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Robustness of stability of time-varying index-1 DAEs*

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Abstract

We study exponential stability and its robustness for time-varying linear index-1 differential-algebraic equations. The effect of perturbations in the leading coefficient matrix is investigated. An appropriate class of allowable perturbations is introduced. Robustness of exponential stability with respect to a certain class of perturbations is proved in terms of the Bohl exponent and perturbation operator. Finally, a stability radius involving these perturbations is introduced and investigated. In particular, a lower bound for the stability radius is derived. The results are presented by means of illustrative examples.

Keywords: Time-varying linear differential-algebraic equations, exponential stability, robustness, Bohl exponent, perturbation operator, stability radius

1 Introduction

We study exponential stability and its robustness for time-varying linear differential-algebraic equations (DAEs) of the form

$$E(t)\dot{x} = A(t)x, \tag{1.1}$$

where $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$, $n \in \mathbb{N}$. For brevity, we identify the tuple (E, A) with the DAE (1.1). For the analysis it also important to consider the inhomogeneous system

$$E(t)\dot{x} = A(t)x + f(t), \tag{1.2}$$

where $f \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^n)$.

DAEs have been discovered to be the appropriate tool for modeling a vast variety of problems e.g. in mechanical engineering [2, 24, 50], multibody dynamics [20, 55], electrical networks [19, 48, 53] and chemical engineering [13, 16, 49], which often cannot be modeled by standard ordinary differential equations (ODEs).

In this work we concentrate on linear time-varying index-1 DAEs, which are, roughly speaking, those DAEs which are decomposable into a differential and an algebraic part and no derivatives of the algebraic variables appear in the decomposed system. The consideration of index-1 DAEs[1] is relevant as in a lot of applications the occurring DAEs are naturally of index-1. For instance, it is shown in [21] that any passive electrical circuit containing nonlinear and possibly time-varying elements has index

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[1]Note that in this article the term "index-1" includes all (implicit) ODE systems, that is all DAE systems which are sometimes referred to as index-0.

less than or equal to two - and the index-2 case is exceptional. Furthermore, so called hybrid models of electrical circuits are always index-1 [56, 34]. Therefore, our approach to index-1 DAEs has a wide area of applications e.g. in electrical engineering, as linear DAEs (E, A) arise as linearizations of nonlinear DAEs $F(t, x, \dot{x}) = 0$ along trajectories [11].
Furthermore, we are investigating the perturbation theory of DAEs (1.1), and, as explained in [12, Rem. 3.2], higher index DAEs are very sensitive to perturbations since they contain hidden constraints which involve higher derivatives of the solutions components. As shown in [5, Sec. 5.3] for a time-invariant index-2 example, arbitrary small perturbations of the matrix A in (1.1) may destroy exponential stability. This is why most of the stability results for DAEs are obtained for index-1 systems, see e.g. [1, 10, 12, 17, 18, 23, 27, 39, 43, 51, 57], because in index-1 DAEs (E, A) exponential stability is robust with respect to perturbations in the matrix A. When higher index DAEs are considered, additional assumptions have to be made [28, 44, 46, 57]. It is also possible to reformulate the DAE by applying some index reduction method in order to obtain a lower-index DAE with the same solution set, see e.g. [36, 37, 40]. However, to the author's best knowledge, the only results on the perturbation theory of higher index DAEs (and DAEs where an index cannot be defined, resp.) so far are given in [5] - and only for perturbations in A.
Among all the available index concepts for DAEs [9, 25, 26, 36, 45], the tractability index as introduced in [42] turned out to be the most suitable for dealing with perturbations in the leading coefficient matrix E of (1.1). This is because the way it allows for the decoupling of the DAE in a differential and an algebraic part via certain projectors enables us to reuse the same projectors for the perturbed system under some appropriate assumptions. This makes a proper analysis of the perturbation problem possible. Moreover, in this approach it is not necessary to carry out any state space transformations.
The present paper is concerned with perturbations in the leading coefficient matrix E. In perturbation theory of DAEs it is usually assumed that the leading coefficient E is not perturbed at all, see e.g. [12, 17, 18, 22, 51]. Even in the time-invariant setting, only very few authors have investigated the effects of perturbations in the leading coefficient, see [8, 10, 15]. For time-varying DAEs, the only work where also perturbations in the leading coefficient are allowed is [40]. The main reason why perturbations in the leading term are usually not considered in the DAE community is that even in the time-invariant index-1 case exponential stability is very sensitive with respect to such perturbations, see [10]. Byers and Nichols [10] gave the first systematic approach to this problem by introducing a class of "allowable perturbations". In the present article we will generalize their results to time-varying systems in a certain sense, see Section 6. Bracke [8] also generalized the approach of [10] within the setting of time-invariant DAEs to obtain a better treatment of higher index DAEs.
The present paper was inspired by the work of Chyan et al. [12] and Du et al. [18], who introduced a stability radius and developed a perturbation theory for time-varying DAEs, and also by the work of Hinrichsen et al. [29], who developed a comprehensive perturbation theory for time-varying ODEs. As anticipated in [12, Sec. 6], robustness results for perturbations in the leading coefficient of a DAE may be obtained for a certain class of perturbations. It is the first aim of the present paper to introduce a class of allowable perturbations in the leading coefficient and then prove robustness of exponential stability with respect to these perturbations using the Bohl exponent and perturbation operator. The second aim is to introduce a stability radius for time-varying DAEs. The stability radius defined in [12, 18] is defined only with respect to perturbations in the coefficients of A. On the other hand, [10] give a definition for the stability radius involving perturbations in E for time-invariant DAEs. Our definition of the stability radius can be viewed as both, a generalization of the definition given in [10] for time-varying systems and as a generalization of the definition given in [12, 18] for a larger set of allowable perturbations with respect to the leading coefficient. We then investigate this new stability radius and in particular prove a lower bound. As far as the author is aware, these results are even new for time-invariant systems.

The paper is organized as follows: In Section 2 we state the class of DAEs we consider in this article, that is DAEs of tractability index-1. We further give the fundamental statements about the decomposition of the DAE via the projectors, initial conditions, transition matrix and a variation of constants formula. The perturbation problem is outlined in Section 3 and the class of allowable perturbations defined. By means of an example it is shown that the class of perturbations is sufficiently large. The notion of Bohl exponent for DAEs is recapitulated in Section 4, along with statements about the equivalence of a negative Bohl exponent and exponential stability, and it is shown in Theorem 4.8 that the Bohl exponent is robust with respect to perturbations introduced in Section 3. In Section 5 we introduce the perturbation operator for the DAE (1.1) and, after recapitulating some of its properties, we show in Theorem 5.4 that its norm can be used to determine a bound ρ such that, roughly speaking, exponential stability is preserved for any perturbation with norm less than ρ. We also prove another robustness result which incorporates the norm of the perturbation operator in Theorem 5.8. We close Section 5 with a recapitulation of the results for the important class of semi-explicit index-1 DAEs. In Section 6 we introduce a stability radius for index-1 DAEs and prove essential properties. The main theorem of this section is Theorem 6.11 which provides a lower bound for the stability radius. This lower bound then enables us to prove a statement about certain subsets of exponentially stable index-1 DAEs being open in the respective supersets. We close the paper with some conclusions and open questions in Section 7. Note that the results obtained in this article are new even for time-invariant systems.

Nomenclature

$\mathbb{N},\ \mathbb{N}_0$		the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
$\mathbb{R}_+$	$:=$	$[0, \infty)$
$\operatorname{im} A, \ker A$		the image and kernel of the matrix $A \in \mathbb{R}^{m \times n}$, resp.
$\mathbf{Gl}_n(\mathbb{R})$		the general linear group of degree n, i.e., the set of all invertible $n \times n$ matrices over $\mathbb{R}$
$\|x\|$	$:=$	$\sqrt{x^\top x}$, the Euclidean norm of $x \in \mathbb{R}^n$
$\|A\|$	$:=$	$\sup\{\ \|Ax\| \mid \|x\| = 1\ \}$, induced matrix norm of $A \in \mathbb{R}^{n \times m}$
$\mathcal{C}(\mathcal{I}; \mathcal{S})$		the set of continuous functions $f : \mathcal{I} \to \mathcal{S}$ from a set $\mathcal{I} \subseteq \mathbb{R}$ to a vector space $\mathcal{S}$
$\mathcal{C}^k(\mathcal{I}; \mathcal{S})$		the set of k-times continuously differentiable functions $f : \mathcal{I} \to \mathcal{S}$ from a set $\mathcal{I} \subseteq \mathbb{R}$ to a vector space $\mathcal{S}$
$\mathcal{B}(\mathcal{I}; \mathcal{S})$		the set of continuous and bounded functions $f : \mathcal{I} \to \mathcal{S}$ from a set $\mathcal{I} \subseteq \mathbb{R}$ to a vector space $\mathcal{S}$
$\mathbb{1}_{\mathcal{M}}(t)$	$:=$	$\begin{cases} 1, & \text{if } t \in \mathcal{M}, \\ 0, & \text{otherwise,} \end{cases}$ for $t \in \mathbb{R}_+$ and $\mathcal{M} \subseteq \mathbb{R}_+$
$\operatorname{dom} f$		the domain of the function f
$\|f\|_\infty$	$:=$	$\sup\{\ \|f(t)\| \mid t \in \operatorname{dom} f\ \}$ the infinity norm of the function f
$f\mid_{\mathcal{M}}$		the restriction of the function f on a set $\mathcal{M} \subseteq \operatorname{dom} f$
$L^2(\mathcal{I}; \mathcal{S})$		the set of measurable and square integrable functions $f : \mathcal{I} \to \mathcal{S}$ from a set $\mathcal{I} \subseteq \mathbb{R}$ to a vector space $\mathcal{S}$
$\|f\|_{L^2[t_0,\infty)}$	$:=$	$\left(\int_{t_0}^{\infty} \|f(t)\|^2 \, \mathrm{d}t\right)^2$ the L^2-norm of the function $f \in L^2([t_0, \infty); \mathcal{S})$, $t_0 \in \mathbb{R}$

2 Index-1 DAEs

The property of a DAE (1.1) to be of differentiation index 1 [9], tractability index 1 [42] or strangeness free [36] resp., is always, roughly speaking, the property of the DAE being decomposable into a differential and an algebraic part and no derivatives of the algebraic variables appear in the decomposed system. As indicated in the introduction we use the tractability index setting to define the "index-1 property" used in this paper.
The tractability index has first been introduced for nonlinear DAEs in [25]. Later, the tractability index setting was stated in a more comprehensive way for linear DAEs in [42]. This theory has then been further developed for DAEs with properly stated leading term [3, 4, 45].
In this section we introduce the set of projector functions necessary to introduce the index and derive properties of it. In particular a simple algorithm is given, which checks the index-1 property of a given DAE and, if satisfied, calculates a corresponding projector. We further give the fundamental statements about the decomposition of the DAE via the projectors, initial conditions, transition matrix and a variation of constants formula.
While we chose a different exposition in this article, the theory presented in this section (except for Algorithm 1 and most parts of Sections 2.3 and 2.4) is essentially the same as the one developed in [42], see also [3, 4, 12, 18, 25, 45]. However, we aim for a comprehensive and self-contained presentation.

2.1 Projectors and index property

In order to define the index-1 property of a DAE (E, A) we introduce the set $\mathfrak{Q}_{E,A}$ of special projector functions as follows.

Definition 2.1 (Projector functions)**.** Let $(E, A) \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^{n\times n})^2$ be given. Define

$$\mathfrak{Q}_{E,A} := \left\{ Q \in \mathcal{C}^1(\mathbb{R}_+;\mathbb{R}^{n\times n}) \;\middle|\; \begin{array}{l} \forall\, t \in \mathbb{R}_+ :\ Q(t)^2 = Q(t) \ \wedge\ \ker E(t) = \operatorname{im} Q(t), \\ E + (E\dot{Q} - A)Q \in \mathcal{C}(\mathbb{R}_+;\mathbf{Gl}_n(\mathbb{R})) \end{array} \right\}.$$

◇

Remark 2.2 (Properties of $\mathfrak{Q}_{E,A}$)**.** Let $(E, A) \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^{n\times n})^2$. Then we have the following properties:

(i) $\forall\, Q_1, Q_2 \in \mathfrak{Q}_{E,A} :\ Q_1Q_2 = Q_2 \ \wedge\ Q_2Q_1 = Q_1$.

(ii) $\forall\, Q \in \mathfrak{Q}_{E,A} :\ \frac{\mathrm{d}}{\mathrm{d}t}(\operatorname{rk} Q) = 0$.

While property (i) is immediate from the fact that all elements in $\mathfrak{Q}_{E,A}$ are projectors onto the same set, property (ii) needs a short proof: As $Q(t)$ is idempotent it follows $\operatorname{rk} Q(t) = \operatorname{tr} Q(t)$ for all $t \in \mathbb{R}_+$ and the latter term is continuous, hence Q has constant rank. ◇

Assuming that $\mathfrak{Q}_{E,A} \neq \emptyset$ and incorporating a projector $Q \in \mathfrak{Q}_{E,A}$, we may immediately rewrite (1.2) as

$$E\tfrac{\mathrm{d}}{\mathrm{d}t}((I - Q)x) = (A - E\dot{Q})x + f. \tag{2.1}$$

The next aim is to define a solution of (1.2) in terms of (2.1), but this requires to prove that (the solutions of) equation (2.1) are independent of the choice of Q.

Lemma 2.3 (Independence of the projector)**.** *Let $(E, A) \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^{n\times n})^2$, $f \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^n)$ and assume that $\mathfrak{Q}_{E,A} \neq \emptyset$. Then, for all $Q_1, Q_2 \in \mathfrak{Q}_{E,A}$ and $P_i := I - Q_i$, $i = 1, 2$, we have, for all $x \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^n)$,*

$$\left\{ \begin{array}{l} \forall\, t \in \mathbb{R}_+ :\ E(t)\tfrac{\mathrm{d}}{\mathrm{d}t}(P_1(t))x(t)) = (A(t) + E(t)\dot{P}_1(t))x(t) + f(t) \\ \textit{and } P_1x \in \mathcal{C}^1(\mathbb{R}_+;\mathbb{R}^n) \end{array} \right.$$

$$\iff \left\{ \begin{array}{l} \forall\, t \in \mathbb{R}_+ :\ E(t)\tfrac{\mathrm{d}}{\mathrm{d}t}(P_2(t))x(t)) = (A(t) + E(t)\dot{P}_2(t))x(t) + f(t) \\ \textit{and } P_2x \in \mathcal{C}^1(\mathbb{R}_+;\mathbb{R}^n). \end{array} \right.$$

Proof: It suffices to show one direction. Let $x \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^n)$ such that it solves (2.1) for $Q = Q_1$ and $P_1x \in \mathcal{C}^1(\mathbb{R}_+;\mathbb{R}^n)$. Observe that, since $Q_2Q_1 = Q_1$,

$$P_2P_1 = (I - Q_2)(I - Q_1) = (I - Q_2 - Q_1 + Q_2Q_1) = (I - Q_2) = P_2. \tag{2.2}$$

This immediately yields that $P_2x = P_2P_1x \in \mathcal{C}^1(\mathbb{R}_+;\mathbb{R}^n)$. Now proceeding as in the proof of [25, Lem. 11, p. 32], we show that

$$E\left(\tfrac{\mathrm{d}}{\mathrm{d}t}(P_1x) - \dot{P}_1x\right) = E\left(\tfrac{\mathrm{d}}{\mathrm{d}t}(P_2x) - \dot{P}_2x\right),$$

from which the assertion follows. This however is immediate from

$$\begin{aligned} E\left(\tfrac{\mathrm{d}}{\mathrm{d}t}(P_1x) - \dot{P}_1x\right) = EP_2\left(\tfrac{\mathrm{d}}{\mathrm{d}t}(P_1x) - \dot{P}_1x\right) &= E\left(\tfrac{\mathrm{d}}{\mathrm{d}t}(P_2P_1x) - \dot{P}_2P_1x - P_2\dot{P}_1x\right) \\ &= E\left(\tfrac{\mathrm{d}}{\mathrm{d}t}(P_2P_1x) - \tfrac{\mathrm{d}}{\mathrm{d}t}(P_2P_1)x\right) \overset{(2.2)}{=} E\left(\tfrac{\mathrm{d}}{\mathrm{d}t}(P_2x) - \dot{P}_2x\right). \end{aligned}$$

□

By Lemma 2.3 the following set of solutions of (1.2) is well-defined.

Definition 2.4 (Solution space)**.** Let $(E, A) \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^{n\times n})^2$, $f \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^n)$ and assume that there exists some $Q \in \mathfrak{Q}_{E,A}$. We call a function $x : \mathbb{R}_+ \to \mathbb{R}^n$ a *solution* of (1.2) if, and only if,

$$x \in \mathcal{C}^1_{E,A,f} \quad := \quad \left\{\ x \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^n)\ \middle|\ (I - Q)x \in \mathcal{C}^1(\mathbb{R}_+;\mathbb{R}^n) \text{ and } x \text{ solves (2.1) for all } t \in \mathbb{R}_+\ \right\}.$$

◇

Note that this solution concept does only incorporate global solutions and does not account for possible local solutions, which however must be expected for time-varying DAEs, see e.g. [5, 6]. This is reasonable since any local solution can be uniquely extended to a global solution for the class of DAEs that we will consider, see Definition 2.6. We show this property in Lemma 2.15.

Remark 2.5 (Properly stated leading term)**.** The reformulated DAE (2.1) is a DAE with so called *properly stated leading term* (see e.g. [3, 4, 45]), because the coefficients of the leading term are well matched, that is $\ker E(t) \oplus \operatorname{im}(I - Q)(t) = \mathbb{R}^n$ for all $t \in \mathbb{R}_+$. ◇

We now define the notion of an index-1 DAE.

Definition 2.6 (Index-1 DAE)**.** The DAE $(E, A) \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^{n\times n})^2$ is called *index-1* if, and only if, $\mathfrak{Q}_{E,A} \neq \emptyset$. ◇

Note that by Definition 2.6 the set of index-1 DAEs includes all implicit ODEs, i.e., any system (1.2), where $E \in \mathcal{C}(\mathbb{R}_+;\mathbf{Gl}_n(\mathbb{R}))$, even though such systems are often referred to as index-0 in the literature. In Lemma 2.12 we show that any index-1 DAE is decomposable into a differential and an algebraic part, which then justifies this notion. Another justification is given in the following remark.

Remark 2.7 (Index-1)**.** Let $(E, A) \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^{n\times n})^2$. Then $\mathfrak{Q}_{E,A} \neq \emptyset$ if, and only if, (E, A) is *index-1 tractable* in the sense of the definition on page 154 in [42]. For a discussion of the tractability index concept in relation to other index concepts, such as the differentiation index [9] or the strangeness index [36], see [38, Secs. 2.10 & 3.10] and [47]. ◇

The following proposition is important for later purposes and gives more insight into the set $\mathfrak{Q}_{E,A}$.

Proposition 2.8 (Index-1 and projectors on $\ker E$)**.** *Let* $(E,A) \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^{n\times n})^2$. *Then*

$$\mathfrak{Q}_{E,A} \neq \emptyset \implies \mathfrak{Q}_{E,A} = \left\{ \ Q \in \mathcal{C}^1(\mathbb{R}_+;\mathbb{R}^{n\times n}) \ \middle| \ \forall\, t \in \mathbb{R}_+ : \ Q(t)^2 = Q(t) \ \wedge \ \ker E(t) = \operatorname{im} Q(t) \ \right\}.$$

Proof: Follows from a pointwise application of [25, Thm. A.13]. □

The existence of a projector Q onto $\ker E$ can be checked via the following lemma in the case of differentiable E.

Lemma 2.9 (Projector on $\ker E$ and rank of E)**.** *Let* $(E,A) \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^{n\times n})^2$ *and suppose that* $E \in \mathcal{C}^1(\mathbb{R}_+;\mathbb{R}^{n\times n})$. *Then*

$$\begin{array}{l} \exists\, Q \in \mathcal{C}^1(\mathbb{R}_+;\mathbb{R}^{n\times n}) \ \forall\, t \in \mathbb{R}_+ : \\ Q(t)^2 = Q(t) \ \wedge \ \ker E(t) = \operatorname{im} Q(t) \end{array} \iff \exists\, r \leq n \ \forall\, t \in \mathbb{R}_+ : \ \operatorname{rk} E(t) = r.$$

In any of the above cases, there exists a continuously differentiable projector Q onto $\ker E$ *which is bounded.*

Proof: "$\Rightarrow$": As in Remark 2.2 we may deduce that Q has constant rank and therefore E has constant rank.
"$\Leftarrow$": Since E is continuously differentiable and has constant rank if follows from [33, Thm. A.1] that there exists a projector Q onto $\ker E$.
The boundedness of Q can be inferred from [33, Thm. A.1] as well. □

Unfortunately, there is no algorithm to calculate the bounded projector Q onto $\ker E$ whose existence follows from Lemma 2.9 for continuously differentiable E with constant rank. Nevertheless, if (E,A) is real-analytic, then the calculation of a projector Q is feasible by the following algorithm, which is motivated by Proposition 2.8 and can be used for checking the index-1 property and calculating a corresponding projector $Q \in \mathfrak{Q}_{E,A}$.

Algorithm 1 Calculation of $Q \in \mathfrak{Q}_{E,A}$

1: **function** $Q = \mathbf{getQ}(E,A)$
2: determine minimal $r \leq n := \text{size}(E)$ s.t. $\operatorname{rk} E(t) \leq r$ for all $t \in \mathbb{R}_+$ and real analytic $\tilde{Q} : \mathbb{R}_+ \to \mathbb{R}^{n\times(n-r)}$ with pointwise full column rank s.t. $E\tilde{Q} = 0$;
3: **if** not$(\forall\, t \in \mathbb{R}_+ : \ \operatorname{rk} E(t) = r)$ **then**
4:    **print** "DAE is not index-1!" **STOP**
5: **end if**
6: $Q := \tilde{Q}(\tilde{Q}^\top \tilde{Q})^{-1}\tilde{Q}^\top$;
7: **if** $E + (E\dot{Q} - A)Q \notin \mathcal{C}(\mathbb{R}_+;\mathbf{Gl}_n(\mathbb{R}))$ **then**
8:    **print** "DAE is not index-1!" **STOP**
9: **end if**

Proposition 2.10 (Correctness of the algorithm)**.** *Let* $E, A : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ *be real-analytic. Then Algorithm 1 terminates after finitely many steps with either "DAE is not index-1!" or it returns a real-analytic matrix* $Q \in \mathfrak{Q}_{E,A}$.

