_*}.$$

We form a graph G_j with vertices V_j, $j \geq 1$, and edge relation $\xi \sim_j \eta$ holding if and only if there exists $i \in \{1, 2, \ldots, 6\}$ such that

$$\xi \sim_j \eta \ \text{ in } \ G_j^{(i)}.$$

There are two possibilities for the graph G_0. Either it can consist of vertices

$$V_0' = \bigcup_{i=1}^{3} V_0^{(i)} = \{x_1, x_3, x_5\}$$

or

$$V_0'' = \bigcup_{i=4}^{6} V_0^{(i)} = \{x_2, x_4, x_6\}.$$

The edge relation is defined in the same way as for the graphs G_j, $j \geq 1$.

The energy forms associated with the sets K_i are denoted by $\mathcal{E}^{(i)}$. For any function $u : V_* \to \mathbb{R}$ define graph energies $\mathcal{E}_j$, $j \geq 1$, by

$$\mathcal{E}_j(u) = 2^j \sum_{\xi \sim_j \eta} (u(\xi) - u(\eta))^2 .$$

Let

$$\begin{aligned} \mathcal{E}(u) &= \lim_{j\to\infty} \mathcal{E}_j(u), \\ \mathcal{D} &= \{u \in C(\mathrm{SF}) : \mathcal{E}(u) < \infty\} . \end{aligned}$$

Theorem 4.6. in [9] implies that for $u \in C(\mathrm{SF})$

$$\begin{aligned} \mathcal{E}(u) &= \mathcal{E}^{(1)}(u|_{K_1}) + \mathcal{E}^{(2)}(u|_{K_2}) + \mathcal{E}^{(3)}(u|_{K_3}) \\ &= \mathcal{E}^{(4)}(u|_{K_4}) + \mathcal{E}^{(5)}(u|_{K_5}) + \mathcal{E}^{(6)}(u|_{K_6}). \end{aligned}$$

Since $\mathcal{E}$ is independent of chosen decomposition of SF, further on we assume

$$\mathrm{SF} = \bigcup_{i=1}^{3} K_i$$

and $V_0 = V_0' = \{x_1, x_3, x_5\}$.

Definition 6.8.

(i) A continuous function $h : V_* \to \mathbb{R}$ is called harmonic if it minimizes $\mathcal{E}_j$ at all levels for given boundary values on V_0:

$$\mathcal{E}_j(h) = \min \{\mathcal{E}_j(u) : u|_{V_0} = \rho\} .$$

(ii) A continuous function $\psi : V_* \to \mathbb{R}$ is called piecewise harmonic of level j if it minimizes $\mathcal{E}_m$ at all levels $m \geq j$ for given boundary values on V_j.

The restriction of a harmonic function on SF to K_i is a harmonic function on K_i.

Let ψ_ξ^j, $\xi \in V_j$, be a piecewise harmonic function of level j which equals 1 at ξ and 0 at any other vertex of V_j:

$$\psi_\xi^j(x) = \delta_{\xi x} = \begin{cases} 1, & x = \xi \\ 0, & x \in V_j \setminus \{\xi\}. \end{cases}$$

From the correspondent properties of harmonic functions on the Koch curve follows that

$$\operatorname{supp}\psi_\xi^j \subset B(\xi,\rho^j),$$

where $\rho = \frac{1}{\sqrt{3}}$, and

$$\left|\psi_\xi^j(\gamma) - \psi_\xi^j(\delta)\right| \le c\,|\gamma-\delta|^d,$$

where $d = \frac{\ln 4}{\ln 3}$ is the Hausdorff dimension of SF.

