

Motion and gravitational wave emission of spinning compact binaries

Dissertation

zur Erlangung des akademischen Grades
doctor rerum naturalium (Dr. rer. nat.)

vorgelegt dem Rat der Physikalisch-Astronomischen Fakultät
der Friedrich-Schiller-Universität Jena

von Manuel Tessmer

geboren am 09.01.1981 in Mühlhausen

06. Juni 2011

Gutachter:

1. Professor Dr. Gerhard Schäfer, Friedrich-Schiller-Universität Jena
2. Professor Dr. Michael Soffel, Technische Universität Dresden
3. Professor Andrzej Krolak, Ph. D., Institute of Mathematics
of the Polish Academy of Sciences Warschau

Tag der Disputation: 08. November 2011

Contents

1	Introduction	4
2	Theoretical aspects of gravitational waves from compact binaries	7
2.1	Outside the near zone: harmonic gauge and the relaxed Einstein equations . . .	7
2.2	Outside the near zone: Multipolar post-Minkowski expansion (MPM)	9
2.3	The near-zone: EOM for the source in harmonic coordinates	10
2.4	Matching the region outside the source and the near zone: results	10
2.5	Far-zone flux: Energy carried by gravitational waves	11
2.6	Far-zone flux: Angular momentum carried by gravitational waves	12
2.7	Representations of the far-zone field	13
2.7.1	Extracting the “plus” and “cross” polarization	13
2.7.2	Tensor spherical harmonics	14
2.8	The Hamiltonian approach for the near-zone’s dynamics	15
2.8.1	Canonical formulation of general relativity	15
2.8.2	Matter stress-energy tensor for monopole-dipole particle sources	19
3	Outlook: planning the work	21
4	Aligned spins under point particle and spin interactions through formal 2PN order	22
4.1	Aligned spins: motivation for an analytic solution	22
4.2	Spin and orbital dynamics	23
4.3	The Keplerian-type parameterisation	27
4.4	Gravitational wave forms	34
4.5	Estimate of the current results for aligned spin vectors	42
5	Motion and gravitational waves from compact binaries with arbitrarily oriented spins under leading-order spin-orbit interaction	44
5.1	Spin dynamics	44
5.2	Geometrical setup	46
5.3	Time derivatives of the spin orientation angles	49
5.4	First order perturbative solution to the EOM for the non-equal mass case . . .	51
5.5	Orbital dynamics	54
5.6	Gravitational wave forms	59
5.7	Radiation reaction with spin, perturbation theory and Lie series	61
6	Full-analytic Fourier domain gravitational wave forms for inspiralling eccentric compact binaries	65
6.1	Orbital motion: 2PN accurate Quasi-Keplerian parameterisation	69
6.2	Relevant multipoles and their time derivatives in harmonic coordinates	70

6.3	Gravitational wave forms in terms of tensor spherical harmonics	74
6.4	Relevant Kapteyn Series of irreducible components	75
6.5	Multipole moment decomposition: a brief posting of the results	83
6.6	The effect of radiation reaction and the SPA	84
6.7	The quasi-circular limit	91
6.8	Truncation of infinite series to a finite order of the eccentricity	92
6.9	Some concluding remarks for the eccentric inspiral templates	93
7	Summary and outlook	94
8	Zusammenfassung und Ausblick	95
9	Danksagung	97
A	Appendix: supplementary material	99
A.1	Canonical variables including spin	99
A.2	Integrals appearing in the QKP for aligned spins	100
A.3	Transformation from harmonic to ADM coordinates including change in SSC .	100
A.4	The relevant tensor spherical harmonics through 2PN order	102
A.5	Appell's integral formula in the solution for the elapsed time	105
B	Results: The remaining GW spherical tensor components through 2PN order	105
C	Results: The spherical tensor components in terms of Fourier coefficients	111
D	Proofs, summations and limitations	127
D.1	Trigonometric double series	127
D.2	Proof of important decomposition formulas	127
D.3	Accuracy of finite sums	128
D.4	The solution to the 2PN accurate Kepler Equation: a useful check	130
D.5	Fourier-domain trigonometric functions of the eccentric anomaly through 2PN	133
	References	134
E	Lebenslauf	146
F	Ehrenwörtliche Erklärung	148

Notation: mathematical symbols

Greek indices: $\alpha, \beta, \gamma, \dots, \mu, \nu, \dots$	spacetime indices
Latin indices: $a, b, c, \dots, i, j, \dots$	spatial indices
${}^{(4)}G_{\mu\nu}, {}^{(4)}R_{\mu\nu}, {}^{(4)}R$	Einstein and Ricci tensor, Ricci scalar in spacetime
${}^{(3)}G_{ij}, {}^{(3)}R_{ij}, {}^{(3)}R$	Einstein and Ricci tensor, Ricci scalar on Σ_t
$g_{\mu\nu}, g, \eta_{\mu\nu}$	metric tensor and determinant, Minkowski metric, $\eta_{\mu\nu} \equiv \text{diag}(-1, 1, 1, 1)$
$T_{\mu\nu}, S_{\mu\nu}$	stress-energy tensor of matter, spin tensor
G, c	Newton's gravitational constant and speed of light
\square	D'Alembert operator in flat spacetime, $\square \equiv \eta^{\mu\nu} \partial_\mu \partial_\nu$
$\xi_{\alpha;\beta}$	covariant derivative of vector ξ
\mathcal{L}_ξ	Lie derivative along vector ξ
$\mathbf{L}, \mathbf{S}, \mathbf{J}$:	orbital, spin, and total angular momentum, $L \equiv \mathbf{L} $
a_r	semimajor axis
δ_m	the difference in the masses, $\delta_m \equiv m_1 - m_2$
$ E $	absolute value of the scaled orbital binding energy
e_r, e_t, e_ϕ	radial, time, and phase eccentricity
$\mathcal{E}, \mathcal{M}, v$	eccentric, mean, and true anomaly
ϵ, ε	book keeping parameters, $\epsilon = \mathcal{O}(c^{-1})$ and $\varepsilon = \mathcal{O}(\delta_m)$
η	symmetric mass ratio: $\eta \equiv \frac{m_1 m_2}{m^2}$, $m \equiv m_1 + m_2$ as the sum of the masses
\mathcal{K}	periastron advance parameter, $\mathcal{K} \equiv \frac{\Phi - 2\pi}{2\pi}$
\mathcal{N}	mean motion, $\mathcal{N} \equiv \frac{2\pi}{P}$
\mathcal{P}, Φ	total time and total phase elapsed between 2 consecutive periastron passages
ϕ, φ	orbital (or standard coordinate) phase, phase in orbital frame
$\mathbf{p}_a, \mathbf{r}_a, \mathbf{v}_a$	momentum, separation, and velocity vector of the binary constituents

Abbreviations

EOM, SSC	equation(s) of motion, spin supplementary condition(s)
\mathcal{FP}	Hadamard's finite part, or in French language "parti finie"
GW	gravitational wave(s)
(N)(LO)SO	(next-to) (leading order) spin orbit
xPN:	x^{th} post Newtonian ...
PP, TT	point particle contribution, tranverse traceless part

1 Introduction

For the current gravitational wave detectors of next generation, currently under construction, astrophysical objects become accessible to observation which could not have been observed directly hitherto. Those are time variable objects that do not reflect or emit electromagnetic radiation themselves or whose electromagnetic radiation is too weak for a sound data analysis. Gravitational wave astronomy gives, therefore, an promising future counterpart to other observational techniques as, for example, gamma ray or infrared astronomy.

There are currently planned and operational gravitational wave detectors applying the technique of laser interferometry which are on to extend their sensitivity to furnish usable data for a detailed analysis. Worthwhile to mention are the 600 meter arm-length detector GEO 600 in Hannover, LIGO in Stanford and Livingston, the VIRGO detector in Italy and TAMA in Japan [1].

Paramount examples for sources of non-transient gravitational waves are bound systems consisting of two compact objects, such as black holes or neutron stars, and these will be the object of interest of this thesis because of their extraordinary strong gravitational field and expected gravitational wave amplitudes. The new generation “advanced” LIGO (*advanced Laser Interferometer Ground-based observatory*) is able to detect binary systems of typical 1.4 solar-mass binaries sending waves from a distance of 300Mpc. This is roughly 15 times as far as best initial LIGO might have achieved [2]. Using this, advanced LIGO will extract gravitational wave data from cosmic sources that, from a naive point of view, is hopelessly digged under a plethora of noise sources that are magnitudes stronger than the binary sources themselves.

To analyse the measured data, one needs a comparison with the theory. The analyst contrasts the detector data, via matched filtering, with a model which is provided with a number of parameters. These have to be varied until, through a convolution with the detector data, the most probable set of parameters has been found or the model is found to be ruled out. Depending on the complexity of the model, it covers various effects, and naturally has more parameters, the more effects are included. Such an effect is for example the orbital decay due to the emission of gravitational waves which carry energy and angular momentum away. The diameter of the orbit shrinks and the orbit itself approaches to a spiral track towards a final plunge, which causes a modulation with a large time scale, because the radiation reaction effects may be small in case of interest. In the circular orbit case, this can be compactly described with the help of a parameter called the “chirp mass”. Generalising the objects trajectories to quasi-elliptic tracks, also the periastron advance will get directly visible in a relatively short time scale. Those effects and other as spin precession will modulate the gravitational wave considerably, and will introduce the mass ratio, spin amplitudes and orientation parameters, respectively.

Comparisons of several inspiral models vis-à-vis, as a dry run in a manner of speaking, showed that neglecting orbital eccentricity, which is a search with non-optimal models, would

lead to considerable losses in event rates as one can extract directly from References [3] and [4]. Subject to this consideration were binary systems at the very end of their inspiral process. Such processes are most likely observed by ground-based kilometer-size detectors as advanced LIGO which can see from 10Hz onwards [2], and GEO from 50Hz to 2kHz [5]. Binary objects like those will, depending on their masses, only have minutes or seconds left to live before the final plunge. Subject of LISA (*Laser Interferometer Space Antenna*), in contrast, are the early stages of the inspiral, probed at relatively low characteristic frequencies of 0.03 mHz to 0.1 Hz [6]. As will become clear, LISA will be able to resolve eccentricities at a high sensitivity, which will turn out to be important for the more natural condition of binary stars to be non-circular. In addition to this, one often has to deal with rotating objects having a considerable spin, which is able to increase the efficiency of gravitational wave emission tremendously. Thus, we also like to investigate how the spins evolve in time and how the characteristic gravitational wave turns out to be, and, in case it is possible, move away from the circular orbit. Finally, an analytical wave model (a search template), which is accurate to the highest available level and includes all important effects is the desired goal.

Compact binaries evolving in their periodic and inspiral phase are regarded. The merger to the final compact object and the ringdown that eliminates all the deviations from the external Kerr-metric are excluded and have to be treated in a separate publication. Basing on the results of recent research, it is the goal of this thesis to furnish a useful contribution for the gravitational wave data analysis community.

Let us shortly explain the toolkit for our work. In Newtonian dynamics, we find the typical orbital velocity v in a binary of compact objects to be related to the radial distance r as

$$\left(\frac{v}{c}\right)^2 \sim 2\frac{Gm}{c^2r} \ll 1, \quad (1.1)$$

when average the kinetic energy and the potential energy due to the virial theorem. As non-relativistic objects are regarded, having small velocities compared to the speed of light, the quantity v/c or $Gm/(c^2r)$ respectively will be taken to be the dimensionless expansion parameter for the “post-Newtonian” perturbation theory that is an approach to general relativity. At order $(v/c)^{2n}$, we will speak of the n^{th} post-Newtonian (nPN) order.

Let us give a reasonable argument why we worked at 2PN order. The binary dynamics is known up to 3PN in point particle contributions, but many important spin interactions were known up to 2PN only when we finished the first calculations of this thesis. Therefore, to stay consistent in the orders of magnitude and not to irregularly neglect important terms, we keep only the stage of terms which is completely investigated. Consistently to 2PN, authors of a recent publication [7] investigated certain equal-mass black hole binaries in rotating clusters and stated parameter estimation errors for the initial eccentricity of $\sim 10^{-7}$ for LISA. Therefore we claim that to perform an eccentric gravitational wave analysis especially for LISA is demanded up to 2PN and we announce eccentric systems to play an important role in this thesis.

Let us begin with some basics to understand the gravitational wave emission and explain idea of the Blanchet-Damour approach. The historical background is always referenced in the text. The informed reader may skip this and the explanation of the Hamiltonian approach and jump to the plan and outlook of this thesis in Section 3.

2 Theoretical aspects of gravitational waves from compact binaries

The mathematical treatment of the generation of GW has been discussed extensively over the last few decades. Early attempts have been flawed by the appearance of technical difficulties but, finally, the formalism has been closed in some sense. In this chapter, we want to provide the reader with necessary theoretical issues of gravitational fields from compact binaries. In fact, Einstein's equations are too complicated to be solved analytically, so to obtain a reasonable description of the binary problem, a set of approximation schemes has been constructed in the recent history of research, such as the multipolar decomposition of the far-zone field and the post-Newtonian prescription of the source in the near-zone.

We have to make some assumptions to enable us continue working. We restrict ourselves to sources having a matter stress-energy tensor with compact spatial support, such as binary star systems. We state that the stress-energy tensor vanishes at a typical radial coordinate $r > d$. This will not only be a reasonable assumption, but also makes it possible to transform spatial integrals over total divergences into surface integrals.

The problem to mathematically formulate GWs from compact binaries then divides into two parts: the computation in the near zone and in the region outside the source. The near zone is the region of space with a size much smaller than the typical gravitational wave length λ , $r \ll \lambda$, and its boundary is labelled \mathcal{R} . A typical wave length can be found via $\lambda = (c/v)d$, where v is the typical velocity of the source. We see that, for non-relativistic sources, $R \gg d$. Inside this region, the post-Newtonian formalism can be applied until it finally breaks down at $r > \mathcal{R}$. In the zone outside the source only the gravitational wave field contributes as a source for itself. The metric only deviates slightly from the Minkowskian even at $r > d$, as one assumes weak gravitational fields inside the matter source. Then the metric in $d < r < \infty$ can be iteratively solved using the post-Minkowskian formalism, an expansion in the gravitational coupling constant G . These both parts have to be solved individually and afterwards to be "agglutinated", which means that they have to be matched in a region where both methods are valid. This algorithm is condensed briefly in subsequent sections. Our computation is compiled from [8–10] and uses those author's usual notation. The reader who is already familiar to the Blanchet-Damour-Iyer formalism may skip the next subsections and jump to the results in Section 2.4.

2.1 Outside the near zone: harmonic gauge and the relaxed Einstein equations

The full set of Einstein equations, without cosmological term, reads

$${}^{(4)}G_{\mu\nu}[g, \partial g, \partial^2 g] = {}^{(4)}R_{\mu\nu} - \frac{1}{2}{}^{(4)}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}[g]. \quad (2.1)$$

We start with the part of the space outside of the source, say $r > d$, where the matter stress-energy tensor vanishes, $T^{\mu\nu} = 0$, and assume that gravitational interactions are weak, i.e. the metric deviates weakly from that of flat space as in the previous section. Having this, we further assume that we can decompose the metric tensor into the Minkowski metric and a small perturbation part, viz. $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}| \ll 1$. The trace-free part of $h_{\mu\nu}$ will be denoted with a “bar”, $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h^\gamma{}_\gamma\eta_{\mu\nu}$, where the trace is understood as a full contraction with the Minkowski metric. We like to motivate an alternative variable instead of $g^{\mu\nu}$ itself, which will show off as a convenient one. Our choice will be the variable $\mathfrak{h}_{\mu\nu}$ which we define as

$$\mathfrak{h}^{\mu\nu} \equiv \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}. \quad (2.2)$$

This is not an approximative definition, because no assumption of the relative order of magnitude of $\mathfrak{h}_{\mu\nu}$ is made. It is interesting to note that in linearised theory, where $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + \mathcal{O}(h^2)$, the quantity $\mathfrak{h}^{\mu\nu}$ reduces to $\bar{h}^{\mu\nu}$ (except an overall sign),

$$-\mathfrak{h}^{\mu\nu} = \eta^{\mu\nu} - (1+h)^{1/2}(\eta^{\mu\nu} - h^{\mu\nu}) = h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h. \quad (2.3)$$

Together with the harmonic gauge condition,

$$\partial_\mu \mathfrak{h}^{\mu\nu} = 0, \quad (2.4)$$

inserted into Einstein’s field equations, $\mathfrak{h}_{\mu\nu}$ gives the “relaxed Einstein equations”,

$$\square \mathfrak{h}^{\mu\nu} = \frac{16\pi G}{c^4} \tau^{\mu\nu}, \quad (2.5)$$

with [10, 11]

$$\tau^{\mu\nu} \equiv (-g)T^{\mu\nu} + \frac{c^4}{16\pi G}\Lambda^{\mu\nu}, \quad (2.6)$$

$$\Lambda^{\alpha\beta} \equiv \frac{16\pi G}{c^4}(-g)t_{\text{LL}}^{\alpha\beta} + (\partial_\mu \mathfrak{h}^{\alpha\mu} \partial_\mu \mathfrak{h}^{\beta\nu} - \mathfrak{h}^{\mu\nu} \partial_\mu \partial_\nu \mathfrak{h}^{\alpha\beta}), \quad (2.7)$$

$$\begin{aligned} (-g)t_{\text{LL}}^{\alpha\beta} \equiv & g_{\lambda\mu}g^{\nu\rho}\partial_\nu \mathfrak{h}^{\alpha\lambda}\partial_\rho \mathfrak{h}^{\beta\mu} + \frac{1}{2}g_{\lambda\mu}g^{\alpha\beta}\partial_\rho \mathfrak{h}^{\lambda\nu}\partial_\nu \mathfrak{h}^{\rho\mu} \\ & - g_{\mu\nu}(g^{\lambda\alpha}\partial_\rho \mathfrak{h}^{\beta\nu} + g^{\lambda\beta}\partial_\rho \mathfrak{h}^{\alpha\nu})\partial_\lambda \mathfrak{h}^{\rho\mu} \\ & + \frac{1}{8}(2g^{\alpha\lambda}g^{\beta\mu} - g^{\alpha\beta}g^{\lambda\mu})(2g_{\nu\rho}g_{\sigma\tau} - g_{\rho\sigma}g_{\nu\tau})\partial_\lambda \mathfrak{h}^{\nu\tau}\partial_\mu \mathfrak{h}^{\rho\sigma}, \end{aligned} \quad (2.8)$$

where $t_{\text{LL}}^{\alpha\beta}$ is the Landau-Lifshitz energy-momentum pseudotensor. Without requiring the gauge condition (2.4), Equation (2.5) alone does not fix the evolution of the matter variables; one could imagine any arbitrary time dependence of matter and (2.5) would still hold. Thus, only (2.5) together with (2.4) are fully equivalent to the complete set of Einstein’s equations. Our aim now is to solve the relaxed Einstein equations in the zone outside the source.

2.2 Outside the near zone: Multipolar post-Minkowski expansion (MPM)

Off the source, where $T_{\mu\nu} = 0$, the term on the right hand side of Equation (2.5) reduces to $\Lambda_{\mu\nu}$. The prefactor in (2.5) is small and couples the field $\mathfrak{h}_{\mu\nu}$ weakly to itself. The most general solution \mathfrak{h}_{ij} can therefore be expressed as an infinite sum in terms having exponentials of the gravitational constant G ,

$$\mathfrak{h}^{\mu\nu} = \sum_{n=1}^{\infty} G^n \mathfrak{h}_n^{\mu\nu}. \quad (2.9)$$

It is intuitive to take above expansion as basis for an iterative solution algorithm. Having found the first order in G , we can in principle find all higher orders, setting all known and relevant orders of \mathfrak{h} as source terms in Λ . Since Λ starts at quadratic order in \mathfrak{h} , we start at a homogeneous differential equation for \mathfrak{h}_1 . The second order will only depend on \mathfrak{h}_1 . This expansion gives, schematically,

$$\square \mathfrak{h}_n^{\mu\nu} = \Lambda_n^{\mu\nu}[\mathfrak{h}_1, \dots, \mathfrak{h}_{n-1}] \quad \text{for } r > d. \quad (2.10)$$

Because of the nonlinearity of the above procedure, the general solution outside matter sources will be a source for itself. Thus, from a sufficient high order onwards, integrals coming from the inverse Laplace and D'Alembert operators will diverge. The reason is not that the solution algorithm is wrong, but we are simply not allowed to expand far-zone expansions to the entire space. Blanchet and Damour used Hadamards finite part regularization of those integrals to solve this problem of finiteness [8].

Because at first post-Minkowski order $n = 1$, $\square \mathfrak{h}_{\mu\nu} = 0$, we can expand the most general solution outside the source into multipoles.

$$\mathfrak{h}_1^{\alpha\beta} = \sum_{l=0}^{\infty} \partial_L \left[\frac{1}{r} K_L^{\alpha\beta}(t - r/c) \right], \quad (2.11)$$

with $K_L^{\alpha\beta}$ as some symmetric and trace-free (STF) tensors with respect to all pairs (α, β) , and r is the distance to the source. Taking additionally the harmonic gauge condition into account, the solution for \mathfrak{h}_1 turns out to be

$$\mathfrak{h}_1^{\alpha\beta} = k_1^{\alpha\beta} + \partial^\alpha \varphi_1^\beta + \partial^\beta \varphi_1^\alpha - \eta^{\alpha\beta} \partial_\mu \varphi_1^\mu, \quad (2.12)$$

$$k_1^{00} = \frac{-4}{c^2} \sum_{l \geq 0} \frac{(-)^l}{l!} \partial_L \left[\frac{1}{r} I_L(u) \right], \quad (2.13)$$

$$k_1^{0i} = \frac{4}{c^3} \sum_{l \geq 0} \frac{(-)^l}{l!} \partial_{L-1} \left[\frac{1}{r} I_{iL-1}^{(1)}(u) + \frac{l}{l+1} \epsilon_{iab} \partial_a \left(\frac{1}{r} J_{bL-1}(u) \right) \right], \quad (2.14)$$

$$k_1^{ij} = \frac{-4}{c^4} \sum_{l \geq 0}^{\infty} \frac{(-)^l}{l!} \partial_{L-2} \left[\frac{1}{r} I_{ijL-2}^{(2)} + \frac{2l}{l+1} \epsilon_{ab(i} J_{j)bL-2}^{(1)}(u) \right], \quad (2.15)$$

where $u \equiv t - r/c$ is the retarded time and

$$f^{(n)}(u) \equiv \frac{d^n f}{du^n}. \quad (2.16)$$

The expressions I_L and J_L are some mass-type and current-type moments with multi-index L of length l ($L - 1$ and $L - 2$ are corresponding multi-indices of lengths $l - 1$ and $l - 2$, respectively), whose explicit expressions will be given later. The interested reader can find explicit expression for the gauge vector φ^μ in [8]. This solution can now be inserted as the source in $\Lambda_{\mu\nu}$ for higher orders \mathfrak{h}_2, \dots , and continue finding the solution to Einstein's equations without matter source.

For the time being, we have only considered the relaxed vacuum Einstein equations, thus, the solution given above is not familiar with the motion of the source. We now have to connect the equations of motion in the near-zone to the wave in the outside region. This will foot on computing the post-Newtonian (PN) accurate solution to the field equations in the near-zone and, afterwards, provide a multipole decomposition of the latter, which has to be matched finally to the exterior.

2.3 The near-zone: EOM for the source in harmonic coordinates

In the near-zone, the gravitational fields will be stronger, but we restrict ourselves to the assumption of weakly relativistic sources: $v \ll c$. Having this, we take $\mathfrak{h}^{\mu\nu}$, plug it into Einsteins equations and obtain the perturbative coefficients. They can be used to extract equations of motion for the source on the one hand and to constrain the exterior solution to physical sources on the other to get the full picture of gravitational wave generation. Therefore, we have to provide the perturbative metric coefficients in the near-zone in powers of inverse c and, as well, their multipolar expansion. We therefore call this prescription of the near-zone the multipolar post-Newtonian expansion (MPN). Both have to be compared in a region of overlap that is assumed to exist for the validity of this prescription. One can find computations for the perturbed metric coefficients for the near zone in [10, 12]. We will not list the results and skip to the more interesting task of gluing both results together.

2.4 Matching the region outside the source and the near zone: results

Having found the solution in the two regions, we have to match them in an overlap region where both approaches are valid. The solution, when comparing MPM and MPN solutions

to arbitrary PN order, reads

$$\begin{aligned}
I_L(u) = & \mathcal{FP} \int d^3x \int_{-1}^{+1} dz \left\{ \delta_l(z) \hat{x}_L \Sigma - \frac{4(2l+1)\delta_{l+1}(z)}{c^2(l+1)(2l+1)} \hat{x}_{iL} \Sigma_i^{(1)} \right. \\
& \left. + \frac{2(2l+1)\delta_{l+2}(z)}{c^4(l+1)(l+2)(2l+5)} \hat{x}_{ijL} \Sigma_{ij}^{(2)} \right\} (u + z|\mathbf{x}|/c, x), \tag{2.17}
\end{aligned}$$

$$\begin{aligned}
J_L(u) = & \mathcal{FP} \int d^3x \int_{-1}^{+1} dz \epsilon_{ab\langle i} \left\{ \delta_l(z) \hat{x}_{L-1\rangle a} \Sigma_b \right. \\
& \left. - \frac{(2l+1)\delta_{l+1}(z)}{c^2(l+2)(2l+3)} \hat{x}_{L-1\rangle \Sigma_{bc}^{(1)} \right\} (u + z|\mathbf{x}|/c, x), \tag{2.18}
\end{aligned}$$

$$\delta_l(z) \equiv \frac{(2l+1)!!}{2^{l+1}l!} (1-z^2)^l, \tag{2.19}$$

$$\Sigma \equiv \frac{\tau^{00} + \tau^{ii}}{c^2}, \tag{2.20}$$

$$\Sigma_i \equiv \frac{\tau^{0i}}{c}, \tag{2.21}$$

$$\Sigma_{ij} \equiv \tau^{ij}, \tag{2.22}$$

$$\tau \equiv \delta_{ij} \tau^{ij}. \tag{2.23}$$

where expansion of each quantity appearing above to the desired PN order is understood. At this point, the field is expressed in terms of a combination of matter and field variables. We wish to express this in terms of matter variables only, therefore the field terms of Equation (2.6) have to be eliminated by iteration (for results, see e.g. [13, 14]), and to avoid logarithmic terms in the distance r , have to go to *radiative coordinates at infinity* and express the FZ-field in terms of the *radiative moments*, see, e.g. [15, 16]. The gravitational wave field, expressed in terms of the radiative coordinates will from now on wards be denoted by the symbol h_{ij} .

Equation (2.17) leaves open whether using point particles, point-dipole particles or continuous deformable sources as input for the stress-energy tensor. Depending on our choice, the GW form will have different appearances. In sections 4, 5 and 6, we will give results according to the problem we put for ourselves.

We note that the general form of the gravitational field outside the source, written in terms of moments I_L and J_L , to any PN order is given in [17] and the radiative moments later in [18]. We now turn to dissipative effects of GW emission on the source system.

2.5 Far-zone flux: Energy carried by gravitational waves

The Einstein field equations can be formulated as an action principle, using the action

$$S_E = \frac{c^3}{16\pi G} \int d^4x \sqrt{-g} R. \tag{2.24}$$

When we take again $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, we can obtain a stress-energy tensor of the GW far-zone field, up to quadratic order in h , viz.

$$t^{\mu\nu} = \frac{c^4}{32\pi G} \langle \partial^\mu h^{\alpha\beta} \partial^\nu h_{\alpha\beta} \rangle, \quad (2.25)$$

where $\langle \dots \rangle$ of some quantity denotes a spatial average over some wave lengths λ . Taking into account the speed of the GW, i.e. c , and imagine a volume traveling with c containing the energy density computed from Equation (2.25), one can compute the radiated energy via GW of a source with compact spatial support (we mean the matter source here) as an integral over a large sphere which compounds that source,

$$\frac{dE}{dt} = \frac{c^3 r^2}{32\pi G} \int d\Omega_2 \langle \dot{h}_{ij}^{TT} \dot{h}_{ij}^{TT} \rangle, \quad (2.26)$$

where $dA = r^2 d\Omega_2$ defines the surface element of the two-sphere. The energy that is lost through the sphere can be regarded as lost by the source at retarded time. This is known as the balance argument for the energy loss via gravitational radiation and is one basic element for the evolution of orbital elements.

2.6 Far-zone flux: Angular momentum carried by gravitational waves

It is well-known from Noether's theorem, that under the if the Lagrangian density \mathcal{L} has continuous symmetry properties, there exist conserved quantities. Using Noether's theorem, we can use the conservation of the Lagrangian of the gravitational field [10, 19] under spatial rotations $\in SO(3)$ to obtain the density of angular momentum (related to the ‘‘charge density’’ associated with rotations) l^i transported by the gravitational wave

$$\frac{1}{c} l^i = \frac{c^2}{32\pi G} \langle -\epsilon^{ikl} \dot{h}_{ab}^{TT} x^k \partial^l h_{ab}^{TT} + 2\epsilon^{ikl} h_{al}^{TT} \dot{h}_{ak}^{TT} \rangle, \quad (2.27)$$

and from that, similarly using the thoughts leading to Equation (2.26),

$$\frac{dL^i}{dt} = \frac{c^3}{32\pi G} \int r^2 d\Omega_2 \langle -\epsilon^{ikl} \dot{h}_{ab}^{TT} x^k \partial^l h_{ab}^{TT} + 2\epsilon^{ikl} h_{al}^{TT} \dot{h}_{ak}^{TT} \rangle, \quad (2.28)$$

PN expanded formulae of the energy loss in terms of the source multipole moments can be found in References [14, 20]. Chandrasekhar and Esposito [21] showed at the leading order, that the balance argument holds when one computes the 2.5 PN accurate equations of motion due to the metric in the near-zone. Further calculations can be found in [22]. From dE/dt and dL/dt one can evaluate the perturbation functions of the Keplerian elements. For example, if one parametrises the orbit by the mean motion \mathcal{N} and the time eccentricity e_t [23, 24], one can use the dependencies

$$\mathcal{N} = \mathcal{N}(|E|, L), \quad (2.29)$$

$$e_t = e_t(|E|, L), \quad (2.30)$$

and compute the time differentials via Leibniz rule.¹ Representations with other orbital variables are trivially computed within PN accuracy, and differ only by higher-order correction terms in inverse c , but may have significantly different numerical behaviours. A detailed computation for compact binaries without spin, up to 2PN, can be found in [20] and the 3PN completion in [25]. A further reference [26] uses 2PN accurate conservative orbital dynamics and computes radiation reaction equations for the orbital elements to leading order, including secular and fast-oscillatory variations. A prequel of that, Reference [27], extends this consideration to 3.5 PN order, and these results are applied to nonspinning compact binaries in eccentric orbits in Reference [28]. Reference [29] compares the behaviour of several parameterisations of circular inspiral with respect to LIGO investigations.

2.7 Representations of the far-zone field

2.7.1 Extracting the polarisations h_\times and h_+ from h_{ij}^{TT}

The radiation field, after one performed an appropriate infinitesimal coordinate transformation, is symmetric, as the metric always is, and tracefree. Therefore, acting on a deformable medium (such as a dust cloud falling freely in space), the shape of the cloud will be changed due to the geodesics equation, but its surface area is not changed. Thus, we can speak of a shear transformation. To characterise the action of h_{ij}^{TT} a bit more detailedly, we define the unit line-of-sight-vector \mathbf{N} as pointing from the source (i.e. the binary system) to the observer. Now, let the unit vectors \mathbf{p} and \mathbf{q} span the plane of the sky for the observer and complete the orthonormal basis $(\mathbf{p}, \mathbf{q}, \mathbf{N})$,

$$\mathbf{p} \times \mathbf{q} = \mathbf{N} \quad \text{and cyclic.} \quad (2.31)$$

As for h the equation $h_{ij}^{\text{TT}} N_i = 0$ holds, it can be represented by a 2×2 matrix in the coordinates of the plane orthogonal to \mathbf{N} . We can split these 2×2 matrices in this plane by two different shear contributions: one that *clinches* a circle in the cloud to an ellipse while keeping the two principal axes in the \mathbf{p} and \mathbf{q} directions and another one that *skews* a circle to an ellipse whose semimajor axis has an angle $\pi/2$ measured from \mathbf{p} in case they act alone. Written down in the basis (\mathbf{p}, \mathbf{q}) , the radiation field reads

$$h_{ij}^{\text{TT}} = \begin{pmatrix} h_+ & h_\times \\ h_\times & -h_+ \end{pmatrix}. \quad (2.32)$$

¹In Eq. (2.29), spinning particles are excluded. The spin itself evolves due to RR – using appropriate definitions, it will keep its magnitude, but will change its direction, the major contribution is the secular one. Then the secular evolution equations for spins have to be included to $\dot{\mathcal{N}}$ and \dot{e}_t .

h_+ is the clinch and h_\times is the skew, and they represent the *shear parameters*. They are projected out of h_{ij}^{TT} via

$$h_\times = \frac{1}{2}(p_i q_j + p_j q_i) \mathfrak{P}_{kl}^{\text{TT}ij} h_{kl}, \quad (2.33a)$$

$$h_+ = \frac{1}{2}(p_i p_j - q_j q_i) \mathfrak{P}_{kl}^{\text{TT}ij} h_{kl}, \quad (2.33b)$$

where $\mathfrak{P}_{kl}^{\text{TT}ij}$ is the usual TT projector onto \mathbf{N} ,

$$\mathfrak{P}_{kl}^{\text{TT}ij} \equiv (\delta_k^i - N^i N_k)(\delta_l^j - N^j N_l) - \frac{1}{2}(\delta^{ij} - N^i N^j)(\delta_{kl} - N_k N_l). \quad (2.34)$$

The polarisations play an important role in data analysis considerations, and we will provide explicit formulas in the subsequent sections.

2.7.2 Rotation symmetry: representing the far-zone field via tensor spherical harmonics

Because of the helicity property of the FZ GW field, $h = 2$, which suggests that the GW field represents a spin-2 field, we are able to write it down in terms of pure spin-2 tensor spherical harmonics. For a short explanation, we first define the spin wave functions, where the spin operator \mathbf{S} acts as $\mathbf{S}^2 \xi_{ss_z} = s(s+1) \xi_{ss_z}$, as we measure the spin in units of \hbar . Taking $s = 1$, ξ can have $s_z = \{0, \pm 1\}$ and we can represent it with a vector in 3 dimensions:

$$\xi^{(\pm 1)} = \mp \frac{1}{\sqrt{2}}(\mathbf{e}_x \pm i\mathbf{e}_y), \quad \xi^{(0)} = \mathbf{e}_z. \quad (2.35)$$

Then we couple two spin function vectors to obtain spin-2 STF tensors with $s = 2$

$$t_{ik}^{(s_z)} = \sum_{m_1, m_2 = -1}^{+1} \langle 11m_1 m_2 | 2s_z \rangle \xi_i^{(m_1)} \xi_k^{(m_2)}, \quad (2.36)$$

with $\langle 11m_1 m_2 | 2s_z \rangle$ as the Clebsch-Gordan coefficients, and combine these tensors with the scalar spherical harmonics Y_{lm} to obtain the spin-2 tensor spherical harmonics,

$$(\mathbf{T}_{jj_z}^l)_{ik} \equiv (Y_{jj_z}^{l2}) = \sum_{l_z = -l}^{+l} \sum_{s_z = -2}^{+2} \langle 2l s_z l_z | j j_z \rangle Y_{l_z}(\theta, \phi) t_{ik}^{(s_z)}. \quad (2.37)$$

We used (θ, ϕ) as our standard spherical coordinates. Because, by our choice of gauge, the massless FZ GW field is transverse and traceless, and we can impose further symmetries for the set of our basis tensors. That is, for the representation of the FZ-field, we can pick up a number of combinations of those $(\mathbf{T}_{jj_z}^l)_{ik}$ which get a special sign under parity transformation. Among those obtained terms, we find that only the following two combinations are normal

to the unit normal vector \mathbf{N} :

$$N_i (\mathbf{T}_{lm}^{E2})_{ij} = 0, \quad N_i (\mathbf{T}_{lm}^{B2})_{ij} = 0. \quad (2.38)$$

These $(\mathbf{T}_{lm}^{E2})_{ij}$ and $(\mathbf{T}_{lm}^{B2})_{ij}$ are related to $(\mathbf{T}_{jjz}^l)_{ik}$ by

$$(\mathbf{T}_{jjz}^{E2}) = E_{j+2} (\mathbf{T}_{jjz}^{j+2}) + E_{j+0} (\mathbf{T}_{jjz}^j) + E_{j-2} (\mathbf{T}_{jjz}^{j-2}), \quad (2.39)$$

$$(\mathbf{T}_{jjz}^{B2}) = B_{j+1} (\mathbf{T}_{jjz}^{j+1}) + B_{j-1} (\mathbf{T}_{jjz}^{j-1}), \quad (2.40)$$

$$E_{j+2} \equiv \left(\frac{j(j-1)}{2(2j+1)(2j+3)} \right)^{1/2}, \quad (2.41)$$

$$E_{j+0} \equiv \left(\frac{3(j-1)(j+2)}{(2j-1)(2j+3)} \right)^{1/2}, \quad (2.42)$$

$$E_{j-2} \equiv \left(\frac{(j+1)(j+2)}{2(2j-1)(2j+1)} \right)^{1/2}, \quad (2.43)$$

$$B_{j+1} \equiv - \left(\frac{j-1}{2j+1} \right)^{1/2}, \quad (2.44)$$

$$B_{j-1} \equiv - \left(\frac{j+2}{2j+1} \right)^{1/2}. \quad (2.45)$$

Then we can write down our FZ field according to

$$h_{ij}^{\text{TT}} = \frac{1}{R} \frac{G}{c^4} \sum_{l=2}^{\infty} \sum_{m=-l}^{+l} \left[I_{lm} (\mathbf{T}_{lm}^{E2})_{ij} + S_{lm} (\mathbf{T}_{lm}^{B2})_{ij} \right]. \quad (2.46)$$

The coefficients I_{lm} and S_{lm} can be projected out of h_{ij}^{TT} with the help of the orthogonality relation

$$\int d\Omega_2 (\mathbf{T}_{lm}^J)_{ij} (\mathbf{T}_{l'm'}^{J'})_{ij} = \delta_{JJ'} \delta_{ll'} \delta_{mm'}. \quad (2.47)$$

We come back to that issue in Section 6 and will provide explicit calculations for the full 2PN accurate orbital parameterisation and the also 2PN accurate amplitudes h_{ij}^{TT} . For further information on that issue, read [17], and, for an application to GW “bremsstrahlung”, the paper of Turner and Will [30]. Further introductory material can be found on pages 140 ff. of [10], which we recommend for many of our discussed issues.

2.8 The Hamiltonian approach for the near-zone’s dynamics

2.8.1 Canonical formulation of general relativity

First attempts that obtained the first post-Newtonian order EOM for a non-spinning compact binary system have been done by Lorentz and Droste [31, 32], Eddington [33], later on by Einstein, Infeld and Hoffmann [34] for the non-test mass case. We like to note how to extend

the computation for two sources of comparable mass to higher PN orders. The computation of the EOM in the near zone with the help of iterating Einstein’s equation, in harmonic coordinates, is only one possible alternative. The other one, that enjoys considerable successes in the recent past, and from that this thesis profits equally well, is the derivation of the Hamiltonian formulation of GR. An appropriate method was devised in the beginning sixties by Arnowitt, Deser and Misner (from that the shorthand “ADM” originates) who assigned canonical gravitational field variables for GR [35–37]. Therefore, using their approach, the form of the occurring equations of motion have a simple Hamiltonian appearance. Recent developments in incorporating the inner angular momentum of the compact objects are basis of evaluating the time dependencies of the GW field in this thesis. We wish to present the basic idea for convenience of the reader in the subsequent section. The expert on computing ADM Hamiltonians including spins in the matter source may skip reading this and continue with Section 3. The reader who likes to know more on the basics can read for example References [38, 39].

3+1 splitting of spacetime

To rewrite the Einstein-Hilbert-Action applied to a three-dimensional geometry evolving in time, the spacetime has to be foliated appropriately. The basic idea is to split it into space-like hypersurfaces and a time-like vector congruence [38].

First, we define a congruence of curves γ which intersects the hypersurface at time coordinate t which we call Σ_t . Then the time like vector $t^\alpha = dx^\alpha/dt$ is tangent to γ . At the value of the parameter t , the coordinates on Σ_t are labeled y^α . As we have x^α as the spacetime coordinates, the above construction defines a coordinate system (t, y^α) with $x^\mu = x^\mu(t, y^\alpha)$. We wish to define a set of vectors

$$t^\alpha = \left(\frac{\partial x^\alpha}{\partial t} \right)_{y^\alpha}, \quad (2.48)$$

$$e_a^\alpha = \left(\frac{\partial x^\alpha}{\partial y^a} \right)_t, \quad (2.49)$$

where the latter is tangent to Σ_t , implying that in the chosen coordinates $t^\alpha \stackrel{*}{=} \delta_t^\alpha$ and $e_a^\alpha \stackrel{*}{=} \delta_a^\alpha$. Note that t^α is not necessary orthogonal to Σ_t . With the normal to Σ_t ,

$$n_\alpha = -N\partial_\alpha t, \quad n_\alpha e_a^\alpha = 0, \quad (2.50)$$

we can decompose t^α as

$$t^\alpha = Nn^\alpha + N^a e_a^\alpha, \quad (2.51)$$

labeling N the *lapse function* and N^a as the *shift vector*. We can write the infinitesimal

spacetime differential as

$$\begin{aligned} dx^\alpha &= t^\alpha dt + e_a^\alpha dy^a \\ &= (Ndt)n^\alpha + (dy^a + N^a dt)e_a^\alpha, \end{aligned} \quad (2.52)$$

and the associated line element as

$$ds^2 = -N^2 dt^2 + h_{ab}(dy^a + N^a dt)(dy^b + N^b dt) \quad (2.53)$$

with $h_{ab} \equiv g_{\alpha\beta} e_a^\alpha e_b^\beta$ as the *induced metric* on Σ_t . The three-dimensional surface Σ_t will now be discussed as the volume in that our physical motion (the dynamics of compact stars, fields and spins, ...) takes place. We enfold the volume Σ_t by the two-dimensional boundary S_t which will play an important role in the further computations.

Equation (2.52) will be used as a prototype for expressing the displacements within the surfaces introduced above for the following reason. We like to recast the gravitational action into a more useful form. At the end of the day, it will alternatively be written as a time integral over a Lagrangian which naturally will contain contractions of the extrinsic curvature tensor of Σ_t , which is, in our sense, more appropriate. Those contractions have to be eliminated with the help of the ‘‘field velocities’’, defined via their Lie derivative along the vector t^α , viz. $\dot{h}_{ab} \equiv \mathcal{L}_t h_{ab}$. With this tool, one is able to compute the Hamiltonian with aid of a Legendre transformation which eliminates \dot{h}_{ab} with benefit of its canonical conjugated momentum π^{ab} . With the usual definition of this conjugated momentum,²

$$\pi^{ab} \equiv \frac{\partial}{\partial \dot{h}_{ab}} (\sqrt{-g} \mathcal{L}_G), \quad (2.54)$$

and after vast computations, the Hamiltonian can be cast into the form

$$\begin{aligned} \frac{16\pi G}{c^4} H_G &= \int_{\Sigma_t} [N(K^{ab}K_{ab} - K^2 - {}^{(3)}R) - 2N_a(K^{ab} - Kh^{ab})_{|b}] \sqrt{h} d^3y \\ &\quad - 2 \oint_{S_t} [N(k - k_0) - N_a(K^{ab} - Kh^{ab})r_b] \sqrt{\sigma} d^2\theta \\ &= \int_{\Sigma_t} (N^a \mathcal{H}_a + N\mathcal{H}) \sqrt{h} d^3y + \oint_{S_t} E \sqrt{\sigma} d^2\theta, \end{aligned} \quad (2.55)$$

where in

$$\sqrt{h} K^{ab} = \frac{16\pi G}{c^3} \left(\pi^{ab} - \frac{1}{2} (\pi^{cd} h_{cd}) h^{ab} \right), \quad (2.56)$$

the extrinsic curvature of Σ_t , K^{ab} , which we mentioned above, appears, and we additionally define $K^2 \equiv K^{ab}K_{ab}$. The quantity k is the extrinsic curvature of S_t on the one hand, and k_0 the version of the latter if S_t was embedded in flat space on the other. In a manner of

² In the definition of this momentum, the Lagrangian density does not contain $\sqrt{-g}$, and thus only R . We follow the convention of [38]. It can, of course, be defined the other way, too.

speaking, k_0 is a regularisation term that “cures the divergence of the gravitational action” [38]. It is a nondynamical quantity and only affects the numerical value of the gravitational action. The N^a and N will, while variational procedure, take the role of a Lagrangian multiplier and the surface term, containing E as some energy scalar, will take the role of the true Hamiltonian. The densities \mathcal{H}^μ will naturally contain complicated constructions, such as three-dimensional Riemann tensor ${}^{(3)}R_{ijkl}$ and its contractions respectively as well. Those can also be eliminated successively when equally projecting the left and the right hand side – the matter stress-energy tensor – of Einstein’s equations onto Σ_t . This operation will end up in solving the *constraint equations*.³ A post-Newtonian iteration of those constraints will be basis for further computations which help us express the Hamiltonian completely as functions of the matter source, making use of a special gauge: in the ADMTT gauge [40], the metric is decomposed into an isotropic and a transverse-traceless part (with respect to the Euclidean metric),

$$g_{ij} = \left(1 + \frac{1}{8}\phi\right)^4 \delta_{ij} + h_{ij}^{\text{TT}}, \quad (2.57)$$

$$\partial_j h_{ij}^{\text{TT}} = 0, \quad (2.58)$$

$$\pi^{ii} = 0. \quad (2.59)$$

The potential ϕ at leading order (the reader may apologize the abuse of the variable ϕ in this context), can be associated with the Newtonian gravitational potential. The longitudinal field momentum, $\tilde{\pi}^{ij} \equiv \pi^{ij} - \pi_{\text{TT}}^{ij}$, can be eliminated as one solves the momentum constraint equations. Also, the quantity ϕ can be eliminated as one solves the lower order Hamilton constraint equations. Assuming an asymptotically flat space time, the final form of the Hamiltonian is the following surface integral (see [37, 41]),

$$H = \oint dS_i (g_{ij,j} - g_{jj,i}). \quad (2.60)$$

Having the independent field degrees of freedom h_{ij}^{TT} and π_{TT}^{ij} , which represent the canonical set of variables, the Hamiltonian has the form

$$H = H[\mathbf{x}_a, \mathbf{p}_a, \mathbf{S}_a, h_{ij}^{\text{TT}}, \pi_{\text{TT}}^{ij}], \quad (2.61)$$

where the TT field variables can be eliminated via solving their canonical evolution equations. We stop here summarizing the procedure and recommend reading the standard literature on that topic, e.g. [35, 37, 42], and [40]. There are very recent progresses and breakthroughs to higher-order Hamiltonians and we will refer to them at a later point of time, when we apply important results.

Hitherto, we did not regard the special form of the matter source of the gravitational field.

³ These are those parts of Einstein’s field equations which do not explicitly contain time evolutions.

One may set the right hand side of the Einstein equations, the matter stress-energy tensor, as a sum of delta functions for point masses as a monopole approximation. Assuming that we do not simply have a swarm of point particles (in our case, a swarm of two), which are not allowed to have an internal degree of freedom, we need an appropriate expression that respects the properties of the dipole approximation, which will be introduced briefly below.

2.8.2 Matter stress-energy tensor for monopole-dipole particle sources

As soon as a massive classical object begins to rotate, it will – in contrast to a delta distribution – have a minimum nonzero spatial extent, and non-geodesic effects will enter its EOM as it moves through curved spacetime. This problem, including only test masses for a first approximation, was introduced in [43, 44] and considerable further developments were made in the 1970s [45–47] from a quantum theoretical approach to GR, when the comparable mass case became accessible, and a first classical investigation has been done in [48] in the slow-motion limit for comparable masses, and, remarkably, in recent years as well. Attempts to compute higher-order Hamiltonians of spinning compact binaries have used the following Tulczyjew stress-energy tensor

$$\sqrt{-g}T^{\mu\nu} = \sum_{a=1}^2 \int d\tau \left[u_a^{(\mu} p_a^{\nu)} \delta_{(4)a} + (u_a^{(\mu} S_a^{\nu)\alpha} \delta_{(4)a})_{;\alpha} \right]. \quad (2.62)$$

This stress-energy tensor includes, beyond the monopolar source terms, the dipolar contribution of the extended particle.⁴ In contrast to what we see for example in electrodynamics, when we expand the field of a source in some distance in multipoles, we attempt to *expand the source itself* at this point. $S_a^{\mu\nu}$ is the spin tensor of the a^{th} particle and it is antisymmetric in both spacetime indices. Equation (2.62) can be set up from a formal orthogonal decomposition [53–55],

$$T^{\mu\nu} = \int d\tau \left[\mu^{\mu\nu} \delta_{(4)} - (\mu^{\mu\nu\lambda} \delta_{(4)})_{;\lambda} \right], \quad (2.63)$$

$$\mu^{\mu\nu\lambda} u_\lambda = 0, \quad (2.64)$$

$$\mu^{\mu\nu\lambda} = S^{\mu\nu\lambda} + \frac{1}{2} S^{\mu\lambda} u^\nu + \frac{1}{2} S^{\nu\lambda} u^\mu + S^\lambda u^\mu u^\nu, \quad (2.65)$$

$$\mu^{\mu\nu} = m^{\mu\nu} + m^\mu u^\nu + m^\nu u^\mu + m u^\mu u^\nu, \quad (2.66)$$

$$m^{\mu\nu} = m^{\nu\mu}, \quad m^{\mu\nu} u_\nu = 0, \quad m^\mu u_\mu = 0, \quad (2.67)$$

stating that the static dipole vanishes by choice of world line, $S^\mu = 0$, and clarified by taking the covariant divergence, viz. $T^{\mu\nu}{}_{;\nu} = 0$. This operation gives algebraic relations of

⁴One could, in principle, add also the quadrupolar part, but time derivatives of the quadrupole part can be re-absorbed into relations concerning the dipole order. There is, thus, no dynamical equation for the quadrupole part, see [49]. The treatment of the quadrupole of extended bodies, such as neutron stars, as they are induced by the external gravitational field of the other compact member of the binary system, is reviewed by Poisson et al. [50]. What effect the quadrupole has on gravitational waves from compact binaries can be found in [51], and for its effect on the Kepler equation in e.g. [52]

the decomposition terms and, at dipole order, the set of Mathisson-Papapetrou equations [43, 56],

$$S^{\mu\nu} = -S^{\nu\mu}, \quad S^{\mu\nu\lambda} = 0, \quad m^{\mu\nu} = 0, \quad (2.68)$$

$$2m^\alpha = u_\beta \frac{DS^{\beta\alpha}}{ds}, \quad (2.69)$$

$$0 = \frac{D}{ds} \left(mu^\alpha + \frac{DS^{\lambda\alpha}}{ds} u_\lambda \right) + \frac{1}{2} S^{\mu\nu} u^\beta R_{\beta\mu\lambda}^\alpha. \quad (2.70)$$

To close the system of EOM, we have to provide an appropriate spin supplementary condition (SSC), otherwise the EOM cannot be solved uniquely. There exist a number of SSC, for example the *Pirani SSC* which fixes the spin tensor $S^{\mu\nu}$ in such a way that it is orthogonal to the four-velocity u_μ of the spinning particle, $S^{\mu\nu}u_\nu = 0$. Another one, the *Tulczyjew SSC*, says that the spin tensor is orthogonal to the four-momentum of the particle, $S^{\mu\nu}p_\nu = 0$. This differs from Pirani's SSC by a term of order $\mathcal{O}(S^2)$. The third one we like to mention is the *Corinaldesi-Papapetrou SSC*, which chooses the spin to have no four-component in its rest frame, $S^{i4} = 0$. A very interesting supplementary condition is the *canonical SSC*, $S^{\mu\nu}p_\nu - mS^{\mu 0} = 0$, as this is the only one that leads to variables with canonical Poisson brackets [57]. Different SSC will, in general, lead to different spin EOM at higher post-Newtonian orders. To linear order in spin, Barker and O'Connell [58] showed that the difference in the EOM can be blamed on the difference in the location of the center of mass due to the SSC in use, and that the EOM are physically equivalent to that order as those different locations are taken into account. The result of [58] is, essentially,

$$\mathbf{r}_a^{(\text{CP})} = \mathbf{r}_a^{(\text{P})} + \frac{\mathbf{v}_a \times \mathbf{S}_a}{c^2 m_a} + \mathcal{O}(c^{-4}), \quad (2.71)$$

where on the right hand side no distinction has to be made of (CP=Corinaldesi-Papapetrou SSC) and (P=Pirani's SSC), because they differ to negligible higher orders. The appropriate SSC for the Hamiltonians of the spinning binary system is the canonical one. Usually, in contrast to that, the far-zone radiation field is worked out in harmonic coordinates and other SSC, such as for example the covariant one. To make both compatible it is necessary to perform a position coordinate transformation analog to Equation (2.71). This thesis heavily profits from the success in finding canonical variables (that, by definition, fulfill the Poincaré algebra), also including spin, which made the computation of the higher order Hamiltonians possible within the ADM formalism.

3 Outlook: planning the work

In the previous sections, we have summarised how to construct gravitational wave radiation consistently when the equations of motion of the source are known. The theory of the gravitational wave generation formalism is closed in a sense and the state-of-the-art today is that the instantaneous parts are completed to 3PN order in PP contributions [16, 25, 59], and where additionally care is taken of higher-order hereditary terms, and including in NLO-SO terms [60]. The formalism of a field-theoretical approach of the Hamiltonians for the matter variables $(\mathbf{p}_a, \mathbf{x}_a, \mathbf{S}_a)$ for a binary system is currently developed and expedited by the collaborators of my working group in Jena to NNLO-SO (“next-to-next-to-leading-order spin-orbit”) contributions [61], and also for n -body systems, NLO-SO and NLO-S(a)-S(b) have been recently investigated [62].

In the following, we try to solve the Hamiltonian equations of motion to give – in the most ideal case – an explicit representation of h_\times and h_+ as functions of time or in case of question, a parameterised solution. We will start with the most simple form of configuration, where the solution – even at relatively high PN orders – turns out to be structurally as simple as in the Newtonian case, such that we can go to 2PN order in the representation of the gravitational radiation amplitudes. This will be the case in Section 4 for a binary system with aligned spins, where we will show that the alignment will be a conserved constraint on the binary and relatively simple-typed differential equations have to be solved even for next-to-leading order spin-orbit effects. We compute the conserved orbital elements of the quasi-Keplerian parameterisation and the explicit amplitudes of the gravitational wave, including the instantaneous parts.

In a further Section 5, we will take care of a more complicated case of a binary whose spins are arbitrary in magnitude and orientation. For an introductory treatment, we regard only the leading-order spin-orbit interaction and the point particle interactions accurately treated up to 2PN order. A solution of the equation of motion for the orientation angles is given at first perturbative order around the exactly known solution of a) single spin case, and b) equal-mass case. The difference in mass is the expansion parameter which is assumed to be very small.

In Section 6 we will regard the case of a non-spinning compact binary system, which we will describe up to 2PN point particle contributions accurately, from which the radiated wave forms are provided fully analytic as a time Fourier-domain series. This has to be seen as the foundation for a representation of radiation including spin effects, where the structure of the solution that we obtain here will essentially be the same for the aligned spin case.

A detailed appendix will give supplementary material and computations.

4 Aligned spins under point particle and spin interactions through formal 2PN order

4.1 Aligned spins: motivation for an analytic solution

Observations lead to the assumption that many astrophysical objects carry a non-negligible spin, such that the effect of spin angular momentum on GW data cannot be ignored for detailed data analysis. For the following consideration, we refer to the work [63] and deal with compact binaries having spins that are aligned or antialigned with the orbital angular momentum \mathbf{L} . They belong to an interesting category of sources, because due to angular momentum conservation, young neutron stars use to have a high rate of rotation (the period is in some cases $\sim 10^{-3}$ seconds.) Numerical results of a recent publication indicate that maximum equal-spins aligned with the orbital angular momentum lead to observable volume of up to ~ 30 times larger than the corresponding binaries with the spins anti-aligned to the orbital angular momentum [64]. From Figure 10 in [64], one can also find an observable volume of those binaries up to ~ 8 times larger compared to non-spinning binaries. These authors conclude that those systems are among the most efficient GW sources in the universe. In another recent publication [65] it can be found that in gas-rich environments the spins of two black holes can align with the larger scale accretion disc on a timescale that is short as 1% of the accretion time. Due to the model of those authors, having two black holes interacting independently with an accretion disc, their spins tend to be aligned with each other and with the orbital angular momentum more or less depending on the model parameters. We bed our consideration in the following

Historical background: For non-spinning compact binaries, the post-Newtonian (PN) expansion in the near-zone has been carried out through 3.5PN order [27, 66] and 3.5PN accurate inspiral templates have been established for circular orbits [67, 68]. For numerical performances of these templates see [69, 70]. Apostolatos [4] showed in his analysis of *simple precession*⁵ for “circular” orbits and spinning self-gravitating sources that the form of the GW signal is remarkably affected by the object’s spin and that there is a long-term modulation of the wave form. The amount of the energy radiated by a binary system with spins has been determined by [71]. Linear-in-spin effects of higher order on the motion and the GW amplitude were discussed in [60, 72–75] for the inspiral of compact binaries were the orbits were assumed to be quasi-circular. Quasi-circular in their sense means that the separation of the companions remains constant in conservative dynamics, but the orbital plane is allowed to precess. A recent publication [76] gave a numerical insight into the evolution of binary systems having spins that are parallel to the orbital angular momentum and evolving in quasi-circular orbits due to RR. We only regard the conservative Hamiltonian for the time

⁵ “simple precession” terms the case when the precession frequency of the orbital angular momentum is constant. This is the case, for example, when only one spin and leading-order spin-orbit interaction is included.

being, and restrict our attention to terms up to 2.5PN order overall, assuming maximally spinning holes. This means neglecting both the well-known 3PN PP contributions, and the NLO-S(1)-S(2) [77], as well as the NLO-S(1)² contributions, which have recently been derived for general compact binaries [78]. This latter publication came out at a late stage in our calculations, but it should be a straightforward task to include these terms in a future publication.

If the objects are slowly rotating, the considered leading-order spin-squared contributions are shifted to 3PN order and, for consistency, the 3PN PP Hamiltonian has to be included. The 3PN PP contributions to the orbital elements are available in the literature [79] and simply have to be added to what we are going to present in this section. Anyway, the work we are referring to [63] is consistently worked out to all terms up to 2.5PN, irrespective of how fast the objects rotate, and will list all results in the spins which are counted of 0PN order.

4.2 Spin and orbital dynamics

In the following sections, the dynamics of spinning compact binaries is investigated, where the SO contributions are restricted to NLO and the S₁S₂ and S² to LO. The PP contributions are cut off after the 2PN terms. The Hamiltonian associated therewith reads

$$\hat{H}(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \hat{\mathbf{p}}_1, \hat{\mathbf{p}}_2, \hat{\mathbf{S}}_1, \hat{\mathbf{S}}_2) = \hat{H}_{\text{PP}}^{\text{N}} + \hat{H}_{\text{PP}}^{\text{1PN}} + \hat{H}_{\text{PP}}^{\text{2PN}} + \hat{H}_{\text{SO}}^{\text{LO}} + \hat{H}_{\text{SO}}^{\text{NLO}} + \hat{H}_{\text{S}^2}^{\text{LO}} + \hat{H}_{\text{S}_1\text{S}_2}^{\text{LO}}. \quad (4.1)$$

These are sufficient for maximally rotating black holes up to and including 2.5PN. The variables $\hat{\mathbf{p}}_a$ and $\hat{\mathbf{x}}_a$ are the linear canonical momentum and position vectors, respectively. They commute with the spin vectors $\hat{\mathbf{S}}_a$, where “ a ” denotes the particle label, $a = 1, 2$. H_{PP} is the conservative point-particle ADM Hamiltonian known up to 3PN, see, e.g., [80] and [81]. The LO spin dependent contributions are well-known, see, e.g., [46, 47, 51]. $H_{\text{SO}}^{\text{NLO}}$ was recently found in [82, 83] and $H_{\text{S}_1\text{S}_2}^{\text{NLO}}$ in [77, 83] (the latter was confirmed in [84]). The leading-order S₁² and S₂² Hamiltonians were derived in [48] and [85]. Hamiltonians of cubic and higher order in spin are given in [86, 87], and higher PN orders linear in spin are tackled in [88, 89]. In order to close the system of equations, one has to impose a spin supplementary condition (SSC), which is most conveniently taken to be

$$S_a^{\mu\nu} p_{a\nu} = 0. \quad (4.2)$$

Notice that the matter variables appearing in the Mathisson-Papapetrou equations and the stress-energy tensor are related to the canonical variables appearing in the Hamiltonians by rather complicated redefinitions, see Appendix A.1. We are going to work in the center-of-mass (COM) frame, where the total linear momentum vector is zero, i.e. $\hat{\mathbf{p}}_2 = -\hat{\mathbf{p}}_1 = -\hat{\mathbf{p}}$. The Hamiltonians taken into account depend on $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$ only in the combinations $\hat{\mathbf{x}}_1 - \hat{\mathbf{x}}_2$, so they can be re-expressed in terms of $\mathbf{n}_{12} = -\mathbf{n}_{21} = \hat{\mathbf{x}}/\hat{r} = \mathbf{x}/r$, the normalised direction

from particle 1 to 2, and $\hat{r} = |\hat{\mathbf{x}}_1 - \hat{\mathbf{x}}_2|$ with $\hat{\mathbf{x}} = \hat{\mathbf{x}}_1 - \hat{\mathbf{x}}_2$.

We will make use of the following scalings to convert quantities with hat to dimensionless ones,

$$H \equiv \hat{H}(\mu c^2)^{-1}, \quad (4.3)$$

$$\mathbf{x} \equiv \hat{\mathbf{x}} \left(\frac{Gm}{c^2} \right)^{-1}, \quad (4.4)$$

$$\mathbf{p} \equiv \hat{\mathbf{p}} (\mu c)^{-1}, \quad (4.5)$$

$$\mathbf{S}_a \equiv \hat{\mathbf{S}}_a \left(\frac{Gm_a}{c^2} (m_a c) \right)^{-1}. \quad (4.6)$$

where $\eta \equiv \mu/m$ is the symmetric mass ratio. Without loss of generality we assume that $m_1 > m_2$. Such an assumption is necessary, because the spins are scaled with the individual masses in a non-symmetric way. Explicitly, the contributions to the rescaled version of Equation (4.1) read

$$H_{\text{PP}}^{\text{N}} = \frac{\mathbf{p}^2}{2} - \frac{1}{r}, \quad (4.7)$$

$$H_{\text{PP}}^{\text{1PN}} = \epsilon^2 \left\{ \frac{1}{8} (3\eta - 1) (\mathbf{p}^2)^2 - \frac{1}{2} [(3 + \eta)(\mathbf{p}^2) + \eta (\mathbf{n}_{12} \cdot \mathbf{p})^2] \frac{1}{r} + \frac{1}{2r^2} \right\}, \quad (4.8)$$

$$H_{\text{PP}}^{\text{2PN}} = \epsilon^4 \left\{ \frac{1}{16} (1 - 5\eta + 5\eta^2) (\mathbf{p}^2)^3 + \frac{1}{8} \left[(5 - 20\eta - 3\eta^2) (\mathbf{p}^2)^2 - 2\eta^2 (\mathbf{n}_{12} \cdot \mathbf{p})^2 (\mathbf{p}^2) - 3\eta^2 (\mathbf{n}_{12} \cdot \mathbf{p})^4 \right] \frac{1}{r} + \frac{1}{2} [(5 + 8\eta) (\mathbf{p}^2) + 3\eta (\mathbf{n}_{12} \cdot \mathbf{p})^2] \frac{1}{r^2} - \frac{1}{4} (1 + 3\eta) \frac{1}{r^3} \right\}, \quad (4.9)$$

$$H_{\text{SO}}^{\text{LO}} = \epsilon^2 \delta \frac{\alpha_{\text{so}}}{r^3} \left\{ \left(1 - \frac{\eta}{2} + \sqrt{1 - 4\eta} \right) (\mathbf{L} \cdot \mathbf{S}_1) + \left(1 - \frac{\eta}{2} - \sqrt{1 - 4\eta} \right) (\mathbf{L} \cdot \mathbf{S}_2) \right\}, \quad (4.10)$$

$$H_{\text{SO}}^{\text{NLO}} = \epsilon^4 \delta \frac{\alpha_{\text{so}}}{16r^4} \left\{ (\mathbf{L} \cdot \mathbf{S}_1) \left[12\eta r \left(1 - \eta + \sqrt{1 - 4\eta} \right) (\mathbf{n}_{12} \cdot \mathbf{p})^2 + \eta r \left(9 - 6\eta + 19\sqrt{1 - 4\eta} \right) (\mathbf{p}^2) - 16 \left((\eta + 3)\sqrt{1 - 4\eta} + 3 \right) \right] - (\mathbf{L} \cdot \mathbf{S}_2) \left[12\eta r \left(-1 + \eta + \sqrt{1 - 4\eta} \right) (\mathbf{n}_{12} \cdot \mathbf{p})^2 + \eta r \left(-9 + 6\eta + 19\sqrt{1 - 4\eta} \right) (\mathbf{p}^2) - 16 \left((\eta + 3)\sqrt{1 - 4\eta} - 3 \right) \right] \right\}, \quad (4.11)$$

$$H_{\text{S}_1\text{S}_2}^{\text{LO}} = \epsilon^2 \delta^2 \alpha_{\text{S}_1\text{S}_2} \frac{\eta}{r^3} \left\{ 3 (\mathbf{n}_{12} \cdot \mathbf{S}_1) (\mathbf{n}_{12} \cdot \mathbf{S}_2) - (\mathbf{S}_1 \cdot \mathbf{S}_2) \right\}, \quad (4.12)$$

$$H_{\text{S}_2}^{\text{LO}} = \epsilon^2 \delta^2 \frac{\alpha_{\text{S}_2}}{2r^3} \left\{ \lambda_1 \left(-1 + 2\eta - \sqrt{1 - 4\eta} \right) (3 (\mathbf{n}_{12} \cdot \mathbf{S}_1)^2 - (\mathbf{S}_1 \cdot \mathbf{S}_1)) + \lambda_2 \left(-1 + 2\eta + \sqrt{1 - 4\eta} \right) (3 (\mathbf{n}_{12} \cdot \mathbf{S}_2)^2 - (\mathbf{S}_2 \cdot \mathbf{S}_2)) \right\}, \quad (4.13)$$

We introduced dimensionless ‘‘book-keeping’’ parameters ϵ to count the formal $1/c$ order and δ to count the spin order (linear or quadratic). Evaluating all given quantities, those have to be given the numerical value 1. The parameters α_{so} , $\alpha_{\text{S}_1\text{S}_2}$, α_{S_2} distinguish the spin-orbit,

spin(1)–spin(2) and the spin-squared contributions and can have values 1 or 0, depending on whether the reader likes to incorporate the associated interactions.