Proof: Feasibility of line 2 of Algorithm 1 is due to [54, Thm. 1]. If the test in line 3 fails, then E does not have constant rank and hence (E,A) cannot be index-1. In the case the test does not fail, we have

$$Q \in \left\{ \ Q \in \mathcal{C}^1(\mathbb{R}_+;\mathbb{R}^{n\times n}) \ \middle| \ \forall\, t \in \mathbb{R}_+ : \ Q(t)^2 = Q(t) \ \wedge \ \ker E(t) = \operatorname{im} Q(t) \ \right\},$$

for Q defined in line 6, which can be seen as follows: Clearly $(\tilde{Q}^\top\tilde{Q})^{-1}$ is well-defined and real-analytic and simple calculations yield $Q^2 = Q$. Furthermore, it is easy to observe that $\operatorname{im} Q(t) \subseteq \ker E(t)$ for all $t \in \mathbb{R}_+$. For the opposite inclusion let $x \in \ker E(t)$, then $x = \tilde{Q}(t)y$ for some $y \in \mathbb{R}^{n-r}$ and since $\operatorname{im}\tilde{Q}(t)^\top = \mathbb{R}^{n-r}$ there exists $z \in \mathbb{R}^n$ such that $(\tilde{Q}(t)^\top\tilde{Q}(t))y = \tilde{Q}(t)^\top z$, hence $x = Q(t)z \in \operatorname{im} Q(t)$. If then the test in line 7 fails, the DAE is not index 1, i.e., $\mathfrak{Q}_{E,A} = \emptyset$, because otherwise we would have $Q \in \mathfrak{Q}_{E,A}$ by Proposition 2.8 and thus the test could not fail. If the test in line 7 is affirmative, we clearly have $Q \in \mathfrak{Q}_{E,A}$, i.e., the DAE is index 1, and the algorithm returns the real-analytic matrix Q as the computed projector. □

Remark 2.11.

(i) In practice, it is not easy to implement Algorithm 1 for the whole class of real analytic functions. The main problem is to find $\tilde{Q}$ such that the condition in line 2 of Algorithm 1 is satisfied. However, if (E, A) has polynomial entries, then there are efficient (actually, polynomial time) algorithms which solve this problem; see [52].

(ii) The test for invertibility of $E(t) + (E(t)\dot{Q}(t) - A(t))Q(t)$ for all $t \in \mathbb{R}_+$ in line 7 of Algorithm 1 cannot be reduced to the test of invertibility on a (finite) subset of $\mathbb{R}_+$. Consider for example the system $\left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & a(t)\end{bmatrix}\right)$ for $a \in \mathcal{C}(\mathbb{R}_+;\mathbb{R})$. Then we may choose $Q = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$ and obtain

$$\operatorname{rk}\big(E(t) + (E(t)\dot{Q}(t) - A(t))Q(t)\big) = 1 + \operatorname{rk} a(t).$$

The rank condition must then be checked for all $t \in \mathbb{R}_+$, because a can vanish at any point. ⋄

2.2 Decomposition of the DAE

We show that any index-1 DAE can be decomposed into a differential and an algebraic part.

Lemma 2.12 (Decomposition). *Let $(E, A) \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^{n\times n})^2$ be index-1, $Q \in \mathfrak{Q}_{E,A}$ and $f \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^n)$. Set*

$$P := I - Q, \quad \bar{A} := A - E\dot{Q}, \quad G := E + (E\dot{Q} - A)Q = E - \bar{A}Q. \tag{2.3}$$

Then, for $x \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^n)$, we have $x \in \mathcal{C}^1_{E,A,f}$ if, and only if, $Px \in \mathcal{C}^1(\mathbb{R}_+;\mathbb{R}^n)$ and x solves the following system for all $t \in \mathbb{R}_+$:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(P(t)x) &= (\dot{P}(t) + P(t)G(t)^{-1}\bar{A}(t))P(t)x + P(t)G(t)^{-1}f(t), \\ \quad Q(t)x &= Q(t)G(t)^{-1}\bar{A}(t)P(t)x + Q(t)G(t)^{-1}f(t). \end{cases} \tag{2.4}$$

Proof: First observe that for any $x \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^n)$ with $Px \in \mathcal{C}^1(\mathbb{R}_+;\mathbb{R}^n)$ we have

$$\begin{aligned} E\dot{x} = Ax + f \;\Leftrightarrow\; & E\tfrac{\mathrm{d}}{\mathrm{d}t}(Px) = (A + E\dot{P})x + f \\ \Leftrightarrow\; & (E - \bar{A}Q)(P\tfrac{\mathrm{d}}{\mathrm{d}t}(Px) + Qx) = \bar{A}Px + f \;\Leftrightarrow\; P\tfrac{\mathrm{d}}{\mathrm{d}t}(Px) = (G^{-1}\bar{A}P - Q)x + G^{-1}f. \end{aligned}$$

Now let $x \in \mathcal{C}^1_{E,A,f}$, then clearly $Px \in \mathcal{C}^1(\mathbb{R}_+;\mathbb{R}^n)$. Furthermore, $\frac{\mathrm{d}}{\mathrm{d}t}(Px) = \frac{\mathrm{d}}{\mathrm{d}t}(PPx) = \dot{P}Px + P\frac{\mathrm{d}}{\mathrm{d}t}(Px)$, whence $P\dot{P}Px = 0$ and $Q\frac{\mathrm{d}}{\mathrm{d}t}(Px) = Q\dot{P}Px$. Then, since x solves $P\frac{\mathrm{d}}{\mathrm{d}t}(Px) = (G^{-1}\bar{A}P - Q)x + G^{-1}f$, we find

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}(Px) = P\big(P\tfrac{\mathrm{d}}{\mathrm{d}t}(Px)\big) + Q\tfrac{\mathrm{d}}{\mathrm{d}t}(Px) = & \\ P(G^{-1}\bar{A}P - Q)x + PG^{-1}f + Q\dot{P}Px + P\dot{P}Px = (\dot{P} + PG^{-1}\bar{A})Px + PG^{-1}f. & \end{aligned}$$

Moreover,

$$0 = QP\tfrac{\mathrm{d}}{\mathrm{d}t}(Px) = Q(G^{-1}\bar{A}P - Q)x + QG^{-1}f = QG^{-1}\bar{A}Px - Qx + QG^{-1}f.$$

On the other hand, if $Px \in \mathcal{C}^1(\mathbb{R}_+;\mathbb{R}^n)$ and x solves (2.4), then

$$\begin{aligned} P\tfrac{\mathrm{d}}{\mathrm{d}t}(Px) &= P\dot{P}Px + PG^{-1}\bar{A}Px + PG^{-1}f = PG^{-1}\bar{A}Px - Qx + Qx + PG^{-1}f \\ &= PG^{-1}\bar{A}Px + QG^{-1}\bar{A}Px - Qx + PG^{-1}f + QG^{-1}f = (G^{-1}\bar{A}P - Q)x + G^{-1}f, \end{aligned}$$

that is $x \in \mathcal{C}^1_{E,A,f}$. □

It can be seen from Lemma 2.12 that, roughly speaking, the solutions of the index-1 DAE (E, A) can be calculated by solving an ODE for Px and then Qx (and therefore x) is given in terms of Px. Therefore, all solutions of the DAE (1.2) are fully determined by the solutions of the ODE (first equation) in (2.4). It is also important to note that no derivatives of the so called "algebraic variables" Qx are involved in (2.4), what justifies the use of the notion "index-1", cf. [42].
The first equation in (2.4) gives rise for the following definition.

Definition 2.13 (Inherent ODE)**.** Let $(E, A) \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^{n\times n})^2$ be index-1, $Q \in \mathfrak{Q}_{E,A}$, $P, \bar{A}, G$ as in (2.3) and $f \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^n)$. Then the equation

$$\dot{y} = (\dot{P}(t) + P(t)G(t)^{-1}\bar{A}(t))y + P(t)G(t)^{-1}f(t) \tag{2.5}$$

is called an *inherent ordinary differential equation* of (1.2). ⋄

Note that, of course, (2.5) depends on the choice of $Q \in \mathfrak{Q}_{E,A}$. The inherent ODE has the property that every solution starting in $\operatorname{im} P(t_0)$ for some $t_0 \in \mathbb{R}_+$ remains in $\operatorname{im} P(t)$ for all $t \in \mathbb{R}_+$ as we will show in the following.

Lemma 2.14 (Property of the inherent ODE)**.** *Let $(E, A) \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^{n\times n})^2$ be index-1, $Q \in \mathfrak{Q}_{E,A}$, $P, \bar{A}, G$ as in* (2.3) *and $f \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^n)$. If $t_0 \in \mathbb{R}_+$ and $y \in \mathcal{C}^1(\mathbb{R}_+;\mathbb{R}^n)$ is a solution of* (2.5) *with $y(t_0) \in \operatorname{im} P(t_0)$, then $y(t) \in \operatorname{im} P(t)$ for all $t \in \mathbb{R}_+$.*

Proof: Observe that any solution $y \in \mathcal{C}^1(\mathbb{R}_+;\mathbb{R}^n)$ of (2.5) satisfies

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}(Qy) = \dot{y} - \tfrac{\mathrm{d}}{\mathrm{d}t}(Py) &= (\dot{P} + PG^{-1}\bar{A})y + PG^{-1}f - \dot{P}y - P\big((\dot{P} + PG^{-1}\bar{A})y + PG^{-1}f\big) \\ &= -P\dot{P}y = (\dot{P}P - \dot{P})y = -\dot{P}(I-P)y = -\dot{P}(Qy), \end{aligned}$$

where we used that $\dot{P} = P\dot{P} + \dot{P}P$. Now given an initial condition $y(t_0) = P(t_0)x^0$, $x^0 \in \mathbb{R}^n$, we obtain $(Qy)(t_0) = 0$, thus $Q(t)y(t) = 0$ for all $t \in \mathbb{R}_+$ as Qy solves a homogeneous linear differential equation with zero initial condition. Therefore, $y(t) = P(t)y(t)$ for all $t \in \mathbb{R}_+$. □

With the decomposed system and the information about the inherent ODE it is now possible to show that for index-1 DAEs every local solution can be uniquely extended to a global solution.

Lemma 2.15 (Unique extension of solutions)**.** *Let $(E, A) \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^{n\times n})^2$ be index-1, $Q \in \mathfrak{Q}_{E,A}$ and $f \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^n)$. If $x \in \mathcal{C}(\mathcal{J};\mathbb{R}^n)$, $\mathcal{J} \subseteq \mathbb{R}_+$ an interval, is such that $(I - Q)x \in \mathcal{C}^1(\mathcal{J};\mathbb{R}^n)$ and x solves* (2.1) *for all $t \in \mathcal{J}$, then there exists a unique $\tilde{x} \in \mathcal{C}^1_{E,A,f}$ such that $x = \tilde{x}|_{\mathcal{J}}$.*

Proof: Let P, $\bar{A}$, G be as in (2.3). Then, by Lemma 2.12 (clearly, the statement does also hold true for local solutions), $y = Px : \mathcal{J} \to \mathbb{R}^n$ is a local solution of the inherent ODE (2.5) and can thus be extended to a global solution $y : \mathbb{R}_+ \to \mathbb{R}^n$. And as $y(t) \in \operatorname{im} P(t)$ for $t \in \mathcal{J}$ we obtain from Lemma 2.14 that $y(t) \in \operatorname{im} P(t)$ for all $t \in \mathbb{R}_+$. Then $\tilde{x} := (I + QG^{-1}\bar{A})y + QG^{-1}f : \mathbb{R}_+ \to \mathbb{R}^n$ is a global solution of (2.4) as a simple calculation shows and therefore, again using Lemma 2.12, $\tilde{x} \in \mathcal{C}^1_{E,A,f}$. This proves existence, as $\tilde{x}|_{\mathcal{J}} = x$.
Uniqueness follows since assuming there is another solution $\hat{x} \in \mathcal{C}^1_{E,A,F}$ with $\hat{x}(t) = \tilde{x}(t)$ for all $t \in \mathcal{J}$ implies $P(t)\hat{x}(t) = P(t)\tilde{x}(t)$ for all $t \in \mathbb{R}_+$, as $P\hat{x}$ and $P\tilde{x}$ solve the inherent ODE (2.5) with same initial values (in $\mathcal{J}$). Then invoking Lemma 2.12 and the second equation in (2.4) gives

$$Q\hat{x} = QG^{-1}\bar{A}P\hat{x} + QG^{-1}f = QG^{-1}\bar{A}P\tilde{x} + QG^{-1}f = Q\tilde{x}. \qquad \square$$

2.3 Initial value problems

In this subsection we investigate initial value problems. First, we define the set of consistent initial conditions.

Definition 2.16 (Consistent initial values)**.** Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ be index-1 and $f \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^n)$. The *set of all pairs of consistent initial values* of (1.2) and the set of *initial values which are consistent at time* $t^0 \in \mathbb{R}_+$ is denoted by

$$\begin{aligned} \mathcal{V}_{E,A,f} &:= \left\{ (t^0, x^0) \in \mathbb{R}_+ \times \mathbb{R}^n \;\middle|\; \exists\, x \in \mathcal{C}^1_{E,A,f} :\; x(t^0) = x^0 \right\} \\ \mathcal{V}_{E,A,f}(t^0) &:= \left\{ x^0 \in \mathbb{R}^n \;\middle|\; (t^0, x^0) \in \mathcal{V}_{E,A} \right\}, \end{aligned}$$

resp. $\diamond$

Note that if $x : \mathbb{R}_+ \to \mathbb{R}^n$ is a solution of (1.2), then $x(t) \in \mathcal{V}_{E,A,f}(t)$ for all $t \in \mathbb{R}_+$. We may derive the following representation of $\mathcal{V}_{E,A,f}(t_0)$ in terms of the decomposition (2.4) which shows in particular that $\mathcal{V}_{E,A,f}(t_0)$ is a linear affine subspace.

Proposition 2.17 (Representation of $\mathcal{V}_{E,A,f}(t_0)$)**.** *Let* $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ *be index-1,* $Q \in \mathfrak{Q}_{E,A}$*,* $P, \bar{A}, G$ *as in* (2.3) *and* $f \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^n)$*. Then*

$$\forall\, t_0 \in \mathbb{R}_+ :\; \mathcal{V}_{E,A,f}(t_0) = Q(t_0)G(t_0)^{-1}f(t_0) + \operatorname{im}(I + Q(t_0)G(t_0)^{-1}\bar{A}(t_0))P(t_0). \tag{2.6}$$

Proof: "$\supseteq$": If $z^0 = (I + Q(t_0)G(t_0)^{-1}\bar{A}(t_0))P(t_0)x^0 + Q(t_0)G(t_0)^{-1}f(t_0)$, $x^0 \in \mathbb{R}^n$, we have $P(t_0)z^0 = P(t_0)x^0$ and thus there exists a solution $y \in \mathcal{C}^1(\mathbb{R}_+; \mathbb{R}^n)$ of (2.5), $y(t_0) = P(t_0)x^0$. By Lemma 2.14 we obtain $y(t) \in \operatorname{im} P(t)$ for all $t \in \mathbb{R}_+$ and a simple calculation then yields that $x := (I + QG^{-1}\bar{A})y + QG^{-1}f : \mathbb{R}_+ \to \mathbb{R}^n$ solves (2.4) and therefore, invoking Lemma 2.12, $x \in \mathcal{C}^1_{E,A,f}$. This gives $z^0 = x(t_0) \in \mathcal{V}_{E,A,f}(t_0)$.
"$\subseteq$": If $z^0 \in \mathcal{V}_{E,A,f}(t_0)$, then there exists $x \in \mathcal{C}^1_{E,A,f}$ such that $x(t_0) = z^0$, and since x also solves (2.4) by Lemma 2.12 it follows that

$$z^0 = x(t_0) = P(t_0)x(t_0) + Q(t_0)x(t_0) = (I + Q(t_0)G(t_0)^{-1}\bar{A}(t_0))P(t_0)x(t_0) + Q(t_0)G(t_0)^{-1}f(t_0). \quad \square$$

In order to define a transition matrix in the subsequent section we need to consider initial value conditions of the form

$$E(t_0)(x(t_0) - x^0) = 0 \tag{2.7}$$

for $t_0 \in \mathbb{R}_+$ and $x^0 \in \mathbb{R}^n$. The following result clarifies the relation to initial value problems with $x(t_0) = x^0$ for $x^0 \in \mathcal{V}_{E,A,f}(t_0)$.

Proposition 2.18 (Initial value problems). *Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ be index-1 and $f \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^n)$. Then, for any $(t_0, x^0) \in \mathcal{V}_{E,A,f}$ and $x \in \mathcal{C}^1_{E,A,f}$ we have*

$$x(t_0) = x^0 \quad \iff \quad E(t_0)(x(t_0) - x^0) = 0.$$

Proof: "$\Rightarrow$": Clear.
"$\Leftarrow$": Let $Q \in \mathfrak{Q}_{E,A}$ and $P, \bar{A}, G$ as in (2.3). First, we show $P(t_0)(x(t_0) - x^0) = 0$. We have

$$E(t_0)(x(t_0) - x^0) = 0 \;\Rightarrow\; E(t_0)P(t_0)(x(t_0) - x^0) = 0 \;\Rightarrow\; (E(t_0) - \bar{A}(t_0)Q(t_0))P(t_0)(x(t_0) - x^0) = 0,$$

whence $G(t_0)P(t_0)(x(t_0) - x^0) = 0$ and invertibility of $G(t_0)$ yields $P(t_0)(x(t_0) - x^0) = 0$. Now we find, invoking Lemma 2.12, that

$$\begin{aligned} Q(t_0)x(t_0) &= Q(t_0)G(t_0)^{-1}\bar{A}(t_0)P(t_0)x(t_0) + Q(t_0)G(t_0)^{-1}f(t_0) \\ &= Q(t_0)G(t_0)^{-1}\bar{A}(t_0)P(t_0)x^0 + Q(t_0)G(t_0)^{-1}f(t_0), \end{aligned}$$

thus

$$x(t_0) = P(t_0)x(t_0) + Q(t_0)x(t_0) = (I + Q(t_0)G(t_0)^{-1}\bar{A}(t_0))P(t_0)x^0 + Q(t_0)G(t_0)^{-1}f(t_0). \tag{2.8}$$

Let $R(t_0) := (I + Q(t_0)G(t_0)^{-1}\bar{A}(t_0))P(t_0)$, then, by Proposition 2.17, we have $x^0 = R(t_0)z^0 + Q(t_0)G(t_0)^{-1}f(t_0)$ for some $z^0 \in \mathbb{R}^n$. Since $R(t_0)^2 = R(t_0)$ we can infer $R(t_0)x^0 = R(t_0)z^0$ and hence $x^0 = R(t_0)x^0 + Q(t_0)G(t_0)^{-1}f(t_0)$, by which, invoking (2.8), we immediately get $x(t_0) = x^0$. ☐

Note it is easily verified that $R(t_0)$ in the proof of Proposition 2.18 is a projector on $\mathcal{V}_{E,A,f}(t_0)$.
It is now interesting that an initial value problem (1.2), (2.7) may also be considered for arbitrary $x^0 \in \mathbb{R}^n$ and that this problem has a unique solution. The only drawback is that for $x^0 \notin \mathcal{V}_{E,A,f}(t_0)$ the solution does not satisfy $x(t_0) = x^0$ anymore. In fact, different initial values may lead to the same solution. We may study this in terms of the mapping which maps initial values onto the solution of the corresponding initial value problem.

Proposition 2.19 (Solution mapping). *Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ be index-1, $Q \in \mathfrak{Q}_{E,A}$, $P, \bar{A}, G$ as in (2.3) and $f \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^n)$. Then, for every $t_0 \in \mathbb{R}_+$, we have:*

(i) *The map*
$$\varphi_{t_0} : \mathbb{R}^n \to \mathcal{C}^1_{E,A,f},\; x^0 \mapsto x, \text{ where } E(t_0)(x(t_0) - x^0) = 0,$$
is well-defined and surjective.

(ii) *φ_{t_0} satisfies*
$$\forall\, x^0 \in \mathbb{R}^n : \; \big(\varphi_{t_0}(x^0)\big)(t_0) = (I + Q(t_0)G(t_0)^{-1}\bar{A}(t_0))P(t_0)x^0 + Q(t_0)G(t_0)^{-1}f(t_0)$$
and $\big(\varphi_{t_0}(x^0)\big)(t_0) = x^0$ for all $x^0 \in \mathcal{V}_{E,A,f}(t_0)$.

(iii) *The restriction $\varphi_{t_0}|_{\mathcal{V}_{E,A,f}(t_0)}$ is bijective.*

(iv) *If $f = 0$, then φ_{t_0} is linear and $\varphi_{t_0}|_{\mathcal{V}_{E,A,0}(t_0)}$ is a vector space isomorphism.*

Proof: (i): In order to show that φ_{t_0} is well-defined we have to prove that for all $x^0 \in \mathbb{R}^n$ the solution of the initial value problem (1.2), (2.7) is unique. This can be shown along lines similar to the uniqueness part of the proof of Lemma 2.15.
It remains to show that φ_{t_0} is surjective. To this end observe that for $x \in \mathcal{C}^1_{E,A,f}$ it follows from Lemma 2.12 that x solves (2.4) and hence

$$x = Px + Qx = (I + QG^{-1}\bar{A})Px + QG^{-1}f.$$

Setting $x^0 := (I + Q(t_0)G(t_0)^{-1}\bar{A}(t_0))P(t_0)x(t_0) + Q(t_0)G(t_0)^{-1}f(t_0)$ yields $E(t_0)(x(t_0) - x^0) = 0$.
(ii): As shown in the proof of Proposition 2.18, it follows for all $x^0 \in \mathbb{R}^n$ and all $x \in \mathcal{C}^1_{E,A,f}$ such that $E(t_0)(x(t_0) - x^0) = 0$ holds, that

$$(I + Q(t_0)G(t_0)^{-1}\bar{A}(t_0))P(t_0)x^0 + Q(t_0)G(t_0)^{-1}f(t_0) = x(t_0) = \big(\varphi_{t_0}(x^0)\big)(t_0)$$

Furthermore, Proposition 2.18 gives that if $x^0 \in \mathcal{V}_{E,A,f}(t_0)$, then $\big(\varphi_{t_0}(x^0)\big)(t_0) = x(t_0) = x^0$.
(iii): The injectivity of the restriction $\varphi_{t_0}|_{\mathcal{V}_{E,A,f}(t_0)}$ follows from Proposition 2.18.
(iv): If $f = 0$, then the linearity of φ_{t_0} follows from the linearity of (1.1) and the initial condition (2.7). From (i) and (iii) it then follows that $\varphi_{t_0}|_{\mathcal{V}_{E,A,0}(t_0)}$ is a vector space isomorphism. □

An explicit formula for φ_{t_0} is derived in Proposition 2.25 using variation of constants.

Remark 2.20 (Kernel of φ_{t_0})**.** The fact that different initial values may lead to the same solution of the initial value problem (1.2), (2.7) leads to the fact that, for $f = 0$, the map φ_{t_0} may have a non-trivial kernel. Indeed, we may calculate

$$\ker \varphi_{t_0} = \left\{\ x^0 \in \mathbb{R}^n \ \middle|\ \varphi_{t_0}(x^0) = 0\ \right\} = \left\{\ x^0 \in \mathbb{R}^n \ \middle|\ E(t_0)x^0 = 0\ \right\} = \ker E(t_0).$$

◇

2.4 Transition matrix and variation of constants

Next we define a transition matrix for the homogeneous system (1.1) using the result of Proposition 2.19.