The system $\{\psi_\xi^j : \xi \in V_j \setminus V_{j-1}, j \ge 0\}$ is the counterpart of Faber-Schauder basis on the snowflake. Any function $f \in C(\mathrm{SF})$ may be represented as

$$f = \sum_{j=0}^{\infty} \sum_{\xi \in V_j \setminus V_{j-1}} c_\xi(f)\psi_\xi^j,$$

where

$$c_\xi(f) = \begin{cases} f(\xi), & \xi \in V_0, \\ f(\xi) - \frac{1}{2}\sum\limits_{\eta \sim_j \xi} f(\eta), & \xi \in V_j \setminus V_{j-1},\ j \ge 1. \end{cases}$$

The isomorphism between function spaces $B^s_{pq}(\mathrm{SF},\mu)$ and sequence spaces is given by the following theorem.

Theorem 6.9. *Let* SF *be the Koch snowflake. Let*

$$\frac{d}{p} < s < 1,\ \ 1 < p < \infty\ \ \text{and}\ \ 1 \le q < \infty.$$

Then $f \in C(\mathrm{SF})$ belongs to $B^s_{pq}(\mathrm{SF},\mu)$ if and only if it can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{\xi \in V_j \setminus V_{j-1}} c_\xi(f)\psi_\xi^j, \tag{6.11}$$

where

$$C^s_{pq}(f) = \left(\sum_{j=0}^{\infty} \sqrt{3}^{jsq} \left(\frac{1}{2^j} \sum_{\xi \in V_j \setminus V_{j-1}} |c_\xi(f)|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} < \infty,$$

unconditional convergence being in $C(\mathrm{SF})$. Furthermore,

$$\|f|B^s_{pq}(K,\mu)\| \sim C^s_{pq}(f).$$

The proof follows the same lines of Theorem 4.2.

CHAPTER 7

Besov spaces on the Cartesian product of some self-similar sets

Let $Q = (0,1)^n$ be the unit cube in $\mathbb{R}^n$. Anisotropic Besov spaces $B^{s,\alpha}_{pp}(Q)$ measure smoothness of functions depending on the direction of the j-th coordinate. Recall that the norm is given by

$$\|f|B^{s,\alpha}_{pp}(Q)\| = \|f|L_p(Q)\| + \left(\int\limits_Q \int\limits_Q \frac{|f(x)-f(y)|^p}{\varrho_{\alpha,n}(x,y)^{n+sp}}\,dx\,dy\right)^{1/p}, \tag{7.1}$$

where $\varrho_{\alpha,n}(x,y)$ is the anisotropic distance on Q defined by (1.6). This means that to describe the anisotropic Besov spaces we modify the way how we measure the distance between points. The anisotropic distance $\varrho_{\alpha,n}(x,y)$ is a quasimetric. According to the Assouad's embedding theorem the snowflaked version of $(Q, \varrho_{\alpha,n}(x,y))$, that is $(Q, \varrho_{\alpha,n}(x,y)^t)$, $0 < t < t_0 < 1$, can be mapped with the help of some bi-Lipschitz transform to a d-set Γ in some $\mathbb{R}^N$. There are two ways of defining Besov spaces on Γ. We get spaces $B^s_{pp}(\Gamma,\mu)$ by traces or we can define Besov spaces $\mathbb{B}^{s,\alpha}_{pp}(\Gamma,\mu)$ on Γ to be the image of anisotropic Besov spaces on Q. The question is how these spaces are interrelated.

7.1 Quasimetric spaces

Definition 7.1. A quasimetric on a set X is a function $q : X \times X \to [0,\infty)$ that satisfies

1. $q(x,y) = q(y,x)$,
2. $q(x,y) = 0$, if and only if $x = y$,
3. for some $K \geq 1$ and all $x, y, z \in X$ holds

$$q(x,y) \leq K(q(x,z) + q(z,y)). \tag{7.2}$$

A quasimetric space is a set X together with a quasimetric q.

When $K = 1$ in formula (7.2), one gets a metric space.

Balls in the quasimetric space $(X, q(x, y))$ can be defined as usually: for $x \in X$ and $r > 0$ a ball centred at x with radius r is a set

$$B(x, r) = \{y \in X : q(x, y) < r\}.$$

A useful fact about quasimetric spaces is given by the following theorem.