The spins are denoted by \mathbf{S}_1 for object 1 and \mathbf{S}_2 for object 2. Notice that the S_1^2 and S_2^2 Hamiltonians depend on constants λ_1 and λ_2 , respectively, which parametrise the quadrupole deformation of the objects 1 and 2 due to the spin and take different values for, e.g., black holes and neutron stars. For black holes, $\lambda_a = -\frac{1}{2}$ and for neutron stars, λ_a can take continuous values from the interval $[-2, -4]$ [51, 90].⁶

The parallelism condition tells us to set the spins to $\mathbf{S}_a = \chi_a \mathbf{L}/L$, where $-1 < \chi_a < 1$. During our calculations, we insert the condition of maximal rotation ($S_a \sim \epsilon$) to cut off every quantity after 2.5PN, but list our results in formal orders $S_a \sim \epsilon^0$ (for the formal counting, see, e.g., [87] and also Appendix A of [89]). However, for $S_a \sim \epsilon^2$, many spin contributions are of the order $\mathcal{O}(\epsilon^6)$, i.e. 3PN which is beyond our present 2PN PP dynamics. The reader may insert either $S_a \sim \epsilon$ (maximal rotation) or $S_a \sim \epsilon^2$ (slow rotation). The next step is to evaluate the EOM due to these Hamiltonians and to find a parametric solution. As stated, we will restrict ourselves to parallel or anti-parallel angular momenta and will, finally, only have to take care of the motion in the orbital plane.

Conservation of parallelism of \mathbf{L} , \mathbf{S}_1 and \mathbf{S}_2

The motion of binaries with arbitrarily-oriented spins is, in general, chaotic as soon as the spin-spin interaction is included [91, 92]. For special configurations, despite this, it is possible to integrate the EOM analytically, which particularly is the case for aligned spins and orbital angular momentum.

The time derivatives of the spins \mathbf{S}_a and the total angular momentum \mathbf{J} are governed by the Poisson brackets with the total Hamiltonian, given by

$$\begin{aligned}
[\mathbf{S}_1, H] = & \delta^2 \epsilon^2 \left\{ \alpha_{s_1 s_2} \frac{\eta}{r^3} (3 (\mathbf{S}_2 \cdot \mathbf{n}_{12}) (\mathbf{n}_{12} \times \mathbf{S}_1) + (\mathbf{S}_1 \times \mathbf{S}_2)) \right. \\
& \left. + \alpha_{s_2} \frac{(\mathbf{n}_{12} \times \mathbf{S}_1)}{r^3} (\mathbf{S}_1 \cdot \mathbf{n}_{12}) 3\lambda_1 (2\eta - 1 - \sqrt{1 - 4\eta}) \right\} \\
& + \delta \left\{ \alpha_{so} \epsilon^2 \frac{(\mathbf{L} \times \mathbf{S}_1)}{r^3} \left(-\frac{\eta}{2} + \sqrt{1 - 4\eta} + 1 \right) \right. \\
& + \alpha_{so} \epsilon^4 \left[\frac{(\mathbf{L} \times \mathbf{S}_1)}{r^3} \left(\frac{3}{4} (\mathbf{p} \cdot \mathbf{n}_{12})^2 \eta (1 - \eta + \sqrt{1 - 4\eta}) \right) \right. \\
& \left. + \frac{1}{16} (\mathbf{p}^2) \eta (9 - 6\eta + 19\sqrt{1 - 4\eta}) \right) \\
& \left. - \frac{(\mathbf{L} \times \mathbf{S}_1)}{r^4} (3 + (\eta + 3)\sqrt{1 - 4\eta}) \right] \left. \right\}, \tag{4.14}
\end{aligned}$$

$$[\mathbf{S}_2, H] = [\mathbf{S}_1, H] (1 \leftrightarrow 2), \tag{4.15}$$

$$[\mathbf{J}, H] = [\mathbf{L}, H] + [\mathbf{S}_1, H] + [\mathbf{S}_2, H] = 0. \tag{4.16}$$

⁶ Note that the definition of the λ_a depends on the definition of the spin Hamiltonian and, thus, can be arbitrarily normalised. We consistently use the notation mentioned above.

Furthermore, the magnitudes of the spins are conserved, because the spins commute with the linear momentum and the position vector and fulfill the canonical angular momentum algebra. Note that the operation ($1 \leftrightarrow 2$) switches the label indices of the individual particles and goes along with $\mathbf{n}_{12} \leftrightarrow \mathbf{n}_{21} = -\mathbf{n}_{12}$. If we assume parallel spins and orbital angular momentum at $t = 0$, all the above Poisson brackets vanish exactly. Anyway, this is insufficient to conclude the conservation of parallelism of \mathbf{L} and the spins for all times $t > 0$ since

$$\begin{aligned} \mathbf{S}_1(t) &= \mathbf{S}_1|_{t=0} + [\mathbf{S}_1, H]|_{t=0} t + \frac{1}{2} [[\mathbf{S}_1, H], H]|_{t=0} t^2 + \dots \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} [\mathbf{S}_1, H]_n|_{t=0} t^n, \end{aligned} \quad (4.17)$$

where

$$\begin{aligned} [\mathbf{S}_1, H]_n &= [[\mathbf{S}_1, H]_{n-1}, H], \\ [\mathbf{S}_1, H]_0 &= \mathbf{S}_1. \end{aligned} \quad (4.18)$$

Because the system of variables \mathbf{S}_1 and \mathbf{S}_2 has to be completed with \mathbf{r} and \mathbf{p} to characterise the matter system, one has to give clear information about the full system of EOM. It is important that, even with vanishing Poisson brackets of H with \mathbf{S}_1 and \mathbf{S}_2 , \mathbf{r} and \mathbf{p} do change due to the orbital revolution. Thus, one has to clarify if this non-stationary subsystem of the EOM is able to cause violation of the parallelism condition during time evolution. From the stability theory of autonomous ordinary differential equations it is well known that there is a fixed point if *all* time derivatives of the system vanish. In the case of a system starting at such a fixed point at $t = 0$ it will not be able to evolve away from this point. This will be discussed below.

One way to show the non-violation of the initial constraint of $\mathbf{L} \parallel \mathbf{S}_1, \mathbf{S}_2$ due to the motion of the binary is to argue via the time derivatives of the constraints. These should be written as a linear combination of the constraints themselves. Let

$$C_a(x, p, S) = 0, \quad (4.19)$$

be the initial constraints of the system. Dirac [93, p. 36] argued: If one can write

$$\dot{C}_a = \sum_b D_{ab}(x, p, S) C_b, \quad (4.20)$$

for the time derivatives of the constraints, the constraints are conserved. That is due to the fact that every time derivative of Equation 4.20 generates only new time derivatives of the constraints on the one hand, which can be expressed as a linear combination of constraints, or time derivatives of the quantities appearing in D_{ab} times the constraints on the other. In

our case the constraints read

$$\mathbf{S}_1 - \frac{|\mathbf{S}_1|}{L} \mathbf{L} = \mathbf{S}_1 - \tilde{\chi}_1 \mathbf{L} = 0, \quad (4.21)$$

$$\mathbf{S}_2 - \frac{|\mathbf{S}_2|}{L} \mathbf{L} = \mathbf{S}_2 - \tilde{\chi}_2 \mathbf{L} = 0, \quad (4.22)$$

A calculation shows (see Section 3 of [63]) that the time derivatives of the left hand sides can be written as linear combinations of the left hand sides themselves, thus, they do vanish if the constraints are inserted.

4.3 The Keplerian-type parameterisation

In Newtonian dynamics the Keplerian parameterisation of a compact binary is a well-known tool for celestial mechanics, see e.g. [94]. After going to spherical coordinates in the COM, (r, θ, ϕ) with the associated orthonormal vectors $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$ and restricting to the $\theta = \pi/2$ plane, the Keplerian parameterisation has the following form:

$$r = a(1 - e \cos \mathcal{E}), \quad (4.23)$$

$$\phi - \phi_0 = v, \quad (4.24)$$

$$v = 2 \arctan \left[\sqrt{\frac{1+e}{1-e}} \tan \frac{\mathcal{E}}{2} \right]. \quad (4.25)$$

Here, a is the semimajor axis, e is the numerical eccentricity, \mathcal{E} and v are eccentric and true anomaly, respectively. The time dependency of r and ϕ is given by the Kepler equation,

$$\mathcal{M} = \mathcal{N}(t - t_0) = \mathcal{E} - e \sin \mathcal{E}, \quad (4.26)$$

where \mathcal{M} is the mean anomaly and \mathcal{N} the so-called mean motion, defined as $\mathcal{N} \equiv \frac{2\pi}{\mathcal{P}}$ with \mathcal{P} as the orbital period [95]. In these formulae t_0 and ϕ_0 are some initial instant and the associated initial phase. In terms of the conserved quantities $|E|$, which is the scaled energy [see Equation (4.3)] and numerically identical to H , and the magnitude of the orbital angular momentum L , the orbital elements e , a and \mathcal{N} satisfy

$$a = \frac{1}{2|E|}, \quad (4.27)$$

$$e^2 = 1 - 2L^2|E|, \quad (4.28)$$

$$\mathcal{N} = (2|E|)^{3/2}. \quad (4.29)$$

For higher PN accurate EOM it is possible to get a solution in a perturbative way, having the inverse speed of light as the perturbation parameter.

Historical background: The 1PN accurate *Keplerian like parameterisation* (from now on we refer to *quasi-Keplerian parameterisation*, *QKP*) was first found in [96] using action variables, and later, using perturbative techniques for solving the EOM of a 1PN accurate Lagrangian, in

[97]. It has been extended for non-spinning compact binaries in [79, 98] to 2PN and finally 3PN accuracy. Finite size effects due to rotational deformation are considered in [99], where care is taken of the orbital angular momentum aligned spin case as well. In the recent past a number of efforts has been undertaken to obtain a solution to the problem of spinning compact binaries via calculating the EOM for spin-related angular variables in harmonic gauge. For circular orbits, including radiation reaction (RR), the authors of [100] evaluated several contributions to the frequency evolution and the number of accumulated GW cycles up to 2PN, such as from the spin, mass quadrupole and the magnetic dipole moment parts. The gravitational wave form amplitudes as functions of separations and velocities up to and including 1.5PN PP and 1.5PN SO corrections are given in [101], discussed for the extreme mass ratio limit in the Lense-Thirring approximation and later in [102] and [103] for comparable mass binaries. Recently, in [104] a set of independent variables and their EOM, characterising the angular momenta, has been provided.

For circular orbits with arbitrary spin orientations and leading-order spin-orbit interactions, the spin and orbital solutions for slightly differing masses will be provided in section 5, and relies on Reference [105]. Including LO contributions of S^2 , S_1S_2 and SO as well as the Newtonian and 1PN contributions to the EOM, a certain time-averaged orbital parameterisation was found in [52], for a time scale where the spin orientations are almost constant, but arbitrary, and the radial motion has been determined. Symbolically, those solutions suggest the following form for the quasi-Keplerian parameterisation including spin interactions:

$$r = a_r (1 - e_r \cos \mathcal{E}), \quad (4.30)$$

$$\mathcal{N}(t - t_0) = \mathcal{E} - e_t \sin \mathcal{E} + \mathcal{F}_{v-\mathcal{E}}(v - \mathcal{E}) + \mathcal{F}_v \sin v + \mathcal{F}_{2v} \sin 2v \dots, \quad (4.31)$$

$$\frac{2\pi}{\Phi} (\phi - \phi_0) = v + \mathcal{G}_{2v} \sin 2v + \mathcal{G}_{3v} \sin 3v + \mathcal{G}_{4v} \sin 4v + \mathcal{G}_{5v} \sin 5v + \dots, \quad (4.32)$$

$$v = 2 \arctan \left[\sqrt{\frac{1 + e_\phi}{1 - e_\phi}} \tan \frac{\mathcal{E}}{2} \right]. \quad (4.33)$$

The coefficients \mathcal{F}_{\dots} , \mathcal{G}_{\dots} are PN functions of E , L and η . At the end of the calculation for binary dynamics with spin, they will obviously include spin dependencies as well. We will go into details below.

Application of the Keplerian-type parameterisation to aligned spinning compact binaries

Having proven constancy in time of the directions of angular momenta, we can adopt the choice of spherical coordinates with $\mathbf{L} \parallel \mathbf{e}_\theta$ (in the $\theta = \pi/2$ plane) and the basis $(\mathbf{e}_r, \mathbf{e}_\phi)$, where $\mathbf{n}_{12} = \mathbf{e}_r$ holds. With some abuse of notation, Hamilton's equations of motion dictate

$$\dot{\mathbf{r}} = \mathbf{n}_{12} \cdot \dot{\mathbf{r}} = \mathbf{n}_{12} \cdot \frac{\partial H}{\partial \mathbf{p}}, \quad (4.34)$$

$$r\dot{\phi} = \mathbf{e}_\phi \cdot \dot{\mathbf{r}} = \mathbf{e}_\phi \cdot \frac{\partial H}{\partial \mathbf{p}}, \quad (4.35)$$

with $\dot{r} = dr/dt$ and $\dot{\phi} = d\phi/dt$, as usual. The next standard step is to introduce $s \equiv 1/r$, such that $\dot{r} = -\dot{s}/s^2$. Using Equations (4.34) and (4.35), we obtain a relation for \dot{r}^2 and thus \dot{s}^2 , and another one for $\dot{\phi}/\dot{s} = d\phi/ds$, where the polynomial of \dot{s}^2 is of third degree in s . To obtain a formal 2PN accurate parameterisation,⁷ we first concentrate on the radial part and search for the two nonzero roots of $\dot{s}^2 = 0$, namely s_+ and s_- . The results, to Newtonian order, are

$$s_+ = \frac{1}{a_r(1 - e_r)} = \frac{1 + \sqrt{1 - 2L^2|E|}}{L^2} + \mathcal{O}(\epsilon^2), \quad (4.36)$$

$$s_- = \frac{1}{a_r(1 + e_r)} = \frac{1 - \sqrt{1 - 2L^2|E|}}{L^2} + \mathcal{O}(\epsilon^2), \quad (4.37)$$

s_- representing periastron and s_+ as the apastron. Next, we factorise \dot{s}^2 with these roots and obtain the following two integrals for the elapsed time t and the total radial period \mathcal{P} ,

$$\mathcal{P} = 2 \int_{s_-}^{s_+} \frac{\mathcal{P}_3(\tau) d\tau}{\tau^2 \sqrt{(\tau - s_-)(s_+ - \tau)}}, \quad (4.38)$$

which is a linear combination of integrals of the type

$$I'_n = 2 \int_{s_-}^{s_+} \frac{\tau^n d\tau}{\tau^2 \sqrt{(\tau - s_-)(s_+ - \tau)}}. \quad (4.39)$$

The time elapsed from s to s_+ ,

$$t - t_0 = \int_s^{s_+} \frac{\mathcal{P}_3(\tau) d\tau}{\tau^2 \sqrt{(\tau - s_-)(s_+ - \tau)}}, \quad (4.40)$$

is a linear combination of integrals of the type

$$I_n = \int_s^{s_+} \frac{\tau^n d\tau}{\tau^2 \sqrt{(\tau - s_-)(s_+ - \tau)}}. \quad (4.41)$$

Both integrals I_n and I'_n are given in Appendix A.2 in terms of s_+ and s_- for I' and in terms of a_r , e_r , \mathcal{E} and \tilde{v} for I , respectively. The function $\mathcal{P}_3(s)$ is a third order polynomial in s and the factor 2 follows from the fact that from s_- to s_+ it is only a half revolution. With the help of the quasi-Keplerian parameterisation

$$r = a_r (1 - e_r \cos \mathcal{E}), \quad (4.42)$$

⁷ When we talk about a formal solution at 2PN here, we mean that we incorporate all terms up to the order ϵ^4 where the spins are formally counted of order ϵ^0 .

where a_r and e_r are some 2PN accurate semi-major axis and radial eccentricity, respectively, satisfying

$$a_r = \frac{1}{2} \frac{s_+ + s_-}{s_- s_+}, \quad (4.43)$$

$$e_r = \frac{1}{2} \frac{s_+ - s_-}{s_- + s_+}, \quad (4.44)$$

due to Equations (4.36) and (4.37), we obtain a 2PN accurate expression for a_r and e_r in terms of several intrinsic quantities. With (4.40), we get a preliminary expression for the Kepler Equation, as we express $\mathcal{N}(t - t_0) = \frac{2\pi}{P}(t - t_0)$ in terms of \mathcal{E} , and as standard, we introduce an auxiliary variable

$$\tilde{v} \equiv 2 \arctan \left[\sqrt{\frac{1 + e_r}{1 - e_r}} \tan \frac{\mathcal{E}}{2} \right]. \quad (4.45)$$

At this stage, we have

$$\mathcal{M} \equiv \mathcal{N}(t - t_0) = \mathcal{E} + \tilde{\mathcal{F}}_{\mathcal{E}} \sin \mathcal{E} + \tilde{\mathcal{F}}_{\tilde{v}-\mathcal{E}}(\tilde{v} - \mathcal{E}) + \tilde{\mathcal{F}}_{\tilde{v}} \sin \tilde{v}, \quad (4.46)$$

with $\tilde{\mathcal{F}}_{\dots}$ as some 2PN accurate functions of E , L , η , λ_a and χ_a . These functions are lengthy and only temporarily needed in the derivation of later results, so we will not provide them.

Let us now move on to the angular part. As for the time variable, we factorise the polynomial of $d\phi/ds$ with the two roots s_- and s_+ and obtain the elapsed phase at s and the total phase Φ from s_- to s_+ ,

$$\phi - \phi_0 = \int_{s_-}^{s_+} \frac{\mathcal{B}_3(\tau)}{\sqrt{(s_- - \tau)(\tau - s_+)}} d\tau, \quad (4.47)$$

$$\Phi = 2 \int_{s_-}^{s_+} \frac{\mathcal{B}_3(\tau)}{\sqrt{(s_- - \tau)(\tau - s_+)}} d\tau, \quad (4.48)$$

where the function $\mathcal{B}_3(\tau)$ is a polynomial of third order in τ , respectively. Using (4.47) and (4.48), the elapsed phase scaled by the total phase $\frac{2\pi}{\Phi}(\phi - \phi_0)$ in terms of \tilde{v} is computed as

$$\frac{2\pi}{\Phi}(\phi - \phi_0) = \tilde{v} + \tilde{\mathcal{G}}_{\tilde{v}} \sin \tilde{v} + \tilde{\mathcal{G}}_{2\tilde{v}} \sin 2\tilde{v} + \tilde{\mathcal{G}}_{3\tilde{v}} \sin 3\tilde{v}. \quad (4.49)$$

For the following, we change from the auxiliary variable \tilde{v} to the true anomaly due to (4.33) with

$$e_\phi = e_r(1 + \epsilon^2 c_1 + \epsilon^4 c_2), \quad (4.50)$$

differing from the radial eccentricity by some 1PN and 2PN level corrections c_1 and c_2 . These corrections are fixed in such a way that the $\sin v$ contribution in $\frac{2\pi}{\Phi}(\phi - \phi_0)$ vanishes at each order and the lowest formal correction to the phase is shifted to 2PN. Therefore, we eliminate \mathcal{E} in (4.45) with the help of (4.33) and insert the result into (4.49). Requiring the $\sin v$ term

to vanish yields

$$\begin{aligned} \tilde{v} = & v + \epsilon^2 c_1 \frac{e_r}{e_r^2 - 1} \sin v \\ & + \epsilon^4 \left\{ \left(c_2 - c_1^2 \frac{e_r^2}{e_r^2 - 1} \right) \frac{e_r}{e_r^2 - 1} \sin v + \frac{1}{4} c_1^2 \frac{e_r^2}{(e_r^2 - 1)^2} \sin(2v) \right\}, \end{aligned} \quad (4.51)$$

where c_1 and c_2 are at most quadratic functions in δ and depend on the intrinsic quantities of the system. After determining e_ϕ , (4.49) takes the form

$$\frac{2\pi}{\Phi} (\phi - \phi_0) = v + \mathcal{G}_{2v} \sin 2v + \mathcal{G}_{3v} \sin 3v. \quad (4.52)$$

With the help of v , we can re-express the preliminary Kepler equation (4.46) as

$$\mathcal{M} = \mathcal{N}(t - t_0) = \mathcal{E} - e_t \sin \mathcal{E} + \mathcal{F}_{v-\mathcal{E}}(v - \mathcal{E}) + \mathcal{F}_v \sin v. \quad (4.53)$$

Here, e_t is the *time eccentricity* and simply represents the sum of all terms with the factor $\sin \mathcal{E}$ in \mathcal{M} . All the orbital quantities will be detailed in the next section.

Summarising the results

We present all the orbital elements $a_r, e_r, e_t, e_\phi, \mathcal{N}$ and the functions \mathcal{F}_{\dots} and \mathcal{G}_{\dots} of the quasi-Keplerian parameterisation, Equations (4.30) - (4.33), in the following list. For $\delta = 0$ (remember that δ counts the spin order) one recovers the results from, e.g. [98].

$$\begin{aligned} a_r = & \frac{1}{2|E|} + \epsilon^2 \left\{ \frac{\eta - 7}{4} + \frac{\delta}{L} \alpha_{\text{so}} \left[\sqrt{1 - 4\eta} (\chi_1 - \chi_2) + \left(1 - \frac{\eta}{2}\right) (\chi_1 + \chi_2) \right] \right. \\ & + \frac{\delta^2}{L^2} \left[(\chi_1 - \chi_2)^2 \left(\frac{\alpha_{s_1 s_2} \eta}{4} + \alpha_{s^2} \left(\frac{1}{8} \sqrt{1 - 4\eta} (\lambda_1 - \lambda_2) + \frac{1}{8} (1 - 2\eta) (\lambda_1 + \lambda_2) \right) \right) \right. \\ & \quad \left. + (\chi_1 + \chi_2)^2 \left(\alpha_{s^2} \left(\frac{1}{8} \sqrt{1 - 4\eta} (\lambda_1 - \lambda_2) + \frac{1}{8} (1 - 2\eta) (\lambda_1 + \lambda_2) \right) - \frac{\alpha_{s_1 s_2} \eta}{4} \right) \right. \\ & \quad \left. \left. + \alpha_{s^2} (\chi_1 + \chi_2) (\chi_1 - \chi_2) \left(\frac{1}{4} (1 - 2\eta) (\lambda_1 - \lambda_2) + \frac{1}{4} \sqrt{1 - 4\eta} (\lambda_1 + \lambda_2) \right) \right] \right\} \\ & + \epsilon^4 \left\{ \frac{1}{4L^2} (11\eta - 17) + |E| \left(\frac{1}{8} (\eta^2 + 10\eta + 1) \right. \right. \\ & \quad \left. \left. + \frac{\delta}{L} \alpha_{\text{so}} \left[\frac{1}{8} (-6\eta^2 + 19\eta - 8) (\chi_1 + \chi_2) + \frac{1}{8} \sqrt{1 - 4\eta} (5\eta - 8) (\chi_1 - \chi_2) \right] \right) \right. \\ & \quad \left. + \frac{\delta}{L^3} \alpha_{\text{so}} \left[\left(\eta^2 - \frac{39\eta}{4} + 8 \right) (\chi_1 + \chi_2) + \frac{1}{4} (32 - 9\eta) \sqrt{1 - 4\eta} (\chi_1 - \chi_2) \right] \right\}, \end{aligned} \quad (4.54)$$

$$\begin{aligned} \mathcal{N} = & 2\sqrt{2}|E|^{3/2} + \epsilon^2 \frac{|E|^{5/2} (\eta - 15)}{\sqrt{2}} + \epsilon^4 \left\{ \frac{|E|^{7/2}}{8\sqrt{2}} (11\eta^2 + 30\eta + 555) + \frac{4|E|^3}{L} (6\eta - 15) \right. \\ & \left. + \alpha_{\text{so}} \frac{\delta}{L} \left[2(\eta^2 - 8\eta + 6) (\chi_1 + \chi_2) - 4\sqrt{1 - 4\eta} (\eta - 3) (\chi_1 - \chi_2) \right] \right\}, \end{aligned} \quad (4.55)$$

$$e_t^2 = 1 - 2L^2|E| + \epsilon^2 \left\{ |E| (L^2|E|(17 - 7\eta) + 4(\eta - 1)) \right.$$

$$\begin{aligned}
& + \frac{\delta}{L} \alpha_{\text{so}} |E| \left[2(\eta - 2)(\chi_1 + \chi_2) - 4\sqrt{1 - 4\eta}(\chi_1 - \chi_2) \right] \\
& + \frac{\delta^2 |E|}{L^2} \left[(\chi_1 - \chi_2)^2 \left(\alpha_{\text{s}^2} \left(\left(\eta - \frac{1}{2} \right) (\lambda_1 + \lambda_2) - \frac{1}{2} \sqrt{1 - 4\eta} (\lambda_1 - \lambda_2) \right) - \alpha_{\text{s}_1 \text{s}_2} \eta \right) \right. \\
& \quad + (\chi_1 + \chi_2)^2 \left(\alpha_{\text{s}_1 \text{s}_2} \eta + \alpha_{\text{s}^2} \left(\left(\eta - \frac{1}{2} \right) (\lambda_1 + \lambda_2) - \frac{1}{2} \sqrt{1 - 4\eta} (\lambda_1 - \lambda_2) \right) \right) \\
& \quad \left. + \alpha_{\text{s}^2} (\chi_1 + \chi_2) (\chi_1 - \chi_2) \left((2\eta - 1) (\lambda_1 - \lambda_2) - \sqrt{1 - 4\eta} (\lambda_1 + \lambda_2) \right) \right] \Big\} \\
& + \epsilon^4 \left\{ \frac{|E|}{L^2} \left(-11\eta + 17 + L^4 |E|^2 (-16\eta^2 + 47\eta - 112) + 12\sqrt{2} L^3 |E|^{3/2} (5 - 2\eta) \right. \right. \\
& \quad \left. \left. + 2L^2 |E| (5\eta^2 + \eta + 2) + 6\sqrt{2} L \sqrt{|E|} (2\eta - 5) \right) \right. \\
& \quad + \frac{\delta}{L} \alpha_{\text{so}} \frac{|E|}{2L^2} \left[(\chi_1 + \chi_2) \left(-16\sqrt{2} L^3 |E|^{3/2} (\eta^2 - 8\eta + 6) \right. \right. \\
& \quad \left. \left. + L^2 |E| (32\eta^2 - 159\eta + 124) \right. \right. \\
& \quad \left. \left. + 8\sqrt{2} \sqrt{|E|} L (\eta^2 - 8\eta + 6) - 8\eta^2 + 78\eta - 64 \right) \right. \\
& \quad \left. + \sqrt{1 - 4\eta} (\chi_1 - \chi_2) \left(32\sqrt{2} L^3 |E|^{3/2} (\eta - 3) + L^2 |E| (124 - 59\eta) \right. \right. \\
& \quad \left. \left. - 16\sqrt{2} L \sqrt{|E|} (\eta - 3) + 18\eta - 64 \right) \right] \Big\}, \tag{4.56}
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_{v-\mathcal{E}} & = -\epsilon^4 \frac{2\sqrt{2} |E|^{3/2}}{L} \left\{ 3 \left(\eta - \frac{5}{2} \right) \right. \\
& \quad \left. + \frac{\delta}{L} \alpha_{\text{so}} \left[(\eta^2 - 8\eta + 6) (\chi_1 + \chi_2) - 2\sqrt{1 - 4\eta} (\eta - 3) (\chi_1 - \chi_2) \right] \right\}, \tag{4.57}
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_v & = \epsilon^4 \frac{|E|^{3/2}}{2\sqrt{2} L} \sqrt{1 - 2L^2 |E|} \left\{ -\eta(\eta + 4) \right. \\
& \quad \left. - \frac{\delta}{L} \alpha_{\text{so}} \left[\sqrt{1 - 4\eta} (\eta + 8) (\chi_1 - \chi_2) - (13\eta - 8) (\chi_1 + \chi_2) \right] \right\}, \tag{4.58}
\end{aligned}$$

$$\begin{aligned}
\mathcal{G}_{2v} & = \epsilon^4 \frac{(2L^2 |E| - 1)}{4L^4} \left\{ \frac{\eta(3\eta - 1)}{2} \right. \\
& \quad \left. + 3 \frac{\delta}{L} \alpha_{\text{so}} \left[\sqrt{1 - 4\eta} \eta (\chi_1 - \chi_2) - (\eta - 1) \eta (\chi_1 + \chi_2) \right] \right\}, \tag{4.59}
\end{aligned}$$

$$\begin{aligned}
\mathcal{G}_{3v} & = \epsilon^4 \frac{(1 - 2L^2 |E|)^{3/2}}{8L^4} \left\{ -\frac{3\eta^2}{4} \right. \\
& \quad \left. + \frac{\delta}{L} \alpha_{\text{so}} \left[(\eta - 1) \eta (\chi_1 + \chi_2) - \sqrt{1 - 4\eta} \eta (\chi_1 - \chi_2) \right] \right\}, \tag{4.60}
\end{aligned}$$

$$\begin{aligned}
\frac{\mathcal{P}}{2\pi} & = \frac{1}{2\sqrt{2} |E|^{3/2}} \left\{ 1 + \frac{1}{4} \epsilon^2 (15 - \eta) |E| + \epsilon^4 \left[-\frac{3}{32} |E|^2 (3\eta^2 + 30\eta + 35) \right. \right. \\
& \quad \left. \left. + \frac{|E|^{3/2}}{L} \left(\frac{\delta}{L} \alpha_{\text{so}} \left[4\sqrt{2 - 8\eta} (\eta - 3) (\chi_1 - \chi_2) - 2\sqrt{2} (\eta^2 - 8\eta + 6) (\chi_1 + \chi_2) \right] \right. \right. \right. \\
& \quad \left. \left. \left. + 3\sqrt{2} (5 - 2\eta) \right) \right] \right\}. \tag{4.61}
\end{aligned}$$

The ratios of the other eccentricities with respect to e_t read

$$\begin{aligned}
\frac{e_r}{e_t} = & 1 + \epsilon^2 \left\{ (8 - 3\eta)|E| \right. \\
& + \alpha_{\text{so}} \frac{\delta|E|}{L} \left[(\eta - 2)(\chi_1 + \chi_2) - 2\sqrt{1 - 4\eta}(\chi_1 - \chi_2) \right] \\
& + \frac{\delta^2|E|}{L^2} \left[(\chi_1 - \chi_2)^2 \times \right. \\
& \left. \left(\alpha_{\text{s}^2} \left(\frac{1}{4}(2\eta - 1)(\lambda_1 + \lambda_2) - \frac{1}{4}\sqrt{1 - 4\eta}(\lambda_1 - \lambda_2) \right) - \frac{\alpha_{\text{s}_1\text{s}_2}\eta}{2} \right) \right. \\
& + (\chi_1 + \chi_2)^2 \times \\
& \left. \left(\frac{\alpha_{\text{s}_1\text{s}_2}\eta}{2} + \alpha_{\text{s}^2} \left(\frac{1}{4}(2\eta - 1)(\lambda_1 + \lambda_2) - \frac{1}{4}\sqrt{1 - 4\eta}(\lambda_1 - \lambda_2) \right) \right) \right. \\
& + \alpha_{\text{s}^2}(\chi_1 + \chi_2)(\chi_1 - \chi_2) \times \\
& \left. \left. \left. \left(\left(\eta - \frac{1}{2} \right) (\lambda_1 - \lambda_2) - \frac{1}{2}\sqrt{1 - 4\eta}(\lambda_1 + \lambda_2) \right) \right) \right] \right\} \\
& + \epsilon^4 \left\{ \frac{|E|}{2L^2} \left[L^2|E|(6\eta^2 - 63\eta + 56) - 6\sqrt{2}L\sqrt{|E|}(2\eta - 5) - 11\eta + 17 \right] \right. \\
& + \alpha_{\text{so}} \frac{\delta|E|}{L4L^2} \left[\sqrt{1 - 4\eta}(\chi_1 - \chi_2) \times \right. \\
& \left. \left(L^2|E|(23\eta - 84) + 16\sqrt{2}L\sqrt{|E|}(\eta - 3) + 18\eta - 64 \right) \right. \\
& - (\chi_1 + \chi_2) \left(L^2|E|(8\eta^2 - 55\eta + 84) + 8\sqrt{2}L\sqrt{|E|}(\eta^2 - 8\eta + 6) \right. \\
& \left. \left. \left. \left. + 8\eta^2 - 78\eta + 64 \right) \right] \right\}, \tag{4.62}
\end{aligned}$$

$$\begin{aligned}
\frac{e_\phi}{e_t} = & 1 + \epsilon^2 \left\{ -2(\eta - 4)|E| + \alpha_{\text{so}} \frac{\delta|E|}{L} \left[(\eta - 2)(\chi_1 + \chi_2) - 2\sqrt{1 - 4\eta}(\chi_1 - \chi_2) \right] \right. \\
& + \frac{\delta^2|E|}{L^2} \left[(\chi_1 - \chi_2)^2 \left(\alpha_{\text{s}^2} \left(\left(\eta - \frac{1}{2} \right) (\lambda_1 + \lambda_2) - \frac{1}{2}\sqrt{1 - 4\eta}(\lambda_1 - \lambda_2) \right) - \alpha_{\text{s}_1\text{s}_2}\eta \right) \right. \\
& \quad \left. + (\chi_1 + \chi_2)^2 \left(\alpha_{\text{s}^2} \left(\left(\eta - \frac{1}{2} \right) (\lambda_1 + \lambda_2) - \frac{1}{2}\sqrt{1 - 4\eta}(\lambda_1 - \lambda_2) \right) + \alpha_{\text{s}_1\text{s}_2}\eta \right) \right. \\
& \left. + \alpha_{\text{s}^2}(\chi_1 + \chi_2)(\chi_1 - \chi_2) \left((2\eta - 1)(\lambda_1 - \lambda_2) - \sqrt{1 - 4\eta}(\lambda_1 + \lambda_2) \right) \right] \right\} \\
& + \epsilon^4 \left\{ \frac{|E|}{16L^2} \left[2L^2|E|(11\eta^2 - 168\eta + 224) \right. \right. \\
& - 48\sqrt{2}L\sqrt{|E|}(2\eta - 5) - 15\eta^2 - 144\eta + 272 \left. \right] + \alpha_{\text{so}} \frac{\delta|E|}{L4L^2} \left[(\chi_1 + \chi_2) \times \right. \\
& \left. \left(-3L^2|E|(2\eta^2 - 15\eta + 28) - 8\sqrt{2}L\sqrt{|E|}(\eta^2 - 8\eta + 6) + 5\eta^2 + 135\eta - 128 \right) \right. \\
& \left. \left. \left. \left. + \sqrt{1 - 4\eta}(\chi_1 - \chi_2) \left(L^2|E|(13\eta - 84) + 16\sqrt{2}L\sqrt{|E|}(\eta - 3) + 15\eta - 128 \right) \right] \right\}. \tag{4.63}
\end{aligned}$$

For the case that one chooses one of the above other eccentricities as the intrinsic parameter to be searched for in the data analysis investigations, these equations can be inverted

perturbatively.

4.4 Gravitational wave forms

The final form of the GW model will rely on the expressions of the far-zone metric as given in Section 2.4, which will naturally depend on general kinematic quantities describing the binary system. Subsequently, we will compute the GW amplitudes h_{\times} and h_{+} , as will require the associated corrections to the desired order in harmonic coordinates from the literature, see below for references. As well, we will use coordinate transformations from ADM to harmonic coordinates to be able to apply the time evolution of the orbital elements in the previous sections, which we have computed in ADM coordinates only.

Post Newtonian expansion of the gravitational radiation amplitudes

The transverse-traceless (TT) projection of the radiation field and thus h_{\times} and h_{+} , the two polarisations, strongly depend on the observer's position relative to the source. We remember Equations (2.33) and start the calculation by defining an invariant Lorentz frame $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ in which we like to express all other quantities.⁸ The observer frame $(\mathbf{p}, \mathbf{q}, \mathbf{N})$ can be constructed from $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ by a constant rotation around \mathbf{e}_x ,

$$\begin{pmatrix} \mathbf{e}_x \\ \mathbf{e}_y \\ \mathbf{e}_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i_0 & \sin i_0 \\ 0 & -\sin i_0 & \cos i_0 \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \\ \mathbf{N} \end{pmatrix}. \quad (4.64)$$

Figure 1 shows shows a representation of what has been done.⁹ Next, we express the radial separation \mathbf{r} in the orbital plane $(\mathbf{e}_x, \mathbf{e}_y)$ and perform the rotation Equation (4.64) to move to the observer's triad and calculate \mathbf{r} and \mathbf{v} ,

$$\mathbf{r} = r (\mathbf{p} \cos \phi + \mathbf{q} \cos i_0 \sin \phi + \mathbf{N} \sin i_0 \sin \phi), \quad (4.65)$$

$$\begin{aligned} \mathbf{v} = & \mathbf{p} \left(\dot{r} \cos \phi - r \dot{\phi} \sin \phi \right) + \mathbf{q} \left(r \dot{\phi} \cos i_0 \cos \phi + \dot{r} \cos i_0 \sin \phi \right) \\ & + \mathbf{N} \left(r \dot{\phi} \sin i_0 \cos \phi + \dot{r} \sin i_0 \sin \phi \right). \end{aligned} \quad (4.66)$$

This provides the orbital contributions to the field. To compute the spin contributions, we also expand the spins in the orbital triad,

$$\mathbf{S}_1 = \chi_1 \mathbf{e}_z = \chi_1 (\mathbf{N} \cos i_0 - \mathbf{q} \sin i_0), \quad (4.67)$$

$$\mathbf{S}_2 = \chi_2 \mathbf{e}_z = \chi_2 (\mathbf{N} \cos i_0 - \mathbf{q} \sin i_0). \quad (4.68)$$

⁸It is strongly desired to do this, because the far-zone field is computed there by construction. Referencing to rotating frames flaw the character and may introduce artifact modulations, e.g. when the spins are not aligned.

⁹In Reference [105] the caption for Figure 2 should be made precise. The plane of the sky meets the orbital plane at \mathbf{e}_x for $\Upsilon = 0$ only. Generally, at $\mathbf{e}_x = \mathbf{p}$ the plane of the sky meets the invariable plane.

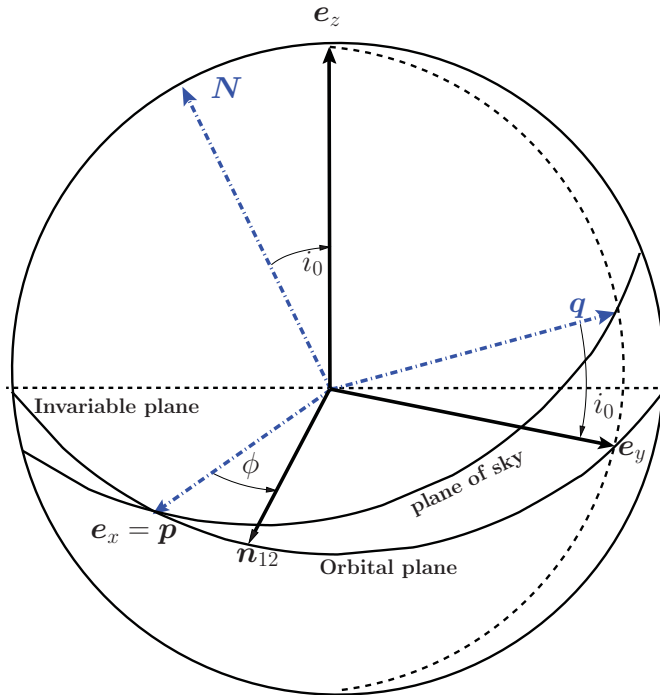


Figure 1: The geometry of the binary. We have added the observer related frame $(\mathbf{p}, \mathbf{q}, \mathbf{N})$ (in dashed and dotted lines) with \mathbf{N} as the line-of-sight vector. \mathbf{N} points from the origin of the invariable frame $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ to the observer. Note that the orbital angular momentum \mathbf{L} lies on the \mathbf{e}_z axis and so do the spins. \mathbf{N} is chosen to lie in the \mathbf{e}_y - \mathbf{e}_z -plane, and measures a constant angle i_0 (associated with the rotation around \mathbf{e}_x) from \mathbf{e}_z , such that $\mathbf{p} = \mathbf{e}_x$, and this is the line where the orbital plane meets the plane of the sky. The angle i_0 is also found to be the angle enclosed by the vector \mathbf{q} [positioned in the $(\mathbf{e}_y, \mathbf{e}_z)$ plane] and \mathbf{e}_y .

To finally obtain h_\times and h_+ , we define

$$\mathfrak{P}_{ij}^{(\times)} \equiv \frac{1}{2}(q_i p_j + p_i q_j), \quad (4.69)$$

$$\mathfrak{P}_{ij}^{(+)} \equiv \frac{1}{2}(p_i p_j - q_i q_j), \quad (4.70)$$

as parts of the projectors in (2.33), which are unaffected by the TT projection operator and which extract both polarisations when contracted with h_{ij} .

The above calculations enable us to compute all the considered contributions to the radiation field polarisations. Following References [20] and [106], we list the lowest order contributions to the gravitational wave form in harmonic coordinates. These are the PP contributions to 2PN, including the NLO-SO and LO-S₁S₂ terms. We also add the terms emerging from

the gauge transformation from ADM to harmonic coordinates,

$$\begin{aligned}
h_{ij}^{\text{TT}} = & \frac{2\eta}{R'} \left[\xi_{ij}^{(0)\text{PP}} + \epsilon \xi_{ij}^{(0.5)\text{PP}} + \epsilon^2 \xi_{ij}^{(1)\text{PP}} + \epsilon^3 \xi_{ij}^{(1.5)\text{PP}} + \epsilon^4 \xi_{ij}^{(2)\text{PP}} \right. \\
& + \epsilon^2 \delta \alpha_{\text{so}} \xi_{ij}^{(1)\text{SO}} + \epsilon^3 \delta \alpha_{\text{so}} \xi_{ij}^{(1.5)\text{SO}} + \epsilon^2 \delta^2 \alpha_{\text{s1s2}} \xi_{ij}^{(1)\text{S1S2}} \\
& \left. + \epsilon^2 \delta \alpha_{\text{so}} \xi_{ij}^{(0+1)\text{PP+SSC}} + \epsilon^3 \delta \alpha_{\text{so}} \xi_{ij}^{(0.5+1)\text{PP+SSC}} + \epsilon^4 \xi_{ij}^{(0+2)\text{PP+g}} \right]. \quad (4.71)
\end{aligned}$$

The terms in the last line of the above equation, labeled “PP+SSC”, denote those coming from corrections responsible for switching the SSC, and “PP+g” denote corrections coming from the gauge transformation from ADM to harmonic coordinates to the desired order [82, 107]. Appendix A.3 gives deeper information about how velocities, distances and normal vectors change within this transformation. We find it convenient to give a hint to their origin by putting the GW multipole order and the order/type of the correction in the label, for example “(0+1)PP + SSC” is the first Taylor correction of the “Newtonian” (PP) quadrupole moment where the coordinates are shifted by a 1PN SSC transformation term (linear in spin).

According to Equation (4.69) and Equation (4.70), one can define the projected components of the ξ via

$$\xi_{\times}^{(\text{order}) \text{ type}} = \mathfrak{P}_{ij}^{(\times)} \xi_{ij}^{(\text{order}) \text{ type}}, \quad (4.72)$$

$$\xi_{+}^{(\text{order}) \text{ type}} = \mathfrak{P}_{ij}^{(+)} \xi_{ij}^{(\text{order}) \text{ type}}, \quad (4.73)$$

where the “cross” and “plus” polarisations read

$$\xi_{\times,+}^{(0)\text{PP}} = 2 \left\{ \mathfrak{P}_{vv}^{(\times,+)} - \frac{1}{r} \mathfrak{P}_{nn}^{(\times,+)} \right\}, \quad (4.74)$$

$$\xi_{\times,+}^{(0.5)\text{PP}} = \frac{\delta m}{m} \left\{ 3 (\mathbf{N} \cdot \mathbf{n}_{12}) \frac{1}{r} [2\mathfrak{P}_{nv}^{(\times,+)} - \dot{r} \mathfrak{P}_{nn}^{(\times,+)}] + (\mathbf{N} \cdot \mathbf{v}) \left[\frac{1}{r} \mathfrak{P}_{nn}^{(\times,+)} - 2\mathfrak{P}_{vv}^{(\times,+)} \right] \right\}, \quad (4.75)$$

$$\begin{aligned}
\xi_{\times,+}^{(1)\text{PP}} = & \frac{1}{3} \left\{ (1 - 3\eta) \left[(\mathbf{N} \cdot \mathbf{n}_{12})^2 \frac{1}{r} \left(\left(3\mathbf{v}^2 - 15\dot{r}^2 + 7\frac{1}{r} \right) \mathfrak{P}_{nn}^{(\times,+)} + 30\dot{r} \mathfrak{P}_{nv}^{(\times,+)} \right. \right. \right. \\
& \left. \left. \left. - 14\mathfrak{P}_{vv}^{(\times,+)} \right) + (\mathbf{N} \cdot \mathbf{n}_{12}) (\mathbf{N} \cdot \mathbf{v}) \frac{1}{r} [12\dot{r} \mathfrak{P}_{nn}^{(\times,+)} - 32\mathfrak{P}_{nv}^{(\times,+)}] \right. \right. \\
& \left. \left. + (\mathbf{N} \cdot \mathbf{v})^2 \left[6\mathfrak{P}_{vv}^{(\times,+)} - 2\frac{1}{r} \mathfrak{P}_{nn}^{(\times,+)} \right] \right] + \left[3(1 - 3\eta)\mathbf{v}^2 - 2(2 - 3\eta)\frac{1}{r} \right] \mathfrak{P}_{vv}^{(\times,+)} \right. \\
& \left. + 4\frac{1}{r} \dot{r} (5 + 3\eta) \mathfrak{P}_{nv}^{(\times,+)} + \frac{1}{r} \left[3(1 - 3\eta)\dot{r}^2 - (10 + 3\eta)\mathbf{v}^2 + 29\frac{1}{r} \right] \mathfrak{P}_{nn}^{(\times,+)} \right\}, \quad (4.76)
\end{aligned}$$

$$\begin{aligned}
\xi_{\times,+}^{(1.5)\text{PP}} = & \frac{\delta m}{m} \left\{ \frac{1}{12} (1 - 2\eta) \left\{ (\mathbf{N} \cdot \mathbf{n}_{12})^3 \frac{1}{r} \left[\left(45\mathbf{v}^2 - 105\dot{r}^2 + 90\frac{1}{r} \right) \dot{r} \mathfrak{P}_{nn}^{(\times,+)} \right. \right. \right. \\
& \left. \left. \left. - 96\dot{r} \mathfrak{P}_{vv}^{(\times,+)} - \left(42\mathbf{v}^2 - 210\dot{r}^2 + 88\frac{1}{r} \right) \mathfrak{P}_{nv}^{(\times,+)} \right] \right. \right. \\
& \left. \left. - (\mathbf{N} \cdot \mathbf{n}_{12})^2 (\mathbf{N} \cdot \mathbf{v}) \frac{1}{r} \left[\left(27\mathbf{v}^2 - 135\dot{r}^2 + 84\frac{1}{r} \right) \mathfrak{P}_{nn}^{(\times,+)} \right. \right. \right.
\end{aligned}$$

$$\begin{aligned}
& +336\dot{\mathfrak{P}}_{nv}^{(\times,+)} - 172\mathfrak{P}_{vv}^{(\times,+)} \Big] \\
& - (\mathbf{N} \cdot \mathbf{n}_{12}) (\mathbf{N} \cdot \mathbf{v})^2 \frac{1}{r} \left[48\dot{\mathfrak{P}}_{nn}^{(\times,+)} - 184\mathfrak{P}_{nv}^{(\times,+)} \right] \\
& + (\mathbf{N} \cdot \mathbf{v})^3 \left[4\frac{1}{r}\mathfrak{P}_{nn}^{(\times,+)} - 24\mathfrak{P}_{vv}^{(\times,+)} \right] \Big\} \\
& - \frac{1}{12} (\mathbf{N} \cdot \mathbf{n}_{12}) \frac{1}{r} \left\{ \left[(69 - 66\eta)\mathbf{v}^2 - (15 - 90\eta)\dot{r}^2 - (242 - 24\eta)\frac{1}{r} \right] \dot{r}\mathfrak{P}_{nn}^{(\times,+)} \right. \\
& - \left[(66 - 36\eta)\mathbf{v}^2 + (138 + 84\eta)\dot{r}^2 \right. \\
& \left. - (256 - 72\eta)\frac{1}{r}\mathfrak{P}_{nv}^{(\times,+)} + (192 + 12\eta)\dot{r}\mathfrak{P}_{vv}^{(\times,+)} \right\} \\
& + \frac{1}{12} (\mathbf{N} \cdot \mathbf{v}) \left\{ \left[(23 - 10\eta)\mathbf{v}^2 - (9 - 18\eta)\dot{r}^2 - (104 - 12\eta)\frac{1}{r} \right] \frac{1}{r}\mathfrak{P}_{nn}^{(\times,+)} \right. \\
& \left. - (88 + 40\eta) \frac{1}{r} \dot{r}\mathfrak{P}_{nv}^{(\times,+)} - \left[(12 - 60\eta)\mathbf{v}^2 - (20 - 52\eta)\frac{1}{r} \right] \mathfrak{P}_{vv}^{(\times,+)} \right\} \Big\}, \quad (4.77)
\end{aligned}$$

$$\begin{aligned}
\xi_{\times,+}^{(2)\text{PP}} &= \frac{1}{120} (1 - 5\eta + 5\eta^2) \left\{ 240 (\mathbf{N} \cdot \mathbf{v})^4 \mathfrak{P}_{vv}^{(\times,+)} - (\mathbf{N} \cdot \mathbf{n}_{12})^4 \right. \\
& \frac{1}{r} \left[\left(90(\mathbf{v}^2)^2 + (318\frac{1}{r} - 1260\dot{r}^2)\mathbf{v}^2 + 344\frac{1}{r^2} + 1890\dot{r}^4 - 2310\frac{1}{r}\dot{r}^2 \right) \mathfrak{P}_{nn}^{(\times,+)} \right. \\
& \left. + \left(1620\mathbf{v}^2 + 3000\frac{1}{r} - 3780\dot{r}^2 \right) \dot{r}\mathfrak{P}_{nv}^{(\times,+)} \right. \\
& \left. - \left(336\mathbf{v}^2 - 1680\dot{r}^2 + 688\frac{1}{r} \right) \mathfrak{P}_{vv}^{(\times,+)} \right] \\
& - (\mathbf{N} \cdot \mathbf{n}_{12})^3 (\mathbf{N} \cdot \mathbf{v}) \frac{1}{r} \left[\left(1440\mathbf{v}^2 - 3360\dot{r}^2 + 3600\frac{1}{r} \right) \dot{r}\mathfrak{P}_{nn}^{(\times,+)} \right. \\
& \left. - \left(1608\mathbf{v}^2 - 8040\dot{r}^2 + 3864\frac{1}{r} \right) \mathfrak{P}_{nv}^{(\times,+)} - 3960\dot{r}\mathfrak{P}_{vv}^{(\times,+)} \right] \\
& + 120 (\mathbf{N} \cdot \mathbf{v})^3 (\mathbf{N} \cdot \mathbf{n}_{12}) \frac{1}{r} \left(3\dot{r}\mathfrak{P}_{nn}^{(\times,+)} - 20\mathfrak{P}_{nv}^{(\times,+)} \right) \\
& + (\mathbf{N} \cdot \mathbf{n}_{12})^2 (\mathbf{N} \cdot \mathbf{v})^2 \frac{1}{r} \left[\left(396\mathbf{v}^2 - 1980\dot{r}^2 + 1668\frac{1}{r} \right) \mathfrak{P}_{nn}^{(\times,+)} \right. \\
& \left. + 6480\dot{r}\mathfrak{P}_{nv}^{(\times,+)} - 3600\mathfrak{P}_{vv}^{(\times,+)} \right] \Big\} - \frac{1}{30} (\mathbf{N} \cdot \mathbf{v})^2 \left\{ \left[(87 - 315\eta + 145\eta^2)\mathbf{v}^2 \right. \right. \\
& - (135 - 465\eta + 75\eta^2)\dot{r}^2 \\
& - (289 - 905\eta + 115\eta^2)\frac{1}{r} \Big] \frac{1}{r} \mathfrak{P}_{nn}^{(\times,+)} \\
& - \left(240 - 660\eta - 240\eta^2 \right) \dot{r}\mathfrak{P}_{nv}^{(\times,+)} \\
& \left. - \left[(30 - 270\eta + 630\eta^2)\mathbf{v}^2 - 60(1 - 6\eta + 10\eta^2)\frac{1}{r} \right] \mathfrak{P}_{vv}^{(\times,+)} \right\} \\
& + \frac{1}{30} (\mathbf{N} \cdot \mathbf{n}_{12}) (\mathbf{N} \cdot \mathbf{v}) \frac{1}{r} \left\{ \left[(270 - 1140\eta + 1170\eta^2)\mathbf{v}^2 \right. \right.
\end{aligned}$$

$$\begin{aligned}
& -\left[(60 - 450\eta + 900\eta^2)\dot{r}^2 - (1270 - 3920\eta + 360\eta^2)\frac{1}{r} \right] \dot{r} \mathfrak{P}_{nn}^{(\times,+)} \\
& - \left[(186 - 810\eta + 1450\eta^2)\mathbf{v}^2 + (990 - 2910\eta - 930\eta^2)\dot{r}^2 \right. \\
& - (1242 - 4170\eta + 1930\eta^2)\frac{1}{r} \left. \mathfrak{P}_{nv}^{(\times,+)} \right. \\
& + \left. \left[1230 - 3810\eta - 90\eta^2 \right] \dot{r} \mathfrak{P}_{vv}^{(\times,+)} \right\} \\
& + \frac{1}{60} (\mathbf{N} \cdot \mathbf{n}_{12})^2 \frac{1}{r} \left\{ \left[(117 - 480\eta + 540\eta^2)(\mathbf{v}^2)^2 \right. \right. \\
& - (630 - 2850\eta + 4050\eta^2)\mathbf{v}^2\dot{r}^2 - (125 - 740\eta + 900\eta^2)\frac{1}{r} \mathbf{v}^2 \\
& + (105 - 1050\eta + 3150\eta^2)\dot{r}^4 + (2715 - 8580\eta + 1260\eta^2)\frac{1}{r} \dot{r}^2 \\
& - (1048 - 3120\eta + 240\eta^2)\frac{1}{r^2} \left. \mathfrak{P}_{nn}^{(\times,+)} \right. \\
& + \left. \left[(216 - 1380\eta + 4320\eta^2)\mathbf{v}^2 + (1260 - 3300\eta - 3600\eta^2)\dot{r}^2 \right. \right. \\
& - (3952 - 12860\eta + 3660\eta^2)\frac{1}{r} \left. \dot{r} \mathfrak{P}_{nv}^{(\times,+)} \right. \\
& - \left. \left[(12 - 180\eta + 1160\eta^2)\mathbf{v}^2 + (1260 - 3840\eta - 780\eta^2)\dot{r}^2 \right. \right. \\
& - (664 - 2360\eta + 1700\eta^2)\frac{1}{r} \left. \mathfrak{P}_{vv}^{(\times,+)} \right\} \\
& - \frac{1}{60} \left\{ \left[(66 - 15\eta - 125\eta^2)(\mathbf{v}^2)^2 \right. \right. \\
& + (90 - 180\eta - 480\eta^2)\mathbf{v}^2\dot{r}^2 - (389 + 1030\eta - 110\eta^2)\frac{1}{r} \mathbf{v}^2 \\
& + (45 - 225\eta + 225\eta^2)\dot{r}^4 + (915 - 1440\eta + 720\eta^2)\frac{1}{r} \dot{r}^2 \\
& + (1284 + 1090\eta)\frac{1}{r^2} \left. \mathfrak{P}_{nn}^{(\times,+)} \right. \\
& - \left. \left[(132 + 540\eta - 580\eta^2)\mathbf{v}^2 + (300 - 1140\eta + 300\eta^2)\dot{r}^2 \right. \right. \\
& + (856 + 400\eta + 700\eta^2)\frac{1}{r} \left. \frac{1}{r} \dot{r} \mathfrak{P}_{nv}^{(\times,+)} \right. \\
& - \left. \left[(45 - 315\eta + 585\eta^2)(\mathbf{v}^2)^2 + (354 - 210\eta - 550\eta^2)\frac{1}{r} \mathbf{v}^2 \right. \right. \\
& - (270 - 30\eta + 270\eta^2)\frac{1}{r} \dot{r}^2 \\
& - (638 + 1400\eta - 130\eta^2)\frac{1}{r^2} \left. \mathfrak{P}_{vv}^{(\times,+)} \right\}, \tag{4.78}
\end{aligned}$$

$$\xi_{\times,+}^{(1)\text{SO}} = -\frac{1}{r^2} \left\{ \left[\mathfrak{P}_{ij}^{(\times,+)} (\boldsymbol{\Delta} \times \mathbf{N})^i n_{12}^j \right] + \sqrt{1-4\eta} \left[\mathfrak{P}_{ij}^{(\times,+)} (\mathbf{S} \times \mathbf{N})^i n_{12}^j \right] \right\}, \quad (4.79)$$

$$\begin{aligned} \xi_{\times,+}^{(1.5)\text{SO}} = & \frac{1}{r^2} \left\{ \sqrt{1-4\eta} \left[6\mathfrak{P}_{nn}^{(\times,+)} [\mathbf{v} \cdot (\boldsymbol{\Delta} \times \mathbf{n}_{12})] - 6\dot{r} \left[\mathfrak{P}_{ij}^{(\times,+)} (\boldsymbol{\Delta} \times \mathbf{n}_{12})^i n_{12}^j \right] \right. \right. \\ & + 4 \left[\mathfrak{P}_{ij}^{(\times,+)} (\boldsymbol{\Delta} \times \mathbf{v})^i n_{12}^j \right] \left. \right] + 6\mathfrak{P}_{nn}^{(\times,+)} [\mathbf{v} \cdot (\mathbf{S} \times \mathbf{n}_{12})] \\ & + \eta \left[\mathfrak{P}_{ij}^{(\times,+)} (\mathbf{S} \times \mathbf{N})^i n_{12}^j \right] (6\dot{r}(\mathbf{N} \cdot \mathbf{n}_{12}) - 4(\mathbf{N} \cdot \mathbf{v})) \\ & - 6\dot{r} \left[\mathfrak{P}_{ij}^{(\times,+)} (\mathbf{S} \times \mathbf{n}_{12})^i n_{12}^j \right] \\ & + \eta \left(4 \left[\mathfrak{P}_{ij}^{(\times,+)} (\mathbf{S} \times \mathbf{n}_{12})^i v^j \right] - 4(\mathbf{N} \cdot \mathbf{n}_{12}) \left[\mathfrak{P}_{ij}^{(\times,+)} (\mathbf{S} \times \mathbf{N})^i v^j \right] \right) \\ & \left. + (2\eta + 4) \left[\mathfrak{P}_{ij}^{(\times,+)} (\mathbf{S} \times \mathbf{v})^i n_{12}^j \right] \right\}, \quad (4.80) \end{aligned}$$

$$\xi_{\times}^{(2)\text{S}_1\text{S}_2} = -\frac{3\eta}{r^3} \chi_1 \chi_2 \cos(i_0) \sin(2\phi), \quad (4.81)$$

$$\xi_{+}^{(2)\text{S}_1\text{S}_2} = -\frac{3\eta}{4r^3} \chi_1 \chi_2 (\cos(2i_0) \cos(2\phi) + 2 \sin^2(i_0) + 3 \cos(2\phi)). \quad (4.82)$$

The remaining contributions are the terms induced by the gauge transformation. Explicitly,

they read

$$\begin{aligned} \xi_{\times,+}^{(0+1)\text{PP+SSC}} = & -\frac{\eta}{r^2} \left\{ 3\mathfrak{P}_{nn}^{(\times,+)} [\mathbf{v} \cdot (\mathbf{S} \times \mathbf{n}_{12})] + 2 \left[\mathfrak{P}_{ij}^{(\times,+)} (\mathbf{S} \times \mathbf{v})^i n_{12}^j \right] \right. \\ & \left. + 2 \left[\mathfrak{P}_{ij}^{(\times,+)} (\mathbf{S} \times \mathbf{n}_{12})^i v^j \right] \right\}, \quad (4.83) \end{aligned}$$

$$\begin{aligned} \xi_{\times,+}^{(0.5+1)\text{PP+SSC}} = & \frac{\delta m}{m} \frac{\eta}{2r^2} \left\{ \mathfrak{P}_{nn}^{(\times,+)} \left[-15\dot{r}(\mathbf{N} \cdot \mathbf{n}_{12}) [\mathbf{v} \cdot (\mathbf{S} \times \mathbf{n}_{12})] - 3\dot{r}[\mathbf{N} \cdot (\mathbf{S} \times \mathbf{v})] \right. \right. \\ & + 3(\mathbf{N} \cdot \mathbf{v}) [\mathbf{v} \cdot (\mathbf{S} \times \mathbf{n}_{12})] - \frac{1}{r} [\mathbf{N} \cdot (\mathbf{S} \times \mathbf{n}_{12})] \left. \right] \\ & + (\mathbf{N} \cdot \mathbf{n}_{12}) \left[-6\dot{r} \left[\mathfrak{P}_{ij}^{(\times,+)} (\mathbf{S} \times \mathbf{v})^i n_{12}^j \right] + 18\mathfrak{P}_{nv}^{(\times,+)} [\mathbf{v} \cdot (\mathbf{S} \times \mathbf{n}_{12})] \right. \\ & \left. \left. - \frac{6}{r} \left[\mathfrak{P}_{ij}^{(\times,+)} (\mathbf{S} \times \mathbf{n}_{12})^i n_{12}^j \right] + 6 \left[\mathfrak{P}_{ij}^{(\times,+)} (\mathbf{S} \times \mathbf{v})^i v^j \right] \right] \right\} \end{aligned}$$

$$\begin{aligned}
& +(\mathbf{N} \cdot \mathbf{v}) \left[2 \left[\mathfrak{P}_{ij}^{(\times,+)} (\mathbf{S} \times \mathbf{v})^i n_{12}^j \right] + 4 \left[\mathfrak{P}_{ij}^{(\times,+)} (\mathbf{S} \times \mathbf{n}_{12})^i v^j \right] \right] \\
& + 6 \mathfrak{P}_{nv}^{(\times,+)} [\mathbf{N} \cdot (\mathbf{S} \times \mathbf{v})] + 2 \mathfrak{P}_{vv}^{(\times,+)} [\mathbf{N} \cdot (\mathbf{S} \times \mathbf{n}_{12})] \Big\}, \tag{4.84}
\end{aligned}$$

$$\begin{aligned}
\xi_{\times,+}^{(0+2) \text{ PP+g}} &= \frac{1}{r} \left\{ \mathfrak{P}_{nv}^{(\times,+)} \dot{r} \left[\frac{1}{2} \eta (3\dot{r}^2 - 7\mathbf{v}^2) - \frac{2(5\eta - 1)}{r} \right] \right. \\
& + \mathfrak{P}_{nn}^{(\times,+)} \left[\frac{5\eta (\mathbf{v}^2 - 11\dot{r}^2)}{4r} + \frac{12\eta + 1}{2r^2} \right] \\
& \left. + \mathfrak{P}_{vv}^{(\times,+)} \left[\frac{1}{2} \eta (17\dot{r}^2 - 13\mathbf{v}^2) + \frac{21\eta + 1}{r} \right] \right\}. \tag{4.85}
\end{aligned}$$

Equation (4.85) shows total agreement with the transformation term in Equation (A2) of [26]. The next block of equations evaluates the scalar products of vectors and projectors containing the spins. First, we list those with the total spin $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$. For those terms with $\mathbf{\Delta} = \mathbf{S}_1 - \mathbf{S}_2$ instead of \mathbf{S} , simply replace $(\mathbf{S} \rightarrow \mathbf{\Delta})$ on the left hand side and $(\chi_1 + \chi_2) \rightarrow (\chi_1 - \chi_2)$ on the right. The used abbreviations are given by

$$[\mathbf{S} \cdot (\mathbf{n}_{12} \times \mathbf{v})] = \dot{\phi} r (\chi_1 + \chi_2), \tag{4.86}$$

$$[\mathbf{N} \cdot (\mathbf{S} \times \mathbf{n}_{12})] = (\chi_1 + \chi_2) \sin(i_0) \cos(\phi), \tag{4.87}$$

$$[\mathbf{N} \cdot (\mathbf{S} \times \mathbf{v})] = (\chi_1 + \chi_2) \sin(i_0) (\dot{r} \cos(\phi) - \dot{\phi} r \sin(\phi)), \tag{4.88}$$

$$\mathfrak{P}_{ij}^{(\times)} v^j (\mathbf{S} \times \mathbf{n}_{12})^i = \frac{1}{2} (\chi_1 + \chi_2) \cos(i_0) \left\{ \dot{r} \cos(2\phi) - \dot{\phi} r \sin(2\phi) \right\}, \tag{4.89}$$

$$\begin{aligned}
\mathfrak{P}_{ij}^{(+)} v^j (\mathbf{S} \times \mathbf{n}_{12})^i &= \frac{1}{8} (\chi_1 + \chi_2) \left\{ -\dot{\phi} r (\cos(2i_0) + 3) \cos(2\phi) + 2\dot{\phi} r \sin^2(i_0) \right. \\
& \left. - \dot{r} (\cos(2i_0) + 3) \sin(2\phi) \right\}, \tag{4.90}
\end{aligned}$$

$$\mathfrak{P}_{ij}^{(\times)} n_{12}^i (\mathbf{S} \times \mathbf{n}_{12})^j = \frac{1}{2} (\chi_1 + \chi_2) \cos(i_0) \cos(2\phi), \tag{4.91}$$

$$\mathfrak{P}_{ij}^{(+)} n_{12}^i (\mathbf{S} \times \mathbf{n}_{12})^j = -\frac{1}{8} (\chi_1 + \chi_2) \{ \cos(2i_0) + 3 \} \sin(2\phi), \tag{4.92}$$

$$\begin{aligned}
\mathfrak{P}_{ij}^{(\times)} v^i (\mathbf{S} \times \mathbf{v})^j &= -\frac{1}{2} (\chi_1 + \chi_2) \cos(i_0) \left\{ 2\dot{\phi} \dot{r} r \sin(2\phi) \right. \\
& \left. + \cos(2\phi) (\dot{\phi} r - \dot{r}) (\dot{\phi} r + \dot{r}) \right\}, \tag{4.93}
\end{aligned}$$

$$\begin{aligned}
\mathfrak{P}_{ij}^{(+)} v^i (\mathbf{S} \times \mathbf{v})^j &= \frac{1}{8} (\chi_1 + \chi_2) (\cos(2i_0) + 3) \left\{ \sin(2\phi) (\dot{\phi} r - \dot{r}) (\dot{\phi} r + \dot{r}) \right. \\
& \left. - 2\dot{\phi} \dot{r} r \cos(2\phi) \right\}, \tag{4.94}
\end{aligned}$$

$$\mathfrak{P}_{ij}^{(\times)} n_{12}^i (\mathbf{S} \times \mathbf{v})^j = \frac{1}{2} (\chi_1 + \chi_2) \cos(i_0) \left\{ \dot{r} \cos(2\phi) - \dot{\phi} r \sin(2\phi) \right\}, \tag{4.95}$$

$$\begin{aligned}
\mathfrak{P}_{ij}^{(+)} n_{12}^i (\mathbf{S} \times \mathbf{v})^j &= \frac{1}{8} (\chi_1 + \chi_2) \left\{ -\dot{\phi} r \cos(2\phi) (3 + \cos(2i_0)) - 2\dot{\phi} r \sin^2(i_0) \right. \\
& \left. - \dot{r} \sin(2\phi) (3 + \cos(2i_0)) \right\}, \tag{4.96}
\end{aligned}$$

$$\mathfrak{P}_{ij}^{(+)} (\mathbf{S} \times \mathbf{N})^i n_{12}^j = -\frac{1}{2}(\chi_1 + \chi_2) \sin(i_0) \cos(\phi), \quad (4.97)$$

$$\mathfrak{P}_{ij}^{(\times)} (\mathbf{S} \times \mathbf{N})^i n_{12}^j = -\frac{1}{4}(\chi_1 + \chi_2) \sin(2i_0) \sin(\phi), \quad (4.98)$$

$$\mathfrak{P}_{ij}^{(+)} (\mathbf{S} \times \mathbf{N})^i v^j = -\frac{1}{2}(\chi_1 + \chi_2) \sin(i_0) (\dot{r} \cos(\phi) - \dot{\phi} r \sin(\phi)), \quad (4.99)$$

$$\mathfrak{P}_{ij}^{(\times)} (\mathbf{S} \times \mathbf{N})^i v^j = -\frac{1}{4}(\chi_1 + \chi_2) \sin(2i_0) (\dot{\phi} r \cos(\phi) + \dot{r} \sin(\phi)). \quad (4.100)$$

The spin-independent projections and the ratio of the difference to the sum of the masses read, using $\mathfrak{P}_{vw}^{(\times)} \equiv \mathfrak{P}_{ij}^{(\times)} v^i w^j$, the contraction of the projector with some arbitrary vectors \mathbf{v} and \mathbf{w} ,

$$(\mathbf{N} \cdot \mathbf{n}_{12}) = \sin(i_0) \sin(\phi), \quad (4.101)$$

$$(\mathbf{N} \cdot \mathbf{v}) = \sin(i_0) \left\{ r \dot{\phi} \cos(\phi) + \dot{r} \sin(\phi) \right\}, \quad (4.102)$$

$$\mathbf{v}^2 = r^2 \dot{\phi}^2 + \dot{r}^2, \quad (4.103)$$

$$\mathfrak{P}_{nn}^{(\times)} = \cos(i_0) \sin(\phi) \cos(\phi), \quad (4.104)$$

$$\mathfrak{P}_{vv}^{(\times)} = \frac{1}{2} \cos(i_0) \left\{ \sin(2\phi) (\dot{r}^2 - \dot{\phi}^2 r^2) + 2 \dot{\phi} \dot{r} r \cos(2\phi) \right\}, \quad (4.105)$$

$$\mathfrak{P}_{nv}^{(\times)} = \frac{1}{2} \cos(i_0) \left\{ \dot{\phi} r \cos(2\phi) + \dot{r} \sin(2\phi) \right\}, \quad (4.106)$$

$$\mathfrak{P}_{nn}^{(+)} = \frac{1}{2} \left\{ \cos^2(\phi) - \cos^2(i_0) \sin^2(\phi) \right\}, \quad (4.107)$$

$$\mathfrak{P}_{vv}^{(+)} = \frac{1}{2} \left\{ (\dot{r} \cos(\phi) - \dot{\phi} r \sin(\phi))^2 - \cos^2(i_0) (\dot{\phi} r \cos(\phi) + \dot{r} \sin(\phi))^2 \right\}, \quad (4.108)$$

$$\mathfrak{P}_{nv}^{(+)} = \frac{1}{8} \left\{ -\dot{\phi} r (\cos(2i_0) + 3) \sin(2\phi) - 4 \dot{r} \cos^2(i_0) \sin^2(\phi) + 4 \dot{r} \cos^2(\phi) \right\}, \quad (4.109)$$

$$\frac{\delta m}{m} \equiv \frac{m_1 - m_2}{m} = \sqrt{1 - 4\eta}. \quad (4.110)$$

In the expression for the emitted gravitational wave amplitudes, Equation (4.71), R' is the rescaled distance from the observer to the binary system,

$$R = R' \frac{Gm}{c^2}. \quad (4.111)$$

We note that it is very important that R' has got the same scaling as r in order to remove the physical dimensions. The common factor c^{-4} of h_{ij}^{TT} will be split in c^{-2} for the distance R' and c^{-2} for the $\xi^{(\dots)}$, in order to make *all* terms dimensionless. Also note that the Equations (4.74)-(4.109) in our special coordinates are valid only when \mathbf{L} is constant in time. In the non-aligned case, additional angular velocity contributions kick in and the expressions become rather impractical. From Reference [75], the reader can extract explicit higher-order spin corrections to the Newtonian quadrupolar field for the case of quasi-circular orbits.

