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Properly optimal elements in vector optimization with variable ordering structures

Gabriele Eichfelder* and Refail Kasimbeyli**

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Abstract

In this paper, proper optimality concepts in vector optimization with variable ordering structures are introduced for the first time and characterization results via scalarizations are given. New type of scalarizing functionals are presented and their properties are discussed. The scalarization approach suggested in the paper does not require convexity and boundedness conditions.

Key Words: Vector optimization, variable ordering structure, proper efficiency, scalarization.

Mathematics subject classifications (MSC 2000): 90C29, 90C30,90C48.

1 Introduction

Decisions depend often on more than one objective and what is preferred may vary on the actual state, i.e. on the actually achieved values. Already in 1974, Yu [32] introduced a definition for optimal solutions of vector optimization problems with a variable ordering structure called nondominated solutions. Thereby, it is assumed that this variable structure is defined by a cone-valued map which associates to each element of the linear space a cone of dominated directions.

Later, Chen and colleagues [5] introduced another optimality notion for variable ordering structures, again based on a cone-valued map but now assuming the negative of the cones to be the set of preferred directions. In this paper, the optimal solutions according to Chen will be called minimal solutions.

In the last years, multiobjective optimization problems with a variable ordering structure have gained interest motivated by several applications [1, 11, 14, 27, 29, 30, 31]. Meanwhile, also the first numerical procedures for solving such optimization

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problems have been proposed, see [29, 21, 13]. Thus, it is of recent interest to built up a comprehensive theory for these types of vector optimization problems.

In view of theoretical examinations for variable ordering structures, Chen and colleagues have presented in [6, 7] some nonlinear scalarization results for minimal solutions based on a generalization of a functional known in the literature as Tammer-Weidner functional or Pascoletti-Serafini functional. Eichfelder has considered in [9, 11] characterizations of minimal and nondominated solutions mainly using linear functionals. In [10], Eichfelder and Ha introduced a new nonlinear scalarization functional. This functional is based on a representation of the images of the cone-valued map that describes the ordering structure as Bishop-Phelps cones. They provided characterization results for minimal and nondominated solutions and also for strongly optimal solutions.

In partially ordered spaces, Kasimbeyli [24] introduced a class of monotone sublinear functionals and with their help suggested the conic scalarization method which is shown to characterize all efficient solutions without any convexity and boundedness conditions (see also [15, 25]). The conic scalarization method is shown to take into account weighting and reference point preferences of decision maker in multiobjective optimization. In this paper, we find a way to generalize these functionals to characterize optimal solutions in vector optimization with variable ordering structures.

However, our main concern in this paper is to generalize proper optimality notions to vector optimization problems with a variable ordering structure. In vector optimization with a partially ordered linear space, the notion of proper optimal elements is very important from a theoretical and practical points of view. For instance, using linear scalarization, properly optimal elements can be completely characterized in the convex case (see for instance [23]), but not the optimal elements. But also in view of applications, decision makers may prefer the proper optimal elements (for instance in the sense of Geoffrion [16]) of a multiobjective optimization problem as they have a bounded trade-off. In the literature, several notions of proper optimality have been proposed in partially ordered spaces but none for variable ordering structures.

It is not a trivial question to find such definitions which generalize the well-known notions of proper optimality in a partially ordered space. The main difficulty is that the original definitions have to be included in the more general definitions as a special case. In addition to that, it is important to be able to characterize the new optimality notions by scalarization functionals.

In this paper we introduce for the first time definitions for proper optimality for variable ordering structures and illustrate in examples why the notions have to be defined in the way we propose it. We obtain the mentioned characterization results for the proper optimality notions by the new scalarization functionals mentioned above. These functionals are defined by using elements of the augmented dual cones [24]. We also examine the relation between augmented dual cones and Bishop-Phelps cones and study the properties of the new functionals. The reason is that the newly introduced functionals have some similarities to the functionals proposed in [10] which are based on the assumption of having Bishop-Phelps cones. We also provide characterization results for other optimality notions as weakly and strongly optimal solutions.

