Stochastic partial differential equations with fractal noises: two different approaches

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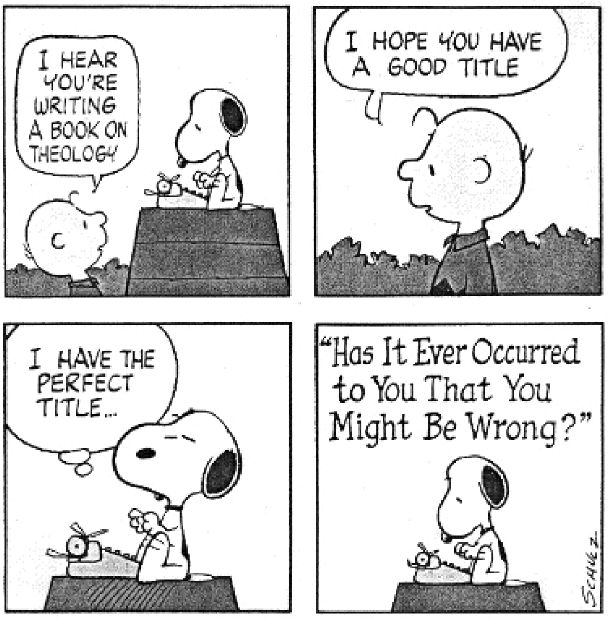

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Zusammenfassung

Diese Arbeit befasst sich mit stochastischen partiellen Differentialgleichungen mit fraktalem Rauschen. In diesem Zusammenhang betrachten und lösen wir verschiedene Probleme aus zwei recht unterschiedlichen Blickwinkeln.

Auf der einen Seite beweisen wir Existenz, Eindeutigkeit und Regularität für milde Lösungen einer parabolischen Transportgleichung mit Diffusion, die nicht-glatte Koeffizienten beinhaltet. Wir untersuchen damit verbundene Cauchy-Probleme auf glatten und beschränkten Gebieten mit Dirichlet-Randbedingungen. Dabei verwenden wir Halbgruppentheorie und Fixpunkt Argumente. Hauptbestandteile sind die Definition eines Produkts einer Funktion und einer (nicht zu unregelmäßigen) Distribution sowie eine zugehörige Norm-Abschätzung. Wir wenden die Theorie auf eine stochastische partielle Transport Differentialgleichung mit fraktalem Brownschen Rauschen an. Dabei wird diese pfadweise betrachtet.

Auf der anderen Seite beschäftigen wir uns mit stochastischen Differentialgleichungen mit gebrochenen Brownschen Prozessen in Banach-Räumen. Genauer gesagt, betrachten wir abstrakte Cauchy-Probleme in Banach-Räumen und suchen nach schwachen und milden Lösungen. Zu diesem Zweck wird eine gebrochene Brownsche Bewegung in separablen Banach-Räume mit Hilfe von zylindrischen Prozessen eingeführt. Wir definieren das damit verbundene stochastische Integral als zylindrischen Prozess und untersuchen seine Eigenschaften. Falls der Banach-Raum einen Funktionenraum darstellt, wird die Gleichung zu einer stochastischen partiellen Differentialgleichung mit fraktalem Rauschen.

Abstract

This thesis deals with stochastic partial differential equations driven by fractional noises. In this work, problems related to this topics are tackled and solved from two fairly different points of view.

On one side we prove existence, uniqueness and regularity for mild solutions to a parabolic transport diffusion type equation that involves a non-smooth coefficient. We investigate related Cauchy problems on bounded smooth domains with Dirichlet boundary conditions by means of semigroup theory and fixed point arguments. Main ingredients are the definition of a product of a function and a (not too irregular) distribution as well as a corresponding norm estimate. As an application, transport stochastic partial differential equations driven by fractional Brownian noises are considered in the pathwise sense.

On the other side we deal with stochastic differential equations driven by fractal noises in Banach spaces. More precisely, we deal with abstract Cauchy problems driven by fractional Brownian processes in Banach spaces and look for weak and mild solutions. To this aim, a fractional Brownian motion in separable Banach spaces is introduced by means of cylindrical processes. The related stochastic integral is then defined as cylindrical stochastic process and its properties are investigated. When the Banach space is a function space then the equation becomes a stochastic partial differential equation driven by a fractional noise.

Contents

List of Notations

$[\cdot,\cdot]$ Scalar product in a Hilbert space

$\langle \nabla \underline{u}(r), \nabla Z \rangle$ Pointwise multiplication between a function and a distribution, page 20

$\mathcal{E}_T, \mathcal{E}_t$ Set of real valued simple functions ϕ on $[0,T]$ and $[0,t]$ respectively, page 46

$\mathscr{E}_T, \mathscr{E}_t$ Set of Hilbert valued simple functions Φ on $[0,T]$ and $[0,t]$ respectively, page 48

Φ Hilbert valued (simple) function (element of $\mathscr{E}_T$ or $\mathscr{H}_T$)

ϕ Real valued (simple) function (element of $\mathcal{E}_T$ or $\mathcal{H}_T$)

$(A, \mathcal{D}(A))$ Infinitesimal generator of a semigroup and its domain, page 13

$(H, [\cdot,\cdot]_H)$ Hilbert space with correspondent scalar product

$(P_t)_{t\geq 0}$ Heat semigroup on $L^2(D)$ generated by Δ_D, page 17

$(T_t)_{t\geq 0}$ Semigroup, page 13

Δ_D Dirichlet Laplacian on D, page 17

$\Gamma_{T,\varphi}$ Decomposition operator for the covariance of the cylindrical integral $\int_0^T \phi \, \mathrm{d}B^H$, page 96

$\langle \cdot,\cdot \rangle_{U,U^*}$ or $\langle \cdot,\cdot \rangle$ Dual pairing (in U, U^*)

$(\mathcal{H}_T, \langle \cdot,\cdot \rangle_{\mathcal{H}_T})$ Hilbert space of real valued integrable (wrt b^H) functions defined as the closure of $\mathcal{E}_T$, page 46

$(\mathscr{H}_T, \langle \cdot,\cdot \rangle_{\mathscr{H}_T})$ Hilbert space of Hilbert space valued integrable (wrt b^H) functions defined as the closure of $\mathscr{E}_T$, page 49

$\mathbb{K}_T^*, \mathbb{K}_t^*$ Isometry between $\mathscr{H}_T$ and $L^2([0,T];H)$ ($\mathscr{H}_t$ and $L^2([0,t];H)$ respectively), page 50

$\mathcal{L}(U,V)$ Space of linear and continuous operators from U to V, page 67

$\mathcal{S}_T$ Space of real valued integrable (wrt b^H) functions on $[0,t]$ for all $t \in [0,T]$, page 50

$\mathscr{B}_T$ Subspace of integrable functions wrt cylindrical fBm in U (with boundedness-type properties), page 93

$\mathscr{I}_T$ Space of integrable functions wrt cylindrical fBm in U, page 93

$\{B^H(t), t \geq 0\}$ Fractional Brownian motion in U, cylindrical and/or classical, page 65

$\{b^H(t), t \geq 0\}$ Real valued fractional Brownian motion

$\{B^H(x)\}_{x \in \mathbb{R}^d}$ Levy fractional Brownian motion, page 36

A^α Fractional power of A, page 16

$C^\gamma([0,T];U)$ Hölder space, page 22

$H \in (0,1)$ Hurst parameter

$H^\alpha(\mathbb{R}^d)$ Fractional Sobolev spaces on $\mathbb{R}$ with $p = 2$

$H_p^\alpha(\mathbb{R}^d)$ Fractional Sobolev spaces on $\mathbb{R}$, page 12

$H_p^\alpha(\mathbb{R}^d; \mathbb{C})$ Fractional Sobolev spaces on $\mathbb{C}$, page 11

H_Q Reproducing kernel Hilbert space (RKHS) associated to Q, page 62

i_Q Inclusion mapping from H_Q to U with decomposition property $Q = i_Q i_Q^*$, page 63

K_T^*, K_t^* Isometry between $\mathcal{H}_T$ and $L^2([0,T])$ (respectively $\mathcal{H}_t$ and $L^2([0,t])$), page 48

$L_\mathbb{P}^0(\Omega;\mathbb{R})$ Space of real valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$

U' Algebraic dual of U

U, V Banach spaces

U^* Topological dual of U

$W_p^m(\mathbb{R}^d)$ Sobolev spaces on $\mathbb{R}$, page 12

Introduction

The main subject of this thesis is the study of stochastic partial differential equations driven by fractional noises. Two different approaches are adopted: a pathwise method is applied to a parabolic transport equation with fractional noise and a statement on existence, uniqueness and regularity of the solution is proven. On the other hand, a more abstract approach is adopted and stochastic partial differential equations with fractal noises are solved as stochastic evolution equations in Banach spaces driven by cylindrical fractional Brownian motions.

The concept of stochastic partial differential equation appears in the literature in the early 1960s. Baklan in [4] proved an existence theorem for a stochastic parabolic equation in a Hilbert space. In the 1970s many researchers, motivated by physical and biological applications, start to become interested in partial differential equations with random parameters. Some examples are Cabana [13], Bensoussan and Temam [8], Pardoux [56], Chow [14], Krylov and Rozovskij [42]. Since then, this area of research has been growing and several theories have been developed to deal with stochastic partial differential equations. One of the main issues is to describe appropriate random fields for which it is possible to construct a stochastic integral and therefore define a meaningful solution.

A classical approach was given by Walsh in [79], where he considered the (famous) example of the vibrating string, a wave equation perturbed by a *space-time white noise*. This noise is a generalized centered Gaussian process $\{\dot{W}(t,x), (t,x) \in [0,T] \times [a,b]\}$ on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with covariance

$$\mathbb{E}[\dot{W}(t,x)\dot{W}(s,y)] = \delta(t-s)\delta(x-y).$$

The integral with respect to such noise is defined by means of *martingale* *measures* and it is of Itô type. This approach is also known as *Brownian sheet approach*. The solution of a partial differential equation perturbed by a space-time white noise is a proper function, but this method fails when the space dimension is bigger than 1. In this case one has to turn to distribution valued concepts or alternatively to consider different type of noises. The random field approach has been successfully applied in higher dimensions for instance by Dalang [17] for spatially homogeneous Gaussian noises and by Balan and Tudor [6] for fractional noises in time and homogeneous in space.

Another approach is proposed by Holden, Øksendal, Ubøe and Zhang in [31] and related papers. Here they make use of *white-noise calculus* and work in a framework which is a priori distribution valued. Moreover they allow space dimension to be higher than 1. Main tools are Wick products, chaos expansions and Hida distributions.

One of the most common approach is the *infinite-dimensional* approach. A comprehensive treatment on stochastic evolution equations in infinite dimensions is given by Da Prato and Zabczyk in the monograph [16]. As in the deterministic case [58], one formulates the stochastic partial differential equation as a stochastic differential equation in Hilbert spaces. Consider for example the heat equation with additive noise

$$\mathrm{d}u(t) = \Delta u(t)\mathrm{d}t + \mathrm{d}W(t) \tag{1}$$

in the Hilbert space $H = L^2(D)$ for some domain $D \subset \mathbb{R}^d$. Here Δ is the Laplacian on H, the function u is H-valued and the noise W is the so called *Q-Wiener process*. W is defined by a series of type

$$W(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} e_k b_k(t) \tag{2}$$

which converges in the underlying Hilbert space H. Here $(e_k)_{k\in\mathbb{N}}$ is an orthonormal basis in H, $(b_k)_{k\in\mathbb{N}}$ is a sequence of independent real Brownian motions and $(\lambda_k)_{k\in\mathbb{N}}$ are the eigenvalues of a symmetric non-negative and trace-class operator Q on H. The solution to (1) with initial condition $u(0) = u_0$ and Dirichlet boundary conditions is given in the mild form by

$$u(t) = P_t u_0 + \int_0^t P_{t-s}\mathrm{d}W(s), \tag{3}$$

where $(P_t)_{t\geq 0}$ is the heat semigroup generated by Δ with Dirichlet boundary conditions. It turns out that one may drop the trace class assumption on Q and still give a meaning to the solution (3) in H, even though the series (2) does not converge any longer in H. In this case the series converges in a bigger Hilbert space $H_1 \supset H$ and the noise is typically called *cylindrical*[1] *Wiener process* in H.

Another, completely different, point of view is the *pathwise approach*. This method is based on the path properties of the noise. The main idea to define a pathwise integral is to fix a path and then perform a *Stieltjes type integral*. We mention in particular Young [80], Russo and Vallois [64], Lyons [47], Zähle [81]. Since the techniques are based only on the regularity properties of the paths, they are successfully applied to noises which are not necessary white. In particular, a big thrust to the development of the pathwise approach was given by the increasing interest in stochastic calculus with respect to different noises, for example with respect to fractional Brownian motion.

The classical *fractional Brownian motion* $\{b^H(t), t \geq 0\}$ with Hurts parameter $H \in (0,1)$ is a real centered Gaussian process with covariance function

$$\mathbb{E}[b^H(t)b^H(s)] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$$

for $s, t \geq 0$. This process was introduced in 1940 by Kolmogorov [41] and was (first) applied by Hurst et al. some years later to model long-term storage capacity in reservoirs, see [34, 35]. In fact fractional Brownian motion is very interesting for many applications because it is stationary, self-similar and in general does not have independent increments. These features make this process suitable for modeling various situations from physics to finance, from engineering to biology, see among others [10, 53, 61, 69].

From the mathematical point of view, fractional Brownian motion is not Markovian and not a (semi)martingale except for $H = 1/2$, is a. s. nowhere differentiable but is a. s. α-Holder continuous for any $0 < \alpha < H$. In particular, this last property has been exploited by many authors to study

[1]In this thesis, the word *cylindrical* will be used in relation to a slightly different concept. Cylindrical processes are defined on Banach spaces and they do not need a bigger Hilbert space (like H_1) to be given a proper meaning. Under suitable assumptions (like the trace class assumption for Q), a cylindrical process reduces to a classical Banach space valued process.

stochastic differential equations driven by fractional Brownian motions (see e. g. Nualart and Răşcanu [55], Zähle [82]) and later stochastic partial differential equations driven by fractional Brownian noises (see e. g. Maslowski and Nualart [49], Gubinelli et al. [27], Hinz and Zähle [29]). For references on stochastic calculus with respect to fractional Brownian motion see the review paper by Nualart [54] or the books by Biagini et al. [9] and Mishura [52].

Partial differential equations with fractional noises have been studied also using the infinite-dimensional approach, mostly in Hilbert spaces. We mention for example Tindel, Tudor, Viens [71], the series of papers by Duncan and coauthors [19, 20, 21, 22, 57], Grecksch et al. [1, 25, 26]. In Banach spaces there are very few works related to equations with fractal noises: Balan [5] considered the heat equation driven by fractional Brownian motion in an L^p setting for $p \geq 2$ and Brzezniak, Van Neerven and Salopek [12] considered stochastic evolution equations driven by Liouville fractional Brownian motion in Banach spaces.

In this thesis we tackle two different problems related to stochastic partial differential equations driven by fractional noises and solve them with different techniques.

In Part I we consider a parabolic transport equation with stochastic velocity field

$$\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + \nabla u(t,x) \cdot \nabla B^H(x), \tag{4}$$

where $\{B^H(x), x \in \mathbb{R}^d\}$ is a centered Gaussian field with covariance

$$\mathbb{E}[B^H(x)B^H(y)] = \frac{1}{2}\left(|x|^{2H} + |y|^{2H} - |x-y|^{2H}\right).$$

This process is nowhere differentiable but possesses some Hölder regularity properties. The first main problem with equation (4) is that the (distributional) derivative of B^H is not a function but a distribution. Therefore the product with ∇u needs some care to be appropriately defined. We do so omegawise using the so-called *paraproducts*: one can define the product between a function and a (not too irregular) distribution using the Fourier transform. We apply this technique to a *general non-differentiable function* Z on $\mathbb{R}^d$ which exhibits the same regularity properties as $B^H(\omega)$ with Hurst parameter $H > \frac{1}{2}$ for almost all $\omega \in \Omega$.

We write the problem in the abstract Cauchy setting, namely we interpret all mappings as functions of time t taking values in some suitable function space U (real function space on $\mathbb{R}^d$, our choice will be specified later). Set $\underline{u} : [0,T] \to U$, $t \mapsto \underline{u}(t) \in U$ and $(\underline{u}(t))(\cdot) := u(t,\cdot)$. The equation (4) with Dirichlet boundary conditions

$$u(t,x) = 0, \quad \text{for } x \in \partial D \tag{5}$$

and initial condition

$$u(0,x) = u_0(x) \quad \text{for } x \in D \tag{6}$$

considered in the *pathwise* sense becomes the following abstract Cauchy problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\underline{u} = \Delta_D \underline{u} + \langle \nabla \underline{u}, \nabla Z \rangle, & t \in (0,T] \\ \underline{u} = u_0, & t = 0, \end{cases} \tag{7}$$

where Δ_D stands for the Dirichlet-Laplace operator and $\langle \cdot, \cdot \rangle$ denotes the pointwise multiplication defined via paraproducts. The product itself will be a distribution. We use a priori estimates on this product which lead to optimal regularity results. This problem is not covered by results in the standard literature for partial differential equations (see for instance [23, 46]).

There are few results on transport diffusion equations with (random) non-smooth drift of the form (7) among the literature. Attanasio and Flandoli [3] consider a stochastic transport equation with non-regular drift but without the diffusion term and with an additive (Brownian) noise. Beck and Flandoli [7] consider a (non-linear) parabolic transport equation with diffusion term but with Brownian noise which is time-dependent.

To our knowledge, the only study regarding a problem of the form (7) is due to Russo and Trutnau [63], who investigate a stochastic equation like (4) but in space dimension one. The authors proceed by freezing the realization of the noise for each ω and overcome the problem of defining the product between a function and a distribution by means of a probabilistic representation: They express the parabolic equation probabilistically through the associated diffusion which is the solution of a stochastic differential equation with generalized drift.

In this thesis we define the mild solution for (7) by

$$\underline{u}(t) = P_t u_0 + \int_0^t P_{t-s}\langle \nabla \underline{u}(s), \nabla Z\rangle \mathrm{d}s. \tag{8}$$

The integral appearing on the right hand side defines an integral operator I on the Hölder space $\mathscr{C}^\gamma([0,T];\tilde{H}^{1+\delta}(D))$. Under suitable conditions on the parameters $\delta, \gamma > 0$ it turns out that I is a contraction in the above mentioned space. This result is given in Theorem 2.2.2. Using this mapping property and a contraction argument we prove the main result (Theorem 2.2.3), that is we prove existence and uniqueness of a global mild solution for (7) in $\mathscr{C}^\gamma([0,T];\tilde{H}^{1+\delta}(D))$. It is relevant that the solution is actually a function, even though we make use of fractional Sobolev spaces of negative index (spaces of distributions) while proving the desired result.

Thanks to how we chose the function Z, this results can be applied to solve in a pathwise sense the stochastic Dirichlet initial value problem (4), (5), (6). Moreover, combining it with a result of Hinz and Zähle [29] we can treat the more general (stochastic) transport equation of the form

$$\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + \langle \nabla u, \nabla Z\rangle(t,x) + \langle F, \frac{\partial}{\partial t}\nabla V\rangle(t,x)$$

for $t \in (0,T], x \in D$ together with (5) and (6), where F is a given vector and $V = V(t,x)$ is a given non-differentiable function.

The content of Part I of this thesis essentially coincides with the paper Issoglio [36]. However the exposition here is a bit more detailed.

In Part II of this thesis we deal with evolution equations driven by fractional Brownian motions in Banach spaces, that is with equations of the form

$$\begin{cases} \mathrm{d}Y(t) = AY(t)\mathrm{d}t + C\mathrm{d}B^H(t), & t \in (0,T] \\ Y(0) = Y_0, \end{cases} \tag{9}$$

where A is the generator of a strongly continuous semigroup on a separable Banach space V, B^H is a fractional Brownian motion on (another) separable Banach space U, C is a bounded linear operator from U to V and Y_0 is a random variable in V.

In order to study and solve the Cauchy problem (9) we need to define many concepts and objects, first of all we need to explain what a *fractional Brownian motion in a Banach space* is. We do so by means of *cylindrical* *processes*. Let us stress the fact that the notion of cylindrical process that is used here is different from the one appearing in the literature (cf. Da Prato and Zabczyk [16]).

In this thesis, a cylindrical random variable X in a Banach space U is defined as a linear map from the topological dual of U to the space of real random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, that is

$$X : U^* \to L^0_{\mathbb{P}}(\Omega, \mathbb{R}).$$

This concept was introduced by Gel'fand in the 1960s, see the monograph by Gel'fand and Wilenkin [24]. A similar object like a cylindrical random variable appears under the name weak distribution in the paper of Segal [68]. See also [38]. Moreover cylindrical random variables and cylindrical measures were extensively considered by Schwartz and his collaborators, see among others [65, 66, 67].

With an idea similar to the one used in Applebaum and Riedle [2] to define (cylindrical) Lévy processes in Banach spaces, we define cylindrical fractional Brownian motions in Banach spaces. Theorem 5.2.3 provides a characterization of this process as a series of the type

$$B^H(t)u^* = \sum_{k=1}^{\infty} \langle Fe_k, u^* \rangle b_k^H(t), \tag{10}$$

where $(b_k^H)_{k\in\mathbb{N}}$ is a sequence of independent real fractional Brownian motions, F is a linear and continuous operator from a Hilbert space H to U, $(e_k)_{k\in\mathbb{N}}$ is an orthonormal basis of H and $\langle\cdot,\cdot\rangle$ denotes the dual pairing between U and U^*. The series converges in $L^2_{\mathbb{P}}(\Omega;\mathbb{R})$. The Hilbert space H is the reproducing kernel Hilbert space of the covariance operator of the Gaussian random variable $B^H(1)$. When the operator F is γ-radonifying [2] then we show (see Theorem 5.2.5) that the series is morally equivalent to

$$B^H(t) = \sum_{k=1}^{\infty} Fe_k b_k^H(t) \tag{11}$$

and it actually converges in $L^2_{\mathbb{P}}(\Omega;U)$, that is B^H is a U-valued fractional Brownian motion.

[2] The notion of γ-radonifying operator is a generalization of the Hilbert-Schmidt property to Banach spaces. It is central in the description of Gaussian random variables in Banach spaces. In fact FF^* is the covariance operator of a Gaussian measure on $\mathcal{B}(U)$ if and only if F is γ-radonifying.

In both cases (10) and (11), if U is a Hilbert space then we recover the classical definition of cylindrical, respectively U-valued, fractional Brownian motion (of the type (2) but with b_k replaced by b_k^H), such as the one used in [22, 25, 49, 71]. Moreover, in the case $U = L^2(D)$ we can recover the space-time fractional noise $\{b^{H,K}(t,x), t \geq 0, x \in D\}$ that was used by Gubinelli Lejay and Tindel [27] or Hinz and Zähle [29].

Let us mention the fact that, even in the Hilbert space case, we allow more general noises than the above-mentioned authors. In fact we consider a wider class of covariance operators FF^*, whereas the (cylindrical) noises given by (2) have covariance Q which is diagonal with respect to the basis, i. e. $Qe_k = \lambda_k e_k$ with λ_k constants (they do not depend on $x \in D$). Some examples of fractional Brownian noises (given as series) in $L^2(D)$ and $L^1(D)$ not yet considered in the literature are explicitly discusses in this thesis.

The second important issue one has to face when trying to solve (9) is the definition of a stochastic integral with respect to a cylindrical fractional Brownian motion. This is crucial in order to define any notion of solution for the problem (9). Since in this case the noise is additive, it is enough to define the stochastic integral for deterministic integrands.

In the Hilbert space case, the definition (based on Wiener integrals) first appeared in [57] for general Hurst parameter $H \in (0, 1)$. It exploits the series representation of the process, as done in the Wiener case by Da Prato and Zabczyk [16], and the integral itself is a random variable which takes values in the underlying Hilbert space.

In Banach spaces we proceed using the same idea, but we define the integral $\int_0^T \varphi(t)\mathrm{d}B^H(t)$ as a cylindrical (in our sense) random variable in U. To do so we exploit the link between Wiener integrals for real valued integrands and Wiener integrals for Hilbert space valued integrands, both with respect to real fractional Brownian motions. In Proposition 6.2.4 we show that the cylindrical integral is a Gaussian cylindrical random variable and we give the explicit decomposition of its covariance operator through a Hilbert space. This result is fundamental to prove Theorem 6.2.5 that provides conditions under which the cylindrical integral is actually a V-valued random variable. We also study the properties of the integral as (cylindrical) stochastic process indexed by time.

With this tools (cylindrical fractional Brownian motion and stochastic integral with respect to it) we can finally study problem (9) for all $H \in$

$(0,1)$ and give a meaning to its solutions. We consider weak (and mild) solutions. Under some assumptions on the semigroup and on the initial condition we prove existence and uniqueness of a cylindrical solution for (9) in Theorem 7.1.4. Moreover we obtain a classical weak solution in V under some additional conditions.

This result is applied to a stochastic parabolic equation with fractal additive noise given by

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = (-Au)(t,x) + G \cdot \frac{\partial}{\partial t}\nabla b^{H,K}(t,x), & t \in (0,T], x \in D \\ u(t,x) = 0, & t \in (0,T], x \in \partial D \\ u(0,x) = u_0(x), & x \in D. \end{cases}$$

This equation has been considered by Hinz and Zähle [29] in space dimension $d = 1$ and has been solved with pathwise techniques. In this example we partly recover their results using the cylindrical approach.

This thesis provides new ideas about possible directions for future research. The author is interested in (stochastic) transport equations like (7) with a non-linear term $G(\nabla \underline{u})$ in place of $\nabla \underline{u}$. This equation can be treated again in a pathwise sense and the main problem is to obtain estimates for the pointwise product $\langle G(\nabla \underline{u}), \nabla Z\rangle$ in terms of $\nabla \underline{u}$. The author believes this can be done using Dirichlet forms.

Another interesting open problem is to extend the definition of the cylindrical stochastic integral in Banach spaces to random integrands $\{X(t), t \geq 0\}$, that is to define $\int_0^T X(t)\,\mathrm{d}B^H(t)$. This would in principle allow to study evolution equations in Banach spaces with multiplicative noise like

$$\begin{cases} \mathrm{d}Y(t) = AY(t)\mathrm{d}t + CY(t)\mathrm{d}B^H(t), & t \in (0,T] \\ Y(0) = Y_0. \end{cases}$$

Clearly, in this situation Wiener integrals are not enough. One should switch to other types of integration, like divergence-type integral or pathwise integrals, and combine it with the cylindrical approach exposed here.

It is also relevant to answer the question of continuity of the cylindrical integral with respect to time T, question which is not yet fully answered even in the one-dimensional case for real fractional Brownian motions.

Part I

Pathwise approach

Chapter 1

The transport equation

In this chapter we introduce the problem which will be solved in Chapter 2. To this aim, in Section 1.1 we start with some preliminaries and fix the notation. We recall and develop some concepts and useful tools that will be fundamental in the proof of existence and uniqueness of a solution. In Section 1.2 we state the problem and rewrite it in an abstract form. We define the concept of solution for this problem by means of mild solutions. Some other technical results such as pointwise multiplication will be explained.

1.1 Preliminaries

1.1.1 Fractional Sobolev spaces

In this first section we deal with fractional Sobolev spaces. It is a family of spaces which are embedded in each other and provide a natural framework for the action of the Dirichlet Laplacian and of its powers. Let us recall the definition of fractional Sobolev spaces on $\mathbb{C}$.

Definition 1.1.1. *Let $\alpha \in \mathbb{R}$ and $1 < p < \infty$. We define the fractional Sobolev space or Bessel potential space as*

$$H_p^\alpha(\mathbb{R}^d; \mathbb{C}) := \left\{ f \in \mathcal{S}'(\mathbb{R}^d; \mathbb{C}) : ((1 + |\xi|^2)^{\alpha/2} \hat{f})^\vee \in L^p(\mathbb{R}^d; \mathbb{C}) \right\}.$$

equipped with the norm [1]

$$\|f | H_p^\alpha(\mathbb{R}^d; \mathbb{C})\| = \|((1 + |\xi|^2)^{\alpha/2} \hat{f})^\vee | L^p(\mathbb{R}^d; \mathbb{C})\|,$$

[1] Sometimes we indicate with $\| \cdot | U\|$ the norm in the space U instead of the usual one $\| \cdot \|_U$. This is done for simplicity of notation when the space U has a long name.

where $\hat{f}$ stands for the Fourier transform of f on $\mathbb{R}^d$ and $(\cdot)^\vee$ denotes the inverse Fourier transform.

We omit the subscript index p when $p = 2$.
We are interested only in real-valued functions and distributions, so we follow [62] and define $\mathcal{S}'(\mathbb{R}^d;\mathbb{R}) := \{f \in \mathcal{S}'(\mathbb{R}^d;\mathbb{C}) : \bar{f} = f\}$ where $\bar{f}$ is defined by $\bar{f}(\phi) = f(\bar{\phi})$ for all $\phi \in \mathcal{S}(\mathbb{R}^d;\mathbb{C})$. For $1 < p < \infty$ and $\alpha \in \mathbb{R}$ we define

$$H_p^\alpha(\mathbb{R}^d;\mathbb{R}) = H_p^\alpha(\mathbb{R}^d;\mathbb{C}) \cap \mathcal{S}'(\mathbb{R}^d;\mathbb{R})$$

and for simplicity of notation we omit the writing of the codomain when it is $\mathbb{R}$.
This family of spaces is also called scale of spaces because $H_p^\beta(\mathbb{R}^d) \subset H_p^\alpha(\mathbb{R}^d)$ for every $\alpha < \beta \in \mathbb{R}$. It is known that when the parameter α is a positive integer $\alpha = m \in \mathbb{N}$ then $H_p^m(\mathbb{R}^d) = W_p^m(\mathbb{R}^d)$ where

$$W_p^m(\mathbb{R}^d) := \Big\{ f \in \mathcal{S}'(\mathbb{R}^d) : \partial^\gamma f \in L^p(\mathbb{R}^d) \text{ for all } |\gamma| \le m \Big\}$$

is the classical Sobolev space on $\mathbb{R}^d$ equipped with the norm

$$\| f \,|\, W_p^m \| := \left(\sum_{|\gamma| \le m} \| \partial^\gamma f \,|\, L^p \|^p \right)^{1/p}.$$

The latter norm and $\| \cdot \|_{H_p^m}$ are equivalent norms.

We are interested in the corresponding fractional Sobolev spaces on domains. There are various ways to define such spaces. Here we recall only one of them, which is suitable for our purposes. For $\alpha > -1/2$ define

$$\tilde{H}_p^\alpha(D) := \Big\{ f \in H_p^\alpha(\mathbb{R}^d) : \operatorname{supp}(f) \subset \bar{D} \Big\}$$

equipped with the norm $\| \cdot \|_{H_p^\alpha(\mathbb{R}^d)}$. The norm itself is not sufficient in order to characterize this space because it is the same as the one used for $H_p^\alpha(\mathbb{R}^d)$. Observe that if $\alpha = 0$ then the space $\tilde{H}_p^0(D)$ is simply $L^p(D)$. Such spaces are embedded in each other in the following way: for all $\alpha > \beta > -\frac{1}{2}$ we have $\tilde{H}_p^\alpha(\mathbb{R}^d) \subset \tilde{H}_p^\beta(\mathbb{R}^d)$.
When $p = 2$ the norm in $\tilde{H}^\alpha(D)$ will be indicated by $\| \cdot \|_\alpha$. Moreover when we have a vector (like ∇Z) we write $\nabla Z \in H_p^\alpha(\mathbb{R}^d)$ (and similarly for spaces on D) to intend that every component of the vector ∇Z belongs to such space. The norm of a d-dimensional vector in the space $(H_p^\alpha(\mathbb{R}^d))^d$ is defined as the square root of the sum of the squared norm of each component in $H_p^\alpha(\mathbb{R}^d)$. For simplicity we will indicate it with the same notation.

1.1.2 Semigroup theory

In this section we recall the theory of semigroups of linear operators. Fore more details and proofs we refer to [58, 78].
We start with the definition of semigroup.

Definition 1.1.2. *Let $(U, \|\cdot\|_U)$ be a Banach space. A family of bounded linear operators $(T_t)_{t\geq 0}$ acting from U into itself is called semigroup of bounded linear operators on U (or simply semigroup) if*

1. $T_0 = \mathrm{Id}$;
2. $T_{t+s} = T_t T_s$ *for every* $t, s \geq 0$.

Furthermore $(T_t)_{t\geq 0}$ is called uniformly continuous if

3. $\lim_{t\downarrow 0} \|T_t - \mathrm{Id}\,\|_{\mathcal{L}(U)} = 0$

and it is called strongly continuous (or also $\mathscr{C}_0$ semigroup) if

4. $\lim_{t\downarrow 0} T_t x = x$ *for every* $x \in U$.

Remark 1.1. Each uniformly continuous semigroup is also a $\mathscr{C}_0$ semigroup, but not the other way around.

For both these two types of semigroups we can define a new operator acting again from U (or from a subspace of U) into itself, called infinitesimal generator. This operator can be either bounded or not, depending on the type of semigroup.

Definition 1.1.3. *Given a semigroup $(T_t)_{t\geq 0}$ on U we define a linear operator A on*

$$\mathcal{D}(A) := \left\{ x \in U \text{ s.t. } \lim_{t\downarrow 0} \frac{T_t x - x}{t} \text{ exists} \right\}$$

as follows

$$Ax := \lim_{t\downarrow 0} \frac{T_t x - x}{t}$$

for every $x \in \mathcal{D}(A)$. Such an operator is called infinitesimal generator of the semigroup.

As we have already mentioned, the generator has different properties depending of the properties of the semigroup. For uniformly continuous semigroup we have the following:

Theorem 1.1.4. *A linear operator $A : U \to U$ is the infinitesimal generator of a uniformly continuous semigroup if and only if A is a bounded linear operator.*

Given a bounded linear operator A we can construct the semigroup by

$$T_t = e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}$$

where the sum on the right-hand side converges in norm for every $t \geq 0$. If we are given the semigroup we know how to construct the generator and moreover if two generators coincide, then the two semigroups coincide as well.

Corollary 1.1.5. *Let $(T_t)_{t\geq 0}$ be a uniformly continuous semigroup. Then*

(a) $\exists\ \omega \geq 0$ *constant such that* $\|T_t\| \leq e^{\omega t}$*;*

(b) There exists a unique bounded linear operator $A : U \to U$ such that $T_t = e^{tA}$;

(c) The operator A is the infinitesimal generator of $(T_t)_{t\geq 0}$;

(d) The mapping $t \mapsto T_t$ is differentiable in norm and

$$\frac{\mathrm{d}T_t}{\mathrm{d}t} = AT_t = T_t A.$$

In many interesting situations though, the semigroup is not uniformly continuous but only strongly continuous. In this case the generator is unbounded and we have the following properties.

Theorem 1.1.6. *Let $(T_t)_{t\geq 0}$ be a $\mathscr{C}_0$ semigroup and let A be its infinitesimal generator defined in Definition 1.1.3. Then*

(a) for every $x \in U$

$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} T_s x \,\mathrm{d}s = T_t x;$$

(b) for every $x \in U$

$$\int_0^t T_s x \,\mathrm{d}s \in \mathcal{D}(A)$$

and

$$A\left(\int_0^t T_s x \,\mathrm{d}s\right) = T_t x - x;$$

(c) for $x \in \mathcal{D}(A)$ then $T_t x \in \mathcal{D}(A)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} T_t x = A T_t x = T_t A x;$$

(d) for every $x \in \mathcal{D}(A)$ and for every $s, t \geq 0$

$$T_t x - T_s x = \int_s^t T_u A x \mathrm{d}u = \int_s^t T_u x \, \mathrm{d}u.$$

Corollary 1.1.7. *If A is the infinitesimal generator of a $\mathscr{C}_0$ semigroup then $\mathcal{D}(A)$ is dense in U and A is a closed linear operator.*

The generator is uniquely defined, meaning that if two $\mathscr{C}_0$ semigroups have the same generator then they coincide. It is possible to have a characterization of the generator of a $\mathscr{C}_0$ semigroup, as in Theorem 1.1.4. For this, recall that if $(T_t)_{t \geq 0}$ is a $\mathscr{C}_0$ semigroup then there exist $\omega \geq 0$ and $M \geq 1$ such that $\|T_t\| \leq M e^{\omega t}$. If $\omega = 0$ then the semigroup is called uniformly bounded since we find a bound M which does not depend on t. Furthermore if $M = 1$ then the semigroup is called $\mathscr{C}_0$ semigroup of contractions since every operator is a contraction.
Given a linear operator $A : U \to U$ we can define the resolvent set of A as follows:

$$\rho(A) = \{\lambda \in \mathbb{C} : \lambda I - A \text{ is invertible}\}.$$

Once we have defined the resolvent set of an operator, we can define the resolvent of A. This is a family of bounded linear operators $\{R(\lambda; A), \lambda \in \rho(A)\}$ each of them defined as $R(\lambda; A) := (\lambda I - A)^{-1}$.

Theorem 1.1.8 (Hille-Yosida)**.** *A linear (unbounded) operator $A : \mathcal{D}(A) \subseteq U \to U$ is the infinitesimal generator of a $\mathscr{C}_0$ semigroup of contractions if and only if*

(a) A is closed and densely defined (i.e. $\overline{\mathcal{D}(A)} = U$);

(b) The resolvent set $\rho(A)$ contains the real positive half-line (i.e. $\rho(A) \supseteq (0, +\infty)$) and for every $\lambda > 0$

$$\|R(\lambda; A)\| \leq \frac{1}{\lambda},$$

where the norm is the usual operator norm.

Fractional powers

Next we introduce fractional powers of a closed operator. In particular we will use them for analytic semigroups. Recall that a semigroup $(T_t)_{t\geq 0}$ on a Banach space U is analytic if there exists $\theta > 0$ such that the map $t \mapsto T_t$ (taking values in $\mathcal{L}(U)$) admits an analytic extension to the sector $S(\theta) = \{\lambda \in \mathbb{C} : |\arg \lambda| < \theta\}$, satisfies the semigroup property there, and is such that $t \mapsto T_t^{(\rho)} = T_{e^{i\rho}t}$ is a strongly continuous semigroup for every $|\rho| < \theta$. A semigroup which is contractive and symmetric is also analytic (see [18], Theorem 1.4.1 or [70], Chapter III). For the generator of an analytic semigroup one can define fractional powers of any order (see for instance [58]).

Definition 1.1.9. *Given an operator $A : \mathcal{D}(A) \subseteq U \to U$ such that $-A$ is the infinitesimal generator of a $\mathscr{C}_0$ semigroup $(T_t)_{t\geq 0}$, we define the negative fractional power of A for any arbitrary positive α by*

$$A^{-\alpha}x := \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} T_t x \, \mathrm{d}t$$

for every $x \in \mathcal{D}(A^{-\alpha})$ where

$$\mathcal{D}(A^{-\alpha}) := \left\{ x \in U \ : \ \int_0^\infty t^{\alpha-1} T(t) x \, \mathrm{d}t \ \textit{is convergent} \right\}.$$

By definition we set $A^{-0} := \mathrm{Id}$ and $\mathcal{D}(A^{-0}) := U$.

We now want to define fractional powers also for positive exponents, and since the operator defined in Definition 1.1.9 under some conditions is one-to-one we can give the following definition.

Definition 1.1.10. *Given an operator $A : \mathcal{D}(A) \subseteq U \to U$ such that $-A$ is the infinitesimal generator of a $\mathscr{C}_0$ semigroup $(T_t)_{t\geq 0}$ with $\|T_t\| \leq Me^{-\omega t}$ for some positive ω, for every $\alpha > 0$ we define*

$$A^\alpha = (A^{-\alpha})^{-1}$$

According to the previous notation we have $A^0 := I$.