Dynamical orbital variables as implicit functions of time

We are now in the position to compute the time domain gravitational wave polarisations with the help of our orbital elements, to be expressed in terms of conserved quantities and the mean anomaly, which is an implicit function of time. Using Equations (4.30) - (4.33), one can express the quantities r , \dot{r} , ϕ , $\dot{\phi}$, appearing in the radiation field expressions, in terms of the eccentric anomaly \mathcal{E} , other orbital elements and several formal 2PN accurate functions. The quantities r and ϕ are computed rather straightforwardly. The coordinate velocities \dot{r} and $\dot{\phi}$ can be obtained using

$$\dot{r} = \left(\frac{\partial r}{\partial \mathcal{E}} \right) \left(\frac{\partial \mathcal{E}}{\partial \mathcal{M}} \right) \left(\frac{\partial \mathcal{M}}{\partial t} \right), \quad (4.112a)$$

$$\dot{\phi} = \left(\frac{\partial \phi}{\partial \mathcal{E}} \right) \left(\frac{\partial \mathcal{E}}{\partial \mathcal{M}} \right) \left(\frac{\partial \mathcal{M}}{\partial t} \right), \quad (4.112b)$$

These expressions are lengthy and easy to obtain, so we skip their presentation. Below, we discuss some qualitative features of what we achieved so far.

4.5 Estimate of the current results for aligned spin vectors

Let us get some numerical insight to the orders of magnitude of the derived quantities. Therefore, we assume a binary of two 1.4 solar mass black holes at an orbital frequency of, say, 216Hz, which advanced LIGO is sensitive to. The following Table 1 lists the individual contributions to selected scaled orbital elements at their formal orders. We remind the reader that all quantities are dimensionless, and that distances for example are given in (half) Schwarzschild radii.

	N	1PN-PP	2PN-PP	LO-SO	NLO-SO	LO-S(1)S(2)	$\frac{LO-S^2}{(\lambda_1+\lambda_2)}$
a_r	+22.4	-1.7	$-5.0 \cdot 10^{-2}$	+0.37	$+8.4 \cdot 10^{-2}$	$-11 \cdot 10^{-2}$	$+1.0 \cdot 10^{-2}$
\mathcal{K}	×	+0.13	$+2.9 \cdot 10^{-2}$	-0.033	$-5.9 \cdot 10^{-4}$	$+1.5 \cdot 10^{-3}$	$-2.0 \cdot 10^{-3}$
\mathcal{F}_v	×	×	$-2.6 \cdot 10^{-5}$	×	$-5.0 \cdot 10^{-5}$	×	×
$\mathcal{F}_{v-\mathcal{E}}$	×	×	$+1.3 \cdot 10^{-2}$	×	$-3.4 \cdot 10^{-3}$	×	×

Table 1: The numerical outcome of the scaled quantities of the QKP, assuming pro-alignment, $\chi_1 = 1 = \chi_2$, equal masses, $m_1 = 1.4M_\odot = m_2$, and the sweetspot of advanced LIGO, $f_{\text{GW}} = 216$ Hz, to be taken for the GW frequency. (Here, we take the quadrupolar part, where $f_{\text{GW}} = 2f_{\text{orbit}}$ holds in the circular limit.) A cross means that there is no formal contribution. We note that at this frequency the radiation reaction effects already have to be regarded.

The reader should be aware that there is a missing term linear in spin at 2PN order in the wave amplitude. Blanchet et al. [60] provided the current and mass multipole moments that are necessary to compute the far-zone fluxes resulting from the next-to-leading order spin-orbit terms in the acceleration, but the wave amplitude at this order was not given. This missing spin-orbit part at 2PN will be given in a forthcoming publication. This decision

is justified by stating that there is a number of relatively complicated terms of higher order due to the transformation from harmonic to ADM coordinates. To this order, the coordinate transformation contains next-to-leading order spin-orbit terms which will result in lengthy expressions in the radiation field. The difficulty of computing the 2PN amplitude itself becomes clear when we keep in mind the errata of reference [60].

There are further outstanding questions: what happens if the spins are misaligned? How will the precessional behaviour of the spins and the time evolution of the enclosed angles look like? How does the wave form look in that case? We showed that if the spins are (anti-) aligned with \mathbf{L} at some instant of time, they will be for all times. In case they are not, the total angular momentum $\mathbf{S}_1 + \mathbf{S}_2 + \mathbf{L}$ will still be conserved, but their orientations are variable in time. Let us tackle this problem and restrict ourselves to the LO-SO contributions to the Hamiltonian and to orbits with constant relative distances for simplicity to get a first impression of how this works.

5 Motion and gravitational waves from compact binaries with arbitrarily oriented spins under leading-order spin-orbit interaction

The next subsections aim to answer the questions of how spins with arbitrary magnitude and direction move under the LO-SO interaction. We therefore generalise the geometrical prescription as given in the work by Königsdörffer and Gopakumar [108], and also an earlier reference [109], which has heavily inspired this work, for single-spin and equal-mass binaries, to unequal-mass binaries and arbitrary spin configurations. The orbital motion is taken to be quasi-circular and the fractional mass difference is assumed to be small against one. The emitted gravitational wave forms are given in analytic form.

Subsequently, we will present the involved interaction terms. In Section 5.2, the geometry and the coordinates relating the generic reference frame with the orientation of the spins and the angular momentum vector are provided and characterised by rotation matrices. The time derivatives of these rotation matrices will be compared by Poisson brackets in Section 5.3 and first order time derivatives of the associated rotation angles will be obtained. A first-order perturbative solution to the EOM for the spins is worked out in Section 5.4. The *orbital* motion will be computed, for quasi-circular orbits (circular orbits in the precessing orbital plane), in Section 5.5. As an application, the resulting GW polarisations, h_{\times} and h_{+} in the quadrupolar restriction, are given in Section 5.6.

5.1 Spin dynamics

The Hamiltonian associated with our problem reads

$$H = H_{\text{PP}}^{\text{N}} + H_{\text{PP}}^{\text{1PN}} + H_{\text{PP}}^{\text{2PN}} + H_{\text{LO}}^{\text{SO}}. \quad (5.1)$$

The SO term can be recast into an “effective” form,

$$H_{\text{LO}}^{\text{SO}}(\mathbf{r}, \mathbf{p}, \mathbf{S}_1, \mathbf{S}_2) = \frac{\epsilon^2}{r^3} (\mathbf{r} \times \mathbf{p}) \cdot \mathbf{S}_{\text{eff}}, \quad (5.2)$$

where \mathbf{S}_{eff} is the so-called *effective spin*,

$$\mathbf{S}_{\text{eff}} \equiv \delta_1^{\text{KG}} \mathbf{S}_1 + \delta_2^{\text{KG}} \mathbf{S}_2, \quad (5.3a)$$

$$\delta_1^{\text{KG}} \equiv \frac{\eta}{2} + \frac{3}{4} \left(1 - \sqrt{1 - 4\eta} \right), \quad (5.3b)$$

$$\delta_2^{\text{KG}} \equiv \frac{\eta}{2} + \frac{3}{4} \left(1 + \sqrt{1 - 4\eta} \right). \quad (5.3c)$$

This combination of numbers in δ_1 and δ_2 shows off as we use the spin scaling of Königsdörffer and Gopakumar [108] (superscript KG), where $\mathbf{S}_a = S_a / (\frac{\mu m G}{c})$. If we had used our own spin

scaling (superscript “ind” for individual mass scaling due to Section 4.2), we would have obtained

$$\delta_1^{\text{ind}} = -\frac{\eta}{2} + \sqrt{1 - 4\eta} + 1, \quad (5.4)$$

$$\delta_2^{\text{ind}} = -\frac{\eta}{2} - \sqrt{1 - 4\eta} + 1. \quad (5.5)$$

Instead, we keep the scaling of [108] for a direct comparison, and we omit the superscript KG from now onwards. *Note:* In the published version of [105], the spin scaling of Equation (2.5) is incorrect and should be replaced with that of Reference [108]. Remembering the aligned-spin case, the orbital plane, perpendicular to \mathbf{L} , did not undergo an internal motion in time. Misalignment, in general, leads to a precession of \mathbf{L} . The EOM for \mathbf{L} , \mathbf{S}_1 & \mathbf{S}_2 can be deduced from their Poisson brackets,

$$\frac{d\mathbf{L}}{dt} = \{\mathbf{L}, H_{\text{LO}}^{\text{SO}}\} = \epsilon^2 \frac{1}{r^3} \mathbf{S}_{\text{eff}} \times \mathbf{L}, \quad (5.6a)$$

$$\frac{d\mathbf{S}_1}{dt} = \{\mathbf{S}_1, H_{\text{LO}}^{\text{SO}}\} = \epsilon^2 \frac{\delta_1}{r^3} \mathbf{L} \times \mathbf{S}_1, \quad (5.6b)$$

$$\frac{d\mathbf{S}_2}{dt} = \{\mathbf{S}_2, H_{\text{LO}}^{\text{SO}}\} = \epsilon^2 \frac{\delta_2}{r^3} \mathbf{L} \times \mathbf{S}_2. \quad (5.6c)$$

Equation (5.6a) describes the precession of \mathbf{L} with respect to $\mathbf{J} \equiv \mathbf{L} + \mathbf{S}_1 + \mathbf{S}_2$. The key idea will be to compute time dependent rotation matrices for \mathbf{L} , \mathbf{S}_1 and \mathbf{S}_2 for a number of rotation axes and angles that are to be introduced below. Let us state that the magnitudes L , S_1 and S_2 of the vectors \mathbf{L} , \mathbf{S}_1 and \mathbf{S}_2 are conserved,

$$\frac{dL^2}{dt} = \frac{d}{dt}(\mathbf{L} \cdot \mathbf{L}) = \epsilon^2 \frac{2}{r^3} \mathbf{L} \cdot (\mathbf{S}_{\text{eff}} \times \mathbf{L}) = 0, \quad (5.7a)$$

$$\frac{dS_1^2}{dt} = \frac{d}{dt}(\mathbf{S}_1 \cdot \mathbf{S}_1) = \epsilon^2 \frac{2\delta_1}{r^3} \mathbf{S}_1 \cdot (\mathbf{L} \times \mathbf{S}_1) = 0, \quad (5.7b)$$

$$\frac{dS_2^2}{dt} = \frac{d}{dt}(\mathbf{S}_2 \cdot \mathbf{S}_2) = \epsilon^2 \frac{2\delta_2}{r^3} \mathbf{S}_2 \cdot (\mathbf{L} \times \mathbf{S}_2) = 0. \quad (5.7c)$$

Thus, it holds that $\dot{L} = 0$. The magnitudes of $\mathbf{S} \equiv \mathbf{S}_1 + \mathbf{S}_2$ and \mathbf{S}_{eff} behave as follows,

$$\frac{dS^2}{dt} = -\epsilon^2 \frac{3\sqrt{1-4\eta}}{r^3} \mathbf{L} \cdot (\mathbf{S}_1 \times \mathbf{S}_2), \quad (5.8a)$$

$$\frac{dS_{\text{eff}}^2}{dt} = -\epsilon^2 \frac{3\sqrt{1-4\eta}(12+\eta)\eta}{4r^3} \mathbf{L} \cdot (\mathbf{S}_1 \times \mathbf{S}_2). \quad (5.8b)$$

Notice the conservation of S_{eff}^2 in both the test-mass ($\eta = 0$) and equal-mass ($\eta = 1/4$) cases. Using above equations, we will be able to compute the evolution equations for the rotation angles. The associated geometry is introduced next.

$\mathbf{j} = \mathbf{k} \times \mathbf{i}$, intersects the invariable plane at the line of nodes \mathbf{i} , with the longitude Υ measured in the invariable plane from \mathbf{e}_x .

The geometry of the binary will be completed by the spin related coordinate system $(\mathbf{i}_s, \mathbf{j}_s, \mathbf{k}_s)$. This frame is constructed from the system $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ to be rotated around the axis \mathbf{i} to point from the top of \mathbf{L} to the top of \mathbf{J} with the new direction \mathbf{k}_s . In other words, this spin coordinate system is chosen in such a way that the total spin, $\mathbf{S}_1 + \mathbf{S}_2$, has only a \mathbf{k}_s component and $\mathbf{i}_s \equiv \mathbf{i}$ holds. If Θ is known, the spins are left with an additional freedom to rotate around \mathbf{k}_s by an angle ϕ_s (the index “s” is a hint for positions in the spin system). This angle is measured from \mathbf{i}_s to the projection of \mathbf{S}_1 to the $(\mathbf{i}_s, \mathbf{j}_s)$ plane, similar to Υ 's function in the reference frame. There exist simple geometrical relations that will reduce the freedom to choose rotation angles arbitrarily, as will be shown subsequently.

Triangular computations

As mentioned already, in this geometry the spins and angular momenta – being fixed in their magnitudes – only have three degrees of freedom: the angles Θ , Υ and ϕ_s . Once Θ is determined, also α_{ks} (the angle between \mathbf{L} and \mathbf{S}) is fixed and so is magnitude S of $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$ by triangular relations. The following equations list the rotation angles and magnitudes as functions of Θ , where use is made of the law of sines,

$$S(\Theta) = \sqrt{J^2 + L^2 - 2JL \cos \Theta}, \quad (5.9a)$$

$$\alpha_{12}(\Theta) = \cos^{-1} \left(\frac{S(\Theta)^2 - S_1^2 - S_2^2}{-2S_1 S_2} \right), \quad (5.9b)$$

$$\alpha_{ks}(\Theta) = \pi - \sin^{-1} \left(\frac{J \sin(\Theta)}{S(\Theta)} \right), \quad (5.9c)$$

$$\tilde{s}(\Theta) = \sin^{-1} \left(\frac{S_2 \sin \alpha_{12}(\Theta)}{S(\Theta)} \right). \quad (5.9d)$$

These relations will be used extensively to simplify the angles evolution equations. How they are incorporated and applied will be shown next.

Coordinate bases and associated transformation matrixes

This section introduces the coordinate transformations from the reference system to the orbital triad and the spin system. To construct the EOM for the 3 physical angles Θ , Υ and ϕ_s , the idea is to compare the evolution of these rotation angles - as arguments for rotation matrices - with the Poisson brackets, Equations (5.6a) - (5.6c). Let us begin with the explicit computation of the transformed coordinate bases.

1. The orbital triad $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ can be, constructed by only 2 rotations from the reference

system. In terms of rotation matrices, we have

$$\begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \Theta & \sin \Theta \\ 0 & -\sin \Theta & \cos \Theta \end{pmatrix} \begin{pmatrix} \cos \Upsilon & \sin \Upsilon & 0 \\ -\sin \Upsilon & \cos \Upsilon & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} \mathbf{e}_X \\ \mathbf{e}_Y \\ \mathbf{e}_Z \end{pmatrix}.$$

2. The spin system is constructed, simply by another rotation of α_{ks} around the vector \mathbf{i} , from the orbital triad,

$$\begin{pmatrix} \mathbf{i}_s \\ \mathbf{j}_s \\ \mathbf{k}_s \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha_{ks} & -\sin \alpha_{ks} \\ 0 & \sin \alpha_{ks} & \cos \alpha_{ks} \end{pmatrix} \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix}, \quad (5.10)$$

such that $\mathbf{i}_s \equiv \mathbf{i}$ holds. *Important note: the angle α_{ks} has negative sign relative to Θ , due to the direction of \mathbf{J} and $\mathbf{J} = \mathbf{L} + \mathbf{S}$.*

Having transformed the unit vectors with these matrices, the coordinates transform by their transposed inverses, which are – in case of rotations – the matrices themselves. Now, we have everything under control to construct the set of all the physical vectors. These are listed below, having defined some shorthands for rotation matrices,

$$[\Theta] \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \Theta & \sin \Theta \\ 0 & -\sin \Theta & \cos \Theta \end{pmatrix}, \quad (5.11)$$

$$[\Upsilon] \equiv \begin{pmatrix} \cos \Upsilon & \sin \Upsilon & 0 \\ -\sin \Upsilon & \cos \Upsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (5.12)$$

$$[\alpha_{ks}] \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha_{ks} & -\sin \alpha_{ks} \\ 0 & \sin \alpha_{ks} & \cos \alpha_{ks} \end{pmatrix}. \quad (5.13)$$

The orbital angular momentum \mathbf{L} in the reference system (indices labeled “inv”) arises from two rotations from the orbital triad (“ot”) where it has only one component:

$$\mathbf{L} = \{[\Theta(t)] [\Upsilon(t)]\}^{-1} (0, 0, L), \quad (5.14)$$

or, in components,

$$(\mathbf{L})_i^{\text{inv}} = \left\{ [\Theta(t)] [\Upsilon(t)] \right\}_{ij}^{-1} (\mathbf{L})_j^{\text{ot}}. \quad (5.15)$$

The spins, in the spin system (s), where the \mathbf{k}_s is aligned with $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$, have the following form,

$$\mathbf{S}_1 = S_1(\cos \phi_s \sin \tilde{s} \mathbf{i}_s + \sin \phi_s \sin \tilde{s} \mathbf{j}_s + \cos \tilde{s} \mathbf{k}_s), \quad (5.16a)$$

$$\mathbf{S}_2 = S \mathbf{k}_s - \mathbf{S}_1, \quad (5.16b)$$

$$\mathbf{S} = S \mathbf{k}_s. \quad (5.16c)$$

5.3 Time derivatives of the spin orientation angles

To obtain an equation of motion for the angle Θ , one possibility is to use the time derivative of $|S|^2 = (\mathbf{S} \cdot \mathbf{S})$, Equation (5.8a), - to apply this for example in the spin system - and to compare the result with the time derivative of Equation (5.9a) with $\Theta = \Theta(t)$. The result is

$$\dot{\Theta} = -\frac{C_S S_1 S}{2J} \sin \alpha_{ks} \cos \phi_s \sin \tilde{s} \csc \Theta \quad (5.17)$$

with

$$C_S = -\epsilon^2 \frac{3\sqrt{1-4\eta}}{r^3}. \quad (5.18)$$

The same result will be obtained by computing the time derivative of \mathbf{L} in the invariable system. Therefore, take Equation (5.14), compute its time derivative and finally compare the result with (5.6a). Because the angular velocities appear in relatively simple relations, it is easy to extract them from the \mathbf{e}_x and \mathbf{e}_y entry. The results are

$$\dot{\Upsilon} = -C_L \csc \Theta [S_1(\delta_2 - \delta_1) \cos \alpha_{ks} \sin \phi_s \sin \tilde{s} + \sin \alpha_{ks}(S_1(\delta_1 - \delta_2) \cos \tilde{s} + S\delta_2)], \quad (5.19)$$

$$\dot{\Theta} = C_L S_1(\delta_1 - \delta_2) \cos \phi_s \sin \tilde{s}, \quad (5.20)$$

with $C_L \equiv \epsilon^2 r^{-3}$. The functional dependencies of α_{ks} , \tilde{s} and S on Θ are implicated. Inserting the geometrical relations, Equations (5.9), it turns out that Equations (5.17) and (5.20) are equivalent. Also, the allegedly worrying asymmetric appearance of the quantity S_1 can be studiously avoided by replacing \tilde{s} by its function of Θ .¹⁰ Also note that, if the relations $\eta = 1/4$ or $S_i = 0$ ($i = 1$ or 2) are inserted in Equation (5.19), one recovers Equation (4.32) of [108].

Now, let us turn to the last quantity to be determined: the angle ϕ_s . The geometry offers various possibilities to calculate the time derivative of this angle. The easy way is to compute \mathbf{S}_1 in the invariable system. In components, we have

$$(\mathbf{S}_1)_i^{\text{inv}} = \left\{ [\alpha_{ks}(t)] [\Theta(t)] [\Upsilon(t)] \right\}_{ij}^{-1} (\mathbf{S}_1)_j^s. \quad (5.21)$$

¹⁰The angular velocities, (5.19) and (5.20), are in complete agreement with Equations (5.11a) and (5.11b) of [110].

The time derivative of (5.21) might be compared with Equation (5.6b). The result will be given in terms of the angles already determined: since we already know $\dot{\Theta}$ and $\dot{\Upsilon}$ on the one hand and α_{ks} as a function of Θ on the other, we have the expression under full control.

The other way is to take the Leibniz product rule for \mathbf{S} , namely $\partial_t \mathbf{S} = \partial_t S^i \mathbf{e}_i + S^i \partial_t \mathbf{e}_i$ with $\mathbf{e}_i = (\mathbf{i}_s, \mathbf{j}_s, \mathbf{k}_s)$. We already know that $\mathbf{i}_s \equiv \mathbf{i}$, $\mathbf{k}_s \parallel \mathbf{S}$ and $\mathbf{j}_s \equiv \mathbf{i}_s \times \mathbf{k}_s$, whose time derivatives are already known. Both considerations result in

$$\dot{\phi}_s = C_\Theta (\dot{\Theta} - \dot{\alpha}_{\text{ks}}) + C_{\tilde{s}} \dot{\tilde{s}} + \Omega_0, \quad (5.22a)$$

$$C_\Theta = \tan \phi_s \cot(\alpha_{\text{ks}} - \Theta) + \sec \phi \cot \tilde{s}, \quad (5.22b)$$

$$C_{\tilde{s}} = -\sec \phi_s \cot(\alpha_{\text{ks}} - \Theta) - \tan \phi \cot \tilde{s}, \quad (5.22c)$$

$$\Omega_0 = -C_1 L \sin \Theta \csc(\alpha_{\text{ks}} - \Theta) \quad (5.22d)$$

with $C_1 \equiv \epsilon^2 \delta_1 / r^3$. For the case of equal masses ($\delta_1 = \delta_2 = \delta = 7/8$), one obtains for $\dot{\phi}_s$, $\dot{\Upsilon}$ and $\dot{\Theta}$ a very simple system of EOM,

$$\dot{\Theta} = 0, \quad (5.23a)$$

$$\dot{\Upsilon} = \epsilon^2 \frac{7J}{8r^3}, \quad (5.23b)$$

$$\dot{\phi}_s = -\epsilon^2 \frac{L\delta \sin \Theta}{r^3} \csc \left\{ \sin^{-1} \left(\frac{J \sin \Theta}{S(\Theta)} \right) + \Theta \right\}, \quad (5.23c)$$

which can be integrated immediately, giving

$$\Theta(t) = \Theta_0, \quad (5.24a)$$

$$\Upsilon(t) = \Omega_\Upsilon t + \Upsilon_0, \quad (5.24b)$$

$$\phi_s(t) = \Omega_\phi t + \phi_{s0}, \quad (5.24c)$$

with the angular velocities

$$\Omega_\Upsilon \equiv \epsilon^2 \frac{7J}{8r^3}, \quad (5.25a)$$

$$\Omega_\phi \equiv -\epsilon^2 \frac{L\delta \sin \Theta_0}{r^3} \csc \left\{ \sin^{-1} \left[\frac{J \sin \Theta_0}{S(\Theta_0)} \right] + \Theta_0 \right\}. \quad (5.25b)$$

Summarizing the EOM for the coordinate transformation angles, Equations (5.19), (5.20) and (5.22), this system of EOM can be written in a compact manner. Labeling \mathbf{C} the vector of constants with $\mathbf{C} \equiv \{E, S_1, S_2, L, \mathbf{k} \cdot \mathbf{S}_{\text{eff}}\}$ – where E and L are closely related in the case of quasi-circular orbits – and \mathbf{X} the vector of dynamic variables, associated with spins and angular momentum, $\mathbf{X} = \{\Theta, \Upsilon, \phi_s\}$, we may write

$$\frac{d}{dt} \mathbf{X} = \mathbf{Y}_{\mathbf{C}}(\mathbf{X}). \quad (5.26)$$

A perturbative solution will be given in the next section.

5.4 First order perturbative solution to the EOM for the non-equal mass case

The EOM for $(\Theta, \Upsilon, \phi_s)$ can also be solved by a simple reduction scheme. We assume that the deviation from the equal-mass case is small compared to unity,

$$\frac{\delta_1 - \delta_2}{\delta_1 + \delta_2} \ll 1. \quad (5.27)$$

Then, having the equal-mass case under full analytic control, we can construct a perturbative solution to the non-equal mass case. The proceeding is as follows: Imagine a system of EOM for a number N of dependent variables \mathbf{X} :

$$\dot{\mathbf{X}} = \mathbf{Y}(\mathbf{X}). \quad (5.28)$$

The time domain solution to this system is denoted by the superscript “0”, viz

$$\mathbf{X}(t) = \mathbf{X}^{(0)}(t). \quad (5.29)$$

Let us assume that the EOM, Equation (5.28), are perturbed by some terms of the order ϵ (ϵ is a dimensionless ordering parameter),

$$\dot{\mathbf{X}} = \mathbf{Y}(\mathbf{X}) + \epsilon \mathbf{P}(\mathbf{X}). \quad (5.30)$$

The solution at the first order in ϵ can be obtained by adding a small perturbing quantity (which is to be determined) to the solution of the homogeneous equation,

$$X_i^{(1)}(t) = X_i^{(0)}(t) + \epsilon S_i(t). \quad (5.31)$$

Inserting this into Equation (5.30), one obtains

$$\begin{aligned} \dot{X}_i^{(1)} &= \dot{X}_i^{(0)} + \epsilon \dot{S}_i \\ &= Y_i(X_j^{(0)} + \epsilon S_j) + \epsilon P_i(X_j^{(0)} + \epsilon S_j) \\ &= Y_i(X_j^{(0)}) + \epsilon \sum_{j=1}^N \frac{\partial Y_i}{\partial X_j} S_j + \epsilon P_i(X_j^{(0)}) + \mathcal{O}(\epsilon^2). \end{aligned} \quad (5.32)$$

Comparing the coefficients of the two orders of ϵ gives

$$0: \quad \dot{X}_i^{(0)} = Y_i(X_j^{(0)}), \quad (5.33)$$

$$1 : \quad \dot{S}_i = \sum_{j=1}^N \frac{\partial Y_i}{\partial X_j} S_j + P_i(X_j^{(0)}). \quad (5.34)$$

The first equation is solved via definition, and what remains is the second, having inserted the unperturbed solution in the perturbing function P . For our purposes, $N = 3$ with $\mathbf{X} = \{\Upsilon, \Theta, \phi_s\}$ is small, but complicated functional dependencies are included. The matrix appearing in Equation (5.34) does not mean a problem to us, because fortunately, the only dependency of the sources is on Θ .

For our computation, we need to divide the EOM into a non-perturbative and a perturbative part. In the following, we use the definitions

$$\chi_1 = \frac{\delta_1 + \delta_2}{2}, \quad (5.35)$$

$$\chi_2 = \frac{\delta_1 - \delta_2}{2}. \quad (5.36)$$

Rewriting the EOM for the angles in terms of χ_1 and χ_2 , labeling all χ_2 contributions with the order parameter ε as well as inserting the non-perturbative solution, Equations (5.24) to these terms, one obtains

$$\dot{\Theta}^{(1)} = \varepsilon \dot{S}_\Theta = \varepsilon C_L S_1 2 \chi_2 \cos(t \Omega_\phi + \phi_0) \sin \tilde{s}(\Theta_0), \quad (5.37a)$$

$$\begin{aligned} \dot{\Upsilon}^{(0)} + \varepsilon \dot{S}_\Upsilon &= \underbrace{C_L S(\Theta) \chi_1 \sin \alpha_{\text{ks}}(\Theta) \csc \Theta}_{\equiv C_L J \chi_1 = \text{const.}} + \\ &\varepsilon \left[C_L \chi_2 \csc \Theta_0 (2S_1 \cos \alpha_{\text{ks}}(\Theta_0) \sin \tilde{s}(\Theta_0) \sin(t \Omega_\phi + \phi_0) \right. \\ &\quad \left. - \sin \alpha_{\text{ks}}(\Theta_0) (S(\Theta_0) - 2S_1 \cos \tilde{s}(\Theta_0))) \right], \end{aligned} \quad (5.37b)$$

$$\begin{aligned} \dot{\phi}_s^{(0)} + \varepsilon \dot{S}_\phi &= -C_1 L \sin \Theta \csc \alpha_{\text{ks}}(\Theta) \\ &+ \varepsilon \left[C_{\tilde{s}}(\Theta, \phi_s) \frac{\partial \tilde{s}}{\partial \Theta} + C_\Theta(\Theta, \phi_s) \left(1 - \frac{\partial \alpha_{\text{ks}}}{\partial \Theta} \right) \right] \bigg|_{\substack{\Theta = \Theta_0 \\ \phi_s = t \Omega_\phi + \phi_{s0}}} \dot{\Theta}, \end{aligned} \quad (5.37c)$$

$$\begin{aligned} &= -(\chi_1 + \varepsilon \chi_2) \varepsilon^2 \frac{L}{r^3} \sin \Theta \csc \alpha_{\text{ks}}(\Theta) \\ &+ \varepsilon \left[C_{\tilde{s}}(\Theta, \phi_s) \frac{\partial \tilde{s}}{\partial \Theta} + C_\Theta(\Theta, \phi_s) \left(1 - \frac{\partial \alpha_{\text{ks}}}{\partial \Theta} \right) \right] \bigg|_{\substack{\Theta = \Theta_0 \\ \phi_s = t \Omega_\phi + \phi_{s0}}} \dot{\Theta}. \end{aligned} \quad (5.37d)$$

The parameter ε simply counts the order of the perturbative contribution and is later set to one. The first term for Υ is constant and thus does not have to be expanded in powers of ε , but the associated first term for ϕ_s does, such that the perturbative solution for Θ has to be included. Taylor expanding this term, removing all contributions to the unperturbed problem, what remains is a system of EOM for $S_\Theta, S_\Upsilon, S_\phi$ that can be simply integrated, because as soon as $S_\Theta(t)$ is known, all the other contributions are straightforwardly evaluated. Requiring that the perturbing solutions vanish at $t = 0$, the solutions are simply given by

$$S_\Theta(t) = \int_0^t \dot{S}_\Theta dt, \quad (5.38a)$$

$$S_{\Upsilon}(t) = \int_0^t \dot{S}_{\Upsilon} dt, \quad (5.38b)$$

$$S_{\phi}(t) = \int_0^t \dot{S}_{\phi} dt, \quad (5.38c)$$

and explicitly read

$$S_{\Theta}(t) = -\frac{C_L S_1 S_2 2\chi_2}{S_{(0)} \Omega_{\phi}} \sin \alpha_{12(0)} (\sin \phi_{s_0} - \sin (t\Omega_{\phi} + \phi_{s_0})), \quad (5.39a)$$

$$S_{\Upsilon}(t) = \frac{C_L \chi_2 \csc \Theta_0}{\Omega_{\phi}} \left[2S_1 \cos \alpha_{\text{ks}(0)} \sin \tilde{s}_{(0)} (\cos \phi_{s_0} - \cos (t\Omega_{\phi} + \phi_{s_0})) \right. \\ \left. - t\Omega_{\phi} \sin \alpha_{\text{ks}(0)} (S_{(0)} - 2S_1 \cos \tilde{s}_{(0)}) \right], \quad (5.39b)$$

$$S_{\phi}(t) = C_{\text{stat}} t + C_0(t) + C_{\tilde{s}}(t) + C_{\Theta}(t) + C_{\alpha}(t), \quad (5.39c)$$

with the shorthands

$$C_{\text{stat}} = \frac{\chi_2 \Omega_{\phi}}{\chi_1}, \quad (5.40a)$$

$$C_0(t) = \frac{C_L J S_1 S_2 \chi_2 \sin \Theta_0}{\chi_1 S_{(0)}^3 \Omega_{\phi} \sqrt{S_{(0)}^2 - J^2 \sin^2 \Theta_0}} (-2\epsilon^{-2} J r^3 \Omega^* - 2\chi_1 S_{(0)}^2) \\ \sin \alpha_{12(0)} (t\Omega_{\phi} \sin \phi_{s_0} + \cos (t\Omega_{\phi} + \phi_{s_0}) - \cos \phi_{s_0}), \quad (5.40b)$$

$$C_{\tilde{s}}(t) = \frac{-2C_L J \chi_2 (S_{(0)}^2 \cot \alpha_{12(0)} - S_1 S_2 \sin \alpha_{12(0)})}{\chi_1 S_{(0)}^2 \Omega_{\phi} \sqrt{S_{(0)}^2 - S_2^2 \sin^2 \alpha_{12(0)}}} \times \\ (\epsilon^{-2} r^3 t \Omega_{\phi} \sin \tilde{s}_{(0)} \Omega^* + L \chi_1 \sin \Theta_0 \cos \tilde{s}_{(0)} (\cos \phi_{s_0} - \cos (t\Omega_{\phi} + \phi_{s_0}))), \quad (5.40c)$$

$$C_{\Theta}(t) = \epsilon^2 \frac{2S_1 t \chi_2 \cos \tilde{s}_{(0)}}{r^3} + \frac{2S_1 \chi_2 \Omega^* \csc \Theta_0 \sin \tilde{s}_{(0)} (\cos \phi_{s_0} - \cos (t\Omega_{\phi} + \phi_{s_0}))}{L \chi_1 \Omega_{\phi}},$$

$$C_{\alpha}(t) = C_{\Theta}(t) \frac{J (S_{(0)}^2 \cos \Theta_0 - J L \sin^2 \Theta_0)}{S_{(0)}^2 \sqrt{S_{(0)}^2 - J^2 \sin^2 \Theta_0}}, \quad (5.40d)$$

the initial values of the functions (5.9a) - (5.9d)

$$S_{(0)} \equiv S(\Theta_0), \quad (5.41a)$$

$$\alpha_{\text{ks}(0)} \equiv \alpha_{\text{ks}}(\Theta_0), \quad (5.41b)$$

$$\alpha_{12(0)} \equiv \alpha_{12}(\Theta_0), \quad (5.41c)$$

$$\tilde{s}_{(0)} \equiv \tilde{s}(\Theta_0), \quad (5.41d)$$

and the definition

$$\Omega^* \equiv \Omega_\phi \sqrt{1 - \epsilon^4 \frac{L^2 \chi_1^2 \sin^2 \Theta_0}{r^6 \Omega_\phi^2}}. \quad (5.41e)$$

Note that $\Omega_\phi = \mathcal{O}(\epsilon^2)$ and the second term in the square root will not vanish due to the PN accuracy.¹¹

The corrections to the angles computed above are of the same PN order as the unperturbed solutions, multiplied by a factor of $F = (\delta_1 - \delta_2)/(\delta_1 + \delta_2)$. If we set $m_2 = m_1(1 + \alpha)$, we obtain following representative pairs $(\alpha, F(\alpha))$: (0.1, 0.04), (0.2, 0.08), (0.5, 0.17), (1.0, 0.29), to give an estimate of the magnitude of the perturbation. For the case of $\alpha < 0.2$, this is below 10 %, in other cases second-order perturbations may be required.

5.5 Orbital dynamics

The motion of the spins is only half of the physical content of the spin-orbit dynamics. Once we fully have the motion of all the spin-related angles under control, we might turn to the orbital dynamics, *i.e.* the motion of the reduced mass in the orbital plane. It will turn out that employing coordinate transformations will be very helpful here, too.

The aim is to solve the *orbital* EOM to the full Hamiltonian,

$$H = H_{\text{PP}}^{\text{N}} + H_{\text{PP}}^{\text{1PN}} + H_{\text{PP}}^{\text{2PN}} + H_{\text{SO}}^{\text{LO}}. \quad (5.42)$$

At this point, we can do a useful simplification. As long as we incorporate only leading order spin dynamics, only Newtonian point particle and spin dependent contributions will mix at the end, higher order PN terms coupling with the spins will be neglected consequently. For the computation of the spin dependent part of the orbital phase, therefore, we only have to take $H_{\text{N,SO}} = H_{\text{PP}}^{\text{N}} + H_{\text{SO}}^{\text{LO}}$ and add the 1PN and 2PN (spinless) terms for the point particle afterwards,

$$H = H^{\text{N,SO}} + H_{\text{PP}}^{\text{1PN}} + H_{\text{PP}}^{\text{2PN}}, \quad (5.43)$$

$$\dot{\phi} = \dot{\phi}_{\text{N,SO}} + \dot{\phi}_{\text{1PN}} + \dot{\phi}_{\text{2PN}}. \quad (5.44)$$

The Newtonian and spin orbit part of Equation (5.42) reads

$$H^{\text{N,SO}} = \frac{\mathbf{p}^2}{2} - \frac{1}{r} + \frac{\epsilon^2}{r^3} (\mathbf{r} \times \mathbf{p}) \cdot \mathbf{S}_{\text{eff}} \quad (5.45)$$

and can be handled with the method described in [108]. The aim there was to introduce advantageous spherical coordinates, (r, θ, ϕ) , with their associated ONS $(\mathbf{n}_{12}, \mathbf{e}_\theta, \mathbf{e}_\phi)$ with $\mathbf{e}_z \cdot \mathbf{n}_{12} = \cos \theta$, $\mathbf{n}_{12} \cdot \mathbf{e}_x = \cos \phi \sin \theta$, as can be seen in Figure (3). First, we define the

¹¹Thanks to the rotation symmetry with respect to the \mathbf{e}_z axis, the EOM do not depend on Υ at all, and decouple nicely. Therefore, no time-ordered product is necessary, at least at first order in ϵ .

normalised relative separation vector according to

$$\mathbf{n}_{12} = \sin \theta \cos \phi \mathbf{e}_x + \sin \theta \sin \phi \mathbf{e}_y + \cos \theta \mathbf{e}_z. \quad (5.46)$$

The time derivative of \mathbf{r} , the linear momentum \mathbf{p} , its decomposition in radial components and the corresponding orthogonal ones can be written as

$$\mathbf{r} = r \mathbf{n}_{12}, \quad (5.47a)$$

$$\dot{\mathbf{r}} = \dot{r} \mathbf{n}_{12} + r \dot{\theta} \mathbf{e}_\theta + r \sin \theta \dot{\phi} \mathbf{e}_\phi, \quad (5.47b)$$

$$\mathbf{p} = p_r \mathbf{n}_{12} + p_\theta \mathbf{e}_\theta + p_\phi \mathbf{e}_\phi, \quad (5.47c)$$

$$\begin{aligned} p^2 &= p_r^2 + p_\theta^2 + p_\phi^2 = (\mathbf{n}_{12} \cdot \mathbf{p})^2 + (\mathbf{n}_{12} \times \mathbf{p})^2 \\ &= p_r^2 + \frac{L^2}{r^2}. \end{aligned} \quad (5.47d)$$

Figure (3) shows in detail how the orbital angular momentum is situated in the reference frame and how the orbital motion takes place in the plane orthogonal to \mathbf{L} which is marked in grey.

Inserting p^2 into Equation (5.45), computing $p_\phi = \mathbf{p} \cdot \mathbf{e}_\phi$ and using the orthogonality relation of the used triad, one obtains

$$p_r^2 = 2|E| + \frac{2}{r} - \frac{L^2}{r^2} - \epsilon^2 \frac{2(\mathbf{L} \cdot \mathbf{S}_{\text{eff}})}{r^3}, \quad (5.48a)$$

$$p_\phi = \frac{L_z}{r \sin \theta}, \quad (5.48b)$$

$$p_\theta^2 = \frac{L^2}{r^2} - p_\phi^2 = \frac{1}{r^2} \left(L^2 - \frac{L_z^2}{\sin^2 \theta} \right), \quad (5.48c)$$

In [108], it was possible to reduce these equations by some algebraic relations and the fact that the angle Θ was constant in time - here, it is more complicated. It is still allowed to express L_z , the projection of \mathbf{L} onto \mathbf{e}_z , in Equation (5.48b) and (5.48c) over Θ with the help of

$$p_\phi = \frac{L \cos \Theta}{r \sin \theta}, \quad (5.49a)$$

$$p_\theta^2 = \frac{L^2}{r^2} \left(1 - \frac{\cos^2 \Theta}{\sin^2 \theta} \right). \quad (5.49b)$$

Above equations are, for our purposes, the most simplified versions of the p components and will enter in the dynamics of the angle φ in their current form.

Our aim is now to connect the coordinate velocities, namely \dot{r} , $\dot{\phi}$ and $\dot{\theta}$, to conserved quantities associated with the Hamiltonian of Equation (5.45). Computation of the velocity in spherical coordinates, Equation (5.46), gives following formulae using Hamilton's EOM,

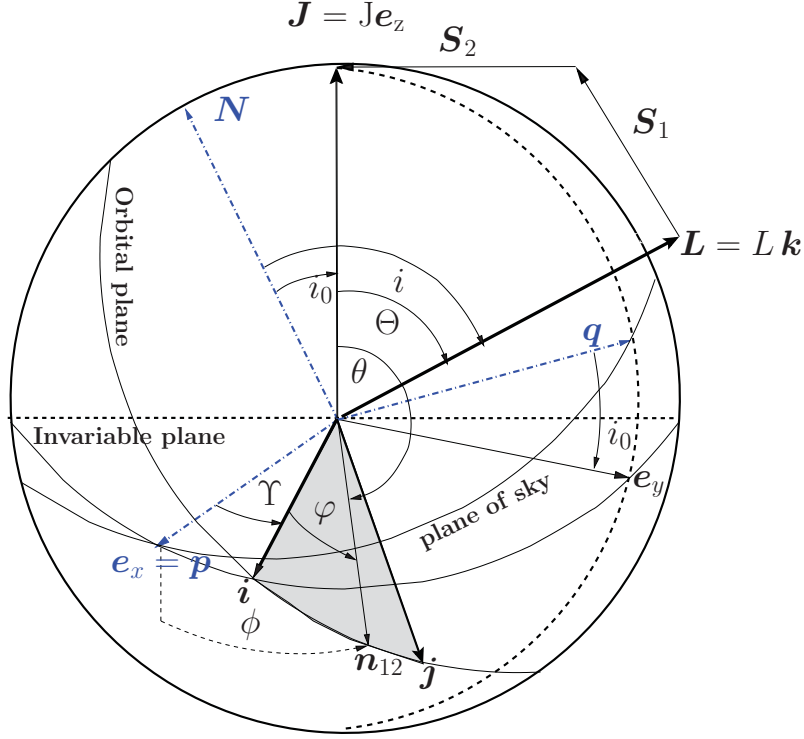


Figure 3: The geometry of the binary, having added the observer-related frame $(\mathbf{p}, \mathbf{q}, \mathbf{N})$ (in dashed and dotted lines) with \mathbf{N} as the line-of-sight vector, after some optical simplifications. Its geometrical properties are chosen according to Section 4.4. The grey area in the graphics completely lies in the orbital plane, spanned by (\mathbf{i}, \mathbf{j}) and φ measures the angle between the separation vector \mathbf{r} and \mathbf{i} . The polarisation vectors \mathbf{p} and \mathbf{q} span the plane of the sky. The inclination of this plane with respect to the orbital plane is the *orbital* inclination i . The inclination of the orbital plane with respect to the invariable plane is denoted by Θ . The reader should note that \mathbf{L} does not lie on the unit sphere, only \mathbf{k} does.

$$\dot{\mathbf{r}} = \partial H^{\text{N,SO}} / \partial \mathbf{p}, \quad \mathbf{n}_{12} \times \mathbf{e}_\theta = \mathbf{e}_\phi \quad \text{and} \quad \mathbf{n}_{12} \times \mathbf{e}_\phi = -\mathbf{e}_\theta.$$

$$\dot{r} = \mathbf{n}_{12} \cdot \dot{\mathbf{r}} = p_r, \quad (5.50a)$$

$$r\dot{\theta} = \mathbf{e}_\theta \cdot \dot{\mathbf{r}} = p_\theta + \epsilon^2 \frac{\mathbf{e}_\phi \cdot \mathbf{S}_{\text{eff}}}{r^2}, \quad (5.50b)$$

$$r \sin \theta \dot{\phi} = \mathbf{e}_\phi \cdot \dot{\mathbf{r}} = p_\phi - \epsilon^2 \frac{\mathbf{e}_\theta \cdot \mathbf{S}_{\text{eff}}}{r^2}. \quad (5.50c)$$

Of course, in the case of quasi-circular motion, $\dot{r} = 0 = p_r$ holds for all times. Remembering the geometry of Figure 3, we recall that \mathbf{r} is lying in the plane orthogonal to \mathbf{L} , which itself is spanned by the vectors \mathbf{i} and \mathbf{j} . Calling φ (the orbital phase) the measure for the angular distance from \mathbf{i} , we can write

$$\mathbf{r} = r \cos \varphi \mathbf{i} + r \sin \varphi \mathbf{j}. \quad (5.51)$$

The comparison of \mathbf{r} , given by Equations (5.47a) and (5.46), with the one in the new angular variables, Equation (5.51) with Equations (5.10), implies the transformation

$$(\theta, \phi) \rightarrow (\Upsilon, \varphi) : \begin{cases} \cos \theta = \sin \varphi \sin \Theta \\ \sin(\phi - \Upsilon) \sin \theta = \sin \varphi \cos \Theta \\ \cos(\phi - \Upsilon) \sin \theta = \cos \varphi. \end{cases} \quad (5.52)$$

Time derivation of the first equation will give an expression for $\dot{\theta}$, which can be simplified using the third one. The final expression is

$$\dot{\theta} = -\sin \Delta \dot{\Theta} - \sqrt{1 - \frac{\cos^2 \Theta}{\sin^2 \theta}} \dot{\varphi} \quad (5.53)$$

with $\Delta \equiv \phi - \Upsilon$. Setting Θ constant, one naturally recovers Equation (4.28a) of [108]. Using this equation to eliminate $\dot{\theta}$ in (5.50b) and after substitution $\pm p_\theta$ from (5.49b), one obtains a solution for $\dot{\varphi}$ and $\dot{\Upsilon}$,

$$\dot{\varphi} = \mp \frac{L}{r^2} - \frac{\tilde{S}_\phi}{\sqrt{1 - \frac{\cos^2 \Theta}{\sin^2 \theta}}} \frac{\epsilon^2}{r^3} - \frac{\sin \Delta}{\sqrt{1 - \frac{\cos^2 \Theta}{\sin^2 \theta}}} \dot{\Theta}, \quad (5.54)$$

where \tilde{S}_ϕ is a shorthand for $\mathbf{S}_{\text{eff}} \cdot \mathbf{e}_\phi$. The ambiguity of the sign in the first term can be removed if one takes the rotation sense of the reduced mass, or equivalently, the direction of \mathbf{L} into account. Having (initially) the vector \mathbf{L} in the northern hemisphere, one should choose “+” in above equation. This condition then holds anytime as long as $S_1 + S_2 < \sqrt{L^2 + J^2}$. The quantity L/r^2 represents only the Newtonian point particle contribution. To express r and L in Equation (5.54) in terms of E , one only needs Newtonian order,

$$r = (2|E|)^{-1}, \quad (5.55)$$

$$L = (2|E|)^{-1/2}. \quad (5.56)$$

Summarizing the evolution for $\dot{\varphi}$, one can separate it into a pure point particle (PP) and the spin orbit part (SO),

$$\dot{\varphi} = \dot{\varphi}_{\text{PP}} + \dot{\varphi}_{\text{SO}}. \quad (5.57)$$

The full 2PN expression for $\dot{\varphi}_{\text{PP}}$ can be extracted from Equations (5.6c), (5.6d) and (5.6k) of [108] without spin dependent terms,

$$\mathcal{N} = (2|E|)^{3/2} \left\{ 1 + \epsilon^2 \frac{(2|E|)}{8} (\eta - 15) + \epsilon^4 \frac{(2|E|)^2}{128} \left[555 + 30\eta + 11\eta^2 - \frac{192}{\sqrt{2|E|L^2}} (5 - 2\eta) \right] \right\}, \quad (5.58a)$$

$$\mathcal{K} = \epsilon^2 \frac{3}{L^2} \left\{ 1 + \epsilon^2 \frac{(2|E|)}{4} \left(-5 + 2\eta + \frac{35 - 10\eta}{2|E|L^2} \right) \right\}, \quad (5.58b)$$

setting $e_t = 0$ in

$$\begin{aligned} e_t^2 = & 1 - 2|E|L^2 + \epsilon^2 \frac{2|E|}{4} \left\{ -8 + 8\eta + (17 - 7\eta)(2|E|L^2) - 8\chi_{\text{so}} \cos \alpha \frac{S}{L} \right\} \\ & + \epsilon^4 \frac{(2|E|)^2}{8} \left\{ 8 + 4\eta + 20\eta^2 - (2|E|L^2)(112 - 47\eta + 16\eta^2) \right. \\ & \left. + 24\sqrt{2|E|L^2}(5 - 2\eta) + \frac{4}{(2|E|L^2)}(17 - 11\eta) - \frac{24}{\sqrt{2|E|L^2}}(5 - 2\eta) \right\} \end{aligned} \quad (5.59)$$

to eliminate L and using $\dot{\varphi}_{\text{PP}} = n(1+k)$ [97], giving

$$\dot{\varphi}_{\text{PP}} = (2|E|)^{3/2} \left\{ 1 + \epsilon^2 \frac{1}{8} (9 + \eta) (2|E|) + \epsilon^4 \left[\frac{891}{128} - \frac{201}{64} \eta + \frac{11}{128} \eta^2 \right] (2|E|)^2 \right\}, \quad (5.60)$$

$$\dot{\varphi}_{\text{SO}} = -3(\mathbf{k} \cdot \mathbf{S}_{\text{eff}})(2|E|)^3 + \frac{(2|E|)^3 \tilde{S}_\phi}{\sqrt{1 - \frac{\cos^2 \Theta}{\sin^2 \theta}}} - \frac{\sin \Delta \dot{\Theta}}{\sqrt{1 - \frac{\cos^2 \Theta}{\sin^2 \theta}}}, \quad (5.61)$$

with

$$\begin{aligned} \tilde{S}_\phi & \equiv \mathbf{S}_{\text{eff}} \cdot \mathbf{e}_\phi \\ & = \cos(\phi - \Upsilon) [S_1 \sin \phi_s \sin s(\delta_1 - \delta_2) \cos(\Theta - \alpha_{\text{ks}}) \\ & \quad - \sin(\Theta - \alpha_{\text{ks}}) (S_1 \cos \tilde{s}(\delta_1 - \delta_2) + S(\Theta) \delta_2)] \\ & \quad + \sin(\phi - \Upsilon) S_1 \cos \phi_s \sin s(\delta_2 - \delta_1). \end{aligned} \quad (5.62)$$

We note that, for further computations including eccentricity, \tilde{S}_ϕ can be simplified essentially and represented with a fast and a slowly evolving term, but this is irrelevant for the time being. The first term in $\dot{\varphi}_{\text{SO}}$, Equation (5.61), comes from spin-orbit contributions to the value of L , as this is obtained from the energy expression (5.1), see Section IV of [108]. The angle ϕ can be computed with the help of Equation (5.52) according to

$$\phi = \Upsilon + \arccos \left(\cos \varphi / \sqrt{1 - \sin^2 \varphi \sin^2 \Theta} \right). \quad (5.63)$$

Inserting the solutions $\Theta(t)$, $\Upsilon(t)$ and $\phi_s(t)$ from Sections 5.3 and 5.4 into Equations (5.60) and (5.61), φ can be obtained by numerical integration,

$$\varphi(t) = \int_0^t \dot{\varphi} dt + \varphi_0 = \dot{\varphi}_{\text{PP}} t + \int_0^t \dot{\varphi}_{\text{SO}}(t) dt + \varphi_0, \quad (5.64)$$

or approximated analytically, after additional investigation have been performed about how much the integrand and its constituents vary in time. The radial separation at 2PN accuracy,

after eliminating L , reads

$$r = \frac{1}{2|E|} \left\{ 1 + \epsilon^2 \frac{(2|E|)}{4} \left[\eta - 7 + 4(\mathbf{k} \cdot \mathbf{S}_{\text{eff}}) \sqrt{(2|E|)} \right] + \epsilon^4 \frac{(2|E|)^2}{16} \left[-67 + \eta(54 + \eta) \right] \right\}. \quad (5.65)$$

We will use these results for the leading-order GW-forms to be constructed.

5.6 Gravitational wave forms

The gravitational wave polarization states, h_+ and h_\times , are given by Equation (2.33). The leading order contribution, $h_{ij}^{\text{TT}}|_Q$, where the subscript Q denotes the quadrupolar approximation, reads (computing (2.33) explicitly, or read e.g. [111])

$$h_{km}^{\text{TT}}|_Q = \epsilon^4 \frac{4\eta}{R} \mathfrak{P}_{km}^{\text{TT}ij}(\mathbf{N}) \left(v_{ij} - \frac{1}{r} n_{ij} \right), \quad (5.66)$$

where \mathcal{P}_{kmij} is the projector defined in Equation (2.34), the shorthands $v_{ij} \equiv v_i v_j$ and $n_{ij} \equiv n_{12i} n_{12j}$, using $\mathbf{v} \equiv d\mathbf{r}/dt$ as the velocity and $\mathbf{n}_{12} \equiv \mathbf{r}/r$, respectively.

Using Equation (5.66), one may express both amplitudes of h_\times and h_+ as

$$\begin{aligned} h_+|_Q &= \epsilon^4 \frac{2\eta}{R} \left[(p_i p_j - q_i q_j) \left(v_{ij} - \frac{1}{r} n_{ij} \right) \right] \\ &= \epsilon^4 \frac{2\eta}{R} \left\{ (\mathbf{p} \cdot \mathbf{v})^2 - (\mathbf{q} \cdot \mathbf{v})^2 - \frac{1}{r} [(\mathbf{p} \cdot \mathbf{n}_{12})^2 - (\mathbf{q} \cdot \mathbf{n}_{12})^2] \right\}, \end{aligned} \quad (5.67a)$$

$$\begin{aligned} h_\times|_Q &= \epsilon^4 \frac{2\eta}{R} \left[(p_i q_j + p_j q_i) \left(v_{ij} - \frac{1}{r} n_{ij} \right) \right] \\ &= \epsilon^4 \frac{2\eta}{R} \left\{ (\mathbf{p} \cdot \mathbf{v})(\mathbf{q} \cdot \mathbf{v}) - \frac{1}{r} (\mathbf{p} \cdot \mathbf{n}_{12})(\mathbf{q} \cdot \mathbf{n}_{12}) \right\}. \end{aligned} \quad (5.67b)$$

To compute the two gravitational wave polarizations, one requires an expression for the radial separation vector \mathbf{r} and its first time derivative. It is efficient to give \mathbf{r} expanded in the observer's triad $(\mathbf{p}, \mathbf{q}, \mathbf{N})$. In [108], this was done by expressing \mathbf{r} in $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ first, and secondly to compute this base from $(\mathbf{p}, \mathbf{q}, \mathbf{N})$ as rotated around \mathbf{p} with the (constant) angle i_0 . The result reads

$$\begin{aligned} \mathbf{r} &= r \left[\{\cos \Upsilon \cos \varphi - C_\Theta \sin \Upsilon \sin \varphi\} \mathbf{p} \right. \\ &\quad + \{C_{i_0} \sin \Upsilon \cos \varphi - (S_{i_0} S_\Theta - C_{i_0} C_\Theta \cos \Upsilon) \sin \varphi\} \mathbf{q} \\ &\quad \left. + \{S_{i_0} \sin \Upsilon \cos \varphi + (C_{i_0} S_\Theta + S_{i_0} C_\Theta \cos \Upsilon) \sin \varphi\} \mathbf{N} \right], \end{aligned} \quad (5.68)$$

where C_{i_0} and S_{i_0} are shorthands for $\cos i_0$ and $\sin i_0$, respectively. The velocity vector $\mathbf{v} = d\mathbf{r}/dt$ is given by

$$\begin{aligned} \mathbf{v} &= r \left[\{\dot{\Theta} \sin \Theta \sin \Upsilon \sin \varphi - \dot{\Upsilon} (\cos \Theta \cos \Upsilon \sin \varphi + \sin \Upsilon \cos \varphi) \right. \\ &\quad \left. - \dot{\varphi} (\cos \Theta \sin \Upsilon \cos \varphi + \cos \Upsilon \sin \varphi) \right] \mathbf{p} \end{aligned}$$

$$\begin{aligned}
& + \{ \dot{\Theta} \sin \varphi (-(C_{i_0} \sin \Theta \cos \Upsilon + S_{i_0} \cos \Theta)) + C_{i_0} \dot{\Upsilon} (\cos \Upsilon \cos \varphi - \cos \Theta \sin \Upsilon \sin \varphi) \\
& \quad + \dot{\varphi} (\cos \varphi (C_{i_0} \cos \Theta \cos \Upsilon - S_{i_0} \sin \Theta) - C_{i_0} \sin \Upsilon \sin \varphi) \} \mathbf{q} \\
& + \{ \dot{\Theta} \sin \varphi (C_{i_0} \cos \Theta - S_{i_0} \sin \Theta \cos \Upsilon) + S_{i_0} \dot{\Upsilon} (\cos \Upsilon \cos \varphi - \cos \Theta \sin \Upsilon \sin \varphi) \\
& \quad + \dot{\varphi} (\cos \varphi (S_{i_0} \cos \Theta \cos \Upsilon + C_{i_0} \sin \Theta) - S_{i_0} \sin \Upsilon \sin \varphi) \} \mathbf{N} \}. \tag{5.69}
\end{aligned}$$

Having inserted above equations into (5.67), the final expressions for h_\times and h_+ with time dependent Θ and the case of quasi-circular orbits are given by

$$\begin{aligned}
h_\times|_{\mathbb{Q}}^{[r \equiv 0]} = & \quad \epsilon^4 \frac{2\eta}{R} \left\{ -\frac{1}{r} [(\cos \Upsilon \cos \varphi - \cos \Theta \sin \Upsilon \sin \varphi) \times \right. \\
& (C_{i_0} (\cos \Theta \cos \Upsilon \sin \varphi + \sin \Upsilon \cos \varphi) - S_{i_0} \sin \Theta \sin \varphi)] \\
& + r^2 [-(\dot{\Theta} \sin \Theta \sin \Upsilon \sin \varphi - \dot{\Upsilon} (\cos \Theta \cos \Upsilon \sin \varphi + \sin \Upsilon \cos \varphi) \\
& \quad - \dot{\varphi} (\cos \Theta \sin \Upsilon \cos \varphi + \cos \Upsilon \sin \varphi))] \times \\
& (\dot{\Theta} \sin \varphi (C_{i_0} \sin(\Theta) \cos \Upsilon + S_{i_0} \cos \Theta) \\
& \quad + C_{i_0} \dot{\Upsilon} (\cos \Theta \sin \Upsilon \sin \varphi - \cos \Upsilon \cos \varphi) \\
& \quad + \dot{\varphi} (-C_{i_0} \cos \Theta \cos \Upsilon \cos(\varphi) + S_{i_0} \sin \Theta \cos \varphi \\
& \quad \left. + C_{i_0} \sin \Upsilon \sin \varphi) \right\}, \tag{5.70}
\end{aligned}$$

$$\begin{aligned}
h_+|_{\mathbb{Q}}^{[r \equiv 0]} = & \quad \epsilon^4 \frac{2\eta}{R} \left\{ \frac{-1}{r} [(\cos \Upsilon \cos \varphi - \cos \Theta \sin \Upsilon \sin \varphi)^2 - (\sin i_0 \sin \Theta \sin \varphi \right. \\
& \quad - C_{i_0} (\cos \Theta \cos \Upsilon \sin \varphi + \sin \Upsilon \cos \varphi))^2] \\
& - r^2 [\dot{\Theta} \sin \varphi (C_{i_0} \sin \Theta \cos \Upsilon + S_{i_0} \cos \Theta) \\
& \quad + C_{i_0} \dot{\Upsilon} (\cos \Theta \sin \Upsilon \sin(\varphi) - \cos \Upsilon \cos \varphi) \\
& \quad + \dot{\varphi} (-C_{i_0} \cos \Theta \cos \Upsilon \cos \varphi + \sin i_0 \sin \Theta \cos \varphi + C_{i_0} \sin \Upsilon \sin \varphi)]^2 \\
& + r^2 [\dot{\Theta} (-\sin \Theta) \sin \Upsilon \sin \varphi + \dot{\Upsilon} (\cos \Theta \cos \Upsilon \sin \varphi + \sin \Upsilon \cos \varphi) \\
& \quad \left. + \dot{\varphi} (\cos \Theta \sin \Upsilon \cos \varphi + \cos \Upsilon \sin \varphi)]^2 \right\}. \tag{5.71}
\end{aligned}$$

If only the conservative dynamics is taken into account, then the above prescription is valid. An analytic investigation on GW from spinning compact binaries that incorporates the precession of the orbital plane *and* the precession of the total angular momentum \mathbf{J} is more involved. A prescription with the help of two additional rotation angles, that rotate the system in which \mathbf{J} is aligned to the \mathbf{e}_z axis will lead to *very* complicated differential equations for all angles, fast and slowly changing equally well, even though one regards only one spin, and the amplitude of \mathbf{L} can be obtained analytically in case the leading order RR Hamiltonian (see [40], and for an application [99]) is taken into account. Next, a possible computation will be proposed that regards also RR in a simple system configuration, where both involved masses are equal.