Definition 2.21 (Transition matrix)**.** Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ be index-1 and φ_{t_0} be the solution map given by Proposition 2.19 for (1.1). Then the *transition matrix* $\Phi(\cdot,\cdot) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ of (E, A) is defined by

$$\Phi(t, t_0) := \left[\big(\varphi_{t_0}(e_1)\big)(t), \dots, \big(\varphi_{t_0}(e_n)\big)(t)\right], \qquad t, t_0 \in \mathbb{R}_+,$$

where e_i is the i-th unit vector. ◇

It is immediate from Definition 2.21 that $\Phi(\cdot, t_0)$ is the unique solution of

$$E(t)\tfrac{\mathrm{d}}{\mathrm{d}t}\Phi(t, t_0) = A(t)\Phi(t, t_0), \quad E(t_0)(\Phi(t_0, t_0) - I) = 0.$$

We may derive the following representation of the transition matrix in terms of the inherent ODE.

Lemma 2.22 (Representation of $\Phi(\cdot,\cdot)$)**.** *Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ be index-1, $Q \in \mathfrak{Q}_{E,A}$ and $P, \bar{A}, G$ as in* (2.3)*. Then*

$$\forall\, t, t_0 \in \mathbb{R}_+ :\ \Phi(t, t_0) = (I + Q(t)G(t)^{-1}\bar{A}(t))\Phi_0(t, t_0)P(t_0), \tag{2.9}$$

where, for all $t_0 \in \mathbb{R}_+$, $\Phi_0(\cdot, t_0)$ is the unique solution of

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\Phi_0(t, t_0) = (\dot{P}(t) + P(t)G(t)^{-1}\bar{A}(t))\Phi_0(t, t_0), \quad \Phi_0(t_0, t_0) = I.$$

Proof: Fix $t_0 \in \mathbb{R}_+$. Then, for all $i = 1, \ldots, n$, we may infer from Lemma 2.12 that

$$\big(\varphi_{t_0}(e_i)\big)(t) = (I + Q(t)G(t)^{-1}\bar{A}(t))y_i(t), \quad t \in \mathbb{R}_+,$$

where $y_i = P\varphi_{t_0}(e_i)$ solves (2.5), $y_i(t_0) = P(t_0)e_i$, with $f = 0$. Hence, $y_i(t) = \Phi_0(t,t_0)P(t_0)e_i$ for all $t \in \mathbb{R}_+$ and this yields the assertion. □

Remark 2.23 (Transition matrix)**.** We use the notation of Lemma 2.22. Note that the columns of $\Phi_0(\cdot,t_0)P(t_0)$ solve (2.5) (for $f = 0$) with initial conditions in $\operatorname{im} P(t_0)$, and hence it follows that they remain in $\operatorname{im} P(t)$ by Lemma 2.14, that is

$$\forall\, t, t_0 \in \mathbb{R}_+ : \ P(t)\Phi_0(t,t_0)P(t_0) = \Phi_0(t,t_0)P(t_0) = P(t)\Phi(t,t_0). \tag{2.10}$$

Using this relation it can easily be deduced that $\Phi(t,t_0)$ has the semi-group property: $\Phi(t,s)\Phi(s,r) = \Phi(t,r)$ for all $t,s,r \in \mathbb{R}_+$. This justifies to call $\Phi(t,t_0)$ a transition matrix for (1.1). And indeed $\Phi(\cdot,\cdot)$ satisfies the definition of a transition matrix in [5, Def. 4.1] except for the smoothness condition, which however holds on $\operatorname{im} P(t)$ by the representation in Lemma 2.22 (i.e., $P(\cdot)\Phi(\cdot,t_0)$ is continuously differentiable). Since the smoothness is not really needed all the results derived in [5] for Bohl exponents and perturbations of A can also be applied to the index-1 DAE (1.1). ⋄

Example 2.24. From the representation of $\Phi(\cdot,\cdot)$ in Lemma 2.22 it can be seen that $\Phi(\cdot,t_0)$ is continuous but not necessarily continuously differentiable. We give an example where $\Phi(\cdot,t_0)$ is only continuous and $P(\cdot)\Phi(\cdot,t_0)$ is continuously differentiable. To this end consider (1.1) with

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A(t) = \begin{bmatrix} 1 & 0 \\ -|t-1| & 1 \end{bmatrix}.$$

By choosing $Q = I - E$ it can be calculated the system is index-1 and the transition matrix is

$$\Phi(t,t_0) = \begin{bmatrix} e^{t-t_0} & 0 \\ |t-1|e^{t-t_0} & 0 \end{bmatrix}.$$

This shows that $\Phi(\cdot,t_0)$ is continuous but not continuously differentiable, whilst $P(\cdot)\Phi(\cdot,t_0) = \left[\begin{smallmatrix} e^{\cdot-t_0} & 0 \\ 0 & 0 \end{smallmatrix}\right]$ is continuously differentiable. ⋄

Now we come back to inhomogeneous problems (1.2) and derive a variation of constants formula for (1.2) using the transition matrix $\Phi(\cdot,\cdot)$.

Proposition 2.25 (Variation of constants)**.** *Let $(E,A) \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^{n\times n})^2$ be index-1, $Q \in \mathfrak{Q}_{E,A}$, $P, \bar{A}, G$ as in* (2.3) *and $f \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^n)$. Furthermore, let $t_0 \in \mathbb{R}_+$, $\Phi(\cdot,\cdot)$ be the transition matrix of (E,A) and φ_{t_0} be as in Proposition 2.19. Then, for all $x^0 \in \mathbb{R}^n$,*

$$\forall\, t \in \mathbb{R}_+ : \ \big(\varphi_{t_0}(x^0)\big)(t) = \Phi(t,t_0)P(t_0)x^0 + \int_{t_0}^{t} \Phi(t,s)P(s)G(s)^{-1}f(s)\ \mathrm{d}s + Q(t)G(t)^{-1}f(t). \tag{2.11}$$

Proof: We may infer from Lemma 2.12 that

$$\big(\varphi_{t_0}(x^0)\big)(t) = (I + Q(t)G(t)^{-1}\bar{A}(t))y_i(t) + Q(t)G(t)^{-1}f(t), \quad t \in \mathbb{R}_+,$$

where $y = P\varphi_{t_0}(x^0)$ solves (2.5), $y(t_0) = P(t_0)x^0$. As a solution of this ODE initial value problem y has the representation

$$\forall\, t \in \mathbb{R}_+ : \ y(t) = \Phi_0(t,t_0)P(t_0)x^0 + \int_{t_0}^{t} \Phi_0(t,s)P(s)G(s)^{-1}f(s)\, ds.$$

The assertion now follows from (2.9). □

Remark 2.26 (Dependencies on the projector)**.** Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ be index-1, $Q \in \mathfrak{Q}_{E,A}$ and $P, \bar{A}, G$ as in (2.3). In several results, for instance in the representation results for the set of consistent initial values and the transition matrix, certain matrices appear which involve products of $Q, P, \bar{A}$ and G. Simple calculations show that neither of the terms QG^{-1}, $QG^{-1}\bar{A}$ and $(I + QG^{-1}\bar{A})P$ depend on the choice of the projector Q. ◇

3 The perturbation problem

In this section we state the class of perturbations considered in this article. For given $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ and perturbation $\Delta_E \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})$ we consider the perturbed system

$$\big(E(t) + \Delta_E(t)\big)\dot{x} = A(t)x, \tag{3.1}$$

i.e., perturbations of the matrix-valued function E. Since exponential stability is very sensitive with respect to arbitrary perturbations in the leading term [10], we do not allow for general perturbations $\Delta_E \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})$, but restrict ourselves to the class of perturbations defined in the following.

Definition 3.1 (Allowable perturbations)**.** Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ be index-1. Then the *set of allowable perturbations (in the leading coefficient)* is defined by

$$\mathcal{P}_{E,A} := \left\{ \Delta_E \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n}) \;\middle|\; \begin{array}{l} \forall\, t \in \mathbb{R}_+ : \ \ker E(t) = \ker(E(t) + \Delta_E(t)), \\ \exists\, Q \in \mathfrak{Q}_{E,A} : \ G + \Delta_E(I + \dot{Q}Q) \in \mathcal{C}(\mathbb{R}_+; \mathbf{Gl}_n(\mathbb{R})) \\ \text{for } G \text{ as in (2.3)} \end{array} \right\}$$

◇

Remark 3.2 (Allowable perturbations)**.**

(i) If $\Delta_E \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})$ is chosen such that $\|\Delta_E\|_\infty$ is sufficiently small, then we can always assure that $G + \Delta_E(I + \dot{Q}Q) \in \mathcal{C}(\mathbb{R}_+; \mathbf{Gl}_n(\mathbb{R}))$ and that $\ker(E + \Delta_E) \subseteq \ker E$ - the latter meaning the kernel of E can only become smaller. If the aforementioned conditions are satisfied, then the condition $\ker E(t) \subseteq \ker \Delta_E(t)$ for all $t \in \mathbb{R}_+$ is equivalent to $\Delta_E \in \mathcal{P}_{E,A}$.

(ii) The matrix $\dot{Q}Q$ is nilpotent and the index of nilpotency is 2 everywhere: As $\dot{Q} = \frac{\mathrm{d}}{\mathrm{d}t}Q^2 = \dot{Q}Q + Q\dot{Q}$ we obtain $Q\dot{Q}Q = 0$ and hence $(\dot{Q}Q)^2 = 0$. Therefore, $I + \dot{Q}Q$ is invertible everywhere with $(I + \dot{Q}Q)^{-1} = I - \dot{Q}Q$. ◇

It may be asked why it is required in the definition of $\mathcal{P}_{E,A}$ that the projector Q must be in $\mathfrak{Q}_{E,A}$ and not in $\mathfrak{Q}_{E+\Delta_E,A}$. In fact, the answer is that it doesn't matter, but re-using the projector from the nominal system is easier than calculating a new one. The following lemma clarifies this.

Lemma 3.3 (Projectors and perturbations)**.** *Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ be index-1 and $\Delta_E \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})$. If $\mathfrak{Q}_{E,A} \neq \emptyset$, $\mathfrak{Q}_{E+\Delta_E,A} \neq \emptyset$ and $\ker E(t) = \ker(E(t) + \Delta_E(t))$ for all $t \in \mathbb{R}_+$, then $\mathfrak{Q}_{E,A} = \mathfrak{Q}_{E+\Delta_E,A}$. Furthermore, we have*

$$\mathcal{P}_{E,A} := \left\{ \Delta_E \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n}) \;\middle|\; \mathfrak{Q}_{E+\Delta_E,A} \neq \emptyset \ \wedge \ \forall\, t \in \mathbb{R}_+ : \ \ker E(t) = \ker(E(t) + \Delta_E(t)) \right\}.$$

Proof: Follows immediately from Proposition 2.8. □

Remark 3.4 (Kernel assumption)**.** The definition of the set $\mathcal{P}_{E,A}$ may seem restrictive, in particular the claim for the kernel of E to be preserved. But on the one hand side, as shown later in this section, perturbations of the algebraic part are still possible. On the other hand side, in the perturbation theory of DAEs it is usually assumed that the leading coefficient E is not perturbed at all, see e.g. [12, 17, 18, 22, 51]. Moreover, the condition on perturbations of the leading term to preserve some kernel is not uncommon, as in [10], where time-invariant systems are considered, it is assumed that the left kernel of E is preserved under the perturbation (see proof of [10, Lem. 3.2]). Furthermore, as argued in [10], in practical applications the set of allowable perturbations is limited anyway, restricted by the physical structure of the considered system. Therefore, as it is widely believed, if the algebraic part of the DAE represents path constraints, then the zero blocks in E are structural and are not subject to disturbances or uncertainties. However, this is not entirely true as it can be deduced from considering a DAE in semi-explicit form:

$$E(t)\dot{x} = \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A(t)x. \tag{3.2}$$

Equation (3.2) consists of n_1 differential equations and $n_2 = n - n_1$ algebraic constraints. Changing any of the zeros in the second column of E would involve derivatives of x_2 and therefore inevitably change the structure of the system - so these zero blocks are structural. However, the zero block in the lower left corner is not. If we change this block to E_{21} for instance, then the second equation now reads $E_{21}(t)\dot{x}_1 = A_{21}(t)x_1 + A_{22}(t)x_2$ and incorporating the first equation gives

$$0 = (A_{21} - E_{21}A_{11})(t)x_1 + (A_{22} - E_{21}A_{12})(t)x_2,$$

so the system has again the same structure as before. This shows that we have to distinguish between perturbations which change the structure of the system and perturbations which change the structure of the matrices E and A. What is desired is that the structure of the system is preserved under perturbations and indeed, in the above example, changing the lower left block in E does not change the kernel of E. This shows that for semi-explicit DAEs, the perturbations which preserve the kernel of E (and may change anything else) are those which preserve the (physical) structure of the system. Note that system (3.2) is index-1 if, and only if, A_{22} is invertible everywhere and hence the perturbed system is index-1 if, and only if, $A_{22} - E_{21}A_{12}$ is invertible everywhere, which holds true if $\|E_{21}\|_\infty$ is sufficiently small. $\diamond$

Lemma 3.5 (Sufficient condition for preserved index)**.** *Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ be index-1, $Q \in \mathfrak{Q}_{E,A}$, G as in (2.3) and $\Delta_E \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})$. Then the following holds true:*

(i) $E - AQ \in \mathcal{C}(\mathbb{R}_+; \mathbf{Gl}_n(\mathbb{R}))$,

(ii) $$\left.\begin{array}{l} \forall\, t \in \mathbb{R}_+ : \ \ker E(t) = \ker(E(t) + \Delta_E(t)), \\ \forall\, t \in \mathbb{R}_+ : \ \|\Delta_E(t)(E(t) - A(t)Q(t))^{-1}\| < 1 \end{array}\right\} \ \Rightarrow \ Q \in \mathfrak{Q}_{E+\Delta_E,A} \ \wedge \ \Delta_E \in \mathcal{P}_{E,A}.$$

Proof: (i): Let $X := (I + \dot{Q}Q)G^{-1}$ and observe that

$$(I + \dot{Q}Q) = XE(I + \dot{Q}Q) - XAQ$$

and hence, as $(I + \dot{Q}Q)^{-1} = I - \dot{Q}Q$ by Remark 3.2 (ii), $XE = I + XAQ(I - \dot{Q}Q) = I + XAQ$, since $Q\dot{Q}Q = 0$. Therefore, $X(E - AQ) = I$ and by invertibility of X we find

$$(E - AQ)^{-1} = X = (I + \dot{Q}Q)G^{-1}. \tag{3.3}$$

(ii): As Δ_E preserves the kernel of E it is clear that Q is a projector on $\ker(E + \Delta_E)$. Hence, it only remains to prove that $E + \Delta_E + ((E + \Delta_E)\dot{Q} - A)Q = G + \Delta_E(I + \dot{Q}Q) \in \mathcal{C}(\mathbb{R}_+; \mathbf{Gl}_n(\mathbb{R}))$. Since $G + \Delta_E(I + \dot{Q}Q) = (I + \Delta_E(E - AQ)^{-1})G$ the invertibility immediately follows from the assumption. $\square$

Lemma 3.5 gives rise for the following definition of subsets of $\mathcal{P}_{E,A}$.

Definition 3.6. Let $(E,A) \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^{n\times n})^2$ be index-1 and $Q \in \mathfrak{Q}_{E,A}$. Then we define

$$\mathcal{P}^{Q}_{E,A} := \left\{ \Delta_E \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^{n\times n}) \;\middle|\; \begin{array}{l} \forall\, t \in \mathbb{R}_+ :\; \ker E(t) = \ker(E(t)+\Delta_E(t)) \text{ and} \\ \|\Delta_E(t)(E(t)-A(t)Q(t))^{-1}\| < 1 \end{array} \right\}$$

$\diamond$

Note that, if $E = 0$, then $I \in \mathfrak{Q}_{E,A}$ and we have $\mathcal{P}^{I}_{E,A} = \{0\} = \mathcal{P}_{E,A}$. In general, we have the following corollary, which is immediate from Lemma 3.5.

Corollary 3.7. *Let $(E,A) \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^{n\times n})^2$ be index-1 and $Q \in \mathfrak{Q}_{E,A}$. Then $\mathcal{P}^{Q}_{E,A} \subseteq \mathcal{P}_{E,A}$.*

For perturbations in $\mathcal{P}^{Q}_{E,A}$ we may also reformulate the perturbed system (3.1) in a form similar to (2.4).

Lemma 3.8 (Decomposition of perturbed system)**.** *Let $(E,A) \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^{n\times n})^2$ be index-1, $Q \in \mathfrak{Q}_{E,A}$, $P, \bar{A}, G$ as in (2.3) and $\Delta_E \in \mathcal{P}^{Q}_{E,A}$. Then $x \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^n)$ is a solution of (3.1) if, and only if, $Px \in \mathcal{C}^1(\mathbb{R}_+;\mathbb{R}^n)$ and x solves the following system for all $t \in \mathbb{R}_+$:*

$$\left\{ \begin{array}{rcl} \frac{\mathrm{d}}{\mathrm{d}t}(P(t)x) &=& (\dot{P}(t) + P(t)G(t)^{-1}\bar{A}(t))P(t)x + P(t)G(t)^{-1}\Delta(t)P(t)x, \\ Q(t)x &=& Q(t)G(t)^{-1}\bar{A}(t)P(t)x + Q(t)G(t)^{-1}\Delta(t)P(t)x, \end{array} \right. \tag{3.4}$$

where

$$\Delta := -(I+\Lambda)^{-1}\Lambda A(I - Q\dot{Q}), \qquad \Lambda = \Delta_E(E - AQ)^{-1}. \tag{3.5}$$

Proof: Using Lemma 2.12 and the fact that $Q \in \mathfrak{Q}_{E+\Delta_E,A}$ by Lemma 3.5, and defining $\tilde{A} := \bar{A} - \Delta_E\dot{Q}$, $\tilde{G} := E + \Delta_E - \tilde{A}Q$, it is immediate that x is a solution of (3.1) if, and only if, $Px \in \mathcal{C}^1(\mathbb{R}_+;\mathbb{R}^n)$ and x solves

$$\left\{ \begin{array}{rcl} \frac{\mathrm{d}}{\mathrm{d}t}(P(t)x) &=& (\dot{P}(t) + P(t)\tilde{G}(t)^{-1}\tilde{A}(t))P(t)x, \\ Q(t)x &=& Q(t)\tilde{G}(t)^{-1}\tilde{A}(t)P(t)x. \end{array} \right. \tag{3.6}$$

Now observe that, by (3.3), $\tilde{G} = G + \Lambda G$ and hence, under the assumption that $\|\Lambda(t)\| = \|\Delta_E(t)(E(t) - A(t)Q(t))^{-1}\| < 1$ for all $t \in \mathbb{R}_+$, it is immediate that

$$\tilde{G}^{-1} = G^{-1}(I+\Lambda)^{-1} = G^{-1}\big(I - \Lambda(I+\Lambda)^{-1}\big).$$

By some simple calculation we then obtain that

$$\tilde{G}^{-1}\tilde{A} = G^{-1}\bar{A} - G^{-1}\big((I+\Lambda)^{-1}\Delta_E\dot{Q} + \Lambda(I+\Lambda)^{-1}\bar{A}\big).$$

Using that $\Delta_E\dot{Q} = \Lambda(E - AQ)\dot{Q}$ and

$$(I+\Lambda)^{-1}\Lambda = \Lambda - \Lambda(I+\Lambda)^{-1}\Lambda = \Lambda(I+\Lambda)^{-1},$$

we get

$$\tilde{G}^{-1}\tilde{A} = G^{-1}\bar{A} - G^{-1}(I+\Lambda)^{-1}\Lambda\big((E - AQ)\dot{Q} + \bar{A}\big).$$

Now $(E - AQ)\dot{Q} + \bar{A} = (E - AQ)\dot{Q} + A - E\dot{Q} = A(I - Q\dot{Q})$, thus

$$\tilde{G}^{-1}\tilde{A} = G^{-1}\bar{A} + G^{-1}\Delta,$$

which yields that (3.6) is equivalent to (3.4). $\square$

In the subsequent sections we will also need the following lemma, the proof of which is straightforward.

Lemma 3.9 (Bound on Δ)**.** *Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ be index-1, $Q \in \mathfrak{Q}_{E,A}$ and $\Delta_E \in \mathcal{P}^Q_{E,A}$. Then, for Δ as in* (3.5) *and all $t \in \mathbb{R}_+$, we have*

$$\|\Delta(t)\| \le \frac{\|\Delta_E(t)(E(t) - A(t)Q(t))^{-1}A(t)(I - Q(t)\dot{Q}(t))\|}{1 - \|\Delta_E(t)(E(t) - A(t)Q(t))^{-1}\|}.$$

From (3.4) it can be seen that the perturbation does not only effect the differential part, but also the algebraic part of the DAE. To make this more clear consider the following example which will serve as a running example in the following.

Example 3.10. Consider the system (1.1) with constant

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The solutions of this system are given by $x_1(\cdot) = c_1 e^{-\cdot}$, $x_2(\cdot) = c_2 e^{-\cdot}$, $x_3(\cdot) = 0$ for $c_1, c_2 \in \mathbb{R}$. Now let

$$\Delta_E = \begin{bmatrix} 0 & \delta & 0 \\ 0 & \delta & 0 \\ \delta & 0 & 0 \end{bmatrix}, \quad \delta \in \mathbb{R},$$

and observe that $\ker E = \ker(E + \Delta_E)$ for all $\delta \in \mathbb{R}$. Furthermore, choosing $Q = I - E \in \mathfrak{Q}_{E,A}$, we have that, for G as in (2.3),

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and hence} \quad G + \Delta_E(I + \dot{Q}Q) = \begin{bmatrix} 1 & \delta & 0 \\ 0 & 1+\delta & 0 \\ \delta & 0 & -1 \end{bmatrix},$$

which is invertible for all $\delta \neq -1$. Hence $\Delta_E \in \mathcal{P}_{E,A}$ for $\delta \neq -1$. As it is easy to calculate that $\|\Delta_E\| = \sqrt{2}|\delta|$, we have $\Delta_E \in \mathcal{P}^Q_{E,A}$ if, and only if, $|\delta| < \frac{\sqrt{2}}{2}$. The perturbed system (3.1) reads, after some rearrangement,

$$\dot{x}_1 = -x_1 + \frac{\delta}{1+\delta}x_2, \quad \dot{x}_2 = -\frac{1}{1+\delta}x_2, \quad x_3 = -\delta x_1 + \frac{\delta^2}{1+\delta}x_2,$$

Therefore, the solutions are

$$x_1(\cdot) = (c_1 - c_2)e^{-\cdot} + c_2 e^{-\frac{1}{1+\delta}\cdot}, \quad x_2(\cdot) = c_2 e^{-\frac{1}{1+\delta}\cdot}, \quad x_3(\cdot) = -\delta(c_1 - c_2)e^{-\cdot} - \frac{\delta c_2}{1+\delta}e^{-\frac{1}{1+\delta}\cdot},$$

for $c_1, c_2 \in \mathbb{R}$, and it is clear that both the differential and the algebraic part of the DAE have been perturbed as all components of the solution have changed. Furthermore, we see that for $\delta > -1$ the perturbed system is exponentially stable (cf. Definition 4.6), whilst it is unstable for $\delta < -1$. For $\delta = -1$ we have $\Delta_E \notin \mathcal{P}_{E,A}$, however the system is still exponentially stable as the equations read, after some rearrangement, $\dot{x}_1 = -x_1, x_2 = 0, x_3 = x_1$ - but this is beyond the scope of this approach because the index of the system did change (it is index-2 tractable in the sense of [42] for $\delta = -1$). ⋄

Remark 3.11. Note that, as shown in Example 3.10, the perturbations may change the algebraic equations, but not the algebraic *structure* of the system as it was pointed out in Remark 3.4. ⋄

4 Bohl exponent

There are two fundamental concepts in the theory of ODEs to investigate asymptotic behaviour of solutions: the Lyapunov and the Bohl exponent. While the Lyapunov exponent, introduced by Aleksandr M. Lyapunov [41], gives a bound for the exponential growth of the solutions of the system, the Bohl exponent, introduced by Piers Bohl [7], describes the uniform exponential growth of the solutions. While the Lyapunov exponent is very useful for time-invariant systems, the Bohl exponent is the appropriate concept when it comes to time-varying ODEs. The Bohl exponent has been successfully used to characterize exponential stability and to derive robustness results, see e.g. [14, 29]. For an excellent summary of the history of the development of the Lyapunov and Bohl exponent see [14, pp. 146–148]. In this section we give the definition for the Bohl exponent as stated in [5] for general DAE systems and derive some formulae for it which hold in the index-1 setting. We also state the equivalence between a negative Bohl exponent and exponential stability and derive a robustness result for the Bohl exponent under the class of perturbations introduced in Section 3. In particular, this shows that exponential stability is robust under these perturbations. The main result of this section is Theorem 4.8 which states the robustness of the Bohl exponent.