We recall now a standard result on semigroups, for a proof we refer to [58] Theorem II.6.13 or [78] Theorem 7.7.2.

Theorem 1.1.11. *Let $-A$ be the infinitesimal generator of an analytic semigroup T_t on a Banach space $(U, \|\cdot\|_U)$. If for each $t \geq 0$ holds $\|T_t\| \leq Me^{-\omega t}$ with $M \geq 1$ and $\omega > 0$, then*

(a) $T_t : U \to \mathcal{D}(A^\alpha)$ *for every* $t > 0, \alpha \geq 0$;

(b) *for every* $\alpha \geq 0$ *and for every* $x \in \mathcal{D}(A^\alpha)$, $T_t A^\alpha x = A^\alpha T_t x$;

(c) *for every* $t > 0$ *and for every* $\alpha \geq 0$ *the operator* $A^\alpha T_t$ *is bounded and linear and there exist constants* M_α *(which depends only on* α*) and* $\theta \in (0, \omega)$ *such that*

$$\|A^\alpha T_t\|_{\mathcal{L}(U)} \leq M_\alpha e^{-\theta t} t^{-\alpha};$$

(d) *for each* $0 < \alpha \leq 1$ *there exists* $C_\alpha > 0$ *such that* $\forall t > 0$ *and for each* $x \in \mathcal{D}(A^\alpha)$ *we have*

$$\|T_t x - x\|_U \leq C_\alpha t^\alpha \|A^\alpha x\|_U.$$

1.1.3 The Dirichlet Laplacian and its powers

The Dirichlet Laplacian is the generator of a particular $\mathscr{C}_0$-semigroup on $L^2(D)$ for a bounded domain $D \subset \mathbb{R}^d$. The action of its power on the frational Sobolev spaces on D will be of vital importance in Chapter 2.
Let us start with the semigroup T_t generated by the negative Laplacian $A = -\Delta$ on the whole space $\mathbb{R}^d$. In this setting the explicit expression for the semigroup applied to a function $u(x)$ is

$$T_t u(x) = \int_{\mathbb{R}^d} p(t, x, y) u(y) \mathrm{d}y$$

where $p(t, x, y) = (2\pi t)^{d/2} \exp\{\frac{-|x-y|^2}{2t}\}$ is the heat kernel. The classical interpretation for this is to consider the Laplacian on $\mathscr{C}^2(\mathbb{R})$ and then we have a semigroup an a Banach space. More generally we can extend it to the Sobolev space $H^0(\mathbb{R})$. Strictly speaking we have a semigroup on a Hilbert space, whose generator is an unbounded operator, since the domain of Δ is dense in $H^0(\mathbb{R}) = L^2(\mathbb{R})$ but strictly included, namely $\mathcal{D}(\Delta) = H^2(\mathbb{R})$.

Consider now the Dirichlet Laplacian Δ_D as the infinitesimal generator of the Dirichlet heat semigroup acting on $L^2(D)$ (see e. g. [78] Section 4.1, [23] Section 7.4.3). Let us indicate it with $\Delta_D = -A$. More precisely $-A$ generates a compact $\mathscr{C}_0$ semigroup of contractions $(P_t)_{t \geq 0}$ in $L^2(D)$ (see [78], Theorem 7.2.5). The semigroup $(P_t)_{t \geq 0}$ is of negative type and symmetric. Moreover since it is also contractive, it is analytic. Thus, one can define fractional powers of A of any order for which we have the following property. For the proof we refer to [75] equations (27.50) and (27.51) or [74] Section 4.9.2.

Proposition 1.1.12. *Let $(P_t)_{t\geq 0}$ be the semigroup generated by the Dirichlet Laplacian $\Delta_D =: -A$. For all $\gamma, \alpha \in \mathbb{R}$ such that $-\frac{1}{2} < \gamma, \gamma - \alpha < \frac{3}{2}$ the fractional power $A^{\frac{\alpha}{2}}$ maps isomorphically $\tilde{H}^{\gamma}(D)$ onto $\tilde{H}^{\gamma-\alpha}(D)$, hence there exist $c_1 > 0$ and $c_2 > 0$ such that for all $f \in \tilde{H}^{\gamma}(D)$*

$$\left\| A^{\frac{\alpha}{2}} f \right\|_{\gamma-\alpha} \leq c_1 \|f\|_{\gamma} \leq c_2 \left\| A^{\frac{\alpha}{2}} f \right\|_{\gamma-\alpha}. \tag{1.1}$$

Furthermore one can prove that $\mathcal{D}(A^{\frac{\alpha}{2}}) = \tilde{H}^{\alpha}(D)$ for all $0 < \alpha < \frac{3}{2}, \alpha \neq \frac{1}{2}$ (for more details see [75]).

Using Theorem 1.1.11 and relation (1.1) we get the following result.

Corollary 1.1.13. *Let $(P_t)_{t\geq 0}$ be the Dirichlet heat semigroup on $L^2(D)$. Then for all positive t and for any $-\frac{1}{2} < \rho, \gamma, \rho + \gamma < \frac{3}{2}$we have*

$$P_t : \tilde{H}^{\gamma}(D) \to \tilde{H}^{\rho+\gamma}(D).$$

In particular if $f \in \tilde{H}^{\gamma}(D)$ then $\mathrm{supp}(P_t f) \subset \bar{D}$.

Proof. Consider first the case when $\gamma > 0$. Let $f \in \tilde{H}^{\gamma}(D)$ so by (1.1) we have $g := A^{\frac{\gamma}{2}} f \in L^2(D)$. We write $P_t f = P_t A^{-\frac{\gamma}{2}} A^{\frac{\gamma}{2}} f = P_t A^{-\frac{\gamma}{2}} g = A^{-\frac{\gamma}{2}} P_t g$ and by Theorem 1.1.11 (a) we know that $P_t g \in \mathcal{D}(A^{\rho})$ for any $\rho \geq 0$. Moreover recall that $\mathcal{D}(A^{2\rho}) = \tilde{H}^{\rho}(D)$ for all $0 \leq \rho < \frac{3}{2}, \rho \neq \frac{1}{2}$, so for this choice of ρ and using (1.1) we get $P_t f = A^{-\frac{\gamma}{2}} P_t g \in \tilde{H}^{\rho+\gamma}(D)$. Observe that this fact is true also if $\rho = \frac{1}{2}$ since $\tilde{H}^{\rho+\gamma}(D) \subset \tilde{H}^{\frac{1}{2}+\gamma}(D)$ for all $\rho > \frac{1}{2}$.
The case when $\gamma < 0$ is proven in the same way, simply write $A^{-\frac{\gamma}{2}} A^{\frac{\gamma}{2}} P_t f$ instead of $P_t A^{-\frac{\gamma}{2}} A^{\frac{\gamma}{2}} f$. □

1.2 The formulation of the problem

We consider the following transport equation on a domain $D \subset \mathbb{R}^d$ with initial and Dirichlet boundary conditions

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + \langle \nabla u, \nabla Z \rangle(t,x), & t \in (0,T], x \in D \\ u(t,x) = 0, & t \in (0,T], x \in \partial D \\ u(0,x) = u_0(x), & x \in D \end{cases} \tag{1.2}$$

where D is a bounded open set of $\mathbb{R}^d$ with C^{∞} boundary, u_0 is a given function in a fractional Sobolev space on D of appropriate order, Z is a given non-differentiable function on $\mathbb{R}^d$ and the derivative is taken in the distributional sense. The gradient ∇ as well as the Laplacian Δ refer to the space variables. The precise definition of the product $\langle \nabla u, \nabla Z \rangle(t,x)$ is given in Section 1.2.2, and it is set by use of the Fourier transform.

1.2.1 The abstract Cauchy problem

We want to rewrite problem (1.2) in another formalism known as abstract Cauchy problem. For this reason we briefly recall some classical results about it (see for instance [58], Chapter 4).
Suppose U is a Banach space and $f : [0,T) \to U$ a given function. The inhomogeneous abstract Cauchy problem is

$$\begin{cases} \frac{\mathrm{d}u(t)}{\mathrm{d}t} = Au(t) + f(t) \qquad t \in (0,T] \\ u(0) = x, \end{cases} \tag{1.3}$$

where u is a U-valued function.
In the particular case when $f \equiv 0$ it holds that for every $x \in \mathcal{D}(A)$, (1.3) has a unique solution given by $u(t) = T_t x$ if and only if A is the generator of a $\mathscr{C}_0$ semigroup $(T_t)_{t\geq 0}$. For "solution" in this contest we mean a *classical solution* and the definition is given below.

Definition 1.2.1. *A function $u : [0,T) \to U$ is a classical solution of (1.3) on $[0,T)$ if u is continuous on $[0,T)$, continuously differentiable on $(0,T)$, $u(t) \in \mathcal{D}(A)$ for $0 < t < T$ and (1.3) is satisfied on $[0,T)$*

Theorem 1.2.2. *If $f \in L^1([0,T])$ then for every $x \in U$ the initial value problem (1.3) has at most one solution. If the solution exists it is of the form*

$$u(t) = T_t x + \int_0^t T_{t-s} f(s)\,\mathrm{d}s. \tag{1.4}$$

Notice that for every $f \in L^1([0,T])$ the right-hand side of (1.4) is a continuous function on $[0,T]$ but it is not necessarily differentiable. In this case, according to Definition 1.2.1, u is not a classical solution. But we can consider it as a generalized solution as follows.

Definition 1.2.3. *Let A be the generator of a $\mathscr{C}_0$ semigroup $(T_t)_{t\geq 0}$, let $x \in U$ and $f \in L^1([0,T];U)$. The function $u \in C([0,T];U)$ given by (1.4) for every $t \in [0,T]$ is called mild solution of the initial value problem (1.3).*

It can be shown that under some conditions the mild solution is also the classical solution, but in general the continuity of f is not sufficient.

Theorem 1.2.4. *Let A be the generator of a $\mathscr{C}_0$ semigroup $(T_t)_{t\geq 0}$. Let $f \in L^1([0,T];U)$ be continuous on $(0,T]$ and set $v(t) = \int_0^t T_{t-s} f(s)\,\mathrm{d}s$ for*

all $0 \le t \le T$.
Then the initial value problem (1.3) has a classical solution on $[0, T)$ *for every* $x \in \mathcal{D}(A)$ *if one of the following conditions hold:*

(1) $v(t)$ *is continuously differentiable on* $(0, T)$*;*

(2) $v(t) \in \mathcal{D}(A)$ *for* $0 < t < T$ *and* $Av(t)$ *is continuous on* $(0, T)$.

Conversely, if (1.3) has a classical solution u *on* $[0, T)$ *for some* $x \in \mathcal{D}(A)$ *then* $v(t)$ *satisfies both (1) and (2).*

The mild solution of the transport equation

To formalize this approach in our case, we interpret all functions appearing in (1.2) as functions only of time t and taking values in some suitable function space U. The choice of the space U is crucial in order to prove existence and uniqueness of the solution, and it turns out that a suitable choice is for instance a fractional Sobolev space on D.
Set $\underline{u} : [0, T] \to U$, $t \mapsto \underline{u}(t) \in U$ and $(\underline{u}(t))(\cdot) := u(t, \cdot)$ and choose $u_0 \in U$. The Dirichlet initial value problem becomes the following problem

$$\begin{cases} \frac{d}{dt}\underline{u} = \Delta_D \underline{u} + \langle \nabla \underline{u}, \nabla Z \rangle, & t \in (0, T] \\ \underline{u} = u_0, & t = 0, \end{cases} \tag{1.5}$$

where Δ_D stands for the Dirichlet-Laplace operator introduced in Section 1.1.3.

A function $\underline{u}$ is a *mild solution* of (1.5) if it satisfies the following integral equation

$$\underline{u}(t) = P_t u_0 + \int_0^t P_{t-r} \langle \nabla \underline{u}(r), \nabla Z \rangle \, \mathrm{d}r. \tag{1.6}$$

To give a formal meaning to the product $\langle \cdot, \cdot \rangle$ we make use of the so called *paraproduct*, see e. g. [62]. We shortly recall its definition and some useful properties in the next section.

1.2.2 A pointwise multiplication

In this section we recall the notion of paraproduct introduced in [62] that allows us to multiply a function and a distribution, provided that they are good enough.
Suppose we are given $f \in \mathcal{S}'(\mathbb{R}^d)$. Choose a function $\psi \in \mathcal{S}(\mathbb{R}^d)$ such that

$0 \le \psi(x) \le 1$ for every $x \in \mathbb{R}^d$, $\psi(x) = 1$ if $|x| \le 1$ and $\psi(x) = 0$ if $|x| \ge \frac{3}{2}$. Then consider the following approximation of f

$$S^j f(x) := \left(\psi \left(\frac{\xi}{2^j} \right) \hat{f} \right)^{\vee} (x)$$

that is in fact the convolution of f with a smoothing function. This approximation is used to define the product of two distributions fg as follows:

$$fg := \lim_{j \to \infty} S^j f S^j g$$

if the limit exists in $\mathcal{S}'(\mathbb{R}^d)$. The convergence in the case we are interested in is part of the assertion below (see [29] appendix C.4, [62] Theorem 4.4.3/1).

Lemma 1.2.5. *Let $1 < p, q < \infty$ and $0 < \beta < \delta$ and assume that $q > \max(p, \frac{d}{\delta})$. Then for every $f \in H_p^\delta(\mathbb{R}^d)$ and $g \in H_q^{-\beta}(\mathbb{R}^d)$ we have*

$$\|fg|H_p^{-\beta}(\mathbb{R}^d)\| \le c\|f|H_p^\delta(\mathbb{R}^d)\| \cdot \|g|H_q^{-\beta}(\mathbb{R}^d)\|. \tag{1.7}$$

The following Lemma regarding a *locality-preserving* property will be used to shift the properties of the product fg from the whole $\mathbb{R}^d$ to the domain D. For the proof see [62] Lemma 4.2.

Lemma 1.2.6. *If $f, g \in \mathcal{S}'(\mathbb{R}^d)$ and $\operatorname{supp}(f) \subset \bar{D}$ then also $\operatorname{supp}(fg) \subset \bar{D}$.*

Our aim now is to apply such product to $\nabla \underline{u}(s)$ and ∇Z. We will denote by $\langle \cdot, \cdot \rangle$ the pointwise product combined with the scalar product in $\mathbb{R}^d$.

Proposition 1.2.7. *Let $\underline{u}(s) \in \tilde{H}_p^{1+\delta}(D)$, $Z \in H_q^{1-\beta}(\mathbb{R}^d)$ for $1 < p, q < \infty$, $q > \max(p, \frac{d}{\delta})$, $0 < \beta < \frac{1}{2}$ and $\beta < \delta$. Then the pointwise multiplication $\langle \nabla \underline{u}(s), \nabla Z \rangle$ is well defined, it belongs to the space $\tilde{H}_p^{-\beta}(D)$ and we have the following bound*

$$\|\langle \nabla \underline{u}(s), \nabla Z \rangle | \tilde{H}_p^{-\beta}(D)\| \le c\|\nabla \underline{u}(s)|\tilde{H}_p^\delta(D)\| \cdot \|\nabla Z|H_q^{-\beta}(\mathbb{R}^d)\|. \tag{1.8}$$

Proof. The idea is to apply first Lemma 1.2.5 to define the product as an element of $H_p^{-\beta}(\mathbb{R}^d)$ and then restrict it to $\tilde{H}_p^{-\beta}(D)$ with the help of Lemma 1.2.6.
Let $f = \nabla \underline{u}(s)$ and $g = \nabla Z$. We should check the conditions in Lemma 1.2.5. Clearly $g \in H_q^{-\beta}(\mathbb{R}^d)$ because $Z \in H_q^{1-\beta}(\mathbb{R}^d)$ and it is easy to show that $(\nabla)_i$ is bounded form $H^\gamma(\mathbb{R}^d)$ to $H^{\gamma-1}(\mathbb{R}^d)$ for every $\gamma \in \mathbb{R}$ and for all $i = 1, \ldots, d$.

The fact that $f \in H_p^\delta(\mathbb{R}^d)$ is also clear since $\tilde{H}_p^{1+\delta}(D) \subset H_p^{1+\delta}(\mathbb{R}^d)$.
Denote $m(s) := \langle \nabla \underline{u}(s), \nabla Z \rangle \in H_p^{-\beta}(\mathbb{R}^d)$ and by Lemma 1.2.5 we get

$$\|m(s)|H_p^{-\beta}(\mathbb{R}^d)\| \le c\|\nabla \underline{u}(s)|H_p^\delta(\mathbb{R}^d)\| \cdot \|\nabla Z|H_q^{-\beta}(\mathbb{R}^d)\| < \infty.$$

Since $\operatorname{supp} \underline{u}(s) \subset \bar{D}$ then $\operatorname{supp} \nabla \underline{u}(s) \subset \bar{D}$ and so by Lemma 1.2.6 it follows $\operatorname{supp} m(s) \subset \bar{D}$ and so $m(s) \in \tilde{H}_p^{-\beta}(D)$ since $\beta < \frac{1}{2}$. Moreover,

$$\begin{aligned} \|\langle \nabla \underline{u}(s), \nabla Z\rangle |\tilde{H}_p^{-\beta}(D)\| &= \|\langle \nabla \underline{u}(s), \nabla Z\rangle |H_p^{-\beta}(\mathbb{R}^d)\| \\ &\le c\|\nabla \underline{u}(s)|H_p^\delta(\mathbb{R}^d)\| \cdot \|\nabla Z|H_q^{-\beta}(\mathbb{R}^d)\| \\ &= c\|\nabla \underline{u}(s)|\tilde{H}_p^\delta(D)\| \cdot \|\nabla Z|H_q^{-\beta}(\mathbb{R}^d)\|. \qquad \square \end{aligned}$$

1.2.3 The integral operator

The notion of mild solution is now formalized. In order to check the convergence of the integral appearing in the definition of the mild solution, we introduce the following operator:

Definition 1.2.8. *Given an $\tilde{H}^{1+\delta}(D)$-valued function $\underline{u}$ on $[0,T]$ we define the integral operator $I_{(\cdot)}(\underline{u})$ for any $t \in [0,T]$ setting*

$$I_t(\underline{u}) := \int_0^t P_{t-r}\langle \nabla \underline{u}(r), \nabla Z\rangle \,\mathrm{d}r.$$

We consider this operator acting on the Hölder space $C^\gamma([0,T];U)$ into itself (this mapping property will be proven later, see Theorem 2.2.2) for some suitable γ and for some infinite dimensional Banach space U. The Hölder space is defined as

$$C^\gamma([0,T];U) := \{h : [0,T] \to U \text{ s.t. } \|h\|_{\gamma,U} < \infty\}$$

where

$$\|h\|_{\gamma,U} := \sup_{t\in[0,T]} \|h(t)\|_U + \sup_{s<t\in[0,T]} \frac{\|h(t) - h(s)\|_U}{(t-s)^\gamma}.$$

When $U = \tilde{H}^{1+\delta}(D)$ the norm will be indicated by $\|\cdot\|_{\gamma,1+\delta}$.

Chapter 2

The main result

In this chapter we prove the contractivity of the operator I in the Hölder space $C^\gamma([0,T];\tilde{H}^{1+\delta}(D))$. From this mapping property we obtain existence, uniqueness and regularity of the solution in the above mentioned space.
In Section 2.1 we consider the local solution. This means the result is valid up to an explosion time $\varepsilon > 0$ which needs to be small enough. In Section 2.2 we introduce a family of equivalent norms in $C^\gamma([0,T];\tilde{H}^{1+\delta}(D))$ thanks to which we can prove the result for any $T > 0$.

Throughout the whole chapter c denotes a finite positive constant whose exact value is not important and may change from line to line.

2.1 The local solution

We start with a result on existence and uniqueness of a local solution. Even though this result is covered by the main theorem of Section 2.2, we give here a detailed proof because in this case the computations are easier.

2.1.1 Preliminary results

Recall that $m(r) := \langle \nabla \underline{u}(r), \nabla Z\rangle$ for all $0 \le r \le T$.

Proposition 2.1.1. *Let $0 < \beta < \frac{1}{2}$ and $\beta < \delta$ and fix a function $Z \in H_q^{1-\beta}(\mathbb{R}^d)$ for some $q > \max(2, \frac{d}{\delta})$. Then for all $0 \le r \le t \le T$ and $\underline{u}(t) \in \tilde{H}^{1+\delta}(D)$ we have*

(1) $\|m(r)|\tilde{H}^{-\beta}(D)\| \le c\|\underline{u}(r)|\tilde{H}^{1+\delta}(D)\|$;

(2) $\|m(t) - m(r)|\tilde{H}^{-\beta}(D)\| \le c\|\underline{u}(t) - \underline{u}(r)|\tilde{H}^{1+\delta}(D)\|$.

Proof. *(1)* Observe that by definition $\nabla \underline{u}(r) \in \tilde{H}^\delta(D)$ means that $\nabla \underline{u}(r) \in H^\delta(\mathbb{R}^d)$ and $\mathrm{supp}(\nabla \underline{u}(r)) \subset \bar{D}$. Also $(\nabla)_j : H^{1+\delta}(\mathbb{R}^d) \to H^\delta(\mathbb{R}^d)$ is bounded for all δ, i.e. for all $f \in H^{1+\delta}(\mathbb{R}^d)$ there exists $c > 0$ such that $\|\nabla f\|_\delta \leq c\|f\|_{1+\delta}$. These results combined with Proposition 1.2.7 (where $p = 2$) lead to (1).
(2) Since $\tilde{H}^{-\beta}(D)$ is a linear space then $m(t) - m(r) \in \tilde{H}^{-\beta}(D)$. The pointwise product and the operator ∇ are linear so we can write $m(t) - m(r) = \langle \nabla \underline{u}(t) - \nabla \underline{u}(r), \nabla Z \rangle = \langle \nabla(\underline{u}(t) - \underline{u}(r)), \nabla Z \rangle$. Clearly $\underline{u}(t) - \underline{u}(r)$ is an element of $\tilde{H}^\delta(D) \subset H^\delta(\mathbb{R}^d)$ so we proceed in the same way as for (1) and we get the wanted result. □

Proposition 2.1.2. *Let* $0 < \beta < \delta < \frac{1}{2}$ *and* $w \in \tilde{H}^{-\beta}(D)$. *Then* $P_t w \in \tilde{H}^{1+\delta}(D)$ *for any* $t > 0$ *and moreover there exists a positive constant c such that*

$$\|P_t w\|_{1+\delta} \leq c \, \|w\|_{-\beta} \, t^{-\frac{1+\delta+\beta}{2}}. \tag{2.1}$$

Proof. Let $w \in \tilde{H}^{-\beta}(D)$. By (1.1) we have

$$\|P_t w\|_{1+\delta} \leq c\|A^{\frac{1+\delta}{2}} P_t w\|_0 = c\|A^{\frac{1+\delta}{2}} A^{\frac{\beta}{2}} A^{-\frac{\beta}{2}} P_t w\|_0 = c\|A^{\frac{1+\delta+\beta}{2}} P_t A^{-\frac{\beta}{2}} w\|_0.$$

Since $w \in \tilde{H}^{-\beta}(D)$ then by Proposition 1.1.12 we have also $A^{-\frac{\beta}{2}} w \in L^2(D)$ and Theorem 1.1.11 part (c) ensures that the following bound holds for all $t > 0$

$$\|A^{\frac{1+\delta+\beta}{2}} P_t\|_{\mathcal{L}(L^2(D))} \leq M e^{-\theta t} t^{-\frac{1+\delta+\beta}{2}}.$$

This fact together with the previous bound imply

$$\|P_t w\|_{1+\delta} \leq c t^{-\frac{1+\delta+\beta}{2}} \|A^{-\frac{\beta}{2}} w\|_0 \leq c t^{-\frac{1+\delta+\beta}{2}} \|w\|_{-\beta} < \infty,$$

having used in the last inequality again equation (1.1). □

2.1.2 Mapping properties of the integral operator

In what follows we state and give the proof of the main mapping property of the integral operator: it is a contraction on a Banach space of function with Hölder-type regularity in time and fractional Sobolev-type regularity in space.

Theorem 2.1.3. *Let* $0 < \beta < \delta < 1/2$ *and* $Z \in H_q^{1-\beta}(D)$ *for some* $q > \max(2, d/\delta)$. *Then for any* γ *such that* $0 < 2\gamma < 1 - \delta - \beta$ *it holds*

$$I_{(\cdot)} : C^\gamma([0, T]; \tilde{H}^{1+\delta}(D)) \to C^\gamma([0, T]; \tilde{H}^{1+\delta}(D))$$

and the following estimate holds for any fixed $\underline{u} \in C^{\gamma}([0,T]; \tilde{H}^{1+\delta}(D))$

$$\|I_{(\cdot)}(\underline{u})\|_{\gamma,1+\delta} \le c(T)\|\underline{u}\|_{\gamma,1+\delta},$$

where $c(T)$ *is a function of* T *not depending on* $\underline{u}$ *and such that*

$$\lim_{T\to 0} c(T) = 0.$$

Proof. Given any $\underline{u} \in C^{\gamma}\left([0,T]; \tilde{H}^{1+\delta}(D)\right)$ our goal is to bound

$$\|I_{(\cdot)}(\underline{u})\|_{\gamma,1+\delta} = \sup_{0\le t\le T}\Bigg(\|I_t(\underline{u})\|_{1+\delta} + \sup_{0\le s<t}\frac{\|I_t(\underline{u}) - I_s(\underline{u})\|_{1+\delta}}{(t-s)^{\gamma}}\Bigg) =: \sup_{0\le t\le T}\big((A)+(B)\big)$$

using the norm of $\underline{u}$, namely using $\|\underline{u}\|_{\gamma,1+\delta}$.
Step 1: consider part (A).
Fix $t \in [0,T]$. By definition of I we have

$$\begin{aligned}(A) = \|I_t(\underline{u})\|_{1+\delta} &= \\ &= \|\int_0^t P_{t-r}m(r)\,\mathrm{d}r\|_{1+\delta} \\ &\le \int_0^t \|P_{t-r}m(r)\|_{1+\delta}\,\mathrm{d}r.\end{aligned}$$

Apply Proposition 2.1.2 with $w = m(s) \in \tilde{H}^{-\beta}(D)$ and afterwards Proposition 2.1.1 part (1) and obtain

$$\begin{aligned}(A) &\le \int_0^t c\|m(r)\|_{-\beta}(t-r)^{-\frac{1+\delta+\beta}{2}}\,\mathrm{d}r \\ &\le \int_0^t c\|\underline{u}(r)\|_{1+\delta}(t-r)^{-\frac{1+\delta+\beta}{2}}\,\mathrm{d}r.\end{aligned}$$

Observe that for any $0 \le r \le t \le T$ we have

$$\|\underline{u}(r)\|_{1+\delta} \le \sup_{0\le r\le T}\|\underline{u}(r)\|_{1+\delta} \le \|\underline{u}\|_{\gamma,1+\delta}$$

and then we obtain

$$\begin{aligned}(A) =&\|I_t(\underline{u})\|_{1+\delta} \le c\|\underline{u}\|_{\gamma,1+\delta}\int_0^t (t-r)^{-\frac{1+\delta+\beta}{2}}\,\mathrm{d}r \\ &\le c\|\underline{u}\|_{\gamma,1+\delta}\frac{2}{1+\delta+\beta}t^{\frac{1-\delta-\beta}{2}}.\end{aligned}$$

Finally we have

$$(A) = \|I_t(\underline{u})\|_{1+\delta} \leq c_1(t)\|\underline{u}\|_{\gamma,1+\delta}. \tag{2.2}$$

where $c_1(t) := ct^{\frac{1-\delta-\beta}{2}}$ and so $\sup_{0\leq t\leq T} c_1(t) = c_1(T)$.

Step 2: consider part (B).

Let for the moment fix our attention only the argument inside the norm in the numerator of (B). Recall that $0 \leq s < t$. We make a change of variable in the middle integral $r' = r - t + s$ and we obtain

$$\begin{aligned}
&\int_0^t P_{t-r}m(r)\,\mathrm{d}r - \int_0^s P_{s-r}m(r)\,\mathrm{d}r \\
&= \int_0^{t-s} P_{t-r}m(r)\,\mathrm{d}r + \int_{t-s}^t P_{t-r}m(r)\,\mathrm{d}r - \int_0^s P_{s-r}m(r)\,\mathrm{d}r \\
&= \int_0^{t-s} P_{t-r}m(r)\,\mathrm{d}r + \int_0^s P_{s-r}m(r+t-s)\,\mathrm{d}r - \int_0^s P_{s-r}m(r)\,\mathrm{d}r \\
&= \int_0^{t-s} P_{t-r}m(r)\,\mathrm{d}r + \int_0^s P_{s-r}\big(m(r+t-s) - m(r)\big)\,\mathrm{d}r.
\end{aligned}$$

Therefore

$$\begin{aligned}
\|I_t(\underline{u}) - I_s(\underline{u})\|_{1+\delta} &= \left\| \int_0^t P_{t-r}m(r)\,\mathrm{d}r - \int_0^s P_{s-r}m(r)\,\mathrm{d}r \right\|_{1+\delta} \\
&= \bigg\| \int_0^{t-s} P_{t-r}m(r)\,\mathrm{d}r \\
&\qquad + \int_0^s P_{s-r}\big(m(r+t-s) - m(r)\big)\,\mathrm{d}r \bigg\|_{1+\delta},
\end{aligned}$$

and these computations enable us to write

$$\begin{aligned}
(B) \leq \sup_{0\leq s<t} &\frac{\| \int_0^{t-s} P_{t-r}m(r)\,\mathrm{d}r\|_{1+\delta}}{(t-s)^\gamma} \\
&+ \sup_{0\leq s<t} \frac{\| \int_0^s P_{s-r}\big(m(r+t-s) - m(r)\big)\,\mathrm{d}r\|_{1+\delta}}{(t-s)^\gamma} =: (C) + (D).
\end{aligned}$$

Step 3: consider term (C).

The numerator is similar to term (A) and therefore we proceed as we did in

Step 1. We have

$$\begin{aligned}
(C) = & \sup_{0\le s<t} \frac{\| \int_0^{t-s} P_{t-r} m(r)\,\mathrm{d}r\|_{1+\delta}}{(t-s)^\gamma}\\
\le & \sup_{0\le s<t} \frac{\int_0^{t-s} c\|\underline{u}(r)\|_{1+\delta}(t-r)^{-\frac{1+\delta+\beta}{2}}\,\mathrm{d}r}{(t-s)^\gamma}\\
\le & c\|\underline{u}\|_{\gamma,1+\delta} \sup_{0\le s<t} \int_0^{t-s} (t-r)^{-\frac{1+\delta+\beta}{2}}\,\mathrm{d}r \cdot (t-s)^{-\gamma}\\
= & c\|\underline{u}\|_{\gamma,1+\delta} \sup_{0\le s<t} (t-s)^{\frac{1-\delta-\beta}{2}-\gamma}\,\mathrm{d}r\\
= & c\|\underline{u}\|_{\gamma,1+\delta}\, t^{\frac{1-\delta-\beta-2\gamma}{2}},
\end{aligned}$$

where the last equality is valid if $1-\delta-\beta-2\gamma>0$.

Step 4: consider term (D).

First apply Proposition 2.1.2 to $w = m(r+t-s)-m(r)$ which is an element of $\tilde{H}^{-\beta}(D)$ thanks to Proposition 1.2.7. Then apply Proposition 2.1.1, part (2) and we get

$$\begin{aligned}
(D) = & \sup_{0\le s<t} \frac{\int_0^s \|P_{s-r}\big(m(r+t-s)-m(r)\big)\|_{1+\delta}\,\mathrm{d}r}{(t-s)^\gamma}\\
\le & \sup_{0\le s<t} \frac{\int_0^s \|m(r+t-s)-m(r)\|_{-\beta}(s-r)^{-\frac{1+\delta+\beta}{2}}\,\mathrm{d}r}{(t-s)^\gamma}\\
\le & \sup_{0\le s<t} \int_0^s \frac{\|\underline{u}(r+t-s)-\underline{u}(r)\|_{1+\delta}}{(t-s)^\gamma}(s-r)^{-\frac{1+\delta+\beta}{2}}\,\mathrm{d}r.
\end{aligned}$$

Fix the attention on the term $\frac{\|\underline{u}(r+t-s)-\underline{u}(r)\|_{1+\delta}}{(t-s)^\gamma}$. Observe that

$$\|\underline{u}\|_{\gamma,1+\delta} = \sup_{0\le t\le T}\|\underline{u}(t)\|_{1+\delta} + \sup_{0\le r<t\le T}\frac{\|\underline{u}(t)-\underline{u}(r)\|_{1+\delta}}{(t-r)^\gamma}$$

and in particular, setting $t-r=h$, the second summand can be rewritten as

$$\sup_{0<h\le r+h\le T}\frac{\|\underline{u}(r+h)-\underline{u}(r)\|_{1+\delta}}{h^\gamma}.$$

Since the parameters r and h are such that $0<h\le r+h\le T$, we then have

$$\begin{aligned}
(D) \le & c\|\underline{u}\|_{\gamma,1+\delta} \sup_{0\le s<t} \int_0^s (s-r)^{-\frac{1+\delta+\beta}{2}}\,\mathrm{d}r\\
\le & c\|\underline{u}\|_{\gamma,1+\delta} \sup_{0\le s<t} s^{\frac{1-\delta-\beta}{2}}\\
= & c\|\underline{u}\|_{\gamma,1+\delta}\, t^{\frac{1-\delta-\beta}{2}} < \infty.
\end{aligned}$$

Clipping the result for part (B) we obtain

$$(B) = (C) + (D) = \sup_{0 \le s < t} \frac{\|I_t(\underline{u}) - I_s(\underline{u})\|_{1+\delta}}{(t-s)^\gamma} \le c_2(t)\|\underline{u}\|_{\gamma,1+\delta}$$

where $c_2(t) = ct^{\frac{1-\delta-\beta-2\gamma}{2}} + ct^{\frac{1-\delta-\beta}{2}}$. Again we have $\sup_{0\le t\le T} c_2(t) = c_2(T)$. In conclusion the bound for (A)+(B) gives

$$\begin{aligned}
\|I_{(\cdot)}(\underline{u})\|_{\gamma,1+\delta} &= \sup_{0<t<T} \big((A) + (B)\big) \\
&\le \sup_{0<t<T} \big(c_1(t) + c_2(t)\big)\|\underline{u}\|_{\gamma,1+\delta} \\
&= c(T)\|\underline{u}\|_{\gamma,1+\delta}
\end{aligned}$$

where $c(t) := c_1(t) + c_2(t)$. Therefore $c(T) = cT^{\frac{1-\delta-\beta-2\gamma}{2}} + cT^{\frac{1-\delta-\beta}{2}}$ and this quantity is finite for every fixed T. Moreover, since $1-\delta-\beta-2\gamma > 0$ with $\gamma > 0$ we have that $c(T) \to 0$ as $T \to 0$. □

2.1.3 Existence and uniqueness of a local solution

Thanks to this mapping property of I we can easily recover the existence and uniqueness of a local mild solution for (1.5) .

Theorem 2.1.4. *Let $0 < \beta < \delta < 1/2$ and $0 < 2\gamma < 1-\delta-\beta$. Fix $Z \in H_q^{1-\beta}(\mathbb{R}^d)$ with $q > \max(2, d/\delta)$. Then for any initial condition $u_0 \in \tilde{H}^{1+\delta+2\gamma}(D)$ there exists a sufficiently small $\varepsilon > 0$ for which (1.5) has a unique local mild solution $\underline{u}$ in $C^\gamma([0,\varepsilon]; \tilde{H}^{1+\delta}(D))$. The solution satisfies the integral equation $\underline{u}(t) = P_t u_0 + I_t(\underline{u})$ for all $0 \le t \le \varepsilon$.*

Proof. From Theorem 2.1.3 we know that if $\underline{u} \in C^\gamma([0,T]; \tilde{H}^{1+\delta}(D))$ then $I_{(\cdot)}(\underline{u}) \in C^\gamma([0,T]; \tilde{H}^{1+\delta}(D))$.
Now we should ensure that for $u_0 \in \tilde{H}^\sigma(D)$, with $\sigma \ge 1+\delta+2\gamma$ then $P_{(\cdot)}u_0 \in C^\gamma([0,T]; \tilde{H}^{1+\delta}(D))$, namely that

$$\sup_{0\le t\le T}\left(\|P_t u_0\|_{1+\delta} + \sup_{0\le s<t} \frac{\|P_t u_0 - P_s u_0\|_{1+\delta}}{(t-s)^\gamma}\right) < \infty.$$

For the second summand use part (d) of Theorem 1.1.11 and relation (1.1) to obtain

$$\begin{aligned}
&\|P_t u_0 - P_s u_0\|_{1+\delta} = \|P_s(P_{t-s} - I)u_0\|_{1+\delta} \\
&\le c\|P_s\|\|(P_{t-s} - I)u_0\|_{1+\delta} \le c\|P_s\|(t-s)^\alpha \|A^\alpha u_0\|_{1+\delta} \\
&\le c\|P_s\|(t-s)^\alpha \|u_0\|_{1+\delta+2\alpha} \le cMe^{-\omega s}(t-s)^\alpha \|u_0\|_{1+\delta+2\alpha}
\end{aligned}$$

for any $0 < \alpha < 1$. Therefore the second summand becomes

$$\sup_{0\le s<t\le T} \frac{\|P_t u_0 - P_s u_0\|_{1+\delta}}{(t-s)^\gamma} \le \sup_{0\le s<t\le T} c_s (t-s)^\alpha \frac{\|u_0\|_{1+\delta+2\alpha}}{(t-s)^\gamma}$$

and if we choose $\alpha = \gamma$ then

$$\sup_{0\le s<t\le T} \frac{\|P_t u_0 - P_s u_0\|_{1+\delta}}{(t-s)^\gamma} \le c_s \|u_0\|_{1+\delta+2\gamma},$$

that is a finite quantity if $u_0 \in \tilde{H}^{1+\delta+2\gamma}(D)$.
So for any fixed $u_0 \in \tilde{H}^{1+\delta+2\gamma}(D)$ the operator $J_{(\cdot)} := P_{(\cdot)} u_0 + I_{(\cdot)}$ is mapping $C^\gamma([0,T]; \tilde{H}^{1+\delta}(D))$ into itself.
It is left to prove that $J_{(\cdot)}$ is a contraction, namely that there exists a constant $k < 1$ such that for all $\underline{u}, \underline{v} \in C^\gamma([0,T]; \tilde{H}^{1+\delta}(D))$

$$\|J_{(\cdot)}(\underline{u}) - J_{(\cdot)}(\underline{v})\|_{\gamma,1+\delta} \le k \|\underline{u} - \underline{v}\|_{\gamma,1+\delta}.$$

To this aim observe that

$$\begin{aligned}
\|J_{(\cdot)}(\underline{u}) - J_{(\cdot)}(\underline{v})\|_{\gamma,1+\delta} &= \|P_{(\cdot)} u_0 + I_{(\cdot)}(\underline{u}) - P_{(\cdot)} u_0 - I_{(\cdot)}(\underline{v})\|_{\gamma,1+\delta} \\
&= \|I_{(\cdot)}(\underline{u}) - I_{(\cdot)}(\underline{v})\|_{\gamma,1+\delta} \\
&= \|I_{(\cdot)}(\underline{u} - \underline{v})\|_{\gamma,1+\delta}
\end{aligned}$$

We clearly have $\underline{w} := \underline{u} - \underline{v} \in C^\gamma([0,T]; \tilde{H}^{1+\delta}(D))$ and then it suffices to apply the result of Theorem 2.1.3 with $\underline{w}$ instead of $\underline{u}$. Then let $k = c(T)$ and choose ε small enough such that for all $T < \varepsilon$ we have $c(T) < 1$. □

2.2 The global solution

We look for a global solution on $[0,T]$ where T is now arbitrary.

