Technische Universität Ilmenau
Institut für Mathematik

Preprint No. M **13/04**

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Thomas Berger

März 2013

Impressum:
Hrsg.: Leiter des Instituts für Mathematik
Weimarer Straße 25
98693 Ilmenau
Tel.: +49 3677 69-3621
Fax: +49 3677 69-3270
http://www.tu-ilmenau.de/math/

Zero dynamics and funnel control of general linear differential-algebraic systems*

Thomas Berger

Institute of Mathematics, Ilmenau University of Technology,
Weimarer Straße 25, 98693 Ilmenau, Germany
thomas.berger@tu-ilmenau.de

March 8, 2013

Abstract

We study linear differential-algebraic multi-input multi-output systems which are not necessarily regular and investigate the zero dynamics and tracking control. We use the concepts of autonomous zero dynamics and (E, A, B)-invariant subspaces to derive the so called zero dynamics form - which decouples the zero dynamics of the system - and exploit it for the characterization of system invertibility. Asymptotic stability of the zero dynamics is characterized and some implications for stabilizability in the behavioral sense are shown. A refinement of the zero dynamics form is then exploited to show that the funnel controller (that is a static nonlinear output error feedback) achieves - for a special class of right-invertible systems with asymptotically stable zero dynamics - tracking of a reference signal by the output signal within a pre-specified performance funnel. It is shown that the results can be applied to a class of passive electrical networks.

Keywords: Differential-algebraic systems, zero dynamics, invariant subspaces, stabilization, system inversion, funnel control, relative degree

1 Introduction

Differential-algebraic equations (DAEs) are a combination of differential equations along with algebraic constraints. They have been discovered as an appropriate tool for modeling many problems e.g. in mechanical multibody dynamics [16], electrical networks [41], and chemical engineering [29]. These problems indeed have in common that the dynamics are algebraically constrained, for instance by tracks, Kirchhoff laws, or conservation laws. As a result of the power in application, DAEs are nowadays an established field in applied mathematics and subject of various monographs and textbooks [12, 13, 14, 15, 20, 30]. In the present work, we consider questions related to the zero dynamics, stabilizability, and closed-loop control of linear constant coefficient DAEs with special emphasis on the non-regular case. The concepts of (E, A, B)-invariance, autonomous and asymptotically stable zero dynamics, stabilizability in the behavioral sense and system inversion are considered for the DAE case. We further show that the 'funnel controller' (developed in [24] for minimum-phase ordinary differential equation systems with strict relative degree one) achieves, for all right-invertible DAE systems with asymptotically stable zero dynamics for which the matrix Γ in (6.5) exists and satisfies $\Gamma = \Gamma^\top \geq 0$, tracking of a reference signal by the output signal within a pre-specified performance funnel.

*This work was supported by DFG grant Il25/9.

We consider linear constant coefficient DAEs of the form

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} Ex(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t)\,, \end{aligned} \tag{1.1}$$

where $E, A \in \mathbb{R}^{l\times n}$, $B \in \mathbb{R}^{l\times m}$, $C \in \mathbb{R}^{p\times n}$. The set of these systems is denoted by $\Sigma_{l,n,m,p}$ and we write $[E, A, B, C] \in \Sigma_{l,n,m,p}$. In the present paper, we put special emphasis on the non-regular case, i.e., we do not assume that $sE - A$ is *regular*, that is $l = n$ and $\det(sE - A) \in \mathbb{R}[s] \setminus \{0\}$.
The functions $u : \mathbb{R} \to \mathbb{R}^m$ and $y : \mathbb{R} \to \mathbb{R}^p$ are called *input* and *output* of the system, resp. A trajectory $(x, u, y) : \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$ is said to be a *solution* of (1.1) if, and only if, it belongs to the *behaviour* of (1.1):

$$\mathfrak{B}_{(1.1)} := \left\{ (x,u,y) \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n) \times \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^m) \times \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^p) \;\middle|\; \begin{array}{l} Ex \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^l) \text{ and } (x,u,y) \\ \text{solves (1.1) for almost all } t \in \mathbb{R} \end{array} \right\}.$$

Recall that any function $z \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^l)$ is in particular continuous and $\mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n)$ consists of equivalence classes of functions which are equal almost everywhere; each equality $f_1 = f_2$ for $f_1, f_2 \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n)$ is to be understood as a (set)-equality of equivalence classes. More smoothness of u and y is required for some results such as funnel control in Section 6.
In the present paper, we provide, in particular, a unified framework for two important classes of differential-algebraic systems which have been investigated in [5, 6]. These two classes encompass regular systems $[E, A, B, C] \in \Sigma_{n,n,m,m}$ for which the transfer function is defined by

$$G(s) = C(sE - A)^{-1}B \in \mathbb{R}(s)^{m\times m}.$$

The notions of properness and strict relative degree are required in the following.

Definition 1.1 (Properness and strict relative degree)**.**
A rational matrix function $G(s) \in \mathbb{R}(s)^{p\times m}$ is called *proper* if, and only if, $\lim_{s\to\infty} G(s) = D$ for some $D \in \mathbb{R}^{p\times m}$.
We say that a square matrix function $G(s) \in \mathbb{R}(s)^{m\times m}$ has *strict relative degree* $\rho \in \mathbb{Z}$ if, and only if,

$$\rho \;=\; \operatorname{sr deg} G(s) \;:=\; \sup\left\{ k \in \mathbb{Z} \;\middle|\; \lim_{s\to\infty} s^k\, G(s) \in \mathbf{Gl}_m(\mathbb{R}) \right\}$$

exists. ◇

Note that for any $G(s) \in \mathbb{R}(s)^{m,m}$ we have (consider the entries)

$$\lim_{s\to\infty} s^k\, G(s) = D \in \mathbb{R}^{m\times m} \quad \text{for some } k \in \mathbb{Z} \qquad \Longrightarrow \qquad \lim_{s\to\infty} s^{k-i}\, G(s) = 0 \quad \text{for all } i \in \mathbb{N}.$$

The notion of strict relative degree generalizes what is known for transfer functions of ODE systems $[I_n, A, B, C] \in \Sigma_{n,n,m,m}$:

$$G(s) = C(sI_n - A)^{-1}B = CBs^{-1} + CABs^{-2} + CA^2Bs^{-3} + \ldots\,,$$

has strict relative degree $\rho \in \mathbb{N}$ if, and only if,

$$CA^iB = 0 \quad \text{for} \quad i = 0, \ldots, \rho - 2 \quad \text{and} \quad CA^{\rho-1}B \in \mathbf{Gl}_m(\mathbb{R})\,.$$

From [6, Prop. 1.2] we have the following.

Lemma 1.2 (Non-positive strict relative degree implies proper inverse)**.**
For $G(s) \in \mathbb{R}(s)^{m,m}$ we have

$$\operatorname{sr\,deg} G(s) \le 0 \quad \underset{\text{i.g.}}{\overset{\Longrightarrow}{\not\Longleftarrow}} \quad G(s) \textit{ has proper inverse over } \mathbb{R}(s).$$

We are now in the position to define the following two system classes: The class

$$\Sigma^{\mathrm{pi}} := \left\{ [E,A,B,C] \in \Sigma_{n,n,m,m} \;\middle|\; \begin{array}{l} \det(sE-A) \in \mathbb{R}[s] \setminus \{0\} \text{ and} \\ \big(C(sE-A)^{-1}B\big)^{-1} \in \mathbb{R}(s)^{m\times m} \text{ exists and is proper} \end{array} \right\}$$

of regular systems with proper inverse transfer function has been investigated in [6]. The class

$$\Sigma^{\mathrm{rd1}} := \left\{ [E,A,B,C] \in \Sigma_{n,n,m,m} \;\middle|\; \begin{array}{l} \det(sE-A) \in \mathbb{R}[s] \setminus \{0\} \text{ and} \\ C(sE-A)^{-1}B \text{ has strict relative degree 1} \end{array} \right\}$$

of regular systems with strict relative degree one has been investigated in [5]. In the present paper, we investigate systems with autonomous zero dynamics. Loosely speaking, the zero dynamics are those dynamics of a system which are not visible at the output; and the zero dynamics are autonomous if any two trajectories coincide on $\mathbb{R}$ whenever they take the same values on an arbitrary small open interval $I \subseteq \mathbb{R}$; see Definition 3.1. Furthermore, right-invertibility of systems is treated, that is, loosely speaking, for any sufficiently smooth output y, the existence of a state x and an input u, such that $(x,u,y) \in \mathfrak{B}_{(1.1)}$; see Definition 5.1. We will show that, for $n,m \in \mathbb{N}_0$,

$$\Sigma^{\mathrm{pi}} \cup \Sigma^{\mathrm{rd1}} \subseteq \Sigma^{\mathrm{azd}} := \left\{ [E,A,B,C] \in \Sigma_{l,n,m,m} \;\middle|\; \begin{array}{l} l \in \mathbb{N}_0,\ [E,A,B,C] \text{ is right-invertible,} \\ \text{has autonomous zero dynamics} \\ \text{and } \Gamma \text{ in (6.5) exists} \end{array} \right\}.$$

We like to stress that regularity of $sE-A$ is no longer required in the class Σ^{azd}. We also show that the class Σ^{azd} includes all regular systems with a vector relative degree which is componentwise smaller or equal to 1, see Appendix B. This in particular encompasses systems with a "mixed relative degree", i.e., a vector relative degree with possibly different components. Remark 6.5 also shows that a class of passive electrical networks is encompassed: systems with invertible and positive real transfer function are included in Σ^{azd}.
We use the class Σ^{azd} to show that funnel control is feasible for a much larger class of systems than considered in [24] for ODEs and in [5, 6] for DAEs. More precise, we show that for

$$\Sigma^{\mathrm{pi}}_{\mathrm{funnel}} := \left\{ [E,A,B,C] \in \Sigma^{\mathrm{pi}} \;\middle|\; [E,A,B,C] \text{ has asymptotically stable zero dynamics} \right\}$$

and

$$\Sigma^{\mathrm{rd1}}_{\mathrm{funnel}} := \left\{ [E,A,B,C] \in \Sigma^{\mathrm{rd1}} \;\middle|\; \begin{array}{l} [E,A,B,C] \text{ has asymptotically stable zero dynamics} \\ \text{and } \Pi = \lim_{s\to\infty} sC(sE-A)^{-1}B \text{ satisfies } \Pi = \Pi^\top > 0 \end{array} \right\}$$

it holds that

$$\Sigma^{\mathrm{pi}}_{\mathrm{funnel}} \cup \Sigma^{\mathrm{rd1}}_{\mathrm{funnel}} \subseteq \Sigma^{\mathrm{azd}}_{\mathrm{funnel}} := \left\{ [E,A,B,C] \in \Sigma^{\mathrm{azd}} \;\middle|\; \begin{array}{l} [E,A,B,C] \text{ has asymptotically stable} \\ \text{zero dynamics and } \Gamma \text{ in (6.5) satisfies} \\ \Gamma = \Gamma^\top \ge 0 \end{array} \right\}.$$

The paper is organized as follows: In Section 2 we collect some preliminary results on matrix pencils, in particular the quasi-Kronecker form. In Section 3 we define the crucial concept of (autonomous)

zero dynamics and derive characterizations of autonomous zero dynamics in terms of a rank condition and the maximal (E, A, B)-invariant subspace included in $\ker C$. The latter also allows to derive the so called zero dynamics form in Theorem 3.7 - one of the main results of the paper - which decouples the zero dynamics of the system. In Section 4 the asymptotic stability of zero dynamics is defined and characterized as well as some implications for stabilizability in the behavioral sense are shown. The zero dynamics form is then refined in Section 5 and exploited for the characterization of system invertibility. The refinement of the zero dynamics form is also used to show feasibility of the funnel controller in Section 6, which is proved to work for the class $\Sigma^{\rm azd}_{\rm funnel}$ in Theorem 6.3 - the second main result of the present paper. In Section 7 we illustrate Theorem 6.3 by a simulation of the funnel controller for a system (1.1). Finally, in Appendix A some results on polynomial matrices and the zero dynamics form are derived, which are crucial for the proof of Theorem 6.3, and in Appendix B systems with a vector relative degree are related to the findings of the paper.
We close the introduction with the nomenclature used in this paper.

Nomenclature

$\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$	set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, set of all integers, resp.
$\ell(\alpha), \lvert\alpha\rvert$	length $\ell(\alpha) = l$ and absolute value $\lvert\alpha\rvert = \sum_{i=1}^{l} \alpha_i$ of a multi-index $\alpha = (\alpha_1, \dots, \alpha_l) \in \mathbb{N}^l$
$\mathbb{R}_{\geq 0}$	$= [0, \infty)$
$\mathbb{C}_+$, $\mathbb{C}_-$	the open set of complex numbers with positive, negative real part, resp.
$\mathbf{Gl}_n(\mathbb{R})$	the set of invertible real $n \times n$ matrices
$\mathbb{R}[s]$	the ring of polynomials with coefficients in $\mathbb{R}$
$\mathbb{R}(s)$	the quotient field of $\mathbb{R}[s]$
$R^{n \times m}$	the set of $n \times m$ matrices with entries in a ring R
$\sigma(A)$	the spectrum of a matrix $A \in \mathbb{R}^{n \times n}$
$\lVert x \rVert$	$= \sqrt{x^\top x}$, the Euclidean norm of $x \in \mathbb{R}^n$
$\lVert A \rVert$	$= \max\{\, \lVert A\,x \rVert \mid x \in \mathbb{R}^m, \lVert x \rVert = 1 \,\}$, induced matrix norm of $A \in \mathbb{R}^{n,m}$
$A^{-1}\mathcal{S}$	$= \{\, x \in \mathbb{R}^m \mid Ax \in \mathcal{S} \,\}$, the pre-image of the set $\mathcal{S} \subseteq \mathbb{R}^n$ under $A \in \mathbb{R}^{n,m}$
$\mathcal{L}^1_{\rm loc}(\mathcal{T}; \mathbb{R}^n)$	the set of locally Lebesgue integrable functions $f : \mathcal{T} \to \mathbb{R}^n$, see [1, Chap. 1]
$\dot f$ $(f^{(i)})$	the (i-th) weak derivative of $f \in \mathcal{L}^1_{\rm loc}(\mathcal{T}; \mathbb{R}^n)$, $i \in \mathbb{N}_0$, see [1, Chap. 1]
$\mathcal{W}^{k,1}_{\rm loc}(\mathcal{T}; \mathbb{R}^n)$	$= \left\{\, x \in \mathcal{L}^1_{\rm loc}(\mathcal{T}; \mathbb{R}^n) \;\middle|\; x^{(i)} \in \mathcal{L}^1_{\rm loc}(\mathcal{T}; \mathbb{R}^n) \text{ for } i = 0, \dots, k \,\right\}$, $k \in \mathbb{N}_0$
$\mathcal{L}^\infty(\mathcal{T}; \mathbb{R}^n)$	the set of essentially bounded functions $f : \mathcal{T} \to \mathbb{R}^n$, see [1, Chap. 2]
$\text{ess-sup}_{\mathcal{I}} \lVert f \rVert$	the essential supremum of the measurable function $f : \mathcal{T} \to \mathbb{R}^n$ over $\mathcal{I} \subseteq \mathcal{T}$
$\mathcal{C}^k(\mathcal{T}; \mathbb{R}^n)$	the set of k-times continuously differentiable functions $f : \mathcal{T} \to \mathbb{R}^n$
$\mathcal{B}^k(\mathcal{T}; \mathbb{R}^n)$	$= \left\{\, f \in \mathcal{C}^k(\mathcal{T}; \mathbb{R}^n) \;\middle|\; \frac{{\rm d}^i}{{\rm d}t^i} f \in \mathcal{L}^\infty(\mathcal{T}; \mathbb{R}^n) \text{ for } i = 0, \dots, k \,\right\}$
$f\vert_{\mathcal{J}}$	the restriction of the function $f : \mathcal{I} \to \mathbb{R}^n$ to $\mathcal{J} \subseteq \mathcal{I}$

2 Preliminaries

For convenience we call the extended matrix pencil $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ the *system pencil* of $[E, A, B, C] \in \Sigma_{l,n,m,p}$. In Section 3 we will derive a so called 'zero dynamics form' of $[E, A, B, C]$ within the equivalence class defined by:

Definition 2.1 (System equivalence).
Two systems $[E_i, A_i, B_i, C_i] \in \Sigma_{l,n,m,p}$, $i = 1, 2$, are called *system equivalent* if, and only if,

$$\exists\, S \in \mathbf{Gl}_l(\mathbb{R}), T \in \mathbf{Gl}_n(\mathbb{R}): \begin{bmatrix} S & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} sE_1 - A_1 & B_1 \\ C_1 & 0 \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} sE_2 - A_2 & B_2 \\ C_2 & 0 \end{bmatrix};$$

we write

$$[\, E_1\,,\, A_1\,,\, B_1\,,\, C_1\,] \overset{S,T}{\sim} [\, E_2\,,\, A_2\,,\, B_2\,,\, C_2\,]\,.$$

$\diamond$

It is easy to see that system equivalence is an equivalence relation on $\Sigma_{l,n,m,p} \times \Sigma_{l,n,m,p}$. The notion of system equivalence goes back to Rosenbrock [42].
We introduce the following notation: For $k \in \mathbb{N}$, we define the matrices

$$N_k = \begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \in \mathbb{R}^{k\times k}, \quad K_k = \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}, \; L_k = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} \in \mathbb{R}^{(k-1)\times k}.$$

For some multi-index $\alpha = (\alpha_1, \ldots, \alpha_l) \in \mathbb{N}^l$, we define

$$\begin{aligned} N_\alpha &= \operatorname{diag}(N_{\alpha_1}, \ldots, N_{\alpha_l}) \in \mathbb{R}^{|\alpha|\times|\alpha|}, \\ K_\alpha &= \operatorname{diag}(K_{\alpha_1}, \ldots, K_{\alpha_l}) \in \mathbb{R}^{(|\alpha|-l)\times|\alpha|}, \\ L_\alpha &= \operatorname{diag}(L_{\alpha_1}, \ldots, L_{\alpha_l}) \in \mathbb{R}^{(|\alpha|-l)\times|\alpha|}. \end{aligned} \tag{2.1}$$

Kronecker proved [28] that any matrix pencil $s\widehat{E} - \widehat{A} \in \mathbb{R}[s]^{\widehat{l}\times\widehat{n}}$ can be put into Kronecker canonical form; for a more comprehensive proof see Gantmacher [18]. In the following we may use the quasi-Kronecker form derived in [9, 10], since in general the Kronecker canonical form is complex-valued even though the given pencil $s\widehat{E} - \widehat{A}$ is real-valued, what we need to avoid. For regular matrix pencils this result has already been derived in [7].

Proposition 2.2 (Quasi-Kronecker form [18, 9, 10]).
For any matrix pencil $s\widehat{E} - \widehat{A} \in \mathbb{R}[s]^{\widehat{l}\times\widehat{n}}$ there exist $S \in \mathbf{Gl}_{\widehat{l}}(\mathbb{R})$, $T \in \mathbf{Gl}_{\widehat{n}}(\mathbb{R})$, $A_s \in \mathbb{R}^{n_s\times n_s}$, and $\alpha \in \mathbb{N}^{n_\alpha}$, $\beta \in \mathbb{N}^{n_\gamma}$, $\gamma \in \mathbb{N}^{n_\gamma}$ such that

$$S(s\widehat{E} - \widehat{A})T \;=\; \begin{bmatrix} sI_{n_s} - A_s & 0 & 0 & 0 \\ 0 & sN_\alpha - I_{|\alpha|} & 0 & 0 \\ 0 & 0 & sK_\beta - L_\beta & 0 \\ 0 & 0 & 0 & sK_\gamma^\top - L_\gamma^\top \end{bmatrix}. \tag{2.2}$$

The multi-indices α, β, γ are uniquely determined by $s\widehat{E} - \widehat{A}$. Further, the matrix A_s is unique up to similarity. $\diamond$

The (entries of the) multi-indices α, β, γ are often called minimal indices and elementary divisors and play an important role in the analysis of matrix pencils, see e.g. [18, 32, 33, 34], where the entries of α are the orders of the infinite elementary divisors, the entries of β are the column minimal indices and the entries of γ are the row minimal indices. $sI_{n_s} - A_s$ may be further transformed into Jordan canonical form to obtain the finite elementary divisors.
Since the multi-indices $\alpha \in \mathbb{N}^{n_\alpha}$, $\beta \in \mathbb{N}^{n_\gamma}$, $\gamma \in \mathbb{N}^{n_\gamma}$ are well-defined by the pencil $s\widehat{E} - \widehat{A}$ and, furthermore, the matrix A_s is unique up to similarity, this justifies the introduction of the following quantities.

Definition 2.3 (Index of $s\widehat{E} - \widehat{A}$)**.**
Let the matrix pencil $s\widehat{E} - \widehat{A} \in \mathbb{R}[s]^{\widehat{l} \times \widehat{n}}$ be given in quasi-Kronecker form (2.2). Then the *index* $\nu \in \mathbb{N}_0$ of $s\widehat{E} - \widehat{A}$ is defined as

$$\nu = \max\{\alpha_1, \ldots, \alpha_{\ell(\alpha)}, \gamma_1, \ldots, \gamma_{\ell(\gamma)}, 0\}.$$

$\diamond$

The index is larger or equal to the index of nilpotency ζ of N_α, i.e., $\zeta \leq \nu$, $N_\alpha^\zeta = 0$ and $N_\alpha^{\zeta-1} \neq 0$. Since each block in $sK_\beta - L_\beta$ ($sK_\gamma^\top - L_\gamma^\top$) causes a single drop of the column (row) rank of $sE - A$, resp., we have

$$\ell(\beta) = \widehat{n} - \mathrm{rk}_{\mathbb{R}(s)}(s\widehat{E} - \widehat{A}), \quad \ell(\gamma) = \widehat{l} - \mathrm{rk}_{\mathbb{R}(s)}(s\widehat{E} - \widehat{A}). \tag{2.3}$$

For later use we collect the following lemma.

Lemma 2.4 (Full column rank and quasi-Kronecker form)**.**
Let $s\widehat{E} - \widehat{A} \in \mathbb{R}[s]^{\widehat{l} \times \widehat{n}}$ and consider any quasi-Kronecker form (2.2) *of $s\widehat{E} - \widehat{A}$. Then $\ell(\beta) = 0$ if, and only if,* $\mathrm{rk}_{\mathbb{R}[s]}\, s\widehat{E} - \widehat{A} = \widehat{n}$.

Proof: The assertion is immediate from (2.3) and $\mathrm{rk}_{\mathbb{R}[s]}\, s\widehat{E} - \widehat{A} = \mathrm{rk}_{\mathbb{R}(s)}\, s\widehat{E} - \widehat{A}$. $\square$

3 Zero dynamics

In this section we introduce the central concept of zero dynamics for DAE systems (1.1) as well as the notion of autonomous zero dynamics. We derive several important characterizations of autonomous zero dynamics and, as the main result of this section, the so called zero dynamics form in Theorem 3.7.