Definition 4.1 (Bohl exponent)**.** Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ be index-1. The *Bohl exponent of* (E, A) is defined as

$$k_B(E, A) := \inf \left\{ \rho \in \mathbb{R} \;\middle|\; \exists\, N_\rho > 0\ \forall\, x \in \mathcal{C}^1_{E,A,0}\ \forall\, t \geq s \geq 0 : \ \|x(t)\| \leq N_\rho e^{\rho(t-s)} \|x(s)\| \right\}.$$

Note that we use the usual convention $\inf \emptyset := +\infty$. ⋄

Next we state a representation of the Bohl exponent for index-1 DAEs which is well-known for ODEs, see e.g. [14, Sec. III.4].

Lemma 4.2 (Representation of the Bohl exponent)**.** *Let* $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ *be index-1 with transition matrix* $\Phi(\cdot, \cdot)$*. Then we have*

$$k_B(E, A) = \inf \left\{ \rho \in \mathbb{R} \;\middle|\; \exists\, N_\rho > 0\ \forall\, t \geq s \geq 0 : \ \|\Phi(t, s)\| \leq N_\rho e^{\rho(t-s)} \right\}$$

and $k_B(E, A) < \infty$ *if, and only if,*

$$\sup_{0 \leq t-s \leq 1} \|\Phi(t, s)\| < \infty.$$

Furthermore, if $k_B(E, A) < \infty$*, then it holds that*

$$k_B(E, A) = \limsup_{s,\, t-s \to \infty} \frac{\ln \|\Phi(t, s)\|}{t - s},$$

where $\ln 0 := -\infty$*.*

Proof: The first statement is immediate from the definition of the Bohl exponent and the second is a special case of [5, Prop. 3.7]. For the last statement see [12, Prop. 4.4]. Note that in the second and last statement a Bohl exponent $k_B(E, A) = -\infty$ is explicitly allowed. □

We stress that the equivalent condition for a Bohl exponent $k_B(E, A) < \infty$ is also valid in the case $k_B(E, A) = -\infty$. Moreover, the formula for the calculation of the Bohl exponent does also hold true in this case. The Bohl exponent can become $-\infty$ if all solutions of (1.1) vanish identically, hence $\Phi(t, s) = 0$ for all $t, s \in \mathbb{R}_+$. However, it is possible that the Bohl exponent is $-\infty$ even in the ODE case, i.e., when we have non-zero solutions. This is illustrated by the following example.

Example 4.3 (Bohl exponent $-\infty$)**.** Consider the system

$$\dot{x} = -2t\, x. \tag{4.1}$$

It is easy to observe that any solution $x(\cdot)$ of (4.1) satisfies $x(t) = e^{-(t^2-s^2)}x(s)$ for all $t \geq s \geq 0$. We show that $k_B(1, -2t) = -\infty$.
First let $\rho \geq 0$ and set $N_\rho := 1$. Then clearly $\|x(t)\| \leq N_\rho e^{\rho(t-s)}\|x(s)\|$ for all $t \geq s \geq 0$.
Now let $\rho < 0$ and set $N_\rho := e^{\rho^2}$. Then let $t \geq s \geq 0$ and observe that

$$0 \leq (-\rho - (t-s))^2 = \rho^2 + 2\rho(t-s) + (t-s)^2 = \left(\rho^2 + \rho(t-s) + t^2 - s^2\right) + \rho(t-s) - 2st + 2s^2,$$

hence

$$\rho^2 + \rho(t-s) + t^2 - s^2 \geq 2s(t-s) - \rho(t-s) \geq 0.$$

Therefore, we have $e^{-(t^2-s^2)} \leq e^{\rho^2}e^{\rho(t-s)}$ which proves $\|x(t)\| \leq N_\rho e^{\rho(t-s)}\|x(s)\|$.
By the above findings we obtain $k_B(1, -2t) = -\infty$ and in particular we see that

$$\limsup_{s,t-s\to\infty} \frac{\ln\|\Phi(t,s)\|}{t-s} = \limsup_{s,t-s\to\infty} \frac{\ln e^{-(t^2-s^2)}}{t-s} = -\infty.$$

◇

Remark 4.4. As shown in Example 4.3, a Bohl exponent of $-\infty$ is not an exceptional case, even for ODEs. For DAEs it is even more common as any equation of the form $0 = A(t)x$ with $A \in \mathcal{C}(\mathbb{R}_+; \mathbf{Gl}_n(\mathbb{R}))$ has Bohl exponent $-\infty$. But compared to a Bohl exponent of $+\infty$ it is of a more "good-natured" kind, as a system with Bohl exponent $-\infty$ is in particular exponentially stable (cf. Definition 4.6). Therefore, we will usually consider the cases of finite Bohl exponent and Bohl exponent $-\infty$ together, i.e., the latter is not excluded when $k_B(E, A) < \infty$ is required, if not stated otherwise. ◇

The Bohl exponent can also be represented in terms of the transition matrix of the inherent ODE (2.5), provided there exists a bounded projector $Q \in \mathfrak{Q}_{E,A}$ - which is guaranteed for any index-1 DAE (E, A) with $E \in \mathcal{C}^1(\mathbb{R}_+; \mathbb{R}^{n\times n})$ by Proposition 2.8 and Lemma 2.9. This result has been proved in [12, Prop. 4.6], but here we show that it indeed holds under very mild assumptions.

Lemma 4.5 (Representation in terms of Φ_0)**.** *Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ be index-1 with transition matrix $\Phi(\cdot,\cdot)$ and let $\Phi_0(\cdot,\cdot)$ be the transition matrix of the inherent ODE* (2.5)*. Suppose that there exits a bounded $Q \in \mathfrak{Q}_{E,A}$ and that $k_B(E, A) < \infty$. Let $P = I - Q$. Then we have*

$$k_B(E, A) = \limsup_{s,t-s\to\infty} \frac{\ln\|\Phi_0(t,s)P(s)\|}{t-s} \leq \limsup_{s,t-s\to\infty} \frac{\ln\|\Phi_0(t,s)\|}{t-s}.$$

Proof: Let $\bar{A}$ and G as in (2.3). By [12, Lem. 4.3] the assumption $k_B(E, A) < \infty$ implies that $I + QG^{-1}\bar{A}$ is bounded. It also follows from the assumption that P is bounded. And indeed this is enough to guarantee that the proof of [12, Prop. 4.6] is feasible. □

Next we state the definition of exponential stability of DAEs (E, A). The definition for general DAEs can be found e.g. in [5, 6]. Here we state the already simplified version derived from [5, Prop. 5.2].

Definition 4.6 (Exponential stability)**.** A linear index-1 DAE $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ with transition matrix $\Phi(\cdot,\cdot)$ is called *exponentially stable* if, and only if,

$$\exists\, \mu, M > 0\ \forall\, t \geq t_0 \geq 0 :\ \|\Phi(t, t_0)\| \leq Me^{-\mu(t-t_0)}. \tag{4.2}$$

◇

Note that usually stability is a property of a particular solution: other existing solutions in a neighborhood of it stay close to it for all time. For linear systems it is sufficient to consider this property only for the trivial solution. However, for (general) DAEs it is at first sight not clear whether this is still true. To this end, it is shown in [6, Thm. 4.3] that also for DAEs (E, A) it suffices to consider the stability properties of the trivial solution.
As shown in [5, Cor. 5.3] we have the following lemma.

Lemma 4.7 (Bohl exponent and exponential stability)**.** *Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ be index-1 with transition matrix $\Phi(\cdot,\cdot)$ and suppose that there exists a bounded $Q \in \mathfrak{Q}_{E,A}$ and that $k_B(E, A) < \infty$. Then the following statements are equivalent:*

(i) $k_B(E, A) < 0$.

(ii) (E, A) *is exponentially stable.*

(iii) $\forall\, p > 0\ \exists\, c > 0\ \forall\, t_0 \in \mathbb{R}_+ :\ \int_{t_0}^{\infty} \|\Phi(t, t_0)\|^p \,\mathrm{d}t \le c$.

In this sense, the next result is a robustness result for exponential stability under perturbations introduced in Section 3. In the proof of Theorem 4.8 we use techniques from the proofs of [5, Lem. 5.8] and [12, Thm. 5.2].

Theorem 4.8 (Robustness of Bohl exponent)**.** *Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ be index-1 and suppose that there exists a bounded $Q \in \mathfrak{Q}_{E,A}$ and that $k_B(E, A) > -\infty$. Further let P and G be as in (2.3). Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\Delta_E \in \mathcal{P}^Q_{E,A}$ which satisfy, for Δ as in (3.5), the condition*

$$\limsup_{t,s\to\infty} \frac{1}{s} \int_t^{t+s} \|P(\tau)G(\tau)^{-1}\Delta(\tau)P(\tau)\|\, d\tau < \delta \tag{4.3}$$

it holds that

$$k_B(E + \Delta_E, A) \le k_B(E, A) + \varepsilon.$$

Proof: Let $\bar{A}$ be as in (2.3), $\varepsilon > 0$, and $\Delta_E \in \mathcal{P}^Q_{E,A}$. Assume that $k_B(E, A) < \infty$, otherwise the inequality is trivially satisfied. Let $\Phi(\cdot,\cdot)$ be the transition matrix of (E, A), $\Phi_0(\cdot,\cdot)$ be the transition matrix of the inherent ODE (2.5) and let $\tilde{\Phi}(\cdot,\cdot)$ be the transition matrix of the perturbed system $(E + \Delta_E, A)$. Fix $s \in \mathbb{R}_+$. Then, invoking $\Delta_E \in \mathcal{P}^Q_{E,A}$ and Lemma 3.8, $\tilde{\Phi}(\cdot,\cdot)$ satisfies the first equation in (3.4) as a matrix equation, that is, for all $t \ge s$,

$$\tfrac{\mathrm{d}}{\mathrm{d}t}(P(t)\tilde{\Phi}(t, s)) = (\dot{P}(t) + P(t)G(t)^{-1}\bar{A}(t))P(t)\tilde{\Phi}(t, s) + P(t)G(t)^{-1}\Delta(t)P(t)\tilde{\Phi}(t, s).$$

This implies that the columns $P(\cdot)\tilde{\Phi}(\cdot, s)e_i$ solve the inherent ODE (2.5) with $f = \Delta P\tilde{\Phi}(\cdot, s)e_i$, $i = 1, \ldots, n$, starting in $\operatorname{im} P(s)$. In fact, the initial values satisfy

$$P(s)\tilde{\Phi}(s, s)e_i = P(s)(I + Q(s)\tilde{G}(s)\tilde{A}(s))P(s)e_i = P(s)e_i$$

and hence an application of the variation of constants formula gives

$$P(t)\tilde{\Phi}(t, s) = \Phi_0(t, s)P(s) + \int_s^t \Phi_0(t, \tau)P(\tau)G(\tau)^{-1}\Delta(\tau)P(\tau)\tilde{\Phi}(\tau, s)\,\mathrm{d}\tau \tag{4.4}$$

for all $t \ge s$. Now invoking the boundedness of Q, $|k_B(E, A)| < \infty$ and Lemma 4.5, we find that for $\mu := -k_B(E, A) - \varepsilon/2$ there exists $M > 0$ such that

$$\|\Phi_0(t, s)P(s)\| \le Me^{-\mu(t-s)}, \quad t \ge s.$$

We obtain from (4.4) that

$$e^{\mu t}\|P(t)\tilde{\Phi}(t,s)\| \le Me^{\mu s} + M\int_s^t \|P(\tau)G(\tau)^{-1}\Delta(\tau)P(\tau)\|e^{\mu\tau}\|P(\tau)\tilde{\Phi}(\tau,s)\|\ \mathrm{d}\tau\,,$$

and an application of Gronwall's inequality (see e.g. [32, Lem. 2.1.18]) yields

$$\|P(t)\tilde{\Phi}(t,s)\| \le Me^{-\mu(t-s)}e^{M\int_s^t \|P(\tau)G(\tau)^{-1}\Delta(\tau)P(\tau)\|\ \mathrm{d}\tau}, \quad t \ge s. \tag{4.5}$$

Now the Condition (4.3) implies existence of $t_0, s_0 \ge 0$ such that

$$\sup_{t\ge t_0}\frac{1}{s_0}\int_t^{t+s_0}\|P(\tau)G(\tau)^{-1}\Delta(\tau)P(\tau)\|\ \mathrm{d}\tau \le 2\delta.$$

We distinguish two cases.
Case 1: $s < t_0$. Let $t \ge s$ and $k \in \mathbb{N}$ such that $s_0(k-1) \le t - t_0 < s_0 k$. Then

$$\int_{t_0}^t \|P(\tau)G(\tau)^{-1}\Delta(\tau)P(\tau)\|\ \mathrm{d}\tau \le ks_0(2\delta) \le 2(t-t_0+s_0)\delta \le 2(t-s+s_0)\delta, \tag{4.6}$$

and therefore, for all $t \ge s$,

$$\begin{aligned}\|P(t)\tilde{\Phi}(t,s)\| &\overset{(4.5)}{\le} Me^{-\mu(t-s)}e^{M\int_s^{t_0}\|P(\tau)G(\tau)^{-1}\Delta(\tau)P(\tau)\|\ \mathrm{d}\tau}e^{M\int_{t_0}^t\|P(\tau)G(\tau)^{-1}\Delta(\tau)P(\tau)\|\ \mathrm{d}\tau}\\ &\overset{(4.6)}{\le} Me^{-\mu(t-s)}N^0e^{2M(t-s+s_0)\delta} = MN^0e^{2s_0M\delta}e^{-(\mu-2M\delta)(t-s)},\end{aligned}$$

where $N^0 = \max\left\{1, e^{M\int_0^{t_0}\|P(\tau)G(\tau)^{-1}\Delta(\tau)P(\tau)\|\ \mathrm{d}\tau}\right\}$.
Case 2: $s \ge t_0$. Let $t \ge s$ and $k \in \mathbb{N}$ such that $s_0(k-1) \le t - s < s_0 k$. Then

$$\int_s^t \|P(\tau)G(\tau)^{-1}\Delta(\tau)P(\tau)\|\ \mathrm{d}\tau \le ks_0(2\delta) \le 2(t-s+s_0)\delta, \tag{4.7}$$

and therefore, (4.5) gives that, for all $t \ge s$,

$$\|P(t)\tilde{\Phi}(t,s)\| \overset{(4.7)}{\le} Me^{-\mu(t-s)}N^0e^{2M(t-s+s_0)\delta} \le MN^0e^{2s_0M\delta}e^{-(\mu-2M\delta)(t-s)}.$$

The two cases together with Lemma 4.5 (taking into account that $P(t)\tilde{\Phi}(t,s) = \tilde{\Phi}_0(t,s)P(s)$) imply that

$$k_B(E+\Delta_E, A) \le -\mu + 2M\delta = k_B(E,A) + \varepsilon/2 + 2M\delta.$$

Choosing $\delta = \frac{\varepsilon}{4M}$ completes the proof of the theorem. □

In the case of bounded perturbations the statement of Theorem 4.8 can, under some further assumptions, be simplified.

Corollary 4.9 (Robustness of Bohl exponent)**.** *Let $(E,A) \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^{n\times n})^2$ be index-1 and suppose that there exists a bounded $Q \in \mathfrak{Q}_{E,A}$ and that $k_B(E,A) > -\infty$. Further let P, G be as in (2.3) and suppose that G^{-1}, $P(E-AQ)^{-1}$ and $P(E-AQ)^{-1}A(P-\dot{Q}P)$ are bounded. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\Delta_E \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ which satisfy* $\ker E(t) = \ker(E(t)+\Delta_E(t))$*, $t \in \mathbb{R}_+$, and $\|\Delta_E\|_\infty < \delta$ it holds that*

$$k_B(E+\Delta_E, A) \le k_B(E,A) + \varepsilon.$$

Proof: First observe that by choosing δ sufficiently small we may assure $\Delta_E \in \mathcal{P}^Q_{E,A}$ and $\|\Delta_E\|_\infty \|P(E-AQ)^{-1}\|_\infty < 1$. Furthermore, for Δ as in (3.5), ΔP is bounded, as from Lemma 3.9

$$\|\Delta P\|_\infty \le \frac{\|P(E-AQ)^{-1}A(P-\dot{Q}P)\|_\infty \|\Delta_E\|_\infty}{1-\|P(E-AQ)^{-1}\|_\infty \|\Delta_E\|_\infty}, \tag{4.8}$$

where it was used that $\Delta_E = \Delta_E P$ and $(I - Q\dot{Q})P = (I - \dot{Q} + \dot{Q}Q)P = P - \dot{Q}P$. It follows that

$$\limsup_{t,s\to\infty} \frac{1}{s} \int_t^{t+s} \|P(\tau)G(\tau)^{-1}\Delta(\tau)P(\tau)\|\, d\tau \le \|PG^{-1}\|_\infty \|\Delta P\|_\infty.$$

Now $\|\Delta_E\|_\infty$ can be chosen sufficiently small so that Theorem 4.8 may be applied to conclude the proof. □

Remark 4.10 (Boundedness assumptions)**.** Note that the boundedness assumptions of Corollary 4.9 are satisfied if Q, $\dot{Q}$, $(E-AQ)^{-1}$ and A are bounded, i.e., boundedness of G^{-1} and $P(E-AQ)^{-1}A(P-\dot{Q}P)$ is implied. Therefore, these assumptions seem appropriate, in particular when we look at the ODE case: If $E = I$, we have $Q = 0$ and $G = I$, hence the assumptions reduce to boundedness of A. This however is (apart from the cases $k_B(E,A) = \pm\infty$) quite natural, as perturbations in the leading term of ODEs correspond to systems

$$\dot{x} = (I + \Delta_E(t))^{-1}A(t)x = A(t)x - \Delta_E(t)(I + \Delta_E(t))^{-1}A(t)x,$$

i.e., perturbations of A of the form $\Delta_E(t)(I + \Delta_E(t))^{-1}A(t)$. Boundedness of this perturbation term is necessary to obtain robustness results, see e.g. [29], and is guaranteed if A is bounded. ⋄

In Theorem 4.8 the case $k_B(E,A) = -\infty$ is excluded. Together with the case $k_B(E,A) = +\infty$, this is treated in the following proposition, which provides a condition under which the Bohl exponent is invariant.

Proposition 4.11 (Equal Bohl exponents)**.** *Let $(E,A) \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^{n\times n})^2$ be index-1 and suppose that there exists a bounded $Q \in \mathfrak{Q}_{E,A}$. Further let P and G be as in* (2.3)*. If $\Delta_E \in \mathcal{P}^Q_{E,A}$ and Δ as in* (3.5) *satisfies*

$$\limsup_{t,s\to\infty} \frac{1}{s} \int_t^{t+s} \|P(\tau)G(\tau)^{-1}\Delta(\tau)P(\tau)\|\, d\tau = 0, \tag{4.9}$$

then $k_B(E+\Delta_E, A) = k_B(E,A)$. This means in particular, if

$$\lim_{t\to\infty} \|P(\tau)G(\tau)^{-1}\Delta(t)P(t)\| = 0 \quad \text{or} \quad \int_0^\infty \|P(\tau)G(\tau)^{-1}\Delta(\tau)P(\tau)\|\, \mathrm{d}\tau < \infty,$$

then $k_B(E+\Delta_E, A) = k_B(E,A)$.