2.2.1 Equivalent norms and mapping properties

Let us introduce a family of equivalent norms $\|\cdot\|^{(\rho)}_{\gamma,U}$, $\rho \ge 1$ in $C^\gamma([0,T]; U)$ with which we can to get a bound for the integral operator which does not depend on T. The (ρ)-norm is defined by

$$\|f\|^{(\rho)}_{\gamma,U} := \sup_{0\le t\le T} e^{-\rho t} \left(\|f(t)\|_U + \sup_{0\le s<t} \frac{\|f(t) - f(s)\|_U}{(t-s)^\gamma} \right). \tag{2.3}$$

Fact 2.1. The (ρ)-norm given by (2.3) is equivalent to $\|\cdot\|_{\gamma,U}$ in $C^\gamma([0,T]; U)$ for all $\rho \ge 1$.

Proof. For any $f \in C^\gamma([0,T];U)$ we have

$$\begin{aligned}
\|f\|^{(\rho)}_{\gamma,U} &= \sup_{0\le t\le T} e^{-\rho t}\left(\|f(t)\|_U + \sup_{0\le s<t}\frac{\|f(t)-f(s)\|_U}{(t-s)^\gamma}\right)\\
&\le \sup_{0\le t\le T} e^{0}\left(\|f(t)\|_U + \sup_{0\le s<t}\frac{\|f(t)-f(s)\|_U}{(t-s)^\gamma}\right)\\
&= \sup_{0\le s<t\le T}\left(\|f(t)\|_U + \frac{\|f(t)-f(s)\|_U}{(t-s)^\gamma}\right)\\
&= \|f\|_{\gamma,U}.
\end{aligned}$$

On the other hand, it holds

$$\begin{aligned}
\|f\|_{\gamma,U} &= \sup_{0\le s<t\le T}\left(\|f(t)\|_U + \frac{\|f(t)-f(s)\|_U}{(t-s)^\gamma}\right)\\
&= \sup_{0\le t\le T}\frac{e^{\rho t}}{e^{\rho t}}\left(\|f(t)\|_U + \sup_{0\le s<t}\frac{\|f(t)-f(s)\|_U}{(t-s)^\gamma}\right)\\
&\le e^{\rho T}\|f\|^{(\rho)}_{\gamma,U}.
\end{aligned}$$

□

The following Lemma gives integral bounds which will be used later. The proof makes use of the Gamma and the Beta functions together with some basic integral estimates. Recall the definition of the gamma function:

$$\Gamma(a) = \int_0^\infty e^{-t}t^{a-1}\mathrm{d}t,$$

and the integral converges for any $a \in \mathbb{C}$ such that $\mathrm{Re}(a) > 0$.

Lemma 2.2.1. *If $0 \le s < t \le T < \infty$ and $0 \le \theta < 1$ then for any $\rho \ge 1$ it holds*

$$\int_s^t e^{-\rho r}r^{-\theta}\mathrm{d}r \le \Gamma(1-\theta)\rho^{\theta-1}. \tag{2.4}$$

Moreover if $\gamma > 0$ is such that $\theta+\gamma<1$ then for any $\rho \ge 1$ there exists a positive constant C such that

$$\int_0^t e^{-\rho(t-r)}(t-r)^{-\theta}r^{-\gamma}\mathrm{d}r \le C\rho^{\theta-1+\gamma}. \tag{2.5}$$

Proof. We start with (2.4). Consider the integral

$$\int_s^t e^{-\rho r}r^{-\theta}\mathrm{d}r = \int_s^t e^{-\rho r}(\rho r)^{-\theta+1-1}\rho^\theta\ \mathrm{d}r$$

now make the change of variable $\tau = \rho r$ and obtain

$$\int_{\rho s}^{\rho t} e^{-\tau}\tau^{-\theta+1-1}\rho^{\theta-1}\,\mathrm{d}\tau \le \rho^{\theta-1}\int_0^{\infty} e^{-\tau}\tau^{-\theta+1-1}\,\mathrm{d}\tau = \rho^{\theta-1}\Gamma(1-\theta).$$

Let us prove (2.5). We make several changes of variable and get

$$\begin{aligned}
&\int_0^t e^{-\rho(t-r)}(t-r)^{-\theta}r^{-\gamma}\mathrm{d}r && \{s=t-r\}\\
&\quad= \int_0^t e^{-\rho s}s^{-\theta}(t-s)^{-\gamma}\mathrm{d}s && \{u=\rho s\}\\
&\quad= \int_0^{\rho t} e^{-u}u^{-\theta}\rho^{\theta}\rho^{\gamma}(\rho t-u)^{-\gamma}\rho^{-1}\mathrm{d}u\\
&\quad= \rho^{\theta+\gamma-1}\int_0^{\rho t} e^{-u}u^{-\theta}(\rho t-u)^{-\gamma}\mathrm{d}u && \{\text{Set } z:=\rho t\}\\
&\quad= \rho^{\theta+\gamma-1}\int_0^{z} e^{-u}u^{-\theta}(z-u)^{-\gamma}\mathrm{d}u\\
&\quad=: \rho^{\theta+\gamma-1}I(z).
\end{aligned}$$

The integral $I(t)$ is finite for all $z \ge 0$. If $z = 0$ then $I(z) = 0$. If $z \in (0, 2]$ we get

$$\begin{aligned}
I(z) &= \int_0^z e^{-u}u^{-\theta}(z-u)^{-\gamma}\mathrm{d}u && \{u=zx\}\\
&= z^{-\theta-\gamma+1}\int_0^1 e^{-zx}x^{-\theta}(1-x)^{-\gamma}\mathrm{d}x\\
&\le 2^{-\theta-\gamma+1}\int_0^1 x^{-\theta}(1-x)^{-\gamma}\mathrm{d}x\\
&\le 2^{-\theta-\gamma+1}B(-\theta+1,-\gamma+1).
\end{aligned}$$

If $z \ge 2$ we can split $I(z)$ into two parts and get

$$\begin{aligned}
I(z) &= \int_0^{z-1} e^{u}u^{-\theta}(z-u)^{-\gamma}\mathrm{d}u + \int_{z-1}^{z} e^{u}u^{-\theta}(z-u)^{-\gamma}\mathrm{d}u\\
&\le \int_0^{z-1} e^{u}u^{-\theta}\mathrm{d}u + \int_{z-1}^{z}(z-u)^{-\gamma}\mathrm{d}u = \Gamma(1-\theta) + \frac{1}{1-\gamma}. \qquad \square
\end{aligned}$$

Theorem 2.2.2. *Let $0 < \beta < \delta < \frac{1}{2}$ and $Z \in H_q^{1-\beta}(\mathbb{R}^d)$ for $q > \max(2, \frac{d}{\delta})$. Then for any γ such that $0 < 2\gamma < 1 - \delta - \beta$ it holds*

$$I : C^{\gamma}([0,T];\tilde{H}^{1+\delta}(D)) \to C^{\gamma}([0,T];\tilde{H}^{1+\delta}(D))$$

and the following estimate holds for any fixed $\underline{u} \in C^{\gamma}([0,T];\tilde{H}^{1+\delta}(D))$

$$\|I_{(\cdot)}(\underline{u})\|^{(\rho)}_{\gamma,1+\delta} \le c(\rho)\|\underline{u}\|^{(\rho)}_{\gamma,1+\delta} \tag{2.6}$$

where $c(\rho)$ is a function of ρ not depending on $\underline{u}$ nor T and such that

$$\lim_{\rho\to\infty} c(\rho) = 0.$$

Proof. In order to prove this result we follow the line of Theorem 2.1.3 but using the equivalent norm $\|\cdot\|^{(\rho)}_{\gamma,1+\delta}$ for some arbitrary ρ.
Given any $\underline{u} \in C^\gamma([0,T];\tilde{H}^{1+\delta}(D))$ our goal is to bound

(2.7)

$$\begin{aligned}\|I_{(\cdot)}(\underline{u})\|^{(\rho)}_{\gamma,1+\delta} = \sup_{0\le t\le T} \Big(& e^{-\rho t}\|I_t(\underline{u})\|_{1+\delta}+ \\ & + e^{-\rho t} \sup_{0\le s<t} \frac{\|I_t(\underline{u}) - I_s(\underline{u})\|_{1+\delta}}{(t-s)^\gamma}\Big) =: \sup_{0\le t\le T} \big((A)+(B)\big)\end{aligned}$$

using the (ρ)-norm of $\underline{u}$, namely using $\|\underline{u}\|^{(\rho)}_{\gamma,1+\delta}$.
Step 1: Consider part (A). Observe that for any $0\le r\le t\le T$

$$e^{-\rho r}\|\underline{u}(r)\|_{1+\delta} \le \sup_{0\le r\le T} e^{-\rho r}\|\underline{u}(r)\|_{1+\delta} \le \|\underline{u}\|^{(\rho)}_{\gamma,1+\delta}.$$

Then we obtain

$$\begin{aligned}(A) &\le e^{-\rho t}\|I_t(\underline{u})\|_{1+\delta} \\ &\le ce^{-\rho t}\int_0^t \|\underline{u}(r)\|_{1+\delta}(t-r)^{-\frac{1+\delta+\beta}{2}}\,\mathrm{d}r \\ &\le c\|\underline{u}\|^{(\rho)}_{\gamma,1+\delta}\int_0^t e^{-\rho(t-r)}(t-r)^{-\frac{1+\delta+\beta}{2}}\,\mathrm{d}r \\ &= c\|\underline{u}\|^{(\rho)}_{\gamma,1+\delta}\int_0^t e^{-\rho r}r^{-\frac{1+\delta+\beta}{2}}\,\mathrm{d}r \\ &\le c\|\underline{u}\|^{(\rho)}_{\gamma,1+\delta}\rho^{\frac{1+\delta+\beta}{2}-1},\end{aligned}$$

having used estimate (2.4) of Lemma 2.2.1 in the last line. Clipping the result together we can state that

$$(A) = e^{-\rho t}\|I_t(\underline{u})\|_{1+\delta} \le c_1(\rho)\|\underline{u}\|^{(\rho)}_{\gamma,1+\delta}$$

where $c_1(\rho) = c\rho^{\frac{\delta+\beta-1}{2}}$ and since $\frac{\delta+\beta-1}{2} < 0$ we have $c_1(\rho)\to 0$ as $\rho\to\infty$.
Step 2: Consider part (B). As done in Step 2 for the proof of Theorem 2.1.3 we get

$$\begin{aligned}(B) \le & e^{-\rho t}\sup_{0\le s<t}\frac{\|\int_0^{t-s} P_{t-r}m(r)\,\mathrm{d}r\|_{1+\delta}}{(t-s)^\gamma} \\ & + e^{-\rho t}\sup_{0\le s<t}\frac{\|\int_0^s P_{s-r}(m(r+t-s)-m(r))\,\mathrm{d}r\|_{1+\delta}}{(t-s)^\gamma} := (C)+(D).\end{aligned}$$

Step 3: Consider term (C).

The numerator is similar to the term (A) and therefore we proceed as we did in Step 1. We have

$$\begin{aligned}(C) =& e^{-\rho t} \sup_{0\le s<t} \frac{\| \int_0^{t-s} P_{t-r}m(r)\,\mathrm{d}r\|_{1+\delta}}{(t-s)^\gamma}\\ \le& e^{-\rho t} \sup_{0\le s<t} \frac{\int_0^{t-s} c\|\underline{u}(r)\|_{1+\delta}(t-r)^{-\frac{1+\delta+\beta}{2}}\,\mathrm{d}r}{(t-s)^\gamma}\\ \le& \sup_{0\le s<t}\int_0^{t-s} e^{-\rho(t-r)}c\|\underline{u}\|^{(\rho)}_{\gamma,1+\delta}(t-r)^{-\frac{1+\delta+\beta}{2}}(t-s)^{-\gamma}\,\mathrm{d}r\\ \le& c\|\underline{u}\|^{(\rho)}_{\gamma,1+\delta}\sup_{0\le s<t}\int_0^{t-s} e^{-\rho(t-r)}(t-r)^{-\frac{1+\delta+\beta}{2}}r^{-\gamma}\,\mathrm{d}r\\ =& c\|\underline{u}\|^{(\rho)}_{\gamma,1+\delta}\int_0^{t} e^{-\rho(t-r)}(t-r)^{-\frac{1+\delta+\beta}{2}}r^{-\gamma}\,\mathrm{d}r\end{aligned}$$

apply estimate (2.5) in Lemma 2.2.1 with $\theta = \frac{1+\delta+\beta}{2}$: since by hypothesis $2\gamma < 1-\delta-\beta$ then $\gamma+\theta<1$. We obtain

$$(C) \le c\|\underline{u}\|^{(\rho)}_{\gamma,1+\delta}\rho^{\frac{1+\delta+\beta+2\gamma}{2}-1} \le c\|\underline{u}\|^{(\rho)}_{\gamma,1+\delta}\rho^{\frac{\delta+\beta+2\gamma-1}{2}}.$$

Clipping the result together

$$(C) = e^{-\rho t} \sup_{0\le s<t} \frac{\| \int_0^{t-s} P_{t-r}m(r)\,\mathrm{d}r\|_{1+\delta}}{(t-s)^\gamma} \le c_1\|\underline{u}\|^{(\rho)}_{\gamma,1+\delta}\rho^{\frac{\delta+\beta+2\gamma-1}{2}}.$$

Step 4: Consider term (D).

First apply Proposition 2.1.2 to $w = m(r+t-s)-m(r)$ which is an element of $\tilde{H}^{-\beta}(D)$ thanks to Proposition 1.2.7. Then apply Proposition 2.1.1, part (2).

$$\begin{aligned}(D) =& e^{-\rho t} \sup_{0\le s<t} \frac{\| \int_0^{s} P_{s-r}(m(r+t-s)-m(r))\,\mathrm{d}r\|_{1+\delta}}{(t-s)^\gamma}\\ \le& e^{-\rho t} \sup_{0\le s<t} \frac{\int_0^{s} \|m(r+t-s)-m(r)\|_{-\beta}(s-r)^{-\frac{1+\delta+\beta}{2}}}{(t-s)^\gamma}\,\mathrm{d}r\\ \le& ce^{-\rho t} \sup_{0\le s<t}\int_0^{s} \frac{e^{-\rho(r+t-s)}}{e^{-\rho(r+t-s)}}\frac{\|\underline{u}(r+t-s)-\underline{u}(r)\|_{1+\delta}(s-r)^{-\frac{1+\delta+\beta}{2}}}{(t-s)^\gamma}\,\mathrm{d}r\\ \le& c\sup_{0\le s<t}\int_0^{s} e^{-\rho(s-r)}e^{-\rho(r+t-s)}\frac{\|\underline{u}(r+t-s)-\underline{u}(r)\|_{1+\delta}}{(t-s)^\gamma}(s-r)^{-\frac{1+\delta+\beta}{2}}\,\mathrm{d}r.\end{aligned}$$

Fix the attention on the term $e^{-\rho(r+t-s)}\frac{\|\underline{u}(r+t-s)-\underline{u}(r)\|_{1+\delta}}{(t-s)^\gamma}$ and set $h=t-s$: we obtain

$$e^{-\rho(r+h)}\frac{\|\underline{u}(r+h)-\underline{u}(r)\|_{1+\delta}}{h^\gamma}. \tag{2.8}$$

Moreover observe that

$$\|\underline{u}\|^{(\rho)}_{\gamma,1+\delta} = \sup_{0\le t\le T} e^{-\rho t}\|\underline{u}(t)\|_{1+\delta} + \sup_{0\le r<t\le T} e^{-\rho t}\frac{\|\underline{u}(t)-\underline{u}(r)\|_{1+\delta}}{(t-r)^\gamma}$$

and in particular, setting again $t-r=h$, the second summand can be rewritten as

$$\sup_{0<h\le r+h\le T} e^{-\rho(r+h)}\frac{\|\underline{u}(r+h)-\underline{u}(r)\|_{1+\delta}}{h^\gamma}.$$

Therefore we can bound (2.8) by $\|\underline{u}\|^{(\rho)}_{\gamma,1+\delta}$ (since the parameters r and h are such that $0<h\le r+h\le T$) and applying once more estimate (2.4) in Lemma 2.2.1 the upper bound for (D) becomes

$$\begin{aligned}(D) \le& c\|\underline{u}\|^{(\rho)}_{\gamma,1+\delta}\sup_{0\le s<t}\int_0^s e^{-\rho(s-r)}(s-r)^{-\frac{1+\delta+\beta}{2}}\,\mathrm{d}r\\ \le& c_2\|\underline{u}\|^{(\rho)}_{\gamma,1+\delta}\rho^{\frac{\delta+\beta-1}{2}}\Gamma\left(\frac{\delta+\beta-1}{2}\right).\end{aligned}$$

Clipping the result for part (B) we obtain

$$(B)=(C)+(D)=e^{-\rho t}\sup_{0\le s<t}\frac{\|I_t(\underline{u})-I_s(\underline{u})\|_{1+\delta}}{(t-s)^\gamma}\le c_2(\rho)\|\underline{u}\|^{(\rho)}_{\gamma,1+\delta} \tag{2.9}$$

where $c_2(\rho)=c\rho^{\frac{\delta+\beta+2\gamma-1}{2}}+c\rho^{\frac{\delta+\beta-1}{2}}$ and since $\frac{\delta+\beta+2\gamma-1}{2}$ and $\frac{\delta+\beta-1}{2}$ are negative we have $c_2(\rho)\to 0$ as $\rho\to\infty$.

Finally observe that the bound for $(A)+(B)$ does not depend on t and then the supremum over $0\le t\le T$ of $(A)+(B)$ is simply bounded by

$$\|I_{(\cdot)}(\underline{u})\|^{(\rho)}_{\gamma,1+\delta}=\sup_{0\le t\le T}\Big((A)+(B)\Big)\le (c_1(\rho)+c_2(\rho))\|\underline{u}\|^{(\rho)}_{\gamma,1+\delta}$$

that is the thesis. □

2.2.2 Existence and uniqueness of a global solution

Now we prove existence and uniqueness of a global mild solution.

Theorem 2.2.3. *Let $0<\beta<\delta<\frac{1}{2}$ and $0<2\gamma<1-\beta-\delta$. Fix $Z\in H^{1-\beta}_q(\mathbb{R}^d)$ for some $q>\max(2,\frac{d}{\delta})$. Then for any initial condition $u_0\in \tilde{H}^{1+\delta+2\gamma}(D)$ and for any positive finite time T there exists a unique mild solution $\underline{u}$ in $C^\gamma([0,T];\tilde{H}^{1+\delta}(D))$ for (1.5) satisfying the integral equation $\underline{u}(t)=P_tu_0+I_t(\underline{u})$.*

Proof. We will follow the line of Theorem 2.1.4. By the proof of the latter theorem, we already know that

- if $\underline{u} \in C^\gamma([0,T];\tilde{H}^{1+\delta}(D))$ then $I_{(\cdot)}(\underline{u}) \in C^\gamma([0,T];\tilde{H}^{1+\delta}(D))$;
- if $u_0 \in \tilde{H}^{1+\delta+2\gamma}(D)$ then $P_{(\cdot)}u_0 \in C^\gamma([0,T];\bar{H}^{1+\delta}(D))$.

So for any fixed $u_0 \in \tilde{H}^{1+\delta+2\gamma}(D)$ the operator $J_{(\cdot)} := P_{(\cdot)}u_0 + I_{(\cdot)}$ is mapping $C^\gamma([0,T];\tilde{H}^{1+\delta}(D))$ into itself. It is left to prove that $J_{(\cdot)}$ is a contraction (for arbitrary $T > 0$), namely that there exists a constant $k < 1$ such that for all $\underline{u}, \underline{v} \in C^\gamma([0,T];\tilde{H}^{1+\delta}(D))$

$$\|J_{(\cdot)}(\underline{u}) - J_{(\cdot)}(\underline{v})\|^{(\rho)}_{\gamma,1+\delta} \leq k\|\underline{u} - \underline{v}\|^{(\rho)}_{\gamma,1+\delta}.$$

For this aim observe that

$$\begin{aligned}
\|J_{(\cdot)}(\underline{u}) - J_{(\cdot)}(\underline{v})\|^{(\rho)}_{\gamma,1+\delta} &= \|P_t u_0 + I_{(\cdot)}(\underline{u}) - P_t u_0 - I_{(\cdot)}(\underline{v})\|^{(\rho)}_{\gamma,1+\delta} \\
&= \left\| \int_0^\cdot P_{\cdot-r}\langle \nabla \underline{u}(r), \nabla Z\rangle \mathrm{d}r - \int_0^\cdot P_{\cdot-r}\langle \nabla \underline{v}(r), \nabla Z\rangle \mathrm{d}r \right\|^{(\rho)}_{\gamma,1+\delta} \\
&\leq \left\| \int_0^\cdot P_{\cdot-r}\left(\langle \nabla(\underline{u}(r) - \underline{v}(r)), \nabla Z\rangle \mathrm{d}r\right) \right\|^{(\rho)}_{\gamma,1+\delta} \leq \|I_{(\cdot)}(\underline{u} - \underline{v})\|^{(\rho)}_{\gamma,1+\delta}.
\end{aligned}$$

We clearly have $\underline{w} := \underline{u} - \underline{v} \in C^\gamma([0,T];\tilde{H}^{1+\delta}(D))$ and then it suffices to apply the result of Theorem 2.2.2 with $\underline{w}$ instead of $\underline{u}$ and choose ρ big enough such that the constant $c(\rho)$ appearing in (2.6) is less than 1. □

Chapter 3

Applications: the stochastic transport equation

In this chapter we will apply the previous results to some stochastic PDEs.

3.1 The stochastic transport equation

Consider the stochastic transport equation given by

(3.1)
$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \sigma^2 \Delta u(t,x) + \langle \nabla u(t,x), \nabla Y(x,\omega) \rangle, & t \in (0,T], x \in D \\ u(t,x) = 0, & t \in (0,T], x \in \partial D \\ u(0,x) = u_0(x), & x \in D \end{cases}$$

where $Y = \{Y(x,\omega)\}_{x\in\mathbb{R}^d}$ is a stochastic field defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$. One suitable example for the noise Y is the *Levy fractional Brownian motion* $\{B^H(x)\}_{x\in\mathbb{R}^d}$ which is the isotropic generalization of the fractional Brownian motion (see [45]). This field is defined to be a centered Gaussian field on $\mathbb{R}^d$ of covariance function

$$\mathbb{E}[B^H(x)B^H(y)] = \frac{1}{2}(|x|_d^{2H} + |y|_d^{2H} - |x-y|_d^{2H}),$$

where $|\cdot|_d$ stands for the Euclidean norm in $\mathbb{R}^d$. The parameter $0 < H < 1$ is called Hurst parameter. In case when $H = \frac{1}{2}$ we recover the *Levy Brownian motion*, whereas if $d = 1$ we get the fractional Brownian motion. Using a Kolmogorov continuity theorem suitable for stochastic fields (see for instance [43], Theorem 1.4.1) and basic properties of Gaussian random variables one can show that there exist $\Omega_1 \subset \Omega$ with $\mathbb{P}(\Omega_1) = 1$ and a modification of $B^H(x), x \in D$ (for simplicity we call it again $B^H(x)$) with $D \subset \mathbb{R}^d$ arbitrary bounded domain of $\mathbb{R}^d$ such that for every $\omega \in \Omega_1$ and for every $x, y \in D$ we have

$$|B^H(x,\omega) - B^H(y,\omega)| \leq K_\omega |x-y|_d^\alpha, \ \forall \alpha < H,$$

where K is a positive random variable with finite moments of every order. In other words, for almost every realization ω the field is α-Hölder continuous on D of any order $\alpha < H$. This fact together with the following property enable us to apply the results presented in the previous section to equation (3.1) in a pathwise sense.

Proposition 3.1.1. *Let h be a compactly supported real valued α-Hölder continuous function on $\mathbb{R}^d$ for some $0 < \alpha < 1$. Then for any $\alpha' < \alpha$ we have $h \in H_p^{\alpha'}(\mathbb{R}^d)$ for all $2 \leq p < \infty$.*

The proof makes use of the equivalent norm

$$\|h\|_{L^p} + \left(\int_{|y|\leq 1} \frac{\|h(\cdot+y) - h(\cdot)\|_{L^p}^2}{|y|^{d+2\alpha'}} \mathrm{d}y \right)^{\frac{1}{2}}$$

for the Besov spaces $B_{p,2}^{\alpha'}(\mathbb{R}^d)$ and of embedding properties between Besov and Sobolev spaces (see [73] for more details).

Proof. First recall the embedding relation between the Besov spaces and the Bessel potential spaces (see [73] for definition, equivalent norms in Section 11.4 and embedding results in Section 2.6.1): $B_{p,2}^{\alpha'}(\mathbb{R}^d) \subset F_p^{\alpha'}(\mathbb{R}^d) = H_p^{\alpha'}(\mathbb{R}^d)$ for all $p \geq 2$ and all $0 < \alpha' < 1$, and an equivalent norm in $B_{p,2}^{\alpha'}(\mathbb{R}^d)$ is

$$\|h\|_{L^p} + \left(\int_{|y|\leq 1} \frac{\|h(\cdot+y) - h(\cdot)\|_{L^p}^2}{|y|^{d+2\alpha'}} \mathrm{d}y \right)^{1/2} \tag{3.2}$$

where $\|\cdot\|_{L^p}$ is the norm in $L^p(\mathbb{R}^d)$. Clearly the first summand of (3.2) is finite, since h is continuous and it has compact support. Consider for a moment only $\|h(\cdot+y) - h(\cdot)\|_{L^p}^2$ for some fixed $y \in \mathbb{R}^d$ such that $|y| \leq 1$. The set $K = \mathrm{supp}(h) \subset \mathbb{R}^d$ is compact by assumption. We define

$$\tilde{K} := \{z = x + y : x \in K, y \in \mathbb{R}^d, |y| \leq 1\}$$

which is obviously also a compact set. Moreover for any $|y| \leq 1$ the function $h(\cdot + y) - h(\cdot)$ is supported in $\tilde{K}$. Using the Hölder continuity we have for all $y \in \mathbb{R}^d, |y| \leq 1$

$$\begin{aligned}\|h(\cdot + y) - h(\cdot)\|_{L^p}^2 &\leq \left(\int_{\tilde{K}} |h(x+y) - h(x)|^p \mathrm{d}x \right)^{2/p} \\ &\leq \left(\int_{\tilde{K}} c^p |y|_d^{\alpha p} \mathrm{d}x \right)^{2/p} = c^2 \lambda(\tilde{K}) |y|_d^{2\alpha} \leq c|y|_d^{2\alpha}.\end{aligned}$$

We can now easily bound the second summand in (3.2). Let $\varepsilon := \alpha - \alpha'$.

$$\begin{aligned}\left(\int_{|y|\leq 1} \frac{\|h(\cdot + y) - f(\cdot)\|_{L^p}^2}{|y|^{d+2\alpha'}} \mathrm{d}y \right)^{1/2} &\leq \left(\int_{|y|\leq 1} \frac{|y|^{2\alpha}}{|y|^{d+2\alpha'}} \mathrm{d}y \right)^{1/2} \\ &\leq \left(\int_{|y|\leq 1} \frac{1}{|y|^{d-2\varepsilon}} \mathrm{d}y \right)^{1/2} \leq C < \infty\end{aligned}$$

because $d - 2\varepsilon < d$ since $\varepsilon > 0$ by assumption. □

In order to apply this to (almost every) path of B^H we should ensure the compactness of the support. This is not true in general. Instead, since (3.1) is considered only on the domain D, let $\psi(x), x \in \mathbb{R}^d$ be a C^∞-function with compact support and such that $\psi(x) = 1\ \forall x \in \bar{D}$. Then for almost every $\omega \in \Omega$ the function $\psi(x)B^H(\omega, x)$ is α-Hölder continuous we have that for all $1 < q < \infty$ and for all $\alpha' < \alpha < H$, $\psi(\cdot)B^H(\omega, \cdot) \in H_q^{\alpha'}(\mathbb{R}^d)$. For consistency of notation call $1-\beta := \alpha'$, and so $1-\beta < H$. In order to match the conditions on the parameter β we have to choose $\frac{1}{2} < H < 1$. Then for every $\omega \in \Omega_1$ we set $Z(x) := \psi(x)B^H(\omega, x)$ and so Theorem 2.2.3 ensures existence and uniqueness of a function solution to the stochastic Dirichlet problem (3.1) with $Y = B^H$.

3.2 A (more) general stochastic transport equation

We combine in this section the main result obtained in Chapter 2 with a result obtained by Hinz and Zähle in [29]. In this paper they consider (among others) a linear equation of the form

$$\begin{cases} \frac{\partial u}{\partial t} = -Au + F\frac{\partial}{\partial t}\nabla V, & t \in (0,T] \\ u(0,x) = f(x), \end{cases} \tag{3.3}$$

where F is a deterministic vector and V is a given non-differentiable function which can be for example the path of a space-time fractional noise. More precisely, the field V is taken to be an element of $C^{1-\alpha}([0,T];H_q^{1-\beta}(\mathbb{R}^d))$. The solution to (3.3) is given in the mild form as

$$\begin{aligned} u(t,\cdot) &= P_t f(\cdot) + \int_0^t P_{t-s}\big(F\cdot\nabla V(s)\big)(\cdot)\,\mathrm{d}s \\ &= P_t f(\cdot) + I_t^{\alpha'}(F, \frac{\partial}{\partial t}\nabla V) \end{aligned}$$

where the integral operator $I_t^{\alpha'}(F, \frac{\partial}{\partial t}\nabla V)$ is defined in Definition 2.1 of [29] (we only need the case $k=1$). Fourier transform is used to perform the integration with respect to the space variable x and fractional derivatives are employed to give a meaning to the derivative with respect to time.
To get a better idea about the definition of this integral operator, let us write it with the help of a mapping $\Psi_t(\cdot)$ given by

$$\Psi_t(s)(w) := P_{t-s}\big(F\cdot w\big),$$

for any $w\in \tilde{H}_q^{-\beta}(D)$. The integral operator becomes then

$$\int_0^t P_{t-s}\big(F\cdot\nabla V(s)\big)\,\mathrm{d}s = \int_0^t \Psi_t(s)\big(F\cdot\nabla V(s)\big)\,\mathrm{d}s.$$

They show that in fact Ψ_t is an operator valued mapping

$$\Psi_t : [0,t] \to \mathcal{L}\big(\tilde{H}^{-\beta}(D); \tilde{H}^{\delta}(D)\big)$$

(for some β) with fractional order of smoothness α' (in time) slightly bigger than α, where α appears in the smoothness of V. By assumptions on V it follows also that ∇V has fractional order of smoothness $1-\alpha'$ therefore one can define

$$\int_0^t \Psi_t(s)\big(\nabla V(s)\big)\,\mathrm{d}s = \int_0^t D_{0+}^{\alpha'}\Psi_t(s)\big(D_{t-}^{1-\alpha'}V_t\big)\,\mathrm{d}s,$$

where $V_t := V - V(t)$ and $D_{0+}^{\alpha'}$ and $D_{t-}^{1-\alpha'}$ are respectively left and right sided fractional derivatives in Banach spaces (for more details see [28]).
The authors exploit also the regularity of this integral and prove in Proposition 7.1 that if $0<\alpha,\beta,\gamma<1$ with $\alpha+\gamma<1$ and $2\gamma+\tilde{\delta}<2-2\alpha-\beta$ then the integral $I_{(\cdot)}^{\alpha'}(F, \frac{\partial}{\partial t}\nabla V)$ (which in fact does not depend on α') belongs to the space $C^{\gamma}([0,T];\tilde{H}^{\tilde{\delta}}(D))$ for any given function $V\in C^{1-\alpha}([0,T];H^{1-\beta}(\mathbb{R}^d))$ and vector $F\in\mathbb{R}^d$.

Taking this into account we are able to give the following existence and uniqueness result.

Corollary 3.2.1. *Let $T > 0$ be fixed, choose $0 < \beta < \delta < \frac{1}{2}$ and $0 < 2\gamma < 1 - \beta - \delta$. Fix $F \in \mathbb{R}^d$, $Z \in H_q^{1-\beta}(\mathbb{R}^d)$ and $V \in C^{1-\alpha}([0,T]; H_q^{1-\beta}(\mathbb{R}^d))$ for some $q > \max(2, \frac{d}{\delta})$ and for some $0 < \alpha < 1$ such that $\alpha + \gamma < 1$. Then given any initial condition $u_0 \in \tilde{H}^{1+\delta+2\gamma}(D)$ there exists a unique global mild solution $u(t,x)$ in the Hölder space $C^\gamma([0,T]; \tilde{H}^{1+\delta}(D))$ for the problem*

$$
(3.4) \qquad \begin{cases} \frac{\partial u}{\partial t}(t,x) = \sigma^2 \Delta u(t,x) + \langle \nabla u(t,x), \nabla Z(x) \rangle \\ \qquad\qquad\qquad + \langle F, \frac{\partial}{\partial t} \nabla V(t,x) \rangle, & t \in (0,T], x \in D \\ u(t,x) = 0, & t \in (0,T], x \in \partial D \\ u(0,x) = u_0(x), & x \in D \end{cases}
$$

and the solution is given by

$$
u(t,\cdot) = P_t u_0 + I_t(\underline{u}) + I_t^\alpha(F, \frac{\partial}{\partial t} \nabla V).
$$

Proof. Set $\tilde{\delta} := 1 + \delta$. Since $2\gamma < 1 - \delta - \beta$ then $2\gamma + \tilde{\delta} < 2 - \beta$ and if one chooses a positive α such that $2\gamma + \tilde{\delta} < 2 - \beta - 2\alpha$ then the condition $\alpha + \gamma < 1$ is satisfied and by Proposition 7.1 in [29] we have $I_{(\cdot)}^\alpha(F, \frac{\partial}{\partial t}\nabla V) \in C^\gamma([0,T]; \tilde{H}^{\tilde{\delta}}(D))$. Finally apply a contraction principle as applied in the proof of Theorem 2.2.3 and recover the thesis. □

With the same technique illustrated in Section 3.1 one can solve (3.4) in the case when Z and V are substituted by stochastic fields, and then the system is solved in the pathwise sense. See [29], Section 6 for a survey on possible noises in place of V.

Part II

Cylindrical approach

Chapter 4

The fractional Brownian motion

This chapter is devoted to fractional Brownian motion (fBm) and related stochastic calculus.

In Section 4.1 we define the fBm in $\mathbb{R}$, $\mathbb{R}^n$ and more generally in a Hilbert space. FBm is a family of Gaussian processes depending on the so-called *Hurst parameter* $H \in (0,1)$ and (except for $H = 1/2$) they are not semimartingales. For this reason a stochastic calculus different from Îto-type calculus has been developed in the last decades, leading to many different approaches, each of them exploiting a different property of fBm.

In Section 4.2 we present Wiener integrals. This integration theory can be performed for deterministic integrands with respect to general Gaussian processes. In Subsection 4.2.1 we recall the theory of Wiener integrals for real valued Gaussian processes, with the main focus on fractional Brownian motion. This theory is classical and we refer to [9, 52] for more details. In Subsection 4.2.2 we present the theory of Wiener integrals for Hilbert space valued integrands with respect to real valued fBm. This results is a generalization of the real case and appeared in some works of Duncan et al. [22, 20, 57]. The integral as stochastic process (indexed by time $t \in [0,T]$) is also considered. Some result on continuity with respect to time are given in some special cases (the answer to this question in general is still unknown). Finally, in Subsection 4.2.4 we give a detailed proof of stochastic Fubini theorem for fractional Brownian motion for Wiener integrals.

4.1 Introduction

The fractional Brownian motion was first introduced in 1940 in a Hilbert space framework by Kolmogorov (see [41]) with the name *Wiener Helix*. Later on, Hurst and coauthors published some papers devoted to long-term storage capacity in reservoirs (see [34, 35]), after which the parameter H was named *Hurst parameter*. In 1968 Mandelbrot and Van Ness provided in [48] a stochastic integral representation of this process in terms of a standard Brownian motion. From this pioneering work originates the name *fractional Brownian motion* and this paper was the starting point for the development of a stochastic calculus for this process. Because of its long-memory property, fBm has been used in various models dealing for instance with teletraffic, finance, climate and weather derivatives.

We start with the definition of fractional Brownian motion in $\mathbb{R}$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space.

Definition 4.1.1. *A fractional Brownian motion with Hurst parameter $H \in (0,1)$ is a continuous and centered Gaussian process $\{b^H(t), t \geq 0\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with covariance function*

$$\mathbb{E}\big[b^H(t) b^H(s)\big] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

The special case $H = 1/2$ correspond to the Brownian motion. For the fBm $b^H := \{b^H(t), t \geq 0\}$ we have the following properties:

1. $b^H(0) = 0$.
2. b^H has homogeneous increments.
3. b^H is a Gaussian process with $\mathbb{E}[b^H(t)] = 0$ and $\mathbb{E}[(b^H(t))^2] = t^{2H}$ for all $t \geq 0$ and for all $H \in (0,1)$.
4. b^H has continuous trajectories.
5. b^H is a self-similar process, i.e. $\mathrm{Law}(b^H(at), t \geq 0) = \mathrm{Law}(a^H b^H(t), t \geq 0)$.
6. b^H admits a version with a.s. α-Hölder continuous trajectories of order $\alpha < H$.
7. b^H is not a semimartingale for $H \neq 1/2$.

Let $(\cdot,\cdot)_n$ denote the Euclidean scalar product in $\mathbb{R}^n$. The definition of fBm can be generalized to $\mathbb{R}^n$ as follows.

Definition 4.1.2. *Let M be a positive symmetric $n \times n$ matrix. A continuous, zero-mean, $\mathbb{R}^n$-valued Gaussian process $\{b^H(t), t \in \mathbb{R}_+\}$ is said to be a fractional Brownian motion in $\mathbb{R}^n$ (or n-dimensional fBm) with Hurst parameter $H \in (0,1)$ if $\mathbb{E}[(v, b^H(t))_n] = 0$ for all $v \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$ and*

$$\mathbb{E}[(v_1, b^H(t))_n (v_2, b^H(s))_n] = (Mv_1, v_2)_n \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$$

for all $t, s \in \mathbb{R}_+$ and $v_1, v_2 \in \mathbb{R}^n$.

Example 4.1. Let $M = \mathrm{Id}$. Then B^H is called *standard fractional Brownian motion in* $\mathbb{R}^n$ since the components of the process are independent.

Example 4.2. If $H = 1/2$ and $M = \mathrm{Id}$ we recover the Brownian motion in $\mathbb{R}^n$ because the covariance is given by

$$\mathbb{E}[(v_1, b^{1/2}(t))_n (v_2, b^{1/2}(s))_n] = (v_1, v_2)_n (t \wedge s).$$

In a similar way as in $\mathbb{R}^n$, one can give the definition of fBm in a Hilbert space. Let H be a separable Hilbert space (possibly infinite dimensional) with scalar product $[\cdot,\cdot]_H$. Let $a \in H$ and Q be a positive, symmetric and trace-class operator on H. A Gaussian measure μ on $(H, \mathcal{B}(H))$ is a measure with mean a, covariance operator Q and Fourier transform given by

$$\hat{\mu}(h) = \exp\left\{ i(a,h) - \frac{1}{2}[Qh, h]_H \right\}, \quad h \in H.$$

A random variable X in H is Gaussian if its law is Gaussian. The following definition of a Hilbert space-valued fBm is verbally given by Duncan et al. [19].