Definition 3.1 (Zero dynamics)**.**
The *zero dynamics* of system (1.1) are defined as the set of trajectories

$$\mathcal{ZD}_{(1.1)} := \left\{ (x, u, y) \in \mathfrak{B}_{(1.1)} \;\middle|\; y = 0 \right\}.$$

The zero dynamics $\mathcal{Z}D_{(1.1)}$ are called *autonomous* if, and only if,

$$\forall\, w_1, w_2 \in \mathcal{Z}D_{(1.1)}\ \forall\, I \subseteq \mathbb{R} \text{ open interval}: \quad w_1|_I = w_2|_I \implies w_1 = w_2\,. \tag{3.1}$$

$\diamond$

Remark 3.2.
By linearity of (1.1), the set $\mathcal{ZD}_{(1.1)}$ is a real vector space. Therefore, the zero dynamics $\mathcal{Z}D_{(1.1)}$ are autonomous if, and only if, for any $w \in \mathcal{Z}D_{(1.1)}$ which satisfies $w|_I = 0$ on some open interval $I \subseteq \mathbb{R}$, it follows that $w = 0$.
The definition of autonomous zero dynamics is a special case of the definition of autonomy, as it has been introduced in [37, Sec. 3.2] for general behaviors. $\diamond$

In order to characterize (autonomous) zero dynamics we introduce the well-known concept of (E, A, B)-invariance, see [3, 4, 31, 34, 36].

Definition 3.3 ((E, A, B)-invariance)**.**
Let $(E, A, B) \in \mathbb{R}^{l\times n} \times \mathbb{R}^{l\times n} \times \mathbb{R}^{l\times m}$ and $\mathcal{V} \subseteq \mathbb{R}^n$ be a linear subspace. Then $\mathcal{V}$ is called (E, A, B)-*invariant* if, and only if,

$$A\mathcal{V} \subseteq E\mathcal{V} + \operatorname{im} B\,. \tag{3.2}$$

$\diamond$

For a system $[E, A, B, C] \in \Sigma_{l,n,m,p}$, we define the set of all (E, A, B)-invariant subspaces included in $\ker C$ by

$$\mathcal{L}(E, A, B; \ker C) := \{\ \mathcal{V} \subseteq \mathbb{R}^n\ \mid \mathcal{V} \text{ is } (E, A, B)\text{-invariant subspace of } \mathbb{R}^n \text{ and } \mathcal{V} \subseteq \ker C\ \}\,.$$

It can easily be verified that $\mathcal{L}(E, A, B; \ker C)$ is closed under subspace addition and thus $\mathcal{L}(E, A, B; \ker C)$ is an upper semi-lattice relative to subspace inclusion and addition. Hence, by [46, Lem. 4.4], there exists a supremal element of $\mathcal{L}(E, A, B; \ker C)$, namely

$$\max(E, A, B; \ker C)\ :=\ \sup \mathcal{L}(E, A, B; \ker C)\ =\ \max \mathcal{L}(E, A, B; \ker C).$$

We show that $\max(E, A, B; \ker C)$ can be derived from a sequence of subspaces which terminates after finitely many steps.

Lemma 3.4 (Subspace sequences leading to $\max(E, A, B; \ker C)$)**.**
Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$ and define $\mathcal{V}_0 := \ker C$ and

$$\forall\, i \in \mathbb{N}:\ \mathcal{V}_i := A^{-1}(E\mathcal{V}_{i-1} + \operatorname{im} B) \cap \ker C.$$

Then the sequence $(\mathcal{V}_i)$ is nested, terminates and satisfies

$$\exists\, k^* \in \mathbb{N}\ \forall\, j \in \mathbb{N}:\ \mathcal{V}_0 \supsetneq \mathcal{V}_1 \supsetneq \cdots \supsetneq \mathcal{V}_{k^*} = \mathcal{V}_{k^*+j} = A^{-1}(E\mathcal{V}_{k^*} + \operatorname{im} B) \cap \ker C. \tag{3.3}$$

Furthermore,

$$\mathcal{V}_{k^*} = \max(E, A, B; \ker C) \tag{3.4}$$

and, if $(x, u, y) \in \mathcal{ZD}_{(1.1)}$, then (for any representative of the equivalence class of x)

$$\textit{for almost all } t \in \mathbb{R}:\ x(t) \in \mathcal{V}_{k^*}.$$

Proof: It is easy to see that (3.3) holds true and (3.4) follows from [36, Lem. 2.1]. For the last statement let $(x, u, y) \in \mathcal{ZD}_{(1.1)}$. Then we have

$$Ax(t) = \tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) - Bu(t) \quad \text{and} \quad x(t) \in \ker C$$

for almost all $t \in \mathbb{R}$. Since, for any subspace $\mathcal{S} \subseteq \mathbb{R}^n$, if $x(t) \in \mathcal{S}$ for almost all $t \in \mathbb{R}$, then $\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) \in E\mathcal{S}$ for almost all $t \in \mathbb{R}$, we conclude

$$x(t) \in A^{-1}(\{\tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t)\} + \operatorname{im} B) \cap \ker C \subseteq \mathcal{V}_1 \text{ for almost all } t \in \mathbb{R}.$$

Inductively, we obtain $x(t) \in \mathcal{V}_{k^*}$ for almost all $t \in \mathbb{R}$. $\square$

The following result is a general version of [6, Prop. 4.3], which follows immediately from Lemma 3.4.

Proposition 3.5 (Characterization of zero dynamics).
Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$*. If* $(x, u, y) \in \mathfrak{B}_{(1.1)}$*, then*

$$(x, u, y) \in \mathcal{ZD}_{(1.1)} \quad \Longleftrightarrow \quad \Big[\ x(t) \in \max(E, A, B; \ker C) \quad \textit{for almost all } t \in \mathbb{R}\ \Big].$$

Next, we state some characterizations of autonomous zero dynamics in terms of a pencil rank condition (exploiting the quasi-Kronecker form) and some conditions involving the largest (E, A, B)-invariant subspace included in $\ker C$.

Proposition 3.6 (Characterization of autonomous zero dynamics).
Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$*. Then the following three statements are equivalent:*

(i) $\mathcal{Z}D_{(1.1)}$ *is autonomous.*

(ii) $\operatorname{rk}_{\mathbb{R}[s]} \begin{bmatrix} sE - A & -B \\ -C & 0 \end{bmatrix} = n + m.$

(iii) **(A1)** $\operatorname{rk} B = m$,

(A2) $\ker E \cap \max(E, A, B; \ker C) = \{0\}$,

(A3) $\operatorname{im} B \cap E \max(E, A, B; \ker C) = \{0\}$.

Proof: In view of Proposition 2.2, there exist $S \in \mathbf{Gl}_{l+p}(\mathbb{R}), T \in \mathbf{Gl}_{n+m}(\mathbb{R})$ such that (using the matrices defined in (2.1))

$$S \begin{bmatrix} sE - A & -B \\ -C & 0 \end{bmatrix} T = \begin{bmatrix} sI_{n_s} - A_s & 0 & 0 & 0 \\ 0 & sN_\alpha - I_{|\alpha|} & 0 & 0 \\ 0 & 0 & sK_\beta - L_\beta & 0 \\ 0 & 0 & 0 & sK_\gamma^\top - L_\gamma^\top \end{bmatrix} \tag{3.5}$$

(i)⇒(ii): Suppose that (ii) does not hold. Then Lemma 2.4 yields $\ell(\beta) > 0$. Therefore, we find $z \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{|\beta|}) \setminus \{0\}$ and $I \subseteq \mathbb{R}$ open interval such that $z|_I = 0$ and $(\frac{\mathrm{d}}{\mathrm{d}t} K_\beta - L_\beta) z = 0$. This implies that

$$\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} E - A & -B \\ -C & 0 \end{bmatrix} T\,(0, z^\top, 0, 0)^\top = 0,$$

which contradicts autonomous zero dynamics.

(ii)⇒(i): By (ii) and Lemma 2.4 it follows that $\ell(\beta) = 0$ in (3.5). Let $w \in \mathcal{Z}D_{(1.1)}$ and $I \subseteq \mathbb{R}$ be an open interval such that $w|_I = 0$. Then, with $(v_1^\top, v_2^\top, v_3^\top)^\top = T^{-1} w$, we have

$$S^{-1} \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} I_{n_s} - A_s & 0 & 0 \\ 0 & \frac{\mathrm{d}}{\mathrm{d}t} N_\alpha - I_{|\alpha|} & 0 \\ 0 & 0 & \frac{\mathrm{d}}{\mathrm{d}t} K_\gamma^\top - L_\gamma^\top \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{bmatrix} sE - A & -B \\ -C & 0 \end{bmatrix} w = 0,$$

and thus $(\frac{\mathrm{d}}{\mathrm{d}t} I_{n_s} - A_s) v_1 = 0$, $(\frac{\mathrm{d}}{\mathrm{d}t} N_\alpha - I_{|\alpha|}) v_2 = 0$, and $(\frac{\mathrm{d}}{\mathrm{d}t} K_\gamma^\top - L_\gamma^\top) v_3 = 0$. Then, successively solving each block in $(\frac{\mathrm{d}}{\mathrm{d}t} N_\alpha - I_{|\alpha|}) v_2 = 0$ and $(\frac{\mathrm{d}}{\mathrm{d}t} K_\gamma^\top - L_\gamma^\top) v_3 = 0$ gives $v_2 = 0$ and $v_3 = 0$. Since $v_1|_I = 0$ it follows that $v_1 = 0$. So we may conclude that $w = 0$, by which the zero dynamics are autonomous.

(i)⇒(iii): *Step 1*: (A1) follows from (ii).

Step 2: We show (A2). Let $V \in \mathbb{R}^{n\times k}$ with full column rank such that $\operatorname{im} V = \max(E, A, B; \ker C)$. By definition of $\max(E, A, B; \ker C)$ there exist $N \in \mathbb{R}^{k\times k}, M \in \mathbb{R}^{m\times k}$ such that $AV = EVN + BM$ and $CV = 0$. Therefore, we have

$$\left(s\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}\right)\begin{bmatrix} V \\ M \end{bmatrix} = \begin{bmatrix} EV \\ 0 \end{bmatrix}(sI_k - N)$$

By (ii) we find $s_0 \in \mathbb{C}$ such that $\left[\begin{smallmatrix} s_0E-A & -B \\ -C & 0 \end{smallmatrix}\right]$ has full column rank and $s_0I_k - N$ is invertible. Let $y \in \ker E \cap \max(E, A, B; \ker C)$. Then there exists $x \in \mathbb{R}^k$ such that $y = Vx$ and $EVx = 0$. Therefore,

$$\begin{bmatrix} s_0E - A & -B \\ -C & 0 \end{bmatrix}\begin{bmatrix} V \\ M \end{bmatrix}(s_0I_k - N)^{-1}x = \begin{bmatrix} EV \\ 0 \end{bmatrix} x = 0.$$

This implies that $\left[\begin{smallmatrix} V \\ M \end{smallmatrix}\right](s_0I_k - N)^{-1}x = 0$ and since V has full column rank we find $x = 0$.
Step 3: We show (A3). Choose $W \in \mathbb{R}^{n\times(n-k)}$ such that $[V, W] \in \mathbf{Gl}_n(\mathbb{R})$. Then

$$\begin{bmatrix} sE - A & -B \\ -C & 0 \end{bmatrix}\begin{bmatrix} [V, W] & 0 \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} s[EV, EW] - [AV, AW] & -B \\ [0, C_2] & 0 \end{bmatrix}$$

and since EV has full column rank by Step 2, there exists $S \in \mathbf{Gl}_l(\mathbb{R})$ such that $SEV = \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$, thus

$$\begin{bmatrix} S & 0 \\ 0 & I_p \end{bmatrix}\begin{bmatrix} sE - A & -B \\ -C & 0 \end{bmatrix}\begin{bmatrix} [V, W] & 0 \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} s\begin{bmatrix} I & E_2 \\ 0 & E_4 \end{bmatrix} - \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} & \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\ [0, C_2] & 0 \end{bmatrix}. \tag{3.6}$$

Since $AV = EVN + BM$, we obtain $SAV = SEVN + SBM$, whence

$$\begin{bmatrix} A_1 \\ A_3 \end{bmatrix} = \begin{bmatrix} N \\ 0 \end{bmatrix} + \begin{bmatrix} B_1M \\ B_2M \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} S & 0 \\ 0 & I_p \end{bmatrix}\begin{bmatrix} sE - A & -B \\ -C & 0 \end{bmatrix}\begin{bmatrix} [V, W] & 0 \\ 0 & I_m \end{bmatrix}\begin{bmatrix} I_n & 0 \\ [M, 0] & I_m \end{bmatrix} = \begin{bmatrix} s\begin{bmatrix} I & E_2 \\ 0 & E_4 \end{bmatrix} - \begin{bmatrix} N & A_2 \\ 0 & A_4 \end{bmatrix} & \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\ [0, C_2] & 0 \end{bmatrix},$$

Now, let $v \in \mathbb{R}^k$ and $w \in \mathbb{R}^m$ be such that $EVv = Bw \in \operatorname{im} B \cap E \max(E, A, B; \ker C)$, hence

$$\begin{pmatrix} v \\ 0 \end{pmatrix} = SEVv = SBw = \begin{pmatrix} B_1w \\ B_2w \end{pmatrix}.$$

For s_0 as in Step 2 we find

$$\begin{bmatrix} s_0I - N & s_0E_2 - A_2 & B_1 \\ 0 & s_0E_4 - A_4 & B_2 \\ 0 & C_2 & 0 \end{bmatrix}\begin{pmatrix} -(s_0I - N)^{-1}v \\ 0 \\ w \end{pmatrix} = 0,$$

and so $v = 0$ and $w = 0$.

(iii)⇒(i): By (A2) we obtain that (3.6) holds. Incorporating (A3) gives

$$\{0\} = \operatorname{im} B \cap E \max(E, A, B; \ker C) = \operatorname{im} SB \cap \operatorname{im} SEV = \operatorname{im}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \cap \operatorname{im}\begin{bmatrix} I_k \\ 0 \end{bmatrix},$$

by which $B_1 = 0$. From (A1) it follows that B_2 has full column rank. Now, let $(x,u,y) \in \mathcal{ZD}_{(1.1)}$ and $I \subseteq \mathbb{R}$ an open interval such that $(x,u)|_I = 0$. Applying the coordinate transformation $(z_1^\top, z_2^\top)^\top = [V,W]^{-1}x$ and observing that by Proposition 3.5 $x(t) \in \operatorname{im} V$ for almost all $t \in \mathbb{R}$, it follows $Wz_2(t) = x(t) - Vz_1(t) \in \operatorname{im} W \cap \operatorname{im} V = \{0\}$ for almost all $t \in \mathbb{R}$. Therefore, $z_2 = 0$ and by (3.6) $z_1 \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R};\mathbb{R}^k$ and z_1 and u solve

$$\begin{array}{rcl} \dot z_1 &=& A_1 z_1, \\ 0 &=& A_3 z_1 + B_2 u, \end{array}$$

Then $z_1|_I = Vx|_I = 0$ gives $z_1 = 0$ and hence $u = 0$. $\square$

The characterization in Proposition 3.6 was observed for ODE systems $(I,A,B,C) \in \Sigma_{n,n,m,m}$ by Ilchmann and Wirth (personal communication, July 4, 2012). The following zero dynamics form for systems with autonomous zero dynamics in Theorem 3.7 was derived for ODE systems (I,A,B,C) by Isidori [26, Rem. 6.1.3]; however, in [26] it is not clear that the assumptions (A1), (A3) are equivalent to autonomous zero dynamics (note that (A2) is superfluous for ODEs).

Theorem 3.7 (Zero dynamics form)**.**
Consider $[E,A,B,C] \in \Sigma_{l,n,m,p}$ and suppose that the zero dynamics $\mathcal{ZD}_{(1.1)}$ are autonomous. Let $V \in \mathbb{R}^{n\times k}$ be such that $\operatorname{im} V = \max(E,A,B;\ker C)$ and $\operatorname{rk} V = k$. Then there exist $W \in \mathbb{R}^{n\times(n-k)}$ and $S \in \mathbf{Gl}_l(\mathbb{R})$ such that $[V,W] \in \mathbf{Gl}_n(\mathbb{R})$ and

$$[E,A,B,C] \overset{S,[V,W]}{\sim} [\tilde E, \tilde A, \tilde B, \tilde C], \tag{3.7}$$

where

$$\tilde E = \begin{bmatrix} I_k & E_2 \\ 0 & E_4 \\ 0 & E_6 \end{bmatrix}, \; \tilde A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \\ 0 & A_6 \end{bmatrix}, \; \tilde B = \begin{bmatrix} 0 \\ I_m \\ 0 \end{bmatrix}, \; \tilde C = [0, C_2] \tag{3.8}$$

such that

$$\mathcal{ZD}_{\left(\left[\begin{smallmatrix} E_4 \\ E_6 \end{smallmatrix}\right], \left[\begin{smallmatrix} A_4 \\ A_6 \end{smallmatrix}\right], \left[\begin{smallmatrix} I_m \\ 0 \end{smallmatrix}\right], C_2\right)} = \{(0,0,0)\}, \tag{3.9}$$

and $A_1 \in \mathbb{R}^{k\times k}$, $E_2 \in \mathbb{R}^{k\times(n-k)}$, $A_2 \in \mathbb{R}^{k\times(n-k)}$, $A_3 \in \mathbb{R}^{m\times k}$ $E_4 \in \mathbb{R}^{m\times(n-k)}$, $A_4 \in \mathbb{R}^{m\times(n-k)}$, $E_6 \in \mathbb{R}^{(l-k-m)\times(n-k)}$, $A_6 \in \mathbb{R}^{(l-k-m)\times(n-k)}$, $C_2 \in \mathbb{R}^{p\times(n-k)}$.
For uniqueness we have: If $[E,A,B,C]$, $[\hat E, \hat A, \hat B, \hat C] \in \Sigma_{l,n,m,p}$ are in the form (3.8) such that (3.9) holds, and

$$[E,A,B,C] \overset{S,T}{\sim} [\hat E, \hat A, \hat B, \hat C] \quad \text{for some } S \in \mathbf{Gl}_l(\mathbb{R}), T \in \mathbf{Gl}_n(\mathbb{R}), \tag{3.10}$$

then

$$S = \begin{bmatrix} S_1 & 0 & S_3 \\ 0 & I_m & S_6 \\ 0 & 0 & S_9 \end{bmatrix}, \quad T = \begin{bmatrix} S_1^{-1} & T_2 \\ 0 & T_4 \end{bmatrix},$$

where $S_1 \in \mathbf{Gl}_k(\mathbb{R}), S_9 \in \mathbf{Gl}_{l-k-m}(\mathbb{R}), T_4 \in \mathbf{Gl}_{n-k}(\mathbb{R})$ and S_3, S_6, T_2 are of appropriate sizes. In particular the dimensions of the matrices in (3.8) are unique and A_1 is unique up to similarity, i.e., $\sigma(A_1)$ is unique.

Proof: *Step 1*: We prove (3.7) and (3.8). By Proposition 3.6, autonomous zero dynamics are equivalent to the conditions (A1)–(A3). These conditions imply that $k+m \le l$. Then we may find $W \in \mathbb{R}^{n\times(n-k)}$ such that $[V,W] \in \mathbf{Gl}_n(\mathbb{R})$. Considering the transformed system $\big(E[V,W], A[V,W], B, C[V,W]\big)$, we find that $CV = 0$, since $\operatorname{im} V \subseteq \ker C$. Further observe that EV has full column rank by (A2) and, since B has full column rank by (A1) and $\operatorname{im} EV \cap \operatorname{im} B = \{0\}$ by (A3), we obtain that $[EV, B]$ has full column rank. Hence, we find $S \in \mathbf{Gl}_l(\mathbb{R})$ such that

$$S\,[EV, B] = \begin{bmatrix} I_k & 0 \\ 0 & I_m \\ 0 & 0 \end{bmatrix}.$$

Therefore,

$$[E, A, B, C] \overset{S,[V,W]}{\sim} \big(SE[V,W], SA[V,W], SB, C[V,W]\big) = \left[\begin{bmatrix} I_k & E_2 \\ 0 & E_4 \\ 0 & E_6 \end{bmatrix}, \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \\ A_5 & A_6 \end{bmatrix}, \begin{bmatrix} 0 \\ I_m \\ 0 \end{bmatrix}, [0, C_2] \right].$$

By (3.2) there exist $N \in \mathbb{R}^{k\times k}$, $M \in \mathbb{R}^{m\times k}$ such that $AV = EVN + BM$, thus

$$S^{-1} \begin{bmatrix} A_1 \\ A_3 \\ A_5 \end{bmatrix} = AV = EVN + BM = S^{-1} \begin{bmatrix} I_k \\ 0 \\ 0 \end{bmatrix} N + S^{-1} \begin{bmatrix} 0 \\ I_m \\ 0 \end{bmatrix} M,$$

which gives $A_5 = 0$.