Theorem 7.2. *Let q be a quasimetric on a set X. Then there is $\varepsilon_0 > 0$, depending only on the constant K in (7.2), such that $q_\varepsilon(x, y) = q(x, y)^\varepsilon$ is bi-Lipschitz equivalent to a metric for each $0 < \varepsilon \le \varepsilon_0$. That is, for each $0 < \varepsilon \le \varepsilon_0$ there is a metric d_ε on X and constants $c, c' > 0$ such that*

$$c' q_\varepsilon(x, y) \le d_\varepsilon(x, y) \le c q_\varepsilon(x, y)$$

for all $x, y \in X$.

For the proof we refer to the Chapter 14 of [11].

Definition 7.3. Let $(M, d(x, y))$ be a metric space. M is called doubling if there is a constant k such that every ball B in M can be covered by at most k balls of half the radius of B.

Theorem 7.4 (Assouad)**.** *Let $(M, d(x, y))$ be a metric space which is doubling. For each $0 < t < 1$ there is $N \in \mathbb{N}$ and a mapping*

$$H : M \to \mathbb{R}^N$$

such that H is bi-Lipschitz as a mapping from $(M, d(x, y)^t)$ into $\mathbb{R}^N$. This means, that there are constants $c, c' > 0$ such that

$$c' d(x, y)^t \le |H(x) - H(y)| \le c d(x, y)^t$$

for all $x, y \in M$.

The dimension N can be chosen to depend only on t and on the doubling constant k in Definition 7.3. The standard reference to the theorem above is [2], some discussion can be found in [11], [36].

In analysis it is often natural to have not just a metric but also a measure.

Definition 7.5. Given a metric space $(M, d(x, y))$, a Borel measure μ on M is said to be doubling if there is a constant c such that

$$\mu(B(x, 2r)) \le c\mu(B(x, r)), \quad \text{for all } \ x \in M, \quad r > 0.$$

The existence of a nonzero doubling measure on M implies that $(M, d(x, y))$ is doubling as a metric space.

Definition 7.6. Let $(M, d(x, y))$ be a metric space and $d > 0$. Then (M, d, μ) is called a d-space if M is complete as a metric space, and if there is a regular Borel measure μ on M such that

$$\mu(B(x, r)) \sim r^d \tag{7.3}$$

for all $x \in M$ and $0 < r < \operatorname{diam} M$.

Distinguished examples of d-spaces are d-sets, which are subsets of $\mathbb{R}^n$ and satisfy (7.3).

The connection between d-spaces and the Hausdorff measure is given by the following theorem.

Theorem 7.7. *Let $(M, d(x, y), \mu)$ be a d-space.*

1. *Then the Hausdorff dimension of M equals d,*

$$\dim_{\mathrm{H}} M = d,$$

and the restriction H^d on M satisfies (7.3).

2. *If μ_1 and μ_2 satisfy (7.3), then $\mu_1 \sim \mu_2$.*

The proof may be found in [17].

If $(M, d(x, y), \mathrm{H}^d)$ is a d-space and $0 < t < 1$, then $(M, d(x, y)^t, \mathrm{H}^{\frac{d}{t}})$ is a $\frac{d}{t}$-space.

If (M, d, μ) is a d-space, then μ is doubling and (M, d) is a doubling metric space.