Definition 4.1.3. *Let Q be a non-negative, nuclear, self-adjoint operator on H. A continuous, zero-mean H-valued Gaussian process $\{X^H(t), t \in \mathbb{R}_+\}$ is said to be a fractional Brownian motion with Hurst parameter $H \in (0,1)$ associated to the covariance operator Q, if $E[[h, X^H(t)]_H] = 0$ for all $h \in H$ and $t \in \mathbb{R}_+$ and*

$$\mathbb{E}\big[[h_1, X^H(t)]_H [h_2, X^H(s)]_H\big] = [Qh_1, h_2]_H \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}) \tag{4.1}$$

for all $t, s \in \mathbb{R}_+$ and $h_1, h_2 \in H$.

4.2 Wiener integral wrt real valued fBm

In this section we recall the theory of Wiener integrals for a real valued fractional Brownian motion $b^H := \{b^H(t), t \geq 0\}$.
In the first subsection we consider real valued integrands and we refer to [9, 52]. In the second subsection we focus on Wiener integrals with respect to real valued fBms for Hilbert space valued integrands. For more details see the works of Duncan and coauthors [22, 20, 57]. In the third subsection we consider the integral as a stochastic process and derive some properties of it. In the last subsection we give the detailed proof of stochastic Fubini theorem for fBm (with respect to Wiener integrals).

4.2.1 Wiener integrals for real valued integrands

In the following we introduce the Wiener integral of a real valued deterministic function ϕ with respect to a one dimensional fBm following [9], Chapter 2. For more details we refer to this book.

Let $b^H := \{b^H(t), t \geq 0\}$ be a real fractional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and denote its covariance function by

$$R_H(t,s) := \mathbb{E}[b^H(t)b^H(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

The covariance function has an integral representation given by

$$R_H(t,s) = \int_0^{t\wedge s} K_H(t,u)K_H(s,u)\,\mathrm{d}u, \tag{4.2}$$

where the kernel $K_H(t,s)$ has different expressions for $H < 1/2$ and $H > 1/2$. If $H > 1/2$, for $t > s$ we have

$$K_H(t,s) = c_H s^{1/2-H} \int_s^t (u-s)^{H-3/2} u^{H-1/2}\,\mathrm{d}u,$$

where $c_H = [H(2H-1)/\beta(2-2H, H-1/2)]^{1/2}$ and $\beta(\alpha,\gamma) := \Gamma(\alpha+\gamma)/(\Gamma(\alpha)\Gamma(\gamma))$ with $\Gamma(\cdot)$ indicating the Gamma function.
If $H < 1/2$, for $t > s$ we have

$$K_H(t,s) = d_H\Big[\Big(\frac{t}{s}\Big)^{H-1/2}(t-s)^{H-1/2} - \Big(H-\frac{1}{2}\Big)s^{1/2-H}\int_s^t (u-s)^{H-1/2}u^{H-3/2}\,\mathrm{d}u\Big],$$

where $d_H = [2H/((1-2H)\beta(1-2H, H+1/2))]^{-1/2}$.

The Wiener integral

Let $\mathcal{E}_T$ be the set of simple functions on $[0,T]$ with values in $\mathbb{R}$, that is

$$\mathcal{E}_T := \{\phi : [0,T] \to \mathbb{R} \text{ such that } \phi(t) = \sum_{i=0}^{L-1} \lambda_i \mathbb{1}_{[t_i,t_{i+1})}(t),$$
$$\text{with } \lambda_i \in \mathbb{R},\ 0 = t_0 < t_1 < \ldots < t_L = T\}.$$

Every function in $\mathcal{E}_T$ can be expressed as a sum of simple functions of the form $\mu \mathbb{1}_{[0,t)}$. This representation is not unique. We define a scalar product on $\mathcal{E}_T$ by

$$(4.3) \qquad \left\langle \sum_{i=0}^{L-1} \lambda_i \mathbb{1}_{[0,t_i)}, \sum_{j=0}^{M-1} \mu_j \mathbb{1}_{[0,s_j)} \right\rangle_{\mathcal{H}_T} := \sum_{i=0}^{L-1} \sum_{j=0}^{M-1} \lambda_i \mu_j R_H(t_i, s_j).$$

It is easy to show that it is well defined in the sense that it does not depend on the representation. Let denote by $\mathcal{H}_T$ the closure of $\mathcal{E}_T$ with respect to $\langle \cdot, \cdot \rangle_{\mathcal{H}_T}$. The Wiener integral of a simple function $\phi \in \mathcal{E}_T$ of the form

$$\phi(t) = \sum_{i=0}^{L-1} \lambda_i \mathbb{1}_{[t_i,t_{i+1})}(t)$$

is defined as

$$(4.4) \qquad \int_0^T \phi(t)\, \mathrm{d}b^H(t) := \sum_{i=0}^{L-1} \lambda_i \left(b^H(t_{i+1}) - b^H(t_i) \right)$$

and it is also sometimes denoted by $b_T^H(\phi)$. The norm of ϕ in $\mathcal{E}_T$ is given by

$$\begin{aligned}
\|\phi\|_{\mathcal{H}_T}^2 = \|\sum_{i=0}^{L-1} \lambda_i \mathbb{1}_{[t_i,t_{i+1})}\|_{\mathcal{H}_T}^2 &= \langle \sum_{i=0}^{L-1} \lambda_i \mathbb{1}_{[t_i,t_{i+1})}, \sum_{j=0}^{L-1} \lambda_j \mathbb{1}_{[t_j,t_{j+1})} \rangle_{\mathcal{H}_T} \\
&= \sum_{i=0}^{L-1} \sum_{j=0}^{L-1} \lambda_i \lambda_j \langle \mathbb{1}_{[0,t_{i+1})} - \mathbb{1}_{[0,t_i)}, \mathbb{1}_{[0,t_{j+1})} - \mathbb{1}_{[0,t_j)} \rangle_{\mathcal{H}_T} \\
&= \sum_{i=0}^{L-1} \sum_{j=0}^{L-1} \lambda_i \lambda_j [R_H(t_{i+1}, t_{j+1}) + R_H(t_i, t_j) - R_H(t_i, t_{j+1}) \\
&\qquad - R_H(t_{i+1}, t_j)].
\end{aligned}$$

The norm[1] of the stochastic integral $b_T^H(\phi)$ in $L^2_{\mathbb{P}}(\Omega;\mathbb{R})$ is

$$\begin{aligned}
\|b_T^H(\phi)\|^2_{L^2_{\mathbb{P}}(\Omega;\mathbb{R})} &= \mathbb{E}\left|\sum_{i=0}^{L-1}\lambda_i\left(b^H(t_{i+1})-b^H(t_i)\right)\right|^2\\
&= \mathbb{E}\left[\sum_{i=0}^{L-1}\sum_{j=0}^{L-1}\lambda_i\lambda_j\left(b^H(t_{i+1})-b^H(t_i)\right)\left(b^H(t_{j+1})-b^H(t_j)\right)\right]\\
&= \sum_{i=0}^{L-1}\sum_{j=0}^{L-1}\lambda_i\lambda_j[R_H(t_{i+1},t_{j+1})+R_H(t_i,t_j)-R_H(t_i,t_{j+1})+\\
&\qquad -R_H(t_{i+1},t_j)].
\end{aligned}$$

Thus, we obtain $\|b_T^H(\phi)\|^2_{L^2_{\mathbb{P}}(\Omega;\mathbb{R})} = \|\varphi\|^2_{\mathcal{H}_T}$. This means that the map $\phi \mapsto \int_0^T \phi\, \mathrm{d}b^H$ defines an isometry between $\mathcal{E}_T$ and $L^2_{\mathbb{P}}(\Omega;\mathbb{R})$ and so one can extend $b_T^H(\cdot)$ as an operator

$$\mathcal{H}_T \to L^2_{\mathbb{P}}(\Omega;\mathbb{R}), \quad \phi \mapsto b_T^H(\phi).$$

This extension defines the Wiener integral of ϕ with respect to b^H.

A characterization of $\mathcal{H}_T$

Consider the linear operator K_T^* for all $H \in (0,1)$ given by

$$(K_T^*\phi)(s) := \phi(s)K_H(T,s) + \int_s^T (\phi(u)-\phi(s))\frac{\partial K_H}{\partial u}(u,s)\,\mathrm{d}u$$

for $\phi \in \mathcal{E}_T$. We have

$$\begin{aligned}
&(K_T^*\mathbb{1}_{[0,t)})(s)\\
&\quad = \mathbb{1}_{[0,t)}(s)K_H(T,s) + \int_s^T (\mathbb{1}_{[0,t)}(u)-\mathbb{1}_{[0,t)}(s))\frac{\partial K_H}{\partial u}(u,s)\,\mathrm{d}u\\
&\quad = \mathbb{1}_{[0,t)}(s)\left(K_H(T,s) + \int_s^T (\mathbb{1}_{[0,t)}(u)-\mathbb{1}_{[0,T)}(u))\frac{\partial K_H}{\partial u}(u,s)\,\mathrm{d}u\right) \qquad (4.5)\\
&\quad = \mathbb{1}_{[0,t)}(s)\left(K_H(T,s) - \int_t^T \frac{\partial K_H}{\partial u}(u,s)\,\mathrm{d}u\right)\\
&\quad = \mathbb{1}_{[0,t)}(s)K_H(t,s).
\end{aligned}$$

[1] We denote by $L^0_{\mathbb{P}}(\Omega;\mathbb{R})$ the space of random variables on $(\Omega,\mathcal{F},\mathbb{P})$ with values in $\mathbb{R}$ and by $L^2_{\mathbb{P}}(\Omega;\mathbb{R})$ the space of square-integrable random variables with values in $\mathbb{R}$.

Therefore, it follows that

$$\begin{aligned}\|K_T^*\mathbb{1}_{[0,t)}\|^2_{L^2([0,T])} &= \|\mathbb{1}_{[0,t)}K_H(t,\cdot)\|^2_{L^2([0,T])}\\ &= \int_0^t K_H(t,s)^2\,\mathrm{d}s = R_H(t,t) < \infty,\end{aligned}$$

and so for each $\phi \in \mathcal{E}_T$, by linearity, K_T^* takes values in $L^2([0,T])$. For this reason it is possible to introduce the following scalar product on $\mathcal{E}_T$

$$\langle \phi, \psi \rangle_{\mathcal{E}_T} := \langle K_T^*\phi, K_T^*\psi \rangle_{L^2([0,T])}. \tag{4.6}$$

In this way the map

$$K_T^* : (\mathcal{E}_T, \langle \cdot, \cdot \rangle_{\mathcal{E}_T}) \to L^2([0,T])$$

is an isometry. Notice that the scalar product defined on $\mathcal{E}_T$ by (4.6) coincides with the one introduced before by (4.3): in fact we have

$$\begin{aligned}\langle \mathbb{1}_{[0,t)}, \mathbb{1}_{[0,s)} \rangle_{\mathcal{E}_T} &= \langle K_T^*\mathbb{1}_{[0,t)}, K_T^*\mathbb{1}_{[0,s)} \rangle_{L^2([0,T])}\\ &= \langle K_H(t,\cdot)\mathbb{1}_{[0,t)}, K_H(s,\cdot)\mathbb{1}_{[0,s)} \rangle_{L^2([0,T])}\\ &= \int_0^{t\wedge s} K_H(t,u)K_H(s,u)\mathrm{d}u\\ &= R_H(t,s) = \langle \mathbb{1}_{[0,t)}, \mathbb{1}_{[0,s)} \rangle_{\mathcal{H}_T}.\end{aligned}$$

Therefore, the closure of $\mathcal{E}_T$ with respect to $\langle \cdot, \cdot \rangle_{\mathcal{E}_T}$ coincides with $\mathcal{H}_T$ and since the operator K_T^* is an isometry, we can extend it to $\mathcal{H}_T$. The isometry property will still hold and combining it with the isometry between $\mathcal{H}_T$ and $L^2_{\mathbb{P}}(\Omega;\mathbb{R})$ we get

$$\mathbb{E}\left|\int_0^T \phi\,\mathrm{d}b^H\right|^2 = \|K_T^*\phi\|^2_{L^2([0,T])} \tag{4.7}$$

for all $\phi \in \mathcal{H}_T$.

4.2.2 Wiener integrals for Hilbert space valued integrands

The same construction of Wiener integrals can be generalised to Hilbert space valued integrands. In this case the integral will be an element of the Hilbert space. Here we recall it briefly, for more details see [19, 20, 57].

Let H be a separable Hilbert space with scalar product $[\cdot,\cdot]_H$ and b^H a real valued fBm on $(\Omega, \mathcal{F}, \mathbb{P})$. Consider the space of simple functions

$\Phi : [0,T] \to H$ and denote it by $\mathscr{E}_T$, that is

$$\mathscr{E}_T := \{\Phi : [0,T] \to H \text{ such that } \Phi(t) = \sum_{i=0}^{L-1} \alpha_i \mathbb{1}_{[t_i,t_{i+1})}(t), \\ \text{with } \alpha_i \in H,\, 0 = t_0 < t_1 < \ldots < t_L = T\}.$$

Then consider the inner product on $\mathscr{E}_T$ given by

$$\left\langle \sum_{i=0}^{L-1} \alpha_i \mathbb{1}_{[0,t_i)}, \sum_{j=0}^{M-1} \gamma_j \mathbb{1}_{[0,s_j)} \right\rangle_{\mathscr{H}_T} = \sum_{i=0}^{L-1}\sum_{j=0}^{M-1} [\alpha_i, \gamma_j]_H R_H(t_i, s_j). \tag{4.8}$$

Let us denote by $\mathscr{H}_T$ the closure of $\mathscr{E}_T$ with respect to $\langle \cdot, \cdot \rangle_{\mathscr{H}_T}$. The Wiener integral of $\Phi = \sum_{i=0}^{L-1} \alpha_i \mathbb{1}_{[t_i,t_{i+1})} \in \mathscr{E}_T$ with respect to b^H is defined as

$$\int_0^T \Phi \, \mathrm{d}b^H := \sum_{i=0}^{L-1} \alpha_i \left(b^H(t_{i+1}) - b^H(t_i) \right), \tag{4.9}$$

also denoted by $b^H(\Phi)$. The integral $b^H(\Phi)$ is a random variable which takes values in H and the map $\Phi \mapsto b^H(\Phi)$ can be extended to all $\Phi \in \mathscr{H}_T$ as in the real valued case because of the isometry between $L^2_{\mathbb{P}}(\Omega; \mathbb{R})$ and $\mathscr{E}_T$

$$\mathbb{E} \left\| \int_0^T \Phi \, \mathrm{d}b^H \right\|_H^2 = \|\Phi\|^2_{\mathscr{E}_T}. \tag{4.10}$$

This isometry is derived with similar computations as in the real valued case: in fact both the right and the left hand side equal the quantity

$$\sum_{i=0}^{L-1}\sum_{j=0}^{L-1} [\alpha_i, \alpha_j]_H (R_H(t_{i+1}, t_{j+1}) + R_H(t_i, t_j) - R_H(t_i, t_{j+1}) - R_H(t_{i+1}, t_j)).$$

Let us introduce the linear operator $\mathbb{K}_T^*$ on $\mathscr{E}_T$ defined by

$$(\mathbb{K}_T^* \Phi)(t) := \Phi(t) K_H(T,t) + \int_t^T (\Phi(s) - \Phi(t)) \frac{\partial K_H}{\partial s}(s,t) \, \mathrm{d}s$$

for $\Phi \in \mathscr{E}_T$. The integral appearing on the right-hand side is a Bochner integral. As for the real valued case, the scalar product on $\mathscr{E}_T$ given by

$$\langle \Phi, \Psi \rangle_{\mathscr{E}_T} := \langle \mathbb{K}_T^* \Phi, \mathbb{K}_T^* \Psi \rangle_{L^2([0,T];H)}$$

coincides with (4.8) because $\mathbb{K}_T^*\left(\alpha \mathbb{1}_{[0,t)}\right)(u) = \alpha K_H(t,u)\mathbb{1}_{[0,t)}(u)$ so that

$$\begin{aligned}
\langle \alpha \mathbb{1}_{[0,t)}, \gamma \mathbb{1}_{[0,s)} \rangle_{\mathscr{E}_T} &= \left\langle \mathbb{K}_T^*\left(\alpha \mathbb{1}_{[0,t)}\right), \mathbb{K}_T^*\left(\gamma \mathbb{1}_{[0,s)}\right) \right\rangle_{L^2([0,T];H)} \\
&= \left\langle \alpha K_H(t,\cdot)\mathbb{1}_{[0,t)}, \gamma K_H(s,\cdot)\mathbb{1}_{[0,s)} \right\rangle_{L^2([0,T];H)} \\
&= [\alpha,\gamma]_H \langle K_H(t,\cdot)\mathbb{1}_{[0,t)}, K_H(s,\cdot)\mathbb{1}_{[0,s)} \rangle_{L^2([0,T])} \\
&= [\alpha,\gamma]_H \int_0^{t\wedge s} K_H(t,u) K_H(s,u)\,\mathrm{d}u \\
&= [\alpha,\gamma]_H R_H(t,s) = \langle \alpha \mathbb{1}_{[0,t)}, \gamma \mathbb{1}_{[0,s)} \rangle_{\mathscr{H}_T}.
\end{aligned}$$

Moreover $\mathbb{K}_T^*$ is an isometry between $\mathscr{E}_T$ and $L^2([0,T];H)$. Therefore, it can be extended to $\mathscr{H}_T$. Combining it with (4.10) we get the following isometry between $L^2_{\mathbb{P}}(\Omega;\mathbb{R})$ and $L^2([0,T];H)$

$$\mathbb{E}\left\| \int_0^T \Phi\, d\beta^H \right\|_H^2 = \|\mathbb{K}_T^*\Phi\|^2_{L^2([0,T];H)} \tag{4.11}$$

for all $\Phi \in \mathscr{H}_T$.

4.2.3 The integral as a stochastic process

In this section we introduce the integral $\int_0^t \phi\,\mathrm{d}b^H$ for $t \in [0,T]$ and consider the process $\{\int_0^t \phi\,\mathrm{d}b^H, t \geq 0\}$. In order for it to be well defined we restrict ourself to the class of $\phi \in \mathcal{H}_T$ such that $\mathbb{1}_{[0,t)}\phi \in \mathcal{H}_T$ for all $t \in [0,T]$.

Definition 4.2.1. *We define*

$$\mathcal{S}_T := \{\phi \in \mathcal{H}_T \textit{ such that } \mathbb{1}_{[0,t)}\phi \in \mathcal{H}_T \textit{ for all } t \in [0,T]\}.$$

Definition 4.2.2. *Let b^H be a fBm in $\mathbb{R}$ and $\phi \in \mathcal{S}_T$. We have*

$$b_T^H(\mathbb{1}_{[0,t)}\phi) := \int_0^T \mathbb{1}_{[0,t)}(s)\phi(s)\,\mathrm{d}b^H(s)$$

for all $t \in [0,T]$.

The integral is well defined because for each $\phi \in \mathcal{S}_T$ then $\mathbb{1}_{[0,t)}\phi \in \mathcal{H}_T$. To show that in this way one defines an integral that is consistent with the integral $b_t^H(\phi) = \int_0^t \phi(s)\,db^H(s)$, we have to introduce some other objects. Let $\mathcal{E}_t$ and $\mathcal{H}_t$ be the analogous spaces to $\mathcal{E}_T$ and $\mathcal{H}_T$ but defined for any fixed $t \in [0,T]$. The operator K_t^* will be the analogous of K_T^*, defined on $\mathcal{H}_t$ with values in $L^2(0,t)$. Clearly it is an isometry.

Proposition 4.2.3. *Let $\phi \in \mathcal{H}_t$. Then for all $s \in [0,T]$ we have*

$$\mathbb{1}_{[0,t)}(s)(K_t^*\phi)(s) = K_T^*(\mathbb{1}_{[0,t)}\phi)(s).$$

Proof. We have

$$\begin{aligned}(K_t^*\phi)(s) =& K_H(t,s)\phi(s) + \int_s^t (\phi(r)-\phi(s))\frac{\partial K_H}{\partial r}(r,s)\,\mathrm{d}r \\ =& K_H(t,s)\phi(s) + \int_s^t \phi(r)\frac{\partial K_H}{\partial r}(r,s)\,\mathrm{d}r \\ & -\phi(s)[K_H(t,s)-K_H(s,s)] \\ =& \int_s^t \phi(r)\frac{\partial K_H}{\partial r}(r,s)\,\mathrm{d}r + \phi(s)K_H(s,s).\end{aligned}$$

It follows that

$$\begin{aligned}\mathbb{1}_{[0,t)}(s)(K_t^*\phi)(s) =& \mathbb{1}_{[0,t)}(s)\int_s^t \phi(r)\frac{\partial K_H}{\partial r}(r,s)\,\mathrm{d}r + \mathbb{1}_{[0,t)}(s)\phi(s)K_H(s,s) \\ =& \int_s^t \mathbb{1}_{[0,t)}(r)\phi(r)\frac{\partial K_H}{\partial r}(r,s)\,\mathrm{d}r + \mathbb{1}_{[0,t)}(s)\phi(s)K_H(s,s) \\ & - \mathbb{1}_{[0,t)}(s)\phi(s)K_H(T,s) + \mathbb{1}_{[0,t)}(s)\phi(s)K_H(T,s) \\ =& \int_s^T \mathbb{1}_{[0,t)}(r)\phi(r)\frac{\partial K_H}{\partial r}(r,s)\,\mathrm{d}r \\ & - \int_s^T \mathbb{1}_{[0,t)}(s)\phi(s)\frac{\partial K_H}{\partial r}(r,s)\,\mathrm{d}r + \mathbb{1}_{[0,t)}(s)\phi(s)K_H(T,s) \\ =& \int_s^T (\mathbb{1}_{[0,t)}(r)\phi(r) - \mathbb{1}_{[0,t)}(s)\phi(s))\frac{\partial K_H}{\partial r}(r,s)\,\mathrm{d}r \\ & + \mathbb{1}_{[0,t)}(s)\phi(s)K_H(T,s) \\ =& K_T^*(\mathbb{1}_{[0,t)}\phi)(s),\end{aligned}$$

which completes the proof. □

Remark 4.1. The integral $b_t^H(\phi)$ defined directly using K_t^* and $\mathcal{H}_t$ coincides with $b_T^H(\mathbb{1}_{[0,t)}\phi)$, as expected.

Proof. Let $\phi \in \mathcal{S}_T$. By Proposition 4.2.3 we have

$$\begin{aligned}\mathbb{E}\left|\int_0^T \mathbb{1}_{[0,t)}(s)\phi(s)\,\mathrm{d}b^H(s)\right|^2 &= \|K_T^*(\mathbb{1}_{[0,t)}\phi)\|_{L^2(0,T)} \\ &= \|\mathbb{1}_{[0,t)}K_t^*\phi\|_{L^2(0,T)} \\ &= \|K_t^*\phi\|_{L^2(0,t)} \\ &= \mathbb{E}\left|\int_0^t \phi(s)\,\mathrm{d}b^H(s)\right|^2.\end{aligned}$$

This tells us that $b_T^H(\mathbb{1}_{[0,t)}\phi) = b_t^H(\phi)$ in $L^2_{\mathbb{P}}(\Omega, \mathbb{R})$. Notice that from the last computations we also get that

$$\|\mathbb{1}_{[0,t)}\phi\|_{\mathcal{H}_T} = \|\phi\|_{\mathcal{H}_t},$$

which ensures that $\|\phi\|_{\mathcal{H}_t} < \infty$ if $\mathbb{1}_{[0,t)}\phi \in \mathcal{H}_T$. In other words, the condition $\phi \in \mathcal{S}_T$ implies that the integral $\int_0^t \phi \, \mathrm{d}b^H$ is well defined as isometry between $\mathcal{H}_t$ and $L^2_{\mathbb{P}}(\Omega; \mathbb{R})$ for all $t \in [0, T]$. □

In what follows, for all $\phi \in \mathcal{S}_T$, we consider the integral as a process, $\{b_t^H(\phi), t \in [0, T]\}$. We prove the existence of a continuous version of the integral process under some suitable conditions. Similar results were proven in [44] for a different kind of integral with respect to fractional Brownian motion.

Lemma 4.2.4. *Let $0 \le t_1 \le t_2 \le T$. It holds*

$$\|K_T^* \mathbb{1}_{[t_1,t_2)}\|_{L^2(0,T)} = |t_2 - t_1|^H.$$

Proof. Recall that the operator K_T^* is linear. It holds

$$\begin{aligned}
\|K_T^* \mathbb{1}_{[t_1,t_2)}\|_{L^2}^2 &= \|K_T^* \mathbb{1}_{[0,t_2)} - K_T^* \mathbb{1}_{[0,t_1)}\|_{L^2(0,T)}^2 \\
&= \int_0^T |K_T^* \mathbb{1}_{[0,t_2)}(t) - K_T^* \mathbb{1}_{[0,t_1)}(t)|^2 \, \mathrm{d}t \\
&= \int_0^T \Big(\big(K_T^* \mathbb{1}_{[0,t_2)}(t)\big)^2 + \big(K_T^* \mathbb{1}_{[0,t_1)}(t)\big)^2 \\
&\qquad - 2K_T^* \mathbb{1}_{[0,t_1)}(t) K_T^* \mathbb{1}_{[0,t_2)}(t) \Big) \, \mathrm{d}t,
\end{aligned}$$

and by (4.5) we get

$$\begin{aligned}
&= \int_0^T \big(\mathbb{1}_{[0,t_2)}(t) K_H(t_2, t)\big)^2 \, \mathrm{d}t + \int_0^T \big(\mathbb{1}_{[0,t_1)}(t) K_H(t_1, t)\big)^2 \, \mathrm{d}t \\
&\qquad - 2 \int_0^T \mathbb{1}_{[0,t_1)}(t) \mathbb{1}_{[0,t_2)}(t) K_H(t_1, t) K_H(t_2, t) \, \mathrm{d}t \\
&= \int_0^{t_2} (K_H(t_2, t))^2 \, \mathrm{d}t + \int_0^{t_1} (K_H(t_1, t))^2 \, \mathrm{d}t \\
&\qquad - 2 \int_0^{t_1 \wedge t_2} K_H(t_1, t) K_H(t_2, t) \, \mathrm{d}t.
\end{aligned}$$

Using the integral representation (4.2) of $R_H(t, s)$ we finally have

$$\begin{aligned}
\|K_T^* \mathbb{1}_{[t_1,t_2)}\|_{L^2}^2 &= R_H(t_2, t_2) + R_H(t_1, t_1) - 2R_H(t_1, t_2) \\
&= t_2^{2H} + t_1^{2H} - (t_1^{2H} + t_2^{2H} - |t_2 - t_1|^{2H}) = |t_2 - t_1|^{2H}. \quad \square
\end{aligned}$$

Proposition 4.2.5. *Let $H \in (1/2, 1)$ and $\phi : [0,T] \to \mathbb{R}$ be such that $\phi \in L^\infty(0,T)$. Then $\phi \in \mathcal{S}_T$ and there exists a version of the integral process $\{b^H_t(\phi), t \in [0,T]\}$ with α-Hölder continuous paths for any $\alpha \in [0, H - \frac{1}{2})$.*

Proof. First of all observe that $\phi \in L^\infty([0,T]) \subset L^2([0,T]) \subset \mathcal{H}_T$ (for the last inclusion see [9], Proposition 2.1.13) and the same holds for $\mathbb{1}_{[0,t)}\phi$ for each $0 \le t \le T$. Thus $\phi \in \mathcal{S}_T$. We have

$$\begin{aligned}
\mathbb{E}\left[\left|b^H_{t_2}(\phi) - b^H_{t_1}(\phi)\right|^2\right] &= \mathbb{E}\left[\left|\int_0^{t_2} \phi(s)\,\mathrm{d}b^H(s) - \int_0^{t_1} \phi(s)\,\mathrm{d}b^H(s)\right|^2\right] \\
&= \mathbb{E}\left[\left(\int_{t_1}^{t_2} \phi(s)\,\mathrm{d}b^H(s)\right)^2\right] \\
&= \mathbb{E}\left[\left(\int_0^T \mathbb{1}_{[t_1,t_2)}(s)\phi(s)\,\mathrm{d}b^H(s)\right)^2\right] \\
&= \|K^*_T(\mathbb{1}_{[t_1,t_2)}\phi)\|^2_{L^2(0,T)}.
\end{aligned}$$

Recall that for $H > 1/2$ the operator K^*_T reduces to

$$K^*_T(\psi)(t) = \int_t^T \psi(s)\frac{\partial K_H}{\partial s}(s,t)\,\mathrm{d}s$$

with $\frac{\partial K_H}{\partial s}(s,t) = c_H\left(\frac{s}{t}\right)^{H-1/2}(s-t)^{H-3/2}$. This function is always positive when $t < s < T$ and so using the fact that ϕ is bounded we have

$$\begin{aligned}
\left|K^*_T(\mathbb{1}_{[t_1,t_2)}\phi)(t)\right| &\le \int_t^T \left|\mathbb{1}_{[t_1,t_2)}(s)\phi(s)\frac{\partial K_H}{\partial s}(s,t)\right|\,\mathrm{d}s \\
&\le \|\phi\|_\infty \int_t^T \mathbb{1}_{[t_1,t_2)}(s)\left|\frac{\partial K_H}{\partial s}(s,t)\right|\,\mathrm{d}s \\
&= \|\phi\|_\infty K^*_T(\mathbb{1}_{[t_1,t_2)})(t).
\end{aligned}$$

By Lemma 4.2.4 we get

$$\begin{aligned}
\mathbb{E}\left[\left|b^H_{t_2}(\phi) - b^H_{t_1}(\phi)\right|^2\right] &= \|K^*_T(\mathbb{1}_{[t_1,t_2)}\phi)\|^2_{L^2(0,T)} \\
&= \int_0^T \left|\|\phi\|_\infty K^*_T(\mathbb{1}_{[t_1,t_2)})(t)\right|^2\mathrm{d}t \\
&\le \|\phi\|^2_\infty \|K^*_T(\mathbb{1}_{[t_1,t_2)})\|^2_{L^2(0,T)} \\
&= \|\phi\|^2_\infty |t_2 - t_1|^{2H}.
\end{aligned}$$

Since $2H > 1$, Kolmogorov's continuity theorem implies that there exists a version with α-Hölder continuous paths for any $\alpha \in [0, H - \frac{1}{2})$. □

4.2.4 Stochastic Fubini theorem for fBm

Here we consider a stochastic Fubini theorem for fBm with Wiener integrals. The result is known in the literature and some more general version of this theorem (for divergence-type integrals) can be found in the literature for instance in [9] or [52]. Here we give a detailed proof of an easier case, namely the case of deterministic integrands, which holds both for $H \in (0, 1/2)$ and for $H \in (1/2, 1)$.

Theorem 4.2.6. *Let $\{b^H(t), t \in [0,T]\}$ be a real fBm and let $f : [0,T] \times [0,T] \to \mathbb{R}$ be element of $\mathcal{S}_T$. Then we have*

$$\int_0^t \int_0^s f(s,r)\, \mathrm{d}b^H(r)\, \mathrm{d}s = \int_0^t \int_r^t f(s,r)\, \mathrm{d}s\, \mathrm{d}b^H(r). \tag{4.12}$$

Proof. Recall that for each fBm $\{b^H(t), t \geq 0\}$ there exists a Bm $\{b(t), t \geq 0\}$ such that

$$\int_0^s \psi(r)\, \mathrm{d}b^H(r) = \int_0^s (K_s^* \psi)(r)\, \mathrm{d}b(r),$$

where K_s^* acts on $\psi \in \mathcal{H}_s$ in the following way

$$(K_s^* \psi)(r) = K_H(s,r)\psi(r) + \int_r^s (\psi(u) - \psi(r)) \frac{\partial K_H}{\partial u}(u,r)\, \mathrm{d}u$$

for (almost) all $r \in [0,s]$. Using this integral representation we can rewrite the inner integral of the LHS of (4.12) as follows

$$\begin{aligned}
\int_0^s f(s,r)\, \mathrm{d}b^H(r) &= \int_0^s \big(K_s^* f(s,\cdot)\big)(r)\, \mathrm{d}b(r) \\
&= \int_0^s \Big(K_H(s,r) f(s,r) \\
&\quad + \int_r^s \big(f(s,u) - f(s,r)\big) \frac{\partial K_H}{\partial u}(u,r)\, \mathrm{d}u\Big)\, \mathrm{d}b(r) \\
&= \int_0^s K_H(s,r) f(s,r)\, \mathrm{d}b(r) \\
&\quad + \int_0^s \int_r^s \big(f(s,u) - f(s,r)\big) \frac{\partial K_H}{\partial u}(u,r)\, \mathrm{d}u\, \mathrm{d}b(r).
\end{aligned}$$

The LHS of (4.12) becomes

$$\begin{aligned}
&\int_0^t \int_0^s f(s,r)\, \mathrm{d}b^H(r)\, \mathrm{d}s = \int_0^t \int_0^s K_H(s,r) f(s,r)\, \mathrm{d}b(r)\, \mathrm{d}s \\
&+ \int_0^t \int_0^s \int_r^s \big(f(s,u) - f(s,r)\big) \frac{\partial K_H}{\partial u}(u,r)\, \mathrm{d}u\, \mathrm{d}b(r)\, \mathrm{d}s =: (A) + (B).
\end{aligned}$$

For the summand (A) we apply stochastic Fubini theorem for Bm and we get

$$(A) = \int_0^t \int_r^t K_H(s,r) f(s,r) \, \mathrm{d}s \, \mathrm{d}b(r).$$

For the summand (B) we first apply stochastic Fubini theorem for Bm and then classical Fubini theorem. We get

$$\begin{aligned}(B) &= \int_0^t \int_r^t \int_r^s \big(f(s,u) - f(s,r)\big) \frac{\partial K_H}{\partial u}(u,r) \, \mathrm{d}u \, \mathrm{d}s \, \mathrm{d}b(r) \\ &= \int_0^t \int_r^t \int_r^s \big(f(s,u) - f(s,r)\big) \frac{\partial K_H}{\partial u}(u,r) \, \mathrm{d}s \, \mathrm{d}u \, \mathrm{d}b(r).\end{aligned}$$

On the other side, the RHS of (4.12) can be written as

$$\int_0^t \int_r^t f(s,r) \, \mathrm{d}s \, \mathrm{d}b^H(r) = \int_0^t K_t^* \Big(\int_\cdot^t f(s,\cdot) \, \mathrm{d}s \Big)(r) \, \mathrm{d}b(r), \tag{4.13}$$

where the integrands is

$$\begin{aligned}K_t^* \Big(\int_\cdot^t f(s,\cdot) \, \mathrm{d}s \Big)(r) =& K_H(t,r) \int_r^t f(s,r) \, \mathrm{d}s \\ &+ \int_r^t \Big(\int_u^t f(s,u) \, \mathrm{d}s - \int_r^t f(s,r) \, \mathrm{d}s \Big) \frac{\partial K_H}{\partial u}(u,r) \, \mathrm{d}u.\end{aligned} \tag{4.14}$$

Observe that $r < u < t$ and we can write

$$\begin{aligned}\int_u^t f(s,u) \, \mathrm{d}s - \int_r^t f(s,r) \, \mathrm{d}s &= \int_u^t f(s,u) \, \mathrm{d}s \pm \int_r^u f(s,r) \, \mathrm{d}s - \int_r^t f(s,r) \, \mathrm{d}s \\ &= \int_u^t \big(f(s,u) - f(s,r)\big) \, \mathrm{d}s - \int_r^u f(s,r) \, \mathrm{d}s.\end{aligned}$$

Therefore (4.14) gives us

$$\begin{aligned}K_t^* \Big(\int_\cdot^t f(s,\cdot) \, \mathrm{d}s \Big)(r) =& K_H(t,r) \int_r^t f(s,r) \, \mathrm{d}s \\ &+ \int_r^t \int_u^t \big(f(s,u) - f(s,r)\big) \, \mathrm{d}s \frac{\partial K_H}{\partial u}(u,r) \, \mathrm{d}u \\ &- \int_r^t \int_r^u f(s,r) \, \mathrm{d}s \frac{\partial K_H}{\partial u}(u,r) \, \mathrm{d}u \\ =& K_H(t,r) \int_r^t f(s,r) \, \mathrm{d}s \\ &+ \int_r^t \int_u^t \big(f(s,u) - f(s,r)\big) \, \mathrm{d}s \frac{\partial K_H}{\partial u}(u,r) \, \mathrm{d}u \\ &- \int_r^t \int_s^t f(s,r) \frac{\partial K_H}{\partial u}(u,r) \, \mathrm{d}u \, \mathrm{d}s,\end{aligned}$$

where in the last summand we applied again Fubini theorem. We get

$$\begin{aligned}
K_t^*\Big(\int_\cdot^t f(s,\cdot)\,\mathrm{d}s\Big)(r) =& K_H(t,r)\int_r^t f(s,r)\,\mathrm{d}s \\
&+\int_r^t\int_u^t \big(f(s,u)-f(s,r)\big)\,\mathrm{d}s\frac{\partial K_H}{\partial u}(u,r)\,\mathrm{d}u \\
&-\int_r^t f(s,r)\big(K_H(t,r)-K_H(s,r)\big)\,\mathrm{d}s \\
=&\int_r^t\int_u^t \big(f(s,u)-f(s,r)\big)\,\mathrm{d}s\frac{\partial K_H}{\partial u}(u,r)\,\mathrm{d}u \\
&+\int_r^t K_H(s,r)f(s,r)\,\mathrm{d}s.
\end{aligned}$$

Using (4.13) we finally get that the RHS of (4.12) equals $(B)+(A)$. □

Chapter 5

Fractional Brownian motion in Banach spaces

In this chapter we introduce the concept of fractional Brownian motion in a separable Banach space as a cylindrical process. The notion of *cylindrical random variable* is therefore a key stone. It was introduced by Gel'fand, see the monograph by Gel'fand and Wilenkin [24]. A similar object as a cylindrical random variable appears under the name weak distribution in the paper of Segal [68]. See also [38]. Moreover cylindrical measures and cylindrical random variables were extensively considered by Schwartz and his collaborators, see among others in [65, 66, 67].
In Section 5.1 we give a brief introduction to probability theory in Banach spaces following the notes [59]. In particular, we focus on the notion of cylindrical measure and cylindrical random variable. We also recall the definition of the reproducing kernel Hilbert space of a covariance operator and some of its properties.
In Section 5.2 we define the cylindrical fBm in a separable Banach space U and we prove some of its properties. Afterwards we define a U-valued fBm as a *classical* process in a Banach space. We then relate them between each other.
Finally, in Section 5.3 we compare the object just introduced with the existing literature. We show that, in the Hilbert space case, the cylindrical fBm we introduced is equivalent to the one that we find in the literature. Moreover the space-time fractional noise is considered and some examples are provided.

5.1 Preliminaries

In this section we introduce cylindrical measures and cylindrical processes on Banach spaces. A very important class of cylindrical processes is the class of Gaussian processes. Related to it, the notion of reproducing kernel Hilbert space is explained. We will follow the notes of Riedle, [59], for more details we refer to it and to the references therein.