Step 2: We show (3.9). Let $(\hat{E}, \hat{A}, \hat{B}, \hat{C}) := \left(\left[\begin{smallmatrix} E_4 \\ E_6 \end{smallmatrix}\right], \left[\begin{smallmatrix} A_4 \\ A_6 \end{smallmatrix}\right], \left[\begin{smallmatrix} I_m \\ 0 \end{smallmatrix}\right], C_2 \right)$ and $(z_2, u, y) \in \mathcal{ZD}_{(\hat{E},\hat{A},\hat{B},\hat{C})}$, i.e.,

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\hat{E} z_2 &= \hat{A} z_2 + \hat{B} u \\ 0 = \quad y &= \hat{C} z_2 \,. \end{aligned} \tag{3.11}$$

Now choose $z_1 \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^k)$ such that $z_1 + E_2 z_2 \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^k)$ and define $\psi := \frac{\mathrm{d}}{\mathrm{d}t}(z_1 + E_2 z_2) - A_2 z_2$. Then we have

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} E[V,W] \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &\overset{(3.7)}{=} S^{-1} \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t}(z_1 + E_2 z_2) \\ \frac{\mathrm{d}}{\mathrm{d}t} E_4 z_2 \\ \frac{\mathrm{d}}{\mathrm{d}t} E_6 z_2 \end{pmatrix} \\ &\overset{(3.11)}{=} S^{-1} \begin{pmatrix} A_1 z_1 + A_2 z_2 \\ A_3 z_1 + A_4 z_2 + u \\ A_6 z_2 \end{pmatrix} - S^{-1} \begin{pmatrix} A_1 z_1 \\ A_3 z_1 \\ 0 \end{pmatrix} + S^{-1} \begin{pmatrix} \psi \\ 0 \\ 0 \end{pmatrix} \overset{(3.7)}{=} A[V,W] \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + Bu - AVz_1 + EV\psi, \end{aligned}$$

and hence

$$\tfrac{\mathrm{d}}{\mathrm{d}t} E[V,W] \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = A[V,W] \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + [AV, EV, B] \begin{pmatrix} -z_1 \\ \psi \\ u \end{pmatrix}, \quad C[V,W] \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = C_2 z_2 = 0.$$

This means that $\left([V,W] \left(\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right), \left(\begin{smallmatrix} -z_1 \\ \psi \\ u \end{smallmatrix}\right), 0 \right) \in \mathcal{ZD}_{(E,A,[AV,EV,B],C)}$ and hence Lemma 3.4 applied to the system $\big[E, A, [AV, EV, B], C\big]$ gives that the limit $\mathcal{W}_{\ell^*} = \bigcap_{i\geq 0} \mathcal{W}_i$ of the sequence

$$\mathcal{W}_0 := \ker C, \quad \mathcal{W}_i := A^{-1}(E\mathcal{W}_{i-1} + \operatorname{im}[AV, EV, B]) \cap \ker C, \quad i \in \mathbb{N}$$

satisfies $[V,W] \left(\begin{smallmatrix} z_1(t) \\ z_2(t) \end{smallmatrix}\right) \in \mathcal{W}_{\ell^*}$ for almost all $t \in \mathbb{R}$. Now, let $\mathcal{V} := \operatorname{im} V + \mathcal{W}_{\ell^*}$ and observe that $\mathcal{V} \subseteq \ker C$ and, since $\mathcal{W}_{\ell^*} = A^{-1}(E\mathcal{W}_{\ell^*} + \operatorname{im}[AV, EV, B]) \cap \ker C$,

$$A\mathcal{V} \subseteq E \operatorname{im} V + \operatorname{im} B + E\mathcal{W}_{\ell^*} + \operatorname{im}[AV, EV, B] \subseteq E\mathcal{V} + \operatorname{im} B,$$

thus $\mathcal{V}$ is (E,A,B)-invariant and included in $\ker C$, which gives $\mathcal{V} \subseteq \max(E,A,B;\ker\, C) = \operatorname{im} V$ and hence $\mathcal{W}_{\ell^*} \subseteq \operatorname{im} V$. Therefore, $[V,W] \binom{z_1(t)}{z_2(t)} \in \mathcal{W}_{\ell^*} \subseteq \operatorname{im} V$ by which $Wz_2(t) \in \operatorname{im} W \cap \operatorname{im} V = \{0\}$ for almost all $t \in \mathbb{R}$. Since W has full column rank, it follows $z_2 = 0$ and therefore $u = 0$.
Step 3: We show the uniqueness property. To this end we first show that

$$\max(SET, SAT, SB; \ker CT) = T^{-1} \max(E,A,B;\ker C).$$

Let $V \in \mathbb{R}^{n \times k}$ with full column rank such that $\operatorname{im} V = \max(E,A,B;\ker\, C)$. By definition of $\max(E,A,B;\ker\, C)$ there exist $N \in \mathbb{R}^{k\times k}, M \in \mathbb{R}^{m \times k}$ such that $AV = EVN + BM$ and $CV = 0$. Then

$$(SAT)(T^{-1}V) = SEVN + SBM = (SET)(T^{-1}V)N + (SB)M$$

and $(CT)(T^{-1}V) = CV = 0$, which proves the assertion. This shows in particular that

$$\dim \max(SET, SAT, SB; \ker CT) = \dim \max(E,A,B;\ker C)$$

and hence the block structures of $[E,A,B,C]$ and $[\hat{E},\hat{A},\hat{B},\hat{C}]$ coincide. Furthermore, since both are in the form (3.8),

$$\operatorname{im} \begin{bmatrix} I_k \\ 0 \end{bmatrix} = \max(SET, SAT, SB; \ker CT) = T^{-1} \max(E,A,B;\ker C) = \operatorname{im} T^{-1} \begin{bmatrix} I_k \\ 0 \end{bmatrix},$$

by which T takes the form $T = \left[\begin{smallmatrix} T_1 & T_2 \\ 0 & T_4 \end{smallmatrix}\right]$, $T_1 \in \mathbf{Gl}_k(\mathbb{R})$, $T_4 \in \mathbf{Gl}_{n-k}(\mathbb{R})$. Moreover,

$$\begin{bmatrix} 0 \\ I_m \\ 0 \end{bmatrix} = \hat{B} = SB = S \begin{bmatrix} 0 \\ I_m \\ 0 \end{bmatrix}, \quad \text{and hence} \quad S = \begin{bmatrix} S_1 & 0 & S_3 \\ S_4 & I_m & S_6 \\ S_7 & 0 & S_9 \end{bmatrix}.$$

Now,

$$\begin{bmatrix} I_k \\ 0 \\ 0 \end{bmatrix} = \hat{E} \begin{bmatrix} I_k \\ 0 \end{bmatrix} = SET \begin{bmatrix} I_k \\ 0 \end{bmatrix} = \begin{bmatrix} S_1T_1 \\ S_4T_1 \\ S_7T_1 \end{bmatrix},$$

by which $T_1 = S_1^{-1}$, $S_4 = 0$ and $S_7 = 0$. This completes the proof of the theorem. □

Remark 3.8 (How close is the zero dynamics form to a canonical form?)**.**
Recall the definition of a canonical form: given a group G, a set $\mathcal{S}$, and a group action $\alpha : G \times \mathcal{S} \to \mathcal{S}$ which defines an equivalence relation $s \overset{\alpha}{\sim} s'$ if, and only if, $\exists\, U \in G : \alpha(U,s) = s'$. Then a map $\gamma : \mathcal{S} \to \mathcal{S}$ is called a *canonical form for* α [11] if, and only if,

$$\forall\, s, s' \in \mathcal{S} \;:\; \gamma(s) \overset{\alpha}{\sim} s \quad \wedge \quad \left[s \overset{\alpha}{\sim} s' \Leftrightarrow \gamma(s) = \gamma(s') \right].$$

Therefore, the set $\mathcal{S}$ is divided into disjoint orbits (i.e., equivalence classes) and the mapping γ picks a unique representative in each equivalence class. In the present setup, the group is $G = \mathbf{Gl}_l(\mathbb{R}) \times \mathbf{Gl}_n(\mathbb{R})$, the considered set is $\mathcal{S} = \Sigma_{l,n,m,p}$ and the group action $\alpha\big((S,T),[E,A,B,C]\big) = [SET, SAT, SB, CT]$ corresponds to $\overset{S,T}{\sim}$.
However, Theorem 3.7 does not provide a mapping γ. That means the zero dynamics form is not a unique representative within the equivalence class and hence it is not a canonical form. The entries E_2, A_2, E_4, A_4 are not even unique up to matrix equivalence (recall that two matrices $M, N \in \mathbb{R}^{l \times n}$ are *equivalent* if, and only if, there exist $S \in \mathbf{Gl}_l(\mathbb{R}), T \in \mathbf{Gl}_n(\mathbb{R})$ such that $SMT = N$): it is easy to construct an example such that (3.10) is satisfied and in the respective forms we have, e.g., $A_2 = 0$ and $\hat{A}_2 \neq 0$. However, the last statement in Theorem 3.7 provides that A_1, A_3, E_6, A_6 and C_2 are unique up to similarity or equivalence, resp. ⋄

Corollary 3.9 (Vector space isomorphism)**.**
Suppose that $[E, A, B, C] \in \Sigma_{l,n,m,p}$ *satisfies the following:*

(i) *The zero dynamics* $\mathcal{ZD}_{(1.1)}$ *are autonomous.*

(ii) *Using the notation from Theorem 3.7 and the form* (3.8) *it holds that*

$$\mathrm{rk}_{\mathbb{R}[s]}\left(s\begin{bmatrix}E_4\\E_6\end{bmatrix}-\begin{bmatrix}A_4\\A_6\end{bmatrix}\right)=n-k.$$

Then the linear mapping, described in terms of Theorem 3.7,

$$\begin{array}{rcl}\varphi\colon\ \max(E,A,B;\ker C) & \to & \mathcal{ZD}_{(1.1)}\cap\big(\mathcal{C}^1(\mathbb{R};\mathbb{R}^n)\times\mathcal{C}(\mathbb{R};\mathbb{R}^m)\times\mathcal{C}(\mathbb{R};\mathbb{R}^p)\big)\\ x^0 & \mapsto & (x(\cdot),\ Fx(\cdot),\ Cx(\cdot))\,,\\ & & \textit{where } F:=[-A_3,0]\,[V,W]^{-1} \textit{ and } x(\cdot) \textit{ solves}\\ & & E\dot x=(A+BF)\,x,\ x(0)=x^0,\end{array}$$

is a vector space isomorphism.

Proof: *Step 1*: We show that φ is well-defined, that means to show that for arbitrary $x^0\in\max(E,A,B;\ker C)$, the (continuously differentiable) solution of

$$E\dot x=(A+BF)\,x,\qquad x(0)=x^0 \tag{3.12}$$

is unique and global and satisfies

$$(x,u,y):=(x,Fx,Cx)\in\mathcal{ZD}_{(1.1)}. \tag{3.13}$$

Applying the coordinate transformation $(z_1^\top,z_2^\top)^\top=[V,W]^{-1}x$ from Theorem 3.7 and invoking

$$BF=S^{-1}\begin{bmatrix}0\\I_m\\0\end{bmatrix}[-A_3,0][V,W]^{-1}=S^{-1}\begin{bmatrix}0&0\\-A_3&0\\0&0\end{bmatrix}[V,W]^{-1},$$

we find

$$\begin{array}{rcl}\dot z_1+E_2\dot z_2&=&A_1z_1+A_2z_2,\\ E_4\dot z_2&=&A_4z_2,\\ E_6\dot z_2&=&A_6z_2,\end{array}$$

and the initial value satisfies

$$Vz_1(0)+Wz_2(0)=x(0)\in\operatorname{im}V.$$

Therefore, $Wz_2(0)=x(0)-Vz_1(0)\in\operatorname{im}W\cap\operatorname{im}V=\{0\}$, by which $Wz_2(0)=0$ and hence, invoking the full column rank of W, $z_2(0)=0$. Now, by (ii), Proposition 2.2, Lemma 2.4 and a straightforward calculation of the solution of the system in quasi-Kronecker form, we find that $\left[\begin{smallmatrix}E_4\\E_6\end{smallmatrix}\right]\dot y=\left[\begin{smallmatrix}A_4\\A_6\end{smallmatrix}\right]y$ satisfies uniqueness, i.e., any local solution $y\in\mathcal{C}^1(I;\mathbb{R}^{n-k})$, $I\subseteq\mathbb{R}$ an interval, can be extended to a unique global solution on all of $\mathbb{R}$. This yields $z_2=0$. Therefore, $x=Vz_1$ and z_1 satisfies $\dot z_1=A_1z_1$, $z_1(0)=[I_k,0][V,W]^{-1}x(0)$, which is a unique and global solution. Finally, $x(t)=Vz_1(t)\in\operatorname{im}V\subseteq\ker C$ for all $t\in\mathbb{R}$ and hence $y=Cx=0$.
Step 2: We show that φ is injective. Let $x^1,x^2\in\max(E,A,B;\ker C)(0)$ so that $\varphi(x^1)(\cdot)=\varphi(x^2)(\cdot)$. Then

$$(x^1,*,*)=\varphi(x^1)(\cdot)\big|_{t=0}=\varphi(x^2)(\cdot)\big|_{t=0}=(x^2,*,*).$$

Step 3: We show that φ is surjective. Let $(x,u,y) \in \mathcal{ZD}_{(1.1)} \cap \big(\mathcal{C}^1(\mathbb{R};\mathbb{R}^n) \times \mathcal{C}(\mathbb{R};\mathbb{R}^m) \times \mathcal{C}(\mathbb{R};\mathbb{R}^p)\big)$. Then Proposition 3.5 yields that $x(t) \in \max(E,A,B;\ker C)$ for all $t \in \mathbb{R}$. Hence, applying the coordinate transformation $(z_1^\top, z_2^\top)^\top = [V,W]^{-1}x$ from Theorem 3.7 to (1.1) gives $Vz_1(t) + Wz_2(t) = x(t) \in \operatorname{im} V$ for all $t \in \mathbb{R}$ and, similar to Step 1, we may conclude $z_2 = 0$. Therefore,

$$\begin{array}{rcl} \dot z_1 &=& A_1 z_1, \\ 0 &=& A_3 z_1 + u. \end{array} \tag{3.14}$$

The second equation in (3.14) now gives

$$u = -A_3 z_1 = -A_3[I,0][V,W]^{-1}x = Fx.$$

Finally, a simple calculation shows that $x = Vz_1$ satisfies $E\dot x = (A+BF)x$ and, clearly, $x(0) = Vz_1(0) \in \max(E,A,B;\ker C)$. □

Remark 3.10 (Zero dynamics form)**.**
The name "zero dynamics form" for the form (3.8) may be justified since the zero dynamics are decoupled in this form. If $(x,u,y) \in \mathcal{ZD}_{(1.1)}$, then, as in Step 3 of the proof of Corollary 3.9, $x = Vz_1 + Wz_2$ and $z_2 = 0$ and z_1, u solve (3.14), i.e., z_1 as the solution of an ODE characterizes the "dynamics" within the zero dynamics and z_2 and u are given by algebraic equations depending on z_1. Another zero dynamics form derived in [6] is a special case of the form (3.8). ⋄

The last result in this section is a characterization of trivial zero dynamics by the left invertibility of the system pencil; this becomes important for a further refinement of the zero dynamics form (3.8) in Theorem 5.6.

Lemma 3.11 (Trivial zero dynamics and system pencil)**.**
Let $[E,A,B,C] \in \Sigma_{l,n,m,p}$. Then

$$\mathcal{Z}D_{(1.1)} = \{(0,0,0)\} \quad \iff \quad \begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix} \text{ is left invertible over } \mathbb{R}[s].$$

Proof: $\Rightarrow$: Since the zero dynamics are trivial, they are autonomous, and by Proposition 3.6 the system pencil has full rank. Hence, invoking Lemma 2.4, in a quasi-Kronecker form (3.5) of the system pencil it holds $\ell(\beta) = 0$. Furthermore, we obtain $n_s = 0$, since otherwise we could find nontrivial solutions of the ODE $\dot z = A_s z$ which would lead to nontrivial trajectories within the zero dynamics. Now,

$$(sN_\alpha - I_{|\alpha|})^{-1} = -I_{|\alpha|} - sN_\alpha - \ldots - s^{\nu-1}N_\alpha^{\nu-1},$$

where ν is the index of $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$. Furthermore, by a permutation of the rows of $sK_\gamma^\top - L_\gamma^\top$ we may achieve that there exists $S \in \mathbf{Gl}_{|\gamma|}(\mathbb{R})$ such that

$$S(sK_\gamma^\top - L_\gamma^\top) = \begin{bmatrix} s\tilde N - I_{|\gamma|-\ell(\gamma)} \\ s\tilde K - \tilde L \end{bmatrix},$$

where $\tilde N \in \mathbb{R}^{(|\gamma|-\ell(\gamma))\times(|\gamma|-\ell(\gamma))}$ is nilpotent and $\tilde K, \tilde L$ are matrices of appropriate sizes. Then $sK_\gamma^\top - L_\gamma^\top$ has left inverse $[(s\tilde N - I_{|\gamma|-\ell(\gamma)})^{-1}, 0]S$ over $\mathbb{R}[s]$.
$\Leftarrow$: From the left invertibility of the system pencil it is immediate that $n_s = 0$ and $\ell(\beta) = 0$ in any quasi-Kronecker form (3.5) of it. The DAEs corresponding to the remaining blocks

$$\tfrac{\mathrm{d}}{\mathrm{d}t} N z_1 = z_1, \quad \tfrac{\mathrm{d}}{\mathrm{d}t} K_\gamma^\top z_2 = L_\gamma^\top z_2,$$

do have only trivial solutions as successive solution of each block quickly shows. Hence the zero dynamics are trivial. □

Remark 3.12 (Zero dynamics and system pencil/Kronecker form).
We stress the difference in the characterization of autonomous and trivial zero dynamics in terms of the system pencil as they arise from Proposition 3.6 and Lemma 3.11: The zero dynamics are autonomous if, and only if, the system pencil has full column rank over $\mathbb{R}[s]$; they are trivial if, and only if, the system pencil is left invertible over $\mathbb{R}[s]$.
Using the quasi-Kronecker form, it follows that the zero dynamics $\mathcal{ZD}_{(1.1)}$ are

(i) autonomous if, and only if, in a quasi-Kronecker form (3.5) of the system pencil no underdetermined blocks are present, i.e., $\ell(\beta) = 0$. The dynamics within the zero dynamics are then characterized by the ODE $\dot{z} = A_s z$.

(ii) trivial if, and only if, in a quasi-Kronecker form (3.5) of the system pencil no underdetermined blocks and no ODE blocks are present, i.e., $\ell(\beta) = 0$ and $n_s = 0$. The remaining nilpotent and overdetermined blocks then have trivial solutions only. $\diamond$

4 Stable zero dynamics and stabilization

In this section we define the asymptotic stability of the zero dynamics in the behavioral sense as in [37, Def. 7.2.1] and give a characterization for it in terms of a rank condition on the system pencil corresponding to the system $[E, A, B, C] \in \Sigma_{l,n,m,p}$. Furthermore, we show that asymptotically stable zero dynamics imply stabilizability in the behavioral sense in the quadratic case (i.e., $l = n$ and $p = m$), but not necessarily in the rectangular case. We also show that for a quadratic system with stable zero dynamics there exists a compatible and stabilizing control in the behavioral sense such that the interconnected system is autonomous.

Definition 4.1 (Asymptotically stable zero dynamics).
For $[E, A, B, C] \in \Sigma_{l,n,m,p}$ the zero dynamics $\mathcal{ZD}_{(1.1)}$ are called *asymptotically stable* if, and only if,

$$\forall\, (x, u, y) \in \mathcal{ZD}_{(1.1)} : \lim_{t\to\infty} \text{ess-sup}_{[t,\infty)} \|(x, u)\| = 0.$$

$\diamond$

In terms of the system pencil we get the following characterization of asymptotically stable zero dynamics.

Lemma 4.2 (Characterization of asymptotically stable zero dynamics).
Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$. Then

$$\mathcal{ZD}_{(1.1)} \textit{ are asymptotically stable} \quad\Longleftrightarrow\quad \forall\, \lambda \in \overline{\mathbb{C}}_+ : \ \mathrm{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E - A & -B \\ -C & 0 \end{bmatrix} = n + m.$$

Proof: $\Rightarrow$: Suppose there exist $\lambda \in \overline{\mathbb{C}}_+$ and $x^0 \in \mathbb{R}^n$, $u^0 \in \mathbb{R}^m$ such that

$$\begin{bmatrix} \lambda E - A & -B \\ -C & 0 \end{bmatrix} \begin{pmatrix} x^0 \\ u^0 \end{pmatrix} = 0.$$

Let $x : \mathbb{R} \to \mathbb{R}^n,\ t \mapsto e^{\lambda t} x^0$ and $u : \mathbb{R} \to \mathbb{R}^m,\ t \mapsto e^{\lambda t} u^0$. Then

$$\tfrac{\mathrm{d}}{\mathrm{d}t} Ex(t) = e^{\lambda t}(\lambda E x^0) = e^{\lambda t}(Ax^0 + Bu^0) = Ax(t) + Bu(t), \quad Cu(t) = 0,$$

hence $(x, u, 0) \in \mathcal{ZD}_{(1.1)}$, which contradicts asymptotic stability of $\mathcal{ZD}_{(1.1)}$.
$\Leftarrow$: The rank condition implies that the system pencil must have full column rank over $\mathbb{R}[s]$. Therefore, by Lemma 2.4, in a quasi-Kronecker form (3.5) of the system pencil it holds that $\ell(\beta) = 0$. It is also immediate that $\sigma(A_s) \subseteq \mathbb{C}_-$. The asymptotic stability of $\mathcal{ZD}_{(1.1)}$ then follows from a consideration of the solutions to each block in (3.5). $\square$

Remark 4.3 (Asymptotically stable zero dynamics are autonomous).
If follows from Lemma 4.2 that for any $[E, A, B, C] \in \Sigma_{l,n,m,p}$ with asymptotically stable zero dynamics, the system pencil must have full column rank for some and hence almost all $s \in \mathbb{C}$. This implies full column rank over $\mathbb{R}[s]$. Therefore, by Proposition 3.6, the zero dynamics $\mathcal{ZD}_{(1.1)}$ are autonomous. ◇

In the following we define the property of stabilizability of a system (1.1), more precisely stabilizability in the behavioral sense. For more details on this concept for linear differential-algebraic equations see [8]; for the general concept in terms of differential behaviors see [37, Def. 5.2.29].

Definition 4.4 (Behavioral stabilizability).
Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$. The system (1.1) or $[E, A, B, C]$ resp., is called *stabilizable in the behavioral sense* if, and only if,

$$\forall\,(x, u, y) \in \mathfrak{B}_{(1.1)}\ \exists\,(\hat{x}, \hat{u}, \hat{y}) \in \mathfrak{B}_{(1.1)} :\ (x, u)|_{(-\infty,0]} = (\hat{x}, \hat{u})|_{(-\infty,0]}\ \wedge\ \lim_{t\to\infty} \text{ess-sup}_{[t,\infty)} \|(\hat{x}, \hat{u})\| = 0.$$

◇

Behavioral stabilizability can be characterized algebraically, which was already stated in [37, Thm. 5.2.30] for differential behaviors. In [8] the special case of linear DAEs is considered, which immediately gives the following result.

Lemma 4.5 (Algebraic characterization of behavioral stabilizability).
Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$. Then $[E, A, B, C]$ is stabilizable in the behavioral sense if, and only if,

$$\forall\,\lambda \in \overline{\mathbb{C}}_+ :\ \mathrm{rk}_{\mathbb{C}}[\lambda E - A, -B] = \mathrm{rk}_{\mathbb{R}(s)}[sE - A, -B].$$

In the following we will show that a system $[E, A, B, C]$ with asymptotically stable zero dynamics is stabilizable in the behavioral sense, provided that $sE - A$ is a quadratic pencil, that is $l = n$, but not necessarily regular, and $p = m$. The following example illustrates that we cannot expect the statement to be true for rectangular pencils.

Example 4.6.
Consider the system $[E, A, B, C] \in \Sigma_{3,2,1,1}$ defined by

$$sE - A = \begin{bmatrix} -1 & 0 \\ s & 0 \\ 0 & s-1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad C = [0, 1].$$

System (1.1) then reads

$$\begin{aligned} 0 &= x_1(t), \\ \dot{x}_1(t) &= u(t), \\ \dot{x}_2(t) &= x_2(t), \\ y(t) &= x_2(t). \end{aligned}$$

and hence the zero dynamics are trivial, that is $\mathcal{ZD}_{(1.1)} = \{(0, 0, 0)\}$, and therefore asymptotically stable. However, for $\lambda = 1$ we have

$$\mathrm{rk}_{\mathbb{C}}[\lambda E - A, -B] = 2 \neq 3 = \mathrm{rk}_{\mathbb{R}(s)}[sE - A, -B],$$

whence the system is not behavioral stabilizable. ◇

We are now in the position to prove the main result of this section. For ODEs, it is has been shown by Isidori [26, Rem. 6.1.3] that asymptotically stable zero dynamics implies stabilizability. For DAEs, various stabilizability concepts are available: complete stabilizability, strong stabilizability and stabilizability in the behavioral sense (see [8]). It turns out that asymptotically stable zero dynamics yields behavioral stabilizability and a careful inspection of the proof of Proposition 4.7 shows that a stronger result (i.e., that strong or complete stabilizability is implied) can, in general, not be expected.