Proof: Let P, $\bar{A}$, G be as in (2.3). If $k_B(E,A) = -\infty$, then it is easy to observe that choosing sequences $\mu_k \to -\infty$ and $\delta_k \searrow 0$ in the proof of Theorem 4.8 shows that $k_B(E+\Delta_E, A) = -\infty$. Suppose now $k_B(E,A) \ne -\infty$. Observe that Theorem 4.8 implies $k_B(E+\Delta_E, A) \le k_B(E,A)$. We now show that it may be applied to $(E+\Delta_E, A)$ with perturbation $-\Delta_E$ as well. To this end, note that $Q \in \mathfrak{Q}_{E,A} \overset{\text{Lem. 3.5}}{=} \mathfrak{Q}_{E+\Delta_E,A}$ and $-\Delta_E \in \mathcal{P}^Q_{E+\Delta_E,A}$. It remains to prove that

$$\begin{aligned} \tilde{G} &:= E + \Delta_E + ((E+\Delta_E)\dot{Q} - A)Q \qquad \text{and} \\ \tilde{\Delta} &:= -(I+\tilde{\Lambda})^{-1}\tilde{\Lambda}A(I - Q\dot{Q}), \qquad \text{where } \tilde{\Lambda} = -\Delta_E(E+\Delta_E - AQ)^{-1}, \end{aligned}$$

satisfy (4.9) as well. First note that $E + \Delta_E - AQ$ is invertible everywhere by Lemma 3.5 since $Q \in \mathfrak{Q}_{E+\Delta_E,A}$.
We will show now that $\tilde{G}^{-1}\tilde{\Delta} = -G^{-1}\Delta$. To this end observe that

$$\tilde{\Lambda} = -\Delta_E(E - AQ)^{-1}\big(I + \Delta_E(E - AQ)^{-1}\big)^{-1} = -\Lambda(I + \Lambda)^{-1} = -\Lambda + \Lambda(I + \Lambda)^{-1}\Lambda = -(I + \Lambda)^{-1}\Lambda,$$

$$(I + \tilde{\Lambda})^{-1} = \big(I - (I + \Lambda)^{-1}\Lambda\big)^{-1} = I + \Lambda,$$

and hence, since $\tilde{G} = G + \Lambda G$,

$$\tilde{G}^{-1}\tilde{\Delta} = -G^{-1}(I + \Lambda)^{-1}(I + \tilde{\Lambda})^{-1}\tilde{\Lambda}A(I - Q\dot{Q}) = G^{-1}(I + \Lambda)^{-1}\Lambda A(I - Q\dot{Q}) = -G^{-1}\Delta.$$

Now Theorem 4.8 implies $k_B(E, A) = k_B((E + \Delta_E) - \Delta_E, A) \leq k_B(E + \Delta_E, A)$. □

Remark 4.12 (Invariance of Bohl exponent $\pm\infty$)**.** Condition (4.9) is a very strong condition on the perturbation in order for the Bohl exponent of $\pm\infty$ to be preserved. For simplicity, let us consider the ODE case for a moment: Example 4.3 shows that a Bohl exponent of $-\infty$ (or, similarly, $+\infty$) is not an exceptional case, however it is usually not treated, even in the standard literature on Bohl exponents for ODEs [14, 32]. Of course, Proposition 4.11 is in particular applicable to ODEs (I, A), but as the system $(I + \Delta_E, A)$ reads, for any $\Delta_E \in \mathcal{P}^0_{I,A}$,

$$\dot{x} = (I + \Delta_E(t))^{-1}A(t)x,$$

the perturbation is multiplicative and hence not the usual kind of perturbations considered for ODEs. Nevertheless, as follows from a careful inspection of the proof of Proposition 4.11, for additive perturbations of the form $(I, A + \Delta)$ we obtain the following: If $A \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})$ and $k_B(I, A) = \pm\infty$, then for all $\Delta \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})$ which satisfy (4.9) (with $P = G = I$) we have $k_B(I, A + \Delta) = \pm\infty$.
It is now immediate that in the scalar case ($n = 1$), in Condition (4.9) we may replace "$= 0$" by "$< \infty$" and the statement still holds true. Moreover, in the general case, we were not able to find any example such that a bounded perturbation Δ could push the Bohl exponent away from $\pm\infty$, while on the other hand side we were not able to prove that it is preserved for such a perturbation. It may be worth noting however, that there are systems with a Lyapunov exponent (mentioned in the beginning of this section) of $-\infty$ which can become $+\infty$ under arbitrary small perturbations.
Thus the invariance of Bohl exponent $\pm\infty$ under an appropriate large class of perturbations is an open problem. If we assume

$$\exists\, f \in \mathcal{C}^1(\mathbb{R}_+; \mathbb{R}) \text{ s.t. } \lim_{t\to\infty} \dot{f}(t) = \infty\ \exists\, M > 0\ \forall\, t \geq s \geq 0 :\ \|\Phi(t, s)\| \leq Me^{-(f(t)-f(s))}, \tag{4.10}$$

it is straightforward to prove (using the mean value theorem) the following:
Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ be index-1 and suppose that there exists a bounded $Q \in \mathfrak{Q}_{E,A}$. Further let P and G be as in (2.3). If (4.10) holds, $\Delta_E \in \mathcal{P}^Q_{E,A}$ and Δ as in (3.5) satisfies (4.9) with "$< \infty$" instead of "$= 0$", then $k_B(E + \Delta_E, A) = k_B(E, A) = -\infty$.
The author conjectures that Condition (4.10) is equivalent to $k_B(E, A) = -\infty$, however it is only clear that (4.10) implies $k_B(E, A) = -\infty$. ◇

We close this section by illustrating the main result by means of our running example.

Example 4.13 (Example 3.10 revisited)**.** It can be immediately seen from the representation of the solutions in Example 3.10 that

$$k_B(E, A) = -1 \quad \text{and} \quad k_B(E + \Delta_E, A) = \max\left\{-1, -\frac{1}{1 + \delta}\right\}$$

for all $\delta \neq -1$. Therefore, given $\varepsilon > 0$ we have that for all $\delta \in \mathbb{R}$ which satisfy

$$\begin{cases} \varepsilon < 1: & \delta \in \left(-1, \frac{\varepsilon}{1-\varepsilon}\right), \\ \varepsilon = 1: & \delta \in (-1, \infty), \\ \varepsilon > 1: & \delta \in \left(-\infty, \frac{\varepsilon}{1-\varepsilon}\right] \cup (-1, \infty), \end{cases}$$

the Bohl exponents satisfy

$$k_B(E + \Delta_E, A) \leq k_B(E, A) + \varepsilon.$$

◇

5 Perturbation operator

In this section we investigate robustness of exponential stability (1.1) in terms of the perturbation operator. As a system (E, A) is exponentially stable if, and only if, its Bohl exponent is negative by Lemma 4.7, Theorem 4.8 states in particular that exponential stability of index-1 DAEs is robust with respect to perturbations in $\mathcal{P}^Q_{E,A}$ for any bounded $Q \in \mathfrak{Q}_{E,A}$. However, Theorem 4.8 does only state that the perturbation has to be sufficiently small in order to preserve exponential stability. In this section we provide a calculable upper bound on the perturbation such that exponential stability is preserved by using the perturbation operator. In [29] it was shown that the perturbation operator is an appropriate tool for investigating perturbations and robustness for ODEs, see also [12, 18] for index-1 DAEs.
Motivated by the variation of constants formula (2.11) the perturbation operator is defined as follows.

Definition 5.1 (Perturbation operator). Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ be index-1 and exponentially stable and let $Q \in \mathfrak{Q}_{E,A}$. Further let $\Phi(\cdot, \cdot)$ be the transition matrix of (E, A), let P and G be as in (2.3) and suppose that PG^{-1} and QG^{-1} are bounded. Then the *perturbation operator* of (E, A) corresponding to Q is defined by

$$L^Q_{t_0} : L^2([t_0, \infty); \mathbb{R}^n) \to L^2([t_0, \infty); \mathbb{R}^n),$$
$$f(\cdot) \mapsto \left(t \mapsto \int_{t_0}^{t} \Phi(t, s)P(s)G(s)^{-1}f(s)\, \mathrm{d}s + Q(t)G(t)^{-1}f(t)\right).$$

◇

Lemma 5.2 (Properties of the perturbation operator). *Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ be index-1 and exponentially stable such that (4.2) holds. Let $Q \in \mathfrak{Q}_{E,A}$, $\Phi(\cdot, \cdot)$ be the transition matrix of (E, A) and let P and G be as in (2.3) and suppose that PG^{-1} and QG^{-1} are bounded. Then we have:*

(i) *For any $t_0 \in \mathbb{R}_+$: $L^Q_{t_0}$ is well-defined, i.e., $L^Q_{t_0}(f) \in L^2([t_0, \infty); \mathbb{R}^n)$ for all $f \in L^2([t_0, \infty); \mathbb{R}^n)$.*

(ii) *For all $t_0 \in \mathbb{R}_+$ the operator $L^Q_{t_0}$ is bounded by*

$$\|L^Q_{t_0}\| \leq \frac{M}{\mu} \left\| PG^{-1}|_{[t_0,\infty)} \right\|_\infty + \left\| QG^{-1}|_{[t_0,\infty)} \right\|_\infty.$$

(iii) *$t_0 \mapsto \|L^Q_{t_0}\|$ is monotonically nonincreasing on $\mathbb{R}_+$, i.e.,*

$$\|L^Q_{t_0}\| \geq \|L^Q_{t_1}\|, \quad 0 \leq t_0 \leq t_1.$$

Proof: See [12, 18]. □

As mentioned before, the perturbation operator is motivated by the variation of constants formula (2.11), but since an introduction of a solution theory for (1.1) involving L^2-inhomogeneities and therefore Sobolev spaces for the solutions would be very technical and not provide any more insight, we restricted ourselves to the class of continuous solutions as introduced in Definition 2.4. Nevertheless, Lemma 5.2 shows that the perturbation operator is well-defined. Furthermore, the dependence on the projector Q is only weak - on the set of continuous L^2-functions all perturbation operators corresponding to different projectors coincide.

Lemma 5.3. *Under the assumptions of Definition 5.1 and for any $t_0 \in \mathbb{R}_+$ we have that: If, for any $f \in \mathcal{C}([t_0,\infty);\mathbb{R}^n) \cap L^2([t_0,\infty);\mathbb{R}^n)$, $\varphi_{t_0}^f$ is the map*

$$\varphi_{t_0}^f : \mathbb{R}^n \to \mathcal{C}([t_0,\infty);\mathbb{R}^n),\ x^0 \mapsto x|_{[t_0,\infty)}, \quad \text{where } x \in \mathcal{C}_{E,A,f} \text{ and } E(t_0)(x(t_0)-x^0) = 0,$$

which is well-defined by Proposition 2.19, then

$$\forall\, f \in \mathcal{C}([t_0,\infty);\mathbb{R}^n) \cap L^2([t_0,\infty);\mathbb{R}^n) : \ L_{t_0}^Q(f) = \varphi_{t_0}^f(0).$$

If we consider perturbations of (1.1) in the leading term as introduced in Section 3, then the perturbed system (3.1) may also be interpreted as a closed-loop system obtained from (1.2) by applying the time-varying *derivative feedback*

$$f(t) = u(t) = -\Delta_E(t)\dot{x}(t).$$

We show now that robustness of exponential stability can be related to the inverse norm of the perturbation operator. In fact, we prove that the latter provides a calculable bound on the perturbation such that exponential stability is preserved. This result is a DAE-version of [29, Cor. 4.3] and to this end we also introduce the notation

$$\ell(E,A,Q) := \lim_{t_0\to\infty} \|L_{t_0}^Q\|^{-1} \overset{\text{Lem. 5.2}}{=} \sup_{t_0\geq 0} \|L_{t_0}^Q\|^{-1}$$

for index-1 $(E,A) \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^{n\times n})^2$ and $Q \in \mathfrak{Q}_{E,A}$. Note that $\ell(E,A,Q) = \infty$ is explicitly allowed. The next theorem states that if the perturbation term Δ as in (3.5) is sufficiently small, then exponential stability is preserved. We like to remark again that if (E,A) is index-1 and $E \in \mathcal{C}^1(\mathbb{R}_+;\mathbb{R}^{n\times n})$, then there always exists a bounded $Q \in \mathfrak{Q}_{E,A}$.

Theorem 5.4 (Exponential stability and perturbation operator)**.** *Let $(E,A) \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^{n\times n})^2$ be index-1 and exponentially stable and suppose that there exists a bounded $Q \in \mathfrak{Q}_{E,A}$. Let P and G be as in* (2.3) *and suppose that G^{-1} is bounded. Furthermore, let $\Delta_E \in \mathcal{P}_{E,A}^Q$ and suppose that for Δ as in* (3.5) *the matrix ΔP is bounded. If*

$$\lim_{t_0\to\infty} \left\| (\Delta P)|_{[t_0,\infty)} \right\|_\infty < \begin{cases} \min\left\{\ell(E,A,Q), \|QG^{-1}\|_\infty^{-1}\right\}, & \text{if } Q \neq 0,\\ \ell(E,A,Q), & \text{if } Q = 0,\end{cases}$$

then the perturbed system (3.1) *is exponentially stable.*

Proof: *Case 1: $Q \neq 0$.* First note that $t_0 \mapsto \left\| (\Delta P)|_{[t_0,\infty)} \right\|_\infty$ is monotonically decreasing on $\mathbb{R}_+$ and hence the limit always exists since ΔP is bounded. Then, it follows from the fact that $t_0 \mapsto \|L_{t_0}^Q\|^{-1}$ is monotonically increasing and the assumption, that there exists $\hat{t} \in \mathbb{R}_+$ such that

$$\left\| (\Delta P)|_{[t_0,\infty)} \right\|_\infty < \min\left\{\|L_{t_0}^Q\|^{-1}, \|QG^{-1}\|_\infty^{-1}\right\}, \quad t_0 \geq \hat{t}. \tag{5.1}$$

By exponential stability of (E,A) we have (4.2), where $\Phi(\cdot,\cdot)$ is the transition matrix of (E,A). In order to show that (3.1) is exponentially stable we will show in Step 1 that $k_B(E+\Delta_E,A)<\infty$ using Lemma 4.2 and then in Step 2, using Lemma 4.7, that $k_B(E+\Delta_E,A)<0$. Note that Lemma 4.7 is applicable since $Q\in\mathfrak{Q}_{E,A}=\mathfrak{Q}_{E+\Delta_E,A}$ is bounded. This means to show that there exist $c_1,c_2>0$ such that for the transition matrix $\tilde\Phi(\cdot,\cdot)$ of $(E+\Delta_E,A)$ it holds that

$$\sup_{0\le t-t_0\le 1}\|\tilde\Phi(t,t_0)\|\le c_1 \quad\text{and}\quad \forall\, t_0\in\mathbb{R}_+:\ \int_{t_0}^{\infty}\|\tilde\Phi(t,t_0)\|^2\ \mathrm{d}t\ \le c_2.$$

Fix $s\ge\hat t$ and let $\bar A$ be as in (2.3). Then $\tilde\Phi(\cdot,\cdot)$ satisfies (3.4) as a matrix equation, which read, for all $t\ge s$,

$$\left\{\begin{array}{rcl}\frac{\mathrm{d}}{\mathrm{d}t}(P(t)\tilde\Phi(t,s)) &=& (\dot P(t)+P(t)G(t)^{-1}\bar A(t))P(t)\tilde\Phi(t,s)+P(t)G(t)^{-1}\Delta(t)P(t)\tilde\Phi(t,s),\\ Q(t)\tilde\Phi(t,s) &=& Q(t)G(t)^{-1}\bar A(t)P(t)\tilde\Phi(t,s)+Q(t)G(t)^{-1}\Delta(t)P(t)\tilde\Phi(t,s).\end{array}\right. \tag{5.2}$$

Proceeding as in the proof of Theorem 4.8, and taking into account that $P(s)\tilde\Phi(s,s)x^0=P(s)x^0$ for $x^0\in\mathbb{R}^n$, we find that applying the variation of constants formula (2.11) yields

$$\tilde\Phi(t,s)x^0=\Phi(t,s)P(s)x^0+\int_s^t\Phi(t,\tau)P(\tau)G(\tau)^{-1}\Delta(\tau)P(\tau)\tilde\Phi(\tau,s)x^0\ \mathrm{d}\tau+Q(t)G(t)^{-1}\Delta(t)P(t)\tilde\Phi(t,s)x^0 \tag{5.3}$$

for all $x^0\in\mathbb{R}^n$.

Step 1: We show that $\sup_{0\le t-t_0\le 1}\|\tilde\Phi(t,t_0)\|\le c_1$. Let $t_0\ge\hat t$ and observe that

$$\left\|(QG^{-1}\Delta P)|_{[t_0,\infty)}\right\|_\infty\le\|QG^{-1}\|_\infty\left\|(\Delta P)|_{[t_0,\infty)}\right\|_\infty<1$$

by (5.1). This gives

$$\begin{aligned}e^{\mu t}\|\tilde\Phi(t,t_0)x^0\|\le\Big(1-\|QG^{-1}\|_\infty\left\|(\Delta P)|_{[t_0,\infty)}\right\|_\infty\Big)^{-1}&Me^{\mu t_0}\|P(t_0)x^0\|\\ &+\|PG^{-1}\|_\infty\left\|(\Delta P)|_{[t_0,\infty)}\right\|_\infty\int_{t_0}^t Me^{\mu\tau}\|\tilde\Phi(\tau,t_0)x^0\|\ \mathrm{d}\tau\end{aligned}$$

for all $x^0\in\mathbb{R}^n$ and an application of Gronwall's inequality (see e.g. [32, Lem. 2.1.18]) yields

$$\|\tilde\Phi(t,t_0)x^0\|\le\kappa_1\|P(t_0)\|e^{-\mu(t-t_0)}\|x^0\|e^{\kappa_2(t-t_0)},$$

where $\kappa_1=\Big(1-\|QG^{-1}\|_\infty\left\|\Delta P|_{[\hat t,\infty)}\right\|_\infty\Big)^{-1}M$ and $\kappa_2=\|PG^{-1}\|_\infty\|\Delta P\|_\infty M$. This immediately implies that

$$\|\tilde\Phi(t,t_0)x^0\|\le c_1\|x^0\|,\quad c_1=\kappa_1\|P\|_\infty e^{\kappa_2},\ t\in[t_0,t_0+1],$$

and c_1 is independent of $t_0\ge\hat t$. It remains to prove that $\sup_{t\in[t_0,t_0+1]}\|\tilde\Phi(t,t_0)\|\le\tilde c_1$ for all $0\le t_0\le\hat t$ and some $\tilde c_1>0$. However, this is clear since the mapping $t_0\mapsto\sup_{t\in[t_0,t_0+1]}\|\tilde\Phi(t,t_0)\|$ is uniformly continuous on $[0,\hat t]$.

Step 2. We show that $\int_{t_0}^\infty\|\tilde\Phi(t,t_0)\|^2\ \mathrm{d}t\ \le c_2$ for all $t_0\in\mathbb{R}_+$. To this end, consider, for $\hat t\le s\le T$, the operator

$$M_{s,T}:\mathbb{R}^n\to L^2([\hat t,\infty);\mathbb{R}^n),\ x^0\mapsto x_{s,T}(\cdot):=\mathbb{1}_{[s,T)}(\cdot)\tilde\Phi(\cdot,s)x^0.$$

Let, for $x^0\in\mathbb{R}^n$, $x_{0,s,T}(\cdot):=\mathbb{1}_{[s,T)}(\cdot)\Phi(\cdot,s)P(s)x^0$ and define the operator

$$L^Q_{s,T}:L^2([s,\infty);\mathbb{R}^n)\to L^2([s,\infty);\mathbb{R}^n),\ f\mapsto\mathbb{1}_{[s,T)}L^Q_s(f).$$

Then we have

$$x_{s,T}(t) = x_{0,s,T}(t) + L^Q_{s,T}(\Delta P x_{s,T})(t), \quad t \geq s. \tag{5.4}$$

Note that $x_{0,s,T}|_{[s,\infty)}, x_{s,T}|_{[s,\infty)} \in L^2([s,\infty);\mathbb{R}^n)$. By (5.1) we find that the operator

$$K : L^2([s,\infty);\mathbb{R}^n) \to L^2([s,\infty);\mathbb{R}^n),\ f \mapsto x_{0,s,T}|_{[s,\infty)} + L^Q_{s,T}(\Delta P f)$$

is a contraction and hence the Banach fixed-point theorem yields that $x_{s,T}$ is the unique solution of (5.4) and

$$\|x_{s,T}\|_{L^2[s,\infty)} \leq \|(I - L^Q_{s,T}\Delta P)^{-1}\|\|x_{0,s,T}\|_{L^2[s,\infty)} \leq \underbrace{\left(1 - \|L^Q_{s,T}\| \left\| (\Delta P)|_{[s,\infty)} \right\|_\infty\right)^{-1}}_{=:\kappa_{s,T}} \|x_{0,s,T}\|_{L^2[s,\infty)},$$

and by exponential stability of (E, A),

$$\|x_{s,T}\|_{L^2[\hat{t},\infty)} = \|x_{s,T}\|_{L^2[s,\infty)} \leq \frac{\kappa_{s,T} M}{\sqrt{2\mu}} \sqrt{1 - e^{-2\mu(T-s)}}\|x^0\|.$$

Now, we have $\|L^Q_{s,T}\| \leq \|L^Q_s\| \leq \|L^Q_{\hat{t}}\|$ and $\left\| (\Delta P)|_{[s,\infty)} \right\|_\infty \leq \left\| (\Delta P)|_{[\hat{t},\infty)} \right\|_\infty$, thus

$$\kappa_{s,T} \leq \left(1 - \|L^Q_{\hat{t}}\| \left\| \Delta P|_{[\hat{t},\infty)} \right\|_\infty\right)^{-1}, \quad \hat{t} \leq s \leq T.$$

Therefore, we find that for all $x^0 \in \mathbb{R}^n$

$$\sup\left\{ \|M_{s,T} x^0\|_{L^2[\hat{t},\infty)} \;\middle|\; (s,T) \in \mathbb{R}^2 \text{ and } \hat{t} \leq s \leq T \right\} < \infty,$$

and hence the uniform boundedness principle yields existence of $K > 0$ such that

$$\forall\, \hat{t} \leq s \leq T : \ \|M_{s,T}\|_{L^2[\hat{t},\infty)} \leq K.$$

This implies that, for all $x^0 \in \mathbb{R}^n$ and $s \geq \hat{t}$, we have

$$\int_s^\infty \|\tilde{\Phi}(t,s)x^0\|^2 \, \mathrm{d}t = \lim_{T\to\infty} \int_s^T \|(M_{s,T}x^0)(t)\|^2 \, \mathrm{d}t \leq K^2\|x^0\|^2,$$

thus $\int_s^\infty \|\tilde{\Phi}(t,s)\|^2 \, \mathrm{d}t \leq K^2$ and K is independent of s. Since we had fixed $s \geq \hat{t}$ it remains to prove the assertion for $t_0 \leq \hat{t}$. The latter follows from

$$\int_{t_0}^\infty \|\tilde{\Phi}(t,t_0)\|^2 \, \mathrm{d}t \leq \int_0^{\hat{t}} \|\tilde{\Phi}(t,0)\|^2 \, \mathrm{d}t \sup_{t_0[0,\hat{t}]} \|\tilde{\Phi}(0,t_0)\|^2 + \int_{\hat{t}}^\infty \|\tilde{\Phi}(t,\hat{t})\|^2 \, \mathrm{d}t \sup_{t_0\in[0,\hat{t}]} \|\tilde{\Phi}(\hat{t},t_0)\|^2 < \infty,$$

which holds by continuity of $\tilde{\Phi}(\cdot,\cdot)$.
Case 2: $Q = 0$. The proof of this case is established along similar lines. □

Note that in Theorem 5.4 the case $Q = 0$ means that E is invertible everywhere and hence (E, A) is an implicit ODE. Furthermore, the boundedness of G^{-1} and ΔP is guaranteed if Q, $\dot{Q}$, $(E - AQ)^{-1}$, A and Δ_E are bounded and $\|\Delta_E(E - AQ)^{-1}\|_\infty < 1$.
Note also that the case $k_B(E, A) = -\infty$ is explicitly allowed in Theorem 5.4.

Remark 5.5 (Structured vs. unstructured)**.** Note that we consider *unstructured* perturbations, in contrast to the *structured* perturbations (of the A matrix) considered in [29] for ODEs, or in [12, 18] for index-1 DAEs. However, it is not easy to incorporate structured perturbations in the setting of perturbations of the leading term E, since the proof of Theorem 5.4 does only work in the unstructured case. So there is no direct motivation for the consideration of the perturbation operator corresponding to structured perturbations of the form $\Delta_E = B\tilde{\Delta}C$ in (3.1). $\diamond$

The following corollary gives a bound directly on the perturbation Δ_E such that exponential stability is preserved for all perturbations within the so defined set.