Example 7.8. Let $\overline{Q} = [0, 1]^n$ be the closed unit cube in $\mathbb{R}^n$, the n-tuple

$$\alpha = (\alpha_1, \ldots, \alpha_n) \text{ with } 0 < \alpha_1 \leq \ldots \alpha_n < \infty, \quad \sum_{j=1}^{n} \alpha_j = n \tag{7.4}$$

be an anisotropy in $\mathbb{R}^n$ and

$$\varrho_{\alpha,n}(x, y) = |x - y|_\alpha \,, \quad \text{where} \quad |x|_\alpha = \max_{k=1,\ldots,n} |x_k|^{\frac{1}{\alpha_k}} \,,$$

be an anisotropic distance. If not all α_k are the same, then from (7.4) follows that there are k such that $\alpha_k < 1$. For these k

$$\varrho_1^k(x_k, y_k) = |x_k - y_k|^{\frac{1}{\alpha_k}} \,, \quad x_k, y_k \in \mathbb{R},$$

is only a quasimetric on $\mathbb{R}$. Hence $\varrho_{\alpha,n}$ is also only a quasimetric. Nevertheless if μ_L is the Lebesgue measure, then

$$\mu_L(B(x, r)) \sim r^n$$

for any ball $B(x, r) \subset \overline{Q}$.

According to Theorem 7.2 there is $\varepsilon_0 > 0$, such that $\varrho_{\alpha,n}^{\varepsilon_0}$ is bi-Lipschitz equivalent to a metric. In this particular case ε_0 is very easy to calculate, it equals α_1. $(\overline{Q}, \varrho_{\alpha,n}(x, y)^{\alpha_1})$

is a metric space. Taking into account the remarks given after Theorem 7.7 and the fact that the Lebesgue measure is equivalent to the n-dimensional Hausdorff measure, we may conclude that $(\overline{Q}, \varrho_{\alpha,n}(x,y)^{\alpha_1}, \mathrm{H}^{\frac{n}{\alpha_1}})$ is an $\frac{n}{\alpha_1}$-space.

$(\overline{Q}, \varrho_{\alpha,n}(x,y)^{\alpha_1}, \mathrm{H}^{n/\alpha_1})$ is doubling as a metric space, thus from Theorem 7.4 it follows that for each $0 < t < 1$ there is $N \in \mathbb{N}$ and a mapping

$$H : \overline{Q} \to \mathbb{R}^N$$

such that

$$c' \varrho_{\alpha,n}(x,y)^{\alpha_1 t} \le |H(x) - H(y)| \le c \varrho_{\alpha,n}(x,y)^{\alpha_1 t}, \quad c, c' > 0,$$

for all $x, y \in \overline{Q}$.

Let $\Gamma = H\overline{Q}$. Then since

$$\mathrm{H}^{\frac{n}{\alpha_1 t}}(\overline{Q}) \sim \mathrm{H}^{\frac{n}{\alpha_1 t}}(\Gamma)$$

we get that $(\Gamma, |\gamma - \delta|, \mathrm{H}^{\frac{n}{\alpha_1 t}})$ is an $\frac{n}{\alpha_1 t}$-set, $0 < t < 1$.

Example 7.9. Let $X = [0,1]^2$ be the unit square and

$$\alpha = (\alpha_1, \alpha_2) \text{ with } 0 < \alpha_1 < \alpha_2 < \infty, \quad \alpha_1 + \alpha_2 = 2, \ \frac{\alpha_1}{\alpha_2} = \frac{\log 3}{\log 4}$$

be an anisotropy. An anisotropic distance on X is defined by

$$\varrho_{\alpha,2}(x,y) = \max\left\{ |x_1 - y_1|^{\frac{1}{\alpha_1}}, |x_2 - y_2|^{\frac{1}{\alpha_2}} \right\}.$$

Then

$$\left(X, \max\left\{|x_1 - y_1|, |x_2 - y_2|^{\frac{\alpha_1}{\alpha_2}}\right\}\right) = ([0,1], |x_1 - y_1|) \otimes \left([0,1], |x_2 - y_2|^{\frac{\alpha_1}{\alpha_2}}\right)$$

is a metric space and $\left(X, \max\left\{|x_1 - y_1|, |x_2 - y_2|^{\frac{\alpha_1}{\alpha_2}}\right\}, \mathrm{H}^{\frac{2}{\alpha_1}}\right)$ is a $\frac{2}{\alpha_1}$-set. Define the transform $H = (H_1, H_2)$ by