5.7 Radiation reaction with spin, perturbation theory and Lie series

A try to solve the problem of the inclusion of RR and reproducing the true evolution of the system with the help of computing order by order of inverse c , up to c^{-5} for the angles, may fail. The reason is that even the fast angles grow and accelerate in velocity in a secular manner as RR kicks in, and they are compact variables. That is why it seems to be difficult to separate the motion into a fast and a slow part finding a practical way to integrate the interaction terms out successively.

Let us, in contrast, do something more promising that also has more formal beauty. Fortunately, Wolfgang Gröbner has already given the solution to our problem some decades ago, as he provided a very general procedure to solve EOM with a large integrable part and a small perturbation.

For problems like this, he used the Lie series formalism [112]. The idea is to associate a linear differential operator \mathfrak{D} to a system of differential equations and to apply this operator in an exponential series to the initial values. Successive computing of the addends will give the perturbed special solution to the required order. Let us suppose \mathfrak{D} to have the explicit form

$$\mathfrak{D} = \alpha_1(\mathbf{x}) \frac{\partial}{\partial x_1} + \dots + \alpha_n(\mathbf{x}) \frac{\partial}{\partial x_n}, \quad (5.72)$$

where n is the number of independent variables and $\mathbf{x} = \{x_i\}$ with $i = (1, \dots, n)$. The α_i are functions of these variables. Then the operator \mathfrak{D} , applied to the variable x_i , will give

$$\mathfrak{D}x_i = \alpha_i(\mathbf{x}). \quad (5.73)$$

Under certain assumptions (holomorphy of the α_i), the series

$$e^{t\mathfrak{D}}f(\mathbf{x}) \equiv \sum_{\nu=0}^{\infty} \frac{t^\nu \mathfrak{D}^\nu}{\nu!} f(\mathbf{x}) = f(\mathbf{x}) + t\mathfrak{D}f(\mathbf{x}) + \frac{t^2}{2!}\mathfrak{D}^2f(\mathbf{x})\dots \quad (5.74)$$

converges absolutely and uniformly for some $|t| < T$. Defining \mathbf{X} to be, in components,

$$X_i \equiv (e^{t\mathfrak{D}}x_i) \text{ with } X_i|_{t=0} = x_i, \quad (5.75)$$

the following ‘‘exchange relation’’

$$F(\mathbf{X}) \equiv F(e^{t\mathfrak{D}}\mathbf{x}) = e^{t\mathfrak{D}}F(\mathbf{x}) \quad (5.76)$$

holds for the region of convergence. Computing the time derivative of the elements X_i , one can use the latter relation for the operator ∂_t as the function F and obtains

$$\frac{dX_i}{dt} = e^{t\mathfrak{D}}[\mathfrak{D}x_i] = e^{t\mathfrak{D}}[\alpha_i(\mathbf{x})] = \alpha_i(\mathbf{X}). \quad (5.77)$$

This shows that the X_i are solutions to Equation (5.77) in the region of convergence for the time t .

The next step is to split the operator \mathfrak{D} into one part \mathfrak{D}_1 , of which the solutions are exactly known, and another part \mathfrak{D}_2 perturbing this system of differential equations, both supposed to be holomorphic functions in the same surrounding of the point $\mathbf{x} = \{x_1, \dots, x_n\}$. Let us define the solution to the operators \mathfrak{D}_1 and \mathfrak{D}_2 as

$$\mathbf{X}^{(0)}(t, \mathbf{x}) \equiv e^{t\mathfrak{D}_1}\mathbf{x}, \quad (5.78)$$

$$\mathbf{X}(t, \mathbf{x}) \equiv e^{t\mathfrak{D}}\mathbf{x} = e^{t(\mathfrak{D}_1+\mathfrak{D}_2)}\mathbf{x}. \quad (5.79)$$

Inside the region of convergence, the series for \mathbf{X} can be re-arranged arbitrarily and cast into another more useful form

$$\mathbf{X}(t, \mathbf{x}) = \mathbf{X}^{(0)}(t, \mathbf{x}) + \sum_{\nu=0}^{\infty} \int_0^t d\tau \frac{(t-\tau)^\nu}{\nu!} (\mathfrak{D}_2 \mathfrak{D}^\nu \mathbf{x})|_{\mathbf{x}=\mathbf{X}^{(0)}}, \quad (5.80)$$

see [113] for further information. The operator \mathfrak{D}_2 can be shifted before the summation, which itself can also be exchanged with the integration, and what remains is

$$\mathbf{X}(t, \mathbf{x}) = \mathbf{X}^{(0)}(t, \mathbf{x}) + \int_0^t d\tau \mathfrak{D}_2 \mathbf{X}(t-\tau, \mathbf{x})|_{\mathbf{x}=\mathbf{X}^{(0)}(\tau, \mathbf{x})}. \quad (5.81)$$

This is an integral relation which can be solved iteratively. To any order $\nu + 1$, the solution reads

$$\begin{aligned} \mathbf{X}^{(1)}(t, \mathbf{x}) &= \mathbf{X}^{(0)}(t, \mathbf{x}) + \int_0^t [\mathfrak{D}_2 \mathbf{X}^{(0)}(t-\tau, \mathbf{x})]_{\mathbf{x}=\mathbf{X}^{(0)}(\tau, \mathbf{x})} d\tau, \\ &\vdots \\ \mathbf{X}^{(\nu+1)}(t, \mathbf{x}) &= \mathbf{X}^{(0)}(t, \mathbf{x}) + \int_0^t [\mathfrak{D}_2 \mathbf{X}^{(\nu)}(t-\tau, \mathbf{x})]_{\mathbf{x}=\mathbf{X}^{(0)}(\tau, \mathbf{x})} d\tau. \end{aligned} \quad (5.82)$$

It turns out that the solution gets a shape as Dyson series in quantum mechanics processes. For convergence issues, we note that this expression converges at least where the double series, Equation (5.79), converges absolutely [112, 113]. For a satisfying application of this algorithm, the operator \mathfrak{D}_2 has to be small; that means that the functions $\alpha_i^{(2)}(\mathbf{x})$ (the superscript 2 stands for the association to the second operator) are smaller in their magnitude in comparison to the coefficients $\alpha_i^{(1)}(\mathbf{x})$.

This algorithm applies excellently to the conservative problem of a binary of arbitrarily configured spins with unequal mass distribution, slightly deviating from the exact equal-mass case, as under the major headline of this section. We note that it has been shown that the perturbation theory of slightly unequal masses which we presented in the previous sections, matches with the results of the Lie series formalism in the appendix of [105]. Equally well, this scheme applies to what we posed recently above: a system of two equal-mass spins

in a circular orbit that evolves far way from the last stable orbit, such that RR will be only a very small correction to the conservative EOM. Let us take a look at the literature to see if the angular momentum vector does not only change its magnitude, but also its direction under RR. The LO-RR Hamiltonian only contains the position variables of the binary [99],

$$H_{\text{react}}^{\text{LO}} = \text{const.} \cdot \mathcal{I}_{ij}^{(3)} \left(p_i p_j - \frac{x_i x_j}{r^3} \right). \quad (5.83)$$

Computing the time derivatives of some quantity with respect to this Hamiltonian, the user should note that the Poisson bracket should be evaluated keeping the mass quadrupole as functions of time first, and, as soon as the bracket with the momentum and position dependent term is computed, replace \mathcal{I}_{ij} by its dependence on the canonical variables. The result of the time derivative of \mathbf{L} therefore reads

$$\frac{dL^i}{dt} = -\text{const.} L^i \left(\frac{L^2}{r^5} + \frac{1}{r^4} \right), \quad (5.84)$$

and \mathbf{L} turns out to decrease only its length, not its direction at LO RR. To lowest order in conservative dynamics and the case of circular orbits, we find

$$\Phi(L) \equiv \frac{dL}{dt} = -\frac{32\eta\epsilon^5}{5L^7}, \quad (5.85)$$

To 2.5 PN order, the motion of the spinning binary can be split into the one-spin spin-orbit evolution completed by the remainder built from the RR loss of \mathbf{L} . Let us choose $\mathbf{X} = \{\mathbf{L}, \mathbf{S}_1, \mathbf{S}_2\}$ as the 9 functions to be evolved. The EOM to be integrated read

$$\frac{d\mathbf{L}}{dt} = \epsilon^2 \frac{\mathbf{S}_{\text{eff}} \times \mathbf{L}}{r^3(L)} + \epsilon^5 \Phi(L) \mathbf{L}, \quad (5.86)$$

$$\frac{d\mathbf{S}_1}{dt} = \epsilon^2 \delta_1 \frac{\mathbf{L} \times \mathbf{S}_1}{r^3(L)} + 0, \quad (5.87)$$

$$\frac{d\mathbf{S}_2}{dt} = \epsilon^2 \delta_2 \frac{\mathbf{L} \times \mathbf{S}_2}{r^3(L)} + 0. \quad (5.88)$$

The operators \mathfrak{D}_1 and \mathfrak{D}_2 , therefore, read

$$\mathfrak{D}_1 \equiv \epsilon^2 \left[\frac{\epsilon_{ijk} (\delta_1 S_{1j} + \delta_2 S_{2j}) L_k \partial_{L_i}}{r^3(L)} - \delta_1 \frac{\epsilon_{ijk} S_{1j} L_k \partial_{S_{1i}}}{r^3(L)} - \delta_2 \frac{\epsilon_{ijk} S_{2j} L_k \partial_{S_{2i}}}{r^3(L)} \right], \quad (5.89)$$

$$\mathfrak{D}_2 \equiv \epsilon^5 \Phi(L) \frac{L^i}{L} \partial_{L_i}. \quad (5.90)$$

Important note: By construction of the solution using Lie series, the operator \mathfrak{D}_2 differentiates with respect to the *initial* values of the components $L_i^{(0)}$, not to the actual ones. For the full motion, given by Equations (5.86), (5.87), and (5.88) $\mathbf{X}(t)$ is given by the Lie series

$$\mathbf{X}(t) = e^{t(\mathfrak{D}_1 + \mathfrak{D}_2)} \mathbf{X}(t=0) = e^{t(\mathfrak{D}_1 + \mathfrak{D}_2)} \mathbf{x}. \quad (5.91)$$

The relation for the perturbative functions can be computed using the unperturbed one,

associated with the conservative equal-mass case.¹² The generic functions therein,

$$\mathbf{X}^{(0)} = \{\mathbf{L}^{(0)}, \mathbf{S}_1^{(0)}, \mathbf{S}_2^{(0)}\}, \text{ read, using the rotation matrices from the beginning of Section 5,}$$

$$\mathbf{L}^{(0)} = \{[\Theta(t)][\Upsilon(t)]^{-1}(0, 0, L|_{t=0}), \quad (5.92)$$

$$\mathbf{S}_1^{(0)} = \{[\alpha_{\text{ks}}(t)][\Theta(t)][\Upsilon(t)]\}^{-1} S_1(\cos \phi_s \sin \tilde{s}, \sin \phi_s \sin \tilde{s}, \cos \tilde{s}), \quad (5.93)$$

$$\mathbf{S}_2^{(0)} = \mathbf{J}^{(0)} - \mathbf{L}^{(0)} - \mathbf{S}_1^{(0)}, \quad (5.94)$$

with *constant* angular velocities, given by Equations (5.25), depending on L_0 . We note that, to Newtonian accuracy, the radial distance r depends on L as

$$r(L) = L^2. \quad (5.95)$$

As L evolves within the perturbation theory, r should be adapted accordingly through the desired order. The first order solutions formally read

$$\mathbf{L}^{(1)}(t) - \mathbf{L}^{(0)}(t) = \int_0^t \left\{ \mathfrak{D}_2 \mathbf{L}^{(0)}(t - \tau, \mathbf{x}) \right\}_{\mathbf{x}=\mathbf{X}^{(0)}(\tau, \mathbf{x})} d\tau, \quad (5.96a)$$

$$\mathbf{S}_1^{(1)}(t) - \mathbf{S}_1^{(0)}(t) = \int_0^t \left\{ \mathfrak{D}_2 \mathbf{S}_1^{(0)}(t - \tau, \mathbf{x}) \right\}_{\mathbf{x}=\mathbf{X}^{(0)}(\tau, \mathbf{x})} d\tau. \quad (5.96b)$$

$$\mathbf{S}_2^{(1)}(t) - \mathbf{S}_2^{(0)}(t) = \int_0^t \left\{ \mathfrak{D}_2 \mathbf{S}_2^{(0)}(t - \tau, \mathbf{x}) \right\}_{\mathbf{x}=\mathbf{X}^{(0)}(\tau, \mathbf{x})} d\tau. \quad (5.96c)$$

To grasp the full picture, we use Equations (5.82) iteratively. We note that this procedure makes no sense if it is expanded to the last stable orbit and beyond respectively. A large amount of the orbital angular momentum will be radiated away, but the final black hole will in general be a Kerr black hole in most cases, especially when the spins and orbital angular momentum are aligned, such that even without any spin, not all the angular momentum can be lost. There exists literature on how much spin the final black hole has [114]. We conclude that, what we can say about our PN prescription in general, it is only valid in a region where the velocities are small and the radiation will not contribute significant kinetic terms to the orbital dynamics.

The next section reduces the difficulties in finding orbital solutions drastically. We like to apply what we have learned about GW emission and try to do some considerations that GW analysts will be interested in. It is, as we will see, nontrivial to find accurate and “end-user friendly” Fourier domain wave forms even in the case of non-spinning compact sources.

¹²The unequal-mass case includes perturbation theory in *two* parameters, namely the difference in masses and the prefactor of the RR term for L . This needs special care and will be task of a future publication.

6 Full-analytic Fourier domain gravitational wave forms for inspiralling eccentric compact binaries

Why do we need time Fourier domain wave forms?

For a successful data analysis consideration, one has to find an appropriate *filter*. A filter is a function, usually provided with some parameters, which the noisy data from a detector is convolved or cross-correlated with and a scalar correlation value is obtained. This operation is necessary as the analyst likes to determine whether the data contains a GW signal whose shape the theorists have computed. A set of parameters, as we have seen in the previous sections, could be the two masses of the binary, the binding energy at the time as the binary's emitted GW frequency enters the detector's bandwidth, the corresponding eccentricity, spin amplitudes and various inclination angles, and so on. In case it *is* contained, it is most likely deeply digged in noises of various types that are much bigger than the signal itself, and has to be "digged out" with the help of special algorithms. One may wonder how it is possible to extract wave forms that are much smaller than the noise, but special characteristics of the noise enable us to do so. For example, one can impose a special distribution on the noise or try to reduce its effect by averaging the signal over time intervals of selected lengths.

Let us come back to the filter for a moment. It depends on the properties of the detector and its sensitivity in a special frequency band, and can be used in such a way that it maximises some preferred quantity, such as for example the conditional probability of the measured data points, represented by the signal s , given that a certain assumed hypothesis is true. A hypothesis could be that the GW signal is assumed to have a specific form, or belongs to a certain template family h_θ provided with some parameters θ . This probability is the frequentist's definition of the so-called *likelihood function* Λ . Assuming that the signal indeed contains the wave, it will be constituted of the theoretical exact wave form h_θ and some noise n , $s = h_\theta + n$, and Λ turns out to be

$$\Lambda(s|h) = \mathcal{N} \exp \left[(h_\theta|s) - \frac{1}{2}(h_\theta|h_\theta) - \frac{1}{2}(s|s) \right], \quad (6.1)$$

where \mathcal{N} is a normalisation factor, $(a|b)$ is an inner product of the form

$$(a|b) = 4\Re \int_0^\infty \frac{\tilde{a}^*(f)\tilde{b}(f)}{S_n(f)} df, \quad (6.2)$$

a and b being some real functions in time domain, and $S_n(f)$ is the one-sided noise-spectral density¹³, which tells us how sensitive the detector is at some frequency f , or equivalently in contrast, how auto-correlated the noise is. This inner product clearly reflects important features of the detector, namely how well it sees/resolves at some frequency. For an overview

¹³One could rewrite the integral to both sides of $f = 0$, and then one would use the two-sided $S_n(f)$.

over the current detector’s model noise densities, see [115] and references therein. A virtue of using the likelihood function to be maximised, one can extract the distance R from the detector to the source in relatively easy mathematical steps. Now, one can make use of Baye’s formula to obtain knowledge of what we desire much more: the conditional probability of the hypothesis to be true, given the data points $s(t)$ were measured. We will not go into details; the interested reader will find the calculation in the first pages of [116]. We also like not to forget to provide an important *Note*: An issue worth being mentioned about parameter estimation at this point is the following. A detector is *unable* to distinguish between a set of frequencies f , masses m_1 and m_2 , the distance to the source D , and their Doppler-shifted versions, i.e. f/λ , λm_1 , λm_2 , and λD , seen by the detector in case the source is in relative motion with Doppler shift factor λ . The reason is that, due to the combinations in what times and distances appear, the wave form is always dimensionless. See, for example, [117]. For more on the topic of GW data analysis, read [116], [118] and [119] as introductory material, [10], and for an application to data analysis with LISA, [120]. A sound quantity, the so-called Fitting Factor (FF)¹⁴, for an estimation of the various GW templates in use, compared vis-a-vis among themselves, and its implication onto the quality and usefulness of a search template, irrespective of the detector, is provided in [4].

By construction, this computation is defined in the Fourier domain only. If one attempts to compare signal and template, one is therefore forced to go to the time Fourier domain, either numerically or analytically. Numerical methods are well-known and advanced to a very fast level. Anyway, to save computing power, one prefers an analytic prescription. Additionally, by that one succeeds in learning more about the structure of GW forms when eccentricity is not negligible. The data analyst is, in general, interested to reduce the system to essential degrees of freedom and to reduce the computational costs as well, such that analytic prescriptions may be the preferred one. Analytic GW templates do not make numerical relativity obsolete, but are advantageous for detectors looking at early stages of the inspiral, where hundreds of cycles would be necessary to be computed with the help of NR. As it could be a very desirable result, an analytic approximation of the Fitting Factor could be an interesting assignment for a future publication. Let us, for the time being, start with the basics and try to calculate analytic time Fourier domain wave forms for non-spinning eccentric compact binaries.

In the last section of this thesis, we discuss full-analytical GW forms for eccentric non-spinning compact binaries of arbitrary mass ratio in the time Fourier domain. Therefore, we go up to 2PN in the harmonic GW amplitude and conservative orbital dynamics.¹⁵

Historical background: The leading-order GW energy flux in terms of its harmonic consti-

¹⁴The FF is, in principle, the value of the integral (6.2), taking some hypothetical signal as a , some template h_θ as b , and maximising this over all appearing parameter entries.

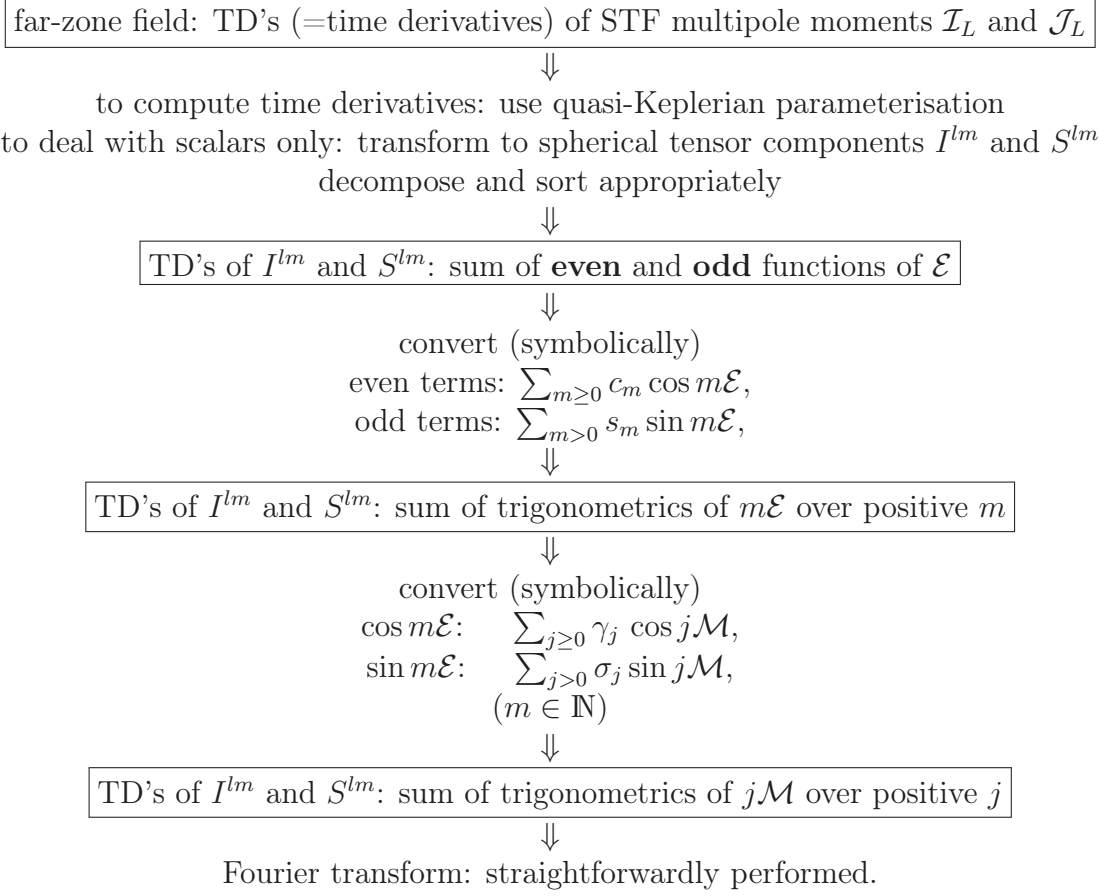
¹⁵The problem of including spin will result in additional GW amplitude corrections, as we have already seen in Section 4, and more difficult solutions to the orbital dynamics. For example, the inclusion of spin without making assumptions about their initial orientation will provoke the additional frequency of one spin precession. Assuming the spins to be aligned with \mathbf{L} will not have this problem, but the additional amplitude and KE corrections remain.

tutive parts, was given through 1PN order in the conservative orbital dynamics, for the first time – to our best knowledge – in [96] in the extreme mass-ratio regime. The GW forms were given the shape of a multiple summation over harmonics with mixed positive and negative frequencies with undetermined coefficients by those authors. First attempts, however, for a time Fourier domain (TFD) of eccentric h_+ and h_\times have been made in Reference [121] where the periastron advance has been incorporated by hand and the stationary phase approximation has come into use. Later computations were done in [122] and [123], where a spectral analysis for steady-state binaries in the simple case of Newtonian motion and amplitude has been performed. Recently in [124], the authors furnished Newtonian accurate TFD wave forms as they incorporated the sine and cosine functions of the eccentric anomaly as Fourier–Bessel series. Taking as starting point h_+ and h_\times themselves, having evaluated all appearing scalars (such as, for example, the scalar product of the basic vectors of the plane of the sky with the orbital velocity vector) being evaluated explicitly, would lead to rather complicated relations of rotation angles connecting the orientation of the binary’s orbital plane with respect to the position of the observer (see, e.g. [63], and – including spin precession – [75] for higher orders of h_+ and h_\times) and this is not well-suited for a systematic TFD representation at higher orders.

The problem of non-spinning compact binaries implies the conservation of the direction of the orbital angular momentum \mathbf{L} , thus, it qualifies for a representation with the help of tensor spherical harmonics [10, 17, 30]. Reference [125] provided 1PN accurate spherical tensor components for non-spinning compact binaries in quasi-elliptic and quasi-hyperbolic orbits that we are on to verify. Irrespective of the observer’s orientation, the form the GW signal can be expressed with the help of these spherical tensor components and applied to give the polarisations h_+ and h_\times afterwards with the help of constant rotations. The considerations rely on [126] and [127].

Schematic prescription of our calculation

For convenience, we show a flow diagram of our computation. Preliminary results have been surrounded by boxes.



The single steps are being detailed below.

6.1 Orbital motion: 2PN accurate Quasi-Keplerian parameterisation

We compute the time derivatives of the multipole moments using the orbital parameterisation through 2PN from [79] in harmonic coordinates to avoid transformation terms as in Section 4.4. For convenience, we recall its symbolical appearance,

$$r = a_r (1 - e_r \cos \mathcal{E}) , \quad (6.3)$$

$$\mathcal{M} \equiv \mathcal{N}(t - t_0) = \mathcal{E} - e_t \sin \mathcal{E} + \epsilon^4 [\mathcal{F}_{v-\mathcal{E}}(v - \mathcal{E}) + \mathcal{F}_v \sin v] , \quad (6.4)$$

$$\frac{2\pi}{\Phi} (\phi - \phi_0) = v + (\epsilon^4 \mathcal{G}_{2\phi}) \sin 2v + (\epsilon^4 \mathcal{G}_{3\phi}) \sin 3v , \quad (6.5)$$

$$v = 2 \arctan \left[\left(\frac{1 + e_\phi}{1 - e_\phi} \right)^{1/2} \tan \frac{\mathcal{E}}{2} \right] . \quad (6.6)$$

The explicit 2PN accurate expressions for the orbital elements and functions of the generalised quasi-Keplerian parameterisation read

$$a_r = \frac{1}{(2|E|)} \left\{ 1 + \epsilon^2 \frac{(2|E|)}{4} (-7 + \eta) + \epsilon^4 \frac{(2|E|)^2}{16} \left[1 + \eta^2 + \frac{16}{(2|E|L^2)} (-4 + 7\eta) \right] \right\} , \quad (6.7)$$

$$e_r^2 = 1 - 2|E|L^2 + \epsilon^2 \frac{(2|E|)}{4} \left\{ 24 - 4\eta + 5(-3 + \eta)(2|E|L^2) \right\} + \epsilon^4 \frac{(2|E|)^2}{8} \left\{ 60 + 148\eta + 2\eta^2 - (2|E|L^2)(80 - 45\eta + 4\eta^2) + \frac{32}{(2|E|L^2)}(4 - 7\eta) \right\} , \quad (6.8)$$

$$\mathcal{N} = (2|E|)^{3/2} \left\{ 1 + \epsilon^2 \frac{(2|E|)}{8} (-15 + \eta) + \epsilon^4 \frac{(2|E|)^2}{128} \left[555 + 30\eta + 11\eta^2 + \frac{192}{\sqrt{(2|E|L^2)}}(-5 + 2\eta) \right] \right\} , \quad (6.9)$$

$$\mathcal{K} = \underbrace{\epsilon^2 \frac{6|E|}{\mathcal{W}_{e_t}^2}}_{\mathcal{K}_{1\text{PN}}} + \epsilon^4 \underbrace{\frac{-3|E|^2((9\eta - 22)e_t^2 + 9\eta - 21)}{\mathcal{W}_{e_t}^4}}_{\mathcal{K}_{2\text{PN}}} , \quad (6.10)$$

$$e_t^2 = 1 - 2|E|L^2 + \epsilon^2 \frac{(2|E|)}{4} \left\{ -8 + 8\eta - (2|E|L^2)(-17 + 7\eta) \right\} + \epsilon^4 \frac{(2|E|)^2}{8} \left\{ 12 + 72\eta + 20\eta^2 - 24\sqrt{(2|E|L^2)}(-5 + 2\eta) - (2|E|L^2)(112 - 47\eta + 16\eta^2) - \frac{16}{(2|E|L^2)}(-4 + 7\eta) + \frac{24}{\sqrt{(2|E|L^2)}}(-5 + 2\eta) \right\} , \quad (6.11)$$

$$\mathcal{F}_{v-\mathcal{E}} = -\frac{3(2|E|)^2}{2} \left\{ \frac{1}{\sqrt{(2|E|L^2)}}(-5 + 2\eta) \right\} , \quad (6.12)$$

$$\mathcal{F}_v = -\frac{(2|E|)^2}{8} \left\{ \frac{\sqrt{1-2|E|L^2}}{\sqrt{(2|E|L^2)}} \eta (-15 + \eta) \right\} , \quad (6.13)$$

$$\Phi = 2\pi \left\{ 1 + \epsilon^2 \frac{3}{L^2} + \epsilon^4 \frac{(2|E|)^2}{4} \left[\frac{3}{(2|E|L^2)}(-5 + 2\eta) - \frac{15}{(2|E|L^2)^2}(-7 + 2\eta) \right] \right\} , \quad (6.14)$$

$$\mathcal{G}_{2\phi} = \frac{(2|E|)^2}{8} \left\{ \frac{1-2|E|L^2}{(2|E|L^2)^2} (1 + 19\eta - 3\eta^2) \right\}, \quad (6.15)$$

$$\mathcal{G}_{3\phi} = -\frac{(2|E|)^2}{32} \left\{ \frac{(1-2|E|L^2)^{3/2}}{(2|E|L^2)^2} \eta (-1 + 3\eta) \right\}, \quad (6.16)$$

$$\begin{aligned} e_\phi^2 &= 1 - 2|E|L^2 + \epsilon^2 \frac{(2|E|)^2}{4} \left\{ 24 + (2|E|L^2)(-15 + \eta) \right\} \\ &+ \epsilon^4 \frac{(2|E|)^2}{16} \left\{ -40 + 34\eta + 18\eta^2 - (2|E|L^2)(160 - 31\eta + 3\eta^2) \right. \\ &\left. - \frac{1}{(2|E|L^2)} (-416 + 91\eta + 15\eta^2) \right\}, \end{aligned} \quad (6.17)$$

where we introduced the abbreviation

$$\mathcal{W}_{e_t} \equiv \sqrt{1 - e_t^2}. \quad (6.18)$$

There are 2PN accurate relations connecting the three eccentricities e_r, e_t and e_ϕ . These relations read

$$\begin{aligned} e_t &= e_r \left\{ 1 + \epsilon^2 \frac{(2|E|)^2}{2} (3\eta - 8) + \epsilon^4 \frac{(2|E|)^2}{4} \frac{1}{(2|E|L^2)} \left[-16 + 28\eta \right. \right. \\ &\left. \left. + (-30 + 12\eta)\sqrt{(2|E|L^2)} + (36 - 19\eta + 6\eta^2)(2|E|L^2) \right] \right\}, \end{aligned} \quad (6.19a)$$

$$\begin{aligned} e_\phi &= e_r \left\{ 1 + \epsilon^2 \frac{(2|E|)^2}{2} \eta + \epsilon^4 \frac{(2|E|)^2}{32} \frac{1}{(2|E|L^2)} \left[160 + 357\eta - 15\eta^2 \right. \right. \\ &\left. \left. + (-\eta + 11\eta^2)(2|E|L^2) \right] \right\}. \end{aligned} \quad (6.19b)$$

Note that the set $(|E|, e_t, \phi_0, t_0 \leftrightarrow \mathcal{M}_0)$ is completely sufficient to describe the initial data of the system in its plane perpendicular to \mathbf{L} , thus, we eliminate L for the benefit of e_t in each expression with the help of Equation (6.11). In the next section, we list the relevant GW moments and their time derivatives.

6.2 Relevant multipoles and their time derivatives in harmonic coordinates

The far-zone regime GW forms can be expressed in terms of the mass- and current-type multipole moments [14, 15, 17, 128],

$$\begin{aligned} h_{km}^{\text{TT}}(\mathbf{R}, t) &= \frac{G}{c^4 R} \mathcal{P}_{kmij}(\mathbf{N}) \sum_{l=2}^{\infty} \left[\left(\frac{1}{c} \right)^{l-2} \left(\frac{4}{l!} \right) \mathcal{I}_{ij\mathcal{A}_{l-2}}^{(l)}(t - R/c) N_{\mathcal{A}_{l-2}} \right. \\ &\left. + \left(\frac{1}{c} \right)^{l-1} \left(\frac{8l}{(l+1)!} \right) \epsilon_{pq(i} \mathcal{J}_{j)p\mathcal{A}_{l-2}}^{(l)}(t - R/c) N_{q\mathcal{A}_{l-2}} \right]. \end{aligned} \quad (6.20)$$

Here, A_l is a multi-index with $A_l \equiv a_1 \dots a_l$, indices with round brackets are symmetrised over, viz. for example $A_{(ij)} \equiv \frac{1}{2}(A_{ij} + A_{ji})$, and the following definitions came to use:

$$\mathbf{R} \equiv \mathbf{N} R, \quad (6.21)$$

$$\mathcal{I}_{A_{l-2}} \equiv \mathcal{I}_{a_1 \dots a_{l-2}}, \quad (6.22)$$

$$N_{A_{l-2}} \equiv N_{a_1} \dots N_{a_{l-2}}, \quad (6.23)$$

$$P_{ijkl}(\mathbf{N}) \equiv (\delta_{ik} - N_{ik})(\delta_{jl} - N_{jl}) - \frac{1}{2}(\delta_{ij} - N_{ij})(\delta_{kl} - N_{kl}). \quad (6.24)$$

R denotes the distance from the observer to the binary. The quantity N with multiple indices is a tensor product of components N_i of \mathbf{N} from N_{a_1} to $N_{a_{l-2}}$, and the superscript (l) denotes the l^{th} time derivative. In the current case of 2PN accurate orbital dynamics, we have to evaluate the GW amplitude, Equation (6.20), consistently to 2PN order relative to the leading term, which kicks in at ϵ^4 . The mass-type multipoles (only their non-tail parts) relevant for the above equation read [20]

$$\begin{aligned} \mathcal{I}_{ij} = & \mu \text{STF}_{ij} \left\{ x^{ij} \left[1 + \frac{1}{42 c^2} \left((29 - 87\eta)v^2 - (30 - 48\eta)\frac{Gm}{r} \right) \right. \right. \\ & + \frac{1}{c^4} \left(\frac{1}{504} (253 - 1835\eta + 3545\eta^2)v^4 + \frac{1}{756} (2021 - 5947\eta - 4883\eta^2)\frac{Gm}{r} v^2 \right. \\ & - \left. \left. \frac{1}{756} (131 - 907\eta + 1273\eta^2)\frac{Gm}{r} \dot{r}^2 - \frac{1}{252} (355 + 1906\eta - 337\eta^2)\frac{G^2 m^2}{r^2} \right) \right] \\ & - x^i v^j \left[\frac{r \dot{r}}{42 c^2} (24 - 72\eta) + \frac{r \dot{r}}{c^4} \left(\frac{1}{63} (26 - 202\eta + 418\eta^2)v^2 \right. \right. \\ & + \left. \left. \frac{1}{378} (1085 - 4057\eta - 1463\eta^2)\frac{Gm}{r} \right) \right] \\ & + v^{ij} \left[\frac{r^2}{21 c^2} (11 - 33\eta) + \frac{r^2}{c^4} \left(\frac{1}{126} (41 - 337\eta + 733\eta^2)v^2 \right. \right. \\ & + \left. \left. \frac{5}{63} (1 - 5\eta + 5\eta^2)\dot{r}^2 + \frac{1}{189} (742 - 335\eta - 985\eta^2)\frac{Gm}{r} \right) \right] \left. \right\}, \quad (6.25) \end{aligned}$$

$$\begin{aligned} \mathcal{I}_{ijk} = & - \left(\mu \frac{\delta_m}{m} \right) \text{STF}_{ijk} \left\{ x^{ijk} \left[1 + \frac{1}{6 c^2} \left((5 - 19\eta)v^2 - (5 - 13\eta)\frac{Gm}{r} \right) \right] \right. \\ & - \left. x^{ij} v^k \left[\frac{r \dot{r}}{c^2} (1 - 2\eta) \right] + x^i v^{jk} \left[\frac{r^2}{c^2} (1 - 2\eta) \right] \right\}, \quad (6.26) \end{aligned}$$

$$\begin{aligned} \mathcal{I}_{ijkl} = & \mu \text{STF}_{ijkl} \left\{ x^{ijkl} \left[(1 - 3\eta) + \frac{1}{110 c^2} \left((103 - 735\eta + 1395\eta^2)v^2 \right. \right. \right. \\ & - \left. \left. (100 - 610\eta + 1050\eta^2)\frac{Gm}{r} \right) \right] - v^i x^{jkl} \left\{ \frac{72 r \dot{r}}{55 c^2} (1 - 5\eta + 5\eta^2) \right\} \\ & + \left. v^{ij} x^{kl} \left\{ \frac{78 r^2}{55 c^2} (1 - 5\eta + 5\eta^2) \right\} \right\}, \quad (6.27) \end{aligned}$$

$$\mathcal{I}_{ijklm} = -\left(\mu \frac{\delta_m}{m}\right) (1 - 2\eta) \text{STF}_{ijklm} \{x^{ijklm}\}, \quad (6.28)$$

$$\mathcal{I}_{ijklmn} = \mu(1 - 5\eta + 5\eta^2) \text{STF}_{ijklmn} \{x^{ijklmn}\}. \quad (6.29)$$

The current-type moments (only their non-tail parts as well) read

$$\begin{aligned} \mathcal{J}_{ij} &= -\left(\mu \frac{\delta_m}{m}\right) \text{STF}_{ij} \epsilon_{jab} \left\{ x^{ia} v^b \left[1 + \frac{1}{28 c^2} \left((13 - 68\eta)v^2 + (54 + 60\eta) \frac{Gm}{r} \right) \right] \right. \\ &\quad \left. + v^{ib} x^a \left[\frac{r \dot{r}}{28 c^2} (5 - 10\eta) \right] \right\}, \end{aligned} \quad (6.30)$$

$$\mathcal{J}_{ijkl} = -\left(\mu \frac{\delta_m}{m} (1 - 2\eta)\right) \text{STF}_{ijkl} \{ \epsilon_{lab} x^{ajk} v^b \}, \quad (6.31)$$

$$\mathcal{J}_{ijklm} = \left(\mu (1 - 5\eta + 5\eta^2)\right) \text{STF}_{ijklm} \{ \epsilon_{mab} x^{ajkl} v^b \}. \quad (6.32)$$

The notation $\text{STF}_{ij\dots}$ denotes the symmetric trace-free part of the tensor with indices $ij\dots$. The GW amplitude, and from that computed, the far zone angular momentum and energy transport, has been completed to 3PN in [25, 59, 129]. From Thorne's paper [17], see his Equation (4.3), we also extract that the GW amplitude can equivalently be expressed in terms of tensor spherical harmonics,

$$\begin{aligned} h_{jk}^{\text{TT}} &= \frac{G}{c^4 R} \sum_{l=2}^{\infty} \sum_{m=-l}^{+l} \left[\left(\frac{1}{c}\right)^{l-2} I^{lm}(t - R/c) T_{jk}^{E2,lm}(\theta, \phi) \right. \\ &\quad \left. + \left(\frac{1}{c}\right)^{l-1} S^{lm}(t - R/c) T_{jk}^{B2,lm}(\theta, \phi) \right]. \end{aligned} \quad (6.33)$$

The components I^{lm} and S^{lm} are projected out of Equation (6.20) with the help of the spherical components $Y_{A_l}^{lm}$. The basis transformation and the explicit representation of the $Y_{A_l}^{lm}$, taking the direction of the orbital angular momentum as the preferred one, $\mathbf{L} = h \mathbf{e}_z$, reads

$$I^{lm}(t) = \frac{16\pi}{(2l+1)!!} \left[\frac{(l+1)(l+2)}{2(l-1)l} \right]^{1/2} \mathcal{I}_{A_l}(t) Y_{A_l}^{lm*}, \quad (6.34)$$

$$S^{lm}(t) = \frac{-32\pi}{(l+1)(2l+1)!!} \left[\frac{(l+1)(l+2)}{2(l-1)l} \right]^{1/2} \mathcal{J}_{A_l}(t) Y_{A_l}^{lm*}, \quad (6.35)$$

$$\begin{aligned} Y_{A_l}^{lm} &\stackrel{A}{=} (-1)^m (2l-1)!! \left[\frac{2l+1}{4\pi(l-m)!(l+m)!} \right]^{1/2} (\delta_{i_1}^1 + i\delta_{i_1}^2) \dots (\delta_{i_m}^1 + i\delta_{i_m}^2) \delta_{i_{m+1}}^3 \dots \delta_{i_l}^3 \\ &\stackrel{B}{=} (-1)^m \left(\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!} \right)^{1/2} \sum_{k=0}^{\lfloor \frac{1}{2}(l-m) \rfloor} \frac{(-1)^k (2l-2k-1)!!}{(l-m-2k)!(2k)!!} \times \\ &\quad \delta_{(i_1 i_2 \dots i_{2k-1} i_{2k})} (\delta_{i_{2k+1}}^1 + i\delta_{i_{2k+1}}^2) \dots (\delta_{i_{2k+m}}^1 + i\delta_{i_{2k+m}}^2) \dots \delta_{i_{2k+m+1}}^3 \dots \delta_{i_l}^3, \end{aligned} \quad (6.36)$$

$$Y_{A_l}^{lm*} = (-1)^m Y_{A_l}^{l|m|*} \quad \text{for } (m < 0), \quad (6.37)$$

where the notation “[x]” denotes the integer number which is the largest smaller than or equal to x . From indices comprised by \langle and \rangle , the STF parts have to be taken. The number l tells us what type of moment we have: quadrupole for $l = 2$, octupole for $l = 3$, hexadecapole for $l = 4$, \dots . From Equation (6.37) we see that we only have to compute the moments for $m = 0 \dots l$. Equal sign B above, taken from Appendix (A3) of [128], may be more practical for programming than A. The relevant pure-spin tensor harmonics, $T_{jk}^{E2,lm}$ and $T_{jk}^{B2,lm}$, are given in our Appendix A.4. The reader can find the 1PN outputs in [125], which we also used for [126]. In case of 2PN accurate conservative dynamics, the GW amplitude, consistently worked out to the same relative order in $c^{-1} = \epsilon$ and having moved to our scaling, explicitly reads

$$\begin{aligned}
h_{ij}^{TT} = & \frac{\epsilon^4 \eta}{R \mu} \left\{ \sum_{m=-2}^2 I^{2m} T_{ij}^{E2,2m} + \epsilon \left[\sum_{m=-2}^2 S^{2m} T_{ij}^{B2,2m} + \sum_{m=-3}^3 I^{3m} T_{ij}^{E2,3m} \right] \right. \\
& + \epsilon^2 \left[\sum_{m=-3}^3 S^{3m} T_{ij}^{B2,3m} + \sum_{m=-4}^4 I^{4m} T_{ij}^{E2,4m} \right] \\
& + \epsilon^3 \left[\sum_{m=-4}^4 S^{4m} T_{ij}^{B2,4m} + \sum_{m=-5}^5 I^{5m} T_{ij}^{E2,5m} \right] \\
& \left. + \epsilon^4 \left[\sum_{m=-5}^5 S^{5m} T_{ij}^{B2,5m} + \sum_{m=-6}^6 I^{6m} T_{ij}^{E2,6m} \right] \right\}. \tag{6.38}
\end{aligned}$$

The unit of the reduced mass μ will always scale out with the prefactors of the tensor components. To reach there, we first have to compute the l^{th} time derivatives of the STF mass and current moments. We obtain them by means of the accelerations for a compact binary in harmonic coordinates to 2PN order, $\mathbf{a} = \mathbf{a}_N + \mathbf{a}_{1\text{PN}} + \mathbf{a}_{2\text{PN}}$, (also taken from [20], written in *their units*),

$$\mathbf{a}_N = -\frac{Gm}{r^2} \mathbf{n}_{12}, \tag{6.39}$$

$$\mathbf{a}_{1\text{PN}} = -\frac{Gm}{c^2 r^2} \left\{ \left[-2(2 + \eta) \frac{Gm}{r} + (1 + 3\eta)v^2 - \frac{3}{2}\eta\dot{r}^2 \right] \mathbf{n}_{12} - 2(2 - \eta)\dot{r}\mathbf{v} \right\}, \tag{6.40}$$

$$\begin{aligned}
\mathbf{a}_{2\text{PN}} = & -\frac{Gm}{c^4 r^2} \left\{ \left[\frac{3}{4}(12 + 29\eta) \frac{G^2 m^2}{r^2} + \eta(3 - 4\eta)v^4 + \frac{15}{8}\eta(1 - 3\eta)\dot{r}^4 \right. \right. \\
& - \frac{3}{2}\eta(3 - 4\eta)v^2\dot{r}^2 - \frac{1}{2}\eta(13 - 4\eta) \frac{Gm}{r} v^2 - (2 + 25\eta + 2\eta^2) \frac{Gm}{r} \dot{r}^2 \left. \right] \mathbf{n}_{12} \\
& \left. - \frac{1}{2} \left[\eta(15 + 4\eta)v^2 - (4 + 41\eta + 8\eta^2) \frac{Gm}{r} - 3\eta(3 + 2\eta)\dot{r}^2 \right] \dot{r}\mathbf{v} \right\}. \tag{6.41}
\end{aligned}$$

To move to unscaled variables, use the scaling of Equations (4.3) - (4.5), such that effectively all the factors Gm are to be replaced by “1” and inverse c by ϵ . The results are lengthy, easily reproducible and needed only temporarily; thus, they are not listed here. We employ the orbital parameterisation, Equations (6.3) - (6.19b), to obtain the normal and the velocity

vector \mathbf{n}_{12} and \mathbf{v} in spherical coordinates, symbolically

$$\mathbf{r} = \mathbf{n}_{12} r, \quad (6.42)$$

$$\mathbf{n}_{12} = \{\cos(\phi), \sin(\phi), 0\}, \quad (6.43)$$

$$\mathbf{v} \equiv \frac{d\mathbf{r}}{dt} = \left[\frac{\partial \mathbf{r}}{\partial \mathcal{E}} \right] \left[\frac{\partial \mathcal{E}}{\partial \mathcal{M}} \right] \left[\frac{\partial \mathcal{M}}{\partial t} \right]. \quad (6.44)$$

Equation (6.44) is to be computed with the help of the KE (6.4). Again, it is not necessary to provide the velocity as functions of \mathcal{E} and ϕ because these terms are easy to be reproduced. With this input we can compute the spherical tensor components of Equation (6.38). The next section gives the results, using the 2PN accurate QKP in harmonic coordinates.

6.3 Gravitational wave forms in terms of tensor spherical harmonics

Defining

$$A(\mathcal{E}) \equiv 1 - e_t \cos \mathcal{E}, \quad (6.45)$$

we get

$$\begin{aligned} I^{20(2)} = & 16\sqrt{\frac{\pi}{15}}|E|\mu \left\{ 1 - \frac{1}{A(\mathcal{E})} + |E|\epsilon^2 \left[-\frac{2(\eta - 26)\mathcal{W}_{e_t}^2}{7A(\mathcal{E})^3} + \frac{3(3\eta - 1)}{14A(\mathcal{E})} + \frac{2(\eta - 26)}{7A(\mathcal{E})^2} \right. \right. \\ & - \frac{3}{14}(3\eta - 1) \left. \right] + \epsilon^4|E|^2 \left[\frac{1}{42}(5 - \eta(11\eta + 16)) + \frac{1}{42\mathcal{W}_{e_t}^2 A(\mathcal{E})} \left(252\mathcal{W}_{e_t}(5 - 2\eta) \right. \right. \\ & - (\eta(11\eta + 16) - 5)(e_t^2 - 1) \left. \right) + \frac{1}{126\mathcal{W}_{e_t}^2 A(\mathcal{E})^2} \left(((1877 - 323\eta)\eta - 3682)e_t^2 \right. \\ & + 756\mathcal{W}_{e_t}(2\eta - 5) + 323\eta^2 + 1651\eta + 1666 \left. \right) + \frac{1}{21A(\mathcal{E})^3} \left((686 - 2\eta(31\eta + 197))e_t^2 \right. \\ & + \eta(41\eta + 625) - 1610 \left. \right) - \frac{\mathcal{W}_{e_t}^2(\eta(1517\eta + 7549) - 8645)}{126A(\mathcal{E})^4} \\ & \left. \left. + \frac{\mathcal{W}_{e_t}^4(4\eta(79\eta + 179) - 217)}{42A(\mathcal{E})^5} \right] \right\}, \quad (6.46) \end{aligned}$$

$$I^{21(2)} = 0, \quad (6.47)$$

$$\begin{aligned} I^{22(2)} = & 8\sqrt{\frac{2\pi}{5}}|E|\mu e^{-2i\phi} \left\{ \frac{1}{A(\mathcal{E})} - 1 - \frac{2\mathcal{W}_{e_t}^2 + 2ie_t \sin(\mathcal{E})\mathcal{W}_{e_t}}{A(\mathcal{E})^2} + \epsilon^2|E| \left[\frac{3(1 - 3\eta)}{14A(\mathcal{E})} \right. \right. \\ & + \frac{3}{14}(3\eta - 1) + \frac{\frac{2}{21}(15\eta - 82)\mathcal{W}_{e_t}^2 + \frac{2}{21}ie_t(6\eta - 79)\sin(\mathcal{E})\mathcal{W}_{e_t}}{A(\mathcal{E})^3} \\ & + A(\mathcal{E})^{-2} \left\{ \frac{1}{14(e_t - 1)}(98\eta - 210 + e_t(35(5 - 3\eta) + 3e_t(5e_t + 2)(3\eta - 1))) \right. \\ & \left. \left. - \frac{1}{14\mathcal{W}_{e_t}}e_t \left(7\sqrt{\frac{1+e_t}{1-e_t}}((3 - 9\eta)e_t^2 + \eta + 5) \right) \right. \right. \end{aligned}$$

$$\begin{aligned}
& +2i \left((9\eta - 3)e_t^2 + 19\eta - 25 \right) \sin(\mathcal{E}) \Bigg\} \Bigg] \\
& + \epsilon^4 |E|^2 \left[i e_t \sin(\mathcal{E}) \left(\frac{\mathcal{W}_{e_t}^3 (614\eta^2 - 3710\eta + 427)}{63A(\mathcal{E})^5} - \frac{\mathcal{W}_{e_t} (\eta(4049\eta + 3901) - 10016)}{189A(\mathcal{E})^4} \right. \right. \\
& \quad \left. \left. - \frac{((\eta(377\eta - 575) + 2200)e_t^2 - \eta(3024\mathcal{W}_{e_t} + 449\eta + 1933) + 7560\mathcal{W}_{e_t} - 1132)}{63\mathcal{W}_{e_t} A(\mathcal{E})^3} \right. \right. \\
& \quad \left. \left. + \frac{1}{21\mathcal{W}_{e_t}^3 A(\mathcal{E})^2} \left((\eta(11\eta + 16) - 5)e_t^4 + 2(\eta(67\eta + 398) + 59)e_t^2 + 252\mathcal{W}_{e_t}(5 - 2\eta) \right. \right. \right. \\
& \quad \left. \left. \left. - 61\eta^2 + 196\eta - 701 \right) \right) + \frac{1}{42} (\eta(11\eta + 16) - 5) + \frac{\frac{6(2\eta-5)}{\mathcal{W}_{e_t}} + \frac{1}{42}(5 - \eta(11\eta + 16))}{A(\mathcal{E})} \right. \\
& \quad \left. + \frac{1}{126\mathcal{W}_{e_t}^2 A(\mathcal{E})^2} \left(6(\eta(11\eta + 16) - 5)e_t^4 + (\eta(2063\eta + 7867) + 5038)e_t^2 \right. \right. \\
& \quad \left. \left. - 3780\mathcal{W}_{e_t}(2\eta - 5) - 2129\eta^2 + 2621\eta - 11056 \right) \right. \\
& \quad \left. + \frac{4(169 - 64\eta)\eta - 766)e_t^2 + 504\mathcal{W}_{e_t}(2\eta - 5) + 373\eta^2 + 1317\eta + 378}{21A(\mathcal{E})^3} \right. \\
& \quad \left. + \frac{\mathcal{W}_{e_t}^2 ((1247 - 12113\eta)\eta + 28061)}{378A(\mathcal{E})^4} + \frac{\mathcal{W}_{e_t}^4 (16\eta(143\eta - 335) + 199)}{126A(\mathcal{E})^5} \right] \Bigg\}. \tag{6.48}
\end{aligned}$$

The remaining projections of current-type and higher-order mass-type moments are lengthy and needed only temporarily and will therefore be presented in Appendix B. It will be necessary to decompose these tensor components in terms of irreducible expressions to get a time Fourier representation. Those will be terms which collect contributions having $\sin \mathcal{E}$ on the one hand and those without $\sin \mathcal{E}$ on the other, and they will be used when we write down the exponential of the orbital phase in such a way that we can use results and representations we already know from the literature or we have to evaluate them from scratch.

6.4 Relevant Kapteyn Series of irreducible components

Series representation for the inverse KE, $\sin m\mathcal{E}$ and $\cos m\mathcal{E}$

We recall the computation of the 1PN version of our considerations in [126]. There we required only a Newtonian accurate expression of the sin- and cos-function of multiples of the eccentric anomaly. As we Taylor expand the argument of the Bessel integral – which will be done below – we are in the position to provide \mathcal{E} as a series in \mathcal{M} up to 2PN. Therefore we need the well-known representation of v and \mathcal{E} in the KE (6.4), where in the 2PN term, we can insert their Newtonian accurate summation surrogates (see Equations (5) on p. 553 and (8) on p. 555 in [130]. In [95] there is a misprint in the definition of the G_n on page 33: the factor $\frac{2}{n}$ should comprise the complete right hand side, and in [126], the “minus” between the Bessel

functions in the summation argument should be a “plus”),

$$v_{(e=e_t)} = \mathcal{M} + \sum_{m=1}^{\infty} G_m(e_t) \sin m\mathcal{M}, \quad (6.49)$$

$$G_m(e) = \frac{2}{m} \left\{ J_m(me) + \sum_{s=1}^{\infty} \alpha^s [J_{m-s}(me) + J_{m+s}(me)] \right\}, \quad (6.50)$$

and α is extractable from

$$e = \frac{2\alpha}{1 + \alpha^2}. \quad (6.51)$$

This result is practicably obtainable by using integration by parts. The series expansions of the functions $(v - \mathcal{E})$ and $\sin v$ at Newtonian accuracy read

$$(v - \mathcal{E}) = \left(\mathcal{M} + \sum_{i=1}^{\infty} G_i(e_t) \sin(i\mathcal{M}) \right) - \left(\mathcal{M} + \sum_{n=1}^{\infty} \frac{2}{n} J_n(ne_t) \sin(n\mathcal{M}) \right) \quad (6.52)$$

$$= \sum_{i=1}^{\infty} \left(G_i(e_t) - \frac{2}{i} J_i(i e_t) \right) \sin(i\mathcal{M}), \quad (6.53)$$

$$\sin v = \sqrt{1 - e_t^2} \sum_{n=1}^{\infty} 2 J'_n(ne_t) \sin(n\mathcal{M}). \quad (6.54)$$

In these equations, $J'_n(ne_t)$ means $\partial_x J_n(x)|_{x=ne_t}$. We take above definitions and write in shorthand notation for further calculations, cf. Equation (6.4),

$$\mathcal{M} = \mathcal{E} - e_t \sin \mathcal{E} + \epsilon^4 \sum_{j=1}^{\infty} \alpha_j \sin(j\mathcal{M}). \quad (6.55)$$

For further considerations, let us call $u = g(\mathcal{M})$ the solution to the 2PN KE. Inserting Equation (6.55) into the KE and solving for the Fourier-Bessel coefficients, we calculate after Taylor expansion in ϵ to 4th order (see Appendix D.4),

$$g(\mathcal{M}) - \mathcal{M} = \sum_{n=1}^{\infty} \mathcal{A}_n \sin(n\mathcal{M}), \quad (6.56)$$

$$\begin{aligned} \mathcal{A}_n &= \frac{2}{n\pi} \int_0^\pi \cos(n\mathcal{M}) dg(\mathcal{M}) \\ &= \frac{2}{n\pi} \int_0^\pi \cos \left(n \left[g_N(\mathcal{M}) - e_t \sin(g_N(\mathcal{M})) + \epsilon^4 \sum_{m=1}^{\infty} \alpha_m \sin(\mathcal{M}m) \right] \right) dg(\mathcal{M}) \\ &= \frac{2}{n\pi} \int_0^\pi \cos(n [g_N(\mathcal{M}) - e_t \sin(g_N(\mathcal{M}))]) dg(\mathcal{M}) \\ &\quad - \frac{2}{\pi} \int_0^\pi \sin(n [g_N(\mathcal{M}) - e_t \sin(g_N(\mathcal{M}))]) \epsilon^4 \sum_{m=1}^{\infty} \alpha_m \sin(\mathcal{M}m) dg(\mathcal{M}). \end{aligned} \quad (6.57)$$

Defining

$$\Theta(j, n) \equiv \begin{cases} 0, & j \leq n \\ 1, & j > n \end{cases}, \quad (6.58)$$

the result reads

$$\begin{aligned} \mathcal{E} &= \mathcal{M} + 2 \left[\sum_{j=1}^{\infty} \frac{\sin(j\mathcal{M}) J_j(je_t)}{j} \right] \\ &\quad - 2\epsilon^4 \mathcal{F}_v \sqrt{1 - e_t^2} \sum_{j=1}^{\infty} \sin(j\mathcal{M}) \times \\ &\quad \left(\left[\sum_{m=1}^{\infty} J_m(e_t m) J'_{j+m}(e_t(j+m)) \right] - \left[\sum_{m=j+1}^{\infty} J_m(e_t m) J'_{m-j}(e_t(m-j)) \right] \right. \\ &\quad \left. + \left[\sum_{m=1}^{j-1} J_m(e_t m) J'_{j-m}(e_t(j-m)) \right] \Theta(j, 1) + J'_j(e_t j) \right) \\ &\quad + \epsilon^4 \mathcal{F}_{v-\mathcal{E}} \sum_{j=1}^{\infty} \sin j\mathcal{M} \times \\ &\quad \left(\left[\sum_{m=1}^{j-1} J_m(me_t) \left(G_{m-j}(e_t) + \frac{2J_{j-m}(e_t(j-m))}{j-m} \right) \right] \Theta(j, 1) \right. \\ &\quad \left. + \left[\sum_{m=j+1}^{\infty} J_m(me_t) \left(G_{m-j}(e_t) + \frac{2J_{m-j}(e_t(m-j))}{j-m} \right) \right] \right. \\ &\quad \left. - \left[\sum_{m=1}^{\infty} J_m(me_t) \left(G_{j+m}(e_t) - \frac{2J_{j+m}(e_t(j+m))}{j+m} \right) \right] \right. \\ &\quad \left. - G_j(e_t) + \frac{2J_j(je_t)}{j} \right). \end{aligned} \quad (6.59)$$

The reader should keep in mind Appendix C and D of [126] where care is taken of products of infinite series of $\sin j\mathcal{M}$ and $\cos j\mathcal{M}$ with arbitrary coefficients α_j and β_j . This consideration is necessary to collect for terms with the same positive frequencies in the above expressions and many more.

Now we proceed with the trigonometrics of $m\mathcal{E}$, $m \in \mathbb{N}$. We know from the symmetry of $\sin m\mathcal{E}$ and $\cos m\mathcal{E}$, that only $\sin j\mathcal{M}$ or $\cos j\mathcal{M}$ can contribute. Thus, we decompose

$$\sin m\mathcal{E} = \sum_{j=1}^{\infty} \bar{\sigma}_j^m \sin j\mathcal{M}, \quad (6.60)$$

$$\cos m\mathcal{E} = \sum_{j=0}^{\infty} \bar{\gamma}_j^m \cos j\mathcal{M}. \quad (6.61)$$

The coefficients $\bar{\sigma}_j^m$ and $\bar{\gamma}_j^m$ can be computed using

$$\bar{\gamma}_j^m = \frac{2}{\pi} \int_0^\pi \sin m\mathcal{E} \sin j\mathcal{M} d\mathcal{M}, \quad (6.62)$$

$$\bar{\sigma}_j^m = \frac{2}{\pi} \int_0^\pi \cos m\mathcal{E} \cos j\mathcal{M} d\mathcal{M}. \quad (6.63)$$

Switching from the integration over $d\mathcal{M}$ to $d\mathcal{E}$ in the above equations using the 2PN accurate KE and Taylor expanding everything to ϵ^4 , we can perform the integration. One technical – but easy to manage – issue is to re-convert the arguments of the integrals for a simple application of the Bessel integral formula,

$$J_y(x) = \frac{1}{\pi} \int_0^\pi d\mathcal{E} \cos(y\mathcal{E} - x \sin \mathcal{E}). \quad (6.64)$$

Appendix D.5 provides the calculation. The results read

$$\sin(m\mathcal{E}) = \sum_{n=1}^{\infty} \bar{\sigma}_j^m \sin j\mathcal{M} + \epsilon^4 \sum_{j=1}^{\infty} \bar{\sigma}_j^m \sin j\mathcal{M}, \quad (6.65)$$

$$\bar{\sigma}_j^m \equiv \frac{m}{j} \{J_{j-m}(je_t) + J_{j+m}(je_t)\}, \quad (6.66)$$

$$\begin{aligned} \bar{\sigma}_j^m \equiv & \frac{1}{2}m \sum_{j=1}^{\infty} \alpha_n \left\{ -J_{j+m-n}(e_t(j-n)) + J_{j-m+n}(e_t(j+n)) \right. \\ & \left. - J_{-j+m+n}(e_t(n-j)) + J_{j+m+n}(e_t(j+n)) \right\}, \end{aligned} \quad (6.67)$$

$$\cos(m\mathcal{E}) = \sum_{j=0}^{\infty} \bar{\gamma}_j^m \cos j\mathcal{M} + \epsilon^4 \sum_{j=0}^{\infty} \bar{\gamma}_j^m \cos j\mathcal{M}, \quad (6.68)$$

$$\bar{\gamma}_j^m \equiv \frac{m}{j} \{J_{j-m}(je_t) - J_{j+m}(je_t)\} \times \Theta(j, 0) + \delta_{m1} \delta_{0j} \left(-\frac{e_t}{2}\right), \quad (6.69)$$

$$\begin{aligned} \bar{\gamma}_j^m \equiv & \frac{1}{2}m \sum_{n=1}^{\infty} \alpha_n \left\{ J_{-j+m-n}(e_t(-(j+n))) + J_{j+m-n}(e_t(j-n)) \right. \\ & \left. - J_{-j+m+n}(e_t(n-j)) - J_{j+m+n}(e_t(j+n)) \right\} \times \Theta(j, 0) \\ & + \delta_{0j} \left\{ \left[\sqrt{\frac{1-e_t}{1+e_t}} (F_v + F_{v-\epsilon}) - F_{v-\epsilon} \right] {}_3F_2^{\text{reg}} \left(\frac{1}{2}, 1, 1; 1-m, m+1; \frac{2e_t}{e_t+1} \right) \right. \\ & \left. + F_{v-\epsilon} \frac{e_t}{e_t+1} {}_3F_2^{\text{reg}} \left(1, \frac{3}{2}, 2; 2-m, m+2; \frac{2e_t}{e_t+1} \right) \right\}. \end{aligned} \quad (6.70)$$

In above expressions, ${}_pF_q^{\text{reg}}(a_1, \dots, a_p; b_1, \dots, b_q; z)$ is the regularised hypergeometric function, see [131]. This prescription is valid in both ADM and harmonic coordinates, where, of course, the appropriate values always have to be included.