5.1.1 Cylindrical measures and cylindrical processes

Throughout this thesis, U indicates a separable Banach space over $\mathbb{R}$ with norm $\|\cdot\|_U$. The (topological) dual space U^* of U is the collection of all linear and continuous functionals on U. We denote by U' the algebraic dual of U, namely all linear functionals on U. For any $u^* \in U^*$ (or $u^* \in U'$) and $u \in U$ we indicate the dual pairing $u^*(u)$ by $\langle u, u^* \rangle = \langle u^*, u \rangle$.
The Borel σ-algebra on U is denoted by $\mathcal{B}(U)$. Let Γ be a subset of U^*, $n \in \mathbb{N}$, $u_1^*, \dots, u_n^* \in \Gamma$ and $B \in \mathcal{B}(\mathbb{R}^n)$. A set of the form

$$Z(u_1^*, \dots, u_n^*, B) := \{u \in U : (\langle u, u_1^* \rangle, \dots, \langle u, u_n^* \rangle) \in B\},$$

is called a cylindrical set. We denote by $\mathcal{Z}(U, \Gamma)$ the set of all cylindrical sets in U for a given Γ. It turns out this is an *algebra*. Let $\mathcal{C}(U, \Gamma)$ be the generated *σ-algebra*. When $\Gamma = U^*$ the notation is $\mathcal{Z}(U)$ and $\mathcal{C}(U)$ respectively. If U is separable then both the Borel σ-algebra $\mathcal{B}(U)$ and the cylindrical σ-algebra $\mathcal{C}(U)$ coincide.
A function $\mu : \mathcal{Z}(U) \to [0, \infty]$ is called a *cylindrical measure* on $\mathcal{Z}(U)$ if for each finite subset $\Gamma \subseteq U^*$ the restriction of μ to the σ-algebra $\mathcal{C}(U, \Gamma)$ is a measure. It is called *finite* if $\mu(\mathcal{Z}(U))$ is finite and *cylindrical probability measure* if $\mu(\mathcal{Z}(U)) = 1$. A cylindrical probability measure μ on $\mathcal{Z}(U)$ is called *weakly Gaussian* if the image measure $\mu \circ (u^*)^{-1}$ is a Gaussian measure on $\mathcal{B}(\mathbb{R})$ for all $u^* \in U^*$. For weakly Gaussian cylindrical measures we have the following result (see [76], Section VI.3.1).

Theorem 5.1.1. *Let μ be a weakly Gaussian cylindrical measure on $\mathcal{C}(U)$. Then its characteristic function $\varphi_\mu : U^* \to \mathbb{C}$ is of the form*

$$\varphi_\mu(u^*) = \exp\{im(u^*) - \frac{1}{2}s(u^*)\} \tag{5.1}$$

where $m : U^* \to \mathbb{R}$ and $s : U^* \to \mathbb{R}_+$ are given by

$$m(u^*) = \int_U \langle u, u^* \rangle \mu(\mathrm{d}u), \quad s(u^*) = \int_U \langle u, u^* \rangle^2 \mu(\mathrm{d}u) - m(u^*)^2.$$

Conversely, if μ is a cylindrical measure with characteristic function of the form (5.1) *for a linear functional $m : U^* \to \mathbb{R}$ and a quadratic form $s : U^* \to \mathbb{R}_+$, then μ is a weakly Gaussian cylindrical measure.*

For a weakly Gaussian cylindrical measure μ one also defines a covariance operator $Q : U^* \to (U^*)'$ given by

$$(Qu^*)\, v^* = \int_U \langle u, u^* \rangle \langle u, v^* \rangle \mu(\mathrm{d}u) - m(u^*) m(v^*)$$

These integrals exist because μ is a Gaussian measure on the σ-algebra generated by u^* and v^*. In general, the covariance operator takes value in the algebraic dual of U^*, that is the map is non necessarily continuous.
In the special case of Gaussian measures, for each $u^* \in U^*$ the operator Qu^* is continuous and moreover can be seen as an element of $U \subset U^{**}$ (see [76], Thm III.2.1). In the framework of cylindrical measures this is not always the case and therefore we introduce a stronger concept of Gaussian cylindrical measures.

Definition 5.1.2. *A centered weakly Gaussian cylindrical measure μ on $\mathcal{Z}(U)$ is called* strongly Gaussian *if the covariance operator $Q : U^* \to (U^*)'$ is U-valued.*

Definition 5.1.3. *A cylindrical random variable Y in U is a linear map*

$$Y : U^* \to L^0_{\mathbb{P}}(\Omega; \mathbb{R}).$$

A cylindrical process X in U is a family $\{X(t), t \geq 0\}$ of cylindrical random variables in U.
A cylindrical process X is said to be adapted to a given filtration $\{\mathcal{F}_t\}_{t \geq 0}$ if $X(t)u^*$ is $\mathcal{F}_t$-measurable for all $u^* \in U^*$ and all $t \geq 0$.

Clearly every classical random variable Y in a Banach space U can be seen as a cylindrical random variable as follows: define

$$X : U^* \to L^0_{\mathbb{P}}(\Omega; \mathbb{R}), \quad X(u^*) := \langle Y, u^* \rangle$$

for all $u^* \in U^*$. The map X is a cylindrical random variable thanks to the linearity of the dual pairing.
With the help of cylindrical random variables we can give a simple example of cylindrical probability measure.

Example 5.1. Let X be a cylindrical random variable. Given any $n \in \mathbb{N}$, any Borel set $F \in \mathcal{B}(\mathbb{R}^n)$ and any $u_1^*, \ldots, u_n^* \in U^*$ we define $\mu : \mathcal{Z}(U) \to [0,1]$ by

$$\mu\left(\{u \in U : (\langle u, u_1^*\rangle, \ldots, \langle u, u_n^*\rangle) \in F\}\right) := \mathbb{P}\left((Xu_1^*, \ldots, Xu_n^*) \in F\right)$$

which is clearly a cylindrical probability measure.

We can introduce the characteristic function associated to a cylindrical process and a cylindrical probability measure.

Definition 5.1.4. *The characteristic function of a cylindrical probability measure μ on $\mathcal{Z}(U)$ is given by*

$$\varphi_\mu : U^* \to \mathbb{C}, \quad \varphi_\mu(u^*) = \int_U \exp\{i\langle u, u^*\rangle\}\mu(\mathrm{d}u).$$

The characteristic function of a cylindrical random variable X in U is given by

$$\varphi_X : U^* \to \mathbb{C}, \quad \varphi_X(u^*) = \mathbb{E}\left[\exp\{iXu^*\}\right].$$

The concepts of cylindrical measure and cylindrical random variable match perfectly. Because the characteristic function of a cylindrical random variable is positive-definite and continuous on finite subspaces, there exists a cylindrical measure μ with the same characteristic function. We call μ the cylindrical distribution of X. Vice versa, for every cylindrical measure μ on $\mathcal{C}(U)$ there exists a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and a cylindrical random variable $X : U^* \to L^0_{\mathbb{P}}(\Omega; \mathbb{R})$ such that μ is the cylindrical distribution of X, see [76], VI.3.2.
In Hilbert spaces we can define a special Gaussian cylindrical measure as follows.

Definition 5.1.5. *Let H be a separable Hilbert space. The measure γ defined by its characteristic function $\varphi_\gamma : H \to \mathbb{C}$ by*

$$\varphi_\gamma(h) = \exp\{-\frac{1}{2}\|h\|_H^2\}$$

is called standard Gaussian cylindrical measure on H.

Noteworthy is the fact that, in contrast to measures on infinite dimensional spaces, there is an analogue of Bochner's theorem for cylindrical measures:

Theorem 5.1.6. *A function $\varphi : U^* \to \mathbb{C}$ is a characteristic function of a cylindrical measure on $\mathcal{Z}(U)$ if and only if*

- $\varphi(0) = 0$
- *φ is positive-definite*
- *the restriction of φ to every finite dimensional subset $\Gamma \subseteq U^*$ is continuous with respect to the norm topology.*

Using the characteristic function it is easy to see that for a centered weakly Gaussian cylindrical random variable X with covariance operator Q, for every $v^*, u^* \in U^*$ we have

$$\mathbb{E}|Xu^* Xv^*| = \langle Qu^*, v^* \rangle.$$

Remark 5.1. If $X : U^* \to L^0_{\mathbb{P}}(\Omega, \mathbb{R})$ is a cylindrical random variable with $E[|Xu^*|^2] < \infty$ and covariance operator $Q : U^* \to U^{*\prime}$ then the following are equivalent

(a) $Q : U^* \to U^{**}$

(b) X is continuous as a mapping from U^* to $L^2_{\mathbb{P}}(\Omega, \mathbb{R})$

Proof. $(a) \Rightarrow (b)$. By Lemma 1.1 in [76], Chapter III there exists a Hilbert space $(H, [\cdot,\cdot]_H)$ (which is nothing but the RKHS of Q, see Section 5.1.2) and a continuous linear operator $F : U^* \to H$ such that $Q = F^*F$. Let $(u_n^*)_{n\in\mathbb{N}} \subset U^*$ be such that $\|u_n^*\| \to 0$. We have

$$\begin{aligned}\mathbb{E}|Xu_n^*|^2 &= \langle Qu_n^*, u_n^* \rangle = \langle F^*Fu_n^*, u_n^* \rangle \\ &= [Fu_n^*, Fu_n^*]_H = \|Fu_n^*\|_H^2.\end{aligned}$$

The continuity of F implies now the continuity of X.
$(b) \Rightarrow (a)$. Let $v^* \in U^*$ be arbitrary and consider a sequence $(v_n^*)_{n\in\mathbb{N}} \subset U^*$ such that $\|u_n^*\| \to 0$, that is such that $\langle v, v_n^* \rangle \to 0$ for all $v \in V$. We show that Q is actually U^{**}-valued because Qv^* is continuous for any v^*. In fact we have

$$|(Qv^*)u_n^*| = \mathbb{E}|Xv^*, Xu_n^*| \leq \mathbb{E}|Xv^*|^2 \mathbb{E}|Xu_n^*|^2$$

and the latter converges to zero by assumption (b). □

Remark 5.2. If $X : U^* \to L^0_{\mathbb{P}}(\Omega;\mathbb{R})$ is a cylindrical random variable with $E[|Xu^*|^2] < \infty$ and covariance operator $Q : U^* \to U^{*\prime}$ then the following are equivalent

(a) $Q : U^* \to U$

(b) X is continuous as a mapping from U^* to $L^2_{\mathbb{P}}(\Omega;\mathbb{R})$ in the weak* topology.

Proof. Let us consider $u^* \in U^*$ and a sequence $(u_n^*)_{n\in\mathbb{N}} \subset U^*$ such that $v_n^* \overset{*}{\rightharpoonup} 0$. Then by [50], Corollary 2.7.10 we have that $Qu^* \in U$ if and only if $\langle Qu^*, v_n^*\rangle \to 0$ and since

$$\langle Qu^*, v_n^*\rangle = \mathbb{E}|Xu^* Xv_n^*| = [Xu^*, Xv_n^*]_{L^2_{\mathbb{P}}(\Omega,\mathbb{R})},$$

we have the equivalence. □

5.1.2 The reproducing kernel Hilbert space

In this section we describe how to factorize a positive and symmetric operator $Q : U^* \to U$ through a Hilbert space (the so-called reproducing kernel Hilbert space). In the special case when U is a Hilbert space, then the operator can be factorized trough the Hilbert space U itself by its square root $Q = Q^{1/2}Q^{1/2}$. This construction is useful in order to characterize covariance operators of Gaussian measures on $\mathcal{B}(U)$. For Hilbert spaces it is well known that Q must be nuclear or equivalently $Q^{1/2}$ must be Hilbert-Schmidt; for Banach spaces an analogous result requires the notion of γ-radonifying operator which we will recall. For more details see [76], Section III.1.

Consider any positive symmetric operator $Q : U^* \to U$. For any $u^*, v^* \in U^*$ let us introduce the following bilinear form on the range of Q

$$[Qu^*, Qv^*]_{H_Q} := \langle Qu^*, v^*\rangle.$$

It is easy to see that this bilinear form $[\cdot,\cdot]_{H_Q}$ is an inner product on $Q(U^*)$ therefore the range of Q is a pre-Hilbert space.

Definition 5.1.7. *Let Q be a symmetric positive operator form U^* to U. The Hilbert space obtained by the completion of $Q(U^*)$ with respect to the inner product $[\cdot,\cdot]_{H_Q}$ is denoted by $(H_Q, [\cdot,\cdot]_{H_Q})$ and is called the reproducing kernel Hilbert space associated to Q.*

The RKHS and its embedding into U have several properties that we recall in what follows (for the proofs see [60], Section 4).

(a) the inclusion mapping from the range of Q into U is continuous with respect to the inner product $[\cdot,\cdot]_{H_Q}$. Thus it extends to a bounded linear operator i_Q from H_Q to U;

(b) the operator Q enjoys the decomposition $Q = i_Q i_Q^*$;

(c) the range of i_Q^* is dense in H_Q;

(d) the inclusion mapping i_Q is injective;

(e) if U is separable then H_Q is separable.

As already mentioned, we are looking for an analogous of the concept of Hilbert-Schmidt operators, and this is actually very much related to the question of the characterization of covariance operators for Gaussian measures.
Recall that a classical result by E. Mourier (see [76], Theorem IV.2.4) about Gaussian measures on Hilbert spaces tells us that $Q : H \to H$ is the covariance operator of a Gaussian measure on $\mathcal{B}(H)$ if and only if it is positive, symmetric and nuclear. In this case one can decompose Q through its square root and the condition "Q is nuclear" is then equivalent to "$Q^{1/2}$ is Hilbert-Schmidt".
The analogous result for Banach spaces is still an open problem. For separable Banach spaces there are some results (see Theorem 5.1.9) which require the notion of γ-radonifying operators.

Definition 5.1.8. *Let H be a separable Hilbert space and U a separable Banach space. Consider the standard Gaussian cylindrical measure γ on H as given in Definition 5.1.5. A linear bounded operator $F \in \mathcal{L}(H,U)$ is called γ-radonifying if the cylindrical measure $\gamma \circ i_Q^{-1}$ extends to a Radon measure on $\mathcal{B}(U)$.*

The proof of the following theorem can be found in [59], Theorem 1.2.26 and in [77], Theorem 5.15. and Theorem 5.16.

Theorem 5.1.9. *Let γ be the standard Gaussian measure on a separable Hilbert space H with orthonormal basis $(e_k)_{k\in\mathbb{N}}$ and let $(G_k)_{k\in\mathbb{N}}$ be a sequence of independent real normal random variables. For an operator $F \in \mathcal{L}(H,U)$ the following are equivalent:*

(a) the operator F is γ-radonifying;

(b) the operator $FF^* : U^* \to U$ is the covariance operator of a Gaussian measure on $\mathcal{B}(U)$;

(c) the series $\sum_{k=1}^{\infty} G_k F e_k$ converges a.s. in U;

(d) the series $\sum_{k=1}^{\infty} G_k F e_k$ converges in $L^p_{\mathbb{P}}(\Omega; U)$ for some $p \in [1, \infty)$;

(e) the series $\sum_{k=1}^{\infty} G_k F e_k$ converges in $L^p_{\mathbb{P}}(\Omega; U)$ for all $p \in [1, \infty)$.

In this situation we have for every $p \in [1, \infty)$:

$$\int_U \|u\|^p \mu(\mathrm{d}u) = \mathbb{E} \left\| \sum_{k=1}^{\infty} G_k F e_k \right\|^p .$$

The following theorem was proved by Itô and Nisio in [39]. The proof can be also found in [76], Theorem V.2.4.

Theorem 5.1.10 (Itô, Nisio)**.** *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent, symmetric U-valued random variables. Then for $S_n := X_1 + \ldots + X_n$ and a U-valued random variable S the following are equivalent:*

(a) $\lim_{n\to\infty} S_n = S$ *$\mathbb{P}$-a.s.;*

(b) $\lim_{n\to\infty} S_n = S$ *in probability;*

(c) $\lim_{n\to\infty} \langle S_n, u^* \rangle = \langle S, u^* \rangle$ *$\mathbb{P}$-a.s. for all $u^* \in U^*$;*

(d) $\lim_{n\to\infty} \langle S_n, u^* \rangle = \langle S, u^* \rangle$ *in probability for all $u^* \in U^*$.*

In this situation, if $S \in L^p_{\mathbb{P}}(\Omega; U)$ then one also has

(e) $\lim_{n\to\infty} S_n = S$ *in $L^p_{\mathbb{P}}(\Omega; U)$.*

The next result gives a straightforward argument of the fact that the class of γ-radonifying operators coincide with the one of Hilbert-Schmidt operators when the underlying space is a Hilbert space. For the proof see [59], Corollary 1.2.27.

Corollary 5.1.11. *If H and U are separable Hilbert spaces then for $F \in \mathcal{L}(H, U)$ the following are equivalent:*

(a) F is γ-radonifying;

(b) F is Hilbert-Schmidt.

5.2 The cylindrical fractional Brownian motion in Banach spaces

In this section we introduce the notion of cylindrical fractional Brownian motion in Banach spaces. The idea is similar to the one adopted by Applebaum and Riedle in [2] to define cylindrical Lévy processes in Banach spaces.

The word *cylindrical* may be misleading. In the literature, it is often referred to infinite sums in a Hilbert space H which do not converge in the space H. Let us explain it with a well known example: consider the cylindrical Wiener process introduced by Da Prato and Zabczyk in [16]. Here the process is of the form

$$W(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n b_n(t),$$

where $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis of H, $(\lambda_n)_{n\in\mathbb{N}}$ is a sequence of positive coefficients and $(b_n)_{n\in\mathbb{N}}$ a sequence of independent real valued Brownian motions. It turns out that this series in general does not converge in $L^2_{\mathbb{P}}(\Omega; H)$. Some additional conditions are necessary for the sum to converge, that is, the sequence $(\lambda_n)_{n\in\mathbb{N}}$ must be summable. In other words, the covariance operator Q associated to W must be nuclear.
Problems arise as soon as one wishes to define (in this way) a process W that generalizes the n-dimensional Bm which has independent components. In this case, one has to require $Q = \mathrm{Id}$ and such operator is not nuclear. To overcome the problem, one formally considers the same series but converging in a bigger Hilbert space (which is not unique and does not even need to be specified).

For this reason, we introduce the cylindrical fractional Brownian motion in a different way, in order to avoid the introduction of the *bigger Hilbert space*. Our approach does not need any Hilbert structure and therefore we consider Banach spaces. When the underlying space is a Hilbert space and the covariance operator is nuclear then the process coincides with the classical one.

5.2.1 The cylindrical fractional Brownian motion

We introduce the cylindrical fBm in U using the definition of n-dimensional fBm. In particular we ask that each n-dimensional projection is a fBm in

$\mathbb{R}^n$.

Definition 5.2.1. *A cylindrical process* $B^H = \{B^H(t), t \geq 0\}$ *in* U *is a cylindrical fractional Brownian motion with Hurst parameter* $H \in (0,1)$ *if*

(i) for any $u_1^*, \dots, u_n^* \in U^*$ *and* $n \in \mathbb{N}$*, the* $\mathbb{R}^n$*-valued stochastic process*

$$\{\big(B^H(t)u_1^*, \dots, B^H(t)u_n^*\big), t \geq 0\}$$

is an n*-dimensional fractional Brownian motion with Hurst parameter* $H \in (0,1)$*;*

(ii) the covariance operator of $B^H(1)$ *is* U*-valued.*

We observe that in case $H = \frac{1}{2}$ then we immediately recover the cylindrical Wiener process as defined in [51] and [59].
This definition involves all possible n-dimensional projections of the process, but since we are dealing with Gaussian processes the condition can be simplified using only 2-dimensional projections.

Lemma 5.2.2. *For a cylindrical process* $B^H = \{B^H(t), t \geq 0\}$ *the following are equivalent:*

(a) B^H *is a cylindrical fractional Brownian motion with Hurst parameter* $H \in (0,1)$*;*

(b) B^H *satisfies:*

  (i) for any $u_1^*, u_2^* \in U^*$ *the vector* $(B^H(t)u_1^*, B^H(t)u_2^*)$ *is a 2-dimensional fBm according to Definition 4.1.2;*

  (ii) the covariance operator of $B^H(1)$ *is* U*-valued.*

Proof. $(a) \Rightarrow (b)$. It is clear by definition.
$(b) \Rightarrow (a)$. We just need to check that for any $n \in \mathbb{N}$ and for any $u_1^*, \dots, u_n^* \in U^*$ the process $\{Y(t), t \geq 0\}$ given by $Y(t) = \big(B^H(t)u_1^*, \dots, B^H(t)u_n^*\big)$ is an n-dimensional fBm according to Definition 4.1.2. It is a Gaussian process because for any $\beta \in \mathbb{R}^n$ the real valued process $(\beta, Y(t))_n = \sum_{i=1}^n \beta_i B^H(t)u_i^* = B^H(t)\sum_{i=1}^n \beta_i u_i^*$ is a Gaussian process. Moreover for any $v \in \mathbb{R}^n$ then

$$\mathbb{E}[(v, Y(t))_n] = \mathbb{E}[\sum_{j=1}^n v^{(j)} Y_j(t)] = \sum_{j=1}^n v^{(j)} \mathbb{E}[Y_j(t)] = 0.$$

Let now $M = (m_{i,j})_{i,j=1}^n$ be the n-dimensional matrix defined by

$$m_{i,j} = \mathbb{E}[B^H(1)u_i^* B^H(1)u_j^*]$$

for each $i, j = 1, \dots, n$. By (i) we have $\mathbb{E}[B^H(t)u_i^* B^H(s)u_j^*] = m_{i,j}\frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$. Then for any $v_1, v_2 \in \mathbb{R}^n$ we get

$$\begin{aligned}
\mathbb{E}[(v_1, Y(t))_n (v_2, Y(s))_n] &= \mathbb{E}\left[\sum_{i=1}^n \left(v_1^{(i)} B^H(t)u_i^*\right) \sum_{j=1}^n \left(v_2^{(j)} B^H(s)u_j^*\right)\right] \\
&= \mathbb{E}\left[\sum_{i=1}^n \sum_{j=1}^n v_1^{(i)} v_2^{(j)} B^H(t)u_i^* B^H(s)u_j^*\right] \\
&= \sum_{i=1}^n \sum_{j=1}^n v_1^{(i)} v_2^{(j)} \mathbb{E}[B^H(t)u_i^* B^H(s)u_j^*] \\
&= \sum_{i=1}^n \sum_{j=1}^n v_1^{(i)} v_2^{(j)} m_{i,j} \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}) \\
&= (Mv_1, v_2)_n \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})
\end{aligned}$$

which concludes the proof. □

The following result provides a representation of the cylindrical fractional Brownian motion in terms of a series.

Theorem 5.2.3. *For a cylindrical process $B^H := (B^H(t), t \geq 0)$ the following are equivalent:*

(a) B^H is a cylindrical fractional Brownian motion with Hurst parameter $H \in (0,1)$;

(b) there exist a Hilbert space H with an orthonormal basis $(e_k)_{k\in\mathbb{N}}$, $F \in \mathcal{L}(H,U)$ and a sequence of independent real valued fractional Brownian motions $(b_k^H)_{k\in\mathbb{N}}$ such that

$$B^H(t)u^* = \sum_{k=1}^\infty \langle Fe_k, u^*\rangle b_k^H(t)$$

in $L^2_{\mathbb{P}}(\Omega;\mathbb{R})$ for all $u^* \in U^*$ and all $t \geq 0$.

Proof. $(a) \Rightarrow (b)$. Since the cylindrical distribution of $B^H(1)$ is strongly Gaussian the covariance operator Q associated to it is a positive and symmetric operator $Q : U^* \to U$. Let H_Q be its RKHS with inclusion mapping $i_Q : H_Q \to U$. Recall that $i_Q^* : U^* \to H_Q$ and $Q = i_Q i_Q^*$. The range of i_Q^* is dense in H_Q and H_Q is separable, so there exists an orthonormal basis $(e_k)_{k\in\mathbb{N}}$ of H_Q such that $(e_k)_{k\in\mathbb{N}} \subset \text{range}(i_Q^*)$.
Choose $w_k^* \in U^*$ such that $i_Q^* w_k^* = e_k$ and define $b_k^H(\cdot) := B^H(\cdot)w_k^*$ for all $k \in \mathbb{N}$. Clearly $\{b_k^H(t), t \geq 0\}$ is a real valued fractional Brownian motion for each $k \in \mathbb{N}$. We obtain

$$\begin{aligned}\mathbb{E}\left|\sum_{k=1}^n \langle i_Q e_k, u^*\rangle b_k^H(t) - B^H(t)u^*\right|^2 =& \mathbb{E}\left[B^H(t)\left(\sum_{k=1}^n \langle i_Q e_k, u^*\rangle w_k^* - u^*\right)\right]^2 \\ =& \mathbb{E}[B^H(t)v^*]^2\end{aligned}$$

having called for simplicity $v^* := \sum_{k=1}^n \langle i_Q e_k, u^*\rangle w_k^* - u^*$. By construction $v^* \in U^*$. Recall that for the Gaussian cylindrical random variable $B^H(1)$ with covariance Q we have $\mathbb{E}[B^H(1)v^*]^2 = \langle Qv^*, v^*\rangle$. Moreover, the fractional Brownian motion is self-similar, meaning that for each $a \in \mathbb{R}$, $B^H(at)$ and $a^H B^H(t)$ are equal in law. It follows

$$\begin{aligned}\mathbb{E}[B^H(t)v^*]^2 &= \mathbb{E}[B^H(1)(t^H v^*)]^2 = t^{2H}\mathbb{E}[B^H(1)v^*]^2 \\ &= t^{2H}\langle Qv^*, v^*\rangle = t^{2H}\langle i_Q i_Q^* v^*, v^*\rangle = t^{2H}\|i_Q^* v^*\|_{H_Q}^2 \\ &= t^{2H}\left\| i_Q^*\left(\sum_{k=1}^n \langle i_Q e_k, u^*\rangle w_k^* - u^*\right)\right\|_{H_Q}^2 \\ &= t^{2H}\left\|\sum_{k=1}^n \langle i_Q e_k, u^*\rangle i_Q^* w_k^* - i_Q^* u^*\right\|_{H_Q}^2 \\ &= t^{2H}\left\|\sum_{k=1}^n [e_k, i_Q^* u^*]_{H_Q} e_k - i_Q^* u^*\right\|_{H_Q}^2 \\ &\to 0 \quad \text{for } n \to \infty.\end{aligned}$$

It is left to prove that the fractional Brownian motions $(b_k^H)_{k\in\mathbb{N}}$ are independent. For any $j, k \in \{1, \ldots, n\}$ consider the 2-dimensional vector $b_{jk}^H(t) := (b_j^H, b_k^H)$. Let $M_{jk} := \begin{pmatrix} m_{j,j} & m_{j,k} \\ m_{k,j} & m_{k,k}\end{pmatrix}$ denote its covariance matrix according to Definition 4.1.2. Then for each $v_1, v_2 \in \mathbb{R}^2$ we have by definition

$$\mathbb{E}[(v_1, b_{jk}^H(t))_2 (v_2, b_{jk}^H(s))_2] = (M_{jk}\, v_1, v_2)_2 \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$$

and choosing $v_1 = (1,0)$ and $v_2 = (0,1)$ we get

$$\mathbb{E}[b_j^H(t)b_k^H(s)] = m_{j,k}\frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

On the other hand, we have

$$\begin{aligned}\mathbb{E}[b_j^H(1)b_k^H(1)] &= \mathbb{E}[B^H(1)w_j^* B^H(1)w_k^*] = \langle Qw_j^*, w_k^*\rangle \\ &= [i_Q^* w_j^*, i_Q^* w_k^*] = [e_j, e_k] = \delta_{j,k}\end{aligned}$$

and for $t = s = 2$ this implies $m_{j,k} = \delta_{j,k}$ and so

$$\mathbb{E}[b_j^H(t)b_k^H(s)] = \delta_{j,k}\frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

$(b) \Rightarrow (a)$. Let $n \in \mathbb{N}$ and $u_1^*, \ldots, u_n^* \in U^*$ be arbitrarily chosen and consider the n-dimensional stochastic process $Y := \{Y(t), t \geq 0\}$ defined by

$$\begin{aligned}Y(t) :&= (B^H(t)u_1^*, \ldots, B^H(t)u_n^*) \\ &= \left(\sum_{k=1}^{\infty}\langle Fe_k, u_1^*\rangle b_k^H(t), \ldots, \sum_{k=1}^{\infty}\langle Fe_k, u_n^*\rangle b_k^H(t)\right)\end{aligned}$$

for all $t \geq 0$. We now check that the stochastic process Y is an n-dimensional fractional Brownian motion according to Definition 4.1.2. It is a Gaussian process because for any $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n$ the real valued process $(\beta, Y(\cdot))_n = \sum_{i=1}^n \beta_i B^H(\cdot)u_i^* = B^H(\cdot)\sum_{i=1}^n \beta_i u_i^*$ is clearly Gaussian. Moreover $\mathbb{E}[(v, Y(t))_n] = 0$ for all $v \in \mathbb{R}^n$ and all $t \geq 0$.
Let $M = (m_{i,j})$ be the n-dimensional covariance matrix of the process Y, that is $m_{i,j} := \mathbb{E}[Y_i(1)Y_j(1)]$. By definition of Y we get

$$\begin{aligned}m_{i,j} &= \mathbb{E}[\sum_{k=1}^{\infty}\langle Fe_k, u_i^*\rangle b_i^H(1)\sum_{l=1}^{\infty}\langle Fe_l, u_j^*\rangle b_j^H(1)] \\ &= \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\langle Fe_k, u_i^*\rangle\langle Fe_l, u_j^*\rangle \mathbb{E}[b_i^H(1)b_j^H(1)] \\ &= \sum_{k=1}^{\infty}\langle Fe_k, u_i^*\rangle\langle Fe_k, u_j^*\rangle.\end{aligned}$$

Using this expression, let us compute for each $v_1, v_2 \in \mathbb{R}^n$ the following expectation

$$\mathbb{E}[(v_1, Y(t))_n(v_2, Y(s))_n] = \mathbb{E}\left[\left(\sum_{i=1}^n v_1^{(i)}Y_i(t)\right)\left(\sum_{j=1}^n v_2^{(j)}Y_j(s)\right)\right]$$

$$
\begin{aligned}
=&\mathbb{E}\left[\left(\sum_{i=1}^{n} v_1^{(i)} \sum_{k=1}^{\infty}\langle Fe_k, u_i^*\rangle b_k^H(t)\right)\left(\sum_{j=1}^{n} v_2^{(j)} \sum_{l=1}^{\infty}\langle Fe_l, u_j^*\rangle b_l^H(s)\right)\right] \\
=&\mathbb{E}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} v_1^{(i)} v_2^{(j)}\left(\sum_{k=1}^{\infty}\langle Fe_k, u_i^*\rangle b_k^H(t)\right)\left(\sum_{l=1}^{\infty}\langle Fe_l, u_j^*\rangle b_l^H(s)\right)\right] \\
=&\sum_{i=1}^{n}\sum_{j=1}^{n} v_1^{(i)} v_2^{(j)} \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\langle Fe_k, u_i^*\rangle\langle Fe_l, u_j^*\rangle \mathbb{E}\left[b_k^H(t) b_l^H(s)\right]
\end{aligned}
$$

and since the b_k^H are independent fBms, we get $\mathbb{E}\left[b_k^H(t) b_l^H(s)\right] = \delta_{k,l}\frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$ and then obtain

$$
\begin{aligned}
& \mathbb{E}[(v_1, Y(t))_n (v_2, Y(s))_n] \\
= & \sum_{i=1}^{n}\sum_{j=1}^{n} v_1^{(i)} v_2^{(j)} \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\langle Fe_k, u_i^*\rangle\langle Fe_l, u_j^*\rangle \delta_{k,l}\frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}) \\
= & \sum_{i=1}^{n}\sum_{j=1}^{n} v_1^{(i)} v_2^{(j)} m_{i,j}\frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}) \\
= & (Mv_1, v_2)_n \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).
\end{aligned}
$$

It is left to prove that the covariance operator $R : U^* \to U^{*\prime}$ of the cylindrical measure μ of $B^H(1)$ is in fact U-valued. The measure μ is centered so we have $\psi_\mu(u^*) = \exp\{-(Ru^*)u^*\}$. On the other hand we have

$$
\begin{aligned}
\psi_\mu(u^*) &= \mathbb{E}[\exp\{iB^H(1)u^*\}] \\
&= \mathbb{E}[\exp\{i\sum_{k=1}^{\infty}\langle Fe_k, u^*\rangle b_k^H(1)\}] \\
&= \lim_{m\to\infty}\mathbb{E}[\exp\{i\sum_{k=1}^{m}\langle Fe_k, u^*\rangle b_k^H(1)\}] \\
&= \lim_{m\to\infty}\mathbb{E}[\prod_{k=1}^{m}\exp\{i\langle Fe_k, u^*\rangle b_k^H(1)\}] \\
&= \lim_{m\to\infty}\prod_{k=1}^{m}\exp\{-\frac{1}{2}\langle Fe_k, u^*\rangle^2\} \\
&= \lim_{m\to\infty}\exp\{-\frac{1}{2}\sum_{k=1}^{m}\langle Fe_k, u^*\rangle^2\} \\
&= \exp\{-\frac{1}{2}\sum_{k=1}^{\infty}\langle Fe_k, u^*\rangle^2\} \\
&= \exp\{-\frac{1}{2}\|F^*u^*\|_H^2\},
\end{aligned}
$$

which implies

$$(Ru^*)u^* = \frac{1}{2}\|F^*u^*\|_H^2 = \frac{1}{2}[F^*u^*, F^*u^*]_H.$$

If one now defines $Q := FF^*$ then $Q : U^* \to U$ and by the previous computations it turns out that $2(Ru^*)u^* = \langle Qu^*, u^*\rangle$ for each $u^* \in U^*$. Thus the proof is complete. $\square$

Remark 5.3. For a cylindrical fBm B^H in U with covariance operator $Q := FF^*$ with F linear and continuous (as appearing in condition (b) of Theorem 5.2.3) we have

$$\mathbb{E}[B^H(t)u^*B^H(s)v^*] = \langle Qu^*, v^*\rangle \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$$

for all $u^*, v^* \in U^*$ and all $s, t \geq 0$.

Proof. For $u^*, v^* \in U^*$ and $s, t \geq 0$ we have

$$\begin{aligned}
\mathbb{E}[B^H(t)u^*B^H(s)v^*] &= \mathbb{E}\left[\sum_{k=1}^{\infty}\langle Fe_k, u^*\rangle b_k^H(t)\sum_{l=1}^{\infty}\langle Fe_l, v^*\rangle b_l^H(s)\right] \\
&= \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\langle Fe_k, u^*\rangle\langle Fe_l, v^*\rangle \mathbb{E}[b_k^H(t)b_l^H(s)] \\
&= \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}[e_k, F^*u^*][e_l, F^*v^*]\delta_{k,l}\frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}) \\
&= \sum_{k=1}^{\infty}[e_k, F^*u^*][e_k, F^*v^*]\frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}) \\
&= [\sum_{k=1}^{\infty}[e_k, F^*u^*]e_k, F^*v^*]\frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}) \\
&= [F^*u^*, F^*v^*]\frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}) \\
&= \langle Qu^*, v^*\rangle\frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).
\end{aligned}$$ $\square$

5.2.2 The U-valued fractional Brownian motion

In this section we focus on the special case of U-valued processes, namely proper processes in Banach spaces. We show their relation to cylindrical processes.

The following definition is similar to the one given by Duncan et al. in [19] (see also Definition 4.1.3 in Section 4.1) but here it is given in the wider framework of Banach spaces. It also can be seen as the direct generalization of Definition 4.1.2.

Definition 5.2.4. *A U-valued stochastic process $B^H = \{B^H(t), t \geq 0\}$ is called a fractional Brownian motion in U if there exists a Gaussian measure on $\mathcal{B}(U)$ with covariance operator* [1] *$Q : U^* \to U$ such that*

(i) $\langle B^H(t), u^* \rangle = 0$ *for all $u^* \in U^*$ and all $t \geq 0$;*

(ii) the covariance function is given by

$$\mathbb{E}[\langle B^H(t), u^* \rangle \langle B^H(s), v^* \rangle] = \langle Qu^*, v^* \rangle \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}) \tag{5.2}$$

for all $u^*, v^* \in U^*$ and all $s, t \geq 0$.

Observe that the definition already requires that the cylindrical distribution of $B^H(1)$ is strongly Gaussian or equivalently the weak* continuity of $B^H(1)$ between U^* and $L^0_{\mathbb{P}}(\Omega; \mathbb{R})$ (see Remark 5.2).

The definition we just stated includes all the known cases in literature, that is the case when U is a finite-dimensional space or a Hilbert space and includes also the Banach space case:

Example 5.2. Let $U = \mathbb{R}^n$. Then it is known by Mourier Theorem that there exists a centered Gaussian measure on $\mathcal{B}(\mathbb{R}^n)$ with covariance operator $Q : \mathbb{R}^n \to \mathbb{R}^n$ if and only if $Q = M$ is a positive and symmetric matrix.

Example 5.3. Let $U = H$ be a Hilbert space. Then it is known that there exists a centered Gaussian measure on $\mathcal{B}(H)$ with covariance operator $Q : H \to H$ if and only if Q is a positive, symmetric and nuclear operator.

Example 5.4. Let U be a separable Banach space. Then by Theorem 5.1.9 there exists a centered Gaussian measure on $\mathcal{B}(U)$ with covariance operator $Q : U^* \to U$ if and only if Q can be factorized through a Hilbert space H by $Q = FF^*$ where $F \in \mathcal{L}(H, U)$ is a γ-radonifying operator.

The next result is the analogous of Theorem 5.2.3 but for U-valued processes.

Theorem 5.2.5. *For a U-valued process $B^H = \{B^H(t), t \geq 0\}$ the following are equivalent:*

[1] For a characterization of covariance operators in Banach spaces see Theorem 5.1.9.

(a) B^H *is a fBm in* U*;*

(b) there exist a Hilbert space H *with an orthonormal basis* $(e_k)_{k\in\mathbb{N}}$*, a* γ*-radonifying operator* $F \in \mathcal{L}(H, U)$ *and independent* $\mathbb{R}$*-valued fBms* $(b_k^H)_{k\in\mathbb{N}}$ *such that*

$$B^H(t) = \sum_{k=1}^{\infty} Fe_k\, b_k^H(t)$$

in $L^2_{\mathbb{P}}(\Omega; U)$*.*

Proof. $(a) \Rightarrow (b)$. Let $(e_k)_{k\in\mathbb{N}} \subset H_Q$ be an orthonormal basis of H_Q and define a sequence $(b_k^H)_{k\in\mathbb{N}}$ as $b_k^H(t) := \langle B^H(t), u_k^*\rangle$ where $(u_k^*)_{k\in\mathbb{N}} \subset U^*$ is arbitrarily chosen such that $i_Q^* u_k^* = e_k$ for all $k \in \mathbb{N}$. Then by Theorem 5.2.3 we have

$$\langle B^H(t), u^*\rangle = \sum_{k=1}^{\infty} \langle i_Q e_k, u^*\rangle b_k^H(t)$$

in $L^2_{\mathbb{P}}(\Omega; \mathbb{R})$ for all $u^* \in U^*$.
Let $B_n^H(t) := \sum_{k=1}^{n} i_Q e_k b_k^H(t)$ so last equation reads

$$\langle B^H(t), u^*\rangle = \lim_{n\to\infty} \sum_{k=1}^{n} \langle i_Q e_k, u^*\rangle b_k^H(t) = \lim_{n\to\infty} \langle B_n^H(t), u^*\rangle \tag{5.3}$$

in $L^2_{\mathbb{P}}(\Omega; \mathbb{R})$ for all $u^* \in U^*$, and so (5.3) holds also in probability for all $u^* \in U^*$. Moreover $B^H(t) \in L^2_{\mathbb{P}}(\Omega; U)$ by Hoffmann-Jørgensen Theorem (see [30]). By Itô-Nisio Theorem (see Theorem 5.1.10) we have

$$\sum_{k=1}^{\infty} i_Q e_k\, b_k^H(t) = \lim_{n\to\infty} B_n^H(t) = B^H(t)$$

in $L^2_{\mathbb{P}}(\Omega; U)$. Moreover, Theorem 5.1.9 verifies i_Q^* as γ-radonifying.
$(b) \Rightarrow (a)$. Our aim is to show that B^H is a fBm according to Definition 5.2.4. By assumption the operator $F \in \mathcal{L}(H, U)$ is γ-radonifying and moreover $(b_k^H)_{k\in\mathbb{N}}$ is a sequence of independent real valued Gaussian random variables. Theorem 5.1.9, part (c) implies that $\sum_{k=1}^{\infty} Fe_k\, b_k^H(t)$ converges a.s. in U for every $t \geq 0$. Therefore, the limit defines a U-valued stochastic process which we denote by $\{B^H(t), t \geq 0\}$. For this process B^H we check that the conditions in Definition 5.2.4 are satisfied.
(i) The process B^H is a zero-mean Gaussian process because it is limit of a sequence of zero-mean Gaussian processes;
(ii) Let $Q := FF^* : U^* \to U$. By Theorem 5.1.9, part (b), Q is the

covariance operator of a Gaussian measure on $\mathcal{B}(U)$. In what follows it is shown that in fact this operator satisfies (5.2) of Definition 5.2.4 for B^H. Let $u^*, v^* \in U^*$. For each $t, s \geq 0$ we have

$$\begin{aligned}
\mathbb{E}[\langle B^H(t), u^*\rangle\langle B^H(s), v^*\rangle] &= \mathbb{E}[\langle \sum_{k=1}^{\infty} Fe_k b_k^H(t), u^*\rangle\langle \sum_{j=1}^{\infty} Fe_j b_j^H(s), v^*\rangle] \\
&= \mathbb{E}[\sum_{k=1}^{\infty} b_k^H(t)\langle Fe_k, u^*\rangle \sum_{j=1}^{\infty} b_j^H(s)\langle Fe_j, v^*\rangle] \\
&= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\langle Fe_k, u^*\rangle\langle Fe_j, v^*\rangle \mathbb{E}[b_k^H(t) b_j^H(s)] \qquad (5.4) \\
&= \sum_{k=1}^{\infty}\langle Fe_k, u^*\rangle\langle Fe_k, v^*\rangle \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}) \\
&= \langle Qu^*, v^*\rangle \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).
\end{aligned}$$

□

In the following proposition we show that the notion of U-valued fBm matches with the notion of cylindrical fBm when the latter is induced by a classical process.