$$H_1 : [0,1] \to [0,1],$$
$$H_2 : [0,1] \to \Gamma_1,$$

where H_1 is the identity map and H_2 is the homeomorphism between $[0,1]$ and the Koch curve K form (1.26). Put $\Gamma = [0,1] \otimes \Gamma_1$. Then

$$H : \left(X, \max\left\{|x_1 - y_1|, |x_2 - y_2|^{\frac{\alpha_1}{\alpha_2}}\right\}\right) \to (\Gamma, |\gamma - \delta|)$$

is bi-Lipschitz.

Properties of Hausdorff measure imply that

$$\mathrm{H}^{\frac{2}{\alpha_1}}(X) \sim \mathrm{H}^{\frac{2}{\alpha_1}}(\Gamma)$$

and it follows that $\left(\Gamma, |\gamma - \delta|, \mathrm{H}^{\frac{2}{\alpha_1}}\right)$ is a $\frac{2}{\alpha_1}$-set.

This example shows that Assouad's embedding theorem applied to the anisotropic Besov spaces is valid also for $t = 1$.

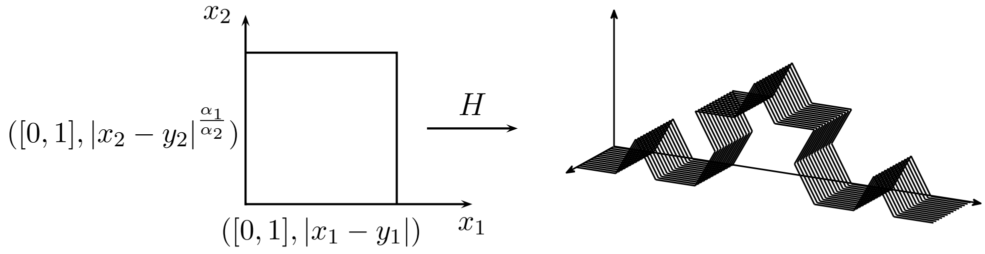


Figure 7.1: *The anisotropic cube and the Cartesian product of the unit interval and the Koch curve*

7.2 Anisotropic spaces as isotropic spaces on fractals

According to the previous Section for every fixed $0 < t \leq 1$ there is a mapping $H : \overline{Q} \to \Gamma$ from the closed unit cube $\overline{Q}$ to an $\frac{n}{\alpha_1 t}$-set Γ such that

$$|H(x) - H(y)| \sim \varrho_{\alpha,n}(x,y)^{\alpha_1 t}$$

and for any $A \subset \Gamma$

$$\mathrm{H}^{\frac{n}{\alpha_1 t}}(A) \sim \mu_L(H^{-1}(A)). \tag{7.5}$$

This means that $\mathrm{H}^{\frac{n}{\alpha_1 t}}$ is equivalent to the image of the Lebesgue measure μ_L under the transform H. According to Theorem 1.19 in [29] for any Borel function $\tilde{f}$ on Γ holds

$$\int_\Gamma |\tilde{f}|\, d\,\mathrm{H}^{\frac{n}{\alpha_1 t}} \sim \int_Q |\tilde{f} \circ H|\, d\mu_L$$

or equivalently

$$\int_\Gamma |f \circ H^{-1}|\, d\,\mathrm{H}^{\frac{n}{\alpha_1 t}} \sim \int_Q |f|\, d\mu_L$$

for any Borel function f on Q.

We simplify the notation and instead of $\mathrm{H}^{\frac{n}{\alpha_1 t}}$ we use μ.