We proceed as follows. Section 2 is devoted to preliminaries as the optimality notions known so far in vector optimization with a variable ordering structure. In Section 3 we propose augmented dual cones and examine their relation to Bishop-Phelps cones. Elements from the augmented dual cones define the nonlinear scalarization functionals which are introduced in Section 4. There, we also examine the properties of these functionals and provide characterization results for (weakly, strongly) minimal and nondominated elements. Finally, in section 5, we propose proper optimality notions for the minimal and for the nondominated elements, generalizing the notions introduced for partially ordered spaces by Henig [20], Benson [2] and Borwein [4], and present characterization results via scalarization.

2 Preliminaries

Throughout the paper let $(Y, \|\cdot\|)$ be a real normed space. For a nonempty set $A \subset Y$, $\mathrm{cl}(A)$, ∂A, $\mathrm{int}(A)$, and $\mathrm{co}(A)$ denote the *closure* (in the norm topology), the *boundary*, the *interior*, and the *convex hull* of the set A, respectively. The closed unit ball and the unit circle of $(Y, \|\cdot\|)$ are denoted by $B(0,1)$ and $U(0,1)$ respectively. $(Y^*, \|\cdot\|_*)$ denotes the topological dual space with the induced norm $\|\cdot\|_*$. For some cone $K \subset Y$,

$$K_U := \{y \in K \mid \|y\| = 1\}$$

denotes the *norm-base* of the cone K. The term *norm-base* can be justified by the obvious assertion that $K = \mathrm{cone}(K_U)$ where $\mathrm{cone}(A) := \{\lambda x \mid \lambda \in \mathbb{R}_+,\ x \in A\}$ denotes the cone generated by some set $A \subset Y$. Thereby, $\mathbb{R}_+$ denotes the set of nonnegative real numbers.

2.1 Variable ordering structure and optimality notions

We assume the variable ordering structure on Y is defined by a set-valued map $\mathcal{D}\colon Y \to 2^Y$ with $\mathcal{D}(y)$ a closed, convex, pointed and nontrivial cone for all $y \in Y$ (or for all elements y of a subset A of Y). Based on this cone-valued map (also called an *ordering map*), one can define two different relations: for $y, \bar{y} \in Y$

$$y \leq_1 \bar{y} \ \text{ if } \ \bar{y} \in \{y\} + \mathcal{D}(y), \tag{1}$$

and

$$y \leq_2 \bar{y} \ \text{ if } \ \bar{y} \in \{y\} + \mathcal{D}(\bar{y}) \ \Leftrightarrow \ y \in \{\bar{y}\} - \mathcal{D}(\bar{y}). \tag{2}$$

The first relation leads to the concept of nondominated elements, which was defined by Yu [32]. The second relation leads to the concept of minimal elements, which was used for instance in [14, 9, 10] and was called nondominated-like solution by Chen in [6, 7]. One speaks here of a variable ordering structure despite these binary relations given above are in general not transitive nor even compatible with positive scalar multiplication, see [13].

The binary relation defined in (1) is based on the idea of domination:

$$\mathcal{D}(y) := \{d \in Y \mid y + d \ \text{ is dominated by } y\} \cup \{0_Y\}.$$

For a meaningful interpretation of the binary relation given in (2), first a set-valued map $\mathcal{P}\colon Y \to 2^Y$ with $\mathcal{P}(y)$ a closed, convex, pointed and nontrivial cone with

$$\mathcal{P}(y) = \{d \in Y \mid y + d \text{ is preferred to } y\} \cup \{0_Y\} \text{ for all } y \in \mathbb{R}^m$$

has to be defined. Then $y \leq_2 \bar{y}$ if $y \in \{\bar{y}\}+\mathcal{P}(\bar{y})$. By simply defining $\mathcal{D}(y) := -\mathcal{P}(y)$ for all $y \in Y$ we obtain the unified notation given in (2) which follows the notation as given for instance in [8].