Proposition 5.2.6. *Let $X = \{X(t), t \geq 0\}$ be a cylindrical fBm which is induced by a U-valued process $\tilde{X} = \{\tilde{X}(t), t \geq 0\}$, i.e.,*

$$X(t)u^* = \langle \tilde{X}(t), u^*\rangle \qquad (5.5)$$

for all $u^* \in U^*$. Then $\tilde{X}$ is a U-valued fBm.
Vice versa, if $\tilde{X}$ is a U-valued fBm then X defined by (5.5) is a cylindrical fBm. Moreover, the covariance operators coincide.

Proof. By hypothesis there exists $\tilde{X}$ such that $X(t)u^* = \langle \tilde{X}(t), u^*\rangle$ for all $u^* \in U^*$. Then the vector

$$(\langle \tilde{X}(t), u_1^*\rangle, \ldots, \langle \tilde{X}(t), u_n^*\rangle) = (X(t)u_1^*, \ldots, X(t)u_n^*)$$

is by Definition 5.2.1 an n-dimensional fBm for all $n \in \mathbb{N}$ and for all $u_1^*, \ldots, u_n^* \in U^*$, and so $\{\tilde{X}(t), t \leq 0\}$ is a zero-mean Gaussian process. Moreover by the same computations as in (5.4) we have for any $u^*, v^* \in U^*$

$$\begin{aligned}
\mathbb{E}[\langle \tilde{X}(t), u^*\rangle\langle \tilde{X}(s), v^*\rangle] &= \mathbb{E}[X(t)u^* X(s)v^*] \\
&= \langle Qu^*, v^*\rangle \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).
\end{aligned}$$

On the other hand, suppose that $\tilde{X}$ is a U-valued fBm. By Theorem 5.2.5 we have that there exist a Hilbert space H with a basis (e_k), a γ-radonifying operator $F \in \mathcal{L}(H, U)$ and a sequence of real valued fBms (b_k^H) such that

$$\tilde{X}(t) = \sum_{k=1}^{\infty} F e_k b_k^H(t)$$

and so we also have the representation

$$X(t)u^* = \langle \tilde{X}(t), u^* \rangle = \langle \sum_{k=1}^{\infty} F e_k b_k^H(t), u^* \rangle = \sum_{k=1}^{\infty} \langle F e_k, u^* \rangle b_k^H(t)$$

which ensures that X is a cylindrical fBm using Theorem 5.2.3.
Finally, since the covariance operator is defined in both cases as $Q = FF^*$, they coincide for X and $\tilde{X}$. □

Remark 5.4. A cylindrical fBm with representation $\sum_{k=1}^{\infty} \langle F e_k, \cdot \rangle b_k^H(t)$ for some $F \in \mathcal{L}(H, U)$ is a classical fBm in U if and only if F is γ-radonifying.

5.3 Comparison with literature and examples

In Section 5.3.1 we compare the cylindrical fractional Brownian motion introduced in the thesis with the existing literature. A first special case that we easily recover is the Hilbert space case which was considered for instance in [16, 19, 49, 71]. There are several definitions of fBm in a Hilbert space: we show how they all coincide and how they can be obtained from our definition.
The second subsection is devoted to the space-time noise which is fractional in time and in space. We show that this process is nothing but a cylindrical fractional Brownian motion in a suitable (function) space. This kind of process is used for instance in the framework of SPDEs driven by fractional noises.
Finally, in Section 5.3.3 we give two examples of fBms in L^1 and L^2. Even in the Hilbert case, the example we provide is a more general noise than the one considered in the literature.

5.3.1 Fractional Brownian motion in Hilbert spaces

Let V be a separable Hilbert space (possibly infinite dimensional) with scalar product $[\cdot, \cdot]_V$. Recall that (see Corollary 5.1.11) in Hilbert spaces an operator $F \in \mathcal{L}(H, U)$ is γ-radonifying if and only if F is Hilbert-Schmidt if and only if FF^* is nuclear.

A possible definition of fBm in Hilbert space is given in Definition 4.1.3.

Remark 5.5. A fractional Brownian motion $\{X^H(t), t \in \mathbb{R}_+\}$ according to Definition 4.1.3 is also a V-valued fractional Brownian motion according to Definition 5.2.4.

An interesting case arises when the components are independent. This would be a generalization of an finite dimensional *standard* fBm and it corresponds to the case $Q = \mathrm{Id}$. More generally, it is interesting to deal with non-nuclear covariance operators. For this reason let us mention the following fact.

Fact 5.1. Consider the formal series

$$X^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n \beta_n^H(t) \tag{5.6}$$

where $(\beta_n^H)_{n\in\mathbb{N}}$ are real independent fractional Brownian motions, $(\lambda_n)_{n\in\mathbb{N}}$ is a bounded sequence of non-negative numbers and $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis in V. In this case the covariance operator Q is given by $Qe_n = \lambda_n e_n, n \in \mathbb{N}$.
If $\sum_{n=1}^{\infty} \lambda_n = \infty$ (that is if Q is not nuclear) then the series (5.6) does not converge in $L^2_{\mathbb{P}}(\Omega; V)$. Nevertheless, one can always consider a Hilbert space V_1 such that $V \subset V_1$ and such that the inclusion is Hilbert-Schmidt. Doing so, we obtain a series which converges in $L^2_{\mathbb{P}}(\Omega; V_1)$.

Several authors, see for instance [16, 49, 71], use (5.6) as a definition for both the V-valued fBm and the cylindrical fBm, depending on the properties of Q. We recall here this definition.

Definition A. Let Q be a self-adjoint positive operator on V. Moreover let Q be nuclear. It is known that in this case, Q admits a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of eigenvalues $\lambda_n > 0$ for all n and $\lambda_n \downarrow 0$ and $\sum_{n=1}^{\infty} \lambda_n < \infty$. The corresponding eigenvectors $(e_n)_{n\in\mathbb{N}}$ form an orthonormal basis in V.
Then we define the *V-valued fractional Brownian motion* with covariance Q by

$$X_Q^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n \beta_n^H(t) \tag{5.7}$$

where $(\beta_n^H)_{n\in\mathbb{N}}$ is a sequence of real independent fractional Brownian motions. The series converges in $L^2_{\mathbb{P}}(\Omega;V)$ if Q is nuclear. (In this case we recover Definition 4.1.3)
If we want to consider a non-nuclear covariance operator Q, then (5.7) defines a *cylindrical fractional Brownian motion in* V to be interpreted as explained in Fact 5.1.

Next we check that in fact our Definition 5.2.4 (and Definition 5.2.1) is a more general notion than the analogous object introduced in the literature, even in the case of Hilbert spaces. We refer to Section 5.3.3 for an example of Gaussian process (a cylindrical fBm) which has not been considered in the classical sense of Da Prato and Zabczyk.

Proposition 5.3.1. *Let* X_Q^H *be a (cylindrical) fBm according to Definition A. Then*

(a) if Q *is nuclear then* X_Q^H *is a* V*-valued fBm according to Definition 5.2.4;*

(b) if Q *is not nuclear then* X_Q^H *is a* V_1*-valued fBm according to Definition 5.2.4 where* V_1 *is chosen according to Fact 5.1;*

(c) if Q *is not nuclear then* X_Q^H *is a cylindrical fBm in* V *according to Definition 5.2.1.*

Proof. (a) We want to use Theorem 5.2.5. We chose $U = H = V$ and $b_k^H = \beta_k^H$ for all $k \in \mathbb{N}$. We define a linear and continuous operator $F : V \to V$ by $Fe_k = \sqrt{\lambda_k} e_k$. It turns out that $F = Q^{1/2}$ is Hilbert-Schmidt because Q is nuclear. Therefore the series

$$B^H(t) = \sum_{k=1}^{\infty} Fe_k b_k^H(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} e_k \beta_k^H(t)$$

converges in $L^2_{\mathbb{P}}(\Omega;V)$ by Theorem 5.2.5.
(b) Consider $Q : V \to V$ and define the separable Hilbert space $V_0 := Q^{1/2}(V)$ with $(g_k)_{k\in\mathbb{N}}$ orthonormal basis of V_0. Then let V_1 be a larger Hilbert space such that the inclusion $J : V_0 \hookrightarrow V_1$ is Hilbert-Schmidt. Clearly V_1 is not unique. A possible construction of the space V_1 is the following: since $(g_k)_{k\in\mathbb{N}} \subset V_0$ is an orthonormal basis of V_0, every $v \in V_0$ admits a representation $v = \sum_{k=1}^{\infty} \alpha_k g_k$ and $\|v\|_0^2 = \sum_{k=1}^{\infty} \alpha_k^2$. Set $V_1 := V_0$ endowed with the norm $\|\cdot\|_1$ given by $\|v\|_1^2 = \sum_{k=1}^{\infty} \frac{1}{k^2}\alpha_k^2$ for any $v \in V_0$.

The embedding operator $J : V_0 \to V_1$ is simply the identity and is Hilbert-Schmidt since $\sum_{k=1}^{\infty} \|Jg_k\|_1^2 = \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$.
By Fact 5.1 we know that

$$B^H(t) = \sum_{k=1}^{\infty} Jg_k \beta_k^H(t)$$

is a V_1-valued fBm according to Definition A. If we set $U = U^* = V_1$, $H = V_0$, $F^* = J^*$ and so $F = J$, $e_k = g_k$ and $b_k^H = \beta_k^H$ for all $k \in \mathbb{N}$, then Theorem 5.2.5, part (b) is satisfied.
(c) Consider as in (a) the space $U = H = V$ and the operator $F = Q^{1/2} \in \mathcal{L}(H)$. Then define the cylindrical process $\{B^H(t), t \geq 0\}$ in V by

$$B^H(t)v = \sum_{j=1}^{\infty} [Fe_j, v]_V \beta_j^H(t).$$

The process B^H is well defined as the series on the right-hand side converges in $L^2_{\mathbb{P}}(\Omega; \mathbb{R})$:

$$\begin{aligned}
\mathbb{E}\left[\left|\sum_{j=m}^{m+n} [Fe_j, v]_V \beta_j^H(t)\right|^2\right] &= \sum_{j=m}^{m+n} [Fe_j, v]_V^2 \mathbb{E}\left[|\beta_j^H(t)|^2\right] \\
&= t^{2H} \sum_{j=m}^{m+n} [Fe_j, v]_V^2 \\
&\leq t^{2H} \sum_{j=m}^{\infty} [Fe_j, v]_V^2 = t^{2H} \sum_{j=m}^{\infty} [e_j, Fv]_V^2,
\end{aligned}$$

which tents to 0 as $m \to \infty$ because $\sum_{j=1}^{\infty} [e_j, Fv]_V^2 = \|Fv\|_V^2 < \infty$.
By Theorem 5.2.3 we have that B^H is a cylindrical fBm in V with covariance operator Q.
(c) alternative proof.
Consider the following cylindrical process $Y = \{Y(t), t \geq 0\}$ in V, defined for any $v \in V$ as

$$Y(t)v := \sum_{j=1}^{\infty} [ig_j, v]_V \beta_j^H(t) \tag{5.8}$$

where $(g_j)_{j \in \mathbb{N}} \subseteq V_0$ is an orthonormal basis of $V_0 := Q^{1/2}(V)$. The space V_0 is endowed with the scalar product $[h_1, h_2]_{V_0} : [Q^{-1/2}h_1, Q^{-1/2}h_2]_V$ and $i : V_0 \hookrightarrow V$ is the natural embedding operator. By the Corollary of Proposition

1.6, Ch III in [76] part (a) and (b) we get that V_0 is dense in V and that $ii^* = Q$ because in this situation the reproducing kernel Hilbert space H_Q is V_0. The process Y is well defined as the series on the right-hand side of (5.8) converges in $L^2_{\mathbb{P}}(\Omega;\mathbb{R})$:

$$\begin{aligned}\mathbb{E}\left[\left|\sum_{j=m}^{m+n}[ig_j,v]_V\beta_j^H(t)\right|^2\right] &= \sum_{j=m}^{m+n}[ig_j,v]_V^2\mathbb{E}\left[|\beta_j^H(t)|^2\right] \\ &= t^{2H}\sum_{j=m}^{m+n}[ig_j,v]_V^2 \\ &\leq t^{2H}\sum_{j=m}^{\infty}[ig_j,v]_V^2 = t^{2H}\sum_{j=m}^{\infty}[g_j,i^*v]_{V_0}^2,\end{aligned}$$

which tents to 0 as $m\to\infty$ because $\sum_{j=1}^{\infty}[g_j,i^*v]_{V_0}^2 = \|i^*v\|_{V_0}^2 < \infty$.
By Theorem 5.2.3 we have that $\{Y(t), t\geq 0\}$ is a cylindrical fBm in V with covariance operator Q. □

Remark 5.6. The Hilbert space used in (a) to decompose the operator is the space V itself and it is not the RKHS. In fact the RKHS would be the closure of $Q^{1/2}(V)$ with respect to the bilinear form $[Q^{1/2}v_1, Q^{1/2}v_2]_{H_Q} := [v_1,v_2]_V$ for all $v_1, v_2 \in V$. Since $Qe_k = \lambda_k e_k$, it turns out that the RKHS H_Q is V endowed with the norm

$$\|v\|_{H_Q}^2 = \|Q^{-1/2}v\|_V^2 = \sum_{k=1}^{\infty}[Q^{-1/2}v,e_k]_V^2 = \sum_{k=1}^{\infty}[v,\lambda_k^{1/2}e_k]_V^2 = \sum_{k=1}^{\infty}\lambda_k[v,e_k]_V^2.$$

The fact that the Hilbert space used in the proof is not the RKHS is not a problem. In fact by Theorem 5.2.5 it just needs to be a Hilbert space with a linear and continuous operator from it to the Banach space that decomposes Q. In this case the operator is exactly the square root of Q. The same holds for the first proof of (c).

5.3.2 The space-time fractional noise

The aim of this section is to show the relation between the cylindrical fBm and a space-time field which would correspond to a *fractional* version of the white noise. Since fBm is not a martingale (except for $H = 1/2$), we cannot expect to have a martingale measure as in the Brownian case. Therefore, in order to relate the fractional field $\{B^H(t,x), t\geq 0, x\in D\}$ and a fractional

Brownian motion (living in a function space) we proceed here inspired by Gubinelli et al. who introduced in [27] a fractional Brownian noise.

Let $D \subset \mathbb{R}^d$ be a bounded and smooth domain and consider the Hilbert space $L^2(D)$ with the usual scalar product denoted by $\langle \cdot, \cdot \rangle$. Suppose we are given a sequence of independent $\mathbb{R}$-valued fBm $(\beta_k^H)_{k \in \mathbb{N}}$ with Hurst index $H \in (0,1)$, an orthonormal system $(e_k)_{k\in\mathbb{N}}$ of $L^2(D)$ and a sequence of real numbers $(q_k)_{k\in\mathbb{N}}$ such that $\sup_{k\in\mathbb{N}} |q_k| < \infty$. Then we can always construct a cylindrical fBm on $L^2(D)$ as follows.

Example 5.5. Let us define an operator F on $L^2(D)$ by $Fe_k := q_k e_k$. Clearly $F \in \mathcal{L}(L^2(D))$. The cylindrical process

$$b^H(t) : L^2(D) \to L^2_{\mathbb{P}}(\Omega; \mathbb{R})$$

defined for all $h \in L^2(D)$ and all $t \geq 0$ by

$$b^H(t)h = \sum_{k=1}^{\infty} \langle Fe_k, h \rangle \beta_k^H(t)$$

is a cylindrical fBm in $L^2(D)$. To see it, simply apply Theorem 5.2.3.

Depending on the properties of F, it might turn out that the cylindrical process is a proper process on $L^2(D)$, as shown in the next example.

Example 5.6. If F is Hilbert-Schmidt, that is if $\sum_{k=1}^{\infty} q_k^2 < \infty$ then b^H is induced by a honest fBm $\tilde{b}^H$ in $L^2(D)$ which has the form

$$\tilde{b}^H(t) = \sum_{k=1}^{\infty} Fe_k \beta_k^H(t).$$

To see it apply Theorem 5.2.5 and Proposition 5.2.6.

Denote by $\|\cdot\|$ the norm in $L^2(D)$. Let P_t be the (heat) semigroup on $L^2(D)$ generated by (the Friedrich extension of) some second order differential operator A with Dirichlet boundary conditions. The simplest example is the Dirichlet Laplacian $-\Delta_D$ on D. Next we introduce a family of Hilbert spaces H_α, $\alpha \in \mathbb{R}$ defined by means of A and of its fractional powers:

- if $\alpha \geq 0$ let $H_\alpha := \mathcal{D}(A^\alpha)$ with scalar product $\langle x, y \rangle_\alpha := \langle A^\alpha x, A^\alpha y \rangle$ and norm $\|x\|_\alpha := \|A^\alpha x\|$. Since $A^{-\alpha}$ is continuous it follows that the norm $\|\cdot\|_\alpha$ is equivalent to the graph norm of A^α. If $\alpha = 0$, then $H_0 = L^2(D)$ and $A^0 = \mathrm{Id}$;

- if $\alpha < 0$ let H_α be the completion of $L^2(D)$ with respect to the norm $\|x\|_\alpha := \|A^\alpha x\|$. Denote by $\langle\cdot,\cdot\rangle_\alpha$ the scalar product. It follows that H_α is a larger space than $L^2(D)$.

The Laplacian acts in a nice way on these spaces, i.e. for any $\nu < 0$ and for any $\gamma \in \mathbb{R}$

$$A^\nu : H_{\gamma+\nu} \to H_\gamma$$

isomorphically.

With the help of these spaces and of the Dirichlet Laplacian we can recover the fractional noise considered by Gubinelli et al. [27] and Hinz and Zähle [29] starting from the noise defined in the two previous examples. To this aim, we report the definition of fractional noise given in terms of Gaussian series that can be found in [29], Section 6 or [27], Section 4.1:

Definition 5.3.2. *Let $(\beta_k)_{k\in\mathbb{N}}$ be a sequence of independent, real valued fractional Brownian motions with Hurst parameter $0 < H < 1$, $\{\lambda_k, k \in \mathbb{N}\} = \sigma(A)$ and let $(q_k)_{k\in\mathbb{N}}$ be a sequence of positive numbers such that $\sum_{k=1}^\infty q_k^2\lambda_k^{-\mu} < \infty$ for some $\mu \in (0,1)$ given. Then define*

$$b^{H,K}(t,x) = \sum_{k=1}^\infty q_k e_k(x)\beta_k^H(t), \tag{5.9}$$

which is shown to be for almost all $\omega \in \Omega$ an element of $C^\alpha([0,T];\bar{H}^{-\beta}(D))$ for $0 < \mu < \beta < 1$ and $0 < \alpha < H$. A number $K \in (0,1)$ slightly bigger than $1-\mu$ can be called Hurst parameter in space.

For more details about the construction of $b^{H,K}$ and its regularity in time and space (fBm behaviour) we refer to the paper of Tindel Tudor and Viens [72], Section 3.2.1.

Fact 5.2. Hinz and Zähle show that for any fixed $t \in [0,T]$ then $b^{H,K}(t,\cdot)$ is an element of $\bar{H}^{-\beta}$ $\mathbb{P}$-a.s. Gubinelli et al. show that it belongs to the space $H_{-\beta/2}$.

Remark 5.7. If we have a fBm with Hurst parameter K, then the fBm is δ-Hölder continuous in space for all $\delta < K$. In this definition, the noise $b^{H,K}(t,\cdot)$ is the derivative of a fBm with Hurst parameter K. In fact since it belongs to $H_{-\beta/2}$, its regularity is of order $-\beta = (1-\beta)-1 = \delta-1$ where the parameter $\delta < K$ is chosen $\delta = 1-\beta$. This is exactly the regularity of the derivative of a function which is $(1-\beta)$-Hölder continuous.

We can recover the result stated in Fact 5.2 by our theory: we consider the noise b^H introduced in Example 5.5 together with the additional condition $\sum_{k=1}^{\infty} q_k^2 \lambda_k^{-\beta} < \infty$. Recall that F is Hilbert-Schmidt if and only if $\sum_{k=1}^{\infty} q_k^2 < \infty$ and this is not necessarily satisfied. Therefore, we do not expect a classical process in $L^2(D)$. On the other hand, we can embed the space $L^2(D)$ in $H_{-\beta/2}$ and consider the process in this bigger space: in this way we obtain a classical process in $H_{-\beta/2}$ (see Theorem 5.3.3). Moreover, this process induces the process b^H in $L^2(D)$ as shown in Corollary 5.3.4.
Let us consider the sequence $(e_k)_{k\in\mathbb{N}}$ of eigenfunctions of A such that they form an orthonormal basis in H_0. Let $(\lambda_k)_{k\in\mathbb{N}}$ be the corresponding eigenvalues. For all $k \in \mathbb{N}$ define $g_k := A^{\beta/2} e_k = \lambda_k^{\beta/2} e_k \in H_{-\beta/2}$ and a linear operator $G : H_{-\beta/2} \to H_{-\beta/2}$ given by $G g_k := \mu_k g_k$, where $\mu_k := q_k \lambda_k^{-\beta/2} \in \mathbb{R}$.

Theorem 5.3.3. *If $\sum_{k=1}^{\infty} \mu_k^2 = \sum_{k=1}^{\infty} q_k^2 \lambda_k^{-\beta} < \infty$, that is if G is Hilbert-Schmidt, then the process $\{\bar{b}^H(t), t \geq 0\}$ defined by*

$$\bar{b}^H(t) = \sum_{k=1}^{\infty} G g_k \beta_k^H(t)$$

is a classical fBm on $H_{-\beta/2}$.

Proof. First observe that the space $H_{-\beta/2}$ is a Hilbert space with scalar product $\langle u, u \rangle_{-\beta/2} = \langle A^{-\beta/2} u, A^{-\beta/2} u \rangle$. Since $(e_k)_{k\in\mathbb{N}}$ is a basis for H_0 then then $(g_k)_{k\in\mathbb{N}}$ is a basis for $H_{-\beta/2}$: to see this let $u \in H_{-\beta/2}$, then

$$\begin{aligned} u &= A^{\beta/2} A^{-\beta/2} u \\ &= A^{\beta/2} \sum_{k=1}^{\infty} \langle A^{-\beta/2} u, e_k \rangle e_k \\ &= A^{\beta/2} \sum_{k=1}^{\infty} \langle A^{-\beta/2} u, A^{-\beta/2} A^{\beta/2} e_k \rangle e_k \\ &= A^{\beta/2} \sum_{k=1}^{\infty} \langle u, A^{\beta/2} e_k \rangle_{-\beta/2} e_k \\ &= \sum_{k=1}^{\infty} \langle u, A^{\beta/2} e_k \rangle_{-\beta/2} A^{\beta/2} e_k \\ &= \sum_{k=1}^{\infty} \langle u, g_k \rangle_{-\beta/2} g_k. \end{aligned}$$

Also $\|g_k\|_{-\beta/2} = \|A^{\beta/2} A^{-\beta/2} g_k\|_{-\beta/2} = \|A^{\beta/2} e_k\|_{-\beta/2} = \|e_k\| = 1$ and for all $j \neq k$ we have $\langle g_k, g_j \rangle_{-\beta/2} = 0$.

The operator G acts on $H_{-\beta/2}$ and we have $Gg_k = \mu_k g_k$ with $\sup_{k\in\mathbb{N}} |\mu_k| < \infty$ which ensures $G \in \mathcal{L}(H_{-\beta/2})$. Moreover G is Hilbert-Schmidt in $H_{-\beta/2}$ and so the process $\{\bar{b}^H(t), t \geq 0\}$ given by

$$\bar{b}^H(t) := \sum_{k=1}^{\infty} Gg_k \beta_k^H(t)$$

is a fBm in $H_{-\beta/2}$. To see this proceed as in proof of Proposition 5.3.1, part (a). □

Corollary 5.3.4. *If* $\sum_{k=1}^{\infty} \mu_k^2 < \infty$ *then the cylindrical process* $\{b^H(t), t \geq 0\}$ *introduced in Example 5.5 is induced by the classical fBm* $\{\bar{b}^H(t), t \geq 0\}$*. Moreover* $b^{H,K}(t, \cdot) = \bar{b}^H$ *in* $H_{-\beta/2}$*, where* $\{b^{H,K}(t, x), t \geq 0, x \in D\}$ *is defined in* (5.9).

Proof. First observe that

$$q_k e_k = \lambda^{-\beta/2} q_k \lambda^{\beta/2} e_k = \lambda^{-\beta/2} q_k g_k = \mu_k g_k,$$

that is $Fe_k = Gg_k$. For all $h \in H$, we have

$$\begin{aligned}
\langle \bar{b}^H(t), h\rangle &= \langle \sum_{k=1}^{\infty} Gg_k \beta_k^H(t), h\rangle \\
&= \langle \sum_{k=1}^{\infty} Fe_k \beta_k^H(t), h\rangle \\
&= \sum_{k=1}^{\infty} \langle Fe_k, h\rangle \beta_k^H(t) = b^H(t)h.
\end{aligned}$$

Moreover we also get

$$b^{H,K}(t, \cdot) = \sum_{k=1}^{\infty} q_k e_k \beta_k^H(t) = \sum_{k=1}^{\infty} Gg_k \beta_k^H(t) = \bar{b}^H(t)$$

as process in $H_{-\beta/2}$. □

5.3.3 Two examples in L^2 and L^1

Here we explicitly construct fractional Brownian noises in $L^2(D)$ and $L^1(D)$ for $D \subseteq \mathbb{R}^d$.

We want to consider noises which are anisotropic, for instance noises localized on a subset $A \subseteq D$ or noises whose action along the kth eigenfunction is restricted to a set A_k. To this aim, we introduce a linear and continuous operator F with a series representation involving the product of a function $q_k(\cdot)$ with an eigenfunction $e_k(\cdot)$. The function q_k will be taken as $\mathbb{1}_{A_k}$ in some specific examples.

The Hilbert space case L^2

Denote by $[\cdot,\cdot]_2$ the scalar product in $L^2(D)$ and consider a complete orthonormal system $(e_k)_{k\in\mathbb{N}}$ in $L^2(D)$. Let $(q_k)_{k\in\mathbb{N}}$ be a sequence of bounded functions $q_k \in L^\infty(D)$ such that

$$\sum_{k=1}^{\infty} \|q_k\|^2_{L^\infty(D)} \leq C < \infty \tag{5.10}$$

for some constant C. Then we define the following function on D for each $h \in L^2(D)$ by setting

$$Fh(x) := \sum_{k=1}^{\infty} [h, e_k]_2 q_k(x) e_k(x). \tag{5.11}$$

Fact 5.3. For each $h \in L^2(D)$ the function Fh belongs to $L^2(D)$.

Proof. Let $h \in L^2(D)$. We have

$$\begin{aligned} \|Fh\|_{L^2(D)} &= \|\sum_{k=1}^{\infty}[h, e_k]_2 q_k e_k\|_{L^2(D)} \\ &\leq \sum_{k=1}^{\infty} |[h, e_k]_2| \|q_k e_k\|_{L^2(D)} \end{aligned}$$

Now observe that

$$\|q_k e_k\|^2_{L^2(D)} = \int_D q_k^2(x) e_k^2(x)\,\mathrm{d}x \leq \|q_k\|^2_{L^\infty(D)} \|e_k\|^2_{L^2(D)} = \|q_k\|^2_{L^\infty(D)}$$

and therefore we get

$$\begin{aligned} \|Fh\|_{L^2(D)} &\leq \sum_{k=1}^{\infty} |[h, e_k]_2| \|q_k\|_{L^\infty(D)} \\ &\leq \left(\sum_{k=1}^{\infty} |[h, e_k]_2|^2\right)^{1/2} \left(\sum_{k=1}^{\infty} \|q_k\|^2_{L^\infty(D)}\right)^{1/2} \qquad (5.12) \\ &= \left(\sum_{k=1}^{\infty} [h, e_k]_2^2\right)^{1/2} \left(\sum_{k=1}^{\infty} \|q_k\|^2_{L^\infty(D)}\right)^{1/2} = \|h\|_{L^2(D)} C^{1/2} < \infty \qquad (5.13) \end{aligned}$$

because of (5.10). □

Therefore the operator

$$F : L^2(D) \to L^2(D), \quad Fh = \sum_{k=1}^{\infty} [h, e_k]_2 q_k e_k \tag{5.14}$$

is well defined. Moreover it turns out it is linear and continuous, as shown in the following proposition.

Proposition 5.3.5. *Under the condition* (5.10)*, the operator F defined in* (5.14) *is linear and continuous.*

Proof. Let us first check linearity. Let $a \in \mathbb{R}$ and $h, g \in L^2(D)$. Then we have

$$\begin{aligned} F(ah+g) &= \sum_{k=1}^{\infty} [ah+g, e_k]_2 q_k(x) e_k(x) \\ &= \sum_{k=1}^{\infty} \left([ah, e_k]_2 q_k(x) e_k(x) + [g, e_k]_2 q_k(x) e_k(x) \right) \\ &= a \sum_{k=1}^{\infty} [h, e_k]_2 q_k(x) e_k(x) + \sum_{k=1}^{\infty} [g, e_k]_2 q_k(x) e_k(x) \\ &= aFh + Fg. \end{aligned}$$

Moreover by the computations in the proof of Fact 5.3 we get that

$$\|Fh\|_{L^2(D)} \le C \|h\|_{L^2(D)}$$

for all $h \in L^2(D)$, which means $F \in \mathcal{L}(L^2(D))$. □

Now we construct a cylindrical fBm in $L^2(D)$ using Theorem 5.2.3. Let $(b_k^H)_{k \in \mathbb{N}}$ be a sequence of independent real fBm. Define the cylindrical process $\{X(t), t \ge 0\}$ by

$$\begin{aligned} X(t): \quad L^2(D) &\to L^2_{\mathbb{P}}(\Omega, \mathbb{R}) \\ f &\mapsto X(t)f \end{aligned}$$

with

$$X(t)f := \sum_{k=1}^{\infty} [Fe_k, f]_2 b_k^H(t) \tag{5.15}$$

which has covariance operator given by $Q = FF^* : L^2(D) \to L^2(D)$. Theorem 5.2.3 yields the following result.

Proposition 5.3.6. *If $(q_k)_{k\in\mathbb{N}} \subset L^2(D)$ satisfies* (5.10) *then the cylindrical noise X given by* (5.15) *is a cylindrical fBm in $L^2(D)$.*

Example 5.7. Let $q_k(x) = \mu_k \mathbb{1}_{A_k}(x)$ where $A_k \subseteq D$ for all $k \in \mathbb{N}$ and such that $\sum \mu_k^2 < \infty$. In this case we clearly have that coefficients $(q_k)_{k\in\mathbb{N}}$ satisfy (5.10) and therefore by Proposition 5.3.6 we can define a cylindrical fBm in $L^2(D)$.

Remark 5.8. The interesting feature of this noise is that it can behave in an asymmetric way in space. This is reflected into the fact that the coefficient q_k depend on x, which is not the case for the classical noises given through series representations that are presented in the literature.

In fact we defined more than just a cylindrical process in L^2, namely under this conditions we actually have a classical fBm which is L^2-valued.

Theorem 5.3.7. *Under the assumption* (5.10)*, the noise $\{\tilde{X}(t), t \in [0,T]\}$ given by*

$$\tilde{X}(t) = \sum_{k=1}^{\infty} F e_k b_k^H(t).$$

is an $L^2(D)$-valued fBm. Moreover, the noise X defined above by (5.15) *is induced by $\tilde{X}$ in the sense that $[\tilde{X}(t), f]_2 = X(t)f$ for all $f \in L^2(D)$.*

Proof. In order to have a classical process in $L^2(D)$ we must verify F as Hilbert-schmidt. If this is the case, by Theorem 5.2.5 we have that $\tilde{X}(t)$ converges in $L^2_{\mathbb{P}}(\Omega; L^2(D))$ and therefore is an $L^2(D)$-valued fBm.
We now check that F is Hilbert-Schmidt. Previously observe that $Fe_k = \sum_{j=1}^{\infty} [e_k, e_j]_2 q_j e_j = q_k e_k$. We have

$$\begin{aligned}
\|F\|^2_{\mathcal{L}_2(L^2(D))} &= \sum_{k=1}^{\infty} \|Fe_k\|^2_{L^2(D)} \\
&= \sum_{k=1}^{\infty} \|q_k e_k\|^2_{L^2(D)} \\
&\leq \sum_{k=1}^{\infty} \|q_k\|^2_{L^\infty(D)} < \infty
\end{aligned}$$

where the last sum is convergent by assumption. The process $\tilde{X}$ induces now the cylindrical process X by Proposition 5.2.6. ∎

Example 5.8. Using this theorem we can write now a fractional Brownian noise whose randomness is localized in space, that is a noise which is not isotropic. Suppose assumptions of Example 5.7 are satisfied and consider the noise is given by

$$\tilde{X}(t,x) = \sum_{k=1}^{\infty} \mu_k \mathbb{1}_{A_k}(x) e_k(x) b_k^H(t).$$

The sum converges in $L^2_{\mathbb{P}}(\Omega; L^2(D))$ as a function of x.

The Banach space case L^1

Now we consider a sequence of weights $(q_k)_{k\in\mathbb{N}} \subset L^2(D)$ such that

$$\sum_{k=1}^{\infty} \|q_k\|^2_{L^2(D)} \leq C < \infty. \tag{5.16}$$

We can show that under this condition then the operator F given by (5.11) for all $h \in L^2(D)$ takes values in $L^1(D)$. It is again linear and continuous and therefore we can define a cylindrical fBm in $L^1(D)$. Since D might be unbounded, we give the explicit proof that F takes actually values in $L^1(D)$.

Fact 5.4. For each $h \in L^2(D)$ the function Fh belongs to $L^1(D)$.

Proof. Let $h \in L^2(D)$. Observe that $\|q_k e_k\|_{L^1(D)} \leq \|q_k\|_{L^2(D)} \|e_k\|_{L^2(D)} = \|q_k\|_{L^2(D)}$ and therefore we get

$$\begin{aligned}\|Fh\|_{L^1(D)} &= \| \sum_{k=1}^{\infty} [h, e_k]_2 q_k e_k \|_{L^1(D)} \\ &\leq \sum_{k=1}^{\infty} |[h, e_k]_2| \|q_k\|_{L^2(D)}.\end{aligned}$$

By similar computations as in (5.12) and using (5.16) we get $\|Fh\|_{L^1(D)} \leq c\|h\|_{L^2(D)} < \infty$. □

We then have an operator

$$F : L^2(D) \to L^1(D), \quad Fh = \sum_{k=1}^{\infty} [h, e_k]_2 q_k e_k \tag{5.17}$$

which is well defined. Moreover we have already shown that it is linear and continuous.

Proposition 5.3.8. *Under the condition* (5.16), *the operator F defined in* (5.17) *is linear and continuous.*

Now we construct a cylindrical fBm in $L^1(D)$ using Theorem 5.2.3. Recall that $(L^1(D))^* = L^\infty(\mathrm{D})$ and let us denote the dual pairing between $L^1(D)$ and $L^\infty(D)$ by $\langle \cdot, \cdot \rangle$. Let $(b_k^H)_{k\in\mathbb{N}}$ be a sequence of independent real fBm. Define the cylindrical process $\{X(t), t \geq 0\}$ by

$$\begin{array}{rccc} X(t): & L^\infty(D) & \to & L^2_{\mathbb{P}}(\Omega, \mathbb{R}) \\ & f & \mapsto & X(t)f \end{array}$$

with

$$X(t)f := \sum_{k=1}^{\infty} \langle Fe_k, f\rangle b_k^H(t) \tag{5.18}$$

which has covariance operator given by $Q = FF^* : L^\infty(D) \to L^1(D)$. Theorem 5.2.3 yields the following result.

Proposition 5.3.9. *If $(q_k)_{k\in\mathbb{N}} \subset L^2(D)$ satisfies* (5.10) *then the cylindrical noise X given by* (5.18) *is a cylindrical fBm in $L^1(D)$.*

Example 5.9. The noise

$$X(t)f = \sum_{k=1}^{\infty} \langle \mu_k \mathbb{1}_{A_k}, f\rangle b_k^H(t)$$

is a cylindrical fBm in $L^1(D)$ if $A_k \subseteq D$ is bounded for all $k \in \mathbb{N}$ and $(\mu_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ are such that $\sum_{k=1}^{\infty} \mu_k^2 < \infty$.

Chapter 6

Stochastic integration with respect to cylindrical fractional Brownian motion in Banach spaces

In this chapter, after giving some preliminary results, we introduce the theory of integration in Banach spaces with respect to the cylindrical fractional Brownian motion introduced in Chapter 5.

The special case when the underlying space is a Hilbert space has already been considered in the literature. The definition, based on Wiener integrals, first appeared in [57] for general Hurst parameter $H \in (0,1)$. We briefly recall it using our setting of cylindrical processes.
Let H be a Hilbert space with scalar product $[\cdot,\cdot]$ and let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal system for H. Let B be a cylindrical standard fBm in H with covariance function $Q = I$, i.e. the process B admits the representation

$$[B(t), h] = \sum_{k=1}^{\infty} [e_k, h]\beta_k(t), \qquad \text{in } L^2_{\mathbb{P}}(\Omega; \mathbb{R})$$

for all $h \in H$ and $\forall t \geq 0$. Recall that $(\beta_k)_{k\in\mathbb{N}}$ are independent real valued fBms.

The stochastic integral $\int_0^T G\,dB$ is defined for an operator-valued function $G : [0,T] \to \mathcal{L}(H)$. The following definition and proposition are cited verbally from [57]. For more details and the proof we refer to it.