Corollary 5.6. *Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ be index-1 and exponentially stable and suppose that there exists a bounded $Q \in \mathfrak{Q}_{E,A}$. Let P and G be as in* (2.3) *and suppose that G^{-1}, $P(E - AQ)^{-1}$ and $P(E - AQ)^{-1}A(P - \dot{Q}P)$ are bounded. Furthermore, let $\Delta_E \in \mathcal{P}^Q_{E,A}$ be bounded and suppose that $\Delta_E \neq 0$, which readily implies $P \neq 0$. Set $\kappa_1 := \|P(E - AQ)^{-1}A(P - \dot{Q}P)\|_\infty \geq 0$ and $\kappa_2 := \|P(E - AQ)^{-1}\|_\infty > 0$. If*

$$\lim_{t_0\to\infty} \left\| \Delta_E|_{[t_0,\infty)} \right\|_\infty < \begin{cases} \frac{\min\left\{\ell(E,A,Q), \|QG^{-1}\|_\infty^{-1}\right\}}{\kappa_1+\kappa_2 \min\left\{\ell(E,A,Q), \|QG^{-1}\|_\infty^{-1}\right\}}, & \text{if } Q \neq 0, \\ \frac{\ell(E,A,Q)}{\|E^{-1}A\|_\infty+\|E^{-1}\|_\infty \ell(E,A,Q)}, & \text{if } Q = 0 \ \wedge\ \ell(E, A, Q) < \infty, \\ \infty, & \text{if } Q = 0 \ \wedge\ \ell(E, A, Q) = \infty, \end{cases}$$

then the perturbed system (3.1) *is exponentially stable.*

Proof: *Case 1*: $Q \neq 0$. First note that by assumption $\left\| \Delta_E|_{[t_0,\infty)} \right\|_\infty < \kappa_2^{-1} = \|P(E - AQ)^{-1}\|_\infty^{-1}$ for t_0 large enough. Furthermore, Lemma 3.9 yields (cf. also (4.8)), for Δ as in (3.5) and t_0 large enough,

$$\left\| (\Delta P)|_{[t_0,\infty)} \right\|_\infty \leq \frac{\|P(E - AQ)^{-1}A(P - \dot{Q}P)\|_\infty \left\| \Delta_E|_{[t_0,\infty)} \right\|_\infty}{1 - \|P(E - AQ)^{-1}\|_\infty \left\| \Delta_E|_{[t_0,\infty)} \right\|_\infty},$$

thus the statement follows from Theorem 5.4.
Case 2: $Q = 0$. In this case, observe that $G = E$ and $P = I$, thus the proof is similar to Case 1. $\square$

Remark 5.7 (ODE case)**.** Consider Corollary 5.6 with $(E, A) = (I, A)$, i.e., an ODE and suppose that $\ell(I, A, 0) < \infty$. In this case the provided bound on the perturbation $\Delta_E \in \mathcal{P}^0_{I,A}$ is

$$\lim_{t_0\to\infty} \left\| \Delta_E|_{[t_0,\infty)} \right\|_\infty < \frac{\ell(I, A, 0)}{\|A\|_\infty + \ell(I, A, 0)}.$$

The latter corresponds to ODE results in the following way: Rewrite the perturbed equation $(I + \Delta_E(t))\dot{x} = A(t)x$ as

$$\dot{x} = (I + \Delta_E(t))^{-1}A(t)x = A(t)x - \Delta_E(t)(I + \Delta_E(t))^{-1}A(t)x,$$

where it is worth noting that $I + \Delta_E$ is invertible everywhere as Δ_E does not change the kernel of the identity by assumption. Now $D := -\Delta_E(I + \Delta_E)^{-1}A$ can be viewed as a perturbation term within the classical ODE theory and the ODE result corresponding to Corollary 5.6 is [29, Cor. 4.3]. We show that the estimate [29, (4.13)] follows from our estimate in Corollary 5.6. To this end denote by

$$\|M\|_{t_0} := \left\| M|_{[t_0,\infty)} \right\|_\infty$$

for any $M \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ and $t_0 \in \mathbb{R}_+$. Then we have, for t_0 sufficiently large,

$$\|D\|_{t_0} \le \frac{\|\Delta_E\|_{t_0}\|A\|_{t_0}}{1-\|\Delta_E\|_{t_0}} < \frac{\|A\|_{t_0}\frac{\ell(I,A,0)}{\|A\|_\infty+\ell(I,A,0)}}{1-\frac{\ell(I,A,0)}{\|A\|_\infty+\ell(I,A,0)}} = \frac{\|A\|_{t_0}\ell(I,A,0)}{\|A\|_\infty} \le \ell(I,A,0),$$

from which the assertion follows. ◇

The next theorem is a version of [29, Prop. 4.5] and [12, Thm. 5.8] for perturbations of the leading coefficient E of an index-1 DAE (E,A). It is a further robustness result under perturbations within the class $\mathcal{P}^Q_{E,A}$, as it shows, for perturbations which converge to zero, that the norm of the difference of the two perturbation operators corresponding to the nominal system and the perturbed system, for the same projector Q, gets arbitrary small for sufficiently large t_0.

Theorem 5.8 (Perturbation operator under perturbations)**.** *Let $(E,A) \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^{n\times n})^2$ be index-1 and exponentially stable and suppose that there exists a bounded $Q \in \mathfrak{Q}_{E,A}$. Let P and G be as in* (2.3) *and suppose that G^{-1}, $P(E-AQ)^{-1}$ and $P(E-AQ)^{-1}A(P-\dot{Q}P)$ are bounded. Furthermore, let $\Delta_E \in \mathcal{P}^Q_{E,A}$ be such that $\|\Delta_E(E-AQ)^{-1}\|_\infty < 1$. If*

$$\lim_{t\to\infty}\|\Delta_E(t)\| = 0,$$

then for the perturbation operator $L^Q_{t_0}$ of (E,A) corresponding to Q and the perturbation operator $\tilde{L}^Q_{t_0}$ of the perturbed system $(E+\Delta_E,A)$ corresponding to Q it holds

$$\lim_{t_0\to\infty}\|L^Q_{t_0}-\tilde{L}^Q_{t_0}\| = 0.$$

In particular

$$\ell(E,A,Q) = \ell(E+\Delta_E,A,Q).$$

Proof: First note that by Lemma 3.5, $Q \in \mathfrak{Q}_{E+\Delta_E,A}$ and hence the perturbation operator $\tilde{L}^Q_{t_0}$ of $(E+\Delta_E,A)$ corresponding to Q exists. Suppose that $P \neq 0$, because otherwise $\Delta_E = \Delta_E P = 0$ and the result is trivially verified. We proceed in several steps and incorporate some ideas of the proof of [12, Thm. 5.8] which treats perturbations of A.
Step 1: Let $t_0 \in \mathbb{R}_+$ and $f \in L^2([t_0,\infty);\mathbb{R}^n)$ be fixed. Denote by $\Phi(\cdot,\cdot)$ the transition matrix of (E,A) and by $\tilde{\Phi}(\cdot,\cdot)$ the transition matrix of $(E+\Delta_E,A)$. Let $\tilde{G} := E+\Delta_E+((E+\Delta_E)\dot{Q}-A)Q \in \mathcal{C}(\mathbb{R}_+;\mathbf{Gl}_n(\mathbb{R}))$. Then, by the definition of the perturbation operator, we have

$$\begin{aligned}(L^Q_{t_0}(f)-\tilde{L}^Q_{t_0}(f))(t) &= \int_{t_0}^t \big(\Phi(t,s)P(s)G(s)^{-1}-\tilde{\Phi}(t,s)P(s)\tilde{G}(s)^{-1}\big)f(s)\,\mathrm{d}s \\ &\quad + Q(t)\big(G(t)^{-1}-\tilde{G}(t)^{-1}\big)f(t) \\ &= \underbrace{\int_{t_0}^t \big(\Phi(t,s)-\tilde{\Phi}(t,s)\big)P(s)\tilde{G}(s)^{-1}f(s)\,\mathrm{d}s}_{=:F_1(t_0,f)(t)} \\ &\quad \underbrace{+\int_{t_0}^t \Phi(t,s)P(s)\big(G(s)^{-1}-\tilde{G}(s)^{-1}\big)f(s)\,\mathrm{d}s}_{=:F_2(t_0,f)(t)} + \underbrace{Q(t)\big(G(t)^{-1}-\tilde{G}(t)^{-1}\big)f(t)}_{=:F_3(t_0,f)(t)}\end{aligned}$$

and therefore

$$\left\|L^Q_{t_0}(f)-\tilde{L}^Q_{t_0}(f)\right\|_{L^2[t_0,\infty)} \le \|F_1(t_0,f)\|_{L^2[t_0,\infty)} + \|F_2(t_0,f)\|_{L^2[t_0,\infty)} + \|F_3(t_0,f)\|_{L^2[t_0,\infty)}.$$

Step 2: We show that $\|F_1(t_0,f)\|_{L^2[t_0,\infty)} \le \frac{\kappa_1 \left\| \Delta_E|_{[t_0,\infty)} \right\|_\infty}{(1-\|\Delta_E(E-AQ)^{-1}\|_\infty)^2} \|f\|_{L^2[t_0,\infty)}$ for some $\kappa_1 > 0$ independent of t_0 and f. Taking into account that by (2.10) it holds $P(t)\Phi(t,t_0)P(t_0) = \Phi_0(t,t_0)P(t_0) = P(t)\Phi(t,t_0)$ for all $t \ge t_0$ we obtain from (5.3) that

$$P(t)\tilde{\Phi}(t,t_0) - P(t)\Phi(t,t_0) = \int_{t_0}^{t} P(t)\Phi(t,s)G(s)^{-1}\Delta(s)P(s)\tilde{\Phi}(s,t_0)\ \mathrm{d}s \tag{5.5}$$

for Δ as in (3.5). Furthermore, by the second equation in (5.2) it holds

$$Q(t)\tilde{\Phi}(t,t_0) - Q(t)\Phi(t,t_0) = Q(t)G(t)^{-1}\bar{A}(t)\big(P(t)\tilde{\Phi}(t,t_0) - P(t)\Phi(t,t_0)\big) + Q(t)G(t)^{-1}\Delta(t)P(t)\tilde{\Phi}(t,t_0) \tag{5.6}$$

for all $t \ge t_0$. Due to Proposition 4.11 and Lemma 3.9 we find that $k_B(E+\Delta_E, A) = k_B(E,A) < 0$, the latter inequality holding by assumption. Lemma 4.2 and the boundedness of Q then yield that there exist $M_1, M_2, \mu > 0$ such that, for all $t \ge t_0$,

$$\begin{aligned} \|\Phi(t,t_0)\| &\le M_1 e^{-\mu(t-t_0)}, & \|\tilde{\Phi}(t,t_0)\| &\le M_1 e^{-\mu(t-t_0)}, \\ \|P(t)\Phi(t,t_0)\| &\le M_2 e^{-\mu(t-t_0)}, & \|P(t)\tilde{\Phi}(t,t_0)\| &\le M_2 e^{-\mu(t-t_0)}. \end{aligned}$$

Applying this to (5.5) and (5.6) and noting that by [12, Lem. 4.3] and $k_B(E,A) < \infty$ the matrix $QG^{-1}\bar{A}$ is bounded, we may calculate

$$\begin{aligned} &\|\tilde{\Phi}(t,t_0) - \Phi(t,t_0)\| \\ \le\ & \|P(t)\tilde{\Phi}(t,t_0) - P(t)\Phi(t,t_0)\| + \|Q(t)\tilde{\Phi}(t,t_0) - Q(t)\Phi(t,t_0)\| \\ \overset{(5.6)}{\le}\ & (1 + \|QG^{-1}\bar{A}\|_\infty)\|P(t)\tilde{\Phi}(t,t_0) - P(t)\Phi(t,t_0)\| + \|Q(t)G(t)^{-1}\Delta(t)P(t)\|\|P(t)\tilde{\Phi}(t,t_0)\| \\ \overset{(5.5)}{\le}\ & (1 + \|QG^{-1}\bar{A}\|_\infty) \int_{t_0}^{t} M_2 e^{-\mu(t-s)}\|G^{-1}\|_\infty \|\Delta(s)P(s)\| M_2 e^{-\mu(s-t_0)}\ \mathrm{d}s \\ & + M_2\|QG^{-1}\|_\infty \|\Delta(t)P(t)\| e^{-\mu(t-t_0)} \\ \le\ & (1 + \|QG^{-1}\bar{A}\|_\infty)\|G^{-1}\|_\infty M_2^2 e^{-\mu(t-t_0)} \int_{t_0}^{t} \|\Delta(s)P(s)\|\ \mathrm{d}s \\ & + M_2\|QG^{-1}\|_\infty \|\Delta(t)P(t)\| e^{-\mu(t-t_0)}. \end{aligned} \tag{5.7}$$

Let $K_1 := (1 + \|QG^{-1}\bar{A}\|_\infty)\|G^{-1}\|_\infty M_2^2$, $K_2 := M_2\|QG^{-1}\|_\infty$. Invoking Young's inequality for convolutions, i.e.,

$$\|f * g\|_{L^2[0,\infty)} \le \|f\|_{L^1[0,\infty)} \|g\|_{L^2[0,\infty)}, \tag{5.8}$$

for $f \in L^1([0,\infty);\mathbb{R}^n), g \in L^2([0,\infty);\mathbb{R}^n), (f * g)(t) = \int_0^t f(t-s)g(s)\ \mathrm{d}s$, we may calculate that

$$\begin{aligned} \|F_1(t_0,f)\|^2_{L^2[t_0,\infty)} =\ & \int_{t_0}^{\infty} \left\| \int_{t_0}^{t} \big(\Phi(t,s) - \tilde{\Phi}(t,s)\big)P(s)\tilde{G}(s)^{-1}f(s)\ \mathrm{d}s \right\|^2 \mathrm{d}t \\ \overset{(5.7)}{\le}\ & K_1^2\|P\tilde{G}^{-1}\|_\infty^2 \int_{t_0}^{\infty} \left(\int_{t_0}^{t} e^{-\mu(t-s)} \int_{s}^{t} \|\Delta(\tau)P(\tau)\|\ \mathrm{d}\tau\, \|f(s)\|\ \mathrm{d}s \right)^2 \mathrm{d}t \\ & + K_2^2\|P\tilde{G}^{-1}\|_\infty^2 \int_{t_0}^{\infty} \left(\int_{t_0}^{t} e^{-\mu(t-s)} \|\Delta(t)P(t)\|\|f(s)\|\ \mathrm{d}s \right)^2 \mathrm{d}t \\ \le\ & K_1^2\|P\tilde{G}^{-1}\|_\infty^2 \left\| \Delta P|_{[t_0,\infty)} \right\|_\infty^2 \int_{0}^{\infty} \left(\int_{0}^{t} e^{-\mu(t-s)}(t-s)\|f(s+t_0)\|\ \mathrm{d}s \right)^2 \mathrm{d}t \\ & + K_2^2\|P\tilde{G}^{-1}\|_\infty^2 \left\| \Delta P|_{[t_0,\infty)} \right\|_\infty^2 \int_{0}^{\infty} \left(\int_{0}^{t} e^{-\mu(t-s)} \|f(s+t_0)\|\ \mathrm{d}s \right)^2 \mathrm{d}t \end{aligned}$$

$$\begin{aligned}
&\overset{(5.8)}{\leq} K_1^2\|P\tilde{G}^{-1}\|_\infty^2 \left\|\Delta P|_{[t_0,\infty)}\right\|_\infty^2 \left(\int_0^\infty t e^{-\mu t}\,\mathrm{d}t\right)^2 \|f\|^2_{L^2[t_0,\infty)} \\
&\quad + K_2^2\|P\tilde{G}^{-1}\|_\infty^2 \left\|\Delta P|_{[t_0,\infty)}\right\|_\infty^2 \left(\int_0^\infty e^{-\mu t}\,\mathrm{d}t\right)^2 \|f\|^2_{L^2[t_0,\infty)} \\
&= \left(\frac{K_1^2}{\mu^4}+\frac{K_2^2}{\mu^2}\right)\|P\tilde{G}^{-1}\|_\infty^2 \left\|\Delta P|_{[t_0,\infty)}\right\|_\infty^2 \|f\|^2_{L^2[t_0,\infty)}.
\end{aligned}$$

Furthermore, we have

$$\|P\tilde{G}^{-1}\|_\infty = \|PG^{-1}(I+\Delta_E(I+\dot{Q}Q)G^{-1})^{-1}\|_\infty \overset{(3.3)}{\leq} \frac{\|PG^{-1}\|_\infty}{1-\|\Delta_E(E-AQ)^{-1}\|_\infty}$$

and by Lemma 3.9

$$\left\|\Delta P|_{[t_0,\infty)}\right\|_\infty \leq \frac{\|P(E-AQ)^{-1}A(P-\dot{Q}P)\|_\infty \left\|\Delta_E|_{[t_0,\infty)}\right\|_\infty}{1-\|\Delta_E(E-AQ)^{-1}\|_\infty},$$

thus it holds

$$\|F_1(t_0,f)\|_{L^2[t_0,\infty)} \leq \left(\frac{K_1}{\mu^2}+\frac{K_2}{\mu}\right)\frac{\|PG^{-1}\|_\infty\|P(E-AQ)^{-1}A(P-\dot{Q}P)\|_\infty \left\|\Delta_E|_{[t_0,\infty)}\right\|_\infty}{(1-\|\Delta_E(E-AQ)^{-1}\|_\infty)^2}\|f\|_{L^2[t_0,\infty)}.$$

Step 3: We show that $\|F_2(t_0,f)\|_{L^2[t_0,\infty)} \leq \frac{\kappa_2\left\|\Delta_E|_{[t_0,\infty)}\right\|_\infty}{1-\|\Delta_E(E-AQ)^{-1}\|_\infty}\|f\|_{L^2[t_0,\infty)}$ for some $\kappa_2>0$ independent of t_0 and f. To this end observe that

$$\begin{aligned}
\left\|(PG^{-1}-P\tilde{G}^{-1})\Big|_{[t_0,\infty)}\right\|_\infty &= \left\|(PG^{-1}\Delta_E(I+\dot{Q}Q)G^{-1})(I+\Delta_E(I+\dot{Q}Q)G^{-1})^{-1}\Big|_{[t_0,\infty)}\right\|_\infty \\
&\overset{(3.3)}{\leq} \frac{\|PG^{-1}\|_\infty\|P(E-AQ)^{-1}\|_\infty \left\|\Delta_E|_{[t_0,\infty)}\right\|_\infty}{1-\|\Delta_E(E-AQ)^{-1}\|_\infty},
\end{aligned}$$

and hence, using the same techniques as in Step 2, we obtain, with $K_3 := \|PG^{-1}\|_\infty\|P(E-AQ)^{-1}\|_\infty$,

$$\begin{aligned}
\|F_2(t_0,f)\|^2_{L^2[t_0,\infty)} &= \int_{t_0}^\infty \left\|\int_{t_0}^t \Phi(t,s)P(s)\big(G(s)^{-1}-\tilde{G}(s)^{-1}\big)f(s)\,\mathrm{d}s\right\|^2 \mathrm{d}t \\
&\leq \left(\frac{M_1K_3\left\|\Delta_E|_{[t_0,\infty)}\right\|_\infty}{1-\|\Delta_E(E-AQ)^{-1}\|_\infty}\right)^2 \int_0^\infty \left(\int_0^t e^{-\mu(t-s)}\|f(s+t_0)\|\,\mathrm{d}s\right)^2 \mathrm{d}t \\
&\leq \left(\frac{M_1K_3\left\|\Delta_E|_{[t_0,\infty)}\right\|_\infty}{\mu(1-\|\Delta_E(E-AQ)^{-1}\|_\infty)}\right)^2 \|f\|^2_{L^2[t_0,\infty)}.
\end{aligned}$$

Step 4: We show that $\|F_3(t_0,f)\|_{L^2[t_0,\infty)} \leq \frac{\kappa_3\left\|\Delta_E|_{[t_0,\infty)}\right\|_\infty}{1-\|\Delta_E(E-AQ)^{-1}\|_\infty}\|f\|_{L^2[t_0,\infty)}$ for some $\kappa_3>0$ independent of t_0 and f. This is straightforward as

$$\begin{aligned}
\|F_3(t_0,f)\|^2_{L^2[t_0,\infty)} &= \int_{t_0}^\infty \left\|Q(t)\big(G(t)^{-1}-\tilde{G}(t)^{-1}\big)f(t)\right\|^2 \mathrm{d}t \\
&\leq \left(\frac{\|QG^{-1}\|_\infty\|P(E-AQ)^{-1}\|_\infty\left\|\Delta_E|_{[t_0,\infty)}\right\|_\infty}{1-\|\Delta_E(E-AQ)^{-1}\|_\infty}\right)^2 \|f\|^2_{L^2[t_0,\infty)}.
\end{aligned}$$

Step 5: The statement of the theorem now follows from Steps 1-4 together with

$$\lim_{t_0\to\infty} \left\| \Delta_E|_{[t_0,\infty)} \right\|_\infty = 0. \qquad \square$$

We illustrate some of the results by means of our running example.