Proposition 4.7 (Asymptotically stable zero dynamics imply stabilizability).
Let $[E,A,B,C] \in \Sigma_{n,n,m,m}$. If the zero dynamics $\mathcal{ZD}_{(1.1)}$ are asymptotically stable, then $[E,A,B,C]$ is stabilizable in the behavioral sense.

Proof: We use the Kalman decomposition [8, Thm. 5] of the system $[E,A,B,C]$: There exist $S,T \in \mathbf{Gl}_n(\mathbb{R})$ such that

$$S(sE-A)T = \begin{bmatrix} sE_{11}-A_{11} & sE_{12}-A_{12} \\ 0 & sE_{22}-A_{22} \end{bmatrix}, \quad SB = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad CT = [C_1, C_2],$$

with $E_{11}, A_{11} \in \mathbb{R}^{k_1 \times n_1}$, $E_{12}, A_{12} \in \mathbb{R}^{k_1 \times n_2}$, $E_{22}, A_{22} \in \mathbb{R}^{k_2 \times n_2}$, $B_1, C_1^\top \in \mathbb{R}^{k_1 \times m}$ and $C_2 \in \mathbb{R}^{m \times n_2}$, where $\mathrm{rk}_{\mathbb{R}}[E_{11}, A_{11}, B_1] = k_1$ and (E_{11}, A_{11}, B_1) is completely controllable in the sense of [8, Def. 1], which implies by [8, Cor. 3] that

$$\forall\, \lambda \in \mathbb{C}: \ \mathrm{rk}_{\mathbb{C}}[\lambda E_{11} - A_{11}, -B_1] = k_1 = \mathrm{rk}_{\mathbb{R}(s)}[sE_{11} - A_{11}, -B_1], \tag{4.1}$$

and $\mathcal{R}_{[E_{22},A_{22},0,C_2]} = \{0\}$ for the reachability space of $(E_{22}, A_{22}, 0, C_2)$; the reachability space of a system $[E,A,B,C] \in \Sigma_{l,n,m,p}$ is defined by

$$\mathcal{R}_{[E,A,B,C]} = \left\{ \, x^0 \in \mathbb{R}^n \;\middle|\; \exists\, t \geq 0 \ \exists\, (x,u,y) \in \mathfrak{B}_{(1.1)} : \ x \in \mathcal{C}(\mathbb{R};\mathbb{R}^n) \ \wedge \ x(0) = 0 \ \wedge \ x(t) = x^0 \, \right\}.$$

$\mathcal{R}_{[E_{22},A_{22},0,C_2]} = \{0\}$ implies, incorporating e.g. [8, Rem. 23], that in a quasi-Kronecker form

$$\tilde{S}(sE_{22} - A_{22})\tilde{T} = \begin{bmatrix} sI_{n_s} - A_s & 0 & 0 & 0 \\ 0 & sN_\alpha - I_{|\alpha|} & 0 & 0 \\ 0 & 0 & sK_\beta - L_\beta & 0 \\ 0 & 0 & 0 & sK_\gamma^\top - L_\gamma^\top \end{bmatrix}, \quad \tilde{S} \in \mathbf{Gl}_{k_2}(\mathbb{R}), \tilde{T} \in \mathbf{Gl}_{n_2}(\mathbb{R}),$$

of $sE_{22} - A_{22}$ it holds $\ell(\beta) = 0$. Furthermore, it follows from Lemma 4.2 that

$$\forall\, \lambda \in \overline{\mathbb{C}}_+ : \ \mathrm{rk}_{\mathbb{C}}(\lambda E_{22} - A_{22}) = k_2,$$

and hence $\ell(\gamma) = 0$, i.e., $sE_{22} - A_{22}$ is regular and in particular $k_2 = n_2$ and $k_1 = n_1$. Therefore, also incorporating (4.1), we find

$$\mathrm{rk}_{\mathbb{R}(s)}[sE - A, -B] = \mathrm{rk}_{\mathbb{R}(s)}[sE_{11} - A_{11}, -B_1] + \mathrm{rk}_{\mathbb{R}(s)}(sE_{22} - A_{22}) = n_1 + n_2 = n.$$

Now assume that $[E,A,B,C]$ is not stabilizable in the behavioral sense. Then there exists $\lambda \in \overline{\mathbb{C}}_+$ such that $\mathrm{rk}_{\mathbb{C}}[\lambda E - A, -B] < n$ and hence, invoking (4.1), $\mathrm{rk}_{\mathbb{C}}(\lambda E_{22} - A_{22}) < n_2$. This gives $\mathrm{rk}_{\mathbb{C}} \left[\begin{smallmatrix} \lambda E - A & -B \\ -C & 0 \end{smallmatrix}\right] < n+m$, which contradicts asymptotically stable zero dynamics by Lemma 4.2. ☐

Proposition 4.7 can be used to show that systems with asymptotically stable zero dynamics can be stabilized by a compatible control in the behavioral sense such that the interconnected system is autonomous. A control in the behavioral sense, or control via interconnection, is the addition of algebraic constraints to the system in the following sense: For given (or to be determined) $K = [K_x, K_u]$ with $K_x \in \mathbb{R}^{q\times n}$, $K_u \in \mathbb{R}^{q\times m}$ and $[E, A, B, C] \in \Sigma_{l,n,m,p}$ we consider

$$\mathfrak{B}^K_{[E,A,B]} = \left\{ (x,u) \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n) \times \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^m) \;\middle|\; \begin{array}{l} Ex \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^l) \text{ and, for almost all } t \in \mathbb{R}, \\ \begin{array}{rcl} \frac{\mathrm{d}}{\mathrm{d}t} Ex(t) &=& Ax(t) + Bu(t) \\ 0 &=& K_x x(t) + K_u u(t) \end{array} \end{array} \right\}.$$

We call K the *control matrix*, since it induces the control law $K_x x(t) + K_u u(t) = 0$. The following definition of compatible and stabilizing control is from [8, Def. 5].

Definition 4.8 (Compatible and stabilizing control).
Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$. The control matrix $K = [K_x, K_u]$, where $K_x \in \mathbb{R}^{q\times n}$ and $K_u \in \mathbb{R}^{q\times m}$, is called

(i) *compatible* with $[E, A, B, C]$ if, and only if,

$$\forall\, x^0 \in \left\{ x^0 \in \mathbb{R}^n \;\middle|\; \exists\, (x,u,y) \in \mathfrak{B}_{(1.1)} : \; Ex(0) = Ex^0 \right\} \; \exists\, (x,u) \in \mathfrak{B}^K_{[E,A,B]} : \; Ex(0) = Ex^0.$$

(ii) *stabilizing* for $[E, A, B, C]$ if, and only if, the system $\left[\left[\begin{smallmatrix} E \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A \\ K_x \end{smallmatrix}\right], \left[\begin{smallmatrix} B \\ K_u \end{smallmatrix}\right], C\right]$ is stabilizable in the behavioral sense. $\diamond$

The following is now immediate from [8, Thm. 3].

Corollary 4.9 (Stabilizing control in behavioral sense).
Let $[E, A, B, C] \in \Sigma_{n,n,m,m}$ with asymptotically stable zero dynamics $\mathcal{ZD}_{(1.1)}$. Then there exists a compatible and stabilizing control matrix $K = [K_x, K_u] \in \mathbb{R}^{q\times(n+m)}$ for $[E, A, B, C]$, such that the interconnected system given by the behavior $\mathfrak{B}^K_{[E,A,B]}$ is autonomous, i.e., in a quasi-Kronecker form (2.2) *of $\left[\begin{smallmatrix} sE-A & -B \\ K_x & K_u \end{smallmatrix}\right]$ it holds that $\ell(\beta) = 0$.*

5 System inversion

In this section we investigate the properties of left-invertibility, right-invertibility, and invertibility of DAE systems. In order to treat these problems we derive a refinement of the zero dynamics form from Theorem 3.7.
In the following we give the definition of left- and right-invertibility of a system, which are from [44, Sec. 8.2] - generalized to the DAE case. A detailed survey of left- and right-invertibility of ODE systems can also be found in [39].

Definition 5.1 (System invertibility).
$[E, A, B, C] \in \Sigma_{l,n,m,p}$ is called

(i) *left-invertible* if, and only if,

$$\forall\, (x_1, u_1, y_1), (x_2, u_2, y_2) \in \mathfrak{B}_{(1.1)} : \; \left[y_1 = y_2 \;\wedge\; Ex_1(0) = Ex_2(0) = 0 \right] \implies u_1 = u_2.$$

(ii) *right-invertible* if, and only if,

$$\forall\, y \in \mathcal{C}^\infty(\mathbb{R};\mathbb{R}^p) \; \exists\, (x,u) \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n) \times \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^m) : \; (x,u,y) \in \mathfrak{B}_{(1.1)}.$$

(iii) *invertible* if, and only if, $[E, A, B, C]$ is left-invertible and right-invertible. $\diamond$

Remark 5.2 (Left-invertibility)**.**
By linearity of the behavior $\mathfrak{B}_{(1.1)}$, left-invertibility of $[E,A,B,C] \in \Sigma_{l,n,m,p}$ is equivalent to

$$\forall\,(x,u,y) \in \mathfrak{B}_{(1.1)}:\ \big[y=0\ \wedge\ Ex(0)=0\big] \implies u=0. \tag{5.1}$$

◇

Remark 5.3 (Inverse system)**.**
For ODE systems, the problem of finding a realization of the inverse of a system $[I,A,B,C] \in \Sigma_{n,n,m,m}$ is usually described as the problem of finding a realization for the inverse of its transfer function $C(sI-A)^{-1}B$, provided it exists, see e.g. [27, p. 557]. This means that in the corresponding behaviors inputs and outputs are interchanged. In the differential-algebraic setting we may generalize this in the following way: For $[E,A,B,C] \in \Sigma_{l,n,m,p}$ we call a system $[\hat{E},\hat{A},\hat{B},\hat{C}] \in \Sigma_{\hat{l},\hat{n},p,m}$ the *inverse* of $[E,A,B,C]$ if, and only if,

$$\begin{aligned}&\forall\,(u,y) \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^m) \times \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^p):\\ &\big[\exists\, x \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n):\ (x,u,y) \in \mathfrak{B}_{[E,A,B,C]}\big] \iff \Big[\exists\, \hat{x} \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^{\hat{n}}):\ (\hat{x},y,u) \in \mathfrak{B}_{[\hat{E},\hat{A},\hat{B},\hat{C}]}\Big].\end{aligned} \tag{5.2}$$

In fact, in the differential-algebraic framework condition (5.2) is so weak that it is possible to show that any system $[E,A,B,C] \in \Sigma_{l,n,m,p}$ has an inverse - thus, the existence of an inverse is in no relation to the notion of invertibility of the system.
Let $[E,A,B,C] \in \Sigma_{l,n,m,p}$ with $\operatorname{rk} B = q \le m$. Then there exist $S_1 \in \mathbb{R}^{q\times l}, S_2 \in \mathbb{R}^{(l-q)\times l}$ and $T \in \mathbf{Gl}_m(\mathbb{R})$ such that $S_1BT = [I_q, 0]$ and $S_2BT = 0$. Let $(x,u,y) \in \mathfrak{B}_{[E,A,B,C]}$ with $\tilde{u} := T^{-1}u = (\tilde{u}_1^\top, \tilde{u}_2^\top)^\top$, where $\tilde{u}_1 \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^q)$, $\tilde{u}_2 \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^{m-q})$. Then

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t} S_1Ex &= S_1Ax + \tilde{u}_1\\ \tfrac{\mathrm{d}}{\mathrm{d}t} S_2Ex &= S_2Ax\\ y &= Cx.\end{aligned}$$

Now, $\tilde{u}_1$ depends on the derivative of S_1Ex, so we introduce the new variable $w = \frac{\mathrm{d}}{\mathrm{d}t}S_1Ex$; and $\tilde{u}_2$ is the vector of free inputs (which are free outputs in the inverse system), so we introduce the new variable $z = \tilde{u}_2$, which will not be restricted in the inverse system. Clearly, adding these equations to the original system does not change it. Switching the roles of inputs and outputs and using the new augmented state $(x^\top, w^\top, z^\top)^\top \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^{n+q+(m-q)})$ we may rewrite the system as follows:

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t} S_1Ex &= w\\ \tfrac{\mathrm{d}}{\mathrm{d}t} S_2Ex &= S_2Ax\\ 0 &= -Cx + y\\ \tilde{u}_1 &= -S_1Ax + w\\ \tilde{u}_2 &= z.\end{aligned}$$

Therefore, an inverse of $[E,A,B,C]$ is

$$\left[\begin{bmatrix} S_1E & 0 & 0\\ S_2E & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, \begin{bmatrix} 0 & I_q & 0\\ S_2A & 0 & 0\\ -C & 0 & 0\end{bmatrix}, \begin{bmatrix} 0\\ 0\\ I_p\end{bmatrix}, T\begin{bmatrix} -S_1A & I_q & 0\\ 0 & 0 & I_{m-q}\end{bmatrix}\right] \in \Sigma_{l+p,n+m,p,m}.$$

Note also that for $(x^\top, w^\top, z^\top)^\top \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^{n+q+(m-q)})$ we have

$$Ex \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n) \iff \begin{bmatrix} S_1E & 0 & 0\\ S_2E & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{pmatrix} x\\ w\\ z\end{pmatrix} \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^{n+m}).$$

◇

Next, we will show that a DAE system with autonomous zero dynamics is left-invertible. However, the converse does, in general, not hold true as the following example illustrates.

Example 5.4.
Consider the system (1.1) with

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C = [0, 0, 1].$$

Let $(x, u, y) \in \mathcal{ZD}_{(1.1)}$ and $x = (x_1, x_2, x_3)^\top$. Then $y = 0$ and hence $x_3 = u = 0$, but x_1 is free and $x_2 = \dot{x}_1$. Therefore, the zero dynamics are not autonomous. However, $[E, A, B, C]$ is left-invertible since (5.1) is satisfied. $\diamond$

Lemma 5.5 (Autonomous zero dynamics imply left-invertibility)**.**
Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$ with autonomous zero dynamics. Then $[E, A, B, C]$ is left-invertible.

Proof: We show that (5.1) is satisfied. To this end let $(x, u, y) \in \mathfrak{B}_{(1.1)}$ with $y = 0$ and $Ex(0) = 0$. Hence, $(x, u, y) \in \mathcal{ZD}_{(1.1)}$ and applying the coordinate transformation $(z_1^\top, z_2^\top)^\top = [V, W]^{-1}x$ from Theorem 3.7 yields $Vz_1(t) + Wz_2(t) = x(t) \overset{\text{Prop. 3.5}}{\in} \operatorname{im} V$ for almost all $t \in \mathbb{R}$. Therefore, $z_2 = 0$ and we have that (3.14) holds. Since $0 = Ex(0) = EVz_1(0)$ we get from (3.8) that $z_1(0) = 0$, and hence it follows from (3.14) that $z_1 = 0$ and thus $u = 0$. $\square$

In the following we investigate right-invertibility for systems with autonomous zero dynamics. In order for $[E, A, B, C] \in \Sigma_{l,n,m,p}$ to be right invertible it is necessary that C has full row rank (i.e., $\operatorname{im} C = \mathbb{R}^p$). This additional assumption leads to the following form for $[E, A, B, C]$ specializing the form (3.8). This is the main result of this section.

Theorem 5.6 (System inversion form)**.**
Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$ with autonomous zero dynamics and $\operatorname{rk} C = p$. Then there exist $S \in \mathbf{Gl}_l(\mathbb{R})$ and $T \in \mathbf{Gl}_n(\mathbb{R})$ such that

$$[E, A, B, C] \overset{S,T}{\sim} [\hat{E}, \hat{A}, \hat{B}, \hat{C}], \tag{5.3}$$

where

$$\hat{E} = \begin{bmatrix} I_k & 0 & E_{13} \\ 0 & E_{22} & E_{23} \\ 0 & E_{32} & N \\ 0 & E_{42} & E_{43} \end{bmatrix}, \; \hat{A} = \begin{bmatrix} Q & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & I_{n_3} \\ 0 & A_{42} & 0 \end{bmatrix}, \; \hat{B} = \begin{bmatrix} 0 \\ I_m \\ 0 \\ 0 \end{bmatrix}, \; \hat{C} = [0, I_p, 0], \tag{5.4}$$

and $N \in \mathbb{R}^{n_3 \times n_3}$, $n_3 = n - k - p$, is nilpotent with $N^\nu = 0$ and $N^{\nu-1} \neq 0$, $\nu \in \mathbb{N}$, $E_{22}, A_{22} \in \mathbb{R}^{m \times p}$ and all other matrices are of appropriate sizes.

Proof: By Theorem 3.7 system $[E, A, B, C]$ is equivalent to $[\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}]$ in (3.8). Since C and therefore C_2 has full row rank, there exists $\tilde{T} \in \mathbf{Gl}_{n-k}(\mathbb{R})$ such that $C_2\tilde{T} = [I_p, 0]$. Hence,

$$[\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}] \overset{I, \left[\begin{smallmatrix} I & 0 \\ 0 & \tilde{T} \end{smallmatrix}\right]}{\sim} \left[\begin{bmatrix} I_k & \tilde{E}_{12} & \tilde{E}_{13} \\ 0 & \tilde{E}_{22} & \tilde{E}_{23} \\ 0 & \tilde{E}_{32} & \tilde{E}_{33} \end{bmatrix}, \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} \\ 0 & \tilde{A}_{32} & \tilde{A}_{33} \end{bmatrix}, \begin{bmatrix} 0 \\ I_m \\ 0 \end{bmatrix}, [0, I_p, 0] \right].$$

Now, since the system $\left[\begin{bmatrix} \tilde{E}_{22} & \tilde{E}_{23} \\ \tilde{E}_{32} & \tilde{E}_{33} \end{bmatrix}, \begin{bmatrix} \tilde{A}_{22} & \tilde{A}_{23} \\ \tilde{A}_{32} & \tilde{A}_{33} \end{bmatrix}, \begin{bmatrix} I_m \\ 0 \end{bmatrix}, [I_p, 0] \right]$ has trivial zero dynamics by Theorem 3.7, we may infer from Lemma 3.11 that there exists $X(s) \in \mathbb{R}[s]^{(n+m-k)\times(l+p-k)}$ such that

$$\begin{bmatrix} X_{11}(s) & X_{12}(s) & X_{13}(s) \\ X_{21}(s) & X_{22}(s) & X_{23}(s) \\ X_{31}(s) & X_{32}(s) & X_{33}(s) \end{bmatrix} \begin{bmatrix} s\tilde{E}_{22} - \tilde{A}_{22} & s\tilde{E}_{23} - \tilde{A}_{23} & I_m \\ s\tilde{E}_{32} - \tilde{A}_{32} & s\tilde{E}_{33} - \tilde{A}_{33} & 0 \\ I_p & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_p & 0 & 0 \\ 0 & I_{n-k-p} & 0 \\ 0 & 0 & I_m \end{bmatrix}.$$

Obviously, $X_{21}(s) = 0$ and hence $X_{22}(s)(s\tilde{E}_{33} - \tilde{A}_{33}) = I_{n-k-p}$, i.e., $s\tilde{E}_{33} - \tilde{A}_{33}$ is left invertible over $\mathbb{R}[s]$. This implies that in a quasi-Kronecker form (2.2) of $s\tilde{E}_{33} - \tilde{A}_{33}$ it holds $n_s = 0$ and $\ell(\beta) = 0$. By a permutation of the rows in the block $sK_\gamma^\top - L_\gamma^\top$ we may achieve that there exists $\hat{S} \in \mathbf{Gl}_{l-k-m}(\mathbb{R})$, $\hat{T} \in \mathbf{Gl}_{n-k-p}(\mathbb{R})$ such that $\hat{S}(s\tilde{E}_{33} - \tilde{A}_{33})\hat{T} = \left[\begin{smallmatrix} sN - I_{n_3} \\ s\hat{E}_{43} - \hat{A}_{43}\end{smallmatrix}\right]$, where N is nilpotent. Hence,

$$[\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}] \overset{\left[\begin{smallmatrix} I & 0 \\ 0 & \left[\begin{smallmatrix} I & 0 \\ 0 & \hat{S}\end{smallmatrix}\right]\end{smallmatrix}\right], \left[\begin{smallmatrix} I & 0 \\ 0 & \tilde{T}\cdot\left[\begin{smallmatrix} I & 0 \\ 0 & \hat{T}\end{smallmatrix}\right]\end{smallmatrix}\right]}{\sim} \left[\begin{bmatrix} I_k & \tilde{E}_{12} & \tilde{E}_{13} \\ 0 & \tilde{E}_{22} & \tilde{E}_{23} \\ 0 & \hat{E}_{32} & N \\ 0 & \hat{E}_{42} & \hat{E}_{43} \end{bmatrix}, \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} \\ 0 & \hat{A}_{32} & I_{n_3} \\ 0 & \hat{A}_{42} & \hat{A}_{43} \end{bmatrix}, \begin{bmatrix} 0 \\ I_m \\ 0 \\ 0 \end{bmatrix}, [0, I_p, 0] \right].$$

It is now clear that the assertion of the proposition follows from additional elementary row and column operations. □

The form derived in Theorem 5.6 is a generalization of the zero dynamics form derived in [6, Thm. 2.3] for system $[E, A, B, C] \in \Sigma_{n,n,m,m}$ with regular $sE - A$ and proper inverse transfer function.