We can introduce the following function spaces on Γ. Let

$$\mathbb{B}^{s,\alpha}_{pp}(\Gamma,\mu) = \left\{ f \circ H^{-1} : f \in B^{s,\alpha}_{pp}(Q) \right\} = B^{s,\alpha}_{pp}(Q) \circ H^{-1}.$$

with

$$\|f \circ H^{-1}|\mathbb{B}^{s,\alpha}_{pp}(\Gamma,\mu)\| = \|f|B^{s,\alpha}_{pp}(Q)\|.$$

Due to our previous remarks we may conclude that the space $\mathbb{B}^{s,\alpha}_{pp}(\Gamma)$ is the set of all $\tilde{f} \in L_p(\Gamma,\mu)$ such that

$$\|\tilde{f}|\mathbb{B}^{s,\alpha}_{pp}(\Gamma,\mu)\|_* = \|\tilde{f}|L_p(\Gamma,\mu)\| + \left(\int_\Gamma \int_\Gamma \frac{\left|\tilde{f}(x) - \tilde{f}(y)\right|^p}{|\gamma - \delta|^{\frac{n}{\alpha_1 t} + \frac{s}{\alpha_1 t} p}}\, \mu(d\gamma)\, \mu(d\delta) \right)^{1/p}. \tag{7.6}$$

The formula (7.6) represents the norm in the Besov space $B_{pp}^{\frac{s}{\alpha_1 t}}(\Gamma, \mu)$ on $\frac{n}{\alpha_1 t}$-set Γ defined by trace. Thus

$$B_{pp}^{s,\alpha}(Q) \circ H^{-1} = B_{pp}^{\frac{s}{\alpha_1 t}}(\Gamma, \mu).$$

Bibliography

[1] D.R. Adams and L.I. Hedberg. *Function spaces and potential theory.* Springer, Berlin, 1996.

[2] P. Assouad. Plongements Lipschitziens dans $\mathbb{R}^n$. *Bull. Soc. Math. France*, 111:429–448, 1983.

[3] M. Bodin. Discrete characterisations of Lipschitz spaces on fractals. *Math. Nachr.*, 282(1):26–43, 2009.

[4] I. Daubechies. Ten lectures on wavelets. In *CBMS-NSF Regional Conf. Series Appl. Math.* SIAM, Philadelphia, 1992.

[5] D.E. Edmunds and H. Triebel. *Function spaces, entropy numbers, differential operators.* Cambridge Univ. Press, Cambridge, 1996.

[6] K. J. Falconer. *Fractal geometry: mathematical foundations and applications.* Wiley, Chichester, 2001.

[7] M. Frazier and B. Jawerth. Decomposition of besov spaces. *Indiana Univ. Math. Journ.*, 34:777–799, 1985.

[8] M. Frazier, B. Jawerth, and G. Weiss. Littlewood-Paley theory and the study of function spaces. In *CBMS-AMS Regional Conf. Ser. 79.* Providence, Amer. Math. Soc., 1991.

[9] U.R. Freiberg and M.R. Lancia. Energy form on a closed fractal curve. *Journal for Analysis and its Applications*, 23(1):115–137, 2004.

[10] A. Haar. Zur Theorie der orthogonalen Funktionensysteme. *Math. Annalen*, 69:331–371, 1910.

[11] J. Heinonen. *Lectures on analysis on metric spaces.* Universitext, Springer-Verlag, New York, 2001.

[12] J. Hu and X. Wang. Domains of Dirichlet forms and effective resistance estimates on p.c.f. fractals. *Studia Math.*, 177:153–172, 2006.

[13] J. E. Hutchinson. Fractals and self similarity. *Indiana Univ. Math. Journ.*, 30:713–747, 1981.

[14] A. Jonsson. Wavelets on fractals and Besov spaces. *J. Fourier Anal. Appl.*, 4:329–340, 1998.

[15] A. Jonsson. Haar wavelets of higher order on fractals and regularity of functions. *J. Math. Anal. Appl.*, 290:86–104, 2004.

[16] A. Jonsson and A. Kamont. Piecewise linear bases and Besov spaces on fractal sets. *Anal. Math.*, 27:77–117, 2001.