Definition 6.0.10. *Let $G : [0,T] \to \mathcal{L}(H)$, $(e_k)_{k\in\mathbb{N}}$ be a complete orthonormal basis in H, $g_n(t) := G(t)e_n$, $g_n \in \mathscr{H}_T$ for $n \in \mathbb{N}$ and B be a cylindrical standard fBm in H. Define*

$$\int_0^T G\mathrm{d}B := \sum_{k=1}^{\infty} \int_0^T g_k \,\mathrm{d}\beta_k \tag{6.1}$$

provided the infinite series converges in $L^2_{\mathbb{P}}(\Omega;\mathbb{R})$.

Proposition 6.0.11. *Let $G : [0,T] \to \mathcal{L}(H)$ and $G(\cdot)h \in \mathscr{H}_T$ for all $h \in H$. Let $\Gamma_T : H \to L^2([0,T];H)$ be given as*

$$(\Gamma_T h)(t) := (\mathbb{K}_T^* G h)(t)$$

for $t \in [0,T]$ and $h \in H$. If $\Gamma_T \in \mathcal{L}_2(H; L^2([0,T];H))$, that is Γ_T is a Hilbert-Schmidt operator, then the stochastic integral (6.1) is a well defined Gaussian H-valued random variable with covariance operator Q_T given by

$$Q_T h = \int_0^T \sum_{k=1}^{\infty} [(\Gamma_T e_k)(s), h](\Gamma_T e_k)(s)\,\mathrm{d}s.$$

This integral does not depend on the choice of the complete orthonormal basis $(e_k)_{k\in\mathbb{N}}$.

Inspired by this result, we define in this chapter the integral with respect to cylindrical fBm in separable Banach spaces. We recover the same type of result and we prove some properties of the integral.

6.1 Preliminary results

In this section we give some preliminary results and definitions. First we concentrate on the the link between Wiener integrals of real valued functions and Wiener integrals of Hilbert space valued functions. These technical results are then used in the second subsection (and later as well) in order to define the integral, its covariance operator and to study their properties.

6.1.1 The link between real- and Hilbert space-valued integrands

Let us recall definition of Bochner integral. Let U be a separable Banach space with norm $\|\cdot\|$, $(A,\mathcal{A},\mu)$ a finite measure space and $f : A \to U$ a measurable function. The Bochner integral is the integral of f with respect to μ and it is defined to be an element of U. It is first defined for simple functions $\xi(a) = \sum_{i=1}^{N} b_i \mathbb{1}_{A_i}(a)$, where $b_i \in U$ and $A_i \in \mathcal{A}$, as

$$\int_A \xi \, \mathrm{d}\mu = \int_A \sum_{i=1}^{N} b_i \mathbb{1}_{A_i}(a) \, \mu(\mathrm{d}a) := \sum_{i=1}^{N} b_i \, \mu(A_i \cap A).$$

By density one extends this definition to all measurable functions $f : A \to U$ such that

$$\int_A \|f(a)\| \, \mu(\mathrm{d}a) < \infty.$$

through

$$\int_A f(a) \, \mu(\mathrm{d}a) := \lim_{n\to\infty} \int_A \xi_n(a) \, \mu(\mathrm{d}a)$$

for simple functions $(\xi_n)_{n\in\mathbb{N}}$ which approximate f appropriately.
The Bochner integral behaves in a nice way with the dual pairing: for $f : [0,T] \to U$ measurable and such that $\int_0^T \|f(t)\| \, dt < \infty$, then for any $b^* \in U^*$ we have

$$\left\langle \int_0^T f(t)\mathrm{d}t, b^* \right\rangle = \int_0^T \langle f(t), b^* \rangle \mathrm{d}t. \tag{6.2}$$

Moreover, for $g : [0,T] \to \mathbb{R}$ measurable and such that $\int_0^T |g(t)| \, \mathrm{d}t < \infty$, then for any $b \in U$ the function $bg : [0,T] \to U$ is Bochner integrable and

$$\int_0^T bg(t)\mathrm{d}t = b \int_0^T g(t)\mathrm{d}t. \tag{6.3}$$

We prove now a technical result that is useful to link the operator $\mathbb{K}_T^*$ with K_T^* and that will be used in this chapter to define the stochastic integral in Banach spaces.
The first obvious result is that $\mathbb{K}_T^*(\alpha \mathbb{1}_{[0,t)}) = \alpha K_T^* \mathbb{1}_{[0,t)}$ for all $\alpha \in H$. This follows from (6.3) applied in the the special case of Hilbert space valued functions.

Proposition 6.1.1. *(i) For any $\phi \in \mathcal{H}_T$ and $\alpha \in H$ then $\Phi : [0,T] \to H$ defined by $\Phi(t) := \alpha\phi(t)$ for all $t \in [0,T]$ is an element of $\mathscr{H}_T$ with*

$$\|\Phi\|_{\mathscr{H}_T}^2 = \|\alpha\|_H^2 \|\phi\|_{\mathcal{H}_T}^2$$

and

$$\mathbb{K}_T^* \alpha\phi = \alpha K_T^* \phi.$$

(ii) For any $\Phi \in \mathscr{H}_T$ and $\alpha \in H$ then $\phi : [0,T] \to \mathbb{R}$ given by $\phi(t) := [\Phi(t), \alpha]$ for all $t \in [0,T]$ is an element of $\mathcal{H}_T$ with

$$\|\phi\|^2_{\mathcal{H}_T} \le \|\alpha\|^2_H \|\Phi\|^2_{\mathscr{H}_T}$$

and

$$[\mathbb{K}^*_T \Phi(\cdot), \alpha]_H = K^*_T \phi(\cdot).$$

Proof. *(i)* Let $\phi \in \mathcal{H}$ and $\alpha \in H$. By linearity of $\mathbb{K}^*_T$ and using equation (6.3) we have $\mathbb{K}^*_H \Phi = \alpha K^*_T \phi$ so we get

$$\begin{aligned}
\|\Phi\|^2_{\mathscr{H}_T} &= \|\alpha\phi\|^2_{\mathscr{H}_T} = \|\mathbb{K}^*_T \alpha\phi\|^2_{L^2([0,T];H)} \\
&= \|\alpha K^*_T \phi\|^2_{L^2([0,T];H)} \\
&= \int_0^T \|\alpha (K^*_T \phi)(t)\|^2_H \, \mathrm{d}t \\
&= \int_0^T \|\alpha\|^2_H |(K^*_T \phi)(t)|^2 \, \mathrm{d}t \\
&= \|\alpha\|^2_H \|K^*_T \phi\|^2_{L^2([0,T])} = \|\alpha\|^2_H \|\phi\|^2_{\mathcal{H}_T} < \infty,
\end{aligned}$$

which guarantees $\Phi \in \mathscr{H}_T$.
(ii) Using equation (6.2) and the linearity of operators K^*_T and $\mathbb{K}^*_T$ we get easily that

$$[\mathbb{K}^*_T \Phi(\cdot), \alpha]_H = K^*_T([\Phi(\cdot), \alpha]_H).$$

Moreover, since $\Phi \in \mathscr{H}_T$ it holds

$$\begin{aligned}
\|\phi\|^2_{\mathcal{H}_T} &= \|K^*_T \phi\|^2_{L^2([0,T])} \\
&= \|[\mathbb{K}^*_T \Phi(\cdot), \alpha]_H\|^2_{L^2([0,T])} \\
&= \int_0^T [\mathbb{K}^*_T \Phi(t), \alpha]^2_H \, \mathrm{d}t \\
&\le \int_0^T \|\mathbb{K}^*_T \Phi(t)\|^2_H \|\alpha\|^2_H \, \mathrm{d}t \\
&= \|\alpha\|^2_H \int_0^T \|\mathbb{K}^*_T \Phi(t)\|^2_H \, \mathrm{d}t \\
&= \|\alpha\|^2_H \|\mathbb{K}^*_T \Phi(\cdot)\|^2_{L^2([0,T];H)} \\
&= \|\alpha\|^2_H \|\Phi\|^2_{\mathscr{H}_T} < \infty.
\end{aligned}$$

□

6.1.2 Some definitions and properties

We indicate by U and V two separable Banach spaces with norm $\|\cdot\|_U$ and $\|\cdot\|_V$, respectively. The dual pairing is denoted by $\langle\cdot,\cdot\rangle_{U,U^*}$ and $\langle\cdot,\cdot\rangle_{V,V^*}$ respectively, or only with $\langle\cdot,\cdot\rangle$ for both of them by abuse of notation when it is clear which one we mean.
Let B^H be a cylindrical fBm in U as introduced in Definition 5.2.1 so that we have the representation

$$B^H(t)u^* = \sum_{k=1}^{\infty}\langle i_Q e_k, u^*\rangle b_k^H(t) \tag{6.4}$$

for all $u^* \in U^*$. Let $\varphi : [0,T] \to \mathcal{L}(U,V)$ be given and denote by $\varphi^* : [0,T] \to \mathcal{L}(V^*,U^*)$ the adjoint of $\varphi(t)$ for all $t \in [0,T]$. The main idea to define a cylindrical integral $\int_0^T \varphi(t)\mathrm{d}B^H(t)$ is by exploiting the representation of B^H as a series, therefore involving one-dimensional fBms. In this spirit one defines the integral as sum of one-dimensional integrals where the integrands are real valued functions defined by means of the dual pairing in V, V^* as follows

$$\sum_{k=1}^{\infty}\int_0^T \langle\varphi(t)i_Q e_k, v^*\rangle_{V,V^*}\,\mathrm{d}b_k^H(t).$$

The series is considered as an element in $L^2_{\mathbb{P}}(\Omega;\mathbb{R})$ (that is it is required to converge). It turns out that under suitable conditions the integral is well defined: each one-dimensional integral makes sense and the sum converges in $L^2_{\mathbb{P}}(\Omega;\mathbb{R})$.

In order to formalize this definition, let us introduce the space of deterministic functions for which the integral is well defined.

Definition 6.1.2. *We denote by $\mathscr{I}_T$ the space of deterministic functions φ that are integrable on $[0,T]$ with respect to B^H, that is*

$$\mathscr{I}_T := \{\varphi : [0,T] \to \mathcal{L}(U,V) \textit{ such that for all } v^* \in V^*,\ i_Q^*\varphi^*(\cdot)v^* \in \mathscr{H}_T\}.$$

Another class of integrable functions is a subset of $\mathscr{I}_T$ and it is given by the following

Definition 6.1.3. *We define the following subset of $\mathscr{I}_T$:*

$$\mathscr{B}_T := \{\varphi \in \mathscr{I}_T \textit{ such that } \exists C > 0 : \\ \|i_Q^*\varphi^*(\cdot)v^*\|_{\mathscr{H}_T} \le C\|v^*\|_{V^*} \textit{ for all } v^* \in V^*\},$$

that is the class of functions φ for which the operator $i_Q^*\varphi^*(\cdot)$ acting on V^* with values in $\mathscr{H}_T$ is continuous.

The next proposition gives an example of a class of functions $\varphi : [0,T] \to \mathcal{L}(U,V)$ which is a subset of $\mathscr{B}_T$. For the proof we need to recall a preliminary known fact about $\mathscr{H}_T$ (see [19], Section 2 or [33], Lemma 5.20).

In the case $H \in (0, 1/2)$ we have $\mathscr{C}^\beta([0,T]; H_Q) \subset \mathscr{H}_T$ for each $\beta > 1/2 - H$ and the inclusion is continuous.

In the case $H \in (1/2, 1)$ then $L^{1/H}([0,T]; H_Q) \subset \mathscr{H}_T$ and the inclusion is continuous. In particular $L^2([0,T]; H_Q) \subset \mathscr{H}_T$, the inclusion being continuous.

Proposition 6.1.4. *Let $\varphi : [0,T] \to \mathcal{L}(U,V)$ be given.*

(i) For $H \in (0, 1/2)$, if $\varphi^* \in \mathscr{C}^\beta([0,T]; \mathcal{L}(V^*, U^*))$ for some $\beta > 1/2 - H$, then $i_Q^*\varphi^*(\cdot)v^* \in \mathscr{H}_T$ for all $v^* \in V^*$ and moreover we have

$$\|i_Q^*\varphi^*(\cdot)v^*\|_{\mathscr{H}_T} \leq c\|v^*\|_{V^*}\, \|i_Q^*\|_{\mathcal{L}(U^*;H_Q)}\, \|\varphi^*\|_{\mathscr{C}^\beta([0,T];\mathcal{L}(V^*,U^*))},$$

that is $\varphi \in \mathscr{B}_T$.

(ii) For $H \in (1/2, 1)$, if $\varphi^* \in L^2([0,T]; \mathcal{L}(V^*, U^*))$ then $i_Q^*\varphi^*(\cdot)v^* \in \mathscr{H}_T$ for all $v^* \in V^*$ and moreover we have

$$\|i_Q^*\varphi^*(\cdot)v^*\|_{\mathscr{H}_T} \leq c\|v^*\|_{V^*}\, \|i_Q^*\|_{\mathcal{L}(U^*;H_Q)}\, \|\varphi^*\|_{L^2([0,T];\mathcal{L}(V^*,U^*))},$$

that is $\varphi \in \mathscr{B}_T$.

Proof. For sake of simplicity, we will sometimes indicate the operator norm of the spaces of linear and continuous functionals with $\|\cdot\|_{\mathcal{L}}$ without specifying the spaces where the operator acts. The same for the norms in $\mathscr{C}^\beta$ and L^2.
(i) Let $H \in (0, 1/2)$ and $\varphi^* \in \mathscr{C}^\beta([0,T]; \mathcal{L}(V^*, U^*))$, i.e.

$$\|\varphi^*\|_\beta = \sup_{0 \leq s < t \leq T} \frac{\|\varphi^*(t) - \varphi^*(s)\|_{\mathcal{L}}}{|t-s|^\beta} < \infty.$$

Let $i_Q^*\varphi^* : [0,T] \to \mathcal{L}(V^*, H_Q)$ be the map defined by $t \mapsto i_Q^*\varphi^*(t)$. We have

$$\begin{aligned}
\|i_Q^*\varphi^*\|_\beta &= \sup_{0\le t\le T} \frac{\|i_Q^*\varphi^*(t) - i_Q^*\varphi^*(s)\|_{\mathcal{L}}}{|t-s|^\beta} \\
&\le \sup_{0\le t\le T} \frac{\|i_Q^*\|_{\mathcal{L}}\|\varphi^*(t) - \varphi^*(s)\|_{\mathcal{L}}}{|t-s|^\beta} \\
&= \|i_Q^*\|_{\mathcal{L}} \sup_{0\le s<t\le T} \frac{\|\varphi^*(t) - \varphi^*(s)\|_{\mathcal{L}}}{|t-s|^\beta} \\
&= \|i_Q^*\|_{\mathcal{L}}\|\varphi^*\|_\beta < \infty
\end{aligned}$$

that is $i_Q^*\varphi^* \in \mathscr{C}^\beta([0,T]; \mathcal{L}(V^*, H_Q))$. With the same kind of computations one proves that the map

$$\begin{aligned}
i_Q^*\varphi^*(\cdot)v^* : [0,T] &\to H_Q \\
t &\mapsto i_Q^*\varphi^*(t)v^*
\end{aligned}$$

is an element of $\mathscr{C}^\beta([0,T]; H_Q)$ and the following bound holds

$$\|i_Q^*\varphi^*(\cdot)v^*\|_\beta \le \|v^*\|_{V^*}\, \|i_Q^*\|_{\mathcal{L}(U^*;H_Q)}\, \|\varphi^*\|_{\mathscr{C}^\beta([0,T];\mathcal{L}(V^*,U^*))}.$$

Since for $H \in (0, 1/2)$ the inclusion $\mathscr{C}^\beta([0,T]; H) \subset \mathscr{H}_T$ is continuous, the claim follows.
(ii) Let $H \in (1/2, 1)$ and $\varphi^* \in L^2([0,T]; \mathcal{L}(V^*, U^*))$. We have

$$\begin{aligned}
\|i_Q^*\varphi^*\|_{L^2}^2 &= \int_0^T \|i_Q^*\varphi^*(t)\|_{\mathcal{L}}^2 \, \mathrm{d}t \\
&\le \int_0^T \|i_Q^*\|_{\mathcal{L}}^2\, \|\varphi^*(t)\|_{\mathcal{L}}^2 \, \mathrm{d}t \\
&\le \|i_Q^*\|_{\mathcal{L}}^2\, \|\varphi^*\|_{L^2}^2 < \infty
\end{aligned}$$

and so the map $i_Q^*\varphi^*$ is an element of $L^2([0,T]; \mathcal{L}(V^*; H_Q))$. In the same way one gets for all $v^* \in V^*$ that $i_Q^*\varphi^*(\cdot)v^* \in L^2([0,T]; H_Q)$ and the following bound holds

$$\|i_Q^*\varphi^*(t)v^*\|_{L^2([0,T];H_Q)} \le \|v^*\|_{V^*}\, \|i_Q^*\|_{\mathcal{L}(U^*;H_Q)}\, \|\varphi^*\|_{L^2([0,T];\mathcal{L}(V^*,U^*))}.$$

Since for $H \in (1/2, 1)$ the inclusion $L^2([0,T]; H_Q) \subset \mathscr{H}_T$ is continuous, the claim follows. □

Next we introduce an operator $\Gamma_{T,\varphi}$ which will be useful to prove that the integral is well defined and to show several properties of it. The operator in fact, plays a crucial role in the description of the random variable as Gaussian process. It will provide the decomposition of the covariance operator of the integral $\int_0^T \varphi \mathrm{d}B^H$, allowing us to give conditions under which the integral is induced by a classical random variable in V.

Definition 6.1.5. *For each $\varphi \in \mathscr{I}_T$ we define the linear operator $\Gamma_{T,\varphi}$ acting on V^* by*

$$\begin{array}{rccl} \Gamma_{T,\varphi}: & V^* & \longrightarrow & L^2([0,T];H_Q) \\ & v^* & \longmapsto & \Gamma_{T,\varphi}(v^*) \end{array}$$

where

$$\Gamma_{T,\varphi}(v^*) := \mathbb{K}_T^*(i_Q^*\varphi^*(\cdot)v^*)$$

with $\mathbb{K}_T^*$ as defined in Section 4.2.2.

By this definition we easily get the following isometry property.

Proposition 6.1.6. *For $\varphi \in \mathscr{I}_T$ and for all $v^* \in V^*$ we have*

$$\|\Gamma_{T,\varphi}(v^*)\|_{L_2([0,T];H_Q)} = \|i_Q^*\varphi^*(\cdot)v^*\|_{\mathscr{H}_T}$$

Proof. Observe that $\varphi \in \mathscr{I}_T$ ensures that $i_Q^*\varphi^*(\cdot)v^* \in \mathscr{H}_T$. We have

$$\begin{aligned} \|\Gamma_{T,\varphi}(v^*)\|^2_{L_2([0,T];H_Q)} &= \int_0^T \|\Gamma_{T,\varphi}(v^*)\|^2_{H_Q}\,\mathrm{d}t \\ &= \int_0^T \|\mathbb{K}_T^*(i_Q^*\varphi^*(t)v^*)\|^2_{H_Q}\mathrm{d}t \\ &= \|\mathbb{K}_T^*(i_Q^*\varphi^*(\cdot)v^*)\|^2_{L^2([0,T];H_Q)} \\ &= \|i_Q^*\varphi^*(\cdot)v^*\|^2_{\mathscr{H}_T}. \end{aligned}$$

The last equality follows from the isometry property of the operator $\mathbb{K}_T^*$. □

6.2 Stochastic integral with respect to fBm in Banach spaces

In this section we define the stochastic integral with respect to a cylindrical fBm. The integral is defined as a cylindrical r.v. in a separable Banach space V. We investigate its properties and give conditions under which it is well defined. The integral turns out to be a centered Gaussian cylindrical random variable. Under suitable assumptions it is actually induced by a classical r.v. in V as shown in the second subsection. Finally we consider the integral as a stochastic process indexed by t.

6.2.1 Definition of the stochastic integral

We can now proceed to give the formal definition of the integral with respect to B^H, which, at a first stage, will be a cylindrical process in V.

Definition 6.2.1. *Let $\varphi \in \mathscr{I}_T$ be given. The stochastic integral of φ on $[0,T]$ with respect to a cylindrical fBm B^H is denoted by $\mathcal{I}_T(\varphi)$ and is defined as a cylindrical random variable in V by*

$$\mathcal{I}_T(\varphi)v^* := \sum_{k=1}^{\infty} \int_0^T \langle \varphi(t) i_Q e_k, v^* \rangle_{V,V^*} \, \mathrm{d}b_k^H(t) \tag{6.5}$$

for all $v^* \in V^*$, where the series converges in $L^2_{\mathbb{P}}(\Omega;\mathbb{R})$.

Theorem 6.2.2. *Let $\varphi \in \mathscr{I}_T$. For the integral $\mathcal{I}_T(\varphi)$ we have:*

(i) the following isometry property

$$\|\mathcal{I}_T(\varphi)v^*\|_{L^2_{\mathbb{P}}(\Omega;\mathbb{R})} = \|\Gamma_{T,\varphi}(v^*)\|_{L^2([0,T];H_Q)}$$

holds for each $v^* \in V^*$;

(ii) the integral is a well-defined cylindrical random variable in V;

(iii) the definition does not depend on the representation of the fBm.

Proof. *(i)* Let $v^* \in V^*$ be fixed. Using the independence of the one-dimensional integrals and the isometry property of K_H^* between $\mathcal{H}_T$ and $L^2([0,T];\mathbb{R})$ we have

$$\begin{aligned}
\|\mathcal{I}_T(\varphi)v^*\|^2_{L^2_{\mathbb{P}}(\Omega;\mathbb{R})} &= \mathbb{E}\left|\sum_{k=1}^{\infty} \int_0^T \langle \varphi(t) i_Q e_k, v^* \rangle \mathrm{d}b_k^H(t)\right|^2 \\
&= \sum_{k=1}^{\infty} \mathbb{E}\left|\int_0^T \langle \varphi(t) i_Q e_k, v^* \rangle \mathrm{d}b_k^H(t)\right|^2 \\
&= \sum_{k=1}^{\infty} \int_0^T |K_T^* \langle \varphi(t) i_Q e_k, v^* \rangle_{V,V^*}|^2 \, \mathrm{d}t \\
&= \sum_{k=1}^{\infty} \int_0^T \left|K_T^* [e_k, i_Q^* \varphi^*(t) v^*]_{H_Q}\right|^2 \mathrm{d}t.
\end{aligned}$$

By Proposition 6.1.1 we get

$$\begin{aligned}\|\mathcal{I}_T(\varphi)v^*\|^2_{L^2_{\mathbb{P}}(\Omega;\mathbb{R})} &= \sum_{k=1}^{\infty}\int_0^T \left|[e_k, \mathbb{K}_T^* i_Q^*\varphi^*(t)v^*]_{H_Q}\right|^2 \mathrm{d}t \\ &= \sum_{k=1}^{\infty}\int_0^T [\Gamma_{T,\varphi}(v^*), e_k]^2_{H_Q}(t)\,\mathrm{d}t \\ &= \int_0^T \sum_{k=1}^{\infty} [\Gamma_{T,\varphi}(v^*), e_k]^2_{H_Q}(t)\,\mathrm{d}t \\ &= \int_0^T \|\Gamma_{T,\varphi}(v^*)\|^2_{H_Q)}\,\mathrm{d}t \\ &= \|\Gamma_{T,\varphi}(v^*)\|^2_{L^2([0,T];H_Q}.\end{aligned}$$

(ii) First observe that $\langle\varphi(t)i_Q e_k, v^*\rangle_{V,V^*} = [e_k, \varphi^*(\cdot)i_Q^*v^*]_{H_Q}$. Since by hypothesis $\varphi^*(\cdot)i_Q^*v^* \in \mathscr{H}_T$, then Proposition 6.1.1 implies $[e_k, \varphi^*(\cdot)i_Q^*v^*]_{H_Q} \in \mathcal{H}_T$ that is the one-dimensional integrals appearing in (6.5) are well defined as Wiener integrals. The sum converges in $L^2_{\mathbb{P}}(\Omega;\mathbb{R})$ thanks to the isometry property (i) and to the assumption that $i_Q^*\varphi^*(\cdot)v^* \in \mathscr{H}_T$. Finally the map $\mathcal{I}_T$ is linear due to the linearity of the Wiener integral and of the dual pairing.

(iii) Let $(f_k)_{k\in\mathbb{N}}$ be another orthonormal system in H_Q, $(w_j^*)_{j\in\mathbb{N}} \subset U^*$ be such that $i_Q^*w_j^* = f_j$ and let $(c_j^H)_{j\in\mathbb{N}}$ be independent one-dimensional fBms such that for all $u^* \in U^*$

$$B^H(t)u^* = \sum_{j=1}^{\infty}\langle i_Q f_j, u^*\rangle c_j^H(t) \tag{6.6}$$

in $L^2_{\mathbb{P}}(\Omega;\mathbb{R})$.

It is easy to show that $\mathbb{E}[b_k^H(t)c_j^H(s)] = R_H(t,s)[e_k, f_j]_{H_Q}$. In fact if we denote by v_k the elements in U^* such that $i_Q^*v_k^* = e_k$ for each $k \in \mathbb{N}$, then we have

$$b_k^H(t) = B^H(t)v_k^* \quad \text{and} \quad c_j^H(t) = B^H(t)w_j^*$$

which, together with the definition of cylindrical fBm, yields

$$\begin{aligned}\mathbb{E}[b_k^H(t)c_j^H(s)] &= \mathbb{E}[B^H(t)v_k^*B^H(s)w_j^*] = R_H(t,s)\langle Qv_k^*, w_j^*\rangle \\ &= R_H(t,s)[i_Q^*v_k^*, i_Q^*w_j^*]_{H_Q} = R_H(t,s)[e_k, f_j]_{H_Q}.\end{aligned}$$

On the other hand, it is known (see for instance [9], Section 2.1) that for each fBm c_j^H there exist a Bm $\tilde{W}_j$ such that $c_j^H(t) = \int_0^t K_H(t,u)\,\mathrm{d}\tilde{W}_j(u)$

and also

$$\int_0^T \varphi(t)\mathrm{d}c_j^H(t) = \int_0^T (K_H^*\varphi)(t)\mathrm{d}\tilde{W}_j(t)$$

for each $\varphi \in \mathcal{H}_T$. Denote by W_k the Bm corresponding to b_k^H. Then we have

$$\begin{aligned}
R_H(t,s)[e_k, f_j]_{H_Q} &= \mathbb{E}[b_k^H(t)c_j^H(s)] \\
&= \mathbb{E}\left[\int_0^t K_H(t,u)\,\mathrm{d}W_k(u)\int_0^s K_H(s,u)\,\mathrm{d}\tilde{W}_j(u)\right] \\
&= \int_0^T \mathbb{1}_{[0,t)}(u)K_H(t,u)\mathbb{1}_{[0,s)}(u)K_H(s,u)\mathrm{d}[W_k,\tilde{W}_j]_u \\
&= \int_0^{s\wedge t} K_H(t,u)K_H(s,u)\mathrm{d}[W_k,\tilde{W}_j]_u
\end{aligned}$$

where $[\cdot,\cdot]_u$ denotes the quadratic variation.
Since $R_H(t,s) = \int_0^{t\wedge s} K_H(t,u)K_H(s,u)\mathrm{d}u$ we have

$$\begin{aligned}
R_H(t,s)[e_k, f_j]_{H_Q} &= \int_0^{t\wedge s} K_H(t,u)K_H(s,u)[e_k, f_j]_{H_Q}\mathrm{d}u \\
&= \int_0^{s\wedge t} K_H(t,u)K_H(s,u)\mathrm{d}[W_k,\tilde{W}_j]_u
\end{aligned}$$

and therefore we get $\mathrm{d}[W_k,\tilde{W}_j]_u = [e_k, f_j]_{H_Q}\mathrm{d}u$. Using this relation we get

$$\begin{aligned}
(6.7)\qquad & \mathbb{E}\left[\int_0^T \langle\varphi(t)i_Q e_k, v^*\rangle\mathrm{d}b_k^H(t)\int_0^T \langle\varphi(t)i_Q f_j, v^*\rangle\mathrm{d}c_j^H(t)\right] \\
&= \mathbb{E}\left[\int_0^T K_H^*\langle\varphi(t)i_Q e_k, v^*\rangle\mathrm{d}W_k(t)\int_0^T K_H^*\langle\varphi(t)i_Q f_j, v^*\rangle\mathrm{d}\tilde{W}_j(t)\right] \\
&= \int_0^T K_T^*\langle\varphi(t)i_Q e_k, v^*\rangle\, K_T^*\langle\varphi(t)i_Q f_j, v^*\rangle[e_k, f_j]_{H_Q}\,\mathrm{d}t \\
&= \int_0^T K_T^*[e_k, i_Q^*\varphi^*(t)v^*]_{H_Q}\, K_T^*[f_j, i_Q^*\varphi^*(t)v^*]_{H_Q}[e_k, f_j]_{H_Q}\,\mathrm{d}t \\
&= \int_0^T [\Gamma_{T,\varphi}(v^*)(t), e_k]_{H_Q}\,[\Gamma_{T,\varphi}(v^*)(t), f_j]_{H_Q}[e_k, f_j]_{H_Q}\,\mathrm{d}t.
\end{aligned}$$

Define in $L^2_{\mathbb{P}}(\Omega;\mathbb{R})$ the integral $\tilde{\mathcal{I}}_T(\varphi)$ of φ with respect to B^H using the representation (6.6), that is define the integral by the following series

$$\tilde{\mathcal{I}}_T(\varphi)v^* := \sum_{j=1}^{\infty}\int_0^T \langle\varphi(t)i_Q f_j, v^*\rangle\,\mathrm{d}c_j^H(t)$$

for all $v^* \in V^*$. Then using the isometry property stated in part (i) and (6.7) we get

$$\begin{aligned}
&\mathbb{E}|\mathcal{I}_T(\varphi)v^* - \tilde{\mathcal{I}}_T(\varphi)v^*|^2 \\
=&\mathbb{E}|\mathcal{I}_T(\varphi)v^*|^2 + \mathbb{E}|\tilde{\mathcal{I}}_T(\varphi)v^*|^2 - 2\mathbb{E}[(\mathcal{I}_T(\varphi)v^*)(\tilde{\mathcal{I}}_T(\varphi)v^*)] \\
=&2\|\Gamma_{T,\varphi}(v^*)\|^2_{L^2([0,T];H_Q)} \\
&- 2\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\int_0^T [\Gamma_{T,\varphi}(v^*)(t), e_k]_{H_Q}\,[\Gamma_{T,\varphi}(v^*)(t), f_j]_{H_Q}\,[e_k, f_j]_{H_Q}\,\mathrm{d}t \\
=&2\|\Gamma_{T,\varphi}(v^*)\|^2_{L^2([0,T];H_Q)} \\
&- 2\int_0^T \|\Gamma_{T,\varphi}(v^*)(t)\|_{H_Q}\,\|\Gamma_{T,\varphi}(v^*)(t)\|_{H_Q}\,\mathrm{d}t \\
=&2\|\Gamma_{T,\varphi}(v^*)\|^2_{L^2([0,T];H_Q)} - 2\|\Gamma_{T,\varphi}(v^*)\|^2_{L^2([0,T];H_Q)} = 0,
\end{aligned}$$

which proves the independence of the definition from the representation. □

Remark 6.1. The isometry property can be expressed also in terms of the space $\mathscr{H}_T$ as follows

$$\|\mathcal{I}_T(\varphi)v^*\|_{L^2_{\mathbb{P}}(\Omega;\mathbb{R})} = \|i_Q^*\varphi^*(\cdot)v^*\|_{\mathscr{H}_T}$$

for all $v^* \in V^*$. The proof follows easily combining Proposition 6.1.6 and part (i) of Theorem 6.2.2.

6.2.2 The stochastic integral as cylindrical random variable and its properties

Next we give a description of the stochastic integral as a cylindrical random variable in V. It turns out that under suitable conditions the covariance function can be factorized through a linear and continuous operator which is given explicitly by $\Gamma_{T,\varphi}$ and therefore the reproducing kernel Hilbert space of the variable $\mathcal{I}_T(\varphi)$ turns out to be $L^2([0,T];H_Q)$.

The following technical lemma will be used for this aim.

Lemma 6.2.3. *Let $w^* \in V^*$ be fixed and let $(v_n^*)_{n\in\mathbb{N}} \subset V^*$ be weakly* convergent to 0. If $\varphi \in \mathscr{B}_T$ then the series*

$$\sum_{k=1}^{\infty}\langle[\Gamma_{T,\varphi}w^*, e_k]_{H_Q}, [\Gamma_{T,\varphi}v_n^*, e_k]_{H_Q}\rangle_{L^2([0,T])}$$

converges in $\mathbb{R}$ uniformly in n.

Proof. Observe that

$$\begin{aligned}&\langle [\Gamma_{T,\varphi} w^*, e_k]_{H_Q}, [\Gamma_{T,\varphi} v_n^*, e_k]_{H_Q} \rangle_{L^2([0,T])}\\ &\le \|[\Gamma_{T,\varphi} w^*, e_k]_{H_Q}\|_{L^2([0,T])} \|[\Gamma_{T,\varphi} v_n^*, e_k]_{H_Q}\|_{L^2([0,T])}\end{aligned}$$

and by Hölder inequality we get

$$\begin{aligned}&\left|\sum_{k=m}^{\infty} \langle [\Gamma_{T,\varphi} w^*, e_k]_{H_Q}, [\Gamma_{T,\varphi} v_n^*, e_k]_{H_Q} \rangle_{L^2([0,T])}\right|\\ &\le \sum_{k=m}^{\infty} \left|\langle [\Gamma_{T,\varphi} w^*, e_k]_{H_Q}, [\Gamma_{T,\varphi} v_n^*, e_k]_{H_Q} \rangle_{L^2([0,T])}\right|\\ &\le \sum_{k=m}^{\infty} \|[\Gamma_{T,\varphi} w^*, e_k]_{H_Q}\|_{L^2([0,T])} \|[\Gamma_{T,\varphi} v_n^*, e_k]_{H_Q}\|_{L^2([0,T])}\\ &\le \left(\sum_{k=m}^{\infty} \|[\Gamma_{T,\varphi} w^*, e_k]_{H_Q}\|^2_{L^2([0,T])}\right)^{1/2} \left(\sum_{k=m}^{\infty} \|[\Gamma_{T,\varphi} v_n^*, e_k]_{H_Q}\|^2_{L^2([0,T])}\right)^{1/2}\end{aligned}$$

$$\le \left(\sum_{k=m}^{\infty} \|[\Gamma_{T,\varphi} w^*, e_k]_{H_Q}\|^2_{L^2([0,T])}\right)^{1/2} \left(\sum_{k=1}^{\infty} \|[\Gamma_{T,\varphi} v_n^*, e_k]_{H_Q}\|^2_{L^2([0,T])}\right)^{1/2}. \tag{6.8}$$

The second factor equals $\|\Gamma_{T,\varphi} v_n^*\|_{L^2([0,T];H_Q)}$ because by Fubini-Tonelli theorem we can write

$$\begin{aligned}\|\Gamma_{T,\varphi} v_n^*\|^2_{L^2([0,T];H_Q)} &= \int_0^T \|\Gamma_{T,\varphi} v_n^*\|^2_{H_Q}\, \mathrm{d}t\\ &= \int_0^T \sum_{k=1}^{\infty} [\Gamma_{T,\varphi} v_n^*, e_k]^2_{H_Q}\, \mathrm{d}t\\ &= \sum_{k=1}^{\infty} \int_0^T [\Gamma_{T,\varphi} v_n^*, e_k]^2_{H_Q}\, \mathrm{d}t\\ &= \sum_{k=1}^{\infty} \|[\Gamma_{T,\varphi} v_n^*, e_k]_{H_Q}\|^2_{L^2([0,T])}.\end{aligned}$$

Using Proposition 6.1.6 and the definition of $\mathscr{B}_T$, the second factor of (6.8) can be bounded as follows

$$\begin{aligned}\|\Gamma_{T,\varphi} v_n^*\|_{L^2(0,T;H_Q)} &= \|i_Q^* \varphi^*(\cdot) v_n^*\rangle\|_{\mathcal{H}_T}\\ &\le C' \|v_n^*\|_{V^*}\\ &\le C' \sup_{n\in\mathbb{N}} \|v_n^*\|_{V^*} = C < \infty,\end{aligned}$$

where the last bound is finite because (v_k^*) is weakly* convergent and therefore the supremum is bounded. Moreover C does not depend on n. On the other hand, we know that $\Gamma_{T,\varphi}w^*$ is an element of $L^2([0,T],H_Q)$ which means

$$\|\Gamma_{T,\varphi}w^*\|^2_{L^2([0,T],H_Q)} = \sum_{k=1}^{\infty} \|[\Gamma_{T,\varphi}w^*, e_k]_{H_Q}\|^2_{L^2([0,T])} < \infty.$$

Therefore, for any $\varepsilon > 0$ given, there exists a positive number N such that for all $m > N$ we have

$$\sum_{k=m}^{\infty} \|[\Gamma_{T,\varphi}w^*, e_k]_{H_Q}\|^2_{L^2([0,T])} < \frac{\varepsilon^2}{C^2}.$$

For this choice of N and for all $m > N$ we therefore have

$$\begin{aligned}&\left|\sum_{k=m}^{\infty} \langle [\Gamma_{T,\varphi}w^*, e_k]_{H_Q}, [\Gamma_{T,\varphi}v_n^*, e_k]_{H_Q}\rangle_{L^2([0,T])}\right| \\ \leq &\left(\sum_{k=m}^{\infty} \|[\Gamma_{T,\varphi}w^*, e_k]_{H_Q}\|^2_{L^2([0,T])}\right)^{1/2} \|\Gamma_{T,\varphi}v_n^*\|_{L^2([0,T];H_Q)} \leq \frac{\varepsilon}{C}C = \varepsilon\end{aligned}$$

for arbitrary ε and N independent of n, that is the series is convergent uniformly in n. □

Proposition 6.2.4. [1] *If $\varphi \in \mathscr{B}_T$ then the integral $\mathcal{I}_T(\varphi)$ is a zero-mean Gaussian cylindrical random variable. Moreover the covariance operator $Q_{T,\varphi} : V^* \to V^{**}$ can be decomposed through $Q_{T,\varphi} = \Gamma^*_{T,\varphi}\Gamma_{T,\varphi}$ and is actually V-valued.*

Proof. The integral $\mathcal{I}_T(\varphi)v^*$ is a zero-mean Gaussian random variable for all $v^* \in V^*$ because it is a linear combination of independent zero-mean Gaussian random variables on $\mathbb{R}$ and it is well defined as random variable in $L^2_{\mathbb{P}}(\Omega;\mathbb{R})$. By similar computations as done in the proof of Theorem 6.2.2, part (iii), we obtain the following expression for the covariance operator $Q_{T,\varphi} : V^* \to V^{**}$

$$\begin{aligned}(Q_{T,\varphi}v^*)w^* &= \mathbb{E}[\mathcal{I}_T(\varphi)v^*\mathcal{I}_T(\varphi)w^*] \\ &= \mathbb{E}\left[\left(\sum_{k=1}^{\infty}\int_0^T \langle\varphi(t)i_Q e_k, v^*\rangle \mathrm{d}b_k^H(t)\right)\left(\sum_{j=1}^{\infty}\int_0^T \langle\varphi(t)i_Q e_j, w^*\rangle \mathrm{d}b_j^H(t)\right)\right]\end{aligned}$$

[1] After the academic defence, the author was informed about an incorrectness in the proof of the fact that $Q_{T,\varphi}$ is V-valued. Fore more details see [37].