According to [32, 6, 9, 10] and based on the two relations defined above, the following definitions of optimal elements (w.r.t. minimization) are known in the literature for variable ordering structures introduced by a cone-valued map $\mathcal{D}$. We characterize these optimal elements in Section 4.2 based on a new scalarization functional. Note that for the following definitions it is sufficient to assume that the images $\mathcal{D}(y)$ of the ordering map are arbitrary nonempty sets.

Definition 2.1. *Let $\bar{y} \in A$. We say that*

(a) The element $\bar{y}$ is a nondominated element *of A w.r.t. the ordering map $\mathcal{D}$ if there is no $y \in A \setminus \{\bar{y}\}$ such that $\bar{y} \in \{y\} + \mathcal{D}(y)$.*

(b) The element $\bar{y}$ is a strongly nondominated *element of A w.r.t. the ordering map $\mathcal{D}$ if $\bar{y} \in \{y\} - \mathcal{D}(y)$ for all $y \in A$.*

(c) Supposing that $\operatorname{int} \mathcal{D}(y) \neq \emptyset$ for all $y \in A$, $\bar{y}$ is a weakly nondominated *element of A w.r.t. the ordering map $\mathcal{D}$ if there is no $y \in A$ such that $\bar{y} \in \{y\}+\operatorname{int}\mathcal{D}(y)$.*

(d) The element $\bar{y}$ is a minimal element *of A w.r.t. the ordering map $\mathcal{D}$ if there is no $y \in A \setminus \{\bar{y}\}$ such that $y \in \{\bar{y}\} - \mathcal{D}(\bar{y})$.*

(e) The element $\bar{y}$ is a strongly minimal *element of A w.r.t. the ordering map $\mathcal{D}$ if $A \subset \{\bar{y}\} + \mathcal{D}(\bar{y})$.*

(f) The element $\bar{y}$ with $\operatorname{int} \mathcal{D}(\bar{y}) \neq \emptyset$ is a weakly minimal *element of A w.r.t. the ordering map $\mathcal{D}$ if there is no $y \in A$ such that $y \in \{\bar{y}\} - \operatorname{int}\mathcal{D}(\bar{y})$.*

If $\mathcal{D}(y) = K$ for all $y \in Y$ then the definitions of a (weakly/strongly) nondominated element w.r.t. $\mathcal{D}$ and of a (weakly/strongly) minimal element w.r.t. $\mathcal{D}$ coincide with the concepts of a (weakly/strongly) optimal element in a space partially ordered by a convex cone K. We will use an efficient element term as a common notion for such elements.

Let X be a real linear space, and let $S \subset X$ be a nonempty set. An element $\bar{x} \in S$ is called a minimal/nondominated/efficient solution of a vector optimization problem

$$\min_{x \in S} f(x)$$

with an objective function $f\colon X \to Y$ if $\bar{y} := f(\bar{x})$ is a minimal/nondominated/efficient element of the set $f(S)$. The analogous definition will be used for the other optimality notions.

2.2 Separability

In this subsection we recall some results from the literature which we need for proving the scalarization results for the properly optimal elements in Section 5.

The following two definitions are given in [24].

Definition 2.2. *Let C and K be nonempty cones of the normed space $(Y, \|\cdot\|)$ with $int(K) \neq \emptyset$. A cone K is called a* conic neighborhood *of C if $C \setminus \{0_Y\} \subset int(K)$. For a positive real number ε, a cone $C_\varepsilon = cone(C_U + \varepsilon B(0,1))$ is called an* ε-conic neighborhood *of C.*

Definition 2.3. *(Separation Property) Let C and K be closed cones of the normed space $(Y, \|\cdot\|)$ with norm-bases C_U and K_U, respectively. Let $K_U^\partial := K_U \cap bd(K)$, and let $\widetilde{C}$ and $\widetilde{K}^\partial$ be the closures of the sets $co(C_U)$ and $co(K_U^\partial \cup \{0_Y\})$, respectively. The cones C and K are said to have the separation property with respect to the norm $\|\cdot\|$ if*

$$\widetilde{C} \cap \widetilde{K}^\partial = \emptyset.$$

In Subsection 3.1, we present a separation theorem for cones which satisfy the separation property. This theorem will use elements from the quasi-interior of the augmented dual cone, which we introduce in the next subsection.