$A(\mathcal{E})^{-n}$ and $\sin \mathcal{E} A(\mathcal{E})^{-n}$ as Fourier-Bessel series

The Fourier domain GW form requires computation of relatively simple structures which will combine in the tensor-spherical harmonics. Let us start with the most fundamental quantities which we compute from the scratch on the one hand and some which we collect from the literature on the other. The inverse scaled radial separation with an arbitrary integer exponent $n > 0$ reads

$$\frac{1}{(1 - e \cos \mathcal{E})^n} = 1 + b_0^{(n)} + \sum_{j=1}^{\infty} b_j^{(n)} \cos j\mathcal{E}, \quad (6.71)$$

$$b_0^{(n)} = \sum_{i=1}^{\infty} \beta_{2i,i}^{(n)}, \quad (6.72)$$

$$b_{j>0}^{(n)} = \sum_{i=0}^{\infty} \beta_{j+2i,i}^{(n)} + \beta_{j+2i,j+i}^{(n)}, \quad (6.73)$$

$$\begin{aligned} \beta_{m,k}^{(n)} &\equiv \frac{(n+m-1)!}{(n-1)!} \frac{1}{m!} \frac{e^m}{2^m} \binom{m}{k} \\ &= e^m \frac{1}{2^m} \binom{m}{k} \frac{1}{m!} \prod_{i=0}^{m-1} (n+i). \end{aligned} \quad (6.74)$$

Equation (6.71) is proven in Appendix D.2. Note that from here onwards, e can be set e_ϕ or e_t , depending on the context.

$$A(\mathcal{E})^{-n} = 1 + b_0^{(n)} + \sum_{m=1}^{\infty} \gamma_0^m b_m^{(n)} + \sum_{j=1}^{\infty} b_j^{(n)} \cos j\mathcal{E}, \quad (6.75)$$

$$\frac{\sin \mathcal{E}}{A(\mathcal{E})^n} = \sum_{j=1}^{\infty} S_j^{(n)} \sin j\mathcal{E}. \quad (6.76)$$

From Equations (6.65) and (6.68) we learn that the Fourier-Bessel representation of the above two series get 2PN corrections,

$$\begin{aligned} A(\mathcal{E})^{-n} &= 1 + b_0^{(n)} + \sum_{m=1}^{\infty} \bar{\gamma}_0^m b_m^{(n)} + \sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} \bar{\gamma}_j^k b_k^{(n)} \right) \cos j\mathcal{M} \\ &= \sum_{j=0}^{\infty} \mathcal{A}_j^{(n)} \cos j\mathcal{M}, \end{aligned} \quad (6.77)$$

$$\begin{aligned} \sin \mathcal{E} A(\mathcal{E})^{-n} &= \sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} \bar{\sigma}_j^k S_k^{(n)} \right) \sin j\mathcal{M} \\ &= \sum_{j=1}^{\infty} \mathcal{S}_j^{(n)} \sin j\mathcal{M}, \end{aligned} \quad (6.78)$$

$$\mathcal{A}_j^{(n)} = \delta_{j0} \left(1 + b_0^{(n)} + \sum_{m=1}^{\infty} \left[\bar{\gamma}_0^m + (\epsilon^4) \bar{\gamma}_0^m \right] b_m^{(n)} \right) + \Theta(j, 0) \sum_{k=1}^{\infty} \left[\bar{\gamma}_j^k + (\epsilon^4) \bar{\gamma}_j^k \right] b_k^{(n)}, \quad (6.79)$$

$$\mathcal{S}_j^{(n)} = \sum_{k=1}^{\infty} \left[\bar{\sigma}_j^k + (\epsilon^4) \bar{\sigma}_j^k \right] S_k^{(n)}. \quad (6.80)$$

Decomposition of $\exp\{-im\phi\}$

What we have done at 1PN accuracy has to be extended to 2PN especially at the orbital dynamics. It is helpful to find a special decomposition of $e^{-im\phi}$ in such a way that the mode decomposition of any 1PN function of \mathcal{E} (to be performed exactly) is not required at this point of calculation. We will combine the terms in such a way that we can use results known from the previous sections,

$$(\phi - \phi_0) = (1 + \mathcal{K})v + \epsilon^4 (f_{4\phi} \sin 2v + g_{4\phi} \sin 3v), \quad (6.81)$$

$$\begin{aligned} v = v(\mathcal{E}) &\stackrel{1}{=} 2 \arctan \left\{ \sqrt{\frac{1+e_\phi}{1-e_\phi}} \tan \left[\frac{\mathcal{E}}{2} \right] \right\} \\ &\stackrel{2}{=} v_N + \epsilon^2 v_{1\text{PN}} + \epsilon^4 v_{2\text{PN}} \\ &\stackrel{3}{=} (\mathcal{M} + \tilde{v}_N) + \epsilon^2 v_{1\text{PN}} + \epsilon^4 v_{2\text{PN}}, \quad (6.82) \\ e^{-im(\phi-\phi_0)} &\stackrel{4}{=} e^{-im[(1+\mathcal{K})v + (\epsilon^4 f_{4\phi}) \sin 2v + (\epsilon^4 g_{4\phi}) \sin 3v]} \\ &\stackrel{5}{=} e^{-imv} e^{-im(\epsilon^2 \mathcal{K}_{1\text{PN}} + \epsilon^4 \mathcal{K}_{2\text{PN}})[v_N + \epsilon^2 v_{1\text{PN}} + \epsilon^4 v_{2\text{PN}}]} e^{-im\epsilon^4 [f_{4\phi} \sin 2v + g_{4\phi} \sin 3v]} \\ &\stackrel{6}{=} e^{-imv} e^{-imv_N \epsilon^2 (\mathcal{K}_{1\text{PN}} + \mathcal{K}_{2\text{PN}} \epsilon^2)} e^{-im\epsilon^4 [\mathcal{K}_{1\text{PN}} v_{1\text{PN}} + f_{4\phi} \sin 2v + g_{4\phi} \sin 3v]} \\ &\stackrel{7}{=} e^{-im(\epsilon^2 \mathcal{K}_{1\text{PN}} + \epsilon^4 \mathcal{K}_{2\text{PN}})} \mathcal{M}_\times \\ &\quad \underbrace{e^{-imv}}_{\text{Part}_A(m)} \underbrace{\left[1 - im\mathcal{K}_{1\text{PN}}\tilde{v}_N\epsilon^2 - \frac{1}{2}m\tilde{v}_N\epsilon^4 (m\mathcal{K}_{1\text{PN}}^2\tilde{v}_N + 2i\mathcal{K}_{2\text{PN}}) \right]}_{\text{Part}_B(m)} \times \\ &\quad \underbrace{\left[1 - im\epsilon^4 (\mathcal{K}_{1\text{PN}}v_{1\text{PN}} + f_{4\phi} \sin(2v_N) + g_{4\phi} \sin(3v_N)) \right]}_{\text{Part}_C(m)}. \quad (6.83) \end{aligned}$$

In step 3, v_N decomposes into a purely secular part, namely \mathcal{M} , and a purely oscillatory one with zero average over the interval $\mathcal{M} = [0, 2\pi]$, viz. \tilde{v}_N [see Equation (6.49)]. Therefore, in step 7, the middle term in edgy brackets (“Part_B”) can be written in terms of single and double summations of terms in the right hand side of Equation (6.49), almost without computational cost. Part_A and Part_C will contribute $A(\mathcal{E})^{-n}$ and also $A(\mathcal{E})^{-n} \sin \mathcal{E}$ terms that will multiply with the series of Part_B. These contributions are lengthy and we again skip explicit expressions. In principle, other decompositions are valid as well, but we find it convenient to choose the above one because of its structural clearness. Part_B is essential and

we give it explicitly, ¹⁶

$$\begin{aligned}
\text{Part}_B(q) &\equiv \left[1 - iq\mathcal{K}_{1\text{PN}}\tilde{v}_N\epsilon^2 - \frac{1}{2}q\tilde{v}_N\epsilon^4 (q\mathcal{K}_{1\text{PN}}^2\tilde{v}_N + 2i\mathcal{K}_{2\text{PN}}) \right] \\
&= 1 - iq\epsilon^2 \mathcal{K}_{1\text{PN}} \left[\sum_{j=1}^{\infty} \sin(j\mathcal{M}) G_j(e_t) \right] - \epsilon^4 \frac{1}{4}q \left\{ \left[\sum_{j=1}^{\infty} q\mathcal{K}_{1\text{PN}}^2 G_j(e_t) \right]^2 \right. \\
&\quad + \sum_{k=1}^{\infty} \left[q\mathcal{K}_{1\text{PN}}^2 \cos(k\mathcal{M}) \times \left(- \left[\sum_{j=1}^{k-1} G_j(e_t) G_{k-j}(e_t) \right] \times \Theta(k, 1) \right. \right. \\
&\quad \left. \left. + \left[\sum_{j=k+1}^{\infty} G_j(e_t) G_{j-k}(e_t) \right] + \left[\sum_{j=1}^{\infty} G_j(e_t) G_{j+k}(e_t) \right] \right) \right. \\
&\quad \left. \left. + 4i\mathcal{K}_{2\text{PN}} \sin(k\mathcal{M}) G_k(e_t) \right] \right\} . \tag{6.84}
\end{aligned}$$

For a clear understanding, we sometimes have added auxiliary indices to the brackets. This simply helps to see how deep the bracket in the current expression is. To 1PN order we recognise what we computed in [126], Section IV. We will face products of Part_B with powers of $A(\mathcal{E})$. They can be put into the form

$$\begin{aligned}
\frac{\text{Part}_B(q)}{A(\mathcal{E})^n} &= \sum_{j=1}^{\infty} \frac{1}{2} iq\epsilon^2 (\mathcal{K}_{1\text{PN}} + \mathcal{K}_{2\text{PN}}\epsilon^2) \sin(j\mathcal{M}) \times \\
&\quad \left\{ \sum_{m=1}^{j-1} \left[-\mathcal{A}_m^{(n)} \Theta(j, 1) G_{j-m}(e_t) \right] + \sum_{m=j+1}^{\infty} \left[\mathcal{A}_m^{(n)} G_{m-j}(e_t) \right] \right. \\
&\quad \left. + \left[\sum_{m=1}^{\infty} -\mathcal{A}_m^{(n)} G_{j+m}(e_t) \right] - 2\mathcal{A}_0^{(n)} G_j(e_t) \right\} + \\
&\quad \frac{-\frac{1}{8} \sum_{j=1}^{\infty} \cos(j\mathcal{M}) \left\{ \mathcal{K}_{1\text{PN}}^2 q^2 \epsilon^4 \left[\sum_{k=j+1}^{\infty} \left[\mathcal{A}_{k-j}^{(n)} \left(- \sum_{m=1}^{k-1} [G_m(e_t) G_{k-m}(e_t)] + \sum_{n=k+1}^{\infty} [G_n(e_t) G_{n-k}(e_t)] \right. \right. \right. \right. \\
&\quad \left. \left. \left. + \sum_{n=1}^k [G_n(e_t) G_{k+n}(e_t)] \right) \right] \right. \right. \\
&\quad \left. \left. + \sum_{k=1}^{\infty} \left[\mathcal{A}_{j+k}^{(n)} \left(- \sum_{m=1}^{k-1} [G_m(e_t) G_{k-m}(e_t)] + \sum_{n=k+1}^{\infty} [G_n(e_t) G_{n-k}(e_t)] \right. \right. \right. \right. \\
&\quad \left. \left. \left. + \sum_{n=1}^k [G_n(e_t) G_{k+n}(e_t)] \right) \right] \right. \\
&\quad \left. \left. + \Theta(j, 1) \sum_{s=1}^{j-1} \left[\mathcal{A}_{j-s}^{(n)} \left(- \sum_{m=1}^{s-1} [G_m(e_t) G_{s-m}(e_t)] + \sum_{n=s+1}^{\infty} [G_n(e_t) G_{n-s}(e_t)] \right) \right] \right. \right.
\end{aligned}$$

¹⁶In Equation (C5) of [126], the very last term should get a factor $\Theta(j, 1)$ to make it consistent with our notation and for convenience of the reader.

$$\begin{aligned}
& + \sum_{n=1}^s [G_n(e_t) G_{n+s}(e_t)] \Bigg] \\
& + 2\mathcal{A}_0^{(n)} \left(- \sum_{m=1}^{j-1} [G_m(e_t) G_{j-m}(e_t)] + \sum_{n=j+1}^{\infty} [G_n(e_t) G_{n-j}(e_t)] \right. \\
& \left. + \left[\sum_{n=1}^j G_n(e_t) G_{j+n}(e_t) \right] \right) + 2\mathcal{A}_j^{(n)} \left(\mathcal{K}_{\text{1PN}}^2 q^2 \epsilon^4 \left[\sum_{k=1}^{\infty} G_k(e_t)^2 \right] - 4 \right) \Bigg\}_1 \\
\hline
& + \left\{ \sum_{j=1}^{\infty} \left[\frac{1}{8} \mathcal{K}_{\text{1PN}}^2 q^2 \epsilon^4 \left(\mathcal{A}_j^{(n)} \left(- \sum_{k=j+1}^{\infty} [G_k(e_t) G_{k-j}(e_t)] \right. \right. \right. \right. \\
& \left. \left. \left. + \sum_{k=1}^{j-1} [G_k(e_t) G_{j-k}(e_t)] - \sum_{k=1}^j [G_k(e_t) G_{j+k}(e_t)] \right) \right. \right. \\
& \left. \left. - 2\mathcal{A}_0^{(n)} G_j(e_t)^2 \right) \right] + \mathcal{A}_0^{(n)} \right\}. \tag{6.85}
\end{aligned}$$

We inserted lines that help distinguish clearly between sin, cos, and constant contributions. The part including $\sin \mathcal{E}$ reads

$$\begin{aligned}
\frac{\text{Part}_B(q)}{A(\mathcal{E})^n} \sin \mathcal{E} &= \sum_{j=1}^{\infty} \frac{1}{2} \sin(j\mathcal{M}) \times \\
& \left\{ 2\mathcal{S}_j^{(n)} + \sum_{m=1}^{j-1} \left[-\frac{1}{4} \mathcal{K}_{\text{1PN}}^2 q^2 \epsilon^4 \Theta(j, 1) \mathcal{S}_{j-m}^{(n)} \left(\sum_{k=m+1}^{\infty} [G_k(e_t) G_{k-m}(e_t)] \right. \right. \right. \\
& \left. \left. \left. + \sum_{k=1}^{m-1} [-G_k(e_t) G_{m-k}(e_t)] + \sum_{k=1}^m [G_k(e_t) G_{k+m}(e_t)] \right) \right] \right\} \\
& + \sum_{m=j+1}^{\infty} \left[\frac{1}{4} \mathcal{K}_{\text{1PN}}^2 q^2 \epsilon^4 \mathcal{S}_{m-j}^{(n)} \left(\sum_{k=m+1}^{\infty} [G_k(e_t) G_{k-m}(e_t)] \right. \right. \\
& \left. \left. \left. + \sum_{k=1}^{m-1} [-G_k(e_t) G_{m-k}(e_t)] + \sum_{k=1}^m [G_k(e_t) G_{k+m}(e_t)] \right) \right] \\
& + \sum_{m=1}^{\infty} \left[-\frac{1}{4} \mathcal{K}_{\text{1PN}}^2 q^2 \epsilon^4 \mathcal{S}_{j+m}^{(n)} \left(\left[\sum_{k=m+1}^{\infty} G_k(e_t) G_{k-m}(e_t) \right] \right. \right. \\
& \left. \left. \left. + \left[\sum_{k=1}^{m-1} -G_k(e_t) G_{m-k}(e_t) \right] + \left[\sum_{k=1}^m G_k(e_t) G_{k+m}(e_t) \right] \right) \right] \Bigg\} \\
& + \left[\sum_{j=1}^{\infty} G_j(e_t)^2 \right] \left[\sum_{j=1}^{\infty} -\frac{1}{4} \mathcal{K}_{\text{1PN}}^2 q^2 \epsilon^4 \mathcal{S}_j^{(n)} \sin(j\mathcal{M}) \right] \\
\hline
& - \frac{1}{2} i q \epsilon^2 (\mathcal{K}_{\text{1PN}} + \mathcal{K}_{\text{2PN}} \epsilon^2) \sum_{j=1}^{\infty} \cos(j\mathcal{M}) \times \\
& \left\{ \left(\sum_{m=j+1}^{\infty} [\mathcal{S}_{m-j}^{(n)} G_m(e_t)] + \sum_{m=1}^{\infty} [\mathcal{S}_{j+m}^{(n)} G_m(e_t)] \right) \right\}
\end{aligned}$$

$$\frac{-\Theta(j, 1) \sum_{m=1}^{j-1} \left[\mathcal{S}_{j-m}^{(n)} G_m(e_t) \right]}{-\frac{1}{2} i q \epsilon^2 (\mathcal{K}_{1\text{PN}} + \mathcal{K}_{2\text{PN}} \epsilon^2) \left[\sum_{j=1}^{\infty} \mathcal{S}_j^{(n)} G_j(e_t) \right]} . \quad (6.86)$$

This will be shortened by writing

$$\frac{\text{Part}_B(q)}{A(\mathcal{E})^n} = \left\{ \sum_{j=0}^{\infty} (\bar{\mathcal{A}}_c)_j^{[n,q]} \cos j\mathcal{M} + \sum_{j=1}^{\infty} (\bar{\mathcal{A}}_s)_j^{[n,q]} \sin j\mathcal{M} \right\} , \quad (6.87)$$

$$\sin \mathcal{E} \frac{\text{Part}_B(q)}{A(\mathcal{E})^n} = \left\{ \sum_{j=0}^{\infty} (\bar{\mathcal{S}}_c)_j^{[n,q]} \cos j\mathcal{M} + \sum_{j=1}^{\infty} (\bar{\mathcal{S}}_s)_j^{[n,q]} \sin j\mathcal{M} \right\} , \quad (6.88)$$

with $(\bar{\mathcal{A}}_c)^{[n,q]}$, $(\bar{\mathcal{A}}_s)^{[n,q]}$, $(\bar{\mathcal{S}}_c)^{[n,q]}$ and $(\bar{\mathcal{S}}_s)^{[n,q]}$ to be easily extracted from Equations (6.85) and (6.86) and as well remembering that $\mathcal{A}_j^{(n)}$ and $\mathcal{S}_j^{(n)}$ have 2PN terms. Equations (6.85) and (6.86) appear to be relatively complicated, but they are simply expanded applications of the product formulas for sin and cos series, Equations (D.1), (D.2) and (D.3). We can now decompose the wave form into the above irreducible components, from whose we have extracted the time Fourier series representation. To simplify matters, we introduce

$$\frac{\text{Part}_B(q)}{A(\mathcal{E})^n} =: \mathcal{F}_{[nq]}(\mathcal{E}) , \quad (6.89)$$

$$\sin \mathcal{E} \frac{\text{Part}_B(q)}{A(\mathcal{E})^n} =: \mathcal{F}_{S[nq]}(\mathcal{E}) . \quad (6.90)$$

6.5 Multipole moment decomposition: a brief posting of the results

Using the results of the previous section, we decompose the multipole coefficients I and S as

$$I^{(a)m} = \mathcal{C}_{I;am} e^{-mi\mathcal{K}\mathcal{M}} e^{-mi\phi_0} \times \left\{ \sum_k^{a-2} \alpha_{[km]} \mathcal{F}_{[km]}(\mathcal{E}) + \sum_{k'}^{a-2} \tilde{\alpha}_{[k'm]} \mathcal{F}_{S[k'm]}(\mathcal{E}) \right\} , \quad (6.91)$$

$$S^{(b)m} = \mathcal{C}_{S;bm} e^{-mi\mathcal{K}\mathcal{M}} e^{-mi\phi_0} \times \left\{ \sum_k^{b-1} \beta_{[km]} \mathcal{F}_{[km]}(\mathcal{E}) + \sum_{k'}^{b-1} \tilde{\beta}_{[k'm]} \mathcal{F}_{S[k'm]}(\mathcal{E}) \right\} , \quad (6.92)$$

symbolically, where $\mathcal{C}_{I;am}$ and $\mathcal{C}_{S;bm}$ are some pre-factors and α , $\tilde{\alpha}$, β and $\tilde{\beta}$ are coefficients to be determined, k and k' are some summation dummy indices with boundaries depending on a or b , m , and depending on the type (I or S), and $a-2$ and $b-1$ are labels for α , $\tilde{\alpha}$ and for β , $\tilde{\beta}$, counting the order of c^{-1} . The associated components of I and S , including Part_A and Part_C , are given in Appendix C. Some “pre – Fourier” domain reads

$$I_{I;am}^{(a)} = \mathcal{C}_{I;am} e^{-mi\mathcal{K}\mathcal{M}} e^{-mi\phi_0} \times \left\{ \sum_j \sin j\mathcal{M} I_{S_j}^{(a)am} + \sum_j \cos j\mathcal{M} I_{C_j}^{(a)am} \right\}, \quad (6.93)$$

$$S_{S;bm}^{(b)} = \mathcal{C}_{S;bm} e^{-mi\mathcal{K}\mathcal{M}} e^{-mi\phi_0} \times \left\{ \sum_j \sin j\mathcal{M} S_{S_j}^{(b)bm} + \sum_j \cos j\mathcal{M} S_{C_j}^{(b)bm} \right\}, \quad (6.94)$$

with

$$I_{S_j}^{(a)am} \equiv \left(\sum_k^{a-2} \alpha_{[km]} (\bar{\mathcal{A}}_s)_j^{[k,m]} + \sum_{k'}^{a-2} \tilde{\alpha}_{[k'm]} (\bar{\mathcal{S}}_s)_j^{[k',m]} \right), \quad (6.95)$$

$$I_{C_j}^{(a)am} \equiv \left(\sum_k^{a-2} \alpha_{[km]} (\bar{\mathcal{A}}_c)_j^{[k,m]} + \sum_{k'}^{a-2} \tilde{\alpha}_{[k'm]} (\bar{\mathcal{S}}_c)_j^{[k',m]} \right), \quad (6.96)$$

$$S_{S_j}^{(b)bm} \equiv \left(\sum_k^{b-1} \beta_{[km]} (\bar{\mathcal{A}}_s)_j^{[k,m]} + \sum_{k'}^{b-1} \tilde{\beta}_{[k'm]} (\bar{\mathcal{S}}_s)_j^{[k',m]} \right), \quad (6.97)$$

$$S_{C_j}^{(b)bm} \equiv \left(\sum_k^{b-1} \beta_{[km]} (\bar{\mathcal{A}}_c)_j^{[k,m]} + \sum_{k'}^{b-1} \tilde{\beta}_{[k'm]} (\bar{\mathcal{S}}_c)_j^{[k',m]} \right), \quad (6.98)$$

for $a \in [2, 6]$, $b \in [2, 5]$,

and for extracting the pure Fourier domain representation with delta distributions – (and not the one mixed in exponential and trigonometric representation as in Equations (6.93) and (6.94)) – we take the Fourier transformation of the $\sin j\mathcal{M}$ and $\cos j\mathcal{M}$ terms,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-im\mathcal{K}\mathcal{M}} \sin j\mathcal{M} e^{i\omega t} dt = i\sqrt{\frac{\pi}{2}} \delta(j\mathcal{N} + \mathcal{K}m\mathcal{N} - \omega) - i\sqrt{\frac{\pi}{2}} \delta(j\mathcal{N} - \mathcal{K}m\mathcal{N} + \omega), \quad (6.99)$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-im\mathcal{K}\mathcal{M}} \cos j\mathcal{M} e^{i\omega t} dt = \sqrt{\frac{\pi}{2}} \delta(j\mathcal{N} + \mathcal{K}m\mathcal{N} - \omega) + \sqrt{\frac{\pi}{2}} \delta(j\mathcal{N} - \mathcal{K}m\mathcal{N} + \omega). \quad (6.100)$$

This is valid only in the purely conservative orbital dynamics case.

6.6 The effect of radiation reaction and the stationary phase approximation of the GW field

Radiative dynamics

Having the conservative evolution under full control, we can apply the Fourier decomposition to the radiative evolution. This is done by separating the full orbital evolution into two time scales: the *orbital* and the *reactive* time scale. The latter is – for our calculation – assumed to be much larger than the orbital scale. This is equivalent to the assumption that the rate of change of the orbital frequency is small measured over one orbit. The time dependence of the orbital frequency is closely related to the loss of energy and orbital

angular momentum due to the GW emission. Peters and Mathews [132, 133] proposed a relatively simple model of the binary inspiral in the approximation of slow motion and weak gravitational interaction for arbitrary eccentricities $0 \leq e < 1$. Blanchet, Damour, Iyer, and Thorne made endeavors to obtain higher-order corrections to the quadrupole formula [15, 17, 128, 134]. The far-zone fluxes of energy and angular momentum to 2PN order [14] have shown up to be unrenounceable for the data analysis community. For the time being, we will restrict ourselves to the Peters-Mathews model for an exemplary calculation. Higher-order corrections can be included in a forthcoming publication.

In the conservative case, \mathcal{N} and e_t were constants of motion, but when the orbit shrinks due to RR, both elements will follow coupled EOM connected to the loss of orbital energy and angular momentum [26, 28],

$$\langle \dot{\mathcal{N}} \rangle_{\text{orbit}} = \mathfrak{N}(\mathcal{N}, e_t), \quad (6.101)$$

$$\langle \dot{e}_t \rangle_{\text{orbit}} = \mathfrak{E}(\mathcal{N}, e_t), \quad (6.102)$$

symbolically. Note that above Equations hold after averaging over one orbit. Exactly, we denote $\langle \dots \rangle_{\text{orbit}}$ the average over the eccentric anomaly \mathcal{E} . Thus, \mathcal{N} will satisfy

$$\mathcal{M}(t) = \int_{t_0}^t \mathcal{N}(t') dt', \quad (6.103)$$

$$\dot{\mathcal{M}}(t) = \mathcal{N}(t). \quad (6.104)$$

We will apply the SPA at the first order to approximate the frequency-domain wave form first to each single frequency separately and will later sum up all the terms of the discrete decomposition as a virtue of the linearity of the Fourier integral with respect to the integrand.

Stationary phase approximation of the gravitational wave signal

Suppose an integral of the form

$$\tilde{h}(f) = \int_{-\infty}^{+\infty} \mathcal{A}(t) e^{i(2\pi ft - \phi(t))} dt. \quad (6.105)$$

Then, having found a stationary point t^* , where the phase defined as $\Phi \equiv 2\pi ft - \phi(t)$ has zero ascent, $\dot{\Phi}(t^*) = 0$, with the assumption of the right behaviour at the boundary, the integral takes the form

$$\tilde{h}(f) \approx \sqrt{2\pi} \frac{A(t^*)}{\sqrt{|\ddot{\phi}(t^*)|}} e^{-i(\phi(t^*) - 2\pi ft^* + \pi/4)}. \quad (6.106)$$

In the case of our GW, the integral turns out to be

$$\tilde{h}^{(n)}(f)_{nm} = \frac{1}{\sqrt{2\pi}} e^{-mi\phi_0} \int_{-\infty}^{+\infty} \sum_{j=0}^{\infty} (S_j \sin j\mathcal{M} + C_j \cos j\mathcal{M}) e^{-im\mathcal{K}\mathcal{M}} e^{i2\pi ft} dt, \quad (6.107)$$

$$\begin{aligned}
\tilde{h}^{(n)}(f)_{C,nm} &= \frac{1}{\sqrt{2\pi}} e^{-mi\phi_0} \int_{-\infty}^{+\infty} \frac{C_j}{2} (e^{ij\mathcal{M}-im\mathcal{K}\mathcal{M}} + e^{-ij\mathcal{M}-im\mathcal{K}\mathcal{M}}) e^{i2\pi ft} dt \\
&= \frac{1}{\sqrt{2\pi}} e^{-mi\phi_0} \int_{-\infty}^{+\infty} \frac{C_j}{2} (e^{i(j\mathcal{M}+2\pi ft-m\mathcal{K}\mathcal{M})} + e^{i(-j\mathcal{M}+2\pi ft-m\mathcal{K}\mathcal{M})}) dt, \tag{6.108}
\end{aligned}$$

where the subscript “ C ” shall denote the contribution of $\cos j\mathcal{M}$. The reader should carefully note that C and S are functions of the elapsed time, as the orbit decays due to RR. We assume that the binary system evolves far away from the last stable orbit, such that the periastron advance parameter is much smaller than unity, $\mathcal{K} \ll 1$. Thus, $\mathcal{N}\mathcal{K} \ll j$ with $j = 1, 2, 3, \dots$. The case $j = 0$ will be discussed in one of the upcoming subsections. In the first part of Equation (6.108), therefore, there exist no points of stationary phase for $j > 0$ and this term vanishes. The second term in (6.108), having the exponential argument $i(-j\mathcal{M} + 2\pi ft - m\mathcal{K}\mathcal{M})$ will contribute, since the phase Φ_{mj} defined as

$$\Phi_{mj} \equiv 2\pi ft - j\mathcal{M} - m\mathcal{K}\mathcal{M} \tag{6.109}$$

has a stationary point at t^* where

$$\dot{\Phi}_{mj} = 2\pi f - j\mathcal{N} - m\mathcal{K}\mathcal{N} = 0. \tag{6.110}$$

Here we note that $\dot{\mathcal{K}} = \mathcal{O}(\epsilon^7)$ and terms of this order will be consistently neglected. We want to compute t^* and its contribution in the following lines, as the question arises how to find out t^* without solving Equations (6.101) and (6.102) numerically. It is answered in a simple manner. We search for the solution $\mathcal{N} = \mathcal{N}(e_t, e_{t_0})$, e_{t_0} being the eccentricity at $t = t_0$, of

$$\frac{d\mathcal{N}}{de_t} = \frac{\dot{\mathcal{N}}}{\dot{e}_t}, \tag{6.111}$$

insert it into Equation (6.102),

$$\dot{e}_t = \mathcal{E}(\mathcal{N}(e_t), e_t, e_{t_0}), \tag{6.112}$$

$$\Rightarrow e_t = e_t(e_{t_0}, t - t_0), \tag{6.113}$$

and invert Equation (6.113) to find $(t - t_0)$ as a function of e_t and e_{t_0} , where for the stationary point the value e_t^* corresponds to the solution of the equation

$$0 = 2\pi f - j\mathcal{N}(e_t^*, e_{t_0}) - m\mathcal{K}(\mathcal{N}(e_t^*, e_{t_0}), e_t^*)\mathcal{N}(e_t^*, e_{t_0}). \tag{6.114}$$

We see that e_t^* and thus t^* depends only on the magnetic number m , the summation index j , the eccentricity and mean motion at t_0 , e_{t_0} and \mathcal{N}_0 , and of course the frequency f . The sin term of all those calculations will have the same stationary point t^* , such that the full Fourier domain wave form will have the following appearance:

$$\tilde{h}^{(n)}(f)_{n,m>0} = \epsilon^4 \frac{G}{R} \sum_{j=0}^{\infty} \frac{1}{2} (iS_{m(j>0)}^* + (1 + \delta_{0j})C_{mj}^*) \left[\frac{e^{i(\Phi_{mj}(t_{mj}^*) - \pi/4)}}{\sqrt{\ddot{\phi}_{mj}(t_{mj}^*)}} \right] e^{-im\phi_0}, \quad (6.115)$$

$$\tilde{h}^{(n)}(f)_{n,m=0} = \epsilon^4 \frac{G}{R} \sum_{j=1}^{\infty} \frac{1}{2} (iS_{0j}^* + C_{0j}^*) \left[\frac{e^{i(\Phi_{0j}(t_{0j}^*) - \pi/4)}}{\sqrt{\ddot{\phi}_{0j}(t_{0j}^*)}} \right], \quad (6.116)$$

$$\ddot{\phi}_{mj}(t_{mj}^*) \equiv \dot{\mathcal{N}}(j + m\mathcal{K})|_{t=t_{mj}^*} \quad (6.117)$$

$$S_{mj}^* \equiv S_{mj}|_{t=t_{mj}^*}, \quad (6.118)$$

$$C_{mj}^* \equiv C_{mj}|_{t=t_{mj}^*}. \quad (6.119)$$

The addend δ_{0j} appears only at one special place in Equation (6.115) because in Equation (6.108) both terms, the first *and* the second, contribute to the cos terms for $j = 0$ and the sin terms cancel each other. For computing the energy E which is a usual prefactor with some exponent in all the multipoles, use of the QKP has to be made when the “stationary” e_t and \mathcal{N} are determined. This approximation is justified as we simply use the leading-order SPA and is valid in a regime where the orbit evolves *slowly* towards coalescence. It is, thus, an approximation around the point (\mathcal{N}_0, e_{t0}) with the aforementioned requirements.

These are the ideas so far. Here are the explicit equations up to 1PN in conservative and only to leading-order in radiative dynamics, where \mathcal{N} is unscaled (that means it has unit second⁻¹),

$$\dot{\mathcal{N}} = \mathfrak{N}(\mathcal{N}, e_t) = c^{-5} \frac{\mathcal{N}^2 \eta (37e_t^4 + 292e_t^2 + 96) (Gm\mathcal{N})^{5/3}}{5(1 - e_t^2)^{7/2}}, \quad (6.120)$$

$$\dot{e}_t = \mathfrak{E}(\mathcal{N}, e_t) = -c^{-5} \frac{\mathcal{N} \eta e_t (121e_t^2 + 304) (Gm\mathcal{N})^{5/3}}{15(1 - e_t^2)^{5/2}}. \quad (6.121)$$

We clearly see units $[\dot{\mathcal{N}}] = s^{-2}$ and $[\dot{e}_t] = s^{-1}$. In terms of combined quantities, related formulas for the simple Peters and Matthews approach have been published by Pierro and Pinto [135] and Appell’s 2-variable hypergeometric function AppellF₁ has come to use. We will keep our own expressions for \mathcal{N} and e_t and will express unscaled elapsed times as functions of the latter. Note that by an appropriate scaling the factor Gm is re-absorbed in the time unit. The solution to $d\mathcal{N}/de_t = (d\mathcal{N}/dt)/(de_t/dt)$ reads

$$\mathcal{N}(\mathcal{N}_0, e_t, e_{t0}) = \mathcal{N}_0 \left(\frac{(e_t^2 - 1)}{(e_{t0}^2 - 1)} \right)^{3/2} \left(\frac{e_t}{e_{t0}} \right)^{18/19} \left(\frac{121e_{t0}^2 + 304}{121e_t^2 + 304} \right)^{1305/2299}, \quad (6.122)$$

with e_{t0} and \mathcal{N}_0 as the value of e_t and \mathcal{N} at the initial instant of time t_0 , respectively. The elapsed time as a function of e_t and e_{t0} reads

$$t - t_0 = \epsilon^{-5} \frac{9519^{1181/2299}}{22^{2173/2299} c_0^{8/3} \eta (Gm)^{5/3}} \times$$

$$\left\{ e_{t0}^{48/19} \text{AppellF}_1 \left(\frac{24}{19}; \frac{3}{2}, -\frac{1181}{2299}; \frac{43}{19}; e_{t0}^2, -\frac{121e_{t0}^2}{304} \right) - e_t^{48/19} \text{AppellF}_1 \left(\frac{24}{19}; \frac{3}{2}, -\frac{1181}{2299}; \frac{43}{19}; e_t^2, -\frac{121e_t^2}{304} \right) \right\}. \quad (6.123)$$

$$c_0 \equiv \frac{e_{t0}^{18/19} (121e_{t0}^2 + 304)^{1305/2299} \mathcal{N}_0}{(1 - e_{t0}^2)^{3/2}}. \quad (6.124)$$

A check will show that the right hand side has the dimension of time. Below, perturbative analytic solutions to the several SPA conditions are provided.

Solution to the SPA condition equation, $j > 0$

It showed up that it is easier to solve Equation (6.114) for \mathcal{N} instead of e_t and then to express $e_t - e_{t0}$ in terms of $\mathcal{N} - \mathcal{N}_0$. Expressed fully in terms of \mathcal{N} , it reads

$$0 = 2\pi f - j\mathcal{N}^* - m\mathcal{K}(\mathcal{N}^*, e_t(\mathcal{N}^*))\mathcal{N}^*. \quad (6.125)$$

It is solved in two steps: first, solve the Newtonian and second: solve the 1PN equation with the help of step 1,

$$\text{Step I :} \quad \mathcal{N}_N^* = \frac{2\pi f}{j}, \quad (6.126)$$

$$\text{Step II :} \quad \mathcal{N}_{1\text{PN}}^* = \frac{2\pi f - m\mathcal{K}(\mathcal{N}_N^*, e_t(\mathcal{N}_N^*))\mathcal{N}_N^*}{j}. \quad (6.127)$$

Step II drops out in case $m = 0$. The Newtonian ‘‘stationary eccentricity’’ for the frequency f , i.e. $e_t(\mathcal{N}_N^*)$, can be found numerically or with the help of a perturbative solution scheme. It is a rather numerical issue to apply fixpoint-method-like iterative algorithms a la Danby and Burkhard’s method [136] to solve the Kepler equation, and a detailed analysis could indicate how many steps are necessary and reasonable towards the solution. It should be noted that there may exist better algorithms to be found in the common literature of approximative solving methods. The function to be inverted for $e_t - e_{t0}$ is the following,

$$\mathcal{N}(\mathcal{N}_0, e_t, e_{t0}) = \mathcal{N}_0 \left(\frac{(e_t^2 - 1)}{(e_{t0}^2 - 1)} \right)^{3/2} \left(\frac{e_{t0}}{e_t} \right)^{18/19} \left(\frac{121e_{t0}^2 + 304}{121e_t^2 + 304} \right)^{1305/2299} \quad (6.128)$$

$$= \mathcal{N}_0 + g^{(1)}(e_t - e_{t0}) + \tilde{\epsilon} g^{(2)}(e_t - e_{t0})^2 + \tilde{\epsilon}^2 g^{(3)}(e_t - e_{t0})^3 + \dots, \quad (6.129)$$

$$g^{(p)} \equiv \frac{1}{p!} \frac{\partial^p}{\partial e_t^p} \mathcal{N}(\mathcal{N}_0, e_t, e_{t0})|_{e_{t0}}, \quad (6.130)$$

having introduced some smallness parameter $\tilde{\epsilon}$ which will be set 1 after the calculation. The solution algorithm reads (defining κ as the difference of e_t^* and e_{t0} and leaving out the ‘‘star’’)

$$\kappa^{[N]} \equiv (e_t - e_{t0})^{[N]}, \quad (6.131)$$

$$\kappa^{[1]} = \frac{(\mathcal{N} - \mathcal{N}_0)}{g^{(1)}}, \quad (6.132)$$

$$\kappa^{[2]} = \frac{1}{g^{(1)}} \{ (\mathcal{N} - \mathcal{N}_0) - \tilde{\epsilon} g^{(2)} (\kappa^{[1]})^2 \}, \quad (6.133)$$

$$\kappa^{[N]} = \frac{1}{g^{(1)}} \left\{ (\mathcal{N} - \mathcal{N}_0) - \sum_{p=2}^N \tilde{\epsilon}^{p-1} g^{(p)} (\kappa^{[N+1-p]})^p \right\}, \quad (6.134)$$

with some current solution order $[N]$. For convenience, we will give the first four orders of $e_t - e_{t0}$ in terms of $\mathcal{N} - \mathcal{N}_0$. With the definitions

$$f_1 \equiv -121e_{t0}^4 - 183e_{t0}^2 + 304, \quad (6.135)$$

$$f_2 \equiv 37e_{t0}^4 + 292e_{t0}^2 + 96, \quad (6.136)$$

we obtain

$$\begin{aligned} \kappa^{[4]} = & -(\mathcal{N} - \mathcal{N}_0) \frac{f_1 e_{t0}}{3f_2 \mathcal{N}_0} \\ & - \tilde{\epsilon} (\mathcal{N} - \mathcal{N}_0)^2 \frac{f_1 e_{t0} (370e_{t0}^8 + 34401e_{t0}^6 - 131844e_{t0}^4 - 26720e_{t0}^2 - 56832)}{18f_2^3 \mathcal{N}_0^2} \\ & - \tilde{\epsilon}^2 (\mathcal{N} - \mathcal{N}_0)^3 \left\{ f_1 e_{t0} \left(410700e_{t0}^{16} + 76370220e_{t0}^{14} + (3257592723 - 48470f_2) e_{t0}^{12} \right. \right. \\ & - 3(4124011f_2 + 9090903688) e_{t0}^{10} + 168(658637f_2 + 276828486) e_{t0}^8 + (9406764288 \\ & - 76265528f_2) e_{t0}^6 + 768(131251f_2 + 61327636) e_{t0}^4 + 18432(267f_2 + 494320) e_{t0}^2 \\ & \left. \left. - 7274496(7f_2 - 1332) \right) \right\} (162f_2^5 \mathcal{N}_0^3)^{-1} \\ & - \tilde{\epsilon}^3 (\mathcal{N} - \mathcal{N}_0)^4 \left\{ f_1 e_{t0} (+759795000e_{t0}^{24} + 211927360500e_{t0}^{22} - 1850(96940f_2 \right. \\ & - 10211818689) e_{t0}^{20} + (459468074902815 - 62450686800f_2) e_{t0}^{18} + 90(135716f_2^2 \\ & - 42030994737f_2 - 75142160154162) e_{t0}^{16} + 8(755339319f_2^2 + 6763450991635f_2 \\ & + 3192333977427390) e_{t0}^{14} - 3(31474158721f_2^2 + 56138837764240f_2 \\ & + 8745121825435200) e_{t0}^{12} + 24(7251390883f_2^2 + 4696313592840f_2 \\ & + 143898051631680) e_{t0}^{10} - 8(35529727041f_2^2 + 21737777348800f_2 \\ & + 5492015496046080) e_{t0}^8 + 384(94956913f_2^2 - 19783158080f_2 \\ & - 34649027828480) e_{t0}^6 + 30720(2707169f_2^2 + 277826496f_2 - 683225807616) e_{t0}^4 \\ & + 18186240(2319f_2^2 + 594368f_2 - 213546240) e_{t0}^2 - 1745879040(35f_2^2 - 16576f_2 \\ & \left. \left. + 1577088) \right\} (1944f_2^7 \mathcal{N}_0^4)^{-1} + \mathcal{O}(\tilde{\epsilon}^4 (\mathcal{N} - \mathcal{N}_0)^5). \quad (6.137) \end{aligned}$$

It is up to the reader to truncate this to some required order in e_{t0} or to extend it in orders of $\tilde{\epsilon}$. Having found the “stationary” \mathcal{N}_{1PN}^* from Equation (6.127) and from $e_t^*(\mathcal{N}_{1PN}^*)$, one can obtain the associated t^* by inserting this into Equation (6.123). This in turn can be inserted into Equation (6.109) to get the value of the phase at the stationary point t^* . The reader should also note that the solution to Equation (6.114) will introduce new 1PN correction terms to the multipole moments that are listed in C.

Solution to the SPA condition equation, $j=0$ and $m>0$. The pure periastron phase shift

The stationary phase condition for the pure periastron-dependent terms (those with $j = 0$) reads

$$\begin{aligned}\dot{\Phi} &= 2\pi f - m\mathcal{KN} \\ &= 2\pi f - m\mathcal{N}(e_t^*) 3 \frac{\mathcal{N}(e_t^*)^{2/3}}{1 - (e_t^*)^2} = 0,\end{aligned}\tag{6.138}$$

$$\Rightarrow g(e_t) \equiv 2\pi f - m\mathcal{N}(e_t^*) 3 \frac{\mathcal{N}(e_t^*)^{2/3}}{1 - (e_t^*)^2} = 0,\tag{6.139}$$

with $\mathcal{N}(e_t)$ taken from Equation (6.122). Here, we proceed presenting all quantities expressed in terms of e_t . To find the solution to this equation analytically, we find it convenient to consider the perturbation algorithm from Danby & Burkhardt [136] to the fourth order. We need to have a nice initial guess for e_t , which we take from the first-order expansion of Equation (6.138) in $e_t - e_{t0}$,

$$e_t^{[0]} = \frac{\frac{\pi f}{3m\mathcal{N}_0^{5/3}} (242e_{t0}^7 + 124e_{t0}^5 - 974e_{t0}^3 + 608e_{t0}) + 178e_{t0}^5 - 669e_{t0}^3 - 784e_{t0}}{3(19e_{t0}^4 - 284e_{t0}^2 - 160)},\tag{6.140}$$

noting that the case $m = 0$ is excluded. This can be inserted into an iterative solution algorithm, which solves for δ in the expression

$$e_t^* = e_t^{[0]} + \delta,\tag{6.141}$$

$$g(e_t^{[0]} + \delta) = 0.\tag{6.142}$$

This δ is found with the help of the following procedure,

$$\delta_1 = -\frac{g}{g'},\tag{6.143}$$

$$\delta_2 = -\frac{g}{g' + \frac{1}{2}\delta_1 g''},\tag{6.144}$$

$$\delta_3 = -\frac{g}{g' + \frac{1}{2}\delta_2 g'' + \frac{1}{6}\delta_2^2 g'''},\tag{6.145}$$

$$\delta_4 = -\frac{g}{g' + \frac{1}{2}\delta_3 g'' + \frac{1}{6}\delta_3^2 g''' + \frac{1}{24}\delta_3^3 g^{(4)}},\tag{6.146}$$

$$g^p \equiv \frac{\partial^p}{\partial e_t^p} g(e_t).\tag{6.147}$$

We have $e_t^{[4]} = e_t^{[0]} + \delta_4$ as the fourth-order solution to Equation (6.138) with quintic convergence, and again extract $\mathcal{N}(e_t^{[4]})$, $t - t_0$ and so on. The case $m = 0$ will be discussed below.

The case $j=0$ and $m=0$. Fourier transformation of a slow-in-time signal

In Equation (6.105), there is no fast oscillating term but only a slow variable of time to be Fourier transformed,

$$\tilde{h}(f) = \int_{-\infty}^{+\infty} \mathcal{A}(t) e^{i(-2\pi ft)} dt. \quad (6.148)$$

The term $\mathcal{A}(t)$ depends on time only due to RR. Those terms are nontrivially dependent on time and have to be treated individually when they are requested analytically. In principle, one would have to express $|E|$, \mathcal{N} and e_t as explicit functions of time. That would include inversion of Appell functions or perturbation theory.

However, for the case of inspiralling compact binaries, they will not be able to significantly contribute to frequencies in comparison to those with fast oscillating exponents as we compare typical time scales for one orbit and for the inspiral. We will therefore impose the following relation.

$$\tilde{h}(f)_{\text{static}} \ll \tilde{h}(f)_{\text{stationary}}, \quad (6.149)$$

where $\tilde{h}_{\text{static}}$ means all Fourier integrals over terms where the mean anomaly – or equivalently, the time – does not appear explicitly. We state that the $(j = 0, m = 0)$ Fourier domain terms almost vanish:

$$\tilde{h}(f)_{[j=0, m=0]} \approx 0, \quad (6.150)$$

for the frequency domain of interest for the regarded detector. LISA for example, will hardly see those terms operating near $\mathcal{N}_0 \approx 0.001\text{Hz}$. Let us give an exemplary number to support this statement. The rate of change of the GW frequency f over one year will be [137]

$$\Delta f_{\text{RR}} \sim \frac{1.6 \times 10^{-9}}{(1 - e_t^2)^{7/2}} \left(\frac{m}{2.8M_\odot} \right)^{\frac{5}{3}} \left(\frac{\eta}{0.25} \right) \left(\frac{f_r}{10^{-3}\text{Hz}} \right)^{\frac{11}{3}} \left(1 + \frac{73}{24}e_t^2 + \frac{37}{96}e_t^4 \right) \text{Hz}, \quad (6.151)$$

where f_r is the radial frequency, given by $f_r = \mathcal{N}(2\pi)^{-1}$. Let $\mathcal{N} = 10^{-3}$, $m_1 = m_2 = 1.4M_\odot$ and $e_t = 0.1$. Then, $\Delta f_{\text{RR}} \sim 2 \times 10^{-12}\text{Hz}$ and the scaled energy loss is $\Delta E_{\text{RR}} \sim 3 \times 10^{-13}$.

6.7 The quasi-circular limit

The quasi-circular inspiral has been discussed extensively in the literature, especially in [124] which we have often cited. For further information, see e.g. [138, 139]. In the limit $e_t \rightarrow 0$, all elements of our calculation simplify drastically. The following equations,

$$\mathcal{M} \rightarrow \mathcal{E}, \quad (6.152)$$

$$A(\mathcal{E}) \rightarrow 1, \quad (6.153)$$

$$f_{lm}(\mathcal{E}) \rightarrow \mathcal{F}_{lm}(|E|), \quad (6.154)$$

$$g_{lm}(\mathcal{E}) \rightarrow \mathcal{G}_{lm}(|E|), \quad (6.155)$$

$$\phi - \phi_0 \rightarrow (1 + \mathcal{K})\mathcal{M}, \quad (6.156)$$

$$e^{-im\phi} \rightarrow e^{-im\phi_0} e^{-im(1+\mathcal{K})\mathcal{M}}, \quad (6.157)$$

show that we have a simple prototype for the SPA for all the multipole moments. Equation (6.152) is Kepler's equation for quasi-circular orbits. The infinite summation series in Equation (6.115) shrink to one single term, where the phase term has to be replaced by the one in Equation (6.157). This is because the $\sin \mathcal{E}$ terms will always have a factor e_t and vanish in the case in question. The value of the phase and the angular velocity, the elapsed time and the resulting SPA integral can be taken one-to-one from [124]. The interested reader may find SPA results for circular orbits at higher PN orders in various data analysis papers [138–141]. The 3PN tensor spherical harmonics expressions for the gravitational wave polarisations for inspiralling compact binaries in quasi-circular orbits are given in [16].

6.8 Truncation of infinite series to a finite order of the eccentricity

Infinite series have to be restricted to finite ones for practical issues. Reference [126] gave instructions how to limit the 1PN series as we set up the following properties concerning the orders of involved terms:

1. Bessel functions of order n : $J_n(n e_t) \sim \mathcal{O}(e_t^n)$,
2. The $v(\mathcal{E})$ expansion coefficients: $G_n(n e_t) \sim \mathcal{O}(e_t^n)$,
3. Even-in- \mathcal{E} expansion coefficients: $\mathcal{A}_j^{(n)} \sim \mathcal{O}(e_t^j)$,
4. Odd-in- \mathcal{E} expansion coefficients: $\mathcal{S}_j^{(n)} \sim \mathcal{O}(e_t^{j-1})$,
5. The $\sin v$ expansion: $J'_n(e_t) = \frac{1}{2}(J_{n-1}(e_t) - J_{n+1}(e_t)) = \mathcal{O}(e_t^{n-1})$,
6. Double series expansions: $\Pi_j^{CS} \sim \mathcal{O}(e_t^j)$ and $\Pi_j^{SS} \sim \mathcal{O}(e_t^{j-1})$,

also see Appendix D.3. These computations are still valid at 2PN and have to be applied to each series where $e_t^{\text{sum index}}$ plays a role. The result of this is obvious but lengthy, so we skip the provision. The interested reader may take a look at [130] for more information on an estimate of the error when using finite sums. Having double, triple, \dots , maximally n -tuples of summations, each evaluated up to some order $\mathcal{O}(e_t^M)$ and, thus, containing M terms (plus or minus some finite number), we have a computational cost of $\sim M^n$ terms per each time step \mathcal{M}_k and $\text{NSP} \times M^n$ term computations in total, where NSP is the number of sampling points, $k = (0, \dots, \text{NSP} - 1)$. In our case, a typical value is $n = 5$.

6.9 Some concluding remarks for the eccentric inspiral templates

It is interesting to note how many parameters are included in the wave form. For the relatively simple case of circular inspiral, the templates used to have $\vec{\theta} = \{t_c, \mathcal{M}_c, \eta; R, \theta, \phi_c\}$ with the intrinsic parameters time to coalescence, chirp mass, symmetric mass ratio, and external parameters as the distance, inclination of the orbit, and phase at coalescence.¹⁷ Because for eccentric orbits, both e_t and \mathcal{N} will dictate the contribution to infinitely many frequencies already on the purely conservative level of EOM, both have to be regarded as parameters for the template. Thus, we have $\vec{\theta} = \{|E|_{(t=0)} \text{ or } \mathcal{N}_{(t=0)}, e_{t(t=0)}, \mathcal{M}_{(t=0)}, m, \eta; R, \theta, \phi_0\}$.

Both \mathcal{N}_0 and e_{t0} will contribute to the time to coalescence, see [135] for the value of what is called “lifetime”. The parameter space has grown,¹⁸ but the good news is that, for data analysis considerations, the ambiguity function is still maximised in view of ϕ_0 in a considerably simple way (see [3] how to do this).

¹⁷An *intrinsic* parameter is a property of the system itself. An *extrinsic* parameter in contrast is a property of the source which can only be defined relative to the detector.

¹⁸Reference [126] should additionally mention the mean anomaly as a parameter because the orbit is not longer degenerated and the phase will not coincide with \mathcal{M} .

7 Summary and outlook

Let us summarise the previous sections. Section 6 discussed full-analytic time Fourier-domain wave forms for non-spinning compact binaries. We provided the wave forms in terms of tensor spherical harmonics which simplified the optical appearance drastically. We have succeeded in finding their analytic time Fourier-domain, and advanced the results of recent publications from only Newtonian to 2PN orders and to arbitrary accuracy. The price being paid for an analytic expression was the necessity of using infinite multiple summations, which, for convenience, could be limited to finite ones through the needed accuracy with respect to the initial eccentricity. One slight drawback should be mentioned. Without appropriate optimising, the presented routine is far less than quickly computable, for example for a data rate of 4096 points per second for LIGO and several minutes to be observed and a restriction to errors of, say, $< \mathcal{O}(e_t^5)$. A first numerical insight, done in C by *Jan Sperrhake* for the 1PN case, showed a CPU time of approximately five minutes for the case of 128 data points at an error of $\mathcal{O}(e_t^6)$, performed on a 1GHz desktop PC. It can, for example, give an impression of the orders of magnitude of how many harmonics may have to be included for a data analysis investigation. Its CPU time consumption should be improved by atomising the series computation to make it attractive for researchers in data analysis. As this is reached, a future investigation may also use our results and try to convert it to an analytic expression (or at least an approximative expression) for the Fitting Factor. A future publication may also extend the calculation of Section VI of [126] with the help of Appell's integral formula (see e.g. [142]), the orbital averaged 2PN accurate EOM for \mathcal{N} and e_t , to be taken from the expressions (4.32) and (4.34) of reference [20] and converted appropriately. We further have, then, to include the fast oscillatory terms in the evolution equations for \mathcal{N} and e_t .

In section 5 we calculated leading-order gravitational wave amplitudes for spinning compact binaries with almost equal masses moving in quasi-circular orbits. It is planned to extend this investigation to the general case of eccentric orbits. It is to expect that the EOM for the angles will not be solvable fully analytically and in a practical manner unless a further averaging operation, such as an averaging over \mathcal{E} or a purely algebraic one, has been performed. Higher order gravitational wave amplitude corrections will definitely be required for data analysis considerations. It also may be interesting to investigate a canonical transformation of the variables in use, for example, with the help of Deprit's method [143] (a detailed application of this method to the orbital motion can be found in [144]) and the difference in mass as the perturbation parameter.

Section 4 deals with aligned spins under relatively high order post-Newtonian spin interaction terms. We succeeded in finding the orbital parameterisation and the relevant instantaneous gravitational wave terms, but an important SO term is missing therein that has to be investigated to obtain the full picture. We note that this prescription is *not yet ready* for data analysis investigations. It is desirable to include radiation reaction with spin dependent terms up to 2PN order beyond the leading-order term. The authors of [60] furnished the

necessary terms which have to be adapted in a future publication. We wish to complete the picture in a future paper. As in all the other sections, hereditary terms were not investigated, and this needs to be made up for. Therefore, it could be helpful to use Reference [145] as a prototype and to generalise this investigation.

It is also desirable to connect results of all three sections and to compute a full-analytic Fourier-domain wave forms for non-aligned spins under higher-order spin interactions without being restricted to circular orbits or to equal-mass or one-spin case. One may speculate if the spin-precession frequency can be separated as well as this could be done for the periastron advance frequency in the point particle prescription of Section 6.

8 Zusammenfassung und Ausblick

Kommen wir nun zur Zusammenfassung. Abschnitt 6 behandelte die analytische Frequenzdarstellung von Gravitationssignalen, ausgesandt von kompakten Binärsystemen ohne Spin. Um eine maximale Kompaktheit der Darstellung der Gravitationswellenamplituden zu erzielen, erscheint die Basis der Tensor-sphärischen Harmonischen als geeignet. Es hat sich gezeigt, dass auch bei 2PN eine voll-analytische Darstellung nach einheitlichem Schema für alle beteiligten Terme durchführbar ist. Mit den gewonnenen Resultaten ist es uns gelungen, die Ergebnisse aus der Literatur, die für den exzentrischen Fall nur bis zur Newtonschen Ordnung in der Bewegung und auch nur bis zu einer endlichen Genauigkeit bekannt waren, auf 2PN und bis hin zu beliebiger Genauigkeit zu verallgemeinern. Der Preis, den man für dieses Ergebnis zahlen muss, ist die Notwendigkeit unendlicher Mehrfachsummationen. Diese kann man für praktische Belange jedoch auf endliche Summationen reduzieren, wenn nur eine endliche Genauigkeit der Wellenform bis zu einer bestimmten Ordnung in der Exzentrizität beim Zeitpunkt des Eintritts in das sensitive Detektorband gefordert ist. Einen geringfügigen Nachteil sollte man an dieser Stelle erwähnen. Ohne entsprechende Optimierung der Berechnung der Koeffizienten ist der dargelegte Algorithmus weit von einer schnellen numerischen Berechnung entfernt. Einen ersten Einblick konnte ein C-Code liefern, den *Jan Sperrhake* für den 1PN-Fall und zunächst unter Vernachlässigung der Strahlungsrückwirkung implementierte. Hier zeigte sich in einer frühen Version auf einem Desktop-PC mit etwa 1.0GHz Leistung eine Berechnungsdauer von etwa 5 Minuten bei einer Datenmenge von 128 Punkten. Die Summen wurden dabei bei der Ordnung $\mathcal{O}(e_t^6)$ abgeschnitten. Der Algorithmus ist aber auch ohne dies in der Lage, eine Abschätzung darüber zu geben, wie viele Harmonische man bei gegebener Bahnexzentrizität für eine Datenanalyse einbeziehen muss, und liefert einen Einblick in die innere Struktur der Suchmuster exzentrischer Bahnen. Um ihn für den Einsatz bei der Auswertung von Detektordaten attraktiv zu machen, muss die Berechnung der Summanden allerdings optimiert werden.

In einem zukünftigen Projekt könnte man sich das Ziel setzen, einen analytischen oder zumindest approximativen Ausdruck für den “Fitting Factor” zu finden. Es sollte auch die

Erweiterung des Kapitels IV von Referenz [126] auf die Strahlungsrückwirkung bis 2PN relativ zur führenden Ordnung in Betracht gezogen werden, um Konsistenz in den Größenordnungen auf allen Ebenen der Darstellung zu gewährleisten. Nötig hierfür sind Appell’s Integralformel (siehe auch: die Appell-F1-Funktion, Referenz [142]), die orbital – das heißt über \mathcal{E} – gemittelten Bewegungsgleichungen für \mathcal{N} und e_t , die den Gleichungen (4.32) und (4.34) von Referenz [20] zu entnehmen sind. Im weiteren Verlauf sind dann auch die schnell oszillierenden Terme einzubeziehen, die in dieser ersten Approximation herausgemittelt worden sein werden.

In Abschnitt 5 wurden die führenden Ordnungen der Gravitationswellenamplituden berechnet, wie sie von Binären fast gleicher Masse in quasi-Kreisbahnen unter Spin-Bahn-Wechselwirkung führender Ordnung abgestrahlt werden. Es ist geplant, die Analyse auf exzentrische Bahnen auszuweiten. Dabei ist zu erwarten, dass die Bewegungsgleichungen für die Orientierungswinkel nicht ohne eine Mittlungsprozedur analytisch und in praktischer Weise lösbar sind. Natürlich sind auch Gravitationswellenamplituden höherer-Ordnung hierfür zu berechnen. Es mag sich auch als sinnvoll herausstellen, eine kanonische Transformation der benutzten Variablen zu finden, die eventuell auftretende Resonanzen bei höheren Ordnungen der Iteration der Lie-Reihen zu eliminieren. Man findet die prinzipielle Form der Berechnung in Referenz [143] (und eine detaillierte Darlegung der Anwendung in der orbitalen Bahntheorie in Referenz [144]). Entwicklungsparameter für die Störungstheorie wird an dieser Stelle die Massendifferenz oder eine damit verwandte Größe sein.

Der Abschnitt 4 behandelte den Fall parallel oder antiparallel angeordneter Spins und Bahndrehimpuls unter Spin-Bahn-Wechselwirkung höherer Ordnung und Spin-Spin-Wechselwirkung. Eine orbitale Parametrisierung sowie die relevanten instantanen Wellenbeiträge konnten berechnet werden, allerdings fehlt ein wichtiger Spin-Bahn-Beitrag zur Amplitude, der noch ermittelt werden muss. Außerdem ist es dringend erforderlich, die Rückwirkung der Abstrahlung einzubeziehen, um die Datenanalyse möglich zu machen. Die Autoren der Referenz [60] lieferten alle notwendigen noch fehlenden Beiträge, die dafür noch adaptiert und eingebaut werden müssen. Auch dies soll in einer zukünftigen Publikation angegriffen werden. Wie in allen Abschnitten dieser Dissertation wurden keine “Tail”-Terme betrachtet. Auch dieser Rückstand muss noch aufgeholt werden. Hierfür kann es sich als nützlich erweisen, die Referenz [145] als Grundlage zu verwenden und entsprechend auszubauen.

Es ist auch ein wünschenswertes Ziel, alle Abschnitte zu verknüpfen und eine analytische Wellenform für beliebige Spins zu konstruieren, die sowohl exzentrische Bahnen als auch die Energiedissipation berücksichtigt. Man kann darüber spekulieren, ob sich – wie in Abschnitt 6 die Frequenz der Periheldrehung – ebenso die Frequenz der Spinpräzession abseparieren lässt.

9 Danksagung

Für das Gelingen meiner Doktorarbeit möchte ich mich bei vielen Menschen bedanken. Zuerst bedanken möchte ich mich aber ganz herzlich bei meinem Doktorvater, Herrn Professor Gerhard Schäfer. Sie haben mich ermutigt, meine Dissertation zu diesem Forschungsthema zu beginnen. Stets haben Sie großes Interesse an meiner Arbeit gezeigt und brachten mir immer viel Geduld entgegen. Mit wertvollen Ratschlägen haben Sie einen wichtigen Teil zum Gelingen der Arbeit beigetragen, und waren nicht nur in fachlichen, sondern auch in privaten Gesprächen immer für mich da.

Ich danke auch den Kollegen und Mitarbeitern meiner Arbeitsgruppe, die ich heute zu meinem engen Freundeskreis zähle. Auch sie fanden immer Zeit für hilfreiche Gespräche und regten mit ihren fachlichen Kenntnissen und auch konstruktiver Kritik neue Ideen für weitere gemeinschaftliche Projekte an. Herrn Diplomphysiker Johannes Hartung möchte ich für die hervorragende Zusammenarbeit und die vielen nützlichen programmtechnischen Dinge danken, die er mir über Mathematica beigebracht hat. Herrn Diplomphysiker Steven Hergt danke ich besonders für viele Gespräche, die dazu beigetragen haben, dass ich dem Verständnis des Forschungsthemas Spin ein Stück näher gekommen bin und für die viele Zeit, die er sich für Diskussionen rund um meine Dissertation genommen hat. Danken möchte ich beiden noch einmal ganz herzlich für das Korrekturlesen meiner Arbeit.

Herrn Doktor Jan Steinhoff danke ich für Anregungen und hilfreiche Diskussionen während unserer Zusammenarbeit und vor allem der finalen Phase meiner Dissertation. Herrn Spermhake danke ich für die umfangreiche numerische Implementierung meines letzten Artikels und Anregungen, die zur Verbesserung des Manuskripts geführt haben. Mit ihren fundierten Kenntnissen haben sie alle dazu beigetragen, dass ich meine eigenen Ergebnisse kritisch hinterfragte und später so auf eine solide Grundlage stellen konnte. Ich danke daher allen für die exzellente Arbeitsatmosphäre und freue mich sehr auf eine weitere Zusammenarbeit.

Danken möchte ich auch Herrn Doktor Achamvedu Gopakumar für die Einführung in das Forschungsthema, die langjährige Zusammenarbeit und das freundschaftliche Verhältnis, das wir in dieser Zeit zueinander aufgebaut haben.