Remark 5.7 (Uniqueness).
Uniqueness of the entries in the form (5.4) may be analyzed similar to the last statement in Theorem 3.7. It is easy to see that Q is unique up to similarity, and that there are entries which are not even unique up to matrix equivalence (cf. Remark 3.8). In particular, the form (5.4) is not a canonical form. ◇

Remark 5.8 (DAE of system inversion form and inverse system).
Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$ with autonomous zero dynamics and $\operatorname{rk} C = p$. The behaviour of the DAE (1.1) may be interpreted, in terms of the form (5.3), (5.4) in Theorem 5.6, as follows: $(x, u, y) \in \mathfrak{B}_{(1.1)} \cap \big(\mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R}; \mathbb{R}^n) \times \mathcal{L}^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^m) \times \mathcal{W}^{\nu+1,1}_{\text{loc}}(\mathbb{R}; \mathbb{R}^p)\big)$ if, and only if, (Tx, u, y) solves

$$\boxed{\begin{array}{rcl} \dot{x}_1 &=& Qx_1 + A_{12}y - \sum_{k=0}^{\nu-1} E_{13}N^k E_{32} y^{(k+2)} \\ 0 &=& -E_{22}\dot{y} - \sum_{k=0}^{\nu-1} E_{23}N^k E_{32} y^{(k+2)} + A_{21}x_1 + A_{22}y + u \\ x_3 &=& \sum_{k=0}^{\nu-1} N^k E_{32} y^{(k+1)} \\ 0 &=& -E_{42}\dot{y} - \sum_{k=0}^{\nu-1} E_{43}N^k E_{32} y^{(k+2)} + A_{42}y, \end{array}} \tag{5.5}$$

where $Tx = (x_1^\top, y^\top, x_3^\top)^\top \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R}; \mathbb{R}^{k+p+n_3})$; see also Figure 1.
From the form (5.4), also the inverse system can be read off immediately. Introducing the new variables $x_2 = y$ and $x_4 = \frac{\mathrm{d}}{\mathrm{d}t}\big(E_{22}x_2 + E_{23}x_3\big)$, an inverse system, with state $(x_1^\top, x_2^\top, x_3^\top, x_4^\top)^\top$, is given by

$$\left[\begin{bmatrix} I_k & 0 & E_{13} & 0 \\ 0 & E_{22} & E_{23} & 0 \\ 0 & E_{32} & N & 0 \\ 0 & E_{42} & E_{43} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} Q & A_{12} & 0 & 0 \\ 0 & 0 & 0 & I_m \\ 0 & 0 & I_{n_3} & 0 \\ 0 & A_{42} & 0 & 0 \\ 0 & -I_p & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I_p \end{bmatrix}, \begin{bmatrix} -A_{21} \\ -A_{22} \\ 0 \\ I_m \end{bmatrix}^\top \right] \in \Sigma_{l+p,n+m,p,m}.$$

◇

Remark 5.9 (Index of nilpotency).
The index of nilpotency ν of the matrix N arising in the form (5.4) in Theorem 5.6 may be larger than the index of the pencil $sE - A$: Consider

$$sE - A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & s \\ s & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = [1, 0, 0].$$

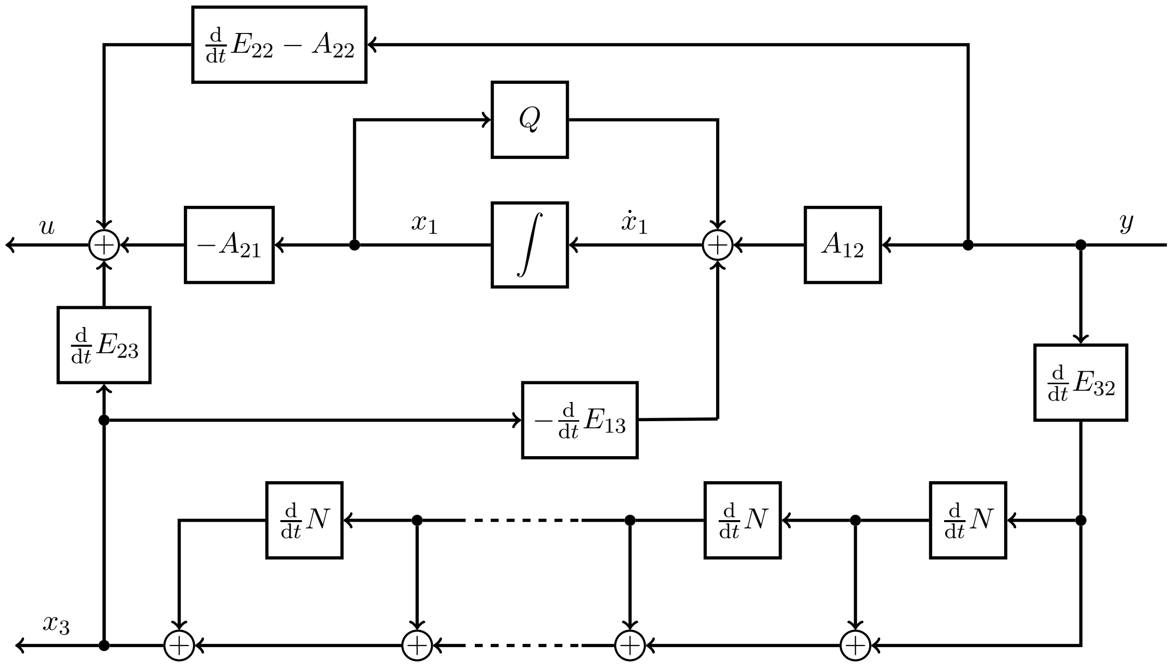


Figure 1: System $[E, A, B, C] \in \Sigma_{l,n,m,p}$ in form (5.4)

It is easy to see, that $[E, A, B, C]$ is in the form (5.4) with $k = 0$, $n_3 = 2$, $E_{32} = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$, $N = \left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]$ and all other entries in $\hat{E}$ in (5.4) are zero or not present. Hence $\nu = 2$, but the index of $sE - A$ is 1, since there exist $S, T \in \mathbf{Gl}_3(\mathbb{R})$ such that

$$S(sE - A)T = \begin{bmatrix} sK_1^\top - L_1^\top & 0 \\ 0 & sK_3 - L_3 \end{bmatrix},$$

i.e., we have an overdetermined block of size 1×0 and an underdetermined block of size 2×3. $\diamond$

The next corollary follows directly from Theorem 5.6 and the representation (5.5).

Corollary 5.10 (Asymptotically stable zero dynamics).
Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$ with autonomous zero dynamics and $\operatorname{rk} C = p$. Then, using the notation from Theorem 5.6, the zero dynamics $\mathcal{ZD}_{(1.1)}$ are asymptotically stable if, and only if, $\sigma(Q) \subseteq \mathbb{C}_-$.

As discussed in Remark 5.8, a realization of the inverse system can be found for $[E, A, B, C] \in \Sigma_{l,n,m,p}$ with autonomous zero dynamics and $\operatorname{rk} C = p$. However, due to the last equation in (5.5), $[E, A, B, C]$ is in general not right-invertible. Necessary and sufficient conditions for the latter are derived next.

Proposition 5.11 (System invertibility).
Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$ with autonomous zero dynamics. Then, in terms of the form (5.4) from Theorem 5.6,

$$[E, A, B, C] \textit{ is invertible} \iff \operatorname{rk} C = p, E_{42} = 0, A_{42} = 0 \textit{ and } E_{43}N^j E_{32} = 0 \textit{ for } j = 0, \ldots, \nu - 1.$$

Proof: By Lemma 5.5 $[E, A, B, C]$ is left-invertible, so it remains to show the equivalence for right-invertibility.
$\Rightarrow$: It is is clear that $\operatorname{rk} C = p$, otherwise we might choose any constant $y \equiv y^0$ with $y^0 \notin \operatorname{im} C$, which cannot be attained by the output of the system. Now, by Theorem 5.6 we may assume, without

loss of generality, that the system is in the form (5.5). Assume that $A_{42} \neq 0$. Hence, there exists $y^0 \in \mathbb{R}^p$ such that $A_{42}y^0 \neq 0$. Then, for $y \equiv y^0$ and all $x \in \mathcal{L}^1_{\text{loc}}(\mathbb{R};\mathbb{R}^n)$, $u \in \mathcal{L}^1_{\text{loc}}(\mathbb{R};\mathbb{R}^m)$ it holds that $(x, u, y) \notin \mathfrak{B}_{(1.1)}$ (since the last equation in (5.5) is not satisfied), which contradicts right-invertibility. Therefore, we have $A_{42} = 0$. Repeating the argument for E_{42} and $E_{43}N^jE_{32}$ with $y(t) = ty^0$ and $y(t) = t^{j+2}y^0$, resp., yields that $E_{42} = 0$ and $E_{43}N^jE_{32} = 0$, $j = 0, \ldots, \nu - 1$.
$\Leftarrow$: This is immediate from (5.5). $\square$

Remark 5.12.
Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$ with autonomous zero dynamics. If $l = n$, $p = m$ and $\operatorname{rk} C = m$, then $[E, A, B, C]$ is invertible. This can be seen using the form (5.4) from Theorem 5.6. $\diamond$

6 Funnel control

In this section we consider funnel control for systems $[E, A, B, C] \in \Sigma_{l,n,m,m}$ with the same number of inputs and outputs. For a motivation of funnel control we consider some classical control strategies: One possibility is constant high-gain control, that is the application of the controller

$$u(t) = -k\, y(t) \tag{6.1}$$

to the system (1.1) in order to achieve stabilization, i.e., that any solution $x \in \mathcal{C}^1(\mathbb{R};\mathbb{R}^n)$ of the closed-loop system (1.1), (6.1) satisfies $\lim_{t\to\infty} x(t) = 0$. Stabilization can be achieved for systems $[E, A, B, C]$ with asymptotically stable zero dynamics and either proper inverse transfer function or positive strict relative degree one by this strategy if the high gain $k > 0$ is chosen sufficiently large, see [5]. The system is then said to have the high-gain property. However, it is not known a priori how large the high gain constant must be.
Another strategy is the "classical" adaptive high-gain controller

$$u(t) = -k(t)y(t), \quad \dot{k}(t) = \|y(t)\|^2, \quad k(0) = k^0, \tag{6.2}$$

which resolves the above mentioned problem by adaptively increasing the high gain. The drawback of the control strategy (6.2) is that, albeit $k(\cdot)$ is bounded, it is monotonically increasing and potentially so large that it is very sensitive to noise corrupting the output measurement. Further drawbacks are that (6.2) does not tolerate mild output perturbations, tracking would require an internal model and, most importantly, transient behaviour is not taken into account. These issues are discussed for ODE systems (with strictly proper transfer function of strict relative degree one and asymptotically stable zero dynamics) in the survey [22].
To overcome these drawbacks, the concept of "funnel control" is introduced (see [22] and the references therein): For any function φ belonging to

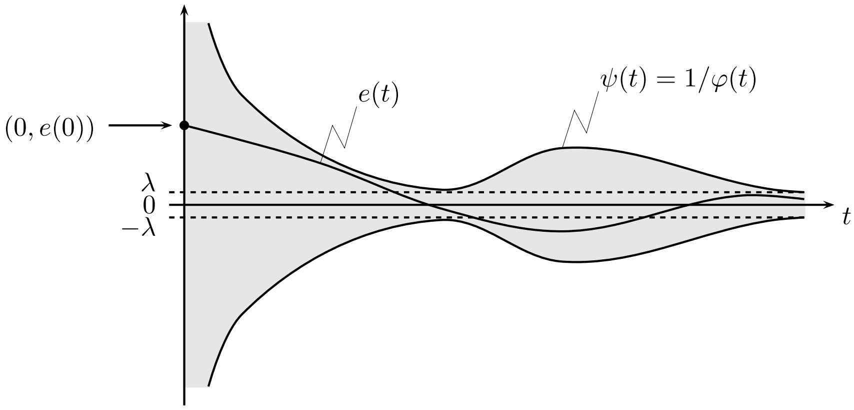
$$\Phi^\mu := \left\{ \varphi \in \mathcal{C}^\mu(\mathbb{R}_{\geq 0};\mathbb{R}) \cap \mathcal{B}^1(\mathbb{R}_{\geq 0};\mathbb{R}) \;\middle|\; \varphi(0) = 0, \quad \varphi(s) > 0 \text{ for all } s > 0 \text{ and } \liminf_{s\to\infty} \varphi(s) > 0 \right\},$$

for $\mu \in \mathbb{N}$, we associate the *performance funnel*

$$\mathcal{F}_\varphi := \left\{ (t, e) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \;\middle|\; \varphi(t)\|e\| < 1 \right\}, \tag{6.3}$$

see Figure 2. The control objective is feedback control so that the tracking error $e(\cdot) = y(\cdot) - y_{\text{ref}}(\cdot)$, where $y_{\text{ref}}(\cdot)$ is the reference signal, evolves within $\mathcal{F}_\varphi$ and all variables are bounded. More specific, the transient behaviour is supposed to satisfy

$$\forall\, t > 0: \; \|e(t)\| < 1/\varphi(t),$$



"Infinite" funnel, that is the funnel defined on $(0, \infty)$ with pole at $t = 0$.

Figure 2: Error evolution in a funnel $\mathcal{F}_\varphi$ with boundary $\psi(t) = 1/\varphi(t)$ for $t > 0$.

and, moreover, if φ is chosen so that $\varphi(t) \geq 1/\lambda$ for all t sufficiently large, then the tracking error remains smaller than λ.

By choosing $\varphi(0) = 0$ we ensure that the width of the funnel is infinity at $t = 0$, see Figure 2. In the following we only treat "infinite" funnels for technical reasons, since if the funnel is finite, that is $\varphi(0) > 0$, then we need to assume that the initial error is within the funnel boundaries at $t = 0$, i.e., $\varphi(0)\|Cx^0 - y_{\mathrm{ref}}(0)\| < 1$, and this assumption suffices.

As indicated in Figure 2, we do not assume that the funnel boundary decreases monotonically. Certainly, in most situations it is convenient to choose a monotone funnel, however there are situations where widening the funnel at some later time might be beneficial, e.g., when it is known that the reference signal varies strongly.

To ensure error evolution within the funnel, we introduce, for $\hat{k} > 0$, the *funnel controller*:

$$\boxed{\begin{array}{lll} u(t) = -k(t)\, e(t), & \text{where} & e(t) = y(t) - y_{\mathrm{ref}}(t) \\ k(t) = \dfrac{\hat{k}}{1 - \varphi(t)^2 \|e(t)\|^2}\,. & & \end{array}} \tag{6.4}$$

If we assume asymptotically stable zero dynamics, we see intuitively that, in order to maintain the error evolution within the funnel, high gain values may only be required if the norm $\|e(t)\|$ of the error is close to the funnel boundary $\varphi(t)^{-1}$: $k(\cdot)$ increases if necessary to exploit the high-gain property of the system and decreases if a high gain is not necessary. This intuition underpins the choice of the gain $k(t)$ in (6.4), where the constant $\hat{k} > 0$ is only of technical importance, see Remark 6.1. The control design (6.4) has two advantages: $k(\cdot)$ is non-monotone and (6.4) is a static time-varying proportional output feedback of striking simplicity.

In the following we show that funnel control for systems (1.1) is feasible under some appropriate assumptions. In [6] it is shown that funnel control works for DAE systems with regular pencil $sE - A$, proper inverse transfer function and asymptotically stable zero dynamics. In [5] it is then shown that funnel control is also feasible if the assumption of proper inverse transfer function is replaced by the existence of a positive strict relative degree, however a filter has to be incorporated in the feedback in this case, see also [25]. What we have presented in the present paper so far is a unified framework for both cases "proper inverse transfer function" and "positive strict relative degree one" and, furthermore, we do not need to assume that $sE - A$ is regular. In fact, we only need the following assumptions for funnel control being feasible for a system $[E, A, B, C] \in \Sigma_{l,n,m,m}$:

- $[E, A, B, C]$ has asymptotically stable zero dynamics,
- $[E, A, B, C]$ is right-invertible,
- $\hat{k}$ in (6.4) is sufficiently large,
- the matrix
$$\Gamma = -\lim_{s\to\infty} s^{-1}[0, I_m]L(s)\begin{bmatrix}0\\ I_m\end{bmatrix} \in \mathbb{R}^{m\times m} \tag{6.5}$$
exists and satisfies $\Gamma = \Gamma^\top \geq 0$, where $L(s)$ denotes a left inverse of $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0\end{smallmatrix}\right]$ over $\mathbb{R}(s)$; by Lemma A.1, Γ is independent of the choice of $L(s)$.

As mentioned above, these assumptions now give a unified approach to two classes of systems which have been treated separately in [5] (cf. also Appendix B): Systems $[E, A, B, C] \in \Sigma_{n,n,m,m}$ with regular $sE - A$, asymptotically stable zero dynamics, $\hat{k}$ sufficiently large and either

- positive strict relative degree one and symmetric, positive definite high frequency gain matrix

or

- proper inverse transfer function.

In [5, Thm. 5.1] it has been proved that funnel control is feasible for the above two classes of systems. Note that for single-input, single-output systems with transfer function $g(s) = \frac{p(s)}{q(s)} \in \mathbb{R}(s) \setminus \{0\}$, the existence of Γ in (6.5) is equivalent to $\deg q(s) - \deg p(s) \leq 1$, i.e., $g(s)$ has strict relative degree smaller or equal to one.

Remark 6.1 (Initial gain)**.**
The condition "$\hat{k}$ sufficiently large" in the above motivated assumptions of Theorem 6.3 is made precise in (6.7). Condition (6.7) is specific for DAEs and already appears in [5, 6], but not in the ODE case, see [24]. A careful inspection of the proof of Theorem 6.3 shows that we have to ensure that the matrix $\hat{A}_{22} - k(t)I_m$ is invertible for all $t \geq 0$, and in order to avoid singularities we choose, as a simple condition, the "minimal value" $\hat{k}$ of $k(\cdot)$ to be larger than $\|\tilde{A}\| \geq \|\hat{A}_{22}\|$. In most cases the lower bound for $\hat{k}$ in (6.7) can be calculated easily. We perform the calculation for some classes of ODEs: Consider the system
$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t)\,, \end{aligned} \tag{6.6}$$
where $A \in \mathbb{R}^{n\times n}, B, C^\top \in \mathbb{R}^{n\times m}, D \in \mathbb{R}^{m\times m}$. System (6.6) can be rewritten in the form (1.1) as
$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} I_n & 0 \\ 0 & 0\end{bmatrix}\begin{pmatrix} x(t) \\ w(t)\end{pmatrix} &= \begin{bmatrix} A & 0 \\ 0 & -I_m\end{bmatrix}\begin{pmatrix} x(t) \\ w(t)\end{pmatrix} + \begin{bmatrix} B \\ D\end{bmatrix} u(t) \\ y(t) &= [C, I]\begin{pmatrix} x(t) \\ w(t)\end{pmatrix}. \end{aligned}$$

Observe that $s\left[\begin{smallmatrix} I_n & 0 \\ 0 & 0\end{smallmatrix}\right] - \left[\begin{smallmatrix} A & 0 \\ 0 & -I_m\end{smallmatrix}\right]$ is regular, and hence applying Remark A.4 gives
$$\Gamma = \lim_{s\to\infty} s^{-1}\left([C, I]\begin{bmatrix} sI - A & 0 \\ 0 & I\end{bmatrix}^{-1}\begin{bmatrix} B \\ D\end{bmatrix}\right)^{-1} = \lim_{s\to\infty} s^{-1}\left(C(sI - A)^{-1}B + D\right)^{-1}.$$

Assume now that $D \in \mathbf{Gl}_m(\mathbb{R})$, i.e., the system has strict relative degree 0. Then

$$\Gamma = \lim_{s\to\infty} s^{-1} D^{-1} \sum_{k=0}^{\infty} \left(-D^{-1}C(sI-A)^{-1}B\right)^k = 0,$$

and

$$\lim_{s\to\infty} \left([0, I_m] L(s) \begin{bmatrix} 0 \\ I_m \end{bmatrix} + s\Gamma \right) = - \lim_{s\to\infty} D^{-1} \sum_{k=0}^{\infty} \left(-D^{-1}C(sI-A)^{-1}B\right)^k = -D^{-1}.$$

Therefore, (6.7) reads $\hat{k} > \|D^{-1}\|$. If $D = 0$ and $CB \in \mathbf{Gl}_m(\mathbb{R})$, i.e., the system has strict relative degree 1, then similar calculations lead to $\Gamma = \big(CB\big)^{-1}$ and (6.7) simply reads $\hat{k} > 0$; the latter is a general condition compared to the choice $\hat{k} = 1$ in [24].
For single-input, single-output systems the above conditions can also be motivated by just looking at the output equation $y = cx + du$, $c^\top \in \mathbb{R}^n, d \in \mathbb{R}$. If a feedback $u = -ky$, $k > 0$ is applied, then $(1 + dk)y = cx$ and in order to solve this equation for y it is sufficient that either $k > 0$ (no further condition) if $d = 0$, or $k > |d^{-1}|$ if $d \neq 0$. $\diamond$

Before we state our main result, we need to define consistency of the initial value of the closed-loop system and solutions of the latter. Compared to the previous sections, here we require more smoothness of the trajectories.

Definition 6.2 (Consistent initial value)**.**
Let $[E, A, B, C] \in \Sigma_{l,n,m,m}$ and $y_{\text{ref}} \in \mathcal{B}^1(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$. An initial value $x^0 \in \mathbb{R}^n$ is called *consistent* for the closed-loop system (1.1), (6.4) if, and only if, there exists a solution of the initial value problem (1.1), (6.4), $x(0) = x^0$, i.e., a function $x \in \mathcal{C}^1([0, \omega); \mathbb{R}^n)$ for some $\omega \in (0, \infty]$, such that $x(0) = x^0$ and x satisfies (1.1), (6.4) for all $t \in [0, \omega)$. $\diamond$

Note that, in practice, consistency of the initial state of the "unknown" system should be satisfied as far as the DAE $[E, A, B, C]$ is the correct model.
We are now in a position to state the main result of this section.

Theorem 6.3 (Funnel control)**.**
Let $[E, A, B, C] \in \Sigma_{l,n,m,m}$ be right-invertible and have asymptotically stable zero dynamics. Suppose that, for a left inverse $L(s)$ of $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ over $\mathbb{R}(s)$, the matrix Γ in (6.5) exists and satisfies $\Gamma = \Gamma^\top \geq 0$. Using the notation from Theorem 5.6, let $\varphi \in \Phi^{\nu+1}$ define a performance funnel $\mathcal{F}_\varphi$. Then, for any reference signal $y_{\text{ref}} \in \mathcal{B}^{\nu+2}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$, any consistent initial value $x^0 \in \mathbb{R}^n$, and initial gain

$$\hat{k} > \left\| \lim_{s\to\infty} \left([0, I_m] L(s) \begin{bmatrix} 0 \\ I_m \end{bmatrix} + s\Gamma \right) \right\|, \tag{6.7}$$

the application of the funnel controller (6.4) to (1.1) yields a closed-loop initial-value problem that has a solution and every solution can be extended to a global solution. Furthermore, for every global solution $x(\cdot)$,

(i) *$x(\cdot)$ is bounded and the corresponding tracking error $e(\cdot) = Cx(\cdot) - y_{\text{ref}}(\cdot)$ evolves uniformly within the performance funnel $\mathcal{F}_\varphi$; more precisely,*

$$\exists\, \varepsilon > 0\ \forall\, t > 0 : \ \|e(t)\| \leq \varphi(t)^{-1} - \varepsilon\,. \tag{6.8}$$

(ii) *the corresponding gain function $k(\cdot)$ given by (6.4) is bounded:*

$$\forall\, t_0 > 0 : \ \sup_{t \geq t_0} |k(t)| \leq \frac{|\hat{k}|}{1 - (1 - \varepsilon\lambda_{t_0})^2},$$

where $\lambda_{t_0} := \inf_{t \geq t_0} \varphi(t) > 0$ for all $t_0 > 0$.