3 Augmented dual and Bishop-Phelps cones

In this section we present the so-called augmented dual cones which are generalizations of dual cones. These cones are introduced in [24], and were used to obtain monotonically increasing sublinear functionals.

By using the elements of the augmented dual cones we will define the nonlinear scalarization functionals considered in this manuscript. It turns out that these nonlinear scalarization functionals are similar to those discussed in [10] in the case the cones are Bishop-Phelps cones. For that reason, we discuss in the following relations between the augmented dual cones and the Bishop-Phelps cones.

3.1 Augmented dual cones

Let $K \subset Y$ be a closed, pointed and convex cone. Let K^* denote the (topological) dual cone, i.e. $K^* := \{\ell \in Y^* \mid \ell(y) \geq 0 \text{ for all } y \in K\}$, and let $K^\#$ denote the quasi-interior of the dual cone, i.e. $K^\# := \{\ell \in Y^* \mid \ell(y) > 0 \text{ for all } y \in K \setminus \{0_Y\}\}$.

Definition 3.1. *Let $K \subset Y$ be a closed, pointed and convex cone with $K^\# \neq \emptyset$.*

(a) The cone

$$K^{a*} = \big\{(\ell, \alpha) \in K^\# \times \mathbb{R}_+ \mid \ell(y) - \alpha\|y\| \geq 0 \ \text{ for all } y \in K\big\}$$

is called the augmented dual cone.

(b) Let $int(K) \neq \emptyset$. The cone

$$K^{a\circ} = \big\{(\ell, \alpha) \in K^\# \times \mathbb{R}_+ \mid \ell(y) - \alpha\|y\| > 0 \ \text{ for all } y \in int(K)\big\}$$

is called the weak augmented dual cone.

(c) *The cone*

$$K^{a\#} = \{(\ell, \alpha) \in K^{\#} \times \mathbb{R}_+ \mid \ell(y) - \alpha\|y\| > 0 \text{ for all } y \in K \setminus \{0_Y\}\}$$

is called the quasi-interior of the augmented dual cone.

Example 3.2. *(a) [24, Example 4.7] For $K := \mathbb{R}^n_+$ in the Euclidean space $\mathbb{R}^n$ we obtain*

$$K^{a*} = \{(y^*, \alpha) \in int(\mathbb{R}^n_+) \times \mathbb{R}^n_+ \mid y^*_i \geq \alpha,\ i = 1, \ldots, n\}\ .$$

(b) Let a polyhedral cone $K := \{y \in \mathbb{R}^n \mid y = Mx,\ x \in \mathbb{R}^m_+\}$ in the Euclidean space $\mathbb{R}^n$ be given, with M some real $n \times m$ matrix with full rank. Then

$$\{(y^*, \alpha) \in \mathbb{R}^n \times \mathbb{R}_+ \mid (M^\top y^*)_i > 0,\ (M^\top y^*)_i \geq \alpha,\ i = 1, \ldots, m\} \subset K^{a*}\ .$$

For instance the famous Krein-Rutman theorem [23, Theorem 3.38] gives conditions ensuring that the quasi-interior of the dual cone $K^{\#}$ is nonempty which is crucial in the definition above: the quasi interior of the topological dual cone of a closed, pointed and convex cone in a separable normed space is nonempty. The pointedness of K is important as for any convex cone K the condition $K^{\#} \neq \emptyset$ already implies the pointedness of K [23, Lemma 1.27]. There is also a strong relation between the base of a cone and the elements of the quasi-interior of the dual cone. A convex cone has a base if and only if the quasi-interior of the dual cone is nonempty and in this case for any element $y^* \in K^{\#}$, $\{y \in K \mid y^*(y) = 1\}$ is a base of K [23, Lemma 1.28].