Herrn Professor Sukanta Bose danke ich besonders für die Gastfreundschaft während meiner Forschungsaufenthalte in Hannover.

Herzlich danken möchte ich auch meinen Freunden in Jena und Erfurt, und auch denen, die mittlerweile in andere Gegenden der Welt gezogen sind, um dort ihr Glück zu versuchen. Sie gaben mir über die ganze Zeit hinweg Rückhalt und das Gefühl, hier zu Hause zu sein und standen mir nicht nur tatkräftig zur Seite, wenn es ein Sofa quer durch die Innenstadt zu tragen galt, sondern sorgten immer für gute Laune, wenn ich sie gebraucht habe.

Außerdem danke ich der Deutschen Forschungsgemeinschaft (DFG), die mich durch den SFB/TR7 “Gravitational Wave Astronomy” finanzierte, und auch der Deutschen Gesellschaft für Luft- und Raumfahrt (DLR) für die finanzielle Unterstützung bei der Fertigstellung dieser Arbeit.

Dank gilt nicht zuletzt meinen Eltern, die immer für mich da gewesen sind. Ohne ihre Hilfe wäre das Studium und die Promotion nicht möglich gewesen.

Alle analytischen Rechnungen, die nur mit maschineller Hilfe zu bewältigen waren, wurden mit *Mathematica* durchgeführt. Alle numerischen Rechnungen, die im Lauf der Promotion angefertigt wurden, wurden in C und C++ implementiert.

A Appendix: supplementary material

A.1 Canonical variables including spin

In Reference [57], the transformation of the spin and position variables to new variables which fulfill canonical Poisson brackets were given. The author uses the Schwinger time gauge [146] that fixes the local basis with respect to the normal vector as

$$e^{(0)\mu} = -n^\mu. \quad (\text{A.1})$$

The new variables that fulfill these canonical relations (with “hat”) read, as expressed in terms of the old ones (those ones that belong to the SSC $S^{\mu\nu}p_\nu = 0$, without “hat”),

$$x^i = \hat{x}^i - \frac{nS^i}{m - np}, \quad (\text{A.2})$$

$$S_{ij} = \hat{S}_{ij} - \frac{p_i n S_j}{m - np} + \frac{p_j n S_i}{m - np}, \quad (\text{A.3})$$

$$nS_i = -\frac{p_k h^{kj} \hat{S}_{ji}}{m}, \quad (\text{A.4})$$

$$\Lambda^{[i](j)} = \hat{\Lambda}^{[i](k)} \left(\delta_{kj} + \frac{p_{(k)} p^{(i)}}{m(m - np)} \right), \quad (\text{A.5})$$

$$\hat{p}_i = p_i + \mathcal{K}_i \quad (\text{A.6})$$

These corrections can be found in Equations (3.42) - (3.44) of Reference [57]. Every quantity with indices (i) in round brackets represents the original quantity projected onto the dreibein, e.g. $p^{(i)} \equiv p^\mu e_\mu^{(i)}$. We follow the author’s notation that the label index of the dreibein vector appears in round brackets. These are the vierbein’s indices on the 3-hypersurface. The Λ are the canonical conjugated variables to the spins S , appearing in the definition of $\Omega^{ij} \equiv \Lambda^{[k]i} \hat{\Lambda}^{[k]j}$, with $[i]$ as indices in the body-fixed frame, $x_a^i(t) = \Lambda^{[j]i}(t) x_a^{[j]}$. The corrections for x^i and S_{ij} are structurally the same as in the special-relativistic case, but one has to enter the dreibein $e_{(i)k}$ and this indeed depends on the induced metric. The correction \mathcal{K}_i to p_i are the only one that are structurally new in the general relativistic case. \mathcal{K}_i can be found in above reference’s Equation (3.45).

Having transformed to the new variables (the field variables suffer a correction, too), the total action reads

$$W = \frac{c^3}{16\pi G} \int d^4x \hat{\pi}_{\text{TT}}^{ij} h_{ij,0}^{\text{TT}} + \int dt \left[\hat{p}_i \dot{\hat{x}}^i + \frac{1}{2} \hat{S}_{(i)(j)} \hat{\Omega}^{(i)(j)} - H_{\text{ADM}} \right], \quad (\text{A.7})$$

which has canonical form in all variables by construction. Note that the field momentum above does not have the prefactor of $16\pi G/c^3$, as it is defined to be a purely geometrical object, and not to be a physical momentum.

A.2 Integrals appearing in the QKP for aligned spins

For the sake of completeness we give the results of the definite integrals I_n and I'_n for different n :

$$I'_0 = \frac{\pi(s_- + s_+)}{(s_- s_+)^{3/2}}, \quad (\text{A.8})$$

$$I'_1 = \frac{2\pi}{\sqrt{s_- s_+}}, \quad (\text{A.9})$$

$$I'_2 = 2\pi, \quad (\text{A.10})$$

$$I'_3 = \pi(s_- + s_+), \quad (\text{A.11})$$

$$I'_4 = \frac{1}{4}\pi(3s_-^2 + 2s_- s_+ + 3s_+^2), \quad (\text{A.12})$$

$$I'_5 = \frac{1}{8}\pi(s_- + s_+)(5s_-^2 - 2s_- s_+ + 5s_+^2). \quad (\text{A.13})$$

The more complicated integrals with boundary s in terms of u , \tilde{v} , e_r and a_r are given by

$$I_0 = a_r^2 \sqrt{1 - e_r^2} (\mathcal{E} - \sin \mathcal{E}), \quad (\text{A.14})$$

$$I_1 = a_r \sqrt{1 - e_r^2} \mathcal{E}, \quad (\text{A.15})$$

$$I_2 = \tilde{v}, \quad (\text{A.16})$$

$$I_3 = \frac{\tilde{v} + e_r \sin(\tilde{v})}{a_r (1 - e_r^2)}, \quad (\text{A.17})$$

$$I_4 = \frac{2(2 + e_r^2)\tilde{v} + 8e_r \sin \tilde{v} + e_r^2 \sin(2\tilde{v})}{4a_r^2 (1 - e_r^2)^2}, \quad (\text{A.18})$$

$$I_5 = \frac{6(2 + 3e_r^2)\tilde{v} + 9e_r(4 + e_r^2)\sin \tilde{v} + 9e_r^2 \sin(2\tilde{v}) + e_r^3 \sin(3\tilde{v})}{12a_r^3 (1 - e_r^2)^3}. \quad (\text{A.19})$$

A.3 Transformation from harmonic to ADM coordinates including change in SSC

We note that Kidder's wave forms [106] have been computed with the help of the *covariant SSC*. The Hamiltonians, separations, velocities from our prescription have been computed, in contrast, within the *canonical SSC*. To avoid confusion about in which coordinates the wave forms are expressed, we give a brief prescription about what to do, that is, we wish to express Kidder's harmonic and covariantly formulated wave form, replacing the covariant harmonic coordinates and velocities by their functions of canonical ADM variables. Let for convenience $\mathbf{Y}_a \equiv \mathbf{x}_a^{\text{hcov}}$ label the harmonic position of the a -th particle with covariant SSC and $\mathbf{x}_a \equiv \mathbf{x}_a^{\text{ADMcan}}$. Then

$$h_{ij}^{\text{TT, hcov}} = h_{ij}^{\text{TT}}(\mathbf{Y}, d_t \mathbf{Y}) = h_{ij}^{\text{TT}}(\mathbf{Y}(\mathbf{x}, \mathbf{p}, \mathbf{S}), d_t \mathbf{Y}(\mathbf{x}, \mathbf{p}, \mathbf{S})), \quad (\text{A.20})$$

From the Appendix of Kidder's paper, we learn that the ath coordinate with canonical SSC can be written as a function of the covariant one - we use the inverse of the following:

$$\mathbf{x}_a^{\text{can}} = \mathbf{x}_a^{\text{cov}} + \frac{1}{2m_a} (\mathbf{v}_a \times \mathbf{S}_a) \quad (\text{A.21})$$

The spin term at leading order is, thus, clarified. From harmonic to ADM coordinates, there is an additional 2PN PP correction, and we take this from the literature. From section IV of [82] and from [107], we collect the contributions for the coordinate transformation from ADM to harmonic coordinates including 2PN PP contributions, which – including those author's spin term – applies to our change of SSC. Then the transformation reads *in their notation*

$$\mathbf{Y}_a(\mathbf{x}_b, \mathbf{p}_b) = \mathbf{x}_a + \epsilon^2 \mathbf{Y}_a^{\text{SO}}(\mathbf{x}_b, \mathbf{p}_b, \mathbf{S}_b) + \epsilon^4 \mathbf{Y}_a^{\text{2PN}}(\mathbf{x}_b, \mathbf{p}_b) \quad (\text{A.22})$$

with

$$\mathbf{Y}_a^{\text{SO}}(\mathbf{x}_b, \mathbf{p}_b, \mathbf{S}_b) = \frac{\mathbf{S}_a \times \mathbf{p}_a}{2m_a^2}, \quad (\text{A.23})$$

$$\begin{aligned} \mathbf{Y}_1^{\text{2PN}}(\mathbf{x}_a, \mathbf{p}_a) = Gm_2 \left\{ \left[\frac{5}{8} \frac{\mathbf{p}_2^2}{m_2^2} - \frac{1}{8} \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)^2}{m_2^2} + \frac{Gm_1}{r_{12}} \left(\frac{7}{4} + \frac{1}{4} \frac{m_2}{m_1} \right) \right] \mathbf{n}_{12} \right. \\ \left. + \frac{1}{2} \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_2} \frac{\mathbf{p}_1}{m_1} - \frac{7}{4} \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_2} \frac{\mathbf{p}_2}{m_2} \right\}, \quad (\text{A.24}) \end{aligned}$$

where $\mathbf{Y}_2^{\text{2PN}}(\mathbf{x}_a, \mathbf{p}_a)$ is simply obtained by exchanging the particle indices ($1 \leftrightarrow 2$). We find it very important to mention some of the rules to obtain the relative separation vector with the scaling introduced in this thesis. The above equations are *not* given in relative coordinates. Thus, we scale every \mathbf{S}_a with m_a^2 . Next, we subtract \mathbf{Y}_2 from \mathbf{Y}_1 to obtain $\mathbf{Y} = \mathbf{Y}_{12} = \mathbf{Y}_1 - \mathbf{Y}_2$ (the order is crucial), setting $\mathbf{p}_2 = -\mathbf{p}_1 = -\mathbf{p}$ for the centre-of-mass frame and scale \mathbf{p} with μ as in Eq. (4.5) to get a dimensionless momentum. Finally, we divide the obtained separation vector with Gm and obtain the separation in terms of the linear momentum and the ADM spin momenta.

The harmonic and covariant velocity is, expressed in terms of ADM and canonical SSC variables, obtained just by plugging the harmonic positions in the Poisson brackets with the total Hamiltonian,

$$\mathbf{v} = [\mathbf{x}, H^{\text{ADM}}], \quad (\text{A.25})$$

$$\mathbf{v}^{\text{hcov}} = [\mathbf{x}^{\text{hcov}}, H^{\text{ADM}}], \quad (\text{A.26})$$

see [147] for the effects of the 1PN time shift part of the coordinate gauge transformation to the velocities and positions. The linear momentum \mathbf{p} can then be expressed in terms of the velocity perturbatively. It is important to express \mathbf{p} in terms of the ADM velocity first and then to plug it into the expression for \mathbf{v}^{hcov} afterwards. To 2PN order, the radial separation,

the velocity and the unit normal vector, r^{hcov} , \mathbf{v}^{hcov} and $\mathbf{n}_{12}^{\text{hcov}}$ transform due to

$$\mathbf{x}^{\text{hcov}} = \mathbf{x} + \frac{1}{2}\epsilon^2\delta\eta(\mathbf{S} \times \mathbf{v}) + \epsilon^4 \left\{ \frac{12\eta + 1}{4r}\mathbf{n}_{12} - \frac{1}{8}\eta(\mathbf{n}_{12}(\dot{r}^2 - 5\mathbf{v}^2) + 18\dot{r}\mathbf{v}) \right\}, \quad (\text{A.27})$$

$$\begin{aligned} \mathbf{v}^{\text{hcov}} &= \mathbf{v} - \epsilon^2\delta\frac{\eta}{2r^2}(\mathbf{S} \times \mathbf{n}_{12}) \\ &\quad + \epsilon^4 \left\{ \frac{\eta}{8r}(\dot{r}\mathbf{n}_{12}(3\dot{r}^2 - 7\mathbf{v}^2) + \mathbf{v}(17\dot{r}^2 - 13\mathbf{v}^2)) \right. \\ &\quad \left. + \frac{1}{4r^2}((21\eta + 1)\mathbf{v} - (19\eta + 2)\dot{r}\mathbf{n}_{12}) \right\}, \end{aligned} \quad (\text{A.28})$$

$$r^{\text{hcov}} = r - \frac{1}{2}\epsilon^2\delta\eta[\mathbf{S} \cdot (\mathbf{n}_{12} \times \mathbf{v})] + \epsilon^4 \left\{ \frac{1}{8}\eta(5\mathbf{v}^2 - 19\dot{r}^2) + \frac{3\eta + \frac{1}{4}}{r} \right\}, \quad (\text{A.29})$$

$$\mathbf{n}_{12}^{\text{hcov}} = \mathbf{n}_{12} + \epsilon^2\delta\frac{\eta}{2r}\{\mathbf{n}_{12}[\mathbf{S} \cdot (\mathbf{n}_{12} \times \mathbf{v})] + (\mathbf{S} \times \mathbf{v})\} + \epsilon^4\frac{9\dot{r}\eta}{4r}\{\dot{r}\mathbf{n}_{12} - \mathbf{v}\}, \quad (\text{A.30})$$

where every quantity on the right hand side is written in ADM coordinates. Note that

$$\mathbf{n}_{12}[\mathbf{S} \cdot (\mathbf{n}_{12} \times \mathbf{v})] + (\mathbf{S} \times \mathbf{v}) = (\mathbf{1} - \mathbf{n}_{12} \otimes \mathbf{n}_{12})(\mathbf{S} \times \mathbf{v}) \quad (\text{A.31})$$

is the part of $(\mathbf{S} \times \mathbf{v})$ which is orthogonal to \mathbf{n}_{12} .

A.4 The relevant tensor spherical harmonics through 2PN order

We take the definitions from [30], Equations (A1) – (A5) therein,

$$T_{LM}^{(m)} = A_{LM}(\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) - iB_{LM}(\hat{\theta}\hat{\phi} + \hat{\phi}\hat{\theta}), \quad (\text{A.32})$$

$$T_{LM}^{(e)} = B_{LM}(\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) - iA_{LM}(\hat{\theta}\hat{\phi} + \hat{\phi}\hat{\theta}), \quad (\text{A.33})$$

$$A_{LM} = 2C_L \left\{ \frac{\partial^2}{\partial\theta^2} + \frac{L(L+1)}{2} \right\} Y_{LM}(\theta, \phi), \quad (\text{A.34})$$

$$B_{LM} = -2C_L \left\{ \frac{\partial}{\partial\theta} - \cot\theta \right\} Y_{LM}(\theta, \phi), \quad (\text{A.35})$$

$$C_L = [2L(L+1)(L+2)(L-1)]^{-1/2}, \quad (\text{A.36})$$

$$T_{LM}^{E2} = T_{LM}^{(m)}, \quad (\text{A.37})$$

$$T_{LM}^{B2} = -iT_{LM}^{(e)}, \quad (\text{A.38})$$

with $\hat{\theta}$ and $\hat{\phi}$ being basis unit vectors in θ and ϕ direction (θ is the angle between the orbital angular momentum and the line-of-sight vector \mathbf{N} , which corresponds to the angle i_0 of our Section 4, and ϕ measures the angle from the x axis to \mathbf{N} projected onto the (x, y) plane, see Figure 3 of [30]). The relevant multipoles read

$$T_{22}^{E2} = \frac{1}{8}\sqrt{\frac{5}{2\pi}}e^{2i\phi} \left[(\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi})\frac{1}{2}(\cos(2\theta) + 3) + 2i\cos(\theta)(\hat{\theta}\hat{\phi} - \hat{\phi}\hat{\theta}) \right], \quad (\text{A.39})$$

$$T_{21}^{E2} = \frac{1}{4} \sqrt{\frac{5}{2\pi}} e^{2i\phi} \left[\cos(\theta) \sin(\theta) (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) + i \sin(\theta) (\hat{\theta}\hat{\phi} - \hat{\phi}\hat{\theta}) \right], \quad (\text{A.40})$$

$$T_{20}^{E2} = \frac{1}{8} \sqrt{\frac{15}{\pi}} \sin^2(\theta) (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}), \quad (\text{A.41})$$

$$T_{22}^{B2} = \frac{-1}{16} \sqrt{\frac{5}{2\pi}} e^{2i\phi} \left[(\hat{\theta}\hat{\phi} - \hat{\phi}\hat{\theta}) (\cos(2\theta) + 3) - 4i (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) \cos(\theta) \right], \quad (\text{A.42})$$

$$T_{21}^{B2} = \frac{1}{4} \sqrt{\frac{5}{2\pi}} i e^{i\phi} \left[(\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) + i (\hat{\theta}\hat{\phi} - \hat{\phi}\hat{\theta}) \cos(\theta) \right] \sin(\theta), \quad (\text{A.43})$$

$$T_{20}^{B2} = \frac{-1}{8} \sqrt{\frac{15}{\pi}} (\hat{\theta}\hat{\phi} - \hat{\phi}\hat{\theta}) \sin^2(\theta), \quad (\text{A.44})$$

$$T_{33}^{E2} = -\frac{1}{32} \sqrt{\frac{21}{\pi}} e^{3i\phi} \left[4i (\hat{\theta}\hat{\phi} - \hat{\phi}\hat{\theta}) \cos(\theta) + (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) (\cos(2\theta) + 3) \right] \sin(\theta), \quad (\text{A.45})$$

$$T_{32}^{E2} = \frac{1}{32} \sqrt{\frac{7}{2\pi}} e^{2i\phi} \left[8i (\hat{\theta}\hat{\phi} - \hat{\phi}\hat{\theta}) \cos(2\theta) + (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) (5 \cos(\theta) + 3 \cos(3\theta)) \right], \quad (\text{A.46})$$

$$T_{31}^{E2} = \frac{1}{32} \sqrt{\frac{35}{\pi}} e^{i\phi} \left[4i (\hat{\theta}\hat{\phi} - \hat{\phi}\hat{\theta}) \cos(\theta) + (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) (3 \cos(2\theta) + 1) \right] \sin(\theta), \quad (\text{A.47})$$

$$T_{30}^{E2} = \frac{1}{8} \sqrt{\frac{105}{\pi}} (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) \cos(\theta) \sin^2(\theta), \quad (\text{A.48})$$

$$T_{33}^{B2} = \frac{1}{32} \sqrt{\frac{21}{\pi}} e^{3i\phi} \left[(\hat{\theta}\hat{\phi} - \hat{\phi}\hat{\theta}) (\cos(2\theta) + 3) - 4i (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) \cos(\theta) \right] \sin(\theta), \quad (\text{A.49})$$

$$T_{32}^{B2} = \frac{-1}{32} \sqrt{\frac{7}{2\pi}} e^{2i\phi} \left[(\hat{\theta}\hat{\phi} - \hat{\phi}\hat{\theta}) (5 \cos(\theta) + 3 \cos(3\theta)) - 8i (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) \cos(2\theta) \right], \quad (\text{A.50})$$

$$T_{31}^{B2} = \frac{-1}{32} \sqrt{\frac{35}{\pi}} e^{i\phi} \left[(\hat{\theta}\hat{\phi} - \hat{\phi}\hat{\theta}) (3 \cos(2\theta) + 1) - 4i (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) \cos(\theta) \right] \sin(\theta), \quad (\text{A.51})$$

$$T_{30}^{B2} = \frac{-1}{8} \sqrt{\frac{105}{\pi}} (\hat{\theta}\hat{\phi} - \hat{\phi}\hat{\theta}) \cos(\theta) \sin^2(\theta), \quad (\text{A.52})$$

$$T_{44}^{E2} = \frac{3}{32} \sqrt{\frac{7}{2\pi}} e^{4i\phi} \left[4i (\hat{\theta}\hat{\phi} - \hat{\phi}\hat{\theta}) \cos(\theta) + (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) (\cos(2\theta) + 3) \right] \sin^2(\theta), \quad (\text{A.53})$$

$$T_{43}^{E2} = \frac{-3}{32} \sqrt{\frac{7}{\pi}} e^{3i\phi} \left[4 (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) \cos^3(\theta) + i (\hat{\theta}\hat{\phi} - \hat{\phi}\hat{\theta}) (3 \cos(2\theta) + 1) \right] \sin(\theta), \quad (\text{A.54})$$

$$T_{42}^{E2} = \frac{3}{64\sqrt{2\pi}} e^{2i\phi} \left[2i (\hat{\theta}\hat{\phi} - \hat{\phi}\hat{\theta}) (\cos(\theta) + 7 \cos(3\theta)) \right. \\ \left. + (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) (4 \cos(2\theta) + 7 \cos(4\theta) + 5) \right], \quad (\text{A.55})$$

$$T_{41}^{E2} = \frac{3}{32\sqrt{\pi}} e^{i\phi} \left[i (\hat{\theta}\hat{\phi} - \hat{\phi}\hat{\theta}) (7 \cos(2\theta) + 5) + (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) (5 \cos(\theta) + 7 \cos(3\theta)) \right] \sin(\theta), \quad (\text{A.56})$$

$$T_{40}^{E2} = \frac{3}{32} \sqrt{\frac{5}{\pi}} (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) (7 \cos(2\theta) + 5) \sin^2(\theta), \quad (\text{A.57})$$

$$T_{44}^{B2} = \frac{-3}{32} \sqrt{\frac{7}{2\pi}} e^{4i\phi} \left[(\hat{\theta}\hat{\phi} - \hat{\phi}\hat{\theta}) (\cos(2\theta) + 3) - 4i (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) \cos(\theta) \right] \sin^2(\theta), \quad (\text{A.58})$$

$$T_{43}^{B2} = \frac{3}{32} \sqrt{\frac{7}{\pi}} e^{3i\phi} \left[4 (\hat{\theta}\hat{\phi} - \hat{\phi}\hat{\theta}) \cos^3(\theta) - i (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) (3 \cos(2\theta) + 1) \right] \sin(\theta), \quad (\text{A.59})$$

$$T_{42}^{B2} = \frac{-3}{64\sqrt{2\pi}} e^{2i\phi} \left[(\hat{\theta}\hat{\phi} - \hat{\phi}\hat{\theta}) (4 \cos(2\theta) + 7 \cos(4\theta) + 5) \right]$$

$$-2i(\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi})(\cos(\theta) + 7\cos(3\theta))], \quad (\text{A.60})$$

$$T_{41}^{B2} = \frac{-3}{32\sqrt{\pi}} e^{i\phi} \left[(\hat{\theta}\hat{\phi} - \hat{\phi}\hat{\theta})(5\cos(\theta) + 7\cos(3\theta)) - i(\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi})(7\cos(2\theta) + 5) \right] \sin(\theta), \quad (\text{A.61})$$

$$T_{40}^{B2} = \frac{-3}{32} \sqrt{\frac{5}{\pi}} (\hat{\theta}\hat{\phi} - \hat{\phi}\hat{\theta})(7\cos(2\theta) + 5) \sin^2(\theta), \quad (\text{A.62})$$

$$T_{55}^{E2} = -\frac{1}{64} \sqrt{\frac{165}{\pi}} e^{5i\phi} \left[4i(\hat{\theta}\hat{\phi} - \hat{\phi}\hat{\theta}) \cos(\theta) + (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi})(\cos(2\theta) + 3) \right] \sin^3(\theta), \quad (\text{A.63})$$

$$T_{54}^{E2} = \frac{1}{64} \sqrt{\frac{33}{2\pi}} e^{4i\phi} \left[8i(\hat{\theta}\hat{\phi} - \hat{\phi}\hat{\theta})(2\cos(2\theta) + 1) + (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi})(19\cos(\theta) + 5\cos(3\theta)) \right] \sin^2(\theta), \quad (\text{A.64})$$

$$T_{53}^{E2} = -\frac{1}{256} \sqrt{\frac{33}{\pi}} e^{3i\phi} \left[4i(\hat{\theta}\hat{\phi} - \hat{\phi}\hat{\theta})(7\cos(\theta) + 9\cos(3\theta)) + (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi})(28\cos(2\theta) + 15\cos(4\theta) + 21) \right] \sin(\theta), \quad (\text{A.65})$$

$$T_{52}^{E2} = \frac{1}{128} \sqrt{\frac{11}{2\pi}} e^{2i\phi} \left[8i(\hat{\theta}\hat{\phi} - \hat{\phi}\hat{\theta})(\cos(2\theta) + 3\cos(4\theta)) + (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi})(14\cos(\theta) + 3(\cos(3\theta) + 5\cos(5\theta))) \right], \quad (\text{A.66})$$

$$T_{51}^{E2} = \frac{1}{128} \sqrt{\frac{77}{2\pi}} e^{i\phi} \left[4i(\hat{\theta}\hat{\phi} - \hat{\phi}\hat{\theta})(5\cos(\theta) + 3\cos(3\theta)) + (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi})(12\cos(2\theta) + 15\cos(4\theta) + 5) \right] \sin(\theta), \quad (\text{A.67})$$

$$T_{50}^{E2} = \frac{1}{64} \sqrt{\frac{1155}{\pi}} (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) [5\cos(\theta) + 3\cos(3\theta)] \sin^2(\theta), \quad (\text{A.68})$$

$$T_{55}^{B2} = \frac{1}{64} \sqrt{\frac{165}{\pi}} e^{5i\phi} \left[(\hat{\theta}\hat{\phi} - \hat{\phi}\hat{\theta})(\cos(2\theta) + 3) - 4i(\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) \cos(\theta) \right] \sin^3(\theta), \quad (\text{A.69})$$

$$T_{54}^{B2} = -\frac{1}{64} \sqrt{\frac{33}{2\pi}} e^{4i\phi} \left[(\hat{\theta}\hat{\phi} - \hat{\phi}\hat{\theta})(19\cos(\theta) + 5\cos(3\theta)) - 8i(\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi})(2\cos(2\theta) + 1) \right] \sin^2(\theta), \quad (\text{A.70})$$

$$T_{53}^{B2} = \frac{1}{256} \sqrt{\frac{33}{\pi}} e^{3i\phi} \left[(\hat{\theta}\hat{\phi} - \hat{\phi}\hat{\theta})(28\cos(2\theta) + 15\cos(4\theta) + 21) - 4i(\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi})(7\cos(\theta) + 9\cos(3\theta)) \right] \sin(\theta), \quad (\text{A.71})$$

$$T_{52}^{B2} = -\frac{1}{128} \sqrt{\frac{11}{2\pi}} e^{2i\phi} \left[(\hat{\theta}\hat{\phi} - \hat{\phi}\hat{\theta})(14\cos(\theta) + 3(\cos(3\theta) + 5\cos(5\theta))) - 8i(\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi})(\cos(2\theta) + 3\cos(4\theta)) \right], \quad (\text{A.72})$$

$$T_{51}^{B2} = -\frac{1}{128} \sqrt{\frac{77}{2\pi}} e^{i\phi} \left[(\hat{\theta}\hat{\phi} - \hat{\phi}\hat{\theta})(12\cos(2\theta) + 15\cos(4\theta) + 5) - 4i(\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi})(5\cos(\theta) + 3\cos(3\theta)) \right] \sin(\theta), \quad (\text{A.73})$$

$$T_{50}^{B2} = -\frac{1}{64} \sqrt{\frac{1155}{\pi}} (\hat{\theta}\hat{\phi} - \hat{\phi}\hat{\theta}) [5\cos(\theta) + 3\cos(3\theta)] \sin^2(\theta), \quad (\text{A.74})$$

$$T_{66}^{E2} = \frac{3}{256} \sqrt{\frac{715}{2\pi}} e^{6i\phi} \left[4i(\hat{\theta}\hat{\phi} - \hat{\phi}\hat{\theta}) \cos(\theta) + (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi})(\cos(2\theta) + 3) \right] \sin^4(\theta), \quad (\text{A.75})$$

$$T_{65}^{E2} = -\frac{1}{256} \sqrt{\frac{2145}{2\pi}} e^{5i\phi} \left[2i(\hat{\theta}\hat{\phi} - \hat{\phi}\hat{\theta})(5 \cos(2\theta) + 3) + (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi})(13 \cos(\theta) + 3 \cos(3\theta)) \right] \sin^3(\theta), \quad (\text{A.76})$$

$$T_{64}^{E2} = \frac{1}{1024} \sqrt{\frac{195}{\pi}} e^{4i\phi} \left[8i(\hat{\theta}\hat{\phi} - \hat{\phi}\hat{\theta})(13 \cos(\theta) + 11 \cos(3\theta)) + (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi})(92 \cos(2\theta) + 33 \cos(4\theta) + 67) \right] \sin^2(\theta), \quad (\text{A.77})$$

$$T_{63}^{E2} = -\frac{3}{1024} \sqrt{\frac{13}{2\pi}} e^{3i\phi} \left[2i(\hat{\theta}\hat{\phi} - \hat{\phi}\hat{\theta})(52 \cos(2\theta) + 55 \cos(4\theta) + 21) + (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi})(126 \cos(\theta) + 75 \cos(3\theta) + 55 \cos(5\theta)) \right] \sin(\theta), \quad (\text{A.78})$$

$$T_{62}^{E2} = \frac{1}{4096} \sqrt{\frac{13}{2\pi}} e^{2i\phi} \left[4i(\hat{\theta}\hat{\phi} - \hat{\phi}\hat{\theta})(10 \cos(\theta) + 81 \cos(3\theta) + 165 \cos(5\theta)) + (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi})(289 \cos(2\theta) + 30(\cos(4\theta) + 7) + 495 \cos(6\theta)) \right], \quad (\text{A.79})$$

$$T_{61}^{E2} = \frac{1}{1024} \sqrt{\frac{65}{\pi}} e^{i\phi} \left[2i(\hat{\theta}\hat{\phi} - \hat{\phi}\hat{\theta})(60 \cos(2\theta) + 33 \cos(4\theta) + 35) + (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi})(70 \cos(\theta) + 87 \cos(3\theta) + 99 \cos(5\theta)) \right] \sin(\theta), \quad (\text{A.80})$$

$$T_{60}^{E2} = \frac{1}{512} \sqrt{\frac{1365}{2\pi}} (\hat{\theta}\hat{\theta} - \hat{\phi}\hat{\phi}) [60 \cos(2\theta) + 33 \cos(4\theta) + 35] \sin^2(\theta). \quad (\text{A.81})$$

A.5 Appell's integral formula in the solution for the elapsed time

The integral in Equation (6.123) in Section 6.6 can be solved with the help of the following integral representation of the AppellF1 function (see <http://dlmf.nist.gov/> or [142] for further information).

$$\int_0^1 du \frac{u^{\alpha-1}(1-u)^{\gamma-\alpha-1}}{(1-ux)^{\beta_1}(1-uy)^{\beta_2}} = \frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)} \text{AppellF}_1(\alpha; \beta_1, \beta_2; \gamma; x, y). \quad (\text{A.82})$$

B Results: The remaining GW spherical tensor components through 2PN order

Below are the spherical tensor contributions coming from 0.5PN to 2PN amplitude corrections to the far-zone field that we did not list explicitly in Section 6.3.

$$S_0^{(2)} = 0, \quad (\text{B.1})$$

$$S_0^{(2)} = \frac{32}{3} \sqrt{\frac{\pi}{5}} |E|^{3/2} \mathcal{W}_{et} \delta_m \eta e^{-i\phi} \left\{ \frac{1}{A(\mathcal{E})^2} \right.$$

$$\begin{aligned}
& +|E|\epsilon^2 \left[\frac{((675 - 3639\eta)e_t^2 + 3695\eta - 731)}{28 \mathcal{W}_{e_t}^2 A(\mathcal{E})^2} + \frac{4(\eta - 1)}{A(\mathcal{E})^3} \right. \\
& \left. + A(\mathcal{E})^{-4} \left\{ \frac{1}{14} \mathcal{W}_{e_t}^2 (2245 - 2096\eta) + \frac{1}{7} i \mathcal{W}_{e_t} (2054\eta - 2077) e_t \sin(\mathcal{E}) \right\} \right] \Bigg\}, \quad (\text{B.2})
\end{aligned}$$

$$S^{22} = 0, \quad (\text{B.3})$$

$$I^{03} = 0, \quad (\text{B.4})$$

$$\begin{aligned}
I^{13} &= 8 \frac{4i \sqrt{\frac{10\pi}{7}} |E|^{3/2} \mathcal{W}_{e_t} \delta_m \eta}{3} e^{-i\phi} \left\{ \frac{-6 - \frac{6ie_t \sin(\mathcal{E})}{\mathcal{W}_{e_t}}}{5A(\mathcal{E})} + \frac{1}{A(\mathcal{E})^2} \right. \\
& + |E|\epsilon^2 \left[-\frac{((5\eta - 7)e_t^2 + 19\eta - 17) - i \mathcal{W}_{e_t} (5\eta - 7) e_t \sin(\mathcal{E})}{10 \mathcal{W}_{e_t}^2 A(\mathcal{E})} \right. \\
& - \frac{(5\eta - 79)e_t^2 - 45\eta + 119}{20 \mathcal{W}_{e_t}^2 A(\mathcal{E})^2} + \frac{-4 \mathcal{W}_{e_t} (\eta - 14) - 2i(4\eta - 17) e_t \sin(\mathcal{E})}{3 \mathcal{W}_{e_t} A(\mathcal{E})^3} \\
& \left. \left. + \frac{3 \mathcal{W}_{e_t}^2 (2\eta - 23) - 2i \mathcal{W}_{e_t} (\eta + 10) e_t \sin(\mathcal{E})}{5A(\mathcal{E})^4} \right] \right\}, \quad (\text{B.5})
\end{aligned}$$

$$I^{23} = 0, \quad (\text{B.6})$$

$$\begin{aligned}
I^{33} &= -20i \sqrt{\frac{2\pi}{21}} |E|^{3/2} \mathcal{W}_{e_t} \delta_m \eta e^{-3i\phi} \left\{ \frac{1}{A(\mathcal{E})^2} - \frac{8 \mathcal{W}_{e_t} (\mathcal{W}_{e_t} + ie_t \sin(\mathcal{E}))}{5A(\mathcal{E})^3} \right. \\
& + \frac{-\frac{6}{5} - \frac{2ie_t \sin(\mathcal{E})}{5 \mathcal{W}_{e_t}}}{A(\mathcal{E})} + |E|\epsilon^2 \left[\frac{i \mathcal{W}_{e_t} (5\eta - 7) e_t \sin(\mathcal{E}) - 3(5\eta - 7) e_t^2 - 57\eta + 51}{30 \mathcal{W}_{e_t}^2 A(\mathcal{E})} \right. \\
& + \frac{(79 - 5\eta) e_t^2 + 45\eta - 119}{20 \mathcal{W}_{e_t}^2 A(\mathcal{E})^2} - \{45 \mathcal{W}_{e_t} A(\mathcal{E})^3\}^{-1} \left\{ 6 \mathcal{W}_{e_t} ((5\eta - 7) e_t^2 + 77\eta - 205) \right. \\
& + 2ie_t \sin(\mathcal{E}) (3(5\eta - 7) e_t^2 + 149\eta - 208) \left. \right\} \\
& \left. \left. + \frac{\mathcal{W}_{e_t} (\mathcal{W}_{e_t} (34\eta - 191) + 14i(\eta - 14) e_t \sin(\mathcal{E}))}{15A(\mathcal{E})^4} \right] \right\}, \quad (\text{B.7})
\end{aligned}$$

$$\begin{aligned}
S^{03} &= 16 \sqrt{\frac{\pi}{105}} |E|^2 m \mathcal{W}_{e_t} \eta (3\eta - 1) e_t \left\{ \frac{\sin(\mathcal{E})}{A(\mathcal{E})^3} \right. \\
& + |E|\epsilon^2 \left[\frac{\sin(\mathcal{E}) ((8\eta^2 - 17\eta + 4) e_t^2 + 10\eta^2 - 7\eta + 2)}{3 \mathcal{W}_{e_t}^2 (3\eta - 1) A(\mathcal{E})^3} \right. \\
& \left. \left. + \frac{2(2\eta^2 - 242\eta + 79) \sin(\mathcal{E})}{(9 - 27\eta) A(\mathcal{E})^4} - \frac{\mathcal{W}_{e_t}^2 (2\eta^2 - 242\eta + 79) \sin(\mathcal{E})}{(3 - 9\eta) A(\mathcal{E})^5} \right] \right\}, \quad (\text{B.8})
\end{aligned}$$

$$S^{13} = 0, \quad (\text{B.9})$$

$$S^{23} = \frac{32}{3} \sqrt{\frac{2\pi}{7}} |E|^2 m \mathcal{W}_{e_t}^2 \eta (3\eta - 1) e^{-2i\phi} \left\{ \frac{i}{A(\mathcal{E})^3} - \frac{e_t \sin(\mathcal{E})}{4 \mathcal{W}_{e_t} A(\mathcal{E})^3} \right\}$$

$$\begin{aligned}
& +|E|\epsilon^2 \left[\sin(\mathcal{E}) \left(-\frac{e_t((8\eta^2 - 17\eta + 4)e_t^2 + 10\eta^2 - 7\eta + 2)}{12\mathcal{W}_{e_t}^3(3\eta - 1)A(\mathcal{E})^3} \right. \right. \\
& + \left. \frac{(2\eta^2 - 242\eta + 79)e_t}{18\mathcal{W}_{e_t}(3\eta - 1)A(\mathcal{E})^4} + \frac{\mathcal{W}_{e_t}(22\eta^2 + 338\eta - 109)e_t}{(12 - 36\eta)A(\mathcal{E})^5} \right) \\
& + \frac{i(15(1 - 3\eta)^2 e_t^2 + 5(19\mathcal{W}_{e_t}^2 - 3)\eta^2 - 10(2\mathcal{W}_{e_t}^2 + 7)\eta - \mathcal{W}_{e_t}^2 + 25)}{10\mathcal{W}_{e_t}^2(3\eta - 1)A(\mathcal{E})^3} \\
& \left. + \frac{i(197\eta^2 - 239\eta + 55)}{(54\eta - 18)A(\mathcal{E})^4} + \frac{i\mathcal{W}_{e_t}^2(59\eta^2 - 89\eta + 25)}{(6 - 18\eta)A(\mathcal{E})^5} \right] \Bigg\}, \tag{B.10}
\end{aligned}$$

$$S^{33} = 0, \tag{B.11}$$

$$\begin{aligned}
I^{04} &= \frac{8}{21} \sqrt{\frac{\pi}{5}} |E|^2 m(1 - 3\eta)\eta \left\{ \frac{5 - 5e_t^2}{A(\mathcal{E})^3} - \frac{6}{A(\mathcal{E})} - \frac{5}{A(\mathcal{E})^2} + 6 \right. \\
& + |E|\epsilon^2 \left[\frac{9\mathcal{W}_{e_t}^4(20\eta(7\eta - 128) + 831)}{110(3\eta - 1)A(\mathcal{E})^5} - \frac{3\mathcal{W}_{e_t}^2(20\eta(7\eta - 128) + 831)}{(66\eta - 22)A(\mathcal{E})^4} \right. \\
& + \frac{\mathcal{W}_{e_t}^2(5\eta(2\eta + 163) - 218)}{55(3\eta - 1)A(\mathcal{E})^3} + \frac{5\eta(82\eta - 1699) + 2711}{55(3\eta - 1)A(\mathcal{E})^2} \\
& \left. \left. + \frac{6(5\eta(8\eta - 41) + 52)}{55(3\eta - 1)A(\mathcal{E})} - \frac{6(5\eta(8\eta - 41) + 52)}{55(3\eta - 1)} \right] \right\}, \tag{B.12}
\end{aligned}$$

$$I^{14} = 0, \tag{B.13}$$

$$\begin{aligned}
I^{24} &= \frac{16}{21} \sqrt{2\pi} |E|^2 m\eta(3\eta - 1) e^{-2i\phi} \left\{ 1 - \frac{1}{A(\mathcal{E})} + \frac{-\mathcal{W}_{e_t}^2 + i\mathcal{W}_{e_t}e_t \sin(\mathcal{E})}{2A(\mathcal{E})^3} \right. \\
& + \frac{2i\mathcal{W}_{e_t}e_t \sin(\mathcal{E}) - 2e_t^2 + \frac{7}{6}}{A(\mathcal{E})^2} \\
& + |E|\epsilon^2 \left[\left\{ 88(3\eta - 1)A(\mathcal{E})^5 \right\}^{-1} \mathcal{W}_{e_t} \left\{ -22\mathcal{W}_{e_t}^4(30\eta^2 - 250\eta + 79) \right. \right. \\
& - \left. \left. 5i\mathcal{W}_{e_t}^2(7\eta^2 - 447\eta + 151)e_t \sin(\mathcal{E}) \right\} \right. \\
& + \frac{\mathcal{W}_{e_t}^2(928\eta^2 - 11620\eta + 3743) + i\mathcal{W}_{e_t}(2147\eta^2 - 10331\eta + 3175)e_t \sin(\mathcal{E})}{132(3\eta - 1)A(\mathcal{E})^4} \\
& + (330\mathcal{W}_{e_t}^2(3\eta - 1)A(\mathcal{E})^2)^{-1} \left(e_t^2 \left(-75(53\mathcal{W}_{e_t}^2 - 66)\eta^2 \right. \right. \\
& + \left. \left. 30(17\mathcal{W}_{e_t}^2 - 22)\eta + 129\mathcal{W}_{e_t}^2 - 330 \right) \right. \\
& + \left. 12i\mathcal{W}_{e_t}e_t \sin(\mathcal{E}) \left((40\eta^2 - 205\eta + 52)e_t^2 + 290\eta^2 - 235\eta + 58 \right) \right. \\
& - \left. 495(1 - 3\eta)^2 e_t^4 + 8345\mathcal{W}_{e_t}^2 \eta^2 \right. \\
& - \left. 14285\mathcal{W}_{e_t}^2 \eta + 3902\mathcal{W}_{e_t}^2 - 495\eta^2 - 2310\eta + 825 \right) \\
& + (660\mathcal{W}_{e_t}^2(3\eta - 1)A(\mathcal{E})^3)^{-1} \left(22(e_t^2(15(29\mathcal{W}_{e_t}^2 - 20)\eta^2 \right. \\
& - \left. 20(75\mathcal{W}_{e_t}^2 - 2)\eta + 459\mathcal{W}_{e_t}^2 + 20) \right. \\
& \left. + 30(1 - 3\eta)^2 e_t^4 + (30 - 675\mathcal{W}_{e_t}^2)\eta^2 + 140(13\mathcal{W}_{e_t}^2 + 1)\eta - 539\mathcal{W}_{e_t}^2 - 50 \right)
\end{aligned}$$

$$\begin{aligned}
& +3i \mathcal{W}_{e_t} e_t \sin(\mathcal{E}) \left((985\eta^2 - 5715\eta + 1781) e_t^2 - 325\eta^2 + 4835\eta - 1561 \right) \\
& + \left. \frac{40\eta^2 - 205\eta + 52}{55(3\eta - 1)A(\mathcal{E})} + \frac{40\eta^2 - 205\eta + 52}{55 - 165\eta} \right\}, \tag{B.14}
\end{aligned}$$

$$I^{34} = 0, \tag{B.15}$$

$$\begin{aligned}
I^{44} = & \frac{8}{3} \sqrt{\frac{2\pi}{7}} |E|^2 m(1 - 3\eta) \eta e^{-4i\phi} \left\{ 1 - \frac{1}{A(\mathcal{E})} - \frac{8o^2(-o^2 - ioe_t \sin(\mathcal{E}))}{A(\mathcal{E})^4} \right. \\
& + \frac{2ioe_t \sin(\mathcal{E}) + 9e_t^2 - 9}{2A(\mathcal{E})^3} + \frac{4i \mathcal{W}_{e_t} e_t \sin(\mathcal{E}) - 8e_t^2 + \frac{43}{6}}{A(\mathcal{E})^2} \\
& + \epsilon^2 |E| \left[\sin(\mathcal{E}) \left(\frac{i \mathcal{W}_{e_t}^3 (923\eta^2 - 13683\eta + 4523) e_t}{44A(\mathcal{E})^5} \right. \right. \\
& \left. \left. - \frac{i \mathcal{W}_{e_t} e_t (48(40\eta^2 - 205\eta + 52) e_t^2 + 56335\eta^2 - 105175\eta + 29219)}{330A(\mathcal{E})^4} \right. \right. \\
& \left. \left. - \frac{ie_t ((985\eta^2 - 5715\eta + 1781) e_t^2 - 325\eta^2 + 4835\eta - 1561)}{110 \mathcal{W}_{e_t} A(\mathcal{E})^3} \right. \right. \\
& \left. \left. - \frac{4ie_t ((40\eta^2 - 205\eta + 52) e_t^2 + 290\eta^2 - 235\eta + 58)}{55 \mathcal{W}_{e_t} A(\mathcal{E})^2} \right) \right. \\
& + \frac{3 \mathcal{W}_{e_t}^4 (716\eta^2 - 4456\eta + 1461)}{44A(\mathcal{E})^5} \\
& \left. - \frac{\mathcal{W}_{e_t}^2 (96(40\eta^2 - 205\eta + 52) e_t^2 + 147740\eta^2 - 496880\eta + 149503)}{660A(\mathcal{E})^4} \right. \\
& + (330 \mathcal{W}_{e_t} A(\mathcal{E})^3 (e_t - 1))^{-1} \left\{ -3e_t^3 \left(90\eta^2 \left(\frac{22(e_t + 1)}{\mathcal{W}_{e_t}} + 5 \mathcal{W}_{e_t} \right) \right. \right. \\
& \left. \left. - 5\eta \left(\frac{6072(e_t + 1)}{\mathcal{W}_{e_t}} - 2363 \mathcal{W}_{e_t} \right) + \frac{9900(e_t + 1)}{\mathcal{W}_{e_t}} - 3826 \mathcal{W}_{e_t} \right) \right. \\
& + 1980(3\eta^2 - 46\eta + 15) \sqrt{\frac{1 + e_t}{1 - e_t}} e_t^4 + 3 \mathcal{W}_{e_t} (2430\eta^2 - 18545\eta + 6074) e_t^2 \\
& \left. + \mathcal{W}_{e_t} (28410\eta^2 - 83795\eta + 25262) e_t + \mathcal{W}_{e_t} (-28410\eta^2 + 83795\eta - 25262) \right\}_3 \\
& + (330 \mathcal{W}_{e_t} A(\mathcal{E})^2 (e_t - 1))^{-1} \left\{ -6e_t^3 \left(5\eta^2 \left(\frac{891(e_t + 1)}{\mathcal{W}_{e_t}} - 827 \mathcal{W}_{e_t} \right) \right. \right. \\
& + \eta \left(1330 \mathcal{W}_{e_t} - \frac{2970(e_t + 1)}{\mathcal{W}_{e_t}} \right) + \frac{495(e_t + 1)}{\mathcal{W}_{e_t}} - 79 \mathcal{W}_{e_t} \left. \right) \\
& - 6e_t^2 \left(5\eta^2 \left(\frac{99(e_t + 1)}{\mathcal{W}_{e_t}} - 64 \mathcal{W}_{e_t} \right) + 10\eta \left(\frac{231(e_t + 1)}{\mathcal{W}_{e_t}} + 164 \mathcal{W}_{e_t} \right) \right. \\
& \left. - \frac{825(e_t + 1)}{\mathcal{W}_{e_t}} - 416 \mathcal{W}_{e_t} \right) + 5e_t \left(\frac{(e_t + 1)}{\mathcal{W}_{e_t}} (594\eta^2 + 2772\eta - 990) \right. \\
& \left. + \mathcal{W}_{e_t} (-6628\eta^2 + 5407\eta - 1165) \right) \left. \right\}
\end{aligned}$$

$$\begin{aligned}
& \left. + 2970(1 - 3\eta)^2 \sqrt{\frac{1 + e_t}{1 - e_t}} e_t^4 + 5 \mathcal{W}_{e_t} (6034\eta^2 - 8179\eta + 2155) \right\}_2 \\
& \left. + \frac{-40\eta^2 + 205\eta - 52}{55A(\mathcal{E})} + \frac{1}{55} (40\eta^2 - 205\eta + 52) \right\}_0, \tag{B.16}
\end{aligned}$$

$$S^{04} = 0, \tag{B.17}$$

$$\begin{aligned}
S^{14} &= \frac{8}{7} \sqrt{2\pi} E^{5/2} (1 - 2\eta) \delta_m \eta e^{-i\phi} \times \\
& \left\{ -\frac{\mathcal{W}_{e_t}^3}{3A(\mathcal{E})^4} + \frac{2\mathcal{W}_{e_t}}{5A(\mathcal{E})^2} - \frac{2ie_t(e_t^2 - 1)\sin(\mathcal{E})}{3A(\mathcal{E})^4} \right\}, \tag{B.18}
\end{aligned}$$

$$S^{24} = 0, \tag{B.19}$$

$$\begin{aligned}
S^{34} &= 40 \sqrt{\frac{2\pi}{7}} E^{5/2} (1 - 2\eta) \delta_m \eta e^{-3i\phi} \times \\
& \left\{ -\frac{\mathcal{W}_{e_t}^3}{3A(\mathcal{E})^4} - \frac{2\mathcal{W}_{e_t}}{75A(\mathcal{E})^2} + \frac{2ie_t(e_t^2 - 1)\sin(\mathcal{E})}{15A(\mathcal{E})^4} \right\}, \tag{B.20}
\end{aligned}$$

$$S^{44} = 0, \tag{B.21}$$

$$I^{05} = 0, \tag{B.22}$$

$$\begin{aligned}
I^{15} &= \frac{10 \sqrt{\frac{7\pi}{11}} E^{5/2} (1 - 2\eta) \delta_m \eta e^{-i\phi}}{9} \left\{ \frac{i\mathcal{W}_{e_t}^3}{A(\mathcal{E})^4} + \frac{8i\mathcal{W}_{e_t}}{7A(\mathcal{E})} - \frac{4i\mathcal{W}_{e_t}}{5A(\mathcal{E})^2} \right. \\
& \left. - \frac{4i\mathcal{W}_{e_t}}{3A(\mathcal{E})^3} + \sin(\mathcal{E}) \left(\frac{2e_t}{3A(\mathcal{E})^3} - \frac{8e_t}{7A(\mathcal{E})} \right) \right\}, \tag{B.23}
\end{aligned}$$

$$I^{25} = 0, \tag{B.24}$$

$$\begin{aligned}
I^{35} &= \sqrt{\frac{2\pi}{33}} E^{5/2} (2\eta - 1) \delta_m \eta e^{-3i\phi} \times \\
& \left\{ + \frac{\frac{8}{3}e_t \sin(\mathcal{E}) - 8i\mathcal{W}_{e_t}}{A(\mathcal{E})} + \frac{28i\mathcal{W}_{e_t}}{5A(\mathcal{E})^2} - \frac{(e_t^2 - 1)(16e_t \sin(\mathcal{E}) - 5i\mathcal{W}_{e_t})}{3A(\mathcal{E})^4} \right. \\
& \left. + \frac{2 \left(e_t (41 - 48e_t^2) \sin(\mathcal{E}) - 6i(e_t - 1) \sqrt{\frac{e_t+1}{1-e_t}} (8e_t^2 - 1) \right)}{9A(\mathcal{E})^3} \right\}, \tag{B.25}
\end{aligned}$$

$$I^{45} = 0, \tag{B.26}$$

$$\begin{aligned}
I^{55} &= 8 \sqrt{\frac{2\pi}{165}} E^{5/2} (2\eta - 1) \delta_m \eta e^{-5i\phi} \times \\
& \left\{ + \frac{16i\mathcal{W}_{e_t}^4 (\mathcal{W}_{e_t} + ie_t \sin(\mathcal{E}))}{A(\mathcal{E})^5} + \frac{3(e_t^2 - 1)(16e_t \sin(\mathcal{E}) + 15i\mathcal{W}_{e_t})}{8A(\mathcal{E})^4} \right. \\
& \left. + \frac{i(-10\mathcal{W}_{e_t}(24e_t^2 - 17) + ie_t(137 - 144e_t^2)\sin(\mathcal{E}))}{12A(\mathcal{E})^3} \right\}
\end{aligned}$$

$$\left. -\frac{7i\mathcal{W}_{e_t}}{2A(\mathcal{E})^2} + \frac{-e_t \sin(\mathcal{E}) + 5i\mathcal{W}_{e_t}}{A(\mathcal{E})} \right\}, \quad (\text{B.27})$$

$$\begin{aligned} S^{05(5)} &= \frac{16}{3} \sqrt{\frac{5\pi}{231}} E^3 \eta(5(\eta-1)\eta+1) m e_t \times \\ &\left\{ \frac{\mathcal{W}_{e_t}^3 \sin(\mathcal{E})}{A(\mathcal{E})^5} - \frac{2\mathcal{W}_{e_t} \sin(\mathcal{E})}{5A(\mathcal{E})^3} - \frac{2\mathcal{W}_{e_t} \sin(\mathcal{E})}{3A(\mathcal{E})^4} \right\}, \end{aligned} \quad (\text{B.28})$$

$$S^{15(5)} = 0, \quad (\text{B.29})$$

$$\begin{aligned} S^{25(5)} &= \frac{16}{9} \sqrt{\frac{2\pi}{11}} E^3 (5(\eta-1)\eta+1) \eta m e^{-2i\phi} \times \\ &\left\{ \frac{i\mathcal{W}_{e_t}^4}{A(\mathcal{E})^5} - \frac{i\mathcal{W}_{e_t}^2}{3A(\mathcal{E})^4} - \frac{6i\mathcal{W}_{e_t}^2}{5A(\mathcal{E})^3} + \sin(\mathcal{E}) e_t \mathcal{W}_{e_t} \left(\frac{\mathcal{W}_{e_t}^2}{A(\mathcal{E})^5} + \frac{2}{5A(\mathcal{E})^3} + \frac{2}{3A(\mathcal{E})^4} \right) \right\} \end{aligned} \quad (\text{B.30})$$

$$S^{35(5)} = 0, \quad (\text{B.31})$$

$$\begin{aligned} S^{45(5)} &= \frac{16}{15} \sqrt{\frac{2\pi}{33}} E^3 \eta(5(\eta-1)\eta+1) m e^{-4i\phi} \times \\ &\left\{ \frac{35i\mathcal{W}_{e_t}^4}{A(\mathcal{E})^5} - \frac{6i(e_t^2-1)}{A(\mathcal{E})^3} - \frac{5i(e_t^2-1)}{3A(\mathcal{E})^4} \right. \\ &\left. + \sin(\mathcal{E}) e_t \mathcal{W}_{e_t} \left(-\frac{35\mathcal{W}_{e_t}^2}{2A(\mathcal{E})^5} - \frac{1}{A(\mathcal{E})^3} - \frac{5}{3A(\mathcal{E})^4} \right) \right\}, \end{aligned} \quad (\text{B.32})$$

$$S^{55(5)} = 0, \quad (\text{B.33})$$

$$\begin{aligned} I^{06(6)} &= \frac{64}{33} \sqrt{\frac{10\pi}{273}} E^3 \eta(5(\eta-1)\eta+1) m \times \\ &\left\{ 1 - \frac{1}{A(\mathcal{E})} - \frac{77}{60A(\mathcal{E})^2} + \frac{7\mathcal{W}_{e_t}^2}{10A(\mathcal{E})^3} + \frac{35\mathcal{W}_{e_t}^2}{24A(\mathcal{E})^4} - \frac{7\mathcal{W}_{e_t}^4}{8A(\mathcal{E})^5} \right\}, \end{aligned} \quad (\text{B.34})$$

$$I^{16(6)} = 0, \quad (\text{B.35})$$

$$\begin{aligned} I^{26(6)} &= -\frac{32}{33} \sqrt{\frac{2\pi}{13}} |E|^3 \mu (5\eta^2 - 5\eta + 1) e^{-2i\phi} \times \\ &\left\{ 1 - \frac{1}{A(\mathcal{E})} \frac{43}{60} - \frac{2e_t^2}{A(\mathcal{E})^2} + \frac{23\mathcal{W}_{e_t}^4}{24A(\mathcal{E})^5} - \frac{\mathcal{W}_{e_t}^2}{10A(\mathcal{E})^3} - \frac{107\mathcal{W}_{e_t}^2}{72A(\mathcal{E})^4} \right. \\ &\left. + i e_t \mathcal{W}_{e_t} \sin(\mathcal{E}) \left(-\frac{\mathcal{W}_{e_t}^2}{12A(\mathcal{E})^5} + \frac{2}{A(\mathcal{E})^2} + \frac{2}{3A(\mathcal{E})^3} - \frac{11}{9A(\mathcal{E})^4} \right) \right\}, \end{aligned} \quad (\text{B.36})$$

$$I^{36(6)} = 0, \quad (\text{B.37})$$

$$\begin{aligned} I^{46(6)} &= \frac{64}{11} \sqrt{\frac{\pi}{195}} |E|^3 \mu (5\eta^2 - 5\eta + 1) e^{-2i\phi} \times \\ &\left\{ 1 - \frac{1}{A(\mathcal{E})} + \frac{403}{60} - \frac{8e_t^2}{A(\mathcal{E})^2} + \frac{(e_t-1)(e_t+1)(576e_t^2+167)}{72A(\mathcal{E})^4} \right\} \end{aligned}$$

$$\begin{aligned}
& + \frac{91 \mathcal{W}_{e_t}^4}{24A(\mathcal{E})^5} - \frac{5 \mathcal{W}_{e_t}^2}{2A(\mathcal{E})^3} + \sin(\mathcal{E}) \left(\frac{4i \mathcal{W}_{e_t} e_t}{A(\mathcal{E})^2} + \frac{4i \mathcal{W}_{e_t} e_t}{3A(\mathcal{E})^3} \right. \\
& \left. - \frac{2i \mathcal{W}_{e_t} e_t (6e_t - 5)(6e_t + 5)}{9A(\mathcal{E})^4} + \frac{35i \mathcal{W}_{e_t}^3 e_t}{6A(\mathcal{E})^5} \right) \Bigg\}, \tag{B.38}
\end{aligned}$$

$$I^{56} = 0, \tag{B.39}$$

$$\begin{aligned}
I^{66} &= -\frac{32}{3} \sqrt{\frac{2\pi}{715}} |E|^3 \mu (5\eta^2 - 5\eta + 1) e^{-2i\phi} \times \\
& \left\{ 1 - \frac{1}{A(\mathcal{E})} + \frac{\frac{1003}{60} - 18e_t^2}{A(\mathcal{E})^2} + \frac{32 \mathcal{W}_{e_t}^6}{A(\mathcal{E})^6} - \frac{3 \mathcal{W}_{e_t}^4}{8A(\mathcal{E})^5} - \frac{13 \mathcal{W}_{e_t}^2}{2A(\mathcal{E})^3} - \frac{\mathcal{W}_{e_t}^2 (1152e_t^2 - 551)}{24A(\mathcal{E})^4} \right. \\
& \left. + i e_t \sin(\mathcal{E}) \left(\frac{32 \mathcal{W}_{e_t}^5}{A(\mathcal{E})^6} + \frac{95 \mathcal{W}_{e_t}^3}{4A(\mathcal{E})^5} - \frac{\mathcal{W}_{e_t} (96e_t^2 - 85)}{3A(\mathcal{E})^4} + \frac{2 \mathcal{W}_{e_t}}{A(\mathcal{E})^3} + \frac{6 \mathcal{W}_{e_t}}{A(\mathcal{E})^2} \right) \right\}. \tag{B.40}
\end{aligned}$$