[17] A. Jonsson and H. Wallin. A Whitney extension theorem in L^p and Besov spaces. *Ann. Inst. Fourier*, 28:139–192, 1978.

[18] A. Jonsson and H. Wallin. *Function spaces on subsets of* $\mathbb{R}^n$. Math. reports 2,1, Harwood acad. publ., London, 1984.

[19] M. Kabanava. Tempered Radon measures. *Rev. Mat. Complut.*, 21(2):553–564, 2008.

[20] M. Kabanava. Function spaces on the Koch curve. *Journal of function spaces and applications*, 8(3):287–299, 2010.

[21] M. Kabanava. Characterization of Besov spaces on nested fractals by piecewise harmonic functions. *Journal for Analysis and its Applications*, accepted.

[22] M. Kabanava. Function spaces on the snowflake. In *Proc. of Function Spaces IX*, Krakow, to appear. Banach Centre Publications.

[23] J. Kigami. *Analysis on fractals.* Cambridge Univ. Press, Cambridge, 2003.

[24] J. Kigami. Harmonic analysis for resistance forms. *J. Funct. Anal.*, 204:399–444, 2003.

[25] T. Kumagai. Estimates of transition densities for Brownian motion on nested fractals. *Probab. Theory Relat. Fields*, 96:205–224, 1993.

[26] T. Lindstrøm. *Brownian motion on nested fratals.* Mem. Amer. Math. Soc. 420, 1990.

[27] T.D. Long and H. Triebel. Equivalent norms and Schauder bases in anisotropic Besov spaces. *Proceedings of the Royal Society of Edinburgh*, 84A:177–183, 1979.

[28] P. Malliavin. *Integration and probability.* Springer, New York, 1995.

[29] P. Mattila. *Geometry of sets and measures in Euclidean spaces.* Cambridge Univ. Press, Cambridge, 1995.

[30] Y. Meyer. *Wavelets and operators.* Cambridge Univ. Press, Cambridge, 1992.

[31] K. Pietruska-Pałuba. Some function spaces related to the Brownian motion on simple nested fractals. *Stochastics*, 67:267–285, 1999.

[32] K. Pietruska-Pałuba and M. Bodin. Harmonic functions representation of Besov-Lipschitz functions on nested fractals. In *Research Report in Mathematics, No. 2, Dept. of Mathematics, Umeå University.* 2010.

[33] J. Schauder. Eine Eigenschaft der Haarschen Orthogonalsysteme. *Math. Zeitschr.*, 28:317–320, 1928.

[34] H.-J. Schmeisser and H. Triebel. *Topics in Fourier analysis and function spaces.* Wiley, Chichester, 1987.

[35] Z. Semadeni. *Schauder bases in Banach spaces of continuous functions*, volume 918 of *Lecture Notes in Mathematics.* Springer, 1982.

[36] S. Semmes. Bilipschitz embedding of metric spaces into Euclidian spaces. *Publ. Mat.*, 43:571–653, 1999.

[37] S.L. Sobolev. *Some applications of functional analysis in mathematical physics (In Russian).* Nauka, Moscow, 1988.

[38] E.M. Stein and G. Weiss. *Introduction to Fourier analysis on Euclidean spaces.* Princeton University Press, Princeton, 1996.

[39] R. Strichartz. *Differential equations on fractals.* Princeton University Press, Princeton, 2006.

[40] H. Triebel. *Interpolation theory, function spaces, differential operators.* North-Holland, Amsterdam, 1978.

[41] H. Triebel. *Theory of function spaces.* Birkhäuser, Basel, 1983.

[42] H. Triebel. *Theory of function spaces II.* Birkhäuser, Basel, 1992.

[43] H. Triebel. *Fractals and spectra.* Birkhäuser, Basel, 1997.

[44] H. Triebel. Refined solutions of integral equations. *Functiones et Approximatio*, XXIX:143–158, 2001.

[45] H. Triebel. *The structure of functions.* Birkhäuser, Basel, 2001.