With the elements from the quasi-interior of the augmented dual cones we can state the following separation theorem [24, Theorem 4.3]:

Theorem 3.3. ***(Separation Theorem)*** *Let C and K be closed cones in a reflexive Banach space $(Y, \|\cdot\|)$ and assume $-C$ and K satisfy the separation property. Then $C^{a\#} \neq \emptyset$ and there exists a pair $(y^*, \alpha) \in C^{a\#}$ such that*

$$y^*(y) + \alpha\|y\| < 0 \leq y^*(z) + \alpha\|z\| \text{ for all } y \in -C \setminus \{0_Y\},\ z \in \partial(K). \tag{3}$$

In this case $-C$ is pointed. Conversely, if there exists a pair $(y^*, \alpha) \in C^{a\#}$ such that (3) holds and if either C is closed and convex or Y is finite dimensional, then the cones $-C$ and K satisfy the separation property.

Lemma 3.4. *Let $K \subset Y$ be a closed, pointed and convex cone and assume there exists $(y^*, \alpha) \in K^{a\#}$. Then*

$$-K \setminus \{0_Y\} \subset int(S(y^*, \alpha)$$

with

$$S(y^*, \alpha) := \{z \in Y \mid y^*(z) + \alpha\|z\| \leq 0\}.$$

Proof. Let $w \in -K \setminus \{0_Y\}$. For $u := -w \in K \setminus \{0_Y\}$, we get by the definition of the quasi-interior of the augmented dual cone, $y^*(u) - \alpha\|u\| > 0$. Thus

$$y^*(w) + \alpha\|w\| < 0,$$

i.e. $-K \setminus \{0_Y\} \subset \{z \in Y \mid y^*(z) + \alpha\|z\| < 0\}$. By [24, Lemma 3.6], $\{z \in Y \mid y^*(z) + \alpha\|z\| < 0\} = \text{int}(S(y^*, \alpha))$. □

Of course, for any $y^* \in K^\#$ it holds $(y^*, 0) \in K^{a*}$, $(y^*, 0) \in K^{a\circ}$ and, if additionally $\mathrm{int}(K) \neq \emptyset$, also $(y^*, 0) \in K^{a\circ}$.

Proposition 3.5. *If $(y^*, \alpha) \in K^{aN}$ for any $N \in \{*, \circ, \#\}$, then either $\alpha = 0$ or also $(\frac{1}{\alpha} y^*, 1) \in K^{aN}$.*

Comparing the quasi-interior with the interior of the dual cone it always holds $\mathrm{int}(K^*) \subset K^\#$ [23, Lemma 1.25] for $K \subset Y$ a convex cone and if K is additionally closed with $\mathrm{int}(K^*) \neq \emptyset$ and Y is a reflexive Banach space, then equality holds. The relation of the cones defined above is $K^{a\#} \subset K^{a\circ} \subset K^{a*}$ [24]. It follows also directly from the definitions that

Proposition 3.6. *Let $K_1, K_2 \subset Y$ be closed, pointed and convex cones with $K_i^\# \neq \emptyset$ for $i = 1, 2$. Then*

$$K_1^{aN} \cap K_2^{aN} = (K_1 \cup K_2)^{aN} \quad \textit{for} \quad N \in \{*, \circ, \#\}.$$

The augmented dual cones for the images $\mathcal{D}(z)$ of the ordering map $\mathcal{D}\colon Y \to 2^Y$ with $\mathcal{D}(z)$ a closed, convex and pointed cone, $(\mathcal{D}(z))^\# \neq \emptyset$ for all $z \in Y$ and $\mathrm{int}(D(z)) \neq \emptyset$ whenever considered are thus