Example 5.9 (Examples 3.10 and 4.13 revisited)**.** First we calculate $\ell(E, A, Q)$ for the system (E, A) and projector Q given in Example 3.10. Simple calculations yield that the transition matrix of (E, A) is given by

$$\Phi(t, s) = \operatorname{diag}(e^{-(t-s)}, e^{-(t-s)}, 1), \qquad t, s \in \mathbb{R}_+,$$

and the perturbation operator by

$$\begin{aligned} L^Q_{t_0} &: L^2([t_0,\infty);\mathbb{R}^3) \to L^2([t_0,\infty);\mathbb{R}^3), \\ &(f_1(\cdot), f_2(\cdot), f_3(\cdot)) \mapsto \left(t \mapsto \left(\int_{t_0}^t e^{-(t-s)} f_1(s)\, \mathrm{d}s\,, \int_{t_0}^t e^{-(t-s)} f_2(s)\, \mathrm{d}s\,, -f_3(t)\right)\right). \end{aligned}$$

We may now calculate that, for any $t_0 \in \mathbb{R}_+$ and $f \in L^2([t_0,\infty);\mathbb{R}^3)$,

$$\begin{aligned} \|L^Q_{t_0} f\|^2_{L^2[t_0,\infty)} &= \int_0^\infty \left(\int_0^t e^{-(t-s)} f_1(s+t_0)\, \mathrm{d}s\right)^2 \mathrm{d}t + \int_0^\infty \left(\int_0^t e^{-(t-s)} f_2(s+t_0)\, \mathrm{d}s\right)^2 \mathrm{d}t \\ &\quad + \int_{t_0}^\infty f_3(t)^2\, \mathrm{d}t \\ &\le \left(\int_0^\infty e^{-t}\, \mathrm{d}t\right)^2 (\|f_1\|^2_{L^2[t_0,\infty)} + \|f_2\|^2_{L^2[t_0,\infty)}) + \|f_3\|^2_{L^2[t_0,\infty)} \\ &= \|f\|^2_{L^2[t_0,\infty)}, \end{aligned}$$

which gives $\|L^Q_{t_0}\| \le 1$. On the other hand side, for $f = \big(t \mapsto (0, 0, e^{-(t-t_0)})\big) \in L^2([t_0,\infty);\mathbb{R}^3)$ we obtain that

$$\|L^Q_{t_0} f\|^2_{L^2[t_0,\infty)} = \|f\|^2_{L^2[t_0,\infty)} = \frac{1}{2},$$

thus it holds $\|L^Q_{t_0}\| = 1$ for all $t_0 \in \mathbb{R}_+$ and hence $\ell(E, A, Q) = 1$. For the constants in Corollary 5.6 we therefore find that $\kappa_1 = 1$, $\kappa_2 = 1$ and $\min\{\ell(E, A, Q), \|QG^{-1}\|^{-1}_\infty\} = 1$ as it can easily be calculated. Now Corollary 5.6 states that, for the perturbations Δ_E that we consider in Example 3.10, if $\|\Delta_E\| < \frac{1}{2}$, then the perturbed system $(E + \Delta_E, A)$ is exponentially stable. As $\|\Delta_E\| = \sqrt{2}|\delta|$ this gives a bound on δ:

$$|\delta| < \frac{1}{2\sqrt{2}}.$$

Indeed, as seen in Example 3.10, the perturbed system is exponentially stable for all $\delta > -1$ so the above statement is true, but not very sharp. ◇

Remark 5.10 (Semi-explicit systems)**.** We revisit the results of the present and the preceding sections for the special class of semi-explicit systems. These systems play an important role, since in a lot of applications the DAE is in semi-explicit form. As already stated in (3.2), a DAE (E, A) is in semi-explicit form if it takes the form

$$E(t)\dot{x} = \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A(t)x. \tag{5.9}$$

We may now choose $Q = \begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} \end{bmatrix} \in \mathfrak{Q}_{E,A}$, thus $P = \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix}$ and $G = \begin{bmatrix} I_{n_1} & -A_{12} \\ 0 & -A_{22} \end{bmatrix}$. Hence, invoking Proposition 2.8, the system is index-1 if, and only if, A_{22} is invertible everywhere. If we look at perturbations Δ_E, then we find that $\Delta_E \in \mathcal{P}_{E,A}$ if, and only if, $\Delta_E = \begin{bmatrix} \Delta_1 & 0 \\ \Delta_2 & 0 \end{bmatrix}$, $\begin{bmatrix} I + \Delta_1(t) \\ \Delta_2(t) \end{bmatrix}$ has full column rank for all $t \in \mathbb{R}_+$ (guarantees the kernel condition) and

$$G(t) + \Delta_E(t) = \begin{bmatrix} I + \Delta_1(t) & -A_{12}(t) \\ \Delta_2(t) & -A_{22}(t) \end{bmatrix}$$

is invertible for all $t \in \mathbb{R}_+$. Using Schur's complement (see e.g. [32, Lem. A.1.17]) and invertibility of A_{22} the latter is equivalent to invertibility of $I + \Delta_1(t) - A_{12}(t)A_{22}(t)^{-1}\Delta_2(t)$ for all $t \in \mathbb{R}_+$, which is, since by Sylvester's determinant theorem (see [32, Lem. A.1.13])

$$\det\left(I_{n_1} + [I, -A_{12}A_{22}^{-1}]\begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix}\right) = \det\left(I_n + \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix}[I, -A_{12}A_{22}^{-1}]\right),$$

equivalent to invertibility of $I + \begin{bmatrix} \Delta_1(t) \\ \Delta_2(t) \end{bmatrix}[I, -A_{12}(t)A_{22}(t)^{-1}]$ for all $t \in \mathbb{R}_+$. The latter is satisfied if $\left\| I + \begin{bmatrix} \Delta_1(t) \\ \Delta_2(t) \end{bmatrix}[I, -A_{12}(t)A_{22}(t)^{-1}] \right\| < 1$, which exactly characterizes the set $\mathcal{P}_{E,A}^Q$.
The perturbed system

$$\begin{bmatrix} I_{n_1} + \Delta_1(t) & 0 \\ \Delta_2(t) & 0 \end{bmatrix}\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{5.10}$$

can also be rewritten in semi-explicit form. To this end observe that the second equation in (5.10) reads $x_2 = A_{22}^{-1}\Delta_1\dot{x}_1 - A_{22}^{-1}A_{21}x_1$ and inserting this in the first equation gives

$$(I + \Delta_1)\dot{x}_1 = A_{11}x_1 + A_{12}A_{22}^{-1}\Delta_2\dot{x}_1 - A_{12}A_{22}^{-1}A_{21}x_1.$$

It is now straightforward to observe that (5.10) is equivalent to

$$\begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} B(t) & 0 \\ A_{22}(t)^{-1}A_{21}(t) - A_{22}(t)^{-1}\Delta_2(t)B(t) & I_{n_2} \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

where $B = (I + \Delta_1 - A_{12}A_{22}^{-1}\Delta_2)^{-1}(A_{11} - A_{12}A_{22}^{-1}A_{21})$.
The boundedness assumptions of Corollary 4.9, Corollary 5.6 and Theorem 5.8 reduce to the boundedness of A_{22}^{-1}, $A_{12}A_{22}^{-1}$ and $A_{11} - A_{12}A_{22}^{-1}A_{21}$, which is satisfied if A_{11}, A_{12}, A_{21} and A_{22}^{-1} are bounded (but not necessarily A_{22}). The boundedness assumptions of Theorem 5.4 reduce to boundedness of A_{22}^{-1} and $A_{12}A_{22}^{-1}$.
Having a closer look at Theorem 5.4 and Corollary 5.6 we may calculate the constant $\|QG^{-1}\|_\infty^{-1}$ appearing in the bound on the perturbations by observing that $G^{-1} = \begin{bmatrix} I & -A_{12}A_{22}^{-1} \\ 0 & -A_{22}^{-1} \end{bmatrix}$ and hence $\|QG^{-1}\|_\infty^{-1} = \|A_{22}^{-1}\|_\infty^{-1}$. Furthermore, the constants κ_1 and κ_2 in the important Corollary 5.6, which determine the lowest bound on the perturbation so that we still have exponential stability, can be calculated as follows:

$$\kappa_1 = \|PG^{-1}AP\|_\infty = \left\|\begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0 \\ 0 & 0 \end{bmatrix}\right\|_\infty = \|A_{11} - A_{12}A_{22}^{-1}A_{21}\|_\infty$$

$$\kappa_2 = \|PG^{-1}\|_\infty = \left\|\begin{bmatrix} I & -A_{12}A_{22}^{-1} \\ 0 & 0 \end{bmatrix}\right\|_\infty = \|[I, -A_{12}A_{22}^{-1}]\|_\infty.$$

◇

6 Stability radius

In Theorem 5.4 and Corollary 5.6 we have derived a bound on the perturbation such that exponential stability is preserved. This rises the question for the distance to instability of an index-1 DAE (E, A). For ODEs this question has been successfully treated by Hinrichsen and Pritchard, who introduced the stability radius as an appropriate measure for robustness [30, 31]. Roughly speaking, the stability radius is the largest bound ρ such that exponential stability and the "algebraic structure" (which is important for DAEs) of the nominal system is preserved for all perturbations of norm less than ρ. After the investigation by Hinrichsen and Pritchard [30, 31] for time-invariant ODEs, the stability radius was generalized to time-varying ODEs, see e.g. [29, 35]. For time-invariant DAEs a stability radius has been defined and investigated in [8, 10, 17, 51], the most general version (in the sense that the set of allowable perturbations is large) is given in [8], and for time-varying DAEs in [12, 18]. In contrast to the definition of the stability radius for time-varying DAEs given in [12, 18], we define the stability radius by also allowing for perturbations in the leading coefficient matrix E.
For time-invariant DAEs the first approach in this direction was undertaken by Byers and Nichols [10] who also introduced a set of allowable perturbations, that is perturbations which preserve regularity and the so called nilpotent part of the matrix pencil $sE - A$, and defined the stability radius with respect to this set. As shown in the proof of [10, Lem. 3.2], the assumption of preserved nilpotent part is, provided that the perturbation preserves the index-1 property, equivalent to a common *left* kernel of the leading coefficient matrices of the perturbed and the nominal matrix pencil. Therefore, it differs from our approach just in the fact that we require the *right* kernel of E to be preserved. In this sense, our definition of the stability radius can be viewed as both a generalization of the definition given in [10] to time-varying system and as a generalization of the definition given in [12, 18] to a larger set of allowable perturbations with respect to the leading coefficient.
When defining the stability radius one might argue about which norm of the perturbations should be taken. As we will consider perturbations in E and A we need to introduce some common measure of the perturbation matrices Δ_E and Δ_A. In [10] the Frobenius norm $\|[\Delta_E, \Delta_A]\|_F$ is considered, while in [8] the norm of the block matrix $\left\| \begin{bmatrix} \Delta_E & 0 \\ 0 & \Delta_A \end{bmatrix} \right\|$ is used, both contributions considering constant matrices. Here we will use the infinity norm of the time-varying perturbation pair $\|[\Delta_E, \Delta_A]\|_\infty$.

Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ be index-1. We introduce the following sets:

$$\begin{aligned} \mathcal{K}(E, A) &:= \left\{ [\Delta_E, \Delta_A] \in \mathcal{B}(\mathbb{R}_+; \mathbb{R}^{n\times 2n}) \;\middle|\; \forall\, t \in \mathbb{R}_+ :\ \ker E(t) = \ker(E(t) + \Delta_E(t)) \right\}, \\ \mathcal{I} &:= \left\{ (E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2 \;\middle|\; (E, A) \text{ is index-1} \right\}, \\ \mathcal{S} &:= \left\{ (E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2 \;\middle|\; (E, A) \text{ is exponentially stable} \right\}. \end{aligned}$$

$\mathcal{K}(E, A)$ is the set of *allowable perturbations.*

Definition 6.1 (Stability radius)**.** Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ be index-1. Then the *stability radius of* (E, A) is the number

$$r(E, A) := \inf \left\{ \|[\Delta_E, \Delta_A]\|_\infty \;\middle|\; [\Delta_E, \Delta_A] \in \mathcal{K}(E, A) \ \wedge\ \big((E + \Delta_E, A + \Delta_A) \notin \mathcal{I} \ \vee\ (E + \Delta_E, A + \Delta_A) \notin \mathcal{S}\big) \right\},$$

for which $r(E, A) \in [0, \infty]$ holds. ◇

Remark 6.2 (Stability radius)**.**

(i) It is immediate that for exponentially stable index-1 $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ and any perturbation $[\Delta_E, \Delta_A] \in \mathcal{K}(E, A)$ with $\|[\Delta_E, \Delta_A]\|_\infty < r(E, A)$ the perturbed system $(E+\Delta_E, A+\Delta_A)$, which corresponds to the equation

$$\big(E(t) + \Delta_E(t)\big)\dot{x} = \big(A(t) + \Delta_A(t)\big)x, \tag{6.1}$$

is exponentially stable and index-1.

(ii) $r(E, A)$ is the measure of the distance to the nearest allowable system that is not exponentially stable. Note that the infimum is taken over the set $\mathcal{K}(E, A)$. If we had taken a larger set, or all of $\mathcal{B}(\mathbb{R}_+; \mathbb{R}^{n\times 2n})$, the infimum would in most cases be zero. This is due to the fact that arbitrarily small perturbations in E can cause the system to become unstable if no further structure of the perturbations is claimed. This is true even in the time-invariant case, see e.g. [10]. Nevertheless, it is still possible that there are exponentially stable systems with stability radius zero, because arbitrary small perturbations can also change the structure of the system, i.e., destroy the index-1 property; this is illustrated in Example 6.3. However, as shown in Lemma 6.4, under some boundedness assumption this cannot happen anymore.

(iii) Note that for time-invariant DAEs the definition of stability radius given in [8] is more general than ours in the sense that the set of allowable perturbations is larger, as it is only required that the index and the degree of the characteristic polynomial are preserved. However, for time-varying DAEs we have no notion like the characteristic polynomial. Concerning the higher index case see the following item.

(iv) It may be possible to define sets of allowable perturbations and the stability radius for higher index DAEs in the following way: If (E, A) is index-μ tractable in the sense of [42], then assume that the perturbation Δ_E is such that in the chain of matrix functions [42, (2.23)] the kernel of A_i (in the notation of [42]) is preserved for $i = 0, \ldots, \mu-1$; note that $A_0 = E$. This might be a proper generalization of the set $\mathcal{K}(E, A)$. The set $\mathcal{I}$ might be generalized in a straightforward manner to the set of all index-μ systems (E, A). Then it is also an interesting question in what way the so generalized stability radius is related to the one defined in [10] in the case of time-invariant DAEs.

(v) For time-varying ODEs (I, A), the stability radius $r(I, A)$ is, in general, much smaller than the stability radius $r(A)$ defined in [29]. In fact, it may even be that $r(A) = \infty$ and $r(I, A) < \infty$: Consider the system $\dot{x} = -tx$. It is easy to see that for any bounded perturbation $\Delta \in \mathcal{B}(\mathbb{R}_+; \mathbb{R})$ the system $\dot{x} = (-t+\Delta(t))x$ is still exponentially stable, thus $r(-t) = \infty$. On the other hand side, let $[\Delta_E, \Delta_A] \in \mathcal{K}(1, -t)$, that is $1 + \Delta_E(t) \neq 0$ for all $t \in \mathbb{R}_+$. Hence the perturbed system (6.1) can be rewritten as

$$\dot{x} = \frac{-t + \Delta_A(t)}{1 + \Delta_E(t)}\, x$$

and by choosing $\Delta_A \equiv 0$ and, for any $\varepsilon > 0$, $\Delta_E \equiv -1 - \varepsilon$, the perturbed system gets unstable, as it reads $\dot{x} = \frac{t}{\varepsilon}\, x$. Thus $r(1, -t) \leq \|[-1 - \varepsilon, 0]\| = 1 + \varepsilon$ and as $\varepsilon > 0$ was arbitrary we get $r(1, -t) \leq 1 < \infty = r(-t)$. ◇

Example 6.3. Consider system (1.1) with $E = 0$ and $A(t) = \frac{1}{t+1}$. Now let $\Delta_E \equiv 0$ and $\Delta_A \equiv -\delta$ for any $\delta > 0$. Then $[\Delta_E, \Delta_A] \in \mathcal{K}(E, A)$. However, there exists some $t > 0$ such that $A(t) + \Delta_A(t) = \frac{1}{t+1} - \delta = 0$ and hence $(E + \Delta_E, A + \Delta_A) \notin \mathcal{I}$. This means $r(E, A) \leq \|[0, -\delta]\| = \delta$ for all $\delta > 0$, i.e., $r(E, A) = 0$. However, the nominal system (E, A) is exponentially stable, as any solution x satisfies $x(t) = 0$ for all $t \in \mathbb{R}_+$. This shows that $r(E, A) = 0$ and $(E, A) \in \mathcal{S}$, but the vanishing stability radius is only due to the structural index-1 property getting weaker and weaker for increasing time t, which may be compensated by appropriate boundedness conditions, see Lemma 6.4. ◇

As stressed in the preceding example, for an index-1 DAE (E, A) the properties $r(E, A) = 0$ and $(E, A) \in \mathcal{S}$ are not equivalent. If however, some boundedness assumptions are satisfied, then this equivalence becomes valid. This and other properties of the stability radius are derived in the following. Note that the stability radius does not have any invariance properties, as we consider an unstructured stability radius. As shown in [29], the unstructured stability radius is not invariant with respect to Bohl transformations (see also [5] for the latter).

Lemma 6.4 (Properties of the stability radius)**.**

(i) *If $Q \in \mathcal{C}^1(\mathbb{R}_+; \mathbb{R}^{n\times n})$ is such that Q and $\dot{Q}$ are bounded and $Q(t)^2 = Q(t)$ for all $t \in \mathbb{R}_+$ and $(E, A) \in \mathcal{B}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ is such that $Q \in \mathfrak{Q}_{E,A}$ and $(E - AQ)^{-1}$ is bounded, then it holds that*

$$r(E, A) = 0 \iff (E, A) \notin \mathcal{S}.$$

(ii) $r(\alpha(E, A)) = r(\alpha E, \alpha A) = \alpha\, r(E, A)$ *for all* $\alpha \geq 0$.

(iii) *Let $\mathcal{V}(t) \subseteq \mathbb{R}^n$ be a time-varying subspace of $\mathbb{R}^n$ with constant dimension, and define*

$$\mathcal{K}_\mathcal{V} := \left\{ [E, A] \in \mathcal{B}(\mathbb{R}_+; \mathbb{R}^{n\times 2n}) \;\middle|\; (E, A) \text{ is index-1 and } \ker E(t) = \mathcal{V}(t) \text{ for all } t \in \mathbb{R}_+ \right\}.$$

Then the map $\mathcal{K}_\mathcal{V} \ni [E, A] \mapsto r(E, A)$ is continuous.

Proof: (i): "$\Leftarrow$" is clear. To show "$\Rightarrow$" we use the result of Theorem 6.11 which will be proved later. So assume that $r(E, A) = 0$ and $(E, A) \in \mathcal{S}$. Observe that, for G as in (2.3), we have $G^{-1} = (I - \dot{Q}Q)(I + \dot{Q}Q)G^{-1}$ and hence the boundedness of $(E - AQ)^{-1}$, Q and $\dot{Q}$ implies, invoking (3.3), boundedness of G^{-1}. This guarantees $\ell(E, A, Q) \in (0, \infty]$. Together with boundedness of E and A it also follows that κ_1 and κ_2 as in Theorem 6.11 are finite. Now Theorem 6.11 implies $r(E, A) > 0$, a contradiction.
(ii): Follows directly from the definition of the stability radius.
(iii): Let $\varepsilon > 0$ and $[E_1, A_1] \in \mathcal{K}_\mathcal{V}$. Choose $\delta = \varepsilon$ and $[E_2, A_2] \in \mathcal{K}_\mathcal{V}$ such that

$$\|[E_1 - E_2, A_1 - A_2]\|_\infty < \delta.$$

Since $[E_1, A_1]$ is bounded we have $r(E_1, A_1) < \infty$, because $[-E_1, -A_1] \in \mathcal{K}(E_1, A_1)$ but $(E_1 - E_1, A_1 - A - 1) = (0, 0) \notin \mathcal{I}$, thus $r(E_1, A_1) \leq \|[E_1, A_1]\|_\infty$. Let $[\Delta_E, \Delta_A] \in \mathcal{K}(E_1, A_1)$ be such that $(E_1 + \Delta_E, A_1 + \Delta_A) \notin \mathcal{I}$ or $(E_1 + \Delta_E, A_1 + \Delta_A) \notin \mathcal{S}$, that is $r(E_1, A_1) \leq \|[\Delta_E, \Delta_A]\|_\infty$. Since

$$(E_1 + \Delta_E, A_1 + \Delta_A) = (E_2 + (E_1 - E_2) + \Delta_E, A_2 + (A_1 - A_2) + \Delta_A),$$

it follows $r(E_2, A_2) \leq \|[E_1 - E_2, A_1 - A_2]\|_\infty + \|[\Delta_E, \Delta_A]\|_\infty$. Now taking the infimum over all such $[\Delta_E, \Delta_A]$ we obtain that $r(E_2, A_2) \leq \|[E_1 - E_2, A_1 - A_2]\|_\infty + r(E_1, A_1)$, thus having $|r(E_2, A_2) - r(E_1, A_1)| < \delta = \varepsilon$. This proves continuity. □

Note that in Lemma 6.4 (iii) we consider the set of bounded functions to get a proper notion of distance between two pairs of matrix functions. Moreover, as can be deduced from the proof, boundedness is essential in order to get a finite stability radius, which is in turn crucial for continuity. Furthermore, the constant dimension of $\mathcal{V}$ is not restrictive as it was shown in Section 2 that if (E, A) is index-1, then E has constant rank, and hence the kernel is of constant dimension. Therefore, it is shown that the stability radius is continuous on every set of bounded pairs of index-1 matrix functions where the leading coefficients share a common kernel. In fact, this is no longer true on sets where the kernel may change, as the following example illustrates.

Example 6.5. Let $\varepsilon \geq 0$ and consider the system (1.1) with $E = \varepsilon$ and $A = -1$. First we consider the case $\varepsilon > 0$. Let $[\Delta_E, \Delta_A] \in \mathcal{K}(\varepsilon, -1)$. Note that $\varepsilon + \Delta_E(t)$ must be always invertible in order to preserve the kernel and hence (6.1) can be rewritten as

$$\dot{x} = \frac{-1 + \Delta_A(t)}{\varepsilon + \Delta_E(t)} x.$$

Now, for any $\gamma > 0$, $\Delta_E \equiv -\varepsilon - \gamma$ and $\Delta_A \equiv 0$ are allowable perturbations and make the system unstable, as it reads $\dot{x} = \frac{1}{\gamma} x$. Hence, $r(\varepsilon, -1) \leq \|[-\varepsilon - \gamma, 0]\| = \varepsilon + \gamma$ for all $\gamma > 0$, thus $0 \leq r(\varepsilon, -1) \leq \varepsilon$. In particular this gives

$$\lim_{\varepsilon \to 0} r(\varepsilon, -1) = 0.$$

Now, for $\varepsilon = 0$ and any $[\Delta_E, \Delta_A] \in \mathcal{K}(0, -1)$ the system (6.1) reads $0 = (-1 + \Delta_A(t))x$. First observe that $[\Delta_E, \Delta_A] \equiv [0, 1] \in \mathcal{K}(0, -1)$ and the resulting perturbed system reads $0 = 0$ which is not index-1 anymore and has any function as a solution. Therefore, it is in particular not exponentially stable, which gives $r(0, -1) \leq 1$. On the other hand side, for any Δ_A with $\|\Delta_A\|_\infty < 1$ the perturbed system stays exponentially stable, so we obtain $r(0, -1) = 1$. Finally we may conclude

$$\lim_{\varepsilon \to 0} r(\varepsilon, -1) = 0 \neq 1 = r(0, -1).$$

◇

In the following we derive a lower bound for the stability radius. In order to do this we further investigate the perturbation structure. Similar to Section 3 we introduce the following.

Definition 6.6 (Pairs of perturbations)**.** Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n \times n})^2$ be index-1 and $Q \in \mathfrak{Q}_{E,A}$. Then we define

$$\widehat{\mathcal{P}}^Q_{E,A} := \left\{ [\Delta_E, \Delta_A] \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n \times 2n}) \;\middle|\; \begin{array}{l} \forall\, t \in \mathbb{R}_+ :\; \ker E(t) = \ker \big(E(t) + \Delta_E(t)\big) \text{ and} \\ \left\| [\Delta_E(t), \Delta_A(t)] \begin{bmatrix} P(t)(E(t) - A(t)Q(t))^{-1} \\ -Q(t)(E(t) - A(t)Q(t))^{-1} \end{bmatrix} \right\| < 1 \end{array} \right\}$$

◇

It is crucial that perturbations in $\widehat{\mathcal{P}}^Q_{E,A}$ preserve the index-1 property of the nominal system. This is stated in the next lemma.