C Results: The spherical tensor components in terms of Fourier coefficients

$$\begin{aligned}
I^{22} &= -8 \sqrt{\frac{2\pi}{5}} |E| \mu e_t^{-2} e^{-2i\mathcal{K}\mathcal{M}} e^{-i2\phi_0} \times \\
& \left\{ \sum_{k=0}^7 {}^0\alpha_{[k2]} \mathcal{F}_{[k2]}(\mathcal{E}) + \sum_{k=1}^7 {}^0\tilde{\alpha}_{[k2]} \mathcal{F}_{S[k2]}(\mathcal{E}) \right\}, \tag{C.1}
\end{aligned}$$

$$\begin{aligned}
{}^0\alpha_{[02]} &\equiv [2 - e_t^2] + \epsilon^2 \frac{1}{14} |E| [(9\eta - 3)e_t^2 + 94\eta - 442] + \epsilon^4 \frac{-|E|^2}{42 \mathcal{W}_{e_t}^2} [(\eta(11\eta + 16) - 5)e_t^4 \\
& + 3(\eta(197\eta - 1216) + 3717)e_t^2 + 32(40 - 63 \mathcal{W}_{e_t})\eta + 5040 \mathcal{W}_{e_t} - 602\eta^2 + 2798],
\end{aligned}$$

$$\begin{aligned}
{}^0\alpha_{[12]} &\equiv [e_t^2 - 2] + \epsilon^2 \frac{1}{14} |E| ((3 - 9\eta)e_t^2 - 94\eta + 778) + \epsilon^4 \frac{-|E|^2}{42 \mathcal{W}_{e_t}^2} \left[(5 - \eta(11\eta + 16))e_t^4 \right. \\
& + 2\eta(1512 \mathcal{W}_{e_t} + 301\eta - 64) - 3e_t^2(\eta(168 \mathcal{W}_{e_t} + 197\eta - 3184) - 420 \mathcal{W}_{e_t} \\
& \left. + 10029) - 7560 \mathcal{W}_{e_t} + 2530 \right], \tag{C.2}
\end{aligned}$$

$$\begin{aligned}
{}^0\alpha_{[22]} &\equiv -2(\mathcal{W}_{e_t}^2) + \epsilon^2 \frac{1}{21} |E| (-9\eta(e_t^2 + 17) - 151e_t^2 + 163) - \epsilon^4 \frac{|E|^2}{126 \mathcal{W}_{e_t}^2} \left[e_t^2(11340 \mathcal{W}_{e_t} \right. \\
& + 45322 - \eta(4536 \mathcal{W}_{e_t} + 1123\eta + 15623)) + (3(951 - 119\eta)\eta + 4188)e_t^4 \\
& \left. + 2\eta(4536 \mathcal{W}_{e_t} + 740\eta - 2435) - 22680 \mathcal{W}_{e_t} - 57574 \right], \tag{C.3}
\end{aligned}$$

$$\begin{aligned}
{}^0\alpha_{[32]} &\equiv 2[e_t^2 - 1]^2 + \epsilon^2 |E| \left[\frac{1}{21} \mathcal{W}_{e_t}^2 (11(3\eta + 13)e_t^2 + 129\eta - 1163) \right] \\
& + \epsilon^4 \frac{|E|^2}{189} \left[\eta(31752 \mathcal{W}_{e_t} + 3433\eta + 13700) - 79380 \mathcal{W}_{e_t} - 60259 \right]
\end{aligned}$$

$$\begin{aligned}
& +e_t^2(\eta(-4536\mathcal{W}_{e_t} - 3481\eta + 10081) + 11340\mathcal{W}_{e_t} - 93512) \\
& +3((830 - 47\eta)\eta + 2117)e_t^4], \tag{C.4}
\end{aligned}$$

$$\begin{aligned}
{}^0\alpha_{[42]} & \equiv \epsilon^2|E| \left[\frac{4}{21}|E| \mathcal{W}_{e_t}^4(15\eta + 121) \right] + \epsilon^4 \frac{-|E|^2}{378} \mathcal{W}_{e_t}^2 \left[9(\eta(11\eta + 3339) - 9453)e_t^2 \right. \\
& \left. + 4\eta(6804\mathcal{W}_{e_t} + 1211\eta + 7423) - 68040\mathcal{W}_{e_t} + 77866 \right], \tag{C.5}
\end{aligned}$$

$$\begin{aligned}
{}^0\alpha_{[52]} & \equiv \epsilon^2|E| \left[\frac{-4}{7}(3\eta - 1)\mathcal{W}_{e_t}^6 \right] + \epsilon^4 \frac{|E|^2}{378} \mathcal{W}_{e_t}^4 \left[3(4\eta(349\eta + 917) - 1117)e_t^2 \right. \\
& \left. - 16\eta(308\eta + 1891) + 95270 \right],
\end{aligned}$$

$${}^0\alpha_{[62]} \equiv \epsilon^4 \frac{|E|^2}{189} \mathcal{W}_{e_t}^6 (\eta(6853\eta + 20333) - 12793), \tag{C.6}$$

$${}^0\alpha_{[72]} \equiv -\epsilon^4 \frac{5}{63}|E|^2 \mathcal{W}_{e_t}^8 (4\eta(53\eta + 103) - 131), \tag{C.7}$$

$$\begin{aligned}
{}^0\tilde{\alpha}_{[12]} & \equiv i2\mathcal{W}_{e_t}e_t + i \frac{|E|\epsilon^2 e_t((109 - 19\eta)e_t^2 + 47\eta - 221)}{7\mathcal{W}_{e_t}} \\
& + i\epsilon^4 \frac{|E|^2}{21\mathcal{W}_{e_t}^6} \left[-252\mathcal{W}_{e_t}^4(2\eta - 5)e_t(e_t^2 - 2) + \mathcal{W}_{e_t}^3 e_t \left((-446\eta^2 + 2144\eta - 4876)e_t^2 \right. \right. \\
& \left. \left. + (61(\eta - 4)\eta + 1445)e_t^4 + \eta(301\eta - 640) - 1399 \right) \right], \tag{C.8}
\end{aligned}$$

$$\begin{aligned}
{}^0\tilde{\alpha}_{[22]} & \equiv -i\epsilon^2 \frac{12|E|e_t(e_t^2 - 2)}{\mathcal{W}_{e_t}} \\
& + i\epsilon^4|E|^2 \left[\frac{6e_t((54\eta - 151)e_t^4 + (600 - 148\eta)e_t^2 - 4(8\eta + 37))}{7\mathcal{W}_{e_t}^3} \right. \\
& \left. + 12(5 - 2\eta)e_t \right], \tag{C.9}
\end{aligned}$$

$$\begin{aligned}
{}^0\tilde{\alpha}_{[32]} & \equiv -i2\mathcal{W}_{e_t}^3 e_t - i\epsilon^2 \frac{1}{21}|E|\mathcal{W}_{e_t}e_t((15\eta + 149)e_t^2 + 153\eta - 667) \\
& + i\epsilon^4 \frac{-|E|^2}{63\mathcal{W}_{e_t}} e_t \left[3024\mathcal{W}_{e_t}(2\eta - 5) + 740\eta^2 - 707\eta - 20795 + \left(-756\mathcal{W}_{e_t}(2\eta - 5) \right. \right. \\
& \left. \left. - 643\eta^2 + 1651\eta - 14042 \right) e_t^2 + 5(\eta(31\eta - 112) + 443)e_t^4 \right], \tag{C.10}
\end{aligned}$$

$$\begin{aligned}
{}^0\tilde{\alpha}_{[42]} & \equiv -i\frac{8}{21}|E|\mathcal{W}_{e_t}^3 \epsilon^2(3\eta + 62)e_t + i\epsilon^4 \frac{|E|^2}{189} \mathcal{W}_{e_t}e_t \left[6804\mathcal{W}_{e_t}(2\eta - 5) + 1213\eta^2 \right. \\
& \left. + 15821\eta + 2126 + ((8627 - 701\eta)\eta - 37414)e_t^2 \right], \tag{C.11}
\end{aligned}$$

$$\begin{aligned}
{}^0\tilde{\alpha}_{[52]} & \equiv i\frac{4}{7}|E|\mathcal{W}_{e_t}^5 \epsilon^2(3\eta - 1)e_t + i\epsilon^4 \frac{|E|^2}{63} \mathcal{W}_{e_t}e_t(e_t^2 - 1) \left[3(4\eta(14\eta + 67) - 77)e_t^2 \right. \\
& \left. + 13\eta(31\eta - 25) + 12269 \right], \tag{C.12}
\end{aligned}$$

$${}^0\tilde{\alpha}_{[62]} \equiv -i\frac{|E|^2}{189} \mathcal{W}_{e_t}^5 (\eta(3673\eta + 14153) - 10828)e_t, \tag{C.13}$$

$${}^0\tilde{\alpha}_{[72]} \equiv i\epsilon^4 \frac{5}{63}|E|^2 \mathcal{W}_{e_t}^7 (4\eta(53\eta + 103) - 131)e_t, \tag{C.14}$$

$$I^{21} = 0, \tag{C.15}$$

$$I^{20(2)} = 16\sqrt{\frac{\pi}{15}}|E|\mu\left\{\sum_{k=0}^5 {}^0\alpha_{[k0]}\mathcal{F}_{[k0]}(\mathcal{E})\right\}, \quad (\text{C.16})$$

$${}^0\alpha_{[00]} \equiv 1 - \frac{3}{14}|E|\epsilon^2(3\eta - 1) + \epsilon^4\frac{1}{42}|E|^2(-11\eta^2 - 16\eta + 5), \quad (\text{C.17})$$

$${}^0\alpha_{[10]} \equiv -1 + \frac{3}{14}|E|\epsilon^2(3\eta - 1) + \epsilon^4|E|^2\left[\frac{30 - 12\eta}{\mathcal{W}_{e_t}} + \frac{1}{42}(\eta(11\eta + 16) - 5)\right], \quad (\text{C.18})$$

$${}^0\alpha_{[20]} \equiv \frac{2}{7}|E|\epsilon^2(\eta - 26) + \frac{|E|^2}{126\mathcal{W}_{e_t}^2}\left[\left((1877 - 323\eta)\eta - 3682\right)e_t^2 + 756\mathcal{W}_{e_t}(2\eta - 5) + 323\eta^2 + 1651\eta + 1666\right], \quad (\text{C.19})$$

$${}^0\alpha_{[30]} \equiv \frac{2}{7}|E|\epsilon^2(26 - \eta)\mathcal{W}_{e_t}^2 + \epsilon^4\frac{1}{21}|E|^2\left[(686 - 2\eta(31\eta + 197))e_t^2 + \eta(41\eta + 625) - 1610\right], \quad (\text{C.20})$$

$${}^0\alpha_{[40]} \equiv -\epsilon^4\frac{1}{126}|E|^2\mathcal{W}_{e_t}^2(\eta(1517\eta + 7549) - 8645), \quad (\text{C.21})$$

$${}^0\alpha_{[50]} \equiv +\epsilon^4\frac{1}{42}|E|^2\mathcal{W}_{e_t}^4(4\eta(79\eta + 179) - 217), \quad (\text{C.22})$$

$$S^{22(2)} = 0, \quad (\text{C.23})$$

$$S^{21(2)} = \frac{32}{3}\sqrt{\frac{\pi}{5}}\sqrt{1 - e_t^2\eta}|E|^{3/2}(m_1 - m_2)e^{-i\mathcal{K}\mathcal{M}}e^{-i\phi_0} \times \left\{\sum_{k=2}^5 {}^1\beta_{[k1]}\mathcal{F}_{[k1]}(\mathcal{E}) + \sum_{k=3}^5 {}^1\tilde{\beta}_{[k1]}\mathcal{F}_{S[k1]}(\mathcal{E})\right\}, \quad (\text{C.24})$$

$${}^1\beta_{[21]} \equiv -\frac{|E|\epsilon^2((899 - 3695\eta)e_t^2 + 3751\eta - 955)}{28\mathcal{W}_{e_t}^2} - 1, \quad (\text{C.25})$$

$${}^1\beta_{[31]} \equiv 1 - e_t^2 + |E|\epsilon^2\left(\frac{1}{28}((675 - 3639\eta)e_t^2 + 3695\eta - 1067) + \frac{1}{7}\mathcal{W}_{e_t}^2(2077 - 2054\eta)\right), \quad (\text{C.26})$$

$${}^1\beta_{[41]} \equiv \frac{1}{14}|E|\mathcal{W}_{e_t}^2\epsilon^2(10340\eta - 10497), \quad (\text{C.27})$$

$${}^1\beta_{[51]} \equiv \frac{3}{14}|E|\mathcal{W}_{e_t}^4\epsilon^2(2133 - 2068\eta), \quad (\text{C.28})$$

$${}^1\tilde{\beta}_{[31]} \equiv -i\mathcal{W}_{e_t}e_t + \frac{i|E|\epsilon^2e_t((3639\eta - 675)e_t^2 - 3751\eta + 955)}{28\mathcal{W}_{e_t}}, \quad (\text{C.29})$$

$${}^1\tilde{\beta}_{[41]} \equiv -\frac{1}{7}i|E|\mathcal{W}_{e_t}\epsilon^2(2068\eta - 2049)e_t, \quad (\text{C.30})$$

$${}^1\tilde{\beta}_{[51]} \equiv \frac{3}{14}i|E|\mathcal{W}_{e_t}^3\epsilon^2(2068\eta - 2133)e_t, \quad (\text{C.31})$$

$$S^{20(2)} = 0, \quad (\text{C.32})$$

$$I^{33(3)} = -4i\sqrt{\frac{2\pi}{21}}\eta\delta_m\sqrt{|E|^3}\mathcal{W}_{e_t}e_t^{-3}e^{-i3\mathcal{K}\mathcal{M}}e^{-i3\phi_0} \times \left\{\sum_{k=0}^7 {}^1\alpha_{[k3]}\mathcal{F}_{[k3]}(\mathcal{E}) + \sum_{k=1}^7 {}^1\tilde{\alpha}_{[k3]}\mathcal{F}_{S[k3]}(\mathcal{E})\right\}, \quad (\text{C.33})$$

$${}^1\alpha_{[03]} \equiv -2e_t^2 + 8 - \frac{|E|\epsilon^2((5\eta - 7)e_t^4 + (191\eta - 829)e_t^2 - 268\eta + 1124)}{6\mathcal{W}_{e_t}^2}, \quad (\text{C.34})$$

$${}^1\alpha_{[13]} \equiv 2(e_t^2 - 4) - \frac{|E|\epsilon^2((7 - 5\eta)e_t^4 + (1477 - 191\eta)e_t^2 + 4(67\eta - 497))}{6\mathcal{W}_{e_t}^2}, \quad (\text{C.35})$$

$${}^1\alpha_{[23]} \equiv 7e_t^2 - 12 - \frac{|E|\epsilon^2((23\eta - 901)e_t^4 + (6305 - 2203\eta)e_t^2 + 2540\eta - 5764)}{36\mathcal{W}_{e_t}^2}, \quad (\text{C.36})$$

$${}^1\alpha_{[33]} \equiv -e_t^2(e_t^2 + 3) + 4 + \frac{1}{36}|E|\epsilon^2(9(23\eta - 13)e_t^4 + (5353 - 395\eta)e_t^2 + 548\eta - 14668), \quad (\text{C.37})$$

$${}^1\alpha_{[43]} \equiv 20\mathcal{W}_{e_t}^4 + \frac{1}{9}|E|\mathcal{W}_{e_t}^2\epsilon^2((211\eta + 1162)e_t^2 + 3(383\eta - 745)), \quad (\text{C.38})$$

$${}^1\alpha_{[53]} \equiv -12\mathcal{W}_{e_t}^6 - \frac{1}{9}|E|\mathcal{W}_{e_t}^4\epsilon^2(9(29\eta + 38)e_t^2 + 331\eta - 4973), \quad (\text{C.39})$$

$${}^1\alpha_{[63]} = -\frac{4}{9}|E|\mathcal{W}_{e_t}^6\epsilon^2(140\eta + 467), \quad (\text{C.40})$$

$${}^1\alpha_{[73]} = \frac{20}{3}|E|\mathcal{W}_{e_t}^8\epsilon^2(4\eta + 1), \quad (\text{C.41})$$

$${}^1\tilde{\alpha}_{[13]} \equiv -\frac{2ie_t(3e_t^2 - 4)}{\mathcal{W}_{e_t}} - \frac{i|E|\epsilon^2e_t(3(19\eta - 89)e_t^2 - 268\eta + 1124)}{6\mathcal{W}_{e_t}}, \quad (\text{C.42})$$

$${}^1\tilde{\alpha}_{[23]} \equiv |E|\epsilon^2 \left[\frac{36i(\eta - 4)e_t(3e_t^4 - 7e_t^2 + 4)}{\mathcal{W}_{e_t}^3} + \frac{36ie_t((3\eta - 13)e_t^2 - 4(\eta - 5))}{\mathcal{W}_{e_t}} \right], \quad (\text{C.43})$$

$${}^1\tilde{\alpha}_{[33]} \equiv 24i\mathcal{W}_{e_t}^3e_t - \frac{ie_t(29e_t^4 - 65e_t^2 + 36)}{\mathcal{W}_{e_t}} + |E|\epsilon^2 \left[2i\mathcal{W}_{e_t}^3(7 - 5\eta)e_t - \frac{30i(\eta - 4)e_t(3e_t^4 - 7e_t^2 + 4)}{\mathcal{W}_{e_t}^3} + \frac{ie_t(9e_t^2((35\eta + 23)e_t^2 - 247\eta + 653) + 2140\eta - 6836)}{36\mathcal{W}_{e_t}} \right], \quad (\text{C.44})$$

$${}^1\tilde{\alpha}_{[43]} \equiv +16i\mathcal{W}_{e_t}^5e_t - 8i\mathcal{W}_{e_t}e_t(2e_t^4 - 5e_t^2 + 3) + |E|\epsilon^2 \left[\frac{2}{3}i\mathcal{W}_{e_t}^5(5\eta - 7)e_t + 2i\mathcal{W}_{e_t}^3e_t((5\eta - 7)e_t^2 + 139\eta - 497) + \frac{2}{3}i\mathcal{W}_{e_t}e_t(2(5\eta - 7)e_t^4 + 37(11\eta - 40)e_t^2 - 505\eta + 1343) \right], \quad (\text{C.45})$$

$${}^1\tilde{\alpha}_{[53]} \equiv 12i\mathcal{W}_{e_t}^5e_t + |E|\epsilon^2 \left(48i\mathcal{W}_{e_t}^5(\eta - 4)e_t + \frac{1}{3}i\mathcal{W}_{e_t}^3e_t((191\eta - 472)e_t^2 + 73\eta - 479) \right), \quad (\text{C.46})$$

$${}^1\tilde{\alpha}_{[63]} = \frac{16}{9}i|E|\mathcal{W}_{e_t}^5\epsilon^2(20\eta + 113)e_t, \quad (\text{C.47})$$

$${}^1\tilde{\alpha}_{[73]} = -\frac{20}{3}i|E|\mathcal{W}_{e_t}^7\epsilon^2(4\eta + 1)e_t, \quad (\text{C.48})$$

$$I^{32} \stackrel{(3)}{=} 0, \quad (\text{C.49})$$

$$I^{31} \stackrel{(3)}{=} e^{-i\mathcal{K}\mathcal{M}} e^{-i\phi_0} 8\sqrt{\frac{2\pi}{35}}|E|^{3/2}(m_1 - m_2)\eta\mathcal{W}_{e_t} \times$$

$$\left\{ \sum_{k=0}^5 {}^1\alpha_{[k1]} \mathcal{F}_{[k1]}(\mathcal{E}) + \sum_{k=1}^5 {}^1\tilde{\alpha}_{[k1]} \mathcal{F}_{S[k1]}(\mathcal{E}) \right\}, \quad (\text{C.50})$$

$${}^1\alpha_{[01]} \equiv i - \frac{1}{12} i |E| \epsilon^2 \left(-\frac{24(\eta-4)}{\mathcal{W}_{e_t}^2} + 5\eta - 7 \right), \quad (\text{C.51})$$

$${}^1\alpha_{[11]} \equiv -i - \frac{i |E| \epsilon^2 ((5\eta-7)e_t^2 + 19\eta - 161)}{12 \mathcal{W}_{e_t}^2}, \quad (\text{C.52})$$

$${}^1\alpha_{[21]} \equiv -\frac{5i}{6} - \frac{5i |E| \epsilon^2 ((5\eta-79)e_t^2 + 19\eta + 55)}{72 \mathcal{W}_{e_t}^2}, \quad (\text{C.53})$$

$${}^1\alpha_{[31]} \equiv \frac{5i \mathcal{W}_{e_t}^2}{6} - \frac{1}{24} i |E| \epsilon^2 ((13\eta+1)e_t^2 - 53\eta + 279), \quad (\text{C.54})$$

$${}^1\alpha_{[41]} \equiv -\frac{1}{18} i |E| \mathcal{W}_{e_t}^2 \epsilon^2 (40\eta - 317), \quad (\text{C.55})$$

$${}^1\alpha_{[51]} \equiv \frac{1}{6} i |E| \mathcal{W}_{e_t}^4 \epsilon^2 (8\eta - 49), \quad (\text{C.56})$$

$${}^1\tilde{\alpha}_{[11]} \equiv -\frac{e_t}{\mathcal{W}_{e_t}} + \frac{|E| \mathcal{W}_{e_t} \epsilon^2 (89 - 19\eta) e_t}{12 \mathcal{W}_{e_t}^2}, \quad (\text{C.57})$$

$${}^1\tilde{\alpha}_{[21]} \equiv -\frac{6|E| \mathcal{W}_{e_t} \epsilon^2 e_t}{1 - e_t^2}, \quad (\text{C.58})$$

$${}^1\tilde{\alpha}_{[31]} \equiv \frac{5 \mathcal{W}_{e_t} e_t}{6} + \frac{|E| \epsilon^2 e_t (3(79 - 5\eta) e_t^2 + 95\eta - 157)}{72 \mathcal{W}_{e_t}}, \quad (\text{C.59})$$

$${}^1\tilde{\alpha}_{[41]} \equiv \frac{1}{9} |E| \mathcal{W}_{e_t} \epsilon^2 (85 - 8\eta) e_t, \quad (\text{C.60})$$

$${}^1\tilde{\alpha}_{[51]} \equiv \frac{1}{6} |E| \epsilon^2 (8\eta - 49) e_t \mathcal{W}_{e_t}^3, \quad (\text{C.61})$$

$${}^3S^{32} = e^{-i2\mathcal{K}\mathcal{M}} e^{-i2\phi_0} \frac{8}{3} \sqrt{\frac{2\pi}{7}} |E|^2 \mathcal{W}_{e_t}^2 \mu (1 - 3\eta) e_t^{-2} \times \left\{ \sum_{k=2}^7 {}^2\beta_{[k2]} \mathcal{F}_{[k2]}(\mathcal{E}) + \sum_{k=3}^7 {}^2\tilde{\beta}_{[k2]} \mathcal{F}_{S[k2]}(\mathcal{E}) \right\}, \quad (\text{C.62})$$

$${}^2\beta_{[22]} \equiv \frac{2i |E| \epsilon^2 ((\eta(10\eta - 61) + 20)e_t^2 + (163 - 46\eta)\eta - 50)}{3 \mathcal{W}_{e_t}^2 (3\eta - 1)} - 2i, \quad (\text{C.63})$$

$${}^2\beta_{[32]} \equiv -2i \mathcal{W}_{e_t}^2 + \frac{2i |E| \epsilon^2}{45 \mathcal{W}_{e_t}^2 (3\eta - 1)} \left[3(5\eta(16\eta - 25) + 28)e_t^4 + 2(5\eta(73\eta - 13) + 4)e_t^2 - 5\eta(194\eta + 223) + 448 \right], \quad (\text{C.64})$$

$${}^2\beta_{[42]} \equiv 10i \mathcal{W}_{e_t}^2 + \frac{2i |E| \epsilon^2 (3(5\eta(71\eta + 370) - 658)e_t^2 + 5\eta(662\eta - 2477) + 3809)}{45(3\eta - 1)}, \quad (\text{C.65})$$

$${}^2\beta_{[52]} \equiv -6i \mathcal{W}_{e_t}^4 - \frac{2i |E| \mathcal{W}_{e_t}^2 \epsilon^2 (3(605\eta^2 + 910\eta - 344)e_t^2 - 5\eta(230\eta + 4147) + 6847)}{45(3\eta - 1)}, \quad (\text{C.66})$$

$${}^2\beta_{[62]} \equiv -\frac{2i |E| \mathcal{W}_{e_t}^4 \epsilon^2 (4\eta(245\eta + 604) - 845)}{27\eta - 9}, \quad (\text{C.67})$$

$${}^2\beta_{[72]} \equiv \frac{2i |E| \mathcal{W}_{e_t}^6 \epsilon^2 (20\eta(7\eta + 8) - 59)}{9\eta - 3}, \quad (\text{C.68})$$

$${}^2\tilde{\beta}_{[32]} \equiv -\frac{e_t(e_t^2 - 2)}{\mathcal{W}_{e_t}} - |E|\epsilon^2 \times \frac{e_t((\eta(8\eta - 17) + 4)e_t^4 + 3(\eta(22\eta - 95) + 30)e_t^2 - 92\eta^2 + 326\eta - 100)}{(9\eta - 3)\mathcal{W}_{e_t}^3}, \quad (\text{C.69})$$

$${}^2\tilde{\beta}_{[42]} \equiv 4\mathcal{W}_{e_t}e_t + \frac{2|E|\epsilon^2 e_t((10(38 - 5\eta)\eta - 157)e_t^2 + 1660\eta^2 - 1330\eta + 302)}{45(3\eta - 1)\mathcal{W}_{e_t}}, \quad (\text{C.70})$$

$${}^2\tilde{\beta}_{[52]} \equiv -6e_t\mathcal{W}_{e_t}^3 + \frac{|E|\epsilon^2(-e_t\mathcal{W}_{e_t})(3(170\eta(\eta + 2) - 131)e_t^2 + 110\eta(10\eta - 67) + 2338)}{15(3\eta - 1)}, \quad (\text{C.71})$$

$${}^2\tilde{\beta}_{[62]} \equiv -\frac{8|E|\epsilon^2(4\eta(35\eta + 121) - 167)e_t\mathcal{W}_{e_t}^3}{27\eta - 9}, \quad (\text{C.72})$$

$${}^2\tilde{\beta}_{[72]} \equiv \frac{2|E|\epsilon^2(20\eta(7\eta + 8) - 59)e_t\mathcal{W}_{e_t}^5}{9\eta - 3}, \quad (\text{C.73})$$

$$S^{(3)30} = -16\sqrt{\frac{\pi}{105}}|E|^2 e_t\sqrt{1 - e_t^2}\mu(1 - 3\eta)\sum_3^5 {}^2\tilde{\beta}_{[k0]}\mathcal{F}_{S[k0]}(\mathcal{E}), \quad (\text{C.74})$$

$${}^2\tilde{\beta}_{[30]} \equiv 1 - \frac{|E|\epsilon^2(((17 - 8\eta)\eta - 4)e_t^2 + (7 - 10\eta)\eta - 2)}{3\mathcal{W}_{e_t}^2(3\eta - 1)}, \quad (\text{C.75})$$

$${}^2\tilde{\beta}_{[40]} \equiv -\frac{2|E|\epsilon^2(2(\eta - 121)\eta + 79)}{9(3\eta - 1)}, \quad (\text{C.76})$$

$${}^2\tilde{\beta}_{[50]} \equiv -\frac{|E|\mathcal{W}_{e_t}^2\epsilon^2(2(\eta - 121)\eta + 79)}{3 - 9\eta}, \quad (\text{C.77})$$

$$I^{(4)44} = e^{-i4\mathcal{K}\mathcal{M}}e^{-i4\phi_0}\frac{8}{3}\sqrt{\frac{2\pi}{7}}|E|^2\mu(1 - 3\eta)e_t^{-4} \times \left\{ \sum_{k=0}^9 {}^2\alpha_{[k4]}\mathcal{F}_{[k4]}(\mathcal{E}) + \sum_{k=1}^9 {}^2\tilde{\alpha}_{[k4]}\mathcal{F}_{S[k4]}(\mathcal{E}) \right\}, \quad (\text{C.78})$$

$${}^2\alpha_{[04]} \equiv e_t^4 - 8e_t^2 + 8 + \frac{|E|\epsilon^2}{55(3\eta - 1)} \left[(5(41 - 8\eta)\eta - 52)e_t^4 - 8(5\eta(124\eta - 531) + 828)e_t^2 + 8(5\eta(256\eta - 1103) + 1708) \right], \quad (\text{C.79})$$

$${}^2\alpha_{[14]} \equiv -e_t^4 + 8e_t^2 - 8 + \frac{|E|\epsilon^2}{55(3\eta - 1)} \left[(5\eta(8\eta - 41) + 52)e_t^4 + 8(5\eta(124\eta - 927) + 1488)e_t^2 - 8(5\eta(256\eta - 1895) + 3028) \right], \quad (\text{C.80})$$

$${}^2\alpha_{[24]} \equiv \frac{1}{6}(-29e_t^4 + 112e_t^2 - 88) + \frac{|E|\epsilon^2}{330(3\eta - 1)} \left[-5(\eta(314\eta - 3185) + 1121)e_t^4 + 20(\eta(3829\eta - 12717) + 3903)e_t^2 - 8(5\eta(2921\eta - 8542) + 12788) \right], \quad (\text{C.81})$$

$${}^2\alpha_{[34]} \equiv \frac{1}{6}(3e_t^6 - 19e_t^4 + 24e_t^2 - 8) + \frac{|E|\epsilon^2}{330(3\eta - 1)} \left[(7610\eta^2 - 73225\eta + 22598)e_t^4 - 3(15\eta(58\eta - 69) + 218)e_t^6 + 40(\eta(411\eta + 10226) - 3407)e_t^2 - 8(5\eta(536\eta + 11245) - 18912) \right], \quad (\text{C.82})$$

$${}^2\alpha_{[44]} \equiv -\frac{16}{3}\mathcal{W}_{e_t}^4(e_t^2 - 6) + \frac{|E|\mathcal{W}_{e_t}^2\epsilon^2}{660(3\eta - 1)} \left[(20\eta(214\eta - 2359) + 15563)e_t^4 - 20(\eta(9705\eta - 64529) + 20707)e_t^2 + 8(5\eta(12953\eta - 54357) + 84244) \right], \quad (\text{C.83})$$

$${}^2\alpha_{[54]} \equiv -\frac{8}{3}\mathcal{W}_{e_t}^6(3e_t^2 - 4) - \frac{|E|\mathcal{W}_{e_t}^4\epsilon^2}{660(3\eta - 1)} \left[(60\eta(824\eta + 67) - 7371)e_t^4 + 220(\eta(107\eta - 363) + 145)e_t^2 - 8(5\eta(7561\eta + 45019) - 77612) \right], \quad (\text{C.84})$$

$${}^2\alpha_{[64]} \equiv -\frac{140\mathcal{W}_{e_t}^8}{3} - \frac{|E|\mathcal{W}_{e_t}^6\epsilon^2}{99\eta - 33} \left[(3\eta(4515\eta + 11017) - 12361)e_t^2 + 4(\eta(9940\eta - 26869) + 8016) \right], \quad (\text{C.85})$$

$${}^2\alpha_{[74]} \equiv 20\mathcal{W}_{e_t}^{10} + \frac{|E|\mathcal{W}_{e_t}^8\epsilon^2}{99\eta - 33} \left[3(15\eta(213\eta + 95) - 703)e_t^2 - 4(\eta(535\eta + 34698) - 11300) \right], \quad (\text{C.86})$$

$${}^2\alpha_{[84]} \equiv \frac{6|E|\mathcal{W}_{e_t}^{10}\epsilon^2(\eta(1225\eta + 2427) - 844)}{33\eta - 11}, \quad (\text{C.87})$$

$${}^2\alpha_{[94]} \equiv -\frac{70|E|\mathcal{W}_{e_t}^{12}\epsilon^2(\eta(35\eta + 9) - 4)}{33\eta - 11}, \quad (\text{C.88})$$

$${}^2\tilde{\alpha}_{[14]} \equiv \frac{4ie_t(e_t^4 - 3e_t^2 + 2)}{\mathcal{W}_{e_t}} + \frac{4i|E|\epsilon^2e_t}{55(3\eta - 1)\mathcal{W}_{e_t}} \left[(5\eta(58\eta - 245) + 388)e_t^4 + (5(2165 - 504\eta)\eta - 3364)e_t^2 + 10\eta(256\eta - 1103) + 3416 \right], \quad (\text{C.89})$$

$${}^2\tilde{\alpha}_{[24]} \equiv \frac{24i|E|\mathcal{W}_{e_t}\epsilon^2e_t(e_t^4 - 8e_t^2 + 8)}{1 - e_t^2}, \quad (\text{C.90})$$

$${}^2\tilde{\alpha}_{[34]} \equiv \frac{ie_t(9e_t^6 - 55e_t^4 + 90e_t^2 - 44)}{3\mathcal{W}_{e_t}} + \frac{i|E|\epsilon^2e_t}{330(3\eta - 1)\mathcal{W}_{e_t}} \left[15(\eta(133\eta - 815) + 273)e_t^6 - 55(\eta(1009\eta - 4619) + 1457)e_t^4 + 4(5\eta(8286\eta - 37381) + 58391)e_t^2 - 8(115\eta(127\eta - 578) + 20708) \right], \quad (\text{C.91})$$

$${}^2\tilde{\alpha}_{[44]} \equiv \frac{4}{3}ie_t\mathcal{W}_{e_t}^3(5e_t^2 - 12) + \frac{i|E|\mathcal{W}_{e_t}\epsilon^2e_t}{330(3\eta - 1)} \left[20(4266\eta^2 - 3452\eta + 829)e_t^2 + (1655(\eta - 1)\eta - 417)e_t^4 - 8(5\eta(3457\eta - 2049) + 1796) \right], \quad (\text{C.92})$$

$${}^2\tilde{\alpha}_{[54]} \equiv -2ie_t\mathcal{W}_{e_t}^5(e_t^2 - 8) + \frac{i|E|\epsilon^2e_t\mathcal{W}_{e_t}^3}{220(3\eta - 1)} \left[(5\eta(581\eta - 1637) + 2473)e_t^4 - 220(2\eta(73\eta - 520) + 333)e_t^2 + 88(915\eta^2 - 7615\eta + 2444) \right], \quad (\text{C.93})$$

$${}^2\tilde{\alpha}_{[64]} \equiv \frac{80}{3}ie_t\mathcal{W}_{e_t}^7 + \frac{4i|E|\epsilon^2e_t\mathcal{W}_{e_t}^5(11(6\eta(15\eta + 37) - 89)e_t^2 + 4(5\eta(340\eta - 131) + 76))}{99\eta - 33}, \quad (\text{C.94})$$

$${}^2\tilde{\alpha}_{[74]} \equiv \frac{-20ie_t\mathcal{W}_{e_t}^9 - |E|\epsilon^2 \times ie_t\mathcal{W}_{e_t}^7(3(10\eta(197\eta + 111) - 563)e_t^2 + 4(\eta(3140\eta - 24249) + 7712))}{99\eta - 33}, \quad (\text{C.95})$$

$${}^2\tilde{\alpha}_{[84]} \equiv -\frac{4i|E|e_t \mathcal{W}_{e_t}^9 \epsilon^2 (\eta(1225\eta + 3483) - 1196)}{33\eta - 11}, \quad (\text{C.96})$$

$${}^2\tilde{\alpha}_{[94]} \equiv \frac{70i|E|e_t \mathcal{W}_{e_t}^{11} \epsilon^2 (\eta(35\eta + 9) - 4)}{33\eta - 11}, \quad (\text{C.97})$$

$$\begin{aligned} I^{42} &= e^{-i2\kappa\mathcal{M}} e^{-i2\phi_0} \frac{16}{21} \sqrt{2\pi} |E|^2 \mu(1 - 3\eta) e_t^{-2} \times \\ &\quad \left\{ \sum_{k=0}^7 {}^2\alpha_{[k2]} \mathcal{F}_{[k2]}(\mathcal{E}) + \sum_{k=1}^7 {}^2\tilde{\alpha}_{[k2]} \mathcal{F}_{S[k2]}(\mathcal{E}) \right\}, \end{aligned} \quad (\text{C.98})$$

$${}^2\alpha_{[02]} \equiv e_t^2 - 2 + \frac{|E|\epsilon^2 ((5(41 - 8\eta)\eta - 52)e_t^2 - 2(5\eta(124\eta - 531) + 828))}{55(3\eta - 1)}, \quad (\text{C.99})$$

$${}^2\alpha_{[12]} \equiv 2 - e_t^2 + \frac{|E|\epsilon^2 ((5\eta(8\eta - 41) + 52)e_t^2 + 10\eta(124\eta - 927) + 2976)}{55(3\eta - 1)}, \quad (\text{C.100})$$

$${}^2\alpha_{[22]} \equiv \frac{8}{3} - \frac{11e_t^2}{6} + \frac{|E|\epsilon^2 ((632 - 5\eta(17\eta + 478))e_t^2 + 5\eta(2047\eta - 3411) + 4797)}{330(3\eta - 1)}, \quad (\text{C.101})$$

$$\begin{aligned} {}^2\alpha_{[32]} &\equiv \frac{1}{6} (-3e_t^4 + 11e_t^2 - 8) + \frac{|E|\epsilon^2}{330(3\eta - 1)} \left[165(1 - 4\eta)\eta e_t^4 \right. \\ &\quad \left. + (5\eta(1186\eta - 9227) + 14738)e_t^2 + 10(7369 - 527\eta)\eta - 23978 \right], \end{aligned} \quad (\text{C.102})$$

$$\begin{aligned} {}^2\alpha_{[42]} &\equiv -\frac{10 \mathcal{W}_{e_t}^4}{3} \\ &\quad + \frac{|E| \mathcal{W}_{e_t}^2 \epsilon^2 ((875\eta^2 - 6815\eta + 2738) e_t^2 - 5\eta(5715\eta + 3613) + 8167)}{660(3\eta - 1)}, \end{aligned} \quad (\text{C.103})$$

$$\begin{aligned} {}^2\alpha_{[52]} &\equiv 2 \mathcal{W}_{e_t}^6 - \frac{|E| \mathcal{W}_{e_t}^4 \epsilon^2}{132(3\eta - 1)} \left[33(\eta(35\eta - 191) + 60)e_t^2 \right. \\ &\quad \left. + (28933 - 6229\eta)\eta - 8915 \right], \end{aligned} \quad (\text{C.104})$$

$${}^2\alpha_{[62]} \equiv -\frac{|E| \mathcal{W}_{e_t}^6 \epsilon^2 (\eta(4375\eta - 29191) + 8993)}{132(3\eta - 1)}, \quad (\text{C.105})$$

$${}^2\alpha_{[72]} \equiv \frac{|E| \mathcal{W}_{e_t}^8 \epsilon^2 (5\eta(125\eta - 653) + 983)}{44(3\eta - 1)}, \quad (\text{C.106})$$

$$\begin{aligned} {}^2\tilde{\alpha}_{[12]} &\equiv -2i \mathcal{W}_{e_t} e_t \\ &\quad + \frac{2i|E|\epsilon^2 e_t ((5\eta(58\eta - 245) + 388)e_t^2 + 5(531 - 124\eta)\eta - 828)}{55(3\eta - 1) \mathcal{W}_{e_t}}, \end{aligned} \quad (\text{C.107})$$

$${}^2\tilde{\alpha}_{[22]} \equiv \frac{12i|E|\epsilon^2 e_t (e_t^2 - 2)}{\mathcal{W}_{e_t}}, \quad (\text{C.108})$$

$$\begin{aligned} {}^2\tilde{\alpha}_{[32]} &\equiv \frac{ie_t (9e_t^4 - 25e_t^2 + 16)}{6 \mathcal{W}_{e_t}} + \frac{i|E|\epsilon^2 e_t}{660(3\eta - 1) \mathcal{W}_{e_t}} \left[15(\eta(133\eta - 815) + 273)e_t^4 \right. \\ &\quad \left. + (5(13579 - 3569\eta)\eta - 21389)e_t^2 + 10\eta(2047\eta - 8163) + 25434 \right], \end{aligned} \quad (\text{C.109})$$

$$\begin{aligned} {}^2\tilde{\alpha}_{[42]} &\equiv \frac{4}{3} ie_t \mathcal{W}_{e_t}^3 - \frac{i|E|\epsilon^2 (-e_t \mathcal{W}_{e_t})}{660(3\eta - 1)} \left[(5\eta(355\eta - 4843) + 7527)e_t^2 \right. \\ &\quad \left. + 10\eta(993\eta + 6575) - 22522 \right], \end{aligned} \quad (\text{C.110})$$

$${}^2\tilde{\alpha}_{[52]} \equiv -2ie_t \mathcal{W}_{e_t}^5 + \frac{i|E|\epsilon^2 e_t \mathcal{W}_{e_t}^3}{88(3\eta - 1)} \left[(\eta(145\eta - 937) + 337)e_t^2 \right.$$

$$-22(\eta(113\eta - 289) + 87) \Big], \quad (\text{C.111})$$

$${}^2\tilde{\alpha}_{[62]} \equiv \frac{i|E|\epsilon^2(\eta(625\eta - 4849) + 1511)e_t \mathcal{W}_{e_t}^5}{99\eta - 33}, \quad (\text{C.112})$$

$${}^2\tilde{\alpha}_{[72]} \equiv -\frac{i|E|\epsilon^2(5\eta(125\eta - 653) + 983)e_t \mathcal{W}_{e_t}^7}{44(3\eta - 1)}, \quad (\text{C.113})$$

$$I^{40} \equiv \frac{8}{21} \sqrt{\frac{\pi}{5}} |E|^2 \mu(1 - 3\eta) \left\{ \sum_{k=0}^5 {}^2\alpha_{[k2]} \mathcal{F}_{[k2]}(\mathcal{E}) \right\}, \quad (\text{C.114})$$

$${}^2\alpha_{[00]} \equiv 6 - \frac{6|E|\epsilon^2(5\eta(8\eta - 41) + 52)}{55(3\eta - 1)}, \quad (\text{C.115})$$

$${}^2\alpha_{[10]} \equiv -6 + \frac{6|E|\epsilon^2(5\eta(8\eta - 41) + 52)}{55(3\eta - 1)}, \quad (\text{C.116})$$

$${}^2\alpha_{[20]} \equiv -5 + \frac{|E|\epsilon^2(5\eta(82\eta - 1699) + 2711)}{55(3\eta - 1)}, \quad (\text{C.117})$$

$${}^2\alpha_{[30]} \equiv -5e_t^2 + 5 + \frac{|E| \mathcal{W}_{e_t}^2 \epsilon^2(5\eta(2\eta + 163) - 218)}{55(3\eta - 1)}, \quad (\text{C.118})$$

$${}^2\alpha_{[40]} \equiv -\frac{3|E| \mathcal{W}_{e_t}^2 \epsilon^2(20\eta(7\eta - 128) + 831)}{66\eta - 22}, \quad (\text{C.119})$$

$${}^2\alpha_{[50]} \equiv \frac{9|E| \mathcal{W}_{e_t}^4 \epsilon^2(20\eta(7\eta - 128) + 831)}{110(3\eta - 1)}, \quad (\text{C.120})$$

$$S^{44} \equiv 0, \quad (\text{C.121})$$

$$S^{43} \equiv e^{-i3\mathcal{K}\mathcal{M}} e^{-i3\phi_0} \frac{16\sqrt{\frac{2\pi}{7}} |E|^{5/2} \mathcal{W}_{e_t} \delta_m(1 - 2\eta)\eta}{15e_t^3} \times \left\{ \sum_{k=2}^7 {}^3\beta_{[k0]} \mathcal{F}_{[k0]}(\mathcal{E}) + \sum_{k=3}^7 {}^3\tilde{\beta}_{[k0]} \mathcal{F}_{S[k2]}(\mathcal{E}) \right\}, \quad (\text{C.122})$$

$${}^3\beta_{[23]} \equiv 4 - 3e_t^2, \quad (\text{C.123})$$

$${}^3\beta_{[33]} \equiv 2(e_t^4 - 5e_t^2 + 4), \quad (\text{C.124})$$

$${}^3\beta_{[43]} \equiv \frac{1}{2}(-e_t^4 + 37e_t^2 - 36), \quad (\text{C.125})$$

$${}^3\beta_{[53]} \equiv \frac{1}{2} \mathcal{W}_{e_t}^4 (33e_t^2 - 68), \quad (\text{C.126})$$

$${}^3\beta_{[63]} \equiv 70 \mathcal{W}_{e_t}^6, \quad (\text{C.127})$$

$${}^3\beta_{[73]} \equiv -30 \mathcal{W}_{e_t}^8, \quad (\text{C.128})$$

$${}^3\tilde{\beta}_{[33]} \equiv -i \mathcal{W}_{e_t} e_t (e_t^2 - 4), \quad (\text{C.129})$$

$${}^3\tilde{\beta}_{[43]} \equiv i \mathcal{W}_{e_t} e_t (12 - 7e_t^2), \quad (\text{C.130})$$

$${}^3\tilde{\beta}_{[53]} \equiv \frac{3}{2} i \mathcal{W}_{e_t} e_t (e_t^4 + 3e_t^2 - 4), \quad (\text{C.131})$$

$${}^3\tilde{\beta}_{[63]} \equiv -40ie_t \mathcal{W}_{e_t}^5, \quad (\text{C.132})$$

$${}^3\tilde{\beta}_{[73]} \equiv 30ie_t \mathcal{W}_{e_t}^7, \quad (\text{C.133})$$

$$S^{42} \equiv 0, \quad (\text{C.134})$$

$$S^{41(4)} = e^{-i\mathcal{K}\mathcal{M}} e^{-i\phi_0} \frac{16\sqrt{2\pi}|E|^{5/2} \mathcal{W}_{e_t} \delta_m (1-2\eta)\eta}{35e_t} \times \left\{ \sum_{k=2}^5 {}^3\beta_{[k1]} \mathcal{F}_{[k0]}(\mathcal{E}) + \sum_{k=3}^5 {}^3\tilde{\beta}_{[k1]} \mathcal{F}_{S[k2]}(\mathcal{E}) \right\}, \quad (\text{C.135})$$

$${}^3\beta_{[21]} \equiv -1, \quad (\text{C.136})$$

$${}^3\beta_{[31]} \equiv -\frac{2\mathcal{W}_{e_t}^2}{3}, \quad (\text{C.137})$$

$${}^3\beta_{[41]} \equiv \frac{25\mathcal{W}_{e_t}^2}{6}, \quad (\text{C.138})$$

$${}^3\beta_{[51]} \equiv -\frac{5\mathcal{W}_{e_t}^4}{2}, \quad (\text{C.139})$$

$${}^3\tilde{\beta}_{[31]} \equiv -i\mathcal{W}_{e_t} e_t, \quad (\text{C.140})$$

$${}^3\tilde{\beta}_{[41]} \equiv -\frac{5}{3}i\mathcal{W}_{e_t} e_t, \quad (\text{C.141})$$

$${}^3\tilde{\beta}_{[51]} \equiv \frac{5}{2}ie_t \mathcal{W}_{e_t}^3, \quad (\text{C.142})$$

$$S^{40(4)} = 0, \quad (\text{C.143})$$

$$I^{55(5)} = e^{-i5\mathcal{K}\mathcal{M}} e^{-i5\phi_0} \frac{8\sqrt{\frac{2\pi}{165}}|E|^{5/2} \delta_m (1-2\eta)\eta \mathcal{W}_{e_t}}{e_t^5} \times \left\{ \sum_{k=0}^9 {}^3\alpha_{[k5]} \mathcal{F}_{[k5]}(\mathcal{E}) + \sum_{k=1}^9 {}^3\tilde{\alpha}_{[k5]} \mathcal{F}_{S[k5]}(\mathcal{E}) \right\}, \quad (\text{C.144})$$

$${}^3\alpha_{[05]} \equiv i(e_t^4 - 12e_t^2 + 16), \quad (\text{C.145})$$

$${}^3\alpha_{[15]} \equiv -i(e_t^4 - 12e_t^2 + 16), \quad (\text{C.146})$$

$${}^3\alpha_{[25]} \equiv -\frac{1}{12}i(73e_t^4 - 396e_t^2 + 400), \quad (\text{C.147})$$

$${}^3\alpha_{[35]} \equiv -\frac{1}{6}i\mathcal{W}_{e_t}^2(3e_t^4 - 48e_t^2 + 80), \quad (\text{C.148})$$

$${}^3\alpha_{[45]} \equiv -\frac{1}{24}i(e_t - 1)(e_t + 1)(83e_t^4 - 1188e_t^2 + 1840), \quad (\text{C.149})$$

$${}^3\alpha_{[55]} \equiv \frac{1}{24}i\mathcal{W}_{e_t}^4(3e_t^4 - 1028e_t^2 + 2096), \quad (\text{C.150})$$

$${}^3\alpha_{[65]} \equiv -\frac{7}{6}i\mathcal{W}_{e_t}^6(23e_t^2 + 112), \quad (\text{C.151})$$

$${}^3\alpha_{[75]} \equiv \frac{95}{6}i\mathcal{W}_{e_t}^8(3e_t^2 - 8), \quad (\text{C.152})$$

$${}^3\alpha_{[85]} \equiv 210i\mathcal{W}_{e_t}^{10}, \quad (\text{C.153})$$

$${}^3\alpha_{[95]} \equiv -70i\mathcal{W}_{e_t}^{12}, \quad (\text{C.154})$$

$${}^3\tilde{\alpha}_{[15]} \equiv \frac{-5e_t^5 + 20e_t^3 - 16e_t}{\mathcal{W}_{e_t}}, \quad (\text{C.155})$$

$${}^3\tilde{\alpha}_{[25]} \equiv 0, \quad (\text{C.156})$$

$${}^3\tilde{\alpha}_{[35]} \equiv \frac{e_t(-42e_t^6 + 341e_t^4 - 692e_t^2 + 400)}{12\mathcal{W}_{e_t}}, \quad (\text{C.157})$$

$${}^3\tilde{\alpha}_{[45]} \equiv \frac{(e_t \mathcal{W}_{e_t})(107e_t^4 - 564e_t^2 + 560)}{12}, \quad (\text{C.158})$$

$${}^3\tilde{\alpha}_{[55]} \equiv -\frac{1}{8}e_t \mathcal{W}_{e_t}^3 (19e_t^4 - 156e_t^2 + 240), \quad (\text{C.159})$$

$${}^3\tilde{\alpha}_{[65]} \equiv \frac{2}{3}e_t \mathcal{W}_{e_t}^5 (31e_t^2 - 176), \quad (\text{C.160})$$

$${}^3\tilde{\alpha}_{[75]} \equiv \frac{5}{6}e_t \mathcal{W}_{e_t}^7 (15e_t^2 + 16), \quad (\text{C.161})$$

$${}^3\tilde{\alpha}_{[85]} \equiv 140e_t \mathcal{W}_{e_t}^9, \quad (\text{C.162})$$

$${}^3\tilde{\alpha}_{[95]} \equiv -70e_t \mathcal{W}_{e_t}^{11}, \quad (\text{C.163})$$

$$I^{54} = 0, \quad (\text{C.164})$$

$$I^{53} = e^{-i3\mathcal{K}\mathcal{M}} e^{-i3\phi_0} \frac{8\sqrt{\frac{2\pi}{33}}|E|^{5/2}\delta_m\eta(1-2\eta)\mathcal{W}_{e_t}(e_t^2-4)}{3e_t^3} \times \left\{ \sum_{k=0}^7 {}^3\alpha_{[k3]}\mathcal{F}_{[k3]}(\mathcal{E}) + \sum_{k=1}^7 {}^3\tilde{\alpha}_{[k3]}\mathcal{F}_{S[k3]}(\mathcal{E}) \right\}, \quad (\text{C.165})$$

$${}^3\alpha_{[03]} \equiv i, \quad (\text{C.166})$$

$${}^3\alpha_{[13]} \equiv -i, \quad (\text{C.167})$$

$${}^3\alpha_{[23]} \equiv -\frac{i(173e_t^2 - 404)}{60(e_t^2 - 4)}, \quad (\text{C.168})$$

$${}^3\alpha_{[33]} \equiv -\frac{i(9e_t^4 - 25e_t^2 + 16)}{30(e_t^2 - 4)}, \quad (\text{C.169})$$

$${}^3\alpha_{[43]} \equiv -\frac{7i(73e_t^4 - 301e_t^2 + 228)}{120(e_t^2 - 4)}, \quad (\text{C.170})$$

$${}^3\alpha_{[53]} \equiv \frac{7i\mathcal{W}_{e_t}^4(9e_t^2 - 4)}{120(e_t^2 - 4)}, \quad (\text{C.171})$$

$${}^3\alpha_{[63]} \equiv \frac{77i\mathcal{W}_{e_t}^6}{6(e_t^2 - 4)}, \quad (\text{C.172})$$

$${}^3\alpha_{[73]} \equiv -\frac{11i\mathcal{W}_{e_t}^8}{2(e_t^2 - 4)}, \quad (\text{C.173})$$

$${}^3\tilde{\alpha}_{[13]} \equiv \frac{4e_t - 3e_t^3}{\mathcal{W}_{e_t}(e_t^2 - 4)}, \quad (\text{C.174})$$

$${}^3\tilde{\alpha}_{[23]} \equiv 0, \quad (\text{C.175})$$

$${}^3\tilde{\alpha}_{[33]} \equiv -\frac{e_t(126e_t^4 - 495e_t^2 + 404)}{60\mathcal{W}_{e_t}(e_t^2 - 4)}, \quad (\text{C.176})$$

$${}^3\tilde{\alpha}_{[43]} \equiv -\frac{e_t(59e_t^4 - 183e_t^2 + 124)}{20\mathcal{W}_{e_t}(e_t^2 - 4)}, \quad (\text{C.177})$$

$${}^3\tilde{\alpha}_{[53]} \equiv -\frac{e_t\mathcal{W}_{e_t}^3(89e_t^2 - 284)}{40(e_t^2 - 4)}, \quad (\text{C.178})$$

$${}^3\tilde{\alpha}_{[63]} \equiv \frac{22e_t\mathcal{W}_{e_t}^5}{3(e_t^2 - 4)}, \quad (\text{C.179})$$

$${}^3\tilde{\alpha}_{[73]} \equiv -\frac{11e_t\mathcal{W}_{e_t}^7}{2(e_t^2 - 4)}, \quad (\text{C.180})$$

$$I^{52} = 0, \quad (C.181)$$

$$I^{51} = e^{-i\mathcal{K}\mathcal{M}} e^{-i\phi_0} \frac{16\sqrt{\frac{\pi}{77}}|E|^{5/2}\mu\delta_m(1-2\eta)\mathcal{W}_{e_t}}{3me_t} \times \left\{ \sum_{k=0}^5 {}^3\alpha_{[k1]}\mathcal{F}_{[k1]}(\mathcal{E}) + \sum_{k=1}^5 {}^3\tilde{\alpha}_{[k1]}\mathcal{F}_{S[k1]}(\mathcal{E}) \right\}, \quad (C.182)$$

$${}^3\alpha_{[01]} \equiv i, \quad (C.183)$$

$${}^3\alpha_{[11]} \equiv -i, \quad (C.184)$$

$${}^3\alpha_{[21]} \equiv -\frac{77i}{60}, \quad (C.185)$$

$${}^3\alpha_{[31]} \equiv \frac{7i\mathcal{W}_{e_t}^2}{10}, \quad (C.186)$$

$${}^3\alpha_{[41]} \equiv \frac{35i\mathcal{W}_{e_t}^2}{24}, \quad (C.187)$$

$${}^3\alpha_{[51]} \equiv -\frac{7i\mathcal{W}_{e_t}^4}{8}, \quad (C.188)$$

$${}^3\tilde{\alpha}_{[11]} \equiv -\frac{e_t}{\mathcal{W}_{e_t}}, \quad (C.189)$$

$${}^3\tilde{\alpha}_{[21]} \equiv 0, \quad (C.190)$$

$${}^3\tilde{\alpha}_{[31]} \equiv \frac{7e_t(11-6e_t^2)}{60\mathcal{W}_{e_t}}, \quad (C.191)$$

$${}^3\tilde{\alpha}_{[41]} \equiv \frac{7}{12}e_t\mathcal{W}_{e_t}, \quad (C.192)$$

$${}^3\tilde{\alpha}_{[51]} \equiv -\frac{7}{8}e_t\mathcal{W}_{e_t}^3, \quad (C.193)$$

$$I^{50} = 0, \quad (C.194)$$

$$S^{55} = 0, \quad (C.195)$$

$$S^{54} = e^{-i4\mathcal{K}\mathcal{M}} e^{-i4\phi_0} \frac{64\sqrt{\frac{2\pi}{33}}|E|^3\mu(5(\eta-1)\eta+1)}{15e_t^4} \times \left\{ \sum_{k=2}^9 {}^4\beta_{[k4]}\mathcal{F}_{[k4]}(\mathcal{E}) + \sum_{k=3}^9 {}^4\tilde{\beta}_{[k4]}\mathcal{F}_{S[k4]}(\mathcal{E}) \right\}, \quad (C.196)$$

$${}^4\beta_{[24]} \equiv i(e_t^4 - 3e_t^2 + 2), \quad (C.197)$$

$${}^4\beta_{[34]} \equiv \frac{1}{6}i(-3e_t^6 + 31e_t^4 - 60e_t^2 + 32), \quad (C.198)$$

$${}^4\beta_{[44]} \equiv -\frac{1}{12}i((e_t^2 + 3)(15e_t^2 - 34)e_t^2 + 76), \quad (C.199)$$

$${}^4\beta_{[54]} \equiv \frac{1}{12}i\mathcal{W}_{e_t}^4(3e_t^4 + 218e_t^2 - 396), \quad (C.200)$$

$${}^4\beta_{[64]} \equiv \frac{7}{6}i\mathcal{W}_{e_t}^6(19e_t^2 + 16), \quad (C.201)$$

$${}^4\beta_{[74]} \equiv -\frac{5}{6}i\mathcal{W}_{e_t}^8(33e_t^2 - 100), \quad (C.202)$$

$${}^4\beta_{[84]} \equiv -105i\mathcal{W}_{e_t}^{10}, \quad (C.203)$$

$${}^4\beta_{[94]} \equiv 35i \mathcal{W}_{e_t}^{12}, \quad (\text{C.204})$$

$${}^4\tilde{\beta}_{[34]} \equiv -\frac{1}{4}e_t \mathcal{W}_{e_t} (e_t^4 - 8e_t^2 + 8), \quad (\text{C.205})$$

$${}^4\tilde{\beta}_{[44]} \equiv \frac{(-e_t \mathcal{W}_{e_t})(29e_t^4 - 112e_t^2 + 88)}{12}, \quad (\text{C.206})$$

$${}^4\tilde{\beta}_{[54]} \equiv -\frac{1}{8}e_t \mathcal{W}_{e_t}^3 (3e_t^4 - 16e_t^2 + 8), \quad (\text{C.207})$$

$${}^4\tilde{\beta}_{[64]} \equiv -\frac{16}{3}e_t \mathcal{W}_{e_t}^5 (e_t^2 - 6), \quad (\text{C.208})$$

$${}^4\tilde{\beta}_{[74]} \equiv -\frac{10}{3}e_t \mathcal{W}_{e_t}^7 (3e_t^2 - 4), \quad (\text{C.209})$$

$${}^4\tilde{\beta}_{[84]} \equiv -70e_t \mathcal{W}_{e_t}^9, \quad (\text{C.210})$$

$${}^4\tilde{\beta}_{[94]} \equiv 35e_t \mathcal{W}_{e_t}^{11}, \quad (\text{C.211})$$

$$S^{53} = 0, \quad (\text{C.212})$$

$$S^{52} = e^{-i2\mathcal{K}\mathcal{M}} e^{-i2\phi_0} \frac{64\sqrt{\frac{2\pi}{11}}|E|^3 \mu(5(\eta-1)\eta+1) \mathcal{W}_{e_t}^2}{45e_t^2} \times \left\{ \sum_{k=2}^7 {}^4\beta_{[k2]} \mathcal{F}_{[k2]}(\mathcal{E}) + \sum_{k=3}^7 {}^4\tilde{\beta}_{[k2]} \mathcal{F}_{S[k2]}(\mathcal{E}) \right\}, \quad (\text{C.213})$$

$${}^4\beta_{[22]} \equiv -i, \quad (\text{C.214})$$

$${}^4\beta_{[32]} \equiv \frac{1}{6}i(3e_t^2 - 10), \quad (\text{C.215})$$

$${}^4\beta_{[42]} \equiv -\frac{7}{12}i(3e_t^2 - 8), \quad (\text{C.216})$$

$${}^4\beta_{[52]} \equiv \frac{7}{12}i(3e_t^4 - 11e_t^2 + 8), \quad (\text{C.217})$$

$${}^4\beta_{[62]} \equiv -\frac{35i \mathcal{W}_{e_t}^4}{3}, \quad (\text{C.218})$$

$${}^4\beta_{[72]} \equiv 5i \mathcal{W}_{e_t}^6, \quad (\text{C.219})$$

$${}^4\tilde{\beta}_{[32]} \equiv -\frac{e_t(e_t^2 - 2)}{2\mathcal{W}_{e_t}}, \quad (\text{C.220})$$

$${}^4\tilde{\beta}_{[42]} \equiv -\frac{e_t(11e_t^2 - 16)}{6\mathcal{W}_{e_t}}, \quad (\text{C.221})$$

$${}^4\tilde{\beta}_{[52]} \equiv -\frac{e_t(3e_t^4 - 11e_t^2 + 8)}{4\mathcal{W}_{e_t}}, \quad (\text{C.222})$$

$${}^4\tilde{\beta}_{[62]} \equiv -\frac{20}{3}e_t \mathcal{W}_{e_t}^3, \quad (\text{C.223})$$

$${}^4\tilde{\beta}_{[72]} \equiv 5e_t \mathcal{W}_{e_t}^5, \quad (\text{C.224})$$

$$S^{51} = 0, \quad (\text{C.225})$$

$$S^{50} = \frac{32}{3} \sqrt{\frac{\pi}{1155}} |E|^3 \mu \mathcal{W}_{e_t} (5(\eta-1)\eta+1) e_t \times \left\{ \sum_{k=3}^5 {}^4\tilde{\beta}_{[k0]} \mathcal{F}_{S[k0]}(\mathcal{E}) \right\}, \quad (\text{C.226})$$

$${}^4\tilde{\beta}_{[32]} \equiv -1, \quad (\text{C.227})$$

$${}^4\tilde{\beta}_{[42]} \equiv -\frac{5}{3}, \quad (\text{C.228})$$

$${}^4\tilde{\beta}_{[52]} \equiv \frac{5\mathcal{W}_{e_t}^2}{2}, \quad (\text{C.229})$$

$$I_{66}^{(6)} = e^{-i6\mathcal{K}\mathcal{M}} e^{-i6\phi_0} \frac{32\sqrt{\frac{2\pi}{715}}|E|^3\mu(5(\eta-1)\eta+1)}{3e_t^6} \times \left\{ \sum_{k=0}^{11} {}^4\alpha_{[k6]}\mathcal{F}_{[k6]}(\mathcal{E}) + \sum_{k=1}^{11} {}^4\tilde{\alpha}_{[k6]}\mathcal{F}_{S[k6]}(\mathcal{E}) \right\}, \quad (\text{C.230})$$

$${}^4\alpha_{[06]} \equiv e_t^6 - 18e_t^4 + 48e_t^2 - 32, \quad (\text{C.231})$$

$${}^4\alpha_{[16]} \equiv -(e_t^2 - 2)(e_t^4 - 16e_t^2 + 16), \quad (\text{C.232})$$

$${}^4\alpha_{[26]} \equiv \frac{1}{60}(-437e_t^6 + 3666e_t^4 - 7536e_t^2 + 4384), \quad (\text{C.233})$$

$${}^4\alpha_{[36]} \equiv \frac{1}{30}(e_t - 1)(e_t + 1)(15e_t^6 - 444e_t^4 + 1648e_t^2 - 1408), \quad (\text{C.234})$$

$${}^4\alpha_{[46]} \equiv -\frac{1}{120}(e_t - 1)(e_t + 1)(283e_t^6 - 7086e_t^4 + 24208e_t^2 - 19680), \quad (\text{C.235})$$

$${}^4\alpha_{[56]} \equiv \frac{1}{120}\mathcal{W}_{e_t}^4(15e_t^6 - 5598e_t^4 + 33488e_t^2 - 37600), \quad (\text{C.236})$$

$${}^4\alpha_{[66]} \equiv -\frac{1}{60}\mathcal{W}_{e_t}^6(719e_t^4 - 1984e_t^2 - 12560), \quad (\text{C.237})$$

$${}^4\alpha_{[76]} \equiv -\frac{1}{60}\mathcal{W}_{e_t}^8(1605e_t^4 + 8336e_t^2 - 44496), \quad (\text{C.238})$$

$${}^4\alpha_{[86]} \equiv -\frac{14}{15}\mathcal{W}_{e_t}^{10}(369e_t^2 + 406), \quad (\text{C.239})$$

$${}^4\alpha_{[96]} \equiv \frac{14}{3}\mathcal{W}_{e_t}^{12}(51e_t^2 - 190), \quad (\text{C.240})$$

$${}^4\alpha_{[106]} \equiv 924\mathcal{W}_{e_t}^{14}, \quad (\text{C.241})$$

$${}^4\alpha_{[116]} \equiv -252\mathcal{W}_{e_t}^{16}, \quad (\text{C.242})$$

$${}^4\tilde{\alpha}_{[16]} \equiv -2ie_t\mathcal{W}_{e_t}(3e_t^4 - 16e_t^2 + 16), \quad (\text{C.243})$$

$${}^4\tilde{\alpha}_{[26]} \equiv 0, \quad (\text{C.244})$$

$${}^4\tilde{\alpha}_{[36]} \equiv \frac{-i(e_t\mathcal{W}_{e_t})(120e_t^6 - 1251e_t^4 + 3152e_t^2 - 2192)}{30}, \quad (\text{C.245})$$

$${}^4\tilde{\alpha}_{[46]} \equiv \frac{-i(e_t\mathcal{W}_{e_t})(329e_t^6 - 2823e_t^4 + 5984e_t^2 - 3600)}{30}, \quad (\text{C.246})$$

$${}^4\tilde{\alpha}_{[56]} \equiv \frac{-i(e_t\mathcal{W}_{e_t}^3)(55e_t^6 - 501e_t^4 + 1216e_t^2 - 880)}{20}, \quad (\text{C.247})$$

$${}^4\tilde{\alpha}_{[66]} \equiv \frac{-i(e_t\mathcal{W}_{e_t}^5)(301e_t^4 - 3296e_t^2 + 5360)}{15}, \quad (\text{C.248})$$

$${}^4\tilde{\alpha}_{[76]} \equiv \frac{i(e_t\mathcal{W}_{e_t}^7)(27e_t^4 + 704e_t^2 - 1776)}{12}, \quad (\text{C.249})$$

$${}^4\tilde{\alpha}_{[86]} \equiv \frac{56i\mathcal{W}_{e_t}^7 e_t(2e_t^4 + 51e_t^2 - 53)}{5}, \quad (\text{C.250})$$

$${}^4\tilde{\alpha}_{[96]} \equiv \frac{28i\mathcal{W}_{e_t}^9 e_t(12e_t^4 - 35e_t^2 + 23)}{3}, \quad (\text{C.251})$$

$${}^4\tilde{\alpha}_{[10\ 6]} \equiv -672i\mathcal{W}_{e_t}^{13}e_t, \quad (\text{C.252})$$

$${}^4\tilde{\alpha}_{[11\ 6]} \equiv 252ie_t\mathcal{W}_{e_t}^{15}, \quad (\text{C.253})$$

$$I^{65} = 0, \quad (C.254)$$

$$I^{64} = e^{-i4\mathcal{KM}} e^{-i4\phi_0} \frac{64\sqrt{\frac{\pi}{195}}|E|^3\mu(5(\eta-1)\eta+1)}{11e_t^4} \times \left\{ \sum_{k=0}^9 {}^4\alpha_{[k4]}\mathcal{F}_{[k4]}(\mathcal{E}) + \sum_{k=1}^9 {}^4\tilde{\alpha}_{[k4]}\mathcal{F}_{S[k4]}(\mathcal{E}) \right\}, \quad (C.255)$$

$${}^4\alpha_{[04]} \equiv 8 - 8e_t^2 + e_t^4, \quad (C.256)$$

$${}^4\alpha_{[14]} \equiv -e_t^4 + 8e_t^2 - 8, \quad (C.257)$$

$${}^4\alpha_{[24]} \equiv \frac{1}{60}(-237e_t^4 + 1096e_t^2 - 936), \quad (C.258)$$

$${}^4\alpha_{[34]} \equiv \frac{1}{90}(-15e_t^6 - 97e_t^4 + 456e_t^2 - 344), \quad (C.259)$$

$${}^4\alpha_{[44]} \equiv -\frac{1}{360}(e_t - 1)(e_t + 1)(1629e_t^4 - 10888e_t^2 + 12584), \quad (C.260)$$

$${}^4\alpha_{[54]} \equiv \frac{1}{360}\mathcal{W}_{e_t}^4(165e_t^4 - 4264e_t^2 + 9384), \quad (C.261)$$

$${}^4\alpha_{[64]} \equiv \frac{7}{45}\mathcal{W}_{e_t}^6(31e_t^2 - 356), \quad (C.262)$$

$${}^4\alpha_{[74]} \equiv \frac{1}{9}\mathcal{W}_{e_t}^8(57e_t^2 - 170), \quad (C.263)$$

$${}^4\alpha_{[84]} \equiv 49\mathcal{W}_{e_t}^{10}, \quad (C.264)$$

$${}^4\alpha_{[94]} \equiv -\frac{49}{3}\mathcal{W}_{e_t}^{12}, \quad (C.265)$$

$${}^4\tilde{\alpha}_{[14]} \equiv -4ie_t\mathcal{W}_{e_t}(e_t^2 - 2), \quad (C.266)$$

$${}^4\tilde{\alpha}_{[24]} \equiv 0, \quad (C.267)$$

$${}^4\tilde{\alpha}_{[34]} \equiv -\frac{i(e_t\mathcal{W}_{e_t})(40e_t^4 - 217e_t^2 + 234)}{15}, \quad (C.268)$$

$${}^4\tilde{\alpha}_{[44]} \equiv -\frac{i(e_t\mathcal{W}_{e_t})(209e_t^4 - 973e_t^2 + 874)}{45}, \quad (C.269)$$

$${}^4\tilde{\alpha}_{[54]} \equiv \frac{i(e_t\mathcal{W}_{e_t}^3)(75e_t^4 - 431e_t^2 + 466)}{30}, \quad (C.270)$$

$${}^4\tilde{\alpha}_{[64]} \equiv -\frac{2i(e_t\mathcal{W}_{e_t}^5)(251e_t^2 - 936)}{45}, \quad (C.271)$$

$${}^4\tilde{\alpha}_{[74]} \equiv \frac{1}{18}i\mathcal{W}_{e_t}^7 e_t(33e_t^2 - 248), \quad (C.272)$$

$${}^4\tilde{\alpha}_{[84]} \equiv -\frac{98}{3}i\mathcal{W}_{e_t}^9 e_t, \quad (C.273)$$

$${}^4\tilde{\alpha}_{[94]} \equiv \frac{49}{3}ie_t\mathcal{W}_{e_t}^{11}, \quad (C.274)$$

$$I^{63} = 0, \quad (C.275)$$

$$I^{62} = e^{-i2\mathcal{KM}} e^{-i2\phi_0} \frac{32\sqrt{\frac{2\pi}{13}}|E|^3\mu(5(\eta-1)\eta+1)}{33e_t^2} \times$$

$$\left\{ \sum_{k=0}^7 {}^4\alpha_{[k2]} \mathcal{F}_{[k2]}(\mathcal{E}) + \sum_{k=1}^7 {}^4\tilde{\alpha}_{[k2]} \mathcal{F}_{S[k2]}(\mathcal{E}) \right\}, \quad (\text{C.276})$$

$${}^4\alpha_{[02]} \equiv e_t^2 - 2, \quad (\text{C.277})$$

$${}^4\alpha_{[12]} \equiv 2 - e_t^2, \quad (\text{C.278})$$

$${}^4\alpha_{[22]} \equiv \frac{97}{30} - \frac{39e_t^2}{20}, \quad (\text{C.279})$$

$${}^4\alpha_{[32]} \equiv \frac{1}{90} (-51e_t^4 + 95e_t^2 - 44), \quad (\text{C.280})$$

$${}^4\alpha_{[42]} \equiv -\frac{7}{72} (27e_t^4 - 89e_t^2 + 62), \quad (\text{C.281})$$

$${}^4\alpha_{[52]} \equiv -\frac{7}{360} \mathcal{W}_{e_t}^4 (21e_t^2 - 26), \quad (\text{C.282})$$

$${}^4\alpha_{[62]} \equiv \frac{175 \mathcal{W}_{e_t}^6}{36}, \quad (\text{C.283})$$

$${}^4\alpha_{[72]} \equiv -\frac{25 \mathcal{W}_{e_t}^8}{12}, \quad (\text{C.284})$$

$${}^4\tilde{\alpha}_{[12]} \equiv -2ie_t \mathcal{W}_{e_t}, \quad (\text{C.285})$$

$${}^4\tilde{\alpha}_{[22]} \equiv 0, \quad (\text{C.286})$$

$${}^4\tilde{\alpha}_{[32]} \equiv \frac{ie_t (40e_t^4 - 137e_t^2 + 97)}{30 \mathcal{W}_{e_t}}, \quad (\text{C.287})$$

$${}^4\tilde{\alpha}_{[42]} \equiv \frac{ie_t (137e_t^4 - 384e_t^2 + 247)}{90 \mathcal{W}_{e_t}}, \quad (\text{C.288})$$

$${}^4\tilde{\alpha}_{[52]} \equiv -\frac{i \mathcal{W}_{e_t} e_t (87e_t^4 - 284e_t^2 + 197)}{60}, \quad (\text{C.289})$$

$${}^4\tilde{\alpha}_{[62]} \equiv -\frac{25}{9} ie_t \mathcal{W}_{e_t}^5, \quad (\text{C.290})$$

$${}^4\tilde{\alpha}_{[72]} \equiv \frac{25}{12} ie_t \mathcal{W}_{e_t}^7, \quad (\text{C.291})$$

$$I^{61} \equiv 0, \quad (\text{C.292})$$

$$I^{60} \equiv \frac{64}{33} \sqrt{\frac{10\pi}{273}} |E|^3 \mu (5(\eta - 1)\eta + 1) \left\{ \sum_{k=0}^5 {}^4\alpha_{[k0]} \mathcal{F}_{[k0]}(\mathcal{E}) \right\}, \quad (\text{C.293})$$

$${}^4\alpha_{[00]} \equiv 1, \quad (\text{C.294})$$

$${}^4\alpha_{[10]} \equiv -1, \quad (\text{C.295})$$

$${}^4\alpha_{[20]} \equiv -\frac{77}{60}, \quad (\text{C.296})$$

$${}^4\alpha_{[30]} \equiv \frac{7 \mathcal{W}_{e_t}^2}{10}, \quad (\text{C.297})$$

$${}^4\alpha_{[40]} \equiv \frac{35 \mathcal{W}_{e_t}^2}{24}, \quad (\text{C.298})$$

$${}^4\alpha_{[50]} \equiv -\frac{7 \mathcal{W}_{e_t}^4}{8}. \quad (\text{C.299})$$