$$\begin{aligned}
(\mathcal{D}(z))^{a*} &= \left\{(\ell_z, \alpha_z) \in (\mathcal{D}(z))^\# \times \mathbb{R}_+ \mid \ell_z(y) - \alpha_z\|y\| \geq 0 \ \text{ for all } y \in \mathcal{D}(z)\right\}, \\
(\mathcal{D}(z))^{a\circ} &= \left\{(\ell_z, \alpha_z) \in (\mathcal{D}(z))^\# \times \mathbb{R}_+ \mid \ell_z(y) - \alpha_z\|y\| > 0 \ \text{ for all } y \in \mathrm{int}(\mathcal{D}(z))\right\}, \\
(\mathcal{D}(z))^{a\#} &= \left\{(\ell_z, \alpha_z) \in (\mathcal{D}(z))^\# \times \mathbb{R}_+ \mid \ell_z(y) - \alpha_z\|y\| > 0 \ \text{ for all } y \in \mathcal{D}(z) \setminus \{0_Y\}\right\}.
\end{aligned}$$

3.2 Bishop-Phelps cones

In 1962, Bishop and Phelps [3] introduced a class of ordering cones which have a rich mathematical structure and already proofed to be very useful in functional analysis and in vector optimization. Based on the definition of these cones we can immediately give elements (y^*, α) of the augmented dual cones, also with $\alpha \neq 0$. We study the relation of Bishop-Phelps cones and augmented dual cones in detail in Subsection 3.3.

We start by recalling the definition of Bishop-Phelps cones as used for instance in [22]:

Definition 3.7. *(a) For an arbitrary continuous linear functional ϕ on the normed space $(Y, \|\cdot\|)$ the cone*

$$C(\phi) := \{y \in Y \mid \|y\| \leq \phi(y)\} \tag{4}$$

is called Bishop-Phelps cone *(BP cone, for short).*

(b) A nontrivial convex cone K is denoted representable as a Bishop-Phelps cone *if there exists a continuous linear functional ϕ on the normed space $(Y, \|\cdot\|)$ and a norm $\|\cdot\|_e$ being equivalent to the norm $\|\cdot\|$ such that*

$$K = \{y \in Y \mid \|y\|_e \leq \phi(y)\}.$$

We collect some properties of BP cones:

Proposition 3.8. *[22] Let $\phi, \phi^1, \phi^2 \in Y^*$ be given.*

(i) $C(\phi)$ is closed, pointed and convex.

(ii) If $\|\phi\|_* > 1$ then $C(\phi)$ is nontrivial; if $\|\phi\|_* < 1$ then $C(\phi) = \{0_Y\}$.

(iii) $\{y \in Y \mid \|y\| < \phi(y)\} \subset int(C(\phi))$. If $\|\phi\|_* > 1$ then the interior of $C(\phi)$ is nonempty and $int(C(\phi)) = \{y \in Y \mid \|y\| < \phi(y)\}$.

(iv) $\phi \in (C(\phi))^\#$.

(v) The set $\{y \in C(\phi) \mid \phi(y) = 1\}$ is a closed and bounded base for the cone $C(\phi)$.

(vi) The dual cone $C(\phi)^*$ can be written as $C(\phi)^* = cl(cone(B(\phi, 1)))$ with $B(\phi, 1) = \{y^* \in Y^* \mid \|y^* - \phi\|_* \leq 1\}$.

(vii) The interior of the dual cone can be written as $int(C(\phi)^*) = cone(\mathring{B}(\phi, 1)) \setminus \{0_{Y^*}\}$ with $\mathring{B}(\phi, 1)) = \{y^* \in Y^* \mid \|y^* - \phi\|_* < 1\}$.

Note that by part (vii) the interior of the dual cone $C(\phi)^*$ always contains the element ϕ and is thus nonempty.

According to [26], cones in a real normed space are representable as BP cone in the sense that they become BP cones when the spaces are equipped with some equivalent norms, if they are nontrivial and convex with a closed and bounded base. Recall that the base of a convex cone $K \neq \{0_Y\}$, $K \subset Y$, is a nonempty convex subset B of the cone such that each $y \in K \setminus \{0_Y\}$ has a unique representation of the form $y = \lambda b$ for some $\lambda > 0$ and some $b \in B$. In $\mathbb{R}^n$ every nontrivial convex cone is representable as a BP cone if and only if it is closed and pointed [26, 22].