Lemma 6.7 (Condition for preserved index)**.** *Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n \times n})^2$ be index-1 and $Q \in \mathfrak{Q}_{E,A}$. Then we have*

$$[\Delta_E, \Delta_A] \in \widehat{\mathcal{P}}^Q_{E,A} \quad \Longrightarrow \quad Q \in \mathfrak{Q}_{E+\Delta_E, A+\Delta_A}.$$

Proof: As we only have to show that $E + \Delta_E + ((E + \Delta_E)\dot{Q} - (A + \Delta_A))Q = G + [\Delta_E, \Delta_A] \begin{bmatrix} I + \dot{Q}Q \\ -Q \end{bmatrix}$ is invertible everywhere, the statement follows immediately from the assumptions and the observations $\Delta_E(I + \dot{Q}Q)G^{-1} \overset{(3.3)}{=} \Delta_E P(E - AQ)^{-1}$ and

$$QG^{-1} = Q(I + \dot{Q}Q)G^{-1} = Q(E - AQ)^{-1}. \tag{6.2}$$

□

We may also reformulate the perturbed system (6.1) in a decomposition as in (3.4).

Lemma 6.8 (Decomposition of perturbed system)**.** *Let* $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ *be index-1,* $Q \in \mathfrak{Q}_{E,A}$*,* P*,* $\bar{A}$*,* G *as in* (2.3) *and* $[\Delta_E, \Delta_A] \in \widehat{\mathcal{P}}^Q_{E,A}$*. Then* $x \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^n)$ *is a solution of* (6.1) *if, and only if,* $Px \in \mathcal{C}^1(\mathbb{R}_+; \mathbb{R}^n)$ *and* x *solves* (3.4) *with*

$$\Delta := (I + \Lambda)^{-1}(\Delta_A - \Lambda A)(I - Q\dot{Q}), \qquad \Lambda = [\Delta_E, \Delta_A] \begin{bmatrix} P(E - AQ)^{-1} \\ -Q(E - AQ)^{-1} \end{bmatrix}. \tag{6.3}$$

Proof: The proof is a straightforward modification of the proof of Lemma 3.8. It is only important to use that

$$[\Delta_E, \Delta_A] \begin{bmatrix} -\dot{Q} \\ I \end{bmatrix} = -\Lambda G(I - \dot{Q}Q)\dot{Q} + \Delta_A(I - Q\dot{Q}).$$ □

In fact, with the new Δ in (6.3), it easy to generalize *all* of the results of Sections 4 and 5 to perturbations $[\Delta_E, \Delta_A]$ in E and A. We state this in the following theorem.

Theorem 6.9 (Results for perturbations in E and A)**.** *The statements of Theorem 4.8, Corollary 4.9, Proposition 4.11, Theorem 5.4, Corollary 5.6 and Theorem 5.8 remain the same for perturbations in* E *and* A*, that is they are true if the following substitutions are applied where possible:*

- $\Delta_E \in \mathcal{P}^Q_{E,A} \;\mapsto\; [\Delta_E, \Delta_A] \in \widehat{\mathcal{P}}^Q_{E,A}$,
- $\Delta_E \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n}) \;\mapsto\; (\Delta_E, \Delta_A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$,
- Δ *as in* (3.5) $\mapsto$ Δ *as in* (6.3),
- $k_B(E + \Delta_E, A) \;\mapsto\; k_B(E + \Delta_E, A + \Delta_A)$,
- $\|\Delta_E\|_\infty \;\mapsto\; \|[\Delta_E, \Delta_A]\|_\infty$,
- *perturbed system* (3.1) $\mapsto$ *perturbed system* (6.1),
- $\left\|\Delta_E|_{[t_0,\infty)}\right\|_\infty \;\mapsto\; \left\|[\Delta_E, \Delta_A]|_{[t_0,\infty)}\right\|_\infty$,
- $\lim_{t\to\infty} \|\Delta_E(t)\| = 0 \;\mapsto\; \lim_{t\to\infty} \|[\Delta_E(t), \Delta_A(t)]\| = 0$,
- *perturbed system* $(E + \Delta_E, A)$ $\mapsto$ *perturbed system* $(E + \Delta_E, A + \Delta_A)$,
- $\|\Delta_E(E - AQ)^{-1}\|_\infty < 1 \;\mapsto\; \left\|[\Delta_E, \Delta_A] \begin{bmatrix} (E - AQ)^{-1} \\ -QG^{-1} \end{bmatrix}\right\|_\infty < 1$,
- $\ell(E + \Delta_E, A) \;\mapsto\; \ell(E + \Delta_E, A + \Delta_A)$.

Furthermore, in Corollary 4.9, Corollary 5.6, and Theorem 5.8 the assumption of boundedness of $(I - QG^{-1}A)(P - \dot{Q}P)$ *has to be added and in Corollary 5.6 the constants* κ_1 *and* κ_2 *have to substituted with the ones defined in Theorem 6.11 and in the second case* $\|E^{-1}A\|_\infty$ *has to be substituted with* $\left\|\begin{bmatrix} -E^{-1}A \\ I \end{bmatrix}\right\|_\infty$.

Proof: Except for slight but obvious modifications the proofs of the results need not to be changed if it is remembered that Δ is another matrix. In particular, at some instances Lemma 6.7 must be used instead of Lemma 3.5. However, in two cases some more comments are warrant.
Corollary 4.9: Equation (4.8) has to be changed to the inequality presented in Step 2 of the proof of Theorem 6.11.
Corollary 5.6: The inequality in Case 1 has to be changed in the same manner.

Theorem 5.8: The matrix $\tilde{G}$ changes to $\tilde{G} = G + [\Delta_E, \Delta_A] \begin{bmatrix} I - \dot{P}Q \\ -Q \end{bmatrix} = G + \Lambda G$, where $\Lambda := [\Delta_E, \Delta_A] \begin{bmatrix} P(E-AQ)^{-1} \\ -Q(E-AQ)^{-1} \end{bmatrix}$. Hence, some of the inequalities change as well. For brevity we use the constants κ_1 and κ_2 defined in Theorem 6.11. Then we obtain

$$
\begin{aligned}
\|P\tilde{G}^{-1}\|_\infty &\le \frac{\|PG^{-1}\|_\infty}{1-\|\Lambda\|_\infty}, \\
\left\| \Delta P|_{[t_0,\infty)} \right\|_\infty &\le \frac{\kappa_1 \left\| [\Delta_E, \Delta_A]|_{[t_0,\infty)} \right\|_\infty}{1-\|\Lambda\|_\infty}, \\
\|F_1(t_0,f)\|_{L^2[t_0,\infty)} &\le \left(\frac{K_1}{\mu^2} + \frac{K_2}{\mu} \right) \frac{\|PG^{-1}\|_\infty \kappa_1 \left\| [\Delta_E, \Delta_A]|_{[t_0,\infty)} \right\|_\infty}{(1-\|\Lambda\|_\infty)^2} \|f\|_{L^2[t_0,\infty)}, \\
\left\| (PG^{-1} - P\tilde{G}^{-1})\Big|_{[t_0,\infty)} \right\|_\infty &\le \frac{\|PG^{-1}\|_\infty \kappa_2 \left\| [\Delta_E, \Delta_A]|_{[t_0,\infty)} \right\|_\infty}{1-\|\Lambda\|_\infty}, \\
\|F_2(t_0,f)\|_{L^2[t_0,\infty)} &\le \frac{M_1 \|PG^{-1}\|_\infty \kappa_2 \left\| [\Delta_E, \Delta_A]|_{[t_0,\infty)} \right\|_\infty}{\mu(1-\|\Lambda\|_\infty)} \|f\|_{L^2[t_0,\infty)}, \\
\|F_3(t_0,f)\|_{L^2[t_0,\infty)} &\le \frac{\|QG^{-1}\|_\infty \kappa_2 \left\| [\Delta_E, \Delta_A]|_{[t_0,\infty)} \right\|_\infty}{1-\|\Lambda\|_\infty} \|f\|_{L^2[t_0,\infty)}.
\end{aligned}
$$

□

Nevertheless, we separately state the following generalized version of Theorem 5.4 which is important in due course.

Proposition 6.10 (Exponential stability and perturbation operator anew). *Let $(E,A) \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^{n\times n})^2$ be index-1 and exponentially stable and suppose that there exists a bounded $Q \in \mathfrak{Q}_{E,A}$. Let P and G be as in* (2.3) *and suppose that G^{-1} is bounded. Furthermore, let $[\Delta_E, \Delta_A] \in \widehat{\mathcal{P}}^Q_{E,A}$ and suppose that for Δ as in* (6.3) *the matrix ΔP is bounded. If*

$$
\lim_{t_0\to\infty} \left\| (\Delta P)|_{[t_0,\infty)} \right\|_\infty < \begin{cases} \min\left\{ \ell(E,A,Q), \|QG^{-1}\|_\infty^{-1} \right\}, & \text{if } Q \neq 0, \\ \ell(E,A,Q), & \text{if } Q = 0, \end{cases}
$$

then the perturbed system (6.1) *is exponentially stable.*

The main theorem of this section essentially relies on the preceding proposition. It gives a lower bound for the stability radius in terms of the norm of the perturbation operator, more precisly the number $\ell(E,A,Q)$ introduced in Section 5.

Theorem 6.11 (Lower bound for the stability radius). *Let $(E,A) \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^{n\times n})^2$ be index-1 and exponentially stable and suppose that there exists a bounded $Q \in \mathfrak{Q}_{E,A}$. Let P and G be as in* (2.3) *and suppose that G^{-1} is bounded. Suppose further that* $\kappa_1 := \left\| \begin{bmatrix} -P(E-AQ)^{-1}A(P-\dot{Q}P) \\ (I-Q(E-AQ)^{-1}A)(P-\dot{Q}P) \end{bmatrix} \right\|_\infty < \infty$

and $\kappa_2 := \left\| \begin{bmatrix} P(E-AQ)^{-1} \\ -Q(E-AQ)^{-1} \end{bmatrix} \right\|_\infty < \infty$*, i.e., the corresponding matrices are bounded. Then* $\kappa_2 > 0$ *and*

$$r(E,A) \geq \begin{cases} \frac{\min\left\{\ell(E,A,Q), \|QG^{-1}\|_\infty^{-1}\right\}}{\kappa_1+\kappa_2 \min\left\{\ell(E,A,Q), \|QG^{-1}\|_\infty^{-1}\right\}}, & \text{if } Q \neq 0, \\ \frac{\ell(E,A,Q)}{\kappa_1+\kappa_2 \ell(E,A,Q)}, & \text{if } Q = 0 \ \wedge \ \ell(E,A,Q) < \infty, \\ \infty, & \text{if } Q = 0 \ \wedge \ \ell(E,A,Q) = \infty. \end{cases}$$

Proof: If $P \neq 0$, then $\kappa_2 > 0$ is obvious. If $P = 0$, then $Q = I$ and hence $\kappa_2 > 0$ as well. Now we show that for any $[\Delta_E, \Delta_A] \in \mathcal{K}(E,A)$ with $\|[\Delta_E, \Delta_A]\|_\infty < \frac{\alpha}{\kappa_1+\kappa_2\alpha}$, where $\alpha := \min\left\{\ell(E,A,Q), \|QG^{-1}\|_\infty^{-1}\right\}$, we have $(E+\Delta_E, A+\Delta_A) \in \mathcal{I}$ and $(E+\Delta_E, A+\Delta_A) \in \mathcal{S}$. We discern two cases and proceed in two steps.
Case 1: $Q \neq 0$.
Step 1: We show $(E+\Delta_E, A+\Delta_A) \in \mathcal{I}$. To this end observe that it follows from the assumption that $\kappa_2 \|[\Delta_E, \Delta_A]\|_\infty < \frac{\kappa_2\alpha}{\kappa_1+\kappa_2\alpha} \leq 1$ and hence

$$\|[\Delta_E, \Delta_A]\|_\infty < \left\| \begin{bmatrix} P(E-AQ)^{-1} \\ -Q(E-AQ)^{-1} \end{bmatrix} \right\|_\infty^{-1},$$

which yields $[\Delta_E, \Delta_A] \in \widehat{\mathcal{P}}_{E,A}^Q$, thus, invoking Lemma 6.7, $(E+\Delta_E, A+\Delta_A)$ is index-1.
Step 2: We show $(E+\Delta_E, A+\Delta_A) \in \mathcal{S}$. By Step 1 we have $[\Delta_E, \Delta_A] \in \widehat{\mathcal{P}}_{E,A}^Q$. Further invoking that, for Δ and Λ as in (6.3),

$$\Delta P = (I+\Lambda)^{-1}(\Delta_A - \Lambda A)(I - Q\dot{Q})P = (I+\Lambda)^{-1}[\Delta_E, \Delta_A] \begin{bmatrix} -P(E-AQ)^{-1}A(P-\dot{Q}P) \\ (I - Q(E-AQ)^{-1}A)(P-\dot{Q}P) \end{bmatrix},$$

we obtain

$$\left\| (\Delta P)|_{[t_0,\infty)} \right\|_\infty \leq \frac{\kappa_1 \|[\Delta_E, \Delta_A]\|_\infty}{1 - \kappa_2 \|[\Delta_E, \Delta_A]\|_\infty} < \alpha$$

for all $t_0 \in \mathbb{R}_+$, hence we may apply Proposition 6.10 to conclude exponential stability.
Case 2: $Q = 0$. The proof is similar and omitted. □

Note that in Theorem 6.11 the boundedness of G^{-1} is still important in order to guarantee that $\ell(E,A,Q) \in (0,\infty]$ exists.

Remark 6.12 (Special cases)**.** We consider Theorem 6.11 for two special cases.
Case 1: $E = I$. In this case we have $Q = 0$, thus $P = I$ and hence $\kappa_1 = \left\| \begin{bmatrix} -A \\ I \end{bmatrix} \right\|_\infty$ and $\kappa_2 = 1$. Suppose that $\ell(I,A,0) < \infty$. Then we obtain from Theorem 6.11 that

$$\frac{\ell(I,A,0)}{1 + \|A\|_\infty + \ell(I,A,0)} \leq \frac{\ell(I,A,0)}{\kappa_1 + \ell(I,A,0)} \leq r(I,A).$$

Note that this does not coincide with any bounds known for the stability radius of an ODE, as still perturbations of the identity and therefore multiplicative perturbations of A are possible. More precisly, A may be perturbed to $(I+\Delta_E)^{-1}(A+\Delta_A)$.
If we considered only perturbations in A, then in κ_1 and κ_2 we neglect the first rows (because these correspond to Δ_E) and thus obtain $\kappa_1 = 1$ and $\kappa_2 = 0$, i.e., $\ell(I,A,0) \leq r(I,A)$, which is just the bound obtained in [29, Prop. 4.1] for ODEs.

Case 2: $E = 0$. In the case of a purely algebraic equation we have $Q = I$. This gives $\kappa_1 = 0$ and, as A must be invertible everywhere, $\kappa_2 = \|A^{-1}\|_\infty$. Now Theorem 6.11 gives

$$\|A^{-1}\|_\infty^{-1} \leq r(0, A).$$

This bound is sharp: Any allowable perturbation $[\Delta_E, \Delta_A]$ with $\|[\Delta_E, \Delta_A]\|_\infty < \|A^{-1}\|_\infty^{-1}$ has $\Delta_E = 0$ and the perturbed system (6.1) reads $0 = (A(t) + \Delta_A(t))x$, or, equivalently, $0 = (I + A(t)^{-1}\Delta_A(t))x$. Then

$$\|A(t)^{-1}\Delta_A(t)\| \leq \|\Delta_A\|_\infty \|A^{-1}\|_\infty < 1$$

for all $t \in \mathbb{R}_+$ and the resulting invertibility of $I + A(t)^{-1}\Delta_A(t)$ yields that the perturbed system (6.1) is exponentially stable (as it only has the trivial solution). Therefore, $r(I, A) = \|A^{-1}\|_\infty^{-1}$.
In fact, $\|A^{-1}\|_\infty^{-1}$ also coincides with the stability radius as defined in [12, 18], see [18, Sec. 5.2], which is reasonable as in this case no perturbations of E are involved. ◇

Remark 6.13 (Semi-explicit systems)**.** Consider a semi-explicit system of the form (5.9). We use the results and notation already obtained in Remark 5.10. With these it is easy calculate

$$\kappa_1 = \left\| \begin{bmatrix} -A_{11} + A_{12}A_{22}^{-1}A_{21} \\ I \\ A_{22}^{-1}A_{21} \end{bmatrix} \right\|_\infty, \qquad \kappa_2 = \left\| \begin{bmatrix} I & -A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix} \right\|_\infty,$$

and obtain the corresponding lower bound on the stability radius via Theorem 6.11. ◇

Remark 6.14 (Structured vs. unstructured reloaded)**.** As pointed out in Remark 5.5 we considered unstructured perturbations in Section 5 because we are unable to give a proof for the structured version of Theorem 5.4. For the same reason we consider unstructured perturbations and hence the unstructured stability radius as in Definition 6.1 in this section. More precisly, the essential result on the stability radius is Theorem 6.11, which itself relies on Proposition 6.10, the generalization of Theorem 5.4. Therefore, there is no proof for the structured version of Theorem 6.11, and it is hard to make any reasonable statement about a structured stability radius. ◇

Corollary 6.15 (Set of stable DAEs is open)**.** *Let $Q \in \mathcal{C}^1(\mathbb{R}_+; \mathbb{R}^{n\times n})$ be such that Q and $\dot{Q}$ are bounded and $Q(t)^2 = Q(t)$ for all $t \in \mathbb{R}_+$. Define*

$$\begin{aligned} \mathcal{K}_Q &:= \left\{ \, [E, A] \in \mathcal{B}(\mathbb{R}_+; \mathbb{R}^{n\times 2n}) \;\middle|\; \forall\, t \in \mathbb{R}_+ : \ker E(t) = \operatorname{im} Q(t) \, \right\}, \\ \mathcal{S}_Q &:= \left\{ \, [E, A] \in \mathcal{B}(\mathbb{R}_+; \mathbb{R}^{n\times 2n}) \;\middle|\; Q \in \mathfrak{Q}_{E,A} \;\wedge\; (E, A) \in \mathcal{S} \;\wedge\; (E - AQ)^{-1} \text{ is bounded} \, \right\}. \end{aligned}$$

Then $\mathcal{S}_Q$ is open in $\mathcal{K}_Q$.

Proof: Observe that clearly $\mathcal{S}_Q \subseteq \mathcal{K}_Q$ and let $[E, A] \in \mathcal{S}_Q$. Since, for G as in (2.3), $G^{-1} = (I - \dot{Q}Q)(I + \dot{Q}Q)G^{-1}$ the boundedness of $(E - AQ)^{-1}$, Q and $\dot{Q}$ implies, invoking (3.3), boundedness of G^{-1}. Together with boundedness of E and A it then follows that κ_1 and κ_2 as in Theorem 6.11 are finite. Set

$$\varepsilon := \frac{\alpha}{\kappa_1 + \kappa_2\alpha}, \qquad \text{where } \alpha := \begin{cases} \min\left\{\ell(E, A, Q), \|QG^{-1}\|_\infty^{-1}\right\}, & \text{if } Q \neq 0, \\ \ell(E, A, Q), & \text{if } Q = 0 \;\wedge\; \ell(E, A, Q) < \infty. \end{cases}$$

If $Q = 0$ and $\ell(E, A, Q) = \infty$, set $\varepsilon = 1$ (any positive real number would be sufficient). If now $[\tilde{E}, \tilde{A}] \in \mathcal{K}_Q$ with $\|[E - \tilde{E}, A - \tilde{A}]\|_\infty < \varepsilon$, then it follows that $[\Delta_E, \Delta_A] := [\tilde{E} - E, \tilde{A} - A] \in \mathcal{K}(E, A)$ and hence, applying Theorem 6.11, we may conclude that $(\tilde{E}, \tilde{A}) \in \mathcal{I} \cap \mathcal{S}$. It remains to prove $Q \in \mathfrak{Q}_{\tilde{E},\tilde{A}}$ and boundedness of $(\tilde{E} - \tilde{A}Q)^{-1}$.

To this end observe that $(\tilde{E},\tilde{A}) \in \mathcal{I} \cap \mathcal{K}_Q$ and an application of Proposition 2.8 imply that $Q \in \mathfrak{Q}_{\tilde{E},\tilde{A}}$. We also calculate that

$$(\tilde{E} - \tilde{A}Q)^{-1} = \big((E - AQ) + (\Delta_E - \Delta_A Q)\big)^{-1} = (E - AQ)^{-1}\big(I + (\Delta_E - \Delta_A Q)(E - AQ)^{-1}\big)^{-1}$$

and since

$$\big\|(\Delta_E - \Delta_A Q)(E - AQ)^{-1}\big\|_\infty = \left\|[\Delta_E, \Delta_A]\begin{bmatrix} P(E - AQ)^{-1} \\ -Q(E - AQ)^{-1} \end{bmatrix}\right\|_\infty \le \|[\Delta_E, \Delta_A]\|_\infty \kappa_2 < \varepsilon\kappa_2 \le 1$$

we find

$$\|(\tilde{E} - \tilde{A}Q)^{-1}\|_\infty \le \frac{\|(E - AQ)^{-1}\|_\infty}{1 - \|(\Delta_E - \Delta_A Q)(E - AQ)^{-1}\|_\infty} < \infty.$$

□

Note that in Corollary 6.15 we again consider the set of bounded functions in order to get a proper notion of distance between two pairs of matrix functions.

7 Conclusion

We have studied exponential stability and its robustness of time-varying index-1 DAEs (1.1). We introduced an appropriate class of perturbations in the leading coefficient of the DAE, and derived that exponential stability is robust with respect to these perturbations using the Bohl exponent. As the Bohl exponent approach does not provide a calculable bound on the perturbation we further proved such a bound incorporating the perturbation operator. Moreover, we introduced a stability radius for time-varying DAEs which incorporates perturbations in the leading coefficient and proved basic properties of it. In a main theorem we derived a lower bound for the stability radius.
There are some open questions:

(i) Is it possible to solve the problem of invariance of Bohl exponent $\pm\infty$ as described in Remark 4.12?

(ii) Is it possible to derive a shift property of the Bohl exponent with respect to the leading coefficient? In [5, Prop. 3.11] a shift property with respect to the coefficient matrix A has been derived.

(iii) Is it possible to derive a relation between $k_B(E, A)$ and $\ell(E, A, Q)$, and between $k_B(E, A)$ and $r(E, A)$ using this shift property?

(iv) Is it possible to define the stability radius for higher index DAEs as indicated in Remark 6.2 (iv)?

(v) It has been mentioned that in [10] the Frobenius norm of $[\Delta_E, \Delta_A]$ is used to define the stability radius. Can we expect better results if we use the supremum of the pointwise Frobenius norms instead of the infinity norm?

(vi) Is it possible to derive a formula for the stability radius as derived by Jacob [35] for ODEs and by Du et al. [18] for DAEs (the latter not incorporating perturbations in E)? To this end, it would be interesting to investigate if the lower bound derived in Theorem 6.11 already is the desired formula, if an appropriate (larger) class of perturbations is considered.

(vii) Is it possible to incorporate structured perturbations? As explained in Remarks 5.5 and 6.14, it is hard to incorporate structured perturbations in the perturbation framework presented in the present article.

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