[46] H. Triebel. *Theory of function spaces III.* Birkhäuser, Basel, 2006.

[47] H. Triebel. *Function spaces and wavelets on domains.* European Math. Soc. Publishing House, Zürich, 2008.

[48] H. Triebel. *Bases in function spaces, sampling, discrepancy, numerical integration.* European Math. Soc. Publishing House, Zürich, 2010.

[49] P. Wojtaszczyk. *A mathematical introduction to wavelets.* Cambridge Univ. Press, Cambridge, 1997.

Curriculum Vitae

Maryia Kabanava

Personal Details

Date of birth:	16.10.1984
Place of birth:	Minsk, Republic of Belarus
Citizenship:	Republic of Belarus

Education

2006–2007	Master of Science in Mathematics, Belarusian State University, Minsk
2001–2006	Diplom in Mathematics, Belarusian State University, Minsk

---

Research

2008–present	PhD student at the Friedrich Schiller University of Jena
2007–2008	DAAD Research grant at the Friedrich Schiller University of Jena, Mathematical Institute, Research Group "Function Spaces"

Teaching and professional activities

2010–2011	LaTeXpresentation of lecture notes *"Mathematik für Geo- und Werkstoffwissenschaftler"*, Friedrich Schiller University of Jena
2010	Exercise course in Mathematics at Friedrich Schiller University of Jena

---

Publications

[1] M. Kabanava. Tempered Radon measures. *Rev. Mat. Complut.*, 21(2):553–564, 2008.

[2] M. Kabanava. Function spaces on the Koch curve. *Journal of function spaces and applications*, 8(3):287–299, 2010.

[3] M. Kabanava. Characterization of Besov spaces on nested fractals by piecewise harmonic functions. *Journal for Analysis and its Applications*, accepted.

[4] M. Kabanava. Function spaces on the snowflake. In *Proc. of Function Spaces IX*, Krakow, to appear. Banach Centre Publications.

Contributed talks

Oct. 2010	"Faber type functions on distinguished fractals", workshop on "Smoothness, Approximation and Function Spaces", Oppurg, Germany
Sep. 2010	"Anisotropic spaces as isotropic spaces on fractals", school on "Nonlinear Analysis, Function Spaces and Applications 9", Trest, Czech Republic
Jul. 2009	"Function spaces on the Koch curve", conference on "Function Spaces IX", Krakow, Poland
Jul. 2008	"Tempered Radon measures", workshop on "Function Spaces and Applications", Freyburg/Unstrut, Germany

Invited talks

Oct. 2010	"Anisotropic Besov spaces as isotropic spaces on fractals", Lomonosov Moscow state university, Moscow, Russia

Ehrenwörtliche Erklärung

Ich erkläre hiermit, dass mir die Promotionsordnung der Friedrich-Schiller-Universität vom 08. 07. 2009 bekannt ist.

Ferner erkläre ich, dass ich die vorliegende Arbeit selbst und ohne Benutzung anderer als der angegebenen Hilfsmittel angefertigt habe.

Weitere Personen waren an der inhaltlich-materiellen Erstellung der Arbeit nicht beteiligt. Insbesondere habe ich hierfür nicht die Hilfe eines Promotionsberaters in Anspruch genommen. Niemand hat von mir unmittelbar oder mittelbar geldwerte Leistungen für Arbeiten erhalten die im Zusammenhang mit dem Inhalt der vorgelegten Arbeit stehen.

Die Arbeit wurde bisher weder im In- noch Ausland in gleicher oder ähnlicher Form einer anderen Prüfungsbehörde vorgelegt. Insbesondere wurde keine in wesentlichen Teilen ähnliche oder andere Abhandlung bei einer anderen Hochschule als Dissertation eingereicht.

Ich versichere, dass ich nach bestem Wissen die reine Wahrheit gesagt und nichts verschwiegen habe.

Jena, den 12. April, 2011 Maryia Kabanava