Proof: Note that Γ is well-defined by Lemma A.1. We proceed in several steps.
Step 1: By Lemma A.3, the closed-loop system (1.1), (6.4) is, without loss of generality, in the form

$$\begin{aligned}
\dot{x}_1(t) &= Q\,x_1(t) + A_{12}\,e(t) - \textstyle\sum_{k=0}^{\nu-1} E_{13}N^k E_{32}\,e^{(k+2)}(t) \\
&\quad + A_{12}\,y_{\mathrm{ref}}(t) - \textstyle\sum_{k=0}^{\nu-1} E_{13}N^k E_{32}\,y_{\mathrm{ref}}^{(k+2)}(t) \\
\Gamma\,\dot{e}(t) &= (\tilde{A} - k(t)I_m)\,e(t) + \tilde{A}\,y_{\mathrm{ref}}(t) - \Gamma\,\dot{y}_{\mathrm{ref}}(t) + \tilde{\Psi}(x_1^0, e)(t) \\
x_3(t) &= \textstyle\sum_{k=0}^{\nu-1} N^k E_{32}\,e^{(k+1)}(t) + \sum_{k=0}^{\nu-1} N^k E_{32}\,y_{\mathrm{ref}}^{(k+1)}(t) \\
k(t) &= \tfrac{\hat{k}}{1-\varphi(t)^2\|e(t)\|^2},
\end{aligned} \tag{6.9}$$

where $\tilde{A} = A_{22} - \sum_{k=0}^{\nu-1} A_{21}Q^{k+1}E_{13}N^kE_{32}$, $x_1^0 = [I_k, 0, 0]T^{-1}x^0$ and

$$\tilde{\Psi}(x_1^0, e)(t) = \Psi(x_1^0, e)(t) + \Psi(x_1^0, y_{\mathrm{ref}})(t) - A_{21}e^{Qt}x_1^0, \quad t \in \mathbb{R}.$$

Note that, as $[0, I_m]L(s)[0, I_m]^\top = X_{45}(s)$ for the representation in (A.2),

$$\hat{k} > \left\| \lim_{s\to\infty} \left([0, I_m]L(s) \begin{bmatrix} 0 \\ I_m \end{bmatrix} + s\Gamma \right) \right\| = \left\| A_{22} - \sum_{k=0}^{\nu-1} A_{21}Q^{k+1}E_{13}N^kE_{32} \right\| = \|\tilde{A}\|.$$

By consistency of the initial value x^0 there exists a local solution $(x_1, e, x_3, k) \in \mathcal{C}^1([0, \rho); \mathbb{R}^{n+1})$ of (6.9) for some $\rho > 0$ and initial data

$$(x_1, e, x_3, k)(0) = \begin{pmatrix} T^{-1}x^0 - \begin{pmatrix} 0 \\ y_{\mathrm{ref}}(0) \\ 0 \end{pmatrix} \\ \hat{k} \end{pmatrix},$$

where the differentiability follows since $y_{\mathrm{ref}} \in \mathcal{B}^{\nu+2}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$ and $\varphi \in \mathcal{C}^{\nu+1}(\mathbb{R}_{\geq 0}; \mathbb{R})$. It is clear that $(t, e(t))$ belongs to the set $\mathcal{F}_\varphi$ for all $t \in [0, \rho)$. Even more so, we have that

$$\forall\, t \in [0, \rho): \ (t, x_1(t), e(t), x_3(t), k(t)) \in \tilde{\mathcal{D}} := \left\{ \ (t, x_1, e, x_3, k) \in [0, \infty) \times \mathbb{R}^{n+1} \ \middle| \ \varphi(t)\|e\| < 1 \ \right\}.$$

We will now, for the time being, ignore the first and third equation in (6.9) and construct an integral-differential equation from the second and fourth equation, which is solved by (e, k). To this end, observe that by $\Gamma = \Gamma^\top \geq 0$, there exists an orthogonal matrix $V \in \mathbf{Gl}_m(\mathbb{R})$ and a diagonal matrix $D \in \mathbb{R}^{m_1 \times m_1}$ with only positive entries for some $0 \leq m_1 \leq m$, such that

$$\Gamma = V^\top \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} V.$$

In order to decouple the second equation in (6.9) into an ODE and an algebraic equation, we introduce the new variables $e_1(\cdot) = [I_{m_1}, 0]Ve(\cdot)$ and $e_2(\cdot) = [0, I_{m-m_1}]Ve(\cdot)$. Rewriting (6.9) and invoking $\|e(t)\|^2 = \|Ve(t)\|^2 = \|e_1(t)\|^2 + \|e_2(t)\|^2$, this leads to the system

$$\begin{aligned}
\dot{e}_1(t) &= [D^{-1}, 0](V\tilde{A}V^\top - k(t)I_m) \begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix} + [D^{-1}, 0]V\,\Theta_1(e_1, e_2)(t) \\
0 &= [0, I_{m-m_1}]V\tilde{A}V^\top \begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix} - k(t)e_2(t) + [0, I_{m-m_1}]V\,\Theta_1(e_1, e_2)(t) \\
k(t) &= \tfrac{\hat{k}}{1-\varphi(t)^2(\|e_1(t)\|^2+\|e_2(t)\|^2)},
\end{aligned} \tag{6.10}$$

where

$$\begin{aligned}\Theta_1 : \mathcal{C}^\nu(\mathbb{R};\mathbb{R}^{m_1}) \times \mathcal{C}^\nu(\mathbb{R};\mathbb{R}^{m-m_1}) &\to \mathcal{C}^{\nu+1}(\mathbb{R};\mathbb{R}^m),\\ (e_1,e_2) &\mapsto \Big(t \mapsto \tilde{A}\, y_{\rm ref}(t) - \Gamma\, \dot{y}_{\rm ref}(t) + \tilde{\Psi}(x_1^0, V^\top(e_1^\top, e_2^\top)^\top)(t)\Big).\end{aligned}$$

Introduce the set

$$\mathcal{D} := \left\{ \ (t,k,e_1,e_2) \in [0,\infty) \times [\hat{k},\infty) \times \mathbb{R}^{m_1} \times \mathbb{R}^{m-m_1} \ \middle| \ \varphi(t)^2(\|e_1\|^2 + \|e_2\|^2) < 1 \ \right\}$$

and define

$$f_1 : \mathcal{D} \times \mathbb{R}^m \to \mathbb{R}^{m_1},\ (t,k,e_1,e_2,\xi) \mapsto [D^{-1},0](V\tilde{A}V^\top - kI_m)\begin{pmatrix} e_1\\ e_2\end{pmatrix} + [D^{-1},0]V\,\xi.$$

By differentiation of the second equation in (6.10), and using

$$\hat{A}_{22} = [0, I_{m-m_1}]V\tilde{A}V^\top \begin{bmatrix} 0\\ I_{m-m_1}\end{bmatrix}, \quad \hat{A}_{21} = [0, I_{m-m_1}]V\tilde{A}V^\top \begin{bmatrix} I_{m_1}\\ 0\end{bmatrix},$$

we get

$$0 = \hat{A}_{21}\dot{e}_1(t) + \hat{A}_{22}\dot{e}_2(t) - \dot{k}(t)e_2(t) - k(t)\dot{e}_2(t) + [0, I_{m-m_1}]V\,\tfrac{\rm d}{{\rm d}t}\Theta_1(e_1,e_2)(t). \tag{6.11}$$

Observe that the derivative of k is given by

$$\begin{aligned}\dot{k}(t) = 2\hat{k}\left(1 - \varphi(t)^2(\|e_1(t)\|^2 + \|e_2(t)\|^2)\right)^{-2}&\\ \times \Big(\varphi(t)\dot{\varphi}(t)(\|e_1(t)\|^2 + \|e_2(t)\|^2) + \varphi(t)^2(e_1(t)^\top\dot{e}_1(t) + e_2(t)^\top\dot{e}_2(t))\Big)&.\end{aligned} \tag{6.12}$$

Now let

$$M : \mathcal{D} \to \mathbf{Gl}_{m-m_1}(\mathbb{R}),\ (t,k,e_1,e_2) \mapsto \left(\hat{A}_{22} - k\left(I_{m-m_1} + 2\varphi(t)^2\left(1 - \varphi(t)^2(\|e_1\|^2 + \|e_2\|^2)\right)^{-2} e_2 e_2^\top\right)\right),$$

$$\begin{aligned}&\Theta_2 : \mathcal{C}^\nu(\mathbb{R};\mathbb{R}^{m_1}) \times \mathcal{C}^\nu(\mathbb{R};\mathbb{R}^{m-m_1}) \to \mathcal{C}^\nu(\mathbb{R};\mathbb{R}^m),\\ &(e_1,e_2) \mapsto \Bigg(t \mapsto \tilde{A}\,\dot{y}_{\rm ref}(t) - \Gamma\,\ddot{y}_{\rm ref}(t) + \tfrac{\rm d}{{\rm d}t}\Psi(x_1^0, y_{\rm ref})(t) + \int_0^t A_{21}Qe^{Q(t-\tau)}A_{12}\,V^\top\begin{pmatrix} e_1(\tau)\\ e_2(\tau)\end{pmatrix}{\rm d}\tau\\ &\qquad + \sum_{k=0}^{\nu-1}\sum_{j=0}^{k+1} A_{21}Q^{j+1}e^{Qt}E_{13}N^kE_{32}\,V^\top\begin{pmatrix} e_1\\ e_2\end{pmatrix}^{(k-j+1)}(0)\\ &\qquad - \sum_{k=0}^{\nu-1}\int_0^t A_{21}Q^{k+3}e^{Q(t-\tau)}E_{13}N^kE_{32}\,V^\top\begin{pmatrix} e_1(\tau)\\ e_2(\tau)\end{pmatrix}{\rm d}\tau\Bigg)\end{aligned}$$

and

$$\begin{aligned}&f_2 : \mathcal{D} \times \mathbb{R}^{m_1} \times \mathbb{R}^m \to \mathbb{R}^{(m-m_1)},\ (t,k,e_1,e_2,\tilde{e}_1,\xi) \mapsto\\ &\quad 2\hat{k}\left(1 - \varphi(t)^2(\|e_1\|^2 + \|e_2\|^2)\right)^{-2}\left(\varphi(t)\dot{\varphi}(t)(\|e_1\|^2 + \|e_2\|^2) + \varphi(t)^2(e_1^\top\tilde{e}_1)\right)e_2\\ &\quad -\hat{A}_{21}\tilde{e}_1 - [0, I_{m-m_1}]V\left(A_{21}A_{12}V^\top\begin{pmatrix} e_1\\ e_2\end{pmatrix} + \textstyle\sum_{k=0}^{\nu-1} A_{21}Q^{k+2}E_{12}N^kE_{32}V^\top\begin{pmatrix} e_1\\ e_2\end{pmatrix} + \xi\right).\end{aligned}$$

We show that M is well-defined. To this end let

$$G : \mathcal{D} \to \mathbb{R}^{(m-m_1)\times(m-m_1)},\ (t,k,e_1,e_2) \mapsto 2\varphi(t)^2 \left(1-\varphi(t)^2(\|e_1\|^2+\|e_2\|^2)\right)^{-2} e_2 e_2^\top$$

and observe that G is symmetric and positive semi-definite everywhere, hence there exist $\hat{V} : \mathcal{D} \to \mathbb{R}^{(m-m_1)\times(m-m_1)}$, $\hat{V}$ orthogonal everywhere, and $\hat{D} : \mathcal{D} \to \mathbb{R}^{(m-m_1)\times(m-m_1)}$, $\hat{D}$ a diagonal matrix with nonnegative entries everywhere, such that $G = \hat{V}^{-1}\hat{D}\hat{V}$. Therefore, $(I+G)^{-1} = \hat{V}^{-1}(I+\hat{D})^{-1}\hat{V}$ and $(I+\hat{D})^{-1}$ is diagonal with entries in $(0,1]$ everywhere, which implies that $\|(I+G)^{-1}\| \le 1$. Then, for all $(t,k,e_1,e_2) \in \mathcal{D}$, we obtain

$$\|k^{-1}(I+G(t,k,e_1,e_2))^{-1}\hat{A}_{22}\| \le \hat{k}\|\hat{A}_{22}\| \le \hat{k}\|\tilde{A}\| < 1.$$

and hence $k^{-1}(I+G(t,e_1,e_2))^{-1}\hat{A}_{22} - I$ is invertible, which gives invertibility of

$$M(t,k,e_1,e_2) = \hat{A}_{22} - k(I+G(t,k,e_1,e_2)).$$

Now, inserting $\dot{k}$ from (6.12) into (6.11) and rearranging according to $\dot{e}_2$ gives

$$M\big(t,k(t),e_1(t),e_2(t)\big)\,\dot{e}_2(t) = f_2\big(t,k(t),e_1(t),\dot{e}_1(t),e_2(t),\Theta_2(e_1,e_2)(t)\big).$$

With

$$\tilde{f}_2 : \mathcal{D}\times\mathbb{R}^m\times\mathbb{R}^m \to \mathbb{R}^{(m-m_1)},\ (t,k,e_1,e_2,\xi_1,\xi_2) \mapsto M(t,k,e_1,e_2)^{-1} f_2(t,k,e_1,e_2,f_1(t,k,e_1,e_2,\xi_1),\xi_2),$$

and

$$\begin{aligned} f_3 : \mathcal{D}\times\mathbb{R}^m\times\mathbb{R}^m \to \mathbb{R},\ (t,k,e_1,e_2,\xi_1,\xi_2) &\mapsto 2\hat{k}\left(1-\varphi(t)^2(\|e_1\|^2+\|e_2\|^2)\right)^{-2} \\ &\times \Big(\varphi(t)\dot{\varphi}(t)(\|e_1\|^2+\|e_2\|^2) + \varphi(t)^2(e_1^\top f_1(t,k,e_1,e_2,\xi_1) + e_2^\top \tilde{f}_2(t,k,e_1,e_2,\xi_1,\xi_2))\Big) \end{aligned}$$

we get the system

$$\begin{aligned} \dot{e}_1(t) &= f_1(t,k(t),e_1(t),e_2(t),\Theta_1(e_1,e_2)(t)) \\ \dot{e}_2(t) &= \tilde{f}_2(t,k(t),e_1(t),e_2(t),\Theta_1(e_1,e_2)(t),\Theta_2(e_1,e_2)(t)) \\ \dot{k}(t) &= f_3(t,k(t),e_1(t),e_2(t),\Theta_1(e_1,e_2)(t),\Theta_2(e_1,e_2)(t)). \end{aligned} \tag{6.13}$$

$(k,e_1,e_2) \in \mathcal{C}^1([0,\rho);\mathbb{R}^{m+1})$ obtained from (e,k) is a local solution of (6.13) with

$$(k,e_1,e_2)(0) = \Big(\hat{k}, V([0,I_m,0]T^{-1}x^0 - y_{\mathrm{ref}}(0))\Big) =: \eta$$

and

$$\forall\, t \in [0,\rho):\ (t,k(t),e_1(t),e_2(t)) \in \mathcal{D}.$$

Step 2: We show that the local solution (x_1,e,x_3,k) can be extended to a maximal solution, the graph of which leaves every compact subset of $\tilde{\mathcal{D}}$.
With $z = (k,e_1^\top,e_2^\top)^\top$ and appropriate $\tilde{F} : \mathcal{D}\times\mathbb{R}^{2m} \to \mathbb{R}^{m+1}$, we may write (6.13) in the form

$$\dot{z}(t) = \tilde{F}(t,z(t),(\tilde{T}z)(t)),$$

where $\tilde{T}z = (\Theta_1(e_1,e_2)^\top,\Theta_2(e_1,e_2)^\top)^\top$. We may further rewrite this equation to achieve that the operator is independent of the initial values $z(0),\dot{z}(0),\ldots,z^{(\nu)}(0)$: we put every term where one of the expressions $z(0), \dot{z}(0), \ldots, z^{(\nu)}(0)$ explicitly appears in the function $\tilde{F}$. This leads to a formulation of the form

$$\dot{z}(t) = F(t, z(t), (Tz)(t), z(0), \dot{z}(0), \ldots, z^{(\nu)}(0)), \quad z(0) = \eta, \tag{6.14}$$

where $F : \mathcal{D} \times \mathbb{R}^{2m} \times \mathbb{R}^{(m+1)(\nu+1)} \to \mathbb{R}^{m+1}$ is a suitable function and $T : \mathcal{C}(\mathbb{R}; \mathbb{R}^{m+1}) \to \mathcal{C}(\mathbb{R}; \mathbb{R}^{2m})$ is an operator with the properties as in [23, Def. 2.1] (note that in [23] only operators with domain $\mathcal{C}(\mathbb{R}; \mathbb{R})$ are considered, but the generalization to domain $\mathcal{C}(\mathbb{R}; \mathbb{R}^q)$ is straightforward). It is immediate that T satisfies properties (i)–(iii) in [23, Def. 2.1]; (iv) follows from the fact that $\sigma(Q) \subseteq \mathbb{C}_-$ by the asymptotically stable zero dynamics (cf. also (A.7)) and $y_{\text{ref}} \in \mathcal{B}^{\nu+2}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$.
Furthermore, for $\mu := \max\{1, \nu\}$ and the functions defined in Step 1, we find that f_1 and f_2 are μ-times continuously differentiable (since $\varphi \in \mathcal{C}^{\nu+1}(\mathbb{R}_{\geq 0}; \mathbb{R})$). Furthermore, M is μ-times continuously differentiable and invertible on $\mathcal{D}$, hence M^{-1} is μ-times continuously differentiable as well. Finally, this gives that $\tilde{f}_2$ and f_3 are μ-times continuously differentiable and hence, since F depends linearly on the initial values $z(0), \dot{z}(0), \ldots, z^{(\nu)}(0)$, we have $F \in \mathcal{C}^{\mu}(\mathcal{D} \times \mathbb{R}^{2m} \times \mathbb{R}^{(m+1)(\nu+1)}; \mathbb{R}^{m+1})$.
Let $\tilde{z} = (k, e_1^\top, e_2^\top)^\top \in \mathcal{C}^1([0, \rho); \mathbb{R}^{m+1})$ be the local solution of (6.13) obtained at the end of Step 1. Then $\tilde{z}$ solves (6.14). Observe that, since F is μ-times continuously differentiable and T is essentially an integral-operator, i.e., it increments the degree of differentiability, we have $\tilde{z} \in \mathcal{C}^{\mu+1}([0, \rho); \mathbb{R}^{m+1})$. Now let $\zeta_j := \tilde{z}^{(j)}(0)$ for $j = 0, \ldots, \nu$. Then [23, Thm. B.1][1] is applicable to the system

$$\dot{z}(t) = F(t, z(t), (Tz)(t), \zeta_0, \zeta_1, \ldots, \zeta_\nu) =: \hat{F}(t, z(t), (Tz)(t)), \quad z(0) = \zeta_0 = \eta, \tag{6.15}$$

and we may conclude that

(a) there exists a solution of (6.15), i.e., a function $z \in \mathcal{C}([0, \rho); \mathbb{R}^{m+1})$ for some $\rho \in (0, \infty]$ such that z is locally absolutely continuous, $z(0) = \zeta_0$, $(t, z(t)) \in \mathcal{D}$ for all $t \in [0, \rho)$ and (6.15) holds for almost all $t \in [0, \rho)$,

(b) every solution can be extended to a maximal solution $z \in \mathcal{C}([0, \omega); \mathbb{R}^{m+1})$, i.e., z has no proper right extension that is also a solution,

(c) if $z \in \mathcal{C}([0, \rho); \mathbb{R}^{m+1})$ is a maximal solution, then the closure of graph z is not a compact subset of $\mathcal{D}$.

(c) follows since $\hat{F}$ is locally essentially bounded, as it is at least continuously differentiable. Clearly $\tilde{z}$ is a solution (in the context of (a)) of (6.15), hence by (b) it can be extended to a maximal solution $\hat{z} \in \mathcal{C}([0, \omega); \mathbb{R}^{m+1})$. Similar to $\tilde{z}$, $\hat{z}$ is $(\mu + 1)$-times continuously differentiable.
We show that the extended solution $\hat{z}$ leads to an extended solution of (6.9). To this end we show that $\hat{z}$ also solves (6.14). This is immediate since, invoking $\hat{z}|_{[0,\rho)} = \tilde{z}$,

$$\forall\, j = 0, \ldots, \nu : \ \hat{z}^{(j)}(0) = \tilde{z}^{(j)}(0) = \zeta_j.$$

This implies that $\hat{z}$ is a solution of (6.14) as well and therefore also solves (6.13). Integrating the equations for k and e_2 in (6.13) and invoking consistency of the initial values gives that (k, e_1, e_2) also solve the problem (6.10) and this leads to a maximal solution $(x_1, e, x_3, k) \in \mathcal{C}^1([0, \omega); \mathbb{R}^{n+1})$, $\omega \in (0, \infty]$, of (6.9) (extension of the original local solution (x_1, e, x_3, k) - for brevity we use the same notation) with graph $(x_1, e, x_3, k) \subseteq \tilde{\mathcal{D}}$. Furthermore, by (c) we have

$$\text{the closure of graph}\,(x_1, e, x_3, k) \text{ is not a compact subset of } \tilde{\mathcal{D}}. \tag{6.16}$$

[1]In [23] a domain $\mathcal{D} \subseteq [0, \infty) \times \mathbb{R}$ is considered, but the generalization to the higher dimensional case is only a technicality.

Step 3: We show that k is bounded. Seeking a contradiction, assume that $k(t) \to \infty$ for $t \to \omega$. Using $e_1(\cdot) = [I_{m_1}, 0]Ve(\cdot)$ and $e_2(\cdot) = [0, I_{m-m_1}]Ve(\cdot)$, we obtain from (6.10) that

$$\|e_2(t)\| \le \|\big(\hat{A}_{22} - k(t)I_{m-m_1}\big)^{-1}\| \left(\|\hat{A}_{21}e_1(t)\| + \|[0, I_{m-m_1}]V\Theta_1(e_1, e_2)(t)\|\right).$$

Observing that, since $\|\hat{A}_{22}\| \le \|\tilde{A}\| < \hat{k}$,

$$\|\big(\hat{A}_{22}-k(t)I_{m-m_1}\big)^{-1}\| = k(t)^{-1}\|\big(I_{m-m_1}-k(t)^{-1}\hat{A}_{22}\big)^{-1}\| \le k(t)^{-1}\frac{1}{1-k(t)^{-1}\|\hat{A}_{22}\|} \le k(t)^{-1}\frac{\hat{k}}{\hat{k}-\|\hat{A}_{22}\|},$$

and invoking boundedness of e_1 (since e evolves within the funnel) and boundedness of $\Theta_1(e_1, e_2)$ (since $y_{\mathrm{ref}} \in \mathcal{B}^{\nu+2}(\mathbb{R}_{\ge 0}; \mathbb{R}^m)$ and (A.7) holds) we obtain

$$\|e_2(t)\| \le k(t)^{-1}\frac{\hat{k}}{\hat{k}-\|\hat{A}_{22}\|}\left(\|\hat{A}_{21}e_1\|_\infty + \|[0, I_{m-m_1}]V\Theta_1(e_1, e_2)\|_\infty\right) \xrightarrow[t\to\omega]{} 0. \tag{6.17}$$

Now, if $m_1 = 0$ then $e = e_2$ and we have $\lim_{t\to\omega} \|e(t)\| = 0$, which implies, by boundedness of φ, $\lim_{t\to\omega} \varphi(t)^2\|e(t)\|^2 = 0$, hence $\lim_{t\to\omega} k(t) = \hat{k}$, a contradiction. Hence, in the following we assume that $m_1 > 0$.