$$= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \mathbb{E}\left[\int_0^T \langle \varphi(t) i_Q e_k, v^*\rangle \mathrm{d}b_k^H(t) \int_0^T \langle \varphi(t) i_Q e_j, w^*\rangle \mathrm{d}b_j^H(t)\right]$$
$$= \sum_{k=1}^{\infty} \int_0^T [\Gamma_{T,\varphi}(v^*), e_k]_{H_Q} [\Gamma_{T,\varphi}(w^*), e_k]\, \mathrm{d}t$$
$$= \langle \Gamma_{T,\varphi} v^*, \Gamma_{T,\varphi} w^*\rangle_{L^2([0,T];H_Q)} = \langle \Gamma_{T,\varphi}^* \Gamma_{T,\varphi} v_n^*, w^*\rangle_{V^*,V^{**}},$$

which gives us the decomposition $Q_{T,\varphi} = \Gamma_{T,\varphi}^* \Gamma_{T,\varphi} : V^* \to V^{**}$.
In order to show that $Q_{T,\varphi}$ is actually V-valued (with V separable Banach space) we use Corollary 2.7.10 in [50] and show, instead, that for each $w^* \in V^*$ and each sequence $(v_n^*)_{n\in\mathbb{N}} \subset V^*$ which is weakly* convergent to zero, then

$$\lim_{n\to\infty} \langle Q_{T,\varphi} w^*, v_n^*\rangle = 0. \tag{6.9}$$

Recall that $v_n^* \overset{*}{\rightharpoonup} 0$ if and only if $\langle v_n^*, z\rangle \to 0$ for all $z \in V$. By the previous computations we have

$$\begin{aligned}\langle Q_{T,\varphi} w^*, v_n^*\rangle =& \langle \Gamma_{T,\varphi}(w^*), \Gamma_{T,\varphi}(v_n^*)\rangle_{L^2([0,T];H_Q)} \\ =& \sum_{k=1}^{\infty} \int_0^T [\Gamma_{T,\varphi}(v_n^*), e_k]_{H_Q} [\Gamma_{T,\varphi}(w^*), e_k]\, \mathrm{d}t,\end{aligned}$$

where the series is uniformly convergent in n by Lemma 6.2.3. Therefore we get

$$\begin{aligned}\lim_{n\to\infty} \langle Q_{T,\varphi} w^*, v_n^*\rangle =& \sum_{k=1}^{\infty} \lim_{n\to\infty} \int_0^T [\Gamma_{T,\varphi}(w^*), e_k]_{H_Q} [\Gamma_{T,\varphi}(v_n^*), e_k]\, \mathrm{d}t \\ \le& \sum_{k=1}^{\infty} \lim_{n\to\infty} \left(\int_0^T [\Gamma_{T,\varphi}(w^*), e_k]_{H_Q}^2\, \mathrm{d}t\right)^{1/2} \\ & \cdot \left(\int_0^T [\Gamma_{T,\varphi}(v_n^*), e_k]_{H_Q}^2\, \mathrm{d}t\right)^{1/2} \\ =& \sum_{k=1}^{\infty} \left(\int_0^T [\Gamma_{T,\varphi}(w^*), e_k]_{H_Q}^2\, \mathrm{d}t\right)^{1/2} \\ & \cdot \left(\lim_{n\to\infty} \int_0^T [\Gamma_{T,\varphi}(v_n^*), e_k]_{H_Q}^2\, \mathrm{d}t\right)^{1/2}.\end{aligned} \tag{6.10}$$

Recall that $[\Gamma_{T,\varphi}(v_n^*), e_k]_{H_Q}^2 \le \|\Gamma_{T,\varphi}(v_n^*)\|_{L^2([0,T];H_Q)}^2 = \|i_Q^* \varphi^*(\cdot) v_n^*\|_{\mathscr{H}_T}^2$. The hypothesis $\varphi \in \mathcal{B}_T$ gives us

$$\|i_Q^* \varphi^*(\cdot) v_n^*\|_{\mathscr{H}_T} \le C' \|v_n^*\|_{V^*} \le C' \sup_{n\in\mathbb{N}} \|v_n^*\|_{V^*},$$

where the supremum is bounded because $v_n^* \overset{*}{\rightharpoonup} 0$. Thus, we have

$$[\Gamma_{T,\varphi}(v_n^*), e_k]_{H_Q}^2 \leq C$$

for some positive constant $C < \infty$. The dominated convergence theorem can be applied and we get

$$\begin{aligned}
\lim_{n\to\infty} \int_0^T [\Gamma_{T,\varphi}(v_n^*), e_k]_{H_Q}^2 \, \mathrm{d}t &= \int_0^T \lim_{n\to\infty} [\mathbb{K}_T^*(i_Q^* \varphi^*(\cdot) v_n^*), e_k]_{H_Q}^2 \, \mathrm{d}t \\
&= \int_0^T \lim_{n\to\infty} \left(K_T^* [i_Q^* \varphi^*(\cdot) v_n^*, e_k]_{H_Q} \right)^2 \, \mathrm{d}t \\
&= \int_0^T \left(K_T^* \lim_{n\to\infty} \langle v_n^*, \underbrace{\varphi(\cdot) i_Q e_k}_{\in V} \rangle \right)^2 \, \mathrm{d}t,
\end{aligned}$$

where the last equality holds by continuity of the operator K_T^*. Since by assumption $(v_n^*)_{n\in\mathbb{N}}$ is weakly* convergent to zero, then

$$\lim_{n\to\infty} \langle v_n^*, \varphi(\cdot) i_Q e_k \rangle = 0$$

and $K_T^* 0 = 0$. Together with (6.10), this gives now the desired result (6.9) and the proof is complete. □

Theorem 6.2.5. *Suppose $\varphi \in \mathscr{B}_T$. Then the integral $\mathcal{I}_T(\varphi)$ is induced by a classical random variable $Z_T(\varphi)$ in V if and only if the operator $\Gamma_{T,\varphi}^*$ is γ-radonifying.*
In this case the random variable $Z_T(\varphi)$ is Gaussian and we have

$$\langle Z_T(\varphi), v^* \rangle = \mathcal{I}_T(\varphi) v^*$$

for all $v^* \in V^*$.

Proof. The assumption $\varphi \in \mathscr{B}_T$ ensures that the operator $\Gamma_{T,\varphi}$ is continuous and that $\Gamma_{T,\varphi}^*$ takes values in V. Another consequence is that $\Gamma_{T,\varphi}^{**} = \Gamma_{T,\varphi}$ and so $\Gamma_{T,\varphi}^* \Gamma_{T,\varphi}^{**} = \Gamma_{T,\varphi}^* \Gamma_{T,\varphi}$.
By Proposition 6.2.4 we have the decomposition $Q_{T,\varphi} = \Gamma_{T,\varphi}^* \Gamma_{T,\varphi}$ and by Theorem 5.1.9 we have that $\Gamma_{T,\varphi}^*$ is γ-radonifying if and only if the operator $\Gamma_{T,\varphi}^* \Gamma_{T,\varphi} : V^* \to V$ is the covariance operator of a Gaussian measure μ on $\mathcal{B}(V)$. Let $Z_T(\varphi)$ be the Gaussian random variable in V with probability distribution μ and assume it is on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Thus we have

$$\mathbb{E}[\langle Z_T(\varphi), v^* \rangle] = 0 = \mathbb{E}[\mathcal{I}_T(\varphi) v^*]$$

for all $v^* \in V^*$, and

$$\mathbb{E}[\langle Z_T(\varphi), v^*\rangle\langle Z_T(\varphi), w^*\rangle] = \langle Q_{T,\varphi}v^*, w^*\rangle = \mathbb{E}[\mathcal{I}_T(\varphi)v^*\mathcal{I}_T(\varphi)w^*]$$

for all $v^*, w^* \in V^*$, which together with the Gaussian property gives us

$$\langle Z_T(\varphi), v^*\rangle = \mathcal{I}_T(\varphi)v^*$$

for all $v^* \in V^*$. □

6.2.3 The cylindrical integral as a stochastic process

In this section we introduce the indefinite cylindrical integral $\int_0^t \varphi(s)dB^H(s)$ for all $t \in [0,T]$. As done for the indefinite integral in $\mathbb{R}$, we need to restrict the set of integrable functions to all those $\varphi \in \mathscr{I}_T$ such that $\mathbb{1}_{[0,t)}\varphi \in \mathscr{I}_T$. But we are interested mainly in the case when the cylindrical integral is induced by a classical random variable, therefore we will consider only the space $\mathscr{B}_T$ instead of the more general one $\mathscr{I}_T$.

Definition 6.2.6. *Let $\varphi \in \mathscr{B}_T$ be such that $\mathbb{1}_{[0,t)}\varphi \in \mathscr{B}_T$ for all $t \in [0,T]$. Then define the integral*

$$\int_0^t \varphi(s)\mathrm{d}B^H(s) := \int_0^T \mathbb{1}_{[0,t)}(s)\varphi(s)\mathrm{d}B^H(s)$$

for all $t \in [0,T]$.

We have that $\mathcal{I}_T(\mathbb{1}_{[0,t)}\varphi)$ is a cylindrical random variable on V.
The integral is well defined because $\mathbb{1}_{[0,t)}\varphi \in \mathscr{B}_T$ and in fact it coincides with the integral $\mathcal{I}_t(\varphi)$ defined directly using $K_t^*, \mathbb{K}_t^*, \mathscr{I}_t$ and $\mathscr{B}_t$ (where they are defined in an analogous way as their correspondents). We see this with the help of the following result.

Proposition 6.2.7. *Let $f \in \mathscr{H}_t$. Then for all $s \in [0,T]$ we have*

$$\mathbb{1}_{[0,t)}(s)(\mathbb{K}_t^* f)(s) = \mathbb{K}_T^*(\mathbb{1}_{[0,t)}f)(s).$$

Proof. The proof is similar to the one of Proposition 4.2.3, with Bochner integrals instead of Lebesgue integrals. □

Proposition 6.2.8. *For φ as in Definition 6.2.6, we have*

$$\mathcal{I}_t(\varphi) = \mathcal{I}_T(\mathbb{1}_{[0,t)}\varphi)$$

as cylindrical random variable in V. In particular it is a strongly Gaussian centered random variable with covariance $Q_{t,\varphi} = Q_{T,\mathbb{1}_{[0,t)}\varphi}$.

Proof. We already know that they are Gaussian random variables with zero mean. Then observe that for a function $\varphi : [0,T] \to \mathcal{L}(U,V)$ we have $(\mathbb{1}_{[0,t)}\varphi)^*(s) = \mathbb{1}_{[0,t)}\varphi^*(s)$ for all $s \in [0,T]$. Let $\varphi \in \mathscr{B}_T$ such that $\mathbb{1}_{[0,t)}\varphi \in \mathscr{B}_T$ for every $t \in [0,T]$. By Proposition 6.2.7 we have that

$$\begin{aligned}
\langle Q_{T,\mathbb{1}_{[0,t)}\varphi}v^*, v^*\rangle &= \mathbb{E}\left|\int_0^T \mathbb{1}_{[0,t)}(s)\varphi(s)\,\mathrm{d}B^H(s)v^*\right|^2 \\
&= \|\mathbb{K}_T^*(i_Q^*(\mathbb{1}_{[0,t)}\varphi)^*(\cdot)v^*)\|^2_{L^2([0,T];H_Q)} \\
&= \|\mathbb{K}_T^*(\mathbb{1}_{[0,t)}i_Q^*\varphi^*(\cdot)v^*)\|^2_{L^2([0,T];H_Q)} \\
&= \|\mathbb{1}_{[0,t)}K_t^*(i_Q^*\varphi^*(\cdot)v^*)\|^2_{L^2([0,T];H_Q)} \\
&= \|K_t^*(i_Q^*\varphi^*(\cdot)v^*)\|^2_{L^2([0,t];H_Q)} \\
&= \mathbb{E}\left|\int_0^t \varphi(s)\,\mathrm{d}B^H(s)v^*\right|^2 \\
&= \langle Q_{t,\varphi}v^*, v^*\rangle.
\end{aligned}$$

The same computations yield $\mathcal{I}_t(\varphi)v^* = \mathcal{I}_T(\mathbb{1}_{[0,t)}\varphi)v^*$ in $L^2_{\mathbb{P}}(\Omega;\mathbb{R})$ for all $v^* \in V^*$ and all $t \in [0,T]$. □

Using this result it is immediate to prove the following corollary.

Corollary 6.2.9. *For any φ such that $\mathbb{1}_{[0,t)}\varphi \in \mathscr{B}_T$ for all $t \in [0,T]$, the cylindrical process $\{\mathcal{I}_t(\varphi), t \in [0,T]\}$ is a family of Gaussian cylindrical random variables in V and for each $t \in [0,T]$ the covariance operator of $\mathcal{I}_t(\varphi)$ is given by $\Gamma_{t,\varphi}^*\Gamma_{t,\varphi}$.*

Chapter 7

Stochastic (partial) differential equations in Banach spaces

In this chapter we apply the integration theory with respect to cylindrical fractional Brownian motions to solve Cauchy problems in Banach spaces. As an application we consider a stochastic parabolic equation in $L^2(D)$ driven by a fractional space-time noise.

7.1 Cylindrical evolution equation

7.1.1 Weak solutions in Banach spaces

The are various notions of solution to an evolution equation in infinite dimensional spaces. In this section we recall the notion of weak solution as this is going to be used later. We refer to [77] for more details.
Consider the problem

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} = Au + f(t) \\ u(0) = x \end{cases} \tag{7.1}$$

where $A : \mathcal{D}(A) \subseteq U \to U$ is a closed densely defined linear operator on U and $f \in L^1([0,T];U)$ is given. Define the adjoint of A, that is a linear operator on U^*, $A^* : \mathcal{D}(A^*) \subseteq U^* \to U^*$ possibly unbounded.

A *strong solution* of (7.1) is a function $u \in L^1([0,T];U)$ such that for all

$t \in [0,T]$ we have $\int_0^t u(s)\mathrm{d}s \in \mathcal{D}(A)$ and

$$u(t) = x + A\int_0^t u(s)\,\mathrm{d}s + \int_0^t f(s)\,\mathrm{d}s.$$

A *weak solution* of (7.1) is a function $u \in L^1([0,T];U)$ such that for all $t \in [0,T]$ and $v^* \in \mathcal{D}(A^*)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u(t), v^*\rangle = \langle u(t), A^*v^*\rangle + \langle f(t), v^*\rangle.$$

Observe that the integral form of this equation from 0 to t is given by

$$\langle u(t), v^*\rangle = \langle x, v^*\rangle + \int_0^t \langle u(r), A^*v^*\rangle\,\mathrm{d}r + \int_0^t \langle f(r), v^*\rangle\,\mathrm{d}r. \tag{7.2}$$

Remark 7.1. Every weak solution of (7.1) is a strong solution.

For the proof of next result we refer to [77], Theorem 7.17.

Theorem 7.1.1. *For each $x \in U$ and $f \in L^1([0,T],U)$ the problem* (7.1) *a unique weak solution u which is given by*

$$u(t) = S_t x + \int_0^t S_{t-s}f(s)\mathrm{d}s, \quad 0 \le t \le T.$$

7.1.2 The Cauchy problem in Banach spaces with fractional noise

In this section we consider evolution equations in Banach spaces driven by fractional Brownian motions. We give a meaning to the equations using cylindrical processes.

Let B^H be a cylindrical fBm in a separable Banach space U, A a linear operator on (another) separable Banach space V such that it is the generator of a strongly continuous semigroup $(S_t)_{t\ge 0}$ on V, $C \in \mathcal{L}(U,V)$ and Y_0 a cylindrical random variable in V such that the map $Y_0 : V^* \to L^0_{\mathbb{P}}(\Omega;\mathbb{R})$ is continuous with the topology of the convergence in probability.

In general, if X is a cylindrical variable in U and $T : D(T) \subset U \to V$ is a closed densely defined linear operator, we can define a linear map with domain $D(T^*)$ by

$$TX : D(T^*) \subset V^* \to L^0_{\mathbb{P}}(\Omega;\mathbb{R}), \quad (TX)(a) := X(T^*a).$$

Clearly if $D(T^*) = V^*$ then the map TX defines a cylindrical random variable in V.

Lemma 7.1.2. *Let B^H be a cylindrical fBm in U and $C \in \mathcal{L}(U,V)$. Then $\{CB^H(t), t \geq 0\}$ is a cylindrical fBm in V with covariance function CQC^*.*

Proof. By definition $CB^H(t)v^* = B^H(C^*v^*)$ for all $v^* \in V^*$, therefore it is well defined as cylindrical process. From the representation $B^H(t)u^* = \sum_{k=1}^{\infty} \langle i_Q e_k, u^* \rangle b_k^H(t)$ we have that

$$\begin{aligned} CB^H(t)v^* &= B^H(C^*v^*) \\ &= \sum_{k=1}^{\infty} \langle i_Q e_k, C^* v^* \rangle b_k^H(t) \\ &= \sum_{k=1}^{\infty} \langle C i_Q e_k, v^* \rangle b_k^H(t) \end{aligned}$$

for all $v^* \in V^*$. The operator $Ci_Q : H_Q \to V$ is linear and continuous so by Theorem 5.2.3 we have that CB^H is a fBm in V and the covariance operator is given by $Ci_Q(Ci_Q)^* = CQC^*$. □

In this setting we give the definition of cylindrical weak solution for the Cauchy problem

$$(7.3) \qquad \begin{cases} dY(t) &= AY(t)\mathrm{d}t + C\mathrm{d}B^H(t), \quad t \in (0,T] \\ Y(0) &= Y_0 \end{cases}$$

inspired by the integral form (7.2) of the weak solution for the deterministic evolution equation.

Definition 7.1.3. *A cylindrical process $\{Y(t), t \in [0,T]\}$ in V is called* weak cylindrical solution *of (7.3) if*

$$Y(t)v^* = Y_0 v^* + \int_0^t AY(r)v^* \, \mathrm{d}r + CB^H(t)v^*$$

for all $v^* \in D(A^*)$ $\mathbb{P}$-a.s.

We can recover the analogous sufficient conditions for the existence of a weak solution given in Theorem 7.1.1, but we need an additional condition for the cylindrical integral to be well defined. Let us make the following assumption:

(A) *For $H \in (0,1/2)$ let the $S^*_{t-\cdot} \in \mathscr{C}^\beta([0,t];\mathcal{L}(V^*))$ for some $\beta > 1/2 - H$ and for all $t \geq 0$. For $H \in (1/2,1)$ let the semigroup $(S_t)_{t\geq 0}$ be uniformly bounded.*

Assumption (A) is satisfied for the Dirichlet heat semigroup $(P_t)_{t\geq 0}$ on $L^2(D)$.

Theorem 7.1.4. *For every Cauchy problem of the form (7.3) such that assumption (A) holds there exists a unique cylindrical weak solution* $\{Y(t), t \in [0,T]\}$ *given by*

$$Y(t) = S_t Y_0 + \int_0^t S_{t-s} C \, \mathrm{d}B^H(s)$$

for all $t \in [0,T]$.

Proof. Let us consider the stochastic convolution integral which is a cylindrical process in V given by

$$\left(\int_0^t S_{t-s} C \mathrm{d}B^H(s)\right) v^* = \left(\int_0^T \mathbb{1}_{[0,t)}(s) S_{t-s} C \mathrm{d}B^H(s)\right) v^*,$$

for all $t \in [0,T]$ and all $v^* \in V^*$.
The integral is well defined if $\mathbb{1}_{[0,t)}(s) S_{t-s} C \in \mathscr{I}_T$ for all $t \in [0,T]$ and this condition is verified under the assumption *(A)*. In fact a stronger condition is verified, that is

$$\mathbb{1}_{[0,t)}(s) S_{t-s} C \in \mathscr{B}_T \text{ for all } t \in [0,T]. \tag{7.4}$$

To see this, let us consider the cases $H < 1/2$ and $H > 1/2$ separately.
Case $H \in (0, 1/2)$. By Proposition 6.1.4 we know that (7.4) is satisfied if $C^* S^*_{t-\cdot} \in \mathscr{C}^\beta([0,t]; \mathcal{L}(V^*, U^*))$ and since C is a linear and continuous operator it is enough to have $S^*_{t-\cdot} \in \mathscr{C}^\beta([0,t]; \mathcal{L}(V^*))$ which is true by assumption.
Case $H \in (1/2, 1)$. By Proposition 6.1.4 we know that (7.4) is satisfied if $C^* S^*_{t-\cdot} \in L^2([0,T]; \mathcal{L}(V^*, U^*))$ which is true if $S^*_{t-\cdot} \in L^2([0,t]; \mathcal{L}(V^*))$ for all $t \in [0,T]$. To verify this condition, observe that since $(S_t)_{t\geq 0}$ is uniformly bounded, so $(S^*_t)_{t\geq 0}$ is, and therefore there exist $M > 0$ such that $\|S^*_t\|_{\mathcal{L}(V^*)} \leq M$. Using this bound we easily compute

$$\int_0^t \|S^*_{t-s}\|^2_{\mathcal{L}(V^*)} \, \mathrm{d}s = \int_0^t \|S^*_s\|^2_{\mathcal{L}(V^*)} \, \mathrm{d}s \leq \int_0^t M^2 \, \mathrm{d}s = tM^2 < \infty.$$

Now set $X(t) := \int_0^t S_{t-s} C \mathrm{d}B^H(s)$ and $Z(t) := S_t Y_0$ and we show that $Y = Z + X$ is a weak cylindrical solution of (7.3), that is Y satisfies

$$(Z(t)+X(t))v^* = Y_0 v^* + \int_0^t AZ(s) v^* \, \mathrm{d}s + \int_0^t AX(s) v^* \, \mathrm{d}s + CB^H(t) v^* \tag{7.5}$$

for all $v^* \in D(A^*)$. The first integral on the RHS of (7.5) is a Bochner integral and gives us

$$\begin{aligned}\int_0^t AZ(s)v^*\,\mathrm{d}s &= \int_0^t AS_sY_0v^*\,\mathrm{d}s\\ &= \int_0^t Y_0(S_s^*A^*v^*)\,\mathrm{d}s\\ &= Y_0\int_0^t S_s^*A^*v^*\,\mathrm{d}s\\ &= Y_0(S_t^*v^* - S_0^*v^*)\\ &= S(t)Y_0v^* - Y_0v^* = Z(t)v^* - Y_0v^*\end{aligned}$$

where we have used the assumption on the continuity of Y_0. The second integral in the RHS of (7.5) is also a Bochner integral, but the integrand is a stochastic integral. Here we use a stochastic Fubini theorem for fBm (see Theorem 4.2.6) and we get

$$\begin{aligned}\int_0^t AX(s)v^*\,\mathrm{d}s &= \int_0^t X(s)(A^*v^*)\,\mathrm{d}s\\ &= \int_0^t \left(\int_0^s S_{s-r}C\,\mathrm{d}B^H(r)\right)(A^*v^*)\,\mathrm{d}s\\ &= \int_0^t \sum_{k=1}^\infty \int_0^s \langle S_{s-r}Ci_Qe_k, A^*v^*\rangle \mathrm{d}b_k^H(r)\mathrm{d}s\\ &= \sum_{k=1}^\infty \int_0^t \int_0^s \langle S_{s-r}Ci_Qe_k, A^*v^*\rangle \mathrm{d}b_k^H(r)\mathrm{d}s\\ &= \sum_{k=1}^\infty \int_0^t \int_r^t \langle S_{s-r}Ci_Qe_k, A^*v^*\rangle\,\mathrm{d}s\,\mathrm{d}b_k^H(r).\end{aligned}$$

For a moment consider only the inner integral. Since $S_{s-r}Ci_Qe_k \in D(A)$ we have

$$\begin{aligned}\int_r^t \langle S_{s-r}Ci_Qe_k, A^*v^*\rangle \mathrm{d}s &= \int_r^t \langle AS_{s-r}Ci_Qe_k, v^*\rangle \mathrm{d}s\\ &= \langle \int_r^t AS_{s-r}Ci_Qe_k\mathrm{d}s, v^*\rangle\\ &= \langle S_{s-r}Ci_Qe_k - S_0Ci_Qe_k, v^*\rangle.\end{aligned}$$

Therefore, we get

$$\begin{aligned}\int_0^t AX(s)v^*\,\mathrm{d}s &= \sum_{k=1}^{\infty}\int_0^t \langle S_{t-r}Ci_Qe_k, v^*\rangle\,\mathrm{d}b_k^H(r)\\ &\qquad\qquad - \sum_{k=1}^{\infty}\int_0^t \langle Ci_Qe_k, v^*\rangle\,\mathrm{d}b_k^H(r)\\ &= \left(\int_0^t S_{t-r}C\,\mathrm{d}B^H(r)\right)(v^*) - \sum_{k=1}^{\infty}\langle i_Qe_k, C^*v^*\rangle b_k^H(t)\\ &= X(t)v^* - B^H(t)(C^*v^*) = X(t)v^* - CB^H(t)v^*.\end{aligned}$$

Clipping the result together we get that the RHS of (7.5) equals

$$Y_0v^* + Z(t)v^* - Y_0v^* + X(t)v^* - CB^H(t)v^* + CB^H(t)v^*. \qquad \square$$

We now investigate the properties of the solution as cylindrical process. Observe that by Corollary 6.2.9 we have that $\mathcal{I}_t(S_{t-\cdot}C)$ is a zero-mean strongly Gaussian cylindrical random variable in V for each $t \in [0, T]$. The covariance operator is given by

$$Q_{t,S_{t-\cdot}C} = Q_{T,\mathbb{1}_{[0,t)}S_{t-\cdot}C} = \Gamma^*_{T,\mathbb{1}_{[0,t)}S_{t-\cdot}C}\Gamma_{T,\mathbb{1}_{[0,t)}S_{t-\cdot}C}.$$

Proposition 7.1.5. *If Y_0 is a centered Gaussian random variable independent of B^H then the process $\{Y(t), t \in [0, T]\}$ given as the weak solution to* (7.3) *is a cylindrical Gaussian family with covariance operator $Q_t^Y = Q_t^{SY_0} + Q_{t,S_{t-\cdot}C}$ where $Q_t^{SY_0}$ denotes the covariance operator of the cylindrical process S_tY_0.*

Proof. First observe that the solution Y is sum of two processes $Y(t) = S_tY_0 + \mathcal{I}_t(S_{t-\cdot}C)$. For each given $t \in [0, T]$ and $v^* \in V^*$, $S_tY_0v^* = Y_0(S_t^*v^*)$ is Gaussian variable because Y_0 is Gaussian and $S_t^* : V^* \to V^*$. Also $\mathcal{I}_t(S_{t-\cdot}C)v^*$ is Gaussian and so for each v^* the random variable $Y(t)v^*$ is Gaussian because linear combination of independent Gaussian variables $S_tY_0v^*$ and $\mathcal{I}_t(S_{t-\cdot}C)$. The mean is clearly zero. The covariance operator at time t for every $v^*, w^* \in V^*$ is given by

$$\begin{aligned}\langle Q_t^Y v^*, w^*\rangle &= \mathbb{E}[Y(t)v^*Y(t)w^*]\\ &= \mathbb{E}[(S_tY_0v^* + \mathcal{I}_t(S_{t-\cdot}C)v^*)(S_tY_0w^* + \mathcal{I}_t(S_{t-\cdot}C)w^*)]\end{aligned}$$

and by independence of Y_0 and B^H we get

$$\begin{aligned}\langle Q_t^Y v^*, w^*\rangle &= \mathbb{E}[S_t Y_0 v^* S_t Y_0 w^*] + \mathbb{E}[\mathcal{I}_t(S_{t-\cdot}C)v^* \mathcal{I}_t(S_{t-\cdot}C)w^*] \\ &= \langle Q_t^{SY_0} v^*, w^*\rangle + \langle Q_{t,S_{t-\cdot}C} v^*, w^*\rangle \\ &= \langle (Q_t^{SY_0} + Q_{t,S_{t-\cdot}C})v^*, w^*\rangle. \qquad \square\end{aligned}$$

We now turn our attention to the question of existence of classical solutions, namely under which conditions the cylindrical solution process Y is induced by a classical process in V. To this aim recall the following fact.

Remark 7.2. A cylindrical random variable $Y = X + Z$ is induced by a classical random variable in V if X and Z are induced by classical random variables in V.

Theorem 7.1.6. *The cylindrical process $\{Y(t), t \in [0,T]\}$ obtained as the weak solution of* (7.3) *is induced by a classical process in V if*

(a) Y_0 is induced by a classical random variable in V;

(b) the operator $\Gamma^*_{t,S_{t-\cdot}C}$ is γ-radonifying for all $t \in [0,T]$.

Proof. By Remark 7.2 we have that Y is induced by a classical random variable if and only if both summands are induced by classical random variables. For the first term we have

$$\langle \xi, v^*\rangle = S_t Y_0 v^* = Y_0(S_t^* v^*) = \langle \eta, S_t^* v^*\rangle = \langle S_t \eta, v^*\rangle$$

where ξ and η are V-valued random variables. We have $\xi = S_t\eta$ induces $S_t Y_0$ if and anly if η induces Y_0. The second summand verifies condition (b) simply by application of Theorem 6.2.5. $\square$

7.2 Applications: a stochastic parabolic equation with fractal noise

In this section we want to consider an example of SPDE driven by fractal noise and write it as an abstract Cauchy problem driven by a cylindrical fractional Brownian motion. After that, we look for cylindrical (mild and weak) solutions and give conditions under which the solution is actually induced by a classical process.

In particular, we consider the linear SPDE considered in [29] of the form

$$(7.6)\quad \begin{cases} \frac{\partial u}{\partial t}(t,x) = (-Au)(t,x) + \langle G, \frac{\partial}{\partial t}\nabla b^{H,K}\rangle(t,x), & t \in (0,T], x \in D \\ u(t,x) = 0, & t \in (0,T], x \in \partial D \\ u(0,x) = u_0(x), & x \in D \end{cases}$$

where here $\langle \cdot, \cdot \rangle$ denotes the pathwise integral introduced by Hinz and Zähle in [29]. In this example the space dimension is taken $d = 1$ therefore the matrix G reduces to a constant $G \in \mathbb{R}$ and $D \subset \mathbb{R}$. The noise $b^{H,K}$ is given by the process introduced in Section 5.3.2. We have proven that this noise can be written as a cylindrical fBm in $L^2(D)$ and we denote it by $\{B^H(t), t \geq 0\}$. The Hurst parameter is taken to be $H \in (1/2, 1)$. With this notation the SPDE (7.6) can be reformulated as follows:

$$(7.7)\quad \begin{cases} \mathrm{d}u(t) = -\Delta_D u(t)\mathrm{d}t + G\mathrm{d}B^H(t) & \text{for } t \in (0,T] \\ u(0) = u_0 & \text{for } t = 0. \end{cases}$$

Recall that we denote by $(P_t)_{t\geq 0}$ the semigroup in $L^2(D)$ generated by the Dirichlet Laplacian Δ_D (see Section 1.1.3 for more details). The semigroup satisfies condition (A), therefore the process given by

$$\begin{aligned} u(t) &= P_t u_0 + \int_0^T P_{t-s} G \,\mathrm{d}B^H(s) \\ &= P_t u_0 + G \int_0^T P_{t-s} \,\mathrm{d}B^H(s) \\ &= P_t u_0 + G\,\mathcal{I}_T(P_{t-\cdot}) \end{aligned}$$

for all $t \in [0,T]$, is a cylindrical process in $L^2(D)$ which is solution (weak and mild) to (7.7) according to Theorem 7.1.4. The initial condition u_0 is an element of $L^2(D)$.

Theorem 7.2.1. *If $\sum_{j=1}^{\infty} q_j^2 \lambda_j^{-\beta} < \infty$ for some $\beta \in (0,1)$ then the solution $\{u(t), t \in [0,T]\}$ is induced by a classical process in $L^2(D)$ for all $t \in [0,T]$.*

Lemma 7.2.2. *In this setting we have*

$$\|P_{t-\cdot} e_j\|_{L^2(D)} \leq \lambda_j^{-\beta/2} \|A^{\beta/2} P_{t-\cdot}\|_{\mathcal{L}(L^2(D))} < \infty$$

for all $\beta \in (0,1)$.

Proof. Using Theorem 1.1.11 part (a) and (b) we have

$$\begin{aligned}\|P_{t-\cdot}e_k\|_{L^2(D)} =&\|A^{\beta/2}A^{-\beta/2}P_{t-\cdot}e_j\|_{L^2(D)}\\ =&\|A^{\beta/2}P_{t-\cdot}A^{-\beta/2}e_j\|_{L^2(D)}\\ =&\|A^{\beta/2}P_{t-\cdot}\lambda_j^{-\beta/2}e_j\|_{L^2(D)}\\ \leq&\|A^{\beta/2}P_{t-\cdot}\|_{\mathcal{L}(L^2(D))}\|\lambda_j^{-\beta/2}e_j\|_{L^2(D)}\\ =&\lambda_j^{-\beta/2}\|A^{\beta/2}P_{t-\cdot}\|_{\mathcal{L}(L^2(D))}\end{aligned}$$

for all $\beta \in (0,1)$. The operator norm of $A^{\beta/2}P_{t-\cdot}$ is bounded by Theorem 1.1.11 part (c). □

Proof of Theorem 7.2.1. According to Theorem 7.1.6 we need to check that u_0 is a cylindrical Gaussian random variable and that $\Gamma^*_{t,P_{t-\cdot}}$ is γ-radonifying. The first condition is clearly verified because u_0 is deterministic. The second condition is equivalent to $\Gamma_{t,P_{t-\cdot}} \in \mathcal{L}_2(L^2(D); L^2([0,t]; H_Q))$ for all $t \in [0,T]$, because the underlying space is a Hilbert space.
We only have to show that $\|\Gamma_{t,P_{t-\cdot}}\|_{\mathcal{L}_2(L^2(D);L^2([0,t];H_Q))} < \infty$ for all $t \in [0,T]$. Let denote by $(e_k)_{k\in\mathbb{N}}$ a complete orthonormal system in $L^2(D)$ formed of eigenfunctions of the Dirichlet Laplacian, that is $\Delta_D e_k = \lambda_k e_k$. Recall that the covariance operator of B^H is decomposed through the reproducing kernel Hilbert space H_Q by i_Q. In this case (see Section 5.3.2) they are given by $H_Q = L^2(D)$ and $i_Q = i_Q^*$ defined by $i_Q e_k = q_k e_k$ for all $k \in \mathbb{N}$. The Hilbert-Schmidt norm gives us

$$\begin{aligned}\|\Gamma_{t,P_{t-\cdot}}\|^2_{\mathcal{L}_2(L^2(D);L^2([0,t];H_Q))} &= \sum_{k=1}^{\infty}\int_0^t \|\mathbb{K}_t^*(i_Q^*P_{t-s}^*e_k)\|^2_{H_Q}\mathrm{d}s\\ &= \int_0^t\sum_{k=1}^{\infty} \|\mathbb{K}_t^*(i_Q^*P_{t-s}^*e_k)\|^2_{H_Q}\mathrm{d}s\end{aligned}$$

for all $t \in [0,T]$, the last equality being true by Fubini-Tonelli theorem. For each fixed $k \in \mathbb{N}$ we have

$$\begin{aligned}
\|\mathbb{K}_t^*(i_Q^* P_{t-s}^* e_k)\|_{H_Q}^2 &= \sum_{j=1}^{\infty} [\mathbb{K}_t^*(i_Q^* P_{t-s}^* e_k), e_j]_{H_Q}^2 \\
&= \sum_{j=1}^{\infty} \left(K_t^* [i_Q^* P_{t-s}^* e_k, e_j]_{H_Q} \right)^2 \\
&= \sum_{j=1}^{\infty} \left(K_t^* [P_{t-s}^* e_k, i_Q e_j]_{H_Q} \right)^2 \\
&= \sum_{j=1}^{\infty} \left(K_t^* [e_k, P_{t-s} q_j e_j]_{H_Q} \right)^2 \\
&= \sum_{j=1}^{\infty} q_j^2 \left(K_t^* [e_k, P_{t-s} e_j]_{H_Q} \right)^2 \\
&= \sum_{j=1}^{\infty} q_j^2 [e_k, \mathbb{K}_t^* \left(P_{t-s} e_j \right)]_{H_Q}^2
\end{aligned}$$

which gives us

$$\begin{aligned}
\int_0^t \sum_{k=1}^{\infty} \|\mathbb{K}_t^*(i_Q^* P_{t-s}^*) e_k)\|_{H_Q}^2 \mathrm{d}s &= \int_0^t \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} q_j^2 [e_k, \mathbb{K}_t^* \left(P_{t-s} e_j \right)]_{H_Q}^2 \, \mathrm{d}s \\
&= \int_0^t \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} q_j^2 \|\mathbb{K}_t^* \left(P_{t-s} e_j \right)\|_{H_Q}^2 \, \mathrm{d}s \\
&\quad \text{Fubini-Tonelli} \\
&= \sum_{j=1}^{\infty} q_j^2 \int_0^t \sum_{k=1}^{\infty} \|\mathbb{K}_t^* \left(P_{t-s} e_j \right)\|_{H_Q}^2 \, \mathrm{d}s.
\end{aligned}$$

We have then

$$\begin{aligned}
\|\Gamma_{t,P_{t-\cdot}}\|_{\mathcal{L}_2(L^2(D);L^2([0,t];H_Q))}^2 &= \sum_{j=1}^{\infty} q_j^2 \|\mathbb{K}_t^* \left(P_{t-\cdot} e_j \right)\|_{L^2([0,T];H_Q)}^2 \\
&= \sum_{j=1}^{\infty} q_j^2 \|P_{t-\cdot} e_j\|_{\mathscr{H}_t}^2.
\end{aligned}$$

Consider for a moment only the norm part. Since $H \in (1/2, 1)$, we can bound the norm in $\mathscr{H}_t$ with $L^2([0,t]; L^2(D))$ (see Section 6.1.2) and get

$$\|P_{t-\cdot} e_j\|_{\mathscr{H}_t}^2 \leq \|P_{t-\cdot} e_j\|_{L^2([0,t];L^2(D))}^2 \leq \int_0^t \|P_{t-s} e_j\|^2 \mathrm{d}s.$$

The inner norm can be bounded using Lemma 7.2.2. Together with Theorem 1.1.11 part (c) we get

$$\begin{aligned}
\|P_{t-\cdot}e_j\|_{\mathscr{H}_t}^2 &\le \left\|\lambda_j^{-\beta/2}\|A^{\beta/2}P_{t-\cdot}\|_{\mathcal{L}(L^2(D))}\right\|_{L^2([0,t])}^2 \\
&= \lambda_j^{-\beta}\int_0^t \|A^{\beta/2}P_{t-s}\|_{\mathcal{L}(L^2(D))}^2 \,\mathrm{d}s \\
&\le \lambda_j^{-\beta}\int_0^t M_\beta^2 e^{-2\theta(t-s)}(t-s)^{-\beta}\,\mathrm{d}s \\
&\le \lambda_j^{-\beta} M_\beta^2 \theta^{\beta-1}\Gamma(1-\beta) = c\lambda_j^{-\beta}
\end{aligned}$$

where in the last inequality we applied Lemma 2.2.1. Combining this estimate with previous ones we get

$$\begin{aligned}
\|\Gamma_{t,P_{t-\cdot}}\|_{\mathcal{L}_2(L^2(D);L^2([0,t];H_Q))}^2 &= \sum_{j=1}^\infty q_j^2 \|P_{t-\cdot}e_j\|_{\mathscr{H}_t}^2 \\
&\le c\sum_{j=1}^\infty q_j^2\lambda_j^{-\beta} < \infty
\end{aligned}$$

where the last series converges by assumption. This proves that the operator $\Gamma_{t,P_{t-\cdot}}$ is Hilbert-Schmidt for all $t \in [0,T]$. □

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