D Proofs, summations and limitations

D.1 Trigonometric double series

In the following, a conversion of products of two infinite series of trigonometric functions to a single series with purely positive frequencies will be provided. The computation is straightforward: one has to re-express products of trigonometrics as sums and differences of trigonometrics with different arguments, and then to collect for the individual positive frequencies. Two sin series will have the following decomposition,

$$\begin{aligned} \left(\sum_{k=1}^{\infty} \mathcal{S}_k^{(n)} \sin k\mathcal{M} \right) \left(\sum_{m=1}^{\infty} G_m \sin m\mathcal{M} \right) &= \sum_{j=0}^{\infty} \Pi_j^{SS}[\mathcal{S}_k^{(n)}; G_m] \cos j\mathcal{M}, \\ \Pi_0^{SS}[\mathcal{S}_k^{(n)}; G_m] &\equiv \frac{1}{2} \sum_{m=1}^{\infty} \mathcal{S}_m^{(n)} G_m, \\ \Pi_{j>0}^{SS}[\mathcal{S}_k^{(n)}; G_m] &\equiv \frac{1}{2} \sum_{k=1}^{\infty} \left(\mathcal{S}_k^{(n)} G_{k-j} \Theta(k, j) + \mathcal{S}_k^{(n)} G_{k+j} \right) - \frac{1}{2} \sum_{k=1}^{j-1} \left(\mathcal{S}_k^{(n)} G_{j-k} \right) \Theta(j, 1). \end{aligned} \quad (\text{D.1})$$

The product of two cos series reads

$$\begin{aligned} \left(\sum_{k=1}^{\infty} \mathcal{A}_k^{(n)} \cos k\mathcal{M} \right) \left(\sum_{m=1}^{\infty} B_m \cos m\mathcal{M} \right) &= \sum_{j=0}^{\infty} \Pi_j^{CC}[\mathcal{A}_k^{(n)}; B_m] \cos j\mathcal{M}, \\ &= \frac{1}{2} \left(\sum_{j=1}^{\infty} \cos j\mathcal{M} \times \left[\sum_{k=j+1}^{\infty} \mathcal{A}_k^{(n)} B_{k-j} + \sum_{k=1}^{\infty} \mathcal{A}_k^{(n)} B_{k+j} + \sum_{k=1}^{j-1} \mathcal{A}_k^{(n)} B_{j-k} \Theta(j, 1) \right] \right. \\ &\quad \left. + \sum_{k=1}^{\infty} \mathcal{A}_k^{(n)} B_k \right). \end{aligned} \quad (\text{D.2})$$

The product of of one sin and one cos series finally reads

$$\begin{aligned} \left(\sum_{k=1}^{\infty} \mathcal{A}_k^{(n)} \cos k\mathcal{M} \right) \left(\sum_{k=1}^{\infty} G_k \sin k\mathcal{M} \right) &= \sum_{j=1}^{\infty} \Pi_j^{CS}[\mathcal{A}_k^{(n)}; G_k] \sin j\mathcal{M}, \\ \Pi_j^{CS}[\mathcal{A}_k^{(n)}; G_k] &\equiv \frac{1}{2} \left\{ \sum_{k=1}^{j-1} \left(\mathcal{A}_k^{(n)} G_{j-k} \right) \Theta(j, 1) - \sum_{k=1}^{\infty} \left(\mathcal{A}_k^{(n)} G_{k-j} \Theta(k, j) - \mathcal{A}_k^{(n)} G_{k+j} \right) \right\}. \end{aligned} \quad (\text{D.3})$$

D.2 Proof of important decomposition formulas

In this appendix, we like to provide some proves of formulas we only listed in the previous sections. Throughout the remaining sections the eccentricity e_t we are using is simply called e . Let us start with the inverse scaled relative separation with some arbitrary positive integer

exponent n . First, we perform a Taylor series expansion in e ,

$$\begin{aligned} \frac{1}{(1 - e \cos \mathcal{E})^n} &= 1 + \sum_{m=1}^{\infty} \frac{(n+m-1)!}{(n-1)!} \frac{e^m}{m!} \cos^m \mathcal{E} \\ &= 1 + \sum_{m=1}^{\infty} \frac{(n+m-1)!}{(n-1)!} \frac{e^m}{m!} \left(\frac{1}{2^m} \sum_{l=0}^m \binom{m}{l} \cos(\mathcal{E}[m-2l]) \right), \end{aligned} \quad (\text{D.4})$$

and list the ‘‘factorial’’ function of the integer number n as

$$\frac{(n+m-1)!}{(n-1)!} \equiv \prod_{k=1}^m (n+k-1). \quad (\text{D.5})$$

To optically simplify this equation, we summarise the terms before $\cos \sim \mathcal{E}$ with $\beta_{m,k}^{(n)}$ as follows,

$$\beta_{m,k}^{(n)} \equiv \frac{(n+m-1)!}{(n-1)!} \frac{1}{m!} \frac{e^m}{2^m} \binom{m}{k}, \quad (\text{D.6})$$

and write the sum with this definition:

$$\frac{1}{(1 - e \cos \mathcal{E})^n} = 1 + \sum_{m=1}^{\infty} \sum_{k=0}^m \beta_{m,k}^{(n)} \cos([m-2k]\mathcal{E}). \quad (\text{D.7})$$

To further markedly reduce the complexity of this double sum, it is the task to find out which pairs of (m, k) lead to the same frequency $j\mathcal{E}$ and which $\beta_{m,k}^{(n)}$ have to be added to this frequency contribution:

$$|m - 2k| = j, \quad (\text{D.8})$$

$$\Rightarrow m_1 = 2k + j, \quad (\text{D.9})$$

$$\Rightarrow m_2 = 2k - j. \quad (\text{D.10})$$

We refer the reader to Appendix D.1 for a completion of our thoughts at this point, and to obtain the terms containing $\sin \mathcal{E}$ as a product as well.

D.3 Accuracy of finite sums

In Section 6.4 we provided decompositions of functions of some ‘‘elementary’’ type in terms of \mathcal{E} which contain infinite summations. Naturally, for numerics it is important to know how many terms are needed to reach some desired accuracy. Considering compact binaries with small eccentricities only, one is allowed to expand the elementary expressions in powers of e and then to look how many terms are needed for the error to be shifted to $\mathcal{O}(e^{M+1})$ with some finite M . For the sake of practical issues, we demonstrate the calculation for terms that typically appear at 1PN. The reader can easily apply the results for the new terms that appear at 2PN doing straightforward calculations.

We start with the basic definitions. The upper limits of the summation has to give a term of order $\mathcal{O}(e^M)$, thus the individual limit has to be matched appropriately,

$$\beta_{m,k}^{(n)} = \frac{(n+m-1)!}{(n-1)!} \frac{1}{m!} \frac{e^m}{2^m} \binom{m}{k} = \mathcal{O}(e^m), \quad (\text{D.11})$$

$$b_0^{(n)} = 1 + \sum_{i=1}^{\infty} \beta_{2i,i}^{(n)} = 1 + \sum_{i=0}^{M/2} \beta_{2i,i}^{(n)} + \mathcal{O}(e^{M+1}), \quad (\text{D.12})$$

$$b_{j>0}^{(n)} = \sum_{i=0}^{\infty} \left(\beta_{j+2i,i}^{(n)} + \beta_{j+2i,j+i}^{(n)} \right) = \sum_{i=0}^{(M-j)/2} \left(\beta_{j+2i,i}^{(n)} + \beta_{j+2i,j+i}^{(n)} \right) + \mathcal{O}(e^{M+1}), \quad (\text{D.13})$$

$$\Rightarrow b_{j>0}^{(n)} = \mathcal{O}(e^j), \quad (\text{D.14})$$

$$A(\mathcal{E})^{-n} = \sum_{j=0}^M b_j^{(n)} \cos j\mathcal{E} + \mathcal{O}(e^{M+1}). \quad (\text{D.15})$$

In the last line we have used that the summation in Equation (D.13) starts with $i = 0$ and leaves no term if $j > M$. The same quantity with $\sin \mathcal{E}$ will also be truncated in the \mathcal{E} domain.

$$\frac{\sin \mathcal{E}}{A(\mathcal{E})^n} = \sum_{j=1}^{\infty} S_j^{(n)} \sin j\mathcal{E}, \quad (\text{D.16})$$

$$\begin{aligned} S_1^{(n)} &= \left(1 + b_0^{(n)} \right) - \frac{1}{2} b_2^{(n)} \\ &= \left(1 + \sum_{i=1}^{M/2} \beta_{2i,i}^{(n)} \right) - \frac{1}{2} \sum_{i=0}^{(M-2)/2} \left(\beta_{2+2i,i}^{(n)} + \beta_{2+2i,2+i}^{(n)} \right) + \mathcal{O}(e^{M+1}), \end{aligned} \quad (\text{D.17})$$

$$\begin{aligned} S_{j>1}^{(n)} &= \frac{1}{2} \left(\sum_{i=0}^{(M-j+1)/2} \left(\beta_{j-1+2i,i}^{(n)} + \beta_{j-1+2i,j-1+i}^{(n)} \right) \right. \\ &\quad \left. - \sum_{i=0}^{(M-j-1)/2} \left(\beta_{j+1+2i,i}^{(n)} + \beta_{j+1+2i,j+1+i}^{(n)} \right) \right) + \mathcal{O}(e^{M+1}), \end{aligned} \quad (\text{D.18})$$

$$\Rightarrow S_{j>1}^{(n)} = \mathcal{O}(e^{j-1}) + \mathcal{O}(e^{j+1}) = \mathcal{O}(e^{j-1}), \quad (\text{D.19})$$

$$\frac{\sin \mathcal{E}}{A(\mathcal{E})^n} = \sum_{j=1}^{M+1} S_j^{(n)} \sin j\mathcal{E} + \mathcal{O}(e^{M+1}). \quad (\text{D.20})$$

For the Fourier representation, we remember Equation (6.65) and take the expansion of the Bessel coefficients [95],

$$J_n(x) = x^n \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k+n} k! (k+n)!}, \quad (\text{D.21})$$

for the determination of the limit for our finite sums,

$$\bar{\gamma}_j^m = \frac{m}{j} (J_{j-m}(je) - J_{j+m}(je)) = \mathcal{O}(e^{j-m}), \quad (\text{D.22})$$

$$\Rightarrow \cos m\mathcal{E} = \sum_{j=1}^{M+m} \bar{\gamma}_j^m \cos j\mathcal{M} - \mathcal{O}(e^{M+1}), \quad (\text{D.23})$$

$$\mathcal{A}_j^{(n)} = \sum_{m=1}^M \bar{\gamma}_j^m b_m^{(n)} = \mathcal{O}(e^{j-m}) \mathcal{O}(e^m) = \mathcal{O}(e^j), \quad (\text{D.24})$$

$$A(\mathcal{E})^{-n} = (1 + b_0^{(n)}) + \left(\sum_{k=1}^M \bar{\gamma}_0^k b_k^{(n)} \right) + \sum_{j=1}^M \mathcal{A}_j^{(n)} \cos j\mathcal{M} + \mathcal{O}(e^{M+1}). \quad (\text{D.25})$$

In the third line above we used Equation (D.15) to truncate the number j in $\cos j\mathcal{E}$. For the other relevant term with $\sin \mathcal{E}$ we consider

$$\bar{\sigma}_j^m = \frac{m}{j} (J_{j-m}(je) + J_{j+m}(je)) = \mathcal{O}(e^{j-m}), \quad (\text{D.26})$$

$$\Rightarrow \sin m\mathcal{E} = \sum_{j=1}^{M+m} \mathcal{S}_j^{(m)} \sin j\mathcal{M} + \mathcal{O}(e^{M+1}), \quad (\text{D.27})$$

$$\mathcal{S}_j^{(n)} = \sum_{m=1}^{M+1} \bar{\sigma}_j^m S_m^{(n)} = \mathcal{O}(e^{j-m}) \mathcal{O}(e^{m-1}) = \mathcal{O}(e^{j-1}), \quad (\text{D.28})$$

$$\frac{\sin \mathcal{E}}{A(\mathcal{E})^n} = \sum_{j=1}^{M+1} \mathcal{S}_j^{(n)} \sin j\mathcal{M} + \mathcal{O}(e^{M+1}). \quad (\text{D.29})$$

In the third line again, we used (D.19) for the index j in $\sin j\mathcal{E}$ and in the last Equation (D.20). We will also consider the expansion coefficients of v from Equation (6.50). With the regular solution to Equation (6.51) at $e = 0$,

$$\alpha = \frac{1 - \sqrt{1 - e^2}}{e} = \mathcal{O}(e^1), \quad (\text{D.30})$$

their order is calculated to be

$$\begin{aligned} G_m(e) &= \frac{2}{m} \left\{ J_m(me) + \sum_{s=1}^{\infty} \alpha^s [J_{m-s}(me) - J_{m+s}(me)] \right\} \\ &= \mathcal{O}(e^m) + \sum_{s=1}^{\infty} \mathcal{O}(e^s) [\mathcal{O}(e^{m-s}) - \mathcal{O}(e^{m+s})] \\ &= \mathcal{O}(e^m). \end{aligned} \quad (\text{D.31})$$

D.4 The solution to the 2PN accurate Kepler Equation: a useful check

The 2PN accurate KE,

$$\begin{aligned} \mathcal{M} &= \mathcal{E} - e_t \sin \mathcal{E} + \epsilon^4 (\mathcal{F}_{v-\mathcal{E}}(v - \mathcal{E}) + \mathcal{F}_v \sin v) \\ &=: \mathcal{E} - e_t \sin \mathcal{E} + \epsilon^4 \mathcal{F}_4(\mathcal{E}), \end{aligned} \quad (\text{D.32})$$

can be inverted by defining

$$g(\mathcal{M} - \epsilon^4 \mathcal{F}_4(\mathcal{E})) = \mathcal{E}_{2\text{PN}}. \quad (\text{D.33})$$

as the appropriate solution, however it will look like. We can Taylor expand it around the Newtonian solution,

$$g_N(\mathcal{M}) - g'_N(\mathcal{M}) \epsilon^4 \mathcal{F}_4(\mathcal{E}) = \mathcal{E}_{2\text{PN}}. \quad (\text{D.34})$$

The Newtonian solution g is known,

$$g_N(\mathcal{M}) = \mathcal{M} + \sum_{n=1}^{\infty} \frac{2}{n} J_n(n e_t) \sin n\mathcal{M}, \quad (\text{D.35})$$

and (D.34) reads

$$\mathcal{E}_{2\text{PN}}(\mathcal{M}) = g_N(\mathcal{M}) - \left(1 + \sum_{n=1}^{\infty} 2J_n(n e_t) \cos n\mathcal{M}\right) \epsilon^4 \mathcal{F}_4(g_N(\mathcal{M})). \quad (\text{D.36})$$

We know

$$v - \mathcal{E} = \sum_{n=1}^{\infty} \left[G_n(n e_t) - \frac{2}{n} J_n(n e_t) \right] \sin n\mathcal{M}, \quad (\text{D.37})$$

$$\sin v = \frac{\sqrt{1 - e_t^2} \sin \mathcal{E}}{1 - e_t \cos \mathcal{E}} = \sqrt{1 - e_t^2} 2 \sum_{n=1}^{\infty} J'_n(n e_t) \sin n\mathcal{M}. \quad (\text{D.38})$$

Inserting (D.37) and (D.38) into (D.36) and applying the product rule for a sin and a cos series, we obtain the result of (6.59). This is a nice calculation, so let us show it in detail.

We abbreviate

$$\mathcal{F}_4(g(\mathcal{M})) = \sum_{n=1}^{\infty} \alpha_n \sin n\mathcal{M}, \quad (\text{D.39})$$

and read the coefficients α from Equations (D.37) and (D.38). In fact, for the comparison, their form does not matter. Equation (D.36) together with the rule for products of a sin and a cos series reads

$$\begin{aligned} \mathcal{E} &= \sum_{n=1}^{\infty} \frac{J_n(n e_t)}{n} \sin n\mathcal{M} \\ &\quad - \epsilon^4 \sum_{n=1}^{\infty} (\alpha_n + 2\Pi_n^{CS}[J_k(k e_t); \alpha_k]) \sin n\mathcal{M}, \end{aligned} \quad (\text{D.40})$$

$$2\Pi_n^{CS}[J_k(k e_t); \alpha_k] \stackrel{\text{!}}{=} \left\{ \Theta(n, 1) \sum_{k=1}^{n-1} \alpha_{n-k} J_k(k e_t) + \sum_{k=1}^{\infty} \alpha_{n+k} J_k(k e_t) - \sum_{k=n+1}^{\infty} \alpha_{k-n} J_k(k e_t) \right\}$$

$$\begin{aligned}
&\stackrel{2}{=} \sum_{k=1}^{n-1} \alpha_k J_{n-k}((n-k)e_t) + \sum_{k=n+1}^{\infty} \alpha_k J_{n-k}((n-k)e_t) \\
&\quad - \sum_{k=1}^{\infty} \alpha_k J_{n+k}((n+k)e_t). \tag{D.41}
\end{aligned}$$

Θ just dropped out as for $n = 1$, α_k for $k \leq 0$ vanish anyway. In summation with α_k the last line gives $\sum_{k=1}^{\infty} \alpha_k (J_{n-k}((n-k)e_t) - J_{n+k}((n+k)e_t))$, remembering that for the $k = n$ term, $J_0(0) = 1$.

Let us, in contrast, directly derive the expansion coefficients via integration and assume that \tilde{g} is the solution to the KE, $\mathcal{E} = \tilde{g}(\mathcal{M})$. Then, at $n\pi$ ($n \in \mathbb{Z}$), there are fixed points of the KE: $n\pi = \mathcal{E} = \mathcal{M}$ and $\tilde{g}(\mathcal{M}) - \mathcal{M}$ can be expressed in sin series,

$$\tilde{g}(\mathcal{M}) - \mathcal{M} = \sum_{n=1}^{\infty} A_n \sin(n\mathcal{M}). \tag{D.42}$$

The expansion coefficients, directly computed via integration read

$$\begin{aligned}
A_n &\stackrel{1}{=} \frac{2}{\pi} \int_0^{\pi} [\tilde{g}(\mathcal{M}) - \mathcal{M}] \sin(n\mathcal{M}) d\mathcal{M} \\
&\stackrel{2}{=} -\frac{2}{n\pi} \int_0^{\pi} [\tilde{g}(\mathcal{M}) - \mathcal{M}] d(\cos n\mathcal{M}) \\
&\stackrel{3}{=} [\text{boundary} = 0] + \frac{2}{n\pi} \int_0^{\pi} \cos(n\mathcal{M}) [\tilde{g}'(\mathcal{M}) - 1] d\mathcal{M} \\
&\stackrel{4}{=} \frac{2}{n\pi} \int_0^{\pi} \cos \{n\mathcal{M}\} d(\tilde{g}(\mathcal{M})) \\
&\stackrel{5}{=} \frac{2}{n\pi} \int_0^{\pi} \cos \{n (\mathcal{E} - e_t \sin \mathcal{E} + \epsilon^4 \mathcal{F}_4(\mathcal{E}))\} d\mathcal{E} \\
&\stackrel{6}{=} \frac{2}{n\pi} \int_0^{\pi} \cos \{n (\mathcal{E} - e_t \sin \mathcal{E})\} d\mathcal{E} - \frac{2}{\pi} \int_0^{\pi} \sin \{n (\mathcal{E} - e_t \sin \mathcal{E})\} \epsilon^4 \sum_{k=1}^{\infty} \alpha_k \sin k\mathcal{M} \\
&\stackrel{7}{=} \frac{2J_n(ne_t)}{n} - \frac{\epsilon^4}{\pi} \int_0^{\pi} \sum_{k=1}^{\infty} \alpha_k (\cos [(k+n)(\mathcal{E} - e_t \sin \mathcal{E})] - \cos [(k-n)(\mathcal{E} - e_t \sin \mathcal{E})]) d\mathcal{E} \\
&\stackrel{8}{=} \frac{2J_n(ne_t)}{n} - \epsilon^4 \sum_{k=1}^{\infty} \alpha_k (J_{k+n}(ne_t) - J_{k-n}(ne_t)), \tag{D.43}
\end{aligned}$$

and we see that the 2PN coefficient shows agreement in both calculations. In step 5 we used Equation (D.32), in step 6 we Taylor expanded the argument of cos around the Newtonian \mathcal{M} and in step 7 we used that only Newtonian \mathcal{M} is required in the sum. The trigonometrics of \mathcal{E} are dealt with equivalently.

D.5 Fourier-domain trigonometric functions of the eccentric anomaly through 2PN

As an exemplary calculation, we determine the expansion coefficients of $\cos m\mathcal{E}$,

$$\cos m\mathcal{E} = \sum_{j=0}^{\infty} \bar{\gamma}_j^m \cos j\mathcal{M}. \quad (\text{D.44})$$

Using integration by parts, the computation turns out to be ¹⁹

$$\begin{aligned} \bar{\gamma}_{j>0}^m &= \frac{2}{\pi} \int_0^\pi \cos m\mathcal{E} \cos j\mathcal{M} \, d\mathcal{M} \\ &= [\cos m\mathcal{E} \sin j\mathcal{M}]_0^\pi - \frac{2}{j\pi} \int_0^\pi (d \cos m\mathcal{E}) \sin j\mathcal{M} \, d\mathcal{M} \\ &= [\dots]_0^\pi - \frac{2m}{j\pi} \int_0^\pi (\sin m\mathcal{E}) \sin j\mathcal{M} \frac{d\mathcal{E}}{d\mathcal{M}} d\mathcal{M} \\ &= \frac{m}{j\pi} \int_0^\pi (\cos[m\mathcal{E} + j\mathcal{M}_{2\text{PN}}] - \cos[m\mathcal{E} - j\mathcal{M}_{2\text{PN}}]) \, d\mathcal{E} \\ &= \frac{m}{j\pi} \int_0^\pi (\cos[m\mathcal{E} + j\mathcal{M}_{\text{N}}] - \cos[m\mathcal{E} - j\mathcal{M}_{\text{N}}]) \, d\mathcal{E} \\ &\quad - \frac{m}{\pi} \int_0^\pi (\sin[m\mathcal{E} + j\mathcal{M}_{\text{N}}] - \sin[m\mathcal{E} - j\mathcal{M}_{\text{N}}]) \epsilon^4 \sum_{f=1}^{\infty} \alpha_f \sin f\mathcal{M}_{\text{N}} \, d\mathcal{E} \\ &= \frac{m}{j} (J_{j+m}(je_t) - J_{j-m}(je_t)) \\ &\quad - \epsilon^4 \frac{m}{2\pi} \sum_{n=1}^{\infty} \int_0^\pi \alpha_n \left[\cos(j\mathcal{M} - m\mathcal{E} - \mathcal{M}n) - \cos(j\mathcal{M} - m\mathcal{E} + \mathcal{M}n) \right. \\ &\quad \left. + \cos(j\mathcal{M} + m\mathcal{E} - \mathcal{M}n) - \cos(j\mathcal{M} + m\mathcal{E} + \mathcal{M}n) \right] d\mathcal{E} \\ &= \frac{m}{j} (J_{j+m}(je_t) - J_{j-m}(je_t)) \\ &\quad - \epsilon^4 \sum_{n=1}^{\infty} \frac{m\alpha_n}{2\pi} \int_0^\pi \left\{ \begin{aligned} &\cos [(j-n)(\mathcal{E} - e_t \sin(\mathcal{E})) - m\mathcal{E}] \\ &- \cos [(j+n)(\mathcal{E} - e_t \sin(\mathcal{E})) - m\mathcal{E}] \\ &- \cos [(j-n)(\mathcal{E} - e_t \sin(\mathcal{E})) + m\mathcal{E}] \\ &+ \cos [(j+n)(\mathcal{E} - e_t \sin(\mathcal{E})) + m\mathcal{E}] \end{aligned} \right\} d\mathcal{E} \end{aligned} \quad (\text{D.45})$$

The task is now to bring these integrals to the form

$$\frac{1}{\pi} \int_0^\pi \cos [\mathbf{x}_i(\mathcal{E} - \mathbf{y}_i \sin \mathcal{E})] = J_{\mathbf{x}_i}(\mathbf{x}_i \mathbf{y}_i). \quad (\text{D.46})$$

¹⁹Note that the zeroth expansion coefficient in Equation (D.44) will not have the overall factor $2/\pi$. Instead, replace 2 by 1 to get the correct result.

with some prefactor \mathbf{x} and “eccentricity” \mathbf{y} to be determined for each special case. In order of appearance in the last equation above, these eccentricities read

$$\mathbf{y}_1 = \frac{e_t(j-n)}{j-m-n}, \quad (\text{D.47})$$

$$\mathbf{y}_2 = \frac{e_t(j+n)}{j-m+n}, \quad (\text{D.48})$$

$$\mathbf{y}_3 = \frac{e_t(j-n)}{j+m-n}, \quad (\text{D.49})$$

$$\mathbf{y}_4 = \frac{e_t(j+n)}{j+m+n}, \quad (\text{D.50})$$

and \mathbf{x}_i is simply the denominator. With this the rest is easy to calculate and so are the coefficients of $\sin m\mathcal{E}$.

References

- [1] S. Rowan and J. Hough, “Gravitational wave detection by interferometry (ground and space),” *Living Rev. Relativity* **3** (2000) 3.
<http://www.livingreviews.org/lrr-2000-3>.
- [2] “advanced LIGO home page.” Electronic adress, May, 2011.
<http://advancedligo.mit.edu>.
- [3] K. Martel and E. Poisson, “Gravitational waves from eccentric compact binaries: Reduction in signal-to-noise ratio due to nonoptimal signal processing,” *Phys. Rev. D* **60** (1999) 124008, [arXiv:gr-qc/9907006](https://arxiv.org/abs/gr-qc/9907006).
- [4] T. A. Apostolatos, “Search templates for gravitational waves from precessing, inspiraling binaries,” *Phys. Rev. D* **52** (1995) 605–620.
- [5] “GEO600 home page.” Electronic adress, May, 2011. <http://www.geo600.org>.
- [6] “LISA home page.” Electronic adress, May, 2011. <http://lisa.jpl.nasa.gov/>.
- [7] P. Amaro-Seoane, C. Eichhorn, E. K. Porter, and R. Spurzem, “Binaries of massive black holes in rotating clusters: dynamics, gravitational waves, detection and the role of eccentricity,” *Mon. Not. R. Astron. Soc.* **401** (2010) , [arXiv:0908.0755v2](https://arxiv.org/abs/0908.0755v2) [[astro-ph](https://arxiv.org/archive/astro)].
- [8] L. Blanchet, “Gravitational radiation from post-Newtonian sources and inspiralling compact binaries,” *Living Rev. Relativity* **9** (2006) 4.
<http://www.livingreviews.org/lrr-2006-4>.
- [9] L. Blanchet and G. Faye, “General relativistic dynamics of compact binaries at the third post-Newtonian order,” *Phys. Rev. D* **63** (2001) 062005, [arXiv:gr-qc/0007051](https://arxiv.org/abs/gr-qc/0007051).

- [10] M. Maggiore, *Gravitational Waves, Volume 1: Theory and Experiments*. Oxford University Press, Oxford, England, 2007.
- [11] L. D. Landau and E. M. Lifshitz, *The classical theory of fields*, vol. 2 of *Course of Theoretical Physics*. Butterworth-Heinemann, Linacre House, Jordan Hill, Oxford OX2 8DP, 4th ed.
- [12] N. Straumann, *General Relativity with Applications to Astrophysics*. Springer, Berlin, 2004.
- [13] L. Blanchet *Phys. Rev. D* **51** (1995) 2559–2583, [arXiv:gr-qc/9501030v1](#).
- [14] L. Blanchet, T. Damour, and B. R. Iyer, “Gravitational waves from inspiralling compact binaries: Energy loss and waveform to second-post-Newtonian order,” *Phys. Rev. D* **51** (1995) no. 5360-5386, , [arXiv:gr-qc/9501029](#).
- [15] L. Blanchet, “Radiative gravitation fields in general relativity II. Asymptotic behaviour at future null infinity,” *Proc. R. Soc. A* **409** (1987) 383–399. <http://www.jstor.org/stable/2398130>.
- [16] L. Blanchet, G. Faye, B. R. Iyer, and S. Sinha, “The third post-Newtonian gravitational wave polarizations and associated spherical harmonic modes for inspiralling compact binaries in quasi-circular orbits,” *Class. Quant. Grav.* **25** (2008) 165003, [arXiv:0802.1249 \[gr-qc\]](#).
- [17] K. S. Thorne, “Multipole expansions of gravitational radiation,” *Rev. Mod. Phys.* **52** (1980) 299–339.
- [18] L. Blanchet and T. Damour, “Post-Newtonian generation of gravitational waves,” *Ann. Inst. H. Poincaré A* **50** (1989) 377–408. http://www.numdam.org/item?id=AIHPA_1989__50_4_377_0.
- [19] C. C. Noack, “Das Noether-Theorem in der klassischen Feldtheorie.” Online lecture note, 1996. <http://www.itp.uni-bremen.de/~noack/>.
- [20] A. Gopakumar and B. R. Iyer, “Gravitational waves from inspiraling compact binaries: Angular momentum flux, evolution of the orbital elements, and the waveform to the second post-Newtonian order,” *Phys. Rev. D* **56** (1997) 7708–7731, [arXiv:gr-qc/0110100](#).
- [21] S. Chandrasekhar and F. P. Esposito, “The 2 1/2-Post-Newtonian Equations of Hydrodynamics and Radiation Reaction in General Relativity,” *ApJ* **160** (1970) 153.
- [22] L. Blanchet and T. Damour, “Multipolar radiation reaction in general relativity,” *Phys. Lett. A* **104** (1984) 82–86.

- [23] A. Gopakumar, “New class of gravitational wave templates for inspiralling compact binaries,” [arXiv:0712.3236 \[gr-qc\]](#).
- [24] M. Tessmer and A. Gopakumar, “On the ability of various circular inspiral templates to capture inspiral gravitational waves from compact binaries having tiny orbital eccentricities,” [arXiv:0812.0549 \[gr-qc\]](#).
- [25] K. G. Arun, L. Blanchet, B. R. Iyer, and S. Sinha, “Third post-Newtonian angular momentum flux and the secular evolution of orbital elements for inspiralling compact binaries in quasi-elliptical orbits,” *Phys. Rev. D* **80** (2009) 124018, [arXiv:0908.3854 \[gr-qc\]](#).
- [26] T. Damour, A. Gopakumar, and B. R. Iyer, “Phasing of gravitational waves from inspiralling eccentric binaries,” *Phys. Rev. D* **70** (2004) 064028, [arXiv:gr-qc/0404128](#).
- [27] S. Nissanke and L. Blanchet, “Gravitational radiation reaction in the equations of motion of compact binaries to 3.5 post-Newtonian order,” *Class. Quant. Grav.* **22** (2005) no. 6, , [arXiv:gr-qc/0412018](#).
- [28] C. Königsdörffer and A. Gopakumar, “Phasing of gravitational waves from inspiralling eccentric binaries at the third-and-a-half post-Newtonian order,” *Phys. Rev. D* **73** (2006) 124012, [arXiv:gr-qc/0603056](#).
- [29] A. Buonanno, B. R. Iyer, E. Ochsner, Y. Pan, and B. S. Sathyaprakash, “Comparison of post-Newtonian templates for compact binary inspiral signals in gravitational-wave detectors,” *Phys. Rev. D* **80** (2009) 084043, [arXiv:0907.0700 \[gr-qc\]](#).
- [30] M. Turner and C. M. Will, “Post-Newtonian gravitation bremsstrahlung,” *ApJ* **220** (1978) 1107.
- [31] H. A. Lorentz and J. Droste, “De beweging van een stelsel lichamen onder den invloed van hunne onderlinge aantrekking, behandeld volgens de theorie van Einstein, I,” *Versl. K. Akad. Wet. (Amsterdam)* **26** (1917) 392.
- [32] H. A. Lorentz and J. Droste, “De beweging van een stelsel lichamen onder den invloed van hunne onderlinge aantrekking, behandeld volgens de theorie van Einstein, II,” *Versl. K. Akad. Wet. (Amsterdam)* **26** (1917) 649.
- [33] A. S. Eddington, A. M. Ostrowski, H. Schmidt, and A. Einstein, *Relativitätstheorie in mathematischer Behandlung*. J. Springer, Berlin, 1925.
- [34] A. Einstein, L. Infeld, and B. Hoffmann, “The gravitational equations and the problem of motion,” *Ann. Math.* **39** (1938) 65–100.

- [35] R. L. Arnowitt, S. Deser, and C. W. Misner, “Canonical variables for general relativity,” *Phys. Rev.* **117** (1960) 1595–1602.
- [36] R. L. Arnowitt, S. Deser, and C. W. Misner, “Finite self-energy of classical point particles,” *Phys. Rev. Lett.* **4** (1960) 375–377.
- [37] R. L. Arnowitt, S. Deser, and C. W. Misner, “The dynamics of general relativity,” in *Gravitation: An Introduction to Current Research*, L. Witten, ed., pp. 227–265. John Wiley, New York, 1962.
- [38] E. Poisson, “An advanced course in general relativity.” Electronic address, 2002. <http://www.physics.uoguelph.ca/poisson/research/agr.pdf>. Draft of lecture notes.
- [39] É.ourgoulhon, “3+1 formalism and bases of numerical relativity,” [arXiv:gr-qc/0703035](https://arxiv.org/abs/gr-qc/0703035).
- [40] G. Schäfer, “The gravitational quadrupole radiation-reaction force and the canonical formalism of ADM,” *Ann. Phys. (N.Y.)* **161** (1985) 81–100.
- [41] R. L. Arnowitt, S. Deser, and C. W. Misner, “Coordinate invariance and energy expressions in general relativity,” *Phys. Rev.* **122** (1961) 997–1006.
- [42] R. L. Arnowitt, S. Deser, and C. W. Misner, “Consistency of the canonical reduction of General Relativity,” *J. Math. Phys.* **1** (1960) 434–439.
- [43] M. Mathisson, “Neue Mechanik materieller Systeme,” *Acta Phys. Pol.* **6** (1937) 163–200.
- [44] E. Corinaldesi and A. Papapetrou, “Spinning test-particles in general relativity. II,” *Proc. R. Soc. A* **209** (1951) 259–268.
- [45] B. M. Barker and R. F. O’Connell, “Derivation of the equations of motion of a gyroscope from the quantum theory of gravitation,” *Phys. Rev. D* **2** (1970) 1428–1435.
- [46] B. M. Barker and R. F. O’Connell, “Gravitational two-body problem with arbitrary masses, spins, and quadrupole moments,” *Phys. Rev. D* **12** (1975) 329–335.
- [47] B. M. Barker and R. F. O’Connell, “The gravitational interaction: Spin, rotation, and quantum effects—a review,” *Gen. Relativ. Gravit.* **11** (1979) 149–175.
- [48] P. D. D’Eath, “Interaction of two black holes in the slow-motion limit,” *Phys. Rev. D* **12** (1975) 2183–2199.
- [49] J. Steinhoff and D. Puetzfeld, “Multipolar equations of motion for extended test bodies in general relativity,” *Phys. Rev. D* **81** (2010) 044019, [arXiv:0909.3756](https://arxiv.org/abs/0909.3756) [gr-qc].

- [50] S. Taylor and E. Poisson, “Nonrotating black hole in a post-Newtonian tidal environment,” *Phys. Rev. D* **78** (2008) 084016, [arXiv:0806.3052 \[gr-qc\]](#).
- [51] E. Poisson, “Gravitational waves from inspiraling compact binaries: The quadrupole-moment term,” *Phys. Rev. D* **57** (1998) 5287–5290, [arXiv:gr-qc/9709032](#).
- [52] Z. Keresztes, B. Mikóczi, and L. Á. Gergely, “The Kepler equation for inspiralling compact binaries,” *Phys. Rev. D* **72** (2005) 104022, [arXiv:astro-ph/0510602](#).
- [53] A. Trautman, “Lectures on general relativity,” *Gen. Relativ. Gravit.* **34** (2002) 721–762.
- [54] W. M. Tulczyjew, “Motion of multipole particles in general relativity theory,” *Acta Phys. Pol.* **18** (1959) 393–409.
- [55] B. Tulczyjew and W. M. Tulczyjew, “On multipole formalism in general relativity,” in *Recent Developments in General Relativity*, pp. 465–472. Pergamon Press, New York, 1962.
- [56] A. Papapetrou, “Spinning test-particles in general relativity. I,” *Proc. R. Soc. A* **209** (1951) 248–258.
- [57] J. Steinhoff, “Canonical formulation of spin in general relativity,” *Ann. Phys. (Berlin)* **523** (2011) 296–353, [arXiv:1106.4203 \[gr-qc\]](#).
- [58] B. M. Barker and R. F. O’Connell, “Nongeodesic motion in general relativity,” *Gen. Relativ. Gravit.* **5** (1974) .
- [59] K. G. Arun, L. Blanchet, B. R. Iyer, and M. S. S. Qusailah, “Tail effects in the third post-Newtonian gravitational wave energy flux of compact binaries in quasi-elliptical orbits,” *Phys. Rev. D* **77** (2008) 064034, [arXiv:0711.0250 \[gr-qc\]](#).
- [60] L. Blanchet, A. Buonanno, and G. Faye, “Higher-order spin effects in the dynamics of compact binaries. II. Radiation field,” *Phys. Rev. D* **74** (2006) 104034, [arXiv:gr-qc/0605140](#).
- [61] J. Hartung and J. Steinhoff, “Next-to-next-to-leading order post-Newtonian spin-orbit Hamiltonian for self-gravitating binaries,” *Ann. Phys. (Berlin)* **523** (2011) 783–790, [arXiv:1104.3079 \[gr-qc\]](#).
- [62] J. Hartung and J. Steinhoff, “Next-to-leading order spin-orbit and spin(a)-spin(b) Hamiltonians for n gravitating spinning compact objects,” *Phys. Rev. D* **83** (2011) 044008, [arXiv:1011.1179 \[gr-qc\]](#).

- [63] M. Tessmer, J. Hartung, and G. Schäfer, “Motion and gravitational wave forms of eccentric compact binaries with orbital-angular-momentum-aligned spins under next-to-leading order in spin-orbit and leading order in spin(1)–spin(2) and spin-squared couplings,” *Class. Quant. Grav.* **27** (2010) 165005, [arXiv:1003.2735 \[gr-qc\]](#).
- [64] C. Reisswig, S. Husa, L. Rezzolla, E. N. Dorband, D. Pollney, and J. Seiler, “Gravitational-wave detectability of equal-mass black-hole binaries with aligned spins,” *Phys. Rev. D* **80** (2009) 124026, [arXiv:0907.0462 \[gr-qc\]](#).
- [65] T. Bogdanović, C. S. Reynolds, and M. C. Miller, “Alignment of the spins of supermassive black holes prior to coalescence,” *ApJ* **661** (2007) L147–L150, [arXiv:gr-qc/0703054](#).
- [66] P. Jaranowski and G. Schäfer, “Radiative 3.5 post-Newtonian ADM Hamiltonian for many-body point-mass systems,” *Phys. Rev. D* **55** (1997) 4712–4722.
- [67] L. Blanchet, G. Faye, B. R. Iyer, and B. Joguet, “Gravitational-wave inspiral of compact binary systems to 7/2 post-Newtonian order,” *Phys. Rev. D* **65** (2002) 061501(R), [arXiv:gr-qc/0105099](#).
- [68] L. Blanchet, G. Faye, B. R. Iyer, and B. Joguet, “Erratum: Gravitational-wave inspiral of compact binary systems to 7/2 post-Newtonian order,” *Phys. Rev. D* **71** (2005) 129902(E).
- [69] P. Ajith, S. Babak, Y. Chen, M. Hewitson, B. Krishnan, A. M. Sintes, J. T. Whelan, B. Brügmann, P. Diener, N. Dorband, J. Gonzalez, M. Hannam, S. Husa, D. Pollney, L. Rezzolla, L. Santamaría, U. Sperhake, and J. Thornburg, “Template bank for gravitational waveforms from coalescing binary black holes: Nonspinning binaries,” *Phys. Rev. D* **77** (2008) 104017, [arXiv:0710.2335 \[gr-qc\]](#).
- [70] P. Ajith, S. Babak, Y. Chen, M. Hewitson, B. Krishnan, A. M. Sintes, J. T. Whelan, B. Brügmann, P. Diener, N. Dorband, J. Gonzalez, M. Hannam, S. Husa, D. Pollney, L. Rezzolla, L. Santamaría, U. Sperhake, and J. Thornburg, “Erratum: Template bank for gravitational waveforms from coalescing binary black holes: Nonspinning binaries,” *Phys. Rev. D* **79** (2009) 129901(E).
- [71] L. E. Kidder, C. M. Will, and A. G. Wiseman, “Spin effects in the inspiral of coalescing compact binaries,” *Phys. Rev. D* **47** (1993) R4183–R4187, [arXiv:gr-qc/9211025](#).
- [72] G. Faye, L. Blanchet, and A. Buonanno, “Higher-order spin effects in the dynamics of compact binaries. I. Equations of motion,” *Phys. Rev. D* **74** (2006) 104033, [arXiv:gr-qc/0605139](#).

- [73] L. Blanchet, A. Buonanno, and G. Faye, “Erratum: Higher-order spin effects in the dynamics of compact binaries. II. Radiation field,” *Phys. Rev. D* **75** (2007) 049903(E).
- [74] L. Blanchet, A. Buonanno, and G. Faye, “Erratum: Higher-order spin effects in the dynamics of compact binaries. II. Radiation field,” *Phys. Rev. D* **81** (2010) 089901(E).
- [75] K. G. Arun, A. Buonanno, G. Faye, and E. Ochsner, “Higher-order spin effects in the amplitude and phase of gravitational waveforms emitted by inspiraling compact binaries: Ready-to-use gravitational waveforms,” *Phys. Rev. D* **79** (2009) 104023, [arXiv:0810.5336 \[gr-qc\]](#).
- [76] M. Hannam, S. Husa, B. Brügmann, and A. Gopakumar, “Comparison between numerical-relativity and post-Newtonian waveforms from spinning binaries: The orbital hang-up case,” *Phys. Rev. D* **78** (2008) 104007, [arXiv:0712.3787 \[gr-qc\]](#).
- [77] J. Steinhoff, S. Hergt, and G. Schäfer, “Next-to-leading order gravitational spin(1)-spin(2) dynamics in Hamiltonian form,” *Phys. Rev. D* **77** (2008) 081501(R), [arXiv:0712.1716 \[gr-qc\]](#).
- [78] S. Hergt, J. Steinhoff, and G. Schäfer, “The reduced Hamiltonian for next-to-leading-order spin-squared dynamics of general compact binaries,” *Class. Quant. Grav.* **27** (2010) 135007, [arXiv:1002.2093 \[gr-qc\]](#).
- [79] R.-M. Memmesheimer, A. Gopakumar, and G. Schäfer, “Third post-Newtonian accurate generalized quasi-Keplerian parametrization for compact binaries in eccentric orbits,” *Phys. Rev. D* **70** (2004) 104011, [arXiv:gr-qc/0407049](#).
- [80] P. Jaranowski and G. Schäfer, “Third post-Newtonian higher order ADM Hamilton dynamics for two-body point-mass systems,” *Phys. Rev. D* **57** (1998) 7274–7291, [arXiv:gr-qc/9712075](#).
- [81] T. Damour, P. Jaranowski, and G. Schäfer, “Dimensional regularization of the gravitational interaction of point masses,” *Phys. Lett. B* **513** (2001) 147–155, [arXiv:gr-qc/0105038](#).
- [82] T. Damour, P. Jaranowski, and G. Schäfer, “Hamiltonian of two spinning compact bodies with next-to-leading order gravitational spin-orbit coupling,” *Phys. Rev. D* **77** (2008) 064032, [arXiv:0711.1048 \[gr-qc\]](#).
- [83] J. Steinhoff, G. Schäfer, and S. Hergt, “ADM canonical formalism for gravitating spinning objects,” *Phys. Rev. D* **77** (2008) 104018, [arXiv:0805.3136 \[gr-qc\]](#).
- [84] M. Levi, “Next-to-leading order gravitational spin1-spin2 coupling with Kaluza-Klein reduction,” *Phys. Rev. D* **82** (2010) 064029, [arXiv:0802.1508 \[gr-qc\]](#).

- [85] K. S. Thorne and J. B. Hartle, “Laws of motion and precession for black holes and other bodies,” *Phys. Rev. D* **31** (1985) 1815–1837.
- [86] S. Hergt and G. Schäfer, “Higher-order-in-spin interaction Hamiltonians for binary black holes from source terms of Kerr geometry in approximate ADM coordinates,” *Phys. Rev. D* **77** (2008) 104001, [arXiv:0712.1515 \[gr-qc\]](#).
- [87] S. Hergt and G. Schäfer, “Higher-order-in-spin interaction Hamiltonians for binary black holes from Poincaré invariance,” *Phys. Rev. D* **78** (2008) 124004, [arXiv:0809.2208 \[gr-qc\]](#).
- [88] J. Steinhoff and G. Schäfer, “Canonical formulation of self-gravitating spinning-object systems,” *Europhys. Lett.* **87** (2009) 50004, [arXiv:0907.1967 \[gr-qc\]](#).
- [89] J. Steinhoff and H. Wang, “Canonical formulation of gravitating spinning objects at 3.5 post-Newtonian order,” *Phys. Rev. D* **81** (2010) 024022, [arXiv:0910.1008 \[gr-qc\]](#).
- [90] W. G. Laarakkers and E. Poisson, “Quadrupole moments of rotating neutron stars,” *Astrophys. J.* **512** (1999) 282–287, [arXiv:gr-qc/9709033](#).
- [91] J. Levin, “Gravity waves, chaos, and spinning compact binaries,” *Phys. Rev. Lett.* **84** (2000) 3515–3518, [arXiv:gr-qc/9910040](#).
- [92] C. Sohr, “Chaos in gravitativ-gebundenen Binärsystemen mit Spin.,” diploma thesis, Friedrich-Schiller-Universität Jena, 2009. unpublished.
- [93] P. A. M. Dirac, *Lectures on Quantum Mechanics*. Yeshiva University Press, New York, 1964.
- [94] H. Goldstein, *Classical Mechanics*. Addison-Wesley, Cambridge, 2nd ed., 1981.
- [95] P. Colwell, *Solving Kepler’s equation over three centuries*. Willman-Bell, Inc., Richmond, VA 23235, 1993.
- [96] D. V. Gal’tsov, A. A. Matiukhin, and V. I. Petukhov, “Relativistic corrections to the gravitational radiation of a binary system and the fine structure of the spectrum,” *Phys. Lett. A* **77** (1980) 387.
- [97] T. Damour and N. Deruelle, “General relativistic celestial mechanics of binary systems. I. the post-Newtonian motion.,” *Ann. Inst. H. Poincaré A* **43** (1985) 107–132. http://www.numdam.org/item?id=AIHPA_1985__43_1_107_0.
- [98] G. Schäfer and N. Wex, “Second post-Newtonian motion of compact binaries,” *Phys. Lett. A* **174** (1993) 196–205.

- [99] D. Hansen, “Next to leading-order gravitational wave emission and the dynamical evolution of close binary systems with spin,” *Class. Quant. Grav.* **25** (2008) 165011, [arXiv:0710.0258 \[gr-qc\]](#).
- [100] B. Mikóczi, M. Vasúth, and L. Á. Gergely, “Self-interaction spin effects in inspiralling compact binaries,” *Phys. Rev. D* **71** (2005) 124043, [arXiv:astro-ph/0504538](#).
- [101] J. Majár and M. Vasúth, “Gravitational waveforms from a Lense-Thirring system,” *Phys. Rev. D* **74** (2006) 124007, [arXiv:gr-qc/0611105](#).
- [102] M. Vasúth and J. Majár, “Gravitational waveforms for finite mass binaries,” *Int. J. Mod. Phys. A* **22** (2007) 2405–2414, [arXiv:0705.3481 \[gr-qc\]](#).
- [103] J. Majár and M. Vasúth, “Gravitational waveforms for spinning compact binaries,” *Phys. Rev. D* **77** (2008) 104005, [arXiv:0806.2273 \[gr-qc\]](#).
- [104] L. Á. Gergely, “Spinning compact binary inspiral: Independent variables and dynamically preserved spin configurations,” *Phys. Rev. D* **81** (2009) 084025, [arXiv:0912.0459](#).
- [105] M. Tessmer, “Gravitational waveforms from unequal-mass binaries with arbitrary spins under leading order spin-orbit coupling,” *Phys. Rev. D* **80** (2009) 124034, [arXiv:0910.5931 \[gr-qc\]](#).
- [106] L. E. Kidder, “Coalescing binary systems of compact objects to (post)^{5/2}-Newtonian order. V. Spin effects,” *Phys. Rev. D* **52** (1995) 821–847, [arXiv:gr-qc/9506022](#).
- [107] T. Damour, P. Jaranowski, and G. Schäfer, “Equivalence between the ADM-Hamiltonian and the harmonic-coordinates approaches to the third post-Newtonian dynamics of compact binaries,” *Phys. Rev. D* **63** (2001) 044021, [arXiv:gr-qc/0010040](#).
- [108] C. Königsdörffer and A. Gopakumar, “Post-Newtonian accurate parametric solution to the dynamics of spinning compact binaries in eccentric orbits: The leading order spin-orbit interaction,” *Phys. Rev. D* **71** (2005) 024039, [arXiv:gr-qc/0501011](#).
- [109] N. Wex, “The second post-Newtonian motion of compact binary-star systems with spin,” *Class. Quant. Grav.* **12** (1995) 983–1005.
- [110] T. Damour and G. Schäfer, “Higher-order relativistic periastron advances and binary pulsars,” *Nuovo Cim. B* **101** (1988) 127–176.
- [111] C. M. Will and A. G. Wiseman, “Gravitational radiation from compact binary systems: Gravitational waveforms and energy loss to second post-Newtonian order,” *Phys. Rev. D* **54** (1996) 4813–4848, [arXiv:gr-qc/9608012](#).

- [112] W. Gröbner, *Die Lie-Reihen und ihre Anwendungen*. VEB Deutscher Verlag der Wissenschaft, Berlin, 1960.
- [113] W. Gröbner and P. Lesky, *Mathematische Methoden der Physik II*. Bibliographisches Institut AG, Mannheim, 1965.
- [114] L. Rezzolla, E. Barausse, N. Dorband, D. Pollney, C. Reisswig, J. Seiler, and S. Husa, “Final spin from the coalescence of two black holes,” *Phys. Rev. D* **78** (2008) 044002, [ArXiv:arXiv:0710.3345v3](#) [[gr-qc](#)].
- [115] S. Bose, A. Gopakumar, and M. Tessmer, “Gravitational wave data analysis implications of TaylorEt inspiral approximants for ground-based detectors: the non-spinning case,” [arXiv:0807.2400](#) [[gr-qc](#)].
- [116] L. S. Finn, “Detection, measurement, and gravitational radiation,” *Phys. Rev. D* **46** (1992) 5236–5249, [arXiv:gr-qc/9609027](#).
- [117] C. Cutler and É. E. Flanagan, “Gravitational waves from merging compact binaries: How accurately can one extract the binary’s parameters from the inspiral waveform?,” *Phys. Rev. D* **49** (1994) 2658–2697, [arXiv:gr-qc/9402014](#).
- [118] P. Jaranowski and A. Królak, “Gravitational-wave data analysis,” *Living Rev. Relativity* **8** (2005) 3. <http://www.livingreviews.org/lrr-2005-3>.
- [119] B. S. Sathyaprakash and B. F. Schutz, “Physics, astrophysics and cosmology with gravitational waves,” *Living Rev. Relativity* **12** (2009) 2, [arXiv:0903.0338](#) [[gr-qc](#)]. <http://www.livingreviews.org/lrr-2009-2>.
- [120] A. Buonanno, Y. Chen, and M. Vallisneri, “Detecting gravitational waves from precessing binaries of spinning compact objects: Adiabatic limit,” *Phys. Rev. D* **67** (2003) 104025, [arXiv:gr-qc/0211087](#).
- [121] C. Moreno-Garrido, J. Buitrago, and E. Mediavilla, “Spectral analysis of the gravitational radiation emitted by binary systems in moderately eccentric orbits - application to coalescing binaries,” *Mon. Not. R. Astron. Soc.* **266** (1994) 16. <http://adsabs.harvard.edu/abs/1994MNRAS.266...16M>.
- [122] C. Moreno-Garrido, E. Mediavilla, and J. Buitrago, “Gravitational radiation from point masses in elliptical orbits: spectral analysis and orbital parameters,” *Mon. Not. R. Astron. Soc.* **274** (1995) 115–126. <http://adsabs.harvard.edu/abs/1995MNRAS.274..115M>.
- [123] V. Pierro, I. M. Pinto, E. Spallicci, E. Laserra, and F. Recano, “Fast and accurate computational tools for gravitational waveforms from binary stars with any orbital eccentricity,” *Mon. Not. R. Astron. Soc.* **325** (2001) 358–372.

- [124] N. Yunes, K. G. Arun, E. Berti, and C. M. Will, “Post-circular expansion of eccentric binary inspirals: Fourier-domain waveforms in the stationary phase approximation,” *Phys. Rev. D* **80** (2009) 084001, [arXiv:0906.0313](https://arxiv.org/abs/0906.0313) [gr-qc].
- [125] W. Junker and G. Schäfer, “Binary systems - higher order gravitational radiation damping and wave emission,” *Mon. Not. R. Astron. Soc.* **254** (1992) 146–164. <http://adsabs.harvard.edu/abs/1992MNRAS.254..146J>.
- [126] M. Tessmer and G. Schäfer, “Full-analytic frequency-domain 1pN-accurate gravitational wave forms from eccentric compact binaries,” *Phys. Rev. D* **82** (2010) 124064, [arXiv:1006.3714v2](https://arxiv.org/abs/1006.3714v2) [gr-qc].
- [127] M. Tessmer and G. Schäfer, “Full-analytic frequency-domain gravitational wave forms from eccentric compact binaries to 2PN accuracy,” *Ann. Phys. (Berlin)* **523** (2011) 813–864, [arXiv:1012.3894](https://arxiv.org/abs/1012.3894) [gr-qc].
- [128] L. Blanchet and T. Damour, “Radiative gravitational fields in general relativity I. General structure of the field outside the source,” *Phil. Trans. R. Soc. A* **320** (1986) 379–430. <http://www.jstor.org/stable/37878>.
- [129] K. G. Arun, L. Blanchet, B. R. Iyer, and M. S. S. Qusailah, “Inspiralling compact binaries in quasi-elliptical orbits: The complete third post-Newtonian energy flux,” *Phys. Rev. D* **77** (2008) 064035, [arXiv:0711.0302](https://arxiv.org/abs/0711.0302) [gr-qc].
- [130] G. N. Watson, *A treatise on the theory of Bessel functions*. Cambridge University Press, Cambridge, 2nd ed., 1980.
- [131] “Wolfram MathWorld.” Electronic adress, 05, 2011. <http://mathworld.wolfram.com>.
- [132] P. C. Peters and J. Mathews, “Gravitational Radiation from Point Masses in a Keplerian Orbit,” *Phys. Rev.* **131** (1963) 435–440.
- [133] P. C. Peters, “Gravitational Radiation and the Motion of Two Point Masses,” *Phys. Rev.* **136** (1964) B1224–B1232.
- [134] T. Damour and B. R. Iyer, “Multipole analysis for electromagnetism and linearized gravity with irreducible Cartesian tensors,” *Phys. Rev. D* **43** (1991) 3259–3272.
- [135] V. Pierro and I. M. Pinto, “Exact solution of Peters-Mathews equations for any orbital eccentricity,” *Nuovo Cim. B* **111** (1996) 631–644.
- [136] J. M. A. Danby and T. M. Burkhardt, “The solution of Kepler’s equation, i,” *Celestial Mechanics* **31** (1983) 95–107.

- [137] M. Tessmer and A. Gopakumar, “Accurate and efficient gravitational waveforms for certain galactic compact binaries,” *Mon. Not. R. Astron. Soc.* **374** (2007) 721–728, [arXiv:gr-qc/0610139](#).
- [138] K. G. Arun, B. R. Iyer, B. S. Sathyaprakash, and P. A. Sundararajan, “Parameter estimation of inspiralling compact binaries using 3.5 post-Newtonian gravitational wave phasing: The non-spinning case,” *Phys. Rev. D* **71** (2005) 084008, [arXiv:gr-qc/0411146](#).
- [139] T. Damour, B. R. Iyer, and B. S. Sathyaprakash, “Frequency-domain p-approximant filters for time-truncated inspiral gravitational wave signals from compact binaries,” *Phys. Rev. D* **62** (2000) 084036, [arXiv:gr-qc/0001023](#).
- [140] T. Damour, B. R. Iyer, and B. S. Sathyaprakash, “Comparison of search templates for gravitational waves from binary inspiral,” *Phys. Rev. D* **63** (2001) 044023, [arXiv:gr-qc/0010009](#).
- [141] T. Damour, B. R. Iyer, and B. S. Sathyaprakash, “Erratum: Comparison of search templates for gravitational waves from binary inspiral,” *Phys. Rev. D* **72** (2005) 029902(E).
- [142] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, *NIST Handbook of Mathematical Functions*. Cambridge University Press, Cambridge, 1st ed., 2010.
- [143] A. Deprit, “Canonical transformations depending on a small parameter,” *Celestial Mechanics and Dynamical Astronomy* **1** (1968) 12–30.
- [144] E. Mai, M. Schneider, and C. Cui, *Zur Entwicklung von Bahntheorien – Methodik und Anwendung*, vol. 122 of *A Theoretische Geodäsie*. Verlag der Bayerischen Akademie der Wissenschaften, München, Germany, 2008.
- [145] L. Blanchet and G. Schäfer, “Gravitational wave tails and binary star systems,” *Class. Quant. Grav.* **10** (1993) 2699.
- [146] J. Schwinger, “Quantized gravitational field. II,” *Phys. Rev.* **132** (1963) 1317–1321.
- [147] T. Damour and G. Schäfer, “Lagrangians for n point masses at the second post-Newtonian approximation of general relativity,” *Gen. Relativ. Gravit.* **17** (1985) 879–905.

E Lebenslauf

Persönliche Daten

Name: Manuel Tessmer
Geburtsdatum: 09. 01.1981
Geburtsort: Mühlhausen, Thüringen
Familienstand: ledig

Schulbildung

Sept. 1987 - Juni 1991 Grundschole in Mühlhausen
Sept. 1991 - Juni 1999 Albert-Schweitzer-Gymnasium in Mühlhausen
Abschluss: Abitur

Wehrdienst

Sept. 1999 - Juli 2001 Grundwehrdienst

Studium

Sept. 2001 - Nov. 2006 Studium der Physik an der
Friedrich-Schiller-Universität (FSU) Jena
Vertiefungsrichtung: Gravitationstheorie
Nov. 2006 Abschluss: Physik Diplom
Thema der Diplomarbeit:
“Computationally efficient gravitational waveforms for
certain galactic compact binaries with arbitrary mass ratio
in eccentric orbits”
Beteuender Hochschullehrer: Prof. Dr. G. Schäfer

Promotionsstudium

seit Nov. 2006 Doktorand und wissenschaftlicher Mitarbeiter
am Theoretisch-Physikalischen Institut der FSU Jena
Beteuender Hochschullehrer: Prof. Dr. G. Schäfer

Lehrtätigkeit

Mathematischer Vorkurs für Physiker, WS 2007/2008
Kosmologie, SS 2008 und SS 2010
Quantentheorie für Lehramt, WS 2008/2009
Theoretische Mechanik, WS 2009/2010

F Ehrenwörtliche Erklärung

Ich erkläre hiermit ehrenwörtlich, dass ich die vorliegende Arbeit selbständig, ohne unzulässige Hilfe Dritter und ohne Benutzung anderer als der angegebenen Hilfsmittel und Literatur angefertigt habe. Die aus anderen Quellen direkt oder indirekt übernommenen Daten und Konzepte sind unter Angabe der Quellen gekennzeichnet.

Bei der Auswahl und Auswertung folgenden Materials haben mir die nachstehenden aufgeführten Personen in der jeweils beschriebenen Weise unentgeltlich geholfen:

- Herr Doktor Jan Steinhoff stellte mir im Rahmen unserer Zusammenarbeit Teile seines *Mathematica*-Verwaltungscodes zur Verfügung, der die Arbeit mit tensoriellen Größen erleichtert. Diesen entwickelte ich weiter, um meine Berechnungen durchzuführen.

Weitere Personen waren an der inhaltlich-materiellen Erstellung der vorliegenden Arbeit nicht beteiligt. Insbesondere habe ich hierfür nicht die entgeltliche Hilfe von Vermittlungs- bzw. Beratungsdiensten (Promotionsberater oder andere Personen) in Anspruch genommen. Niemand hat von mir unmittelbar oder mittelbar geldwerte Leistungen für Arbeiten erhalten, die im Zusammenhang mit dem Inhalt der vorgelegten Dissertation stehen.

Die Arbeit wurde bisher weder im In- noch im Ausland in gleicher oder ähnlicher Form einer anderen Prüfungsbehörde vorgelegt.

Die geltende Promotionsordnung der Physikalisch-Astronomischen Fakultät ist mir bekannt. Ich versichere ehrenwörtlich, dass ich nach bestem Wissen und die reine Wahrheit gesagt und nichts verschwiegen habe.

Jena, 06. Juni 2011