Let $\delta \in (0, \omega)$ be arbitrary but fix and $\lambda := \inf_{t\in(0,\omega)} \varphi(t)^{-1} > 0$. Since $\dot{\varphi}$ is bounded and $\liminf_{t\to\infty} \varphi(t) > 0$ we find that $\frac{\mathrm{d}}{\mathrm{d}t}\, \varphi|_{[\delta,\infty)}(\cdot)^{-1}$ is bounded and hence there exists a Lipschitz bound $L > 0$ of $\varphi|_{[\delta,\infty)}(\cdot)^{-1}$. Furthermore, let $\hat{A}_{11} := [I_{m_1}, 0]V\tilde{A}V^\top[I_{m_1}, 0]^\top$, $\hat{A}_{12} := [I_{m_1}, 0]V\tilde{A}V^\top[0, I_{m-m_1}]^\top$ and

$$\begin{aligned}
\alpha &:= \|D^{-1}\hat{A}_{11}\|\|e_1\|_\infty + \|[D^{-1}, 0]V\Theta_1(e_1, e_2)\|_\infty,\\
\beta &:= \tfrac{2}{\lambda\hat{k}}\|D^{-1}\hat{A}_{12}\|,\\
\gamma &:= \tfrac{\hat{k}}{\hat{k}-\|\hat{A}_{22}\|}\left(\|\hat{A}_{21}e_1\|_\infty + \|[0, I_{m-m_1}]V\Theta_1(e_1, e_2)\|_\infty\right),\\
\kappa &:= \tfrac{\lambda^2\hat{k}}{4\,\sigma_{\max}(\Gamma)} > 0,
\end{aligned}$$

where $\sigma_{\max}(\Gamma)$ denotes the largest eigenvalue of the positive semi-definite matrix Γ and $\sigma_{\max}(\Gamma) > 0$ since $m_1 > 0$.

Choose $\varepsilon > 0$ small enough so that

$$\varepsilon \le \min\left\{\frac{\lambda}{2}, \min_{t\in[0,\delta]} (\varphi(t)^{-1} - \|e_1(t)\|)\right\}$$

and

$$L \le -\alpha - \beta\gamma\varepsilon + \frac{\kappa}{\varepsilon}. \tag{6.18}$$

We show that

$$\forall\, t \in (0, \omega):\ \varphi(t)^{-1} - \|e_1(t)\| \ge \varepsilon. \tag{6.19}$$

By definition of ε this holds on $(0, \delta]$. Seeking a contradiction suppose that

$$\exists\, t_1 \in [\delta, \omega):\ \varphi(t_1)^{-1} - \|e_1(t_1)\| < \varepsilon.$$

Then for

$$t_0 := \max\left\{\ t \in [\delta, t_1)\ \middle|\ \varphi(t)^{-1} - \|e_1(t)\| = \varepsilon\ \right\}$$

we have for all $t \in [t_0, t_1]$ that

$$\varphi(t)^{-1} - \|e_1(t)\| \le \varepsilon \quad \text{and} \quad \|e_1(t)\| \ge \varphi(t)^{-1} - \varepsilon \ge \lambda - \varepsilon \ge \frac{\lambda}{2}$$

and

$$k(t) = \frac{\hat{k}}{1 - \varphi(t)^2\|e(t)\|^2} \ge \frac{\hat{k}}{1 - \varphi(t)^2\|e_1(t)\|^2} = \frac{\hat{k}}{(1 - \varphi(t)\|e_1(t)\|)(1 + \varphi(t)\|e_1(t)\|} \ge \frac{\hat{k}}{2\varepsilon\varphi(t)} \ge \frac{\lambda\hat{k}}{2\varepsilon}.$$

Now we have, for all $t \in [t_0, t_1]$,

$$\begin{aligned}
\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|e_1(t)\|^2 &= e_1(t)^\top \dot{e}_1(t) \\
&= e_1(t)^\top \Big(D^{-1}(\hat{A}_{11} - k(t)I_{m_1})e_1(t) + D^{-1}\hat{A}_{12}e_2(t) + [D^{-1}, 0]V\,\Theta_1(e_1, e_2)(t) \Big) \\
&\le \alpha\|e_1(t)\| + \|D^{-1}\hat{A}_{12}\|\|e_2(t)\|\|e_1(t)\| - \frac{\lambda\hat{k}}{2\varepsilon} e_1(t)^\top D^{-1} e_1(t)^\top \\
&\le \alpha\|e_1(t)\| + \|D^{-1}\hat{A}_{12}\|\|e_2(t)\|\|e_1(t)\| - \frac{\lambda\hat{k}}{2\varepsilon\,\sigma_{\max}(\Gamma)}\|e_1(t)\|^2.
\end{aligned}$$

Moreover, from the inequality in (6.17) we obtain that, for all $t \in [t_0, t_1]$,

$$\|e_2(t)\| \le k(t)^{-1}\gamma \le \frac{2}{\lambda\hat{k}}\gamma\varepsilon.$$

This yields that

$$\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|e_1(t)\|^2 \le \Big(\alpha + \beta\gamma\varepsilon - \frac{\kappa}{\varepsilon}\Big)\|e_1(t)\| \overset{(6.18)}{\le} -L\|e_1(t)\|.$$

Therefore, using

$$\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|e_1(t)\|^2 = \|e_1(t)\|\tfrac{\mathrm{d}}{\mathrm{d}t}\|e_1(t)\|,$$

we find that

$$\begin{aligned}
\|e_1(t_1)\| - \|e_1(t_0)\| = \int_{t_0}^{t_1} \frac{1}{2}\|e_1(t)\|^{-1}\tfrac{\mathrm{d}}{\mathrm{d}t}\|e_1(t)\|^2 \,\mathrm{d}t& \\
\le -L(t_1 - t_0) \le -|\varphi(t_1)^{-1} - \varphi(t_0)^{-1}| \le \varphi(t_1)^{-1} - \varphi(t_0)^{-1},&
\end{aligned}$$

and hence

$$\varepsilon = \varphi(t_0)^{-1} - \|e_1(t_0)\| \le \varphi(t_1)^{-1} - \|e_1(t_1)\| < \varepsilon,$$

a contradiction.
Therefore, (6.19) holds and by (6.17) there exists $\tilde{t} \in [0, \omega)$ such that $\|e_2(t)\| \le \varepsilon$ for all $t \in [\tilde{t}, \omega)$. Then, invoking $\varepsilon \le \frac{\lambda}{2}$, we obtain for all $t \in [\tilde{t}, \omega)$

$$\|e(t)\|^2 = \|e_1(t)\|^2 + \|e_2(t)\|^2 \le (\varphi(t)^{-1} - \varepsilon)^2 + \varepsilon^2 \le \varphi(t)^{-2} - 2\varepsilon\lambda + 2\varepsilon^2 \le \varphi(t)^{-2} - 2\varepsilon^2.$$

This implies boundedness of k, a contradiction.

Step 4: We show that x_1 and x_3 are bounded. To this end, observe that $z = (k, e_1^\top, e_2^\top)^\top$ solves (6.14) and, by Step 3, z is bounded. Using (A.7) and $y_{\text{ref}} \in \mathcal{B}^{\nu+2}(\mathbb{R}_{\ge 0}; \mathbb{R}^m)$ we find that Tz is bounded as well. This implies, since F is continuously differentiable, that $\dot{z}$ is bounded. Then again, we obtain that $\frac{\mathrm{d}}{\mathrm{d}t}(Tz)$ is bounded and differentiating (6.14) gives boundedness of $\ddot{z}$. Iteratively, we have that

$$\forall\, j = 0, \ldots, \nu+1: \ \Big(\exists\, c_0, \ldots, c_j > 0 \ \forall\, t \in [0,\omega): \ \|z(t)\| \le c_0, \ldots, \|z^{(j)}(t)\| \le c_j\Big) \\ \implies \Big(\exists\, C > 0 \ \forall\, t \in [0,\omega): \ \|(Tz)^{(j)}(t)\| \le C\Big)$$

and successive differentiation of (6.14) finally yields that $z, \dot{z}, \ldots, z^{(\nu+1)}$ are bounded. This gives boundedness of $e, \dot{e}, \ldots, e^{(\nu+1)}$. Then, from the first and third equation in (6.9) and the fact that $\sigma(Q) \subseteq \mathbb{C}_-$ and $y_{\mathrm{ref}} \in \mathcal{B}^{\nu+2}(\mathbb{R}_{\ge 0}; \mathbb{R}^m)$, it is immediate that x_1 and x_3 are bounded.

Step 5: We show that $\omega = \infty$. First note that by Step 3 and Step 4 we have that $(x_1, e, x_3, k) : [0,\omega) \to \mathbb{R}^{n+1}$ is bounded. Further noting that boundedness of k is equivalent to (6.8) (for $t \in [0,\omega)$), the assumption $\omega < \infty$ implies existence of a compact subset $\mathcal{K} \subseteq \tilde{\mathcal{D}}$ such that $\operatorname{graph}(x_1, e, x_3, k) \subseteq \mathcal{K}$. This contradicts (6.16).

Step 6: It remains to show (ii). This follows from

$$\forall\, t > 0: \ k(t) = \hat{k} + k(t)\varphi(t)^2 \|e(t)\|^2 \overset{(6.8)}{\le} \hat{k} + k(t)\varphi(t)^2(\varphi(t)^{-1} - \varepsilon)^2 = \hat{k} + k(t)(1 - \varphi(t)\varepsilon)^2. \qquad \square$$

Remark 6.4.

(i) Note that ν in Theorem 6.3 is in general not known explicitly. However, we have, by Theorem 5.6, the estimate $\nu \le n_3 = n-k-m$, where $k = \dim \mathcal{Z}D_{(1.1)}$. Hence, choosing φ and y_{ref} $(n-m+2)$-continuously differentiable will always suffice.

(ii) Theorem 6.3 specifies [6, Rem. 6.4 (i)]: It is shown that, compared [6, Thm. 6.2], regularity is not needed and the assumptions of [6, Thm. 6.2] can be relaxed, while funnel control is still feasible.

(iii) The problem of finding a solution of (6.14) with the properties (a)–(c) as in the proof of Theorem 6.3 is not solved just by the consistency of the initial value, i.e., existence of a local solution, since it is not clear that this solution can be extended to a maximal solution which leaves every compact subset of $\mathcal{D}$. Solvability for any other initial value (for (6.14)) is required for this. $\diamond$

Remark 6.5 (Passive electrical networks)**.**
The findings of the present paper, in particular the application of the funnel controller, can also be applied to a class of passive electrical networks. A common way of modeling electrical networks is the *modified nodal analysis* (MNA), see [17, 21, 40, 45]. This modeling procedure results in a description of the circuit by a system of the form (1.1), where the inputs and outputs are appropriately chosen and the matrices E, A, B, C have specific properties, see also [38]. Omitting the details of this procedure and the circuit theoretic background, we are only interested in the resulting system (1.1) and its properties. From [38] we have that, in a MNA model of a passive electrical circuit,

$$\left.\begin{array}{l} sE - A \text{ is regular,} \\ G(s) := C(sE-A)^{-1}B \text{ has no poles in } \mathbb{C}_+, \\ \forall\, \lambda \in \mathbb{C}_+: \ G(\lambda) + \overline{G(\lambda)}^\top \ge 0. \end{array}\right\} \tag{6.20}$$

The second and third property in (6.20) state that $G(s)$ is *positive real*. Note that in a MNA model we have even more structure than stated in (6.20), such as $C = B^\top$ and a special block structure of E, A, B, C. However, Condition (6.20) is sufficient for our purposes.

We consider the class of systems which satisfy (6.20) and

$$G(s) \text{ is invertible over } \mathbb{R}(s). \tag{6.21}$$

Property (6.21) implies (following the lines of the proof of Proposition B.3) that the zero dynamics are autonomous and $[E,A,B,C]$ is right-invertible. Since $G(s)$ is positive real and invertible over $\mathbb{R}(s)$, we may infer that $G(s)^{-1}$ is positive real as well. Then, by [2, p. 216] (see also [38, Prop. 7]) we obtain that $G(s)^{-1} = G_{\mathrm{p}}(s) + sM$, where $G_{\mathrm{p}}(s) \in \mathbb{R}(s)^{m\times m}$ is proper and $M \in \mathbb{R}^{m\times m}$ satisfies $M = M^\top \geq 0$. As in Remark A.4 we may now conclude that $[0, I_m]L(s)[0, I_m]^\top = -G(s)^{-1}$ and hence we obtain existence of

$$\Gamma = \lim_{s\to\infty} s^{-1}G(s)^{-1} = M.$$

Therefore, $[E,A,B,C]$ satisfies the assumptions of Lemma A.3. If furthermore asymptotically stable zero dynamics are assumed, then by Theorem 6.3 the funnel controller works for the class of systems satisfying (6.20) and (6.21). ◇

7 Simulations

For purposes of illustration we consider an example of a differential-algebraic system (1.1) and apply the funnel controller (6.4). The simulation of the funnel controller for a mechanical system with springs, masses and dampers which has a proper inverse transfer function is performed in [6, Sec. 7.1]. In [6, Sec. 7.1] an academic example of a system with singular matrix pencil $sE - A$ is considered and it is shown that the funnel controller works for this system, however a proof was not included. This was the reason for the conjecture in [6, Rem. 6.4] that the funnel controller works for a much larger class than systems with proper inverse transfer function. It is now clear that funnel control is feasible for this example since it satisfies the assumptions of Theorem 6.3. The simulation of the funnel controller for a differential-algebraic system with strict relative degree one can be found in [5, Sec. 6]. For all of the aforementioned systems, feasibility of funnel control has been proved in Theorem 6.3.

Due to the above reasons, and in order to point out the peculiarities, in the present section we only state an academic example which has neither proper inverse transfer function nor strict relative degree one, but satisfies the assumptions of Theorem 6.3. Consider system (1.1) with

$$[E,A,B,C] := \left[\begin{bmatrix}1&0&0\\0&1&-1\\0&-1&1\end{bmatrix}, \begin{bmatrix}-1&1&-2\\3&1&1\\0&0&1\end{bmatrix}, \begin{bmatrix}0&0\\1&0\\0&1\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&1\end{bmatrix}\right]. \tag{7.1}$$

It is immediate that $[E,A,B,C]$ is in the form (5.4), has asymptotically stable zero dynamics, and the matrix

$$\Gamma = \begin{bmatrix}1&-1\\-1&1\end{bmatrix}$$

satisfies (6.5) and $\Gamma = \Gamma^\top \geq 0$. We set

$$\hat{k} := 2 > \sqrt{2} = \left\| \begin{bmatrix}1&1\\0&1\end{bmatrix} \right\| = \|A_{22}\| = \left\| \lim_{s\to\infty} \left([0, I_m]L(s)[0, I_m]^\top + s\Gamma\right) \right\|, \tag{7.2}$$

where $L(s)$ is an inverse of the system pencil, see also Step 1 in the proof of Theorem 6.3 for the latter equalities. The (consistent) initial value for the closed-loop system (7.1), (6.4) is chosen as

$$x^0 = (-4, 3, -2)^\top. \tag{7.3}$$

As reference signal $y_{\text{ref}} : \mathbb{R}_{\geq 0} \to \mathbb{R}$, we take components of the (chaotic) solution of the following initial-value problem for the Lorenz system

$$\begin{array}{rcll} \dot{\xi}_1(t) & = & 10\,(\xi_2(t) - \xi_1(t)), & \xi_1(0) = 5 \\ \dot{\xi}_2(t) & = & 28\,\xi_1(t) - \xi_1(t)\,\xi_3(t) - \xi_2(t), & \xi_2(0) = 5 \\ \dot{\xi}_3(t) & = & \xi_1(t)\,\xi_2(t) - \frac{8}{3}\,t\xi_3(t), & \xi_3(0) = 5\,. \end{array} \tag{7.4}$$

It is well known that the unique global solution of (7.4) is bounded with bounded derivative on the positive real axis, see for example [43, App. C]. The first and second components of the solution of (7.4) are depicted in Fig. 3.

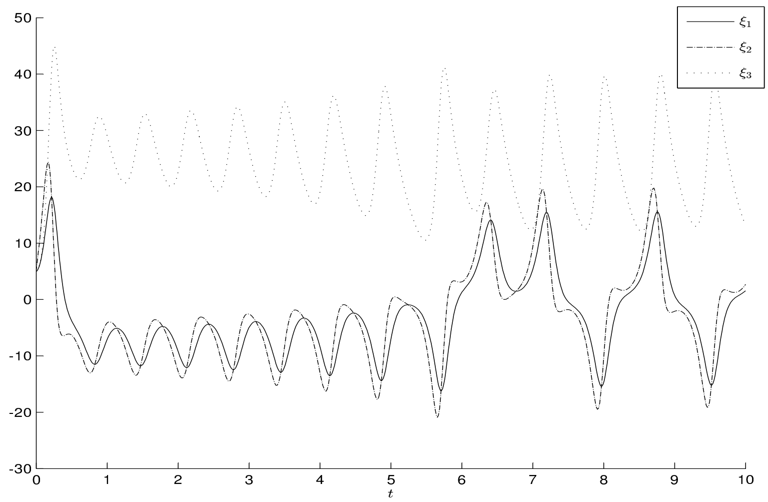


Figure 3: Components $\xi_i(\cdot)$ of the Lorenz system (7.4)

The funnel $\mathcal{F}_\varphi$ is determined by the function

$$\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, \quad t \mapsto 0.5\ te^{-t} + 2\ \arctan t\,. \tag{7.5}$$

Note that this prescribes an exponentially (exponent 1) decaying funnel in the transient phase $[0, T]$, where $T \approx 3$, and a tracking accuracy quantified by $\lambda = 1/\pi$ thereafter, see Fig. 4d.
The simulation has been performed in MATLAB. In Figure 4 the simulation, over the time interval $[0, 10]$, of the funnel controller (6.4) with funnel boundary specified in (7.5) and reference signal $y_{\text{ref}}(\cdot) = (\xi_1(\cdot), \xi_2(\cdot))^\top$ given in (7.4), applied to system (7.1) with initial data (7.2), (7.3) is depicted. Fig. 4a shows the output components $y_1(\cdot)$ and $y_2(\cdot)$ tracking the rather "vivid" reference signal $y_{\text{ref}}(\cdot)$ within the funnel shown in Fig. 4d. Note that an action of the input components $u_1(\cdot)$ and $u_2(\cdot)$ in Fig. 4c and the gain function $k(\cdot)$ in Fig. 4b is required only if the error $\|e(t)\|$ is close to the funnel boundary $\varphi(t)^{-1}$. It can be seen that initially the error is very close to the funnel boundary and hence the gain rises sharply. Then, at approximately $t = 0.2$, the distance between error and funnel boundary gets larger and the gain drops accordingly. After $t = 2$, the error gets close to the funnel boundary again which causes the gain to rise again. This in particular shows that the gain function $k(\cdot)$ is non-monotone.

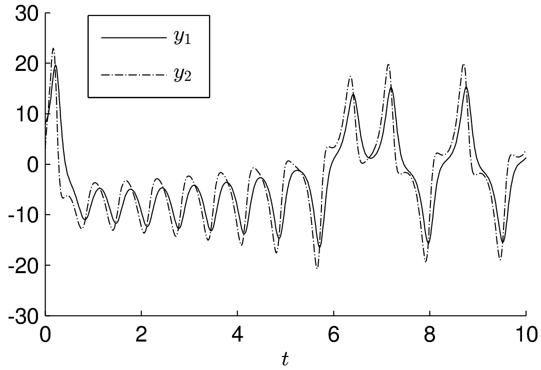


Fig. a: Solution components y_1 and y_2

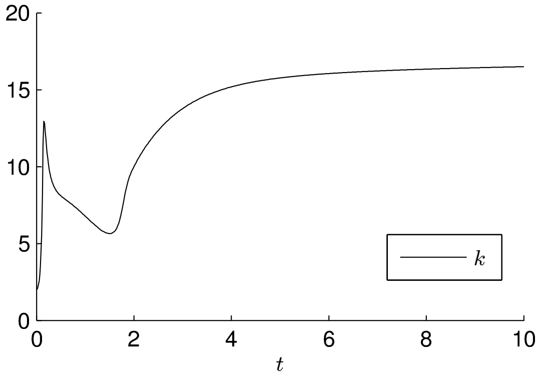


Fig. b: Gain k

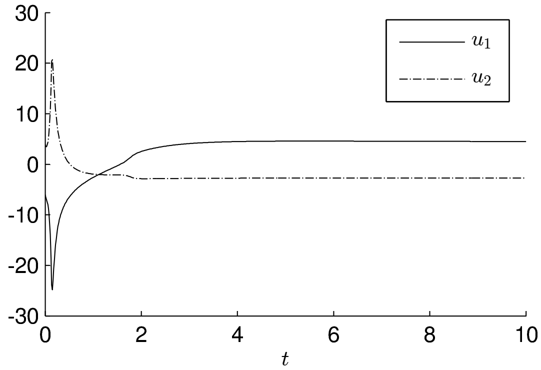


Fig. c: Input components u_1 and u_2

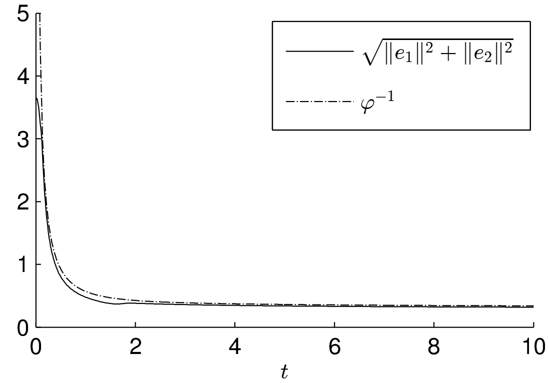


Fig. d: Norm of error $\|e(\cdot)\|$ and funnel boundary $\varphi(\cdot)^{-1}$

Figure 4: Simulation of the funnel controller (6.4) with funnel boundary specified in (7.5) and reference signal $y_{\text{ref}}(\cdot) = (\xi_1(\cdot), \xi_2(\cdot))^\top$ given in (7.4) applied to system (7.1) with initial data (7.2), (7.3).

Appendix A Polynomial matrices

The purpose of this section is, essentially, to derive a simplification of the form (5.4) under the condition that for a left inverse $L(s)$ of $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ the matrix

$$\Gamma = -\lim_{s\to\infty} s^{-1}[0, I_m]L(s)\begin{bmatrix} 0 \\ I_p \end{bmatrix} \in \mathbb{R}^{m\times p} \tag{A.1}$$

exists. This simplified form then provides an operator differential-algebraic equation which is used in the proof of Theorem 6.3 to show feasibility of funnel control.

In the following we parameterize all left inverses of the system pencil for right-invertible systems with autonomous zero dynamics; this is important to read off some properties of the block matrices in the form (5.4). Furthermore, it is shown that the lower right block in any left inverse is well-defined and therefore Γ in (A.1) is well-defined, provided it exists. The existence of a left inverse of the system pencil over $\mathbb{R}(s)$ is clear, since by Proposition 3.6 autonomous zero dynamics lead to a full column rank of the system pencil over $\mathbb{R}[s]$.

Lemma A.1 (Left inverse of system pencil)**.**
Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$ be right-invertible and have autonomous zero dynamics. Then $L(s) \in$

$\mathbb{R}(s)^{(n+m)\times(l+p)}$ is a left inverse of $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ if, and only if, using the notation from Theorem 5.6,

$$L(s) = \begin{bmatrix} T & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} (sI_k - Q)^{-1} & 0 & X_{13}(s) & X_{14}(s) & X_{15}(s) \\ 0 & 0 & 0 & X_{24}(s) & I_p \\ 0 & 0 & (sN - I_{n_3})^{-1} & X_{34}(s) & X_{35}(s) \\ X_{41}(s) & I_m & X_{43}(s) & X_{44}(s) & X_{45}(s) \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & I_p \end{bmatrix}, \tag{A.2}$$

where $[X_{14}(s)^\top, X_{24}(s)^\top, X_{34}(s)^\top, X_{44}(s)^\top]^\top \in \mathbb{R}(s)^{(n+m)\times(l+p-n-m)}$ and

$$\begin{array}{rclrcl} X_{13}(s) & = & -s(sI-Q)^{-1}E_{13}(sN-I)^{-1}, & X_{15}(s) & = & (sI-Q)^{-1}A_{12} - sX_{13}(s)E_{32}, \\ X_{35}(s) & = & -s(sN-I)^{-1}E_{32}, & X_{41}(s) & = & A_{21}(sI-Q)^{-1}, \\ X_{43}(s) & = & -(sX_{41}(s)E_{13} + sE_{23})(sN-I)^{-1}, & X_{45}(s) & = & -(sE_{22} - A_{22}) + X_{41}(s)A_{12} - sX_{43}(s)E_{32}, \end{array}$$

and $L(s)$ is partitioned according to the block structure of (5.4).
If $L_1(s), L_2(s) \in \mathbb{R}(s)^{(n+m)\times(l+p)}$ are two left inverse matrices of $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$, then

$$[0, I_m]L_1(s)\begin{bmatrix} 0 \\ I_p \end{bmatrix} = [0, I_m]L_2(s)\begin{bmatrix} 0 \\ I_p \end{bmatrix}.$$

Furthermore, if Γ in (A.1) exists, then it is well-defined.

Proof: By Proposition 5.11 we have $\operatorname{rk} C = p$ and hence the assumptions of Theorem 5.6 are satisfied. The statements can then be verified by a simple calculation. $\square$

Now we investigate the consequences of the assumption of existence of Γ in (A.1).

Lemma A.2 (Consequences of existence of Γ)**.**
Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$ be right-invertible and have autonomous zero dynamics. Suppose that, for a left inverse $L(s)$ of $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ over $\mathbb{R}(s)$, the matrix Γ in (A.1) exists. Then, using the notation from Theorem 5.6, we have

$$\forall\, i = 0, \ldots, \nu - 2\ \forall\, k = i+1, \ldots, \nu - 1:\ A_{21}Q^iE_{13}N^kE_{32} = 0 \tag{A.3}$$

and

$$\forall\, k = 0, \ldots, \nu - 1:\ E_{23}N^kE_{32} = 0. \tag{A.4}$$

Furthermore,

$$\Gamma = E_{22} + \sum_{k=0}^{\nu-1} A_{21}Q^kE_{13}N^kE_{32}. \tag{A.5}$$

Proof: The left inverse $L(s)$ is given in (A.2) and Γ is independent of the choice of $L(s)$ by Lemma A.1. By existence of Γ the matrix $s^{-1}[0, I_m]L(s)[0, I_p]^\top$ is proper, which implies that

$$s^{-1}X_{45}(s) = -(E_{22} - s^{-1}A_{22}) + s^{-1}A_{21}(sI-Q)^{-1}A_{12} + (sA_{21}(sI-Q)^{-1}E_{13} + sE_{23})(sN-I)^{-1}E_{32}$$

is proper. Hence, invoking that

$$(sI-Q)^{-1} = \sum_{i\geq 1} Q^{i-1}s^{-i}, \qquad (sN-I)^{-1} = -\sum_{i=0}^{\nu-1} N^is^i,$$

we have that, for all $i = 0, \ldots, \nu - 2$, $\sum_{k=0}^{\nu-1} A_{21}Q^iE_{13}N^kE_{32}s^{k-i}$ and $\sum_{k=0}^{\nu-1} E_{23}N^kE_{32}s^{k+1}$ have to be proper. This yields (A.3) and (A.4). The last statement (A.5) is then an immediate consequence of $\Gamma = -\lim_{s\to\infty} s^{-1}X_{45}(s)$. $\square$

The final result of this section, the simplification of the form (5.4), relies on partially solving the equations (5.5) using the conditions (A.3) and (A.4) derived in Lemma A.2.

Lemma A.3 (Behavior and underlying equations)**.**
Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$ be right-invertible and have autonomous zero dynamics. Suppose that, for a left inverse $L(s)$ of $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ over $\mathbb{R}(s)$, the matrix Γ in (A.1) exists. Then, using the notation from Theorem 5.6, for any $(x, u, y) \in \mathfrak{B}_{(1.1)} \cap \big(\mathcal{C}^1(\mathbb{R};\mathbb{R}^n) \times \mathcal{C}^0(\mathbb{R};\mathbb{R}^m) \times \mathcal{C}^{\nu+1}(\mathbb{R};\mathbb{R}^p)\big)$ and $Tx = (x_1^\top, y^\top, x_3^\top)^\top \in \mathcal{C}^1(\mathbb{R};\mathbb{R}^{k+p+n_3})$, (Tx, u, y) solves

$$\boxed{\begin{array}{rcl} \dot{x}_1(t) &=& Q\,x_1(t) + A_{12}\,y(t) - \sum_{k=0}^{\nu-1} E_{13}N^k E_{32}\,y^{(k+2)}(t) \\ \Gamma\,\dot{y}(t) &=& \Big(A_{22} - \sum_{k=0}^{\nu-1} A_{21}Q^{k+1}E_{13}N^k E_{32}\Big)\,y(t) + \Psi(x_1(0), y)(t) + u(t) \\ x_3(t) &=& \sum_{k=0}^{\nu-1} N^k E_{32}\,y^{(k+1)}(t), \\ 0 &=& 0, \end{array}} \tag{A.6}$$

where

$$\begin{aligned} \Psi : \mathbb{R}^k \times \mathcal{C}^\nu(\mathbb{R};\mathbb{R}^m) \to \mathcal{C}^{\nu+1}(\mathbb{R};\mathbb{R}^m),\ (x_1^0, y) \mapsto \Bigg(t \mapsto A_{21}e^{Qt}x_1^0 + \int_0^t A_{21}e^{Q(t-\tau)}A_{12}\,y(\tau)\,\mathrm{d}\tau \\ + \sum_{k=0}^{\nu-1}\sum_{j=0}^{k+1} A_{21}Q^j e^{Qt}E_{13}N^k E_{32}\,y^{(k-j+1)}(0) - \sum_{k=0}^{\nu-1}\int_0^t A_{21}Q^{k+2}e^{Q(t-\tau)}E_{13}N^k E_{32}\,y(\tau)\,\mathrm{d}\tau\Bigg). \end{aligned}$$

Ψ is linear in each argument and, if $\sigma(Q) \subseteq \mathbb{C}_-$, then Ψ has the property

$$\Psi\big(\mathbb{R}^k \times (\mathcal{L}^\infty(\mathbb{R};\mathbb{R}^p) \cap \mathcal{C}^\nu(\mathbb{R};\mathbb{R}^p))\big) \subseteq \mathcal{L}^\infty(\mathbb{R};\mathbb{R}^m) \cap \mathcal{C}^{\nu+1}(\mathbb{R};\mathbb{R}^m). \tag{A.7}$$

Proof: The assumptions of Theorem 5.6 are satisfied and it is clear that the respective first and third equations in (5.5) and (A.6) coincide. By Proposition 5.11 and right-invertibility of $[E, A, B, C]$, the fourth equation in (5.5) reads $0 = 0$. Therefore, it remains to show that under the additional assumption of existence of Γ, the second equation in (A.6) follows from (5.5). To this end, observe that by Lemma A.2, namely (A.4), the second equation in (5.5) reads

$$E_{22}\dot{y}(t) = A_{22}y(t) + A_{21}x_1(t) + u(t). \tag{A.8}$$

From the first equation in (5.5) we may infer, using the variation of constants formula,

$$A_{21}x_1(t) = A_{21}e^{Qt}x_1(0) + \int_0^t A_{21}e^{Q(t-\tau)}A_{12}y(\tau)\,\mathrm{d}\tau - \sum_{k=0}^{\nu-1}\int_0^t A_{21}e^{Q(t-\tau)}E_{13}N^k E_{32}y^{(k+2)}(\tau)\,\mathrm{d}\tau\,. \tag{A.9}$$

Now, using integration by parts, we find that

$$\begin{aligned} &\forall\, i \ge 0\ \forall\, j = 0\ldots,\nu-1\ \forall\, k \ge 1\ \forall\, t > 0 : \int_0^t A_{21}Q^i e^{Q(t-\tau)}E_{13}N^j E_{32}y^{(k)}(\tau)\,\mathrm{d}\tau \\ &= A_{21}Q^i E_{13}N^j E_{32}y^{(k-1)}(t) - A_{21}Q^i e^{Qt}E_{13}N^j E_{32}y^{(k-1)}(0) + \int_0^t A_{21}Q^{i+1}e^{Q(t-\tau)}E_{13}N^j E_{32}y^{(k-1)}(\tau)\,\mathrm{d}\tau\,, \end{aligned}$$

hence, recursively,

$$\forall\, k=0,\ldots,\nu-1: \int_0^t A_{21}e^{Q(t-\tau)}E_{13}N^kE_{32}y^{(k+2)}(\tau)\ \mathrm{d}\tau$$
$$= \sum_{j=0}^{k+1}\Big(A_{21}Q^jE_{13}N^kE_{32}y^{(k-j+1)}(t) - A_{21}Q^je^{Qt}E_{13}N^kE_{32}y^{(k-j+1)}(0)\Big)$$
$$+\int_0^t A_{21}Q^{k+2}e^{Q(t-\tau)}E_{13}N^kE_{32}y(\tau)\ \mathrm{d}\tau\,. \quad \text{(A.10)}$$

Using (A.3) we find that $A_{21}Q^jE_{13}N^kE_{32}=0$ for $k=0,\ldots,\nu-1$ and $j=0,\ldots,k-1$, hence

$$\sum_{j=0}^{k+1}A_{21}Q^jE_{13}N^kE_{32}y^{(k-j+1)}(t) = A_{21}Q^kE_{13}N^kE_{32}\dot{y}(t) + A_{21}Q^{k+1}E_{13}N^kE_{32}y(t).$$

Backward insertion of this into (A.10), (A.10) into (A.9) and (A.9) into (A.8) immediately yields the second equation in (A.6). Statement (A.7) about Ψ is obvious from the representation of Ψ and the fact that if $\sigma(Q)\subseteq\mathbb{C}_-$, then there exist $\mu, M>0$ such that

$$\forall\, t\geq 0:\ \|e^{Qt}\|\leq Me^{-\mu t}.$$

□

Remark A.4 (Regular systems)**.**
Let $[E,A,B,C]\in\Sigma_{n,n,m,m}$ be such that $sE-A$ is regular. If $L(s)$ is a left inverse of the system pencil, then we have

$$\begin{bmatrix} I_n & (sE-A)^{-1}B\\ 0 & I_m\end{bmatrix} = L(s)\begin{bmatrix} sE-A & -B\\ -C & 0\end{bmatrix}\begin{bmatrix} I_n & (sE-A)^{-1}B\\ 0 & I_m\end{bmatrix}$$
$$= L(s)\begin{bmatrix} sE-A & 0\\ -C & -C(sE-A)^{-1}B\end{bmatrix},$$

and therefore $C(sE-A)^{-1}B$ is invertible over $\mathbb{R}(s)$, and

$$H(s) := [0, I_m]L(s)\begin{bmatrix}0\\ I_m\end{bmatrix} = -\big(C(sE-A)^{-1}B\big)^{-1},$$

i.e., $-H(s)$ is exactly the inverse transfer function of the system $[E,A,B,C]$. Note that, if $sE-A$ is not regular, then the transfer function $C(sE-A)^{-1}B$ does not exist. ◇

Appendix B Relative degree

In this section we give the definition of vector relative degree for transfer functions of regular systems $[E,A,B,C]\in\Sigma_{n,n,m,p}$ and relate this property to the findings of the paper.

Definition B.1 (Vector relative degree)**.**
We say that $G(s)\in\mathbb{R}(s)^{p\times m}$ has *vector relative degree* $(\rho_1,\ldots,\rho_p)\in\mathbb{Z}^{1\times p}$ if, and only if, the limit

$$D := \lim_{s\to\infty}\operatorname{diag}(s^{\rho_1},\ldots,s^{\rho_p})\,G(s)\in\mathbb{R}^{p\times m}$$

exists and satisfies $\operatorname{rk} D = p$. ◇

Remark B.2 (Vector relative degree).

(i) It is an easy calculation that if $G(s) \in \mathbb{R}(s)^{p\times m}$ has a vector relative degree, then the vector relative degree is unique. However, a vector relative degree does not necessarily exist, even if $G(s)$ is (strictly) proper; see Example B.4.

(ii) If $G(s) \in \mathbb{R}(s)^{m\times m}$ has vector relative degree $(\rho_1, \ldots, \rho_m) \in \mathbb{Z}^{1\times m}$, then $\rho = \rho_1 = \ldots = \rho_m$ if, and only if, $G(s)$ has strict relative degree ρ.

(iii) Isidori [26, Sec. 5.1] introduced a local version of vector relative degree for nonlinear systems. Definition B.1 coincides with Isidori's definition if strictly proper transfer functions are considered. In this sense, Definition B.1 is a generalization to arbitrary rational transfer functions. For linear ODE systems a global version of the vector relative degree has been stated in [35]. It is straightforward to show that $[I_n, A, B, C] \in \Sigma_{n,n,m,m}$ has vector relative degree $(\rho_1, \ldots, \rho_p)$ in the sense of [35, Def. 2.1] if, and only if, $C(sI - A)^{-1}B$ has vector relative degree $(\rho_1, \ldots, \rho_p)$. $\diamond$

In the following we show that a regular system with transfer function which has componentwise vector relative degree smaller or equal to one, is included in the class of systems investigated in this paper, i.e., in particular satisfies the assumptions of Lemma A.3. If furthermore asymptotically stable zero dynamics and $\Gamma = \Gamma^\top \geq 0$ are assumed, then by Theorem 6.3 funnel control is feasible for this class of systems.

Proposition B.3 (Vector relative degree ≤ 1 implies existence of Γ).
Let $[E, A, B, C] \in \Sigma_{n,n,m,m}$ be such that $sE - A$ is regular and $C(sE-A)^{-1}B$ has vector relative degree $(\rho_1, \ldots, \rho_m)$ with $\rho_i \leq 1$ for all $i = 1, \ldots, m$. Then

(i) *$\mathcal{ZD}_{(1.1)}$ are autonomous,*

(ii) *$[E, A, B, C]$ is right-invertible,*

(iii) $\begin{bmatrix} sE - A & -B \\ -C & 0 \end{bmatrix}$ *has inverse $L(s)$ over $\mathbb{R}(s)$ and the matrix Γ in (6.5) exists and satisfies*

$$\forall\, j = 1, \ldots, m: \ \Gamma e_j = \begin{cases} \left(\lim\limits_{s\to\infty} \operatorname{diag}\left(s^{\rho_1}, \ldots, s^{\rho_p}\right) G(s)\right)^{-1} e_j, & \text{if } \rho_j = 1, \\ 0, & \text{if } \rho_j < 1. \end{cases} \tag{B.1}$$

Proof: *Step 1*: We show that $G(s) := C(sE - A)^{-1}B \in \mathbb{R}(s)^{m\times m}$ is invertible over $\mathbb{R}(s)$. To this end, let $F(s) := \operatorname{diag}\left(s^{\rho_1}, \ldots, s^{\rho_p}\right) G(s)$. Since

$$D := \lim_{s\to\infty} F(s) = \lim_{s\to\infty} \operatorname{diag}\left(s^{\rho_1}, \ldots, s^{\rho_p}\right) G(s) \in \mathbf{Gl}_m(\mathbb{R})$$

exists, $G_{\mathrm{sp}}(s) := F(s) - D \in \mathbb{R}(s)^{m\times m}$ is strictly proper, i.e., $\lim_{s\to\infty} G_{\mathrm{sp}}(s) = 0$. Since D is invertible, $F(s)$ is invertible as well, as by the Sherman-Morrison-Woodbury formula (see [19, p. 50])

$$F(s)^{-1} = D^{-1} - D^{-1}G_{\mathrm{sp}}(s)(I + D^{-1}G_{\mathrm{sp}}(s))^{-1}D^{-1} \in \mathbb{R}(s)^{m\times m}. \tag{B.2}$$

It is then immediate that $G(s)$ has inverse $G(s)^{-1} = F(s)^{-1} \operatorname{diag}\left(s^{\rho_1}, \ldots, s^{\rho_p}\right)$ over $\mathbb{R}(s)$.
Step 2: We show (i). Using invertibility of $G(s)$ we calculate

$$\begin{aligned} &\begin{bmatrix} (sE - A)^{-1} & 0 \\ -G(s)^{-1}C(sE - A)^{-1} & -G(s)^{-1} \end{bmatrix} \begin{bmatrix} sE - A & -B \\ -C & 0 \end{bmatrix} \begin{bmatrix} I_n & (sE - A)^{-1}B \\ 0 & I_m \end{bmatrix} \\ &\qquad = \begin{bmatrix} (sE - A)^{-1} & 0 \\ -G(s)^{-1}C(sE - A)^{-1} & -G(s)^{-1} \end{bmatrix} \begin{bmatrix} sE - A & 0 \\ -C & -G(s) \end{bmatrix} = I_{n+m}, \end{aligned}$$

which gives invertibility of the system pencil and thus the zero dynamics are autonomous by Proposition 3.6.
Step 3: We show (ii). It is clear that $\operatorname{rk} C = m$, since otherwise there exists $x \in \mathbb{R}^n \setminus \{0\}$ such that $x^\top C = 0$ and hence $x^\top G(s) = 0$, which contradicts invertibility of $G(s)$. Therefore, we find that $[E, A, B, C]$ is right-invertible by virtue of Remark 5.12.
Step 4: We show (iii). As in Remark A.4 we may conclude that $[0, I_m]L(s)[0, I_m]^\top = -G(s)^{-1}$ and

$$\begin{aligned} s^{-1}G(s)^{-1} &= \big(\operatorname{diag}(s^{\rho_1}, \ldots, s^{\rho_p})G(s)\big)^{-1}\operatorname{diag}(s^{\rho_1 - 1}, \ldots, s^{\rho_p - 1}) \\ &\overset{(B.2)}{=} \big(D^{-1} - D^{-1}G_{\mathrm{sp}}(s)(I + D^{-1}G_{\mathrm{sp}}(s))^{-1}D^{-1}\big)\operatorname{diag}(s^{\rho_1 - 1}, \ldots, s^{\rho_p - 1}). \end{aligned}$$

Hence, using $\rho_i \le 1$ for $i = 1, \ldots, m$, we obtain existence of

$$\Gamma = \lim_{s\to\infty} s^{-1}G(s)^{-1} \in \mathbb{R}^{m\times m},$$

where $\Gamma e_j = D^{-1}e_j$ if $\rho_j = 1$ and $\Gamma e_j = 0$ if $\rho_j < 1$, for all $j = 1, \ldots, m$. $\square$

We illustrate the vector relative degree and Proposition B.3 by means of an example.

Example B.4.
Consider system (1.1) with

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \ B = C = I_2.$$

It can be seen that $sE - A$ is regular and

$$G(s) = C(sE - A)^{-1}B = \begin{bmatrix} s-1 & -2 \\ 0 & -3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{s-1} & -\frac{2}{3(s-1)} \\ 0 & -\frac{1}{3} \end{bmatrix}.$$

We calculate

$$D := \lim_{s\to\infty} \operatorname{diag}(s, 1)\, G(s) = \begin{bmatrix} 1 & -\frac{2}{3} \\ 0 & -\frac{1}{3} \end{bmatrix} \in \mathbf{Gl}_2(\mathbb{R}),$$

and hence $G(s)$ has vector relative degree $(1, 0)$. Proposition B.3 then implies that $\mathcal{ZD}_{(1.1)}$ are autonomous, $[E, A, B, C]$ is right-invertible, and Γ in (6.5) exists. In fact, it is easy to see that the zero dynamics are asymptotically stable and

$$\Gamma = \lim_{s\to\infty} s^{-1}G(s)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

satisfies (B.1): $\Gamma e_1 = D^{-1}e_1 = \left[\begin{smallmatrix} 1 & -2 \\ 0 & -3 \end{smallmatrix}\right]$ and $\Gamma e_2 = 0$. Since $\Gamma = \Gamma^\top \ge 0$, the assumptions of Theorem 6.3 are satisfied.
We like to stress that, compared to the above, the regular system (7.1) from Section 7 does not have a vector relative degree: while its transfer function $G(s)$ is proper, the limit

$$\lim_{s\to\infty} G(s) = -\frac{1}{3}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

does not have full row rank. Nevertheless, as shown in Section 7, funnel control is feasible. $\diamond$

Remark B.5 (High-frequency gain matrix).
For systems $[E, A, B, C] \in \Sigma_{n,n,m,m}$ with regular $sE - A$ and strict relative degree $\rho \in \mathbb{N}$ the matrix

$$\lim_{s \to \infty} s^{\rho}\, C(sE - A)^{-1}B$$

is called the high-frequency gain matrix, see [5]. If $\rho = 1$, then by Proposition B.3, Γ in (6.5) exists and we have, from the proof of Proposition B.3, $\Gamma = \left(\lim_{s\to\infty} s\, C(sE - A)^{-1}B\right)^{-1}$, i.e., Γ is exactly the inverse of the high-frequency gain matrix. Since, furthermore, Γ is also defined when no high-frequency gain matrix exists, we may view the definition of Γ an appropriate generalization of the high-frequency gain matrix to DAEs which do not have a strict relative degree. In particular, if $C(sE - A)^{-1}B$ has proper inverse, then $\Gamma = 0$. $\diamond$

Acknowledgement: I am indebted to Stefan Brechtken, Achim Ilchmann (Ilmenau University of Technology), and Timo Reis (University of Hamburg) for several constructive discussions.

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