3.3 Relation between augmented dual cones and BP cones

According to Proposition 3.8, BP cones $C(\phi)$ are closed, convex and pointed cones and the quasi-interior of the dual cone is nonempty because $\phi \in C(\phi)^\#$. Hence, for BP cones we get the following elements of the augmented dual cones:

Proposition 3.9. *Let $\phi \in Y^*$ and the BP cone $C(\phi) = \{y \in Y \mid \|y\| \leq \phi(y)\}$ be given. Then*

$$(\phi, \alpha) \in C(\phi)^{a*} \quad \text{for all} \quad \alpha \in [0, 1]$$

and $(\phi, 0) \in C(\phi)^{a\#}$. If $\|\phi\|_* > 1$ then

$$(\phi, \alpha) \in C(\phi)^{a\circ} \quad \text{for all} \quad \alpha \in [0, 1].$$

Proof. According to the definition of the BP cone $\phi(y) - \|y\| \geq 0$ for all $y \in C(\phi)$ and thus also for all $\alpha \in [0, 1]$

$$\phi(y) - \alpha\|y\| \geq 0 \quad \text{for all } y \in C(\phi).$$

This implies $(\phi, \alpha) \in C(\phi)^{a*}$. As $\phi \in C(\phi)^\#$ it holds $\phi(y) > 0$ for all $y \in C(\phi) \setminus \{0_Y\}$ and thus $(\phi, 0) \in C(\phi)^{a\circ}$. For $\|\phi\|_* > 1$, according to Proposition 3.8, $\text{int}(C(\phi)) \neq \emptyset$ and $\text{int}(C(\phi)) = \{y \in Y \mid \phi(y) > \|y\|\}$. Hence $\phi(y) - \alpha\|y\| > 0$ for all $y \in \text{int}(C(\phi))$ and any $\alpha \in [0, 1]$. □

Proposition 3.10. *Let $K \subset Y$ be a nonempty, closed, convex and pointed cone and $C(\phi)$ a BP cone as in (4) with $K \subset C(\phi)$, then $(\phi, 1) \in K^{a*}$ and $(\phi, 0) \in K^{a\#}$. If int$(K) \neq \emptyset$, then $(\phi, 1) \in K^{a\circ}$.*

Proof. It follows directly from the definitions of the augmented dual cones that $(C(\phi))^{aN} \subset K^{aN}$ for any $N \in \{*, \circ, \#\}$. The remaining follows with Proposition 3.9. □

We even can give a representation result for the elements of the augmented dual cone of a BP cone.

Proposition 3.11. *Let $\phi \in Y^*$ and the BP cone $C(\phi) = \{y \in Y \mid \|y\| \leq \phi(y)\}$ be given.*

(i) If $(y^*, \alpha) \in C(\phi)^{a*}$ with $\alpha \neq 0$, then there is some $v \in Y^*$ with $\|v\|_* < 1$,

$$y^* = \alpha\lambda(\phi + v) \quad \text{and } \lambda \geq \frac{1}{\|\phi + v\|_*} \ .$$

If additionally int$(C(\phi)) \neq \emptyset$ and $(y^*, \alpha) \in C(\phi)^{a\circ}$ then it even holds $\lambda > \frac{1}{\|\phi+v\|_*}$.

(ii) If $y^* = \alpha\lambda(\phi+v)$ for some $\alpha > 0$, some $v \in Y^*$ with $\|v\|_* < 1$ and some $\lambda \in \mathbb{R}$ with

$$\lambda \geq \frac{1}{1 - \|v\|_*} \ ,$$

then $(y^*, \alpha) \in C(\phi)^{a*}$. If $\lambda > \frac{1}{1-\|v\|_*}$ then it even holds $(y^*, \alpha) \in C(\phi)^{a\#}$.

Proof. According to the Proposition 3.8,(vii), $\text{int}(C(\phi)^*) = \text{cone}(\mathring{B}(\phi, 1)) \setminus \{0_{Y^*}\}$ and since $\phi \in \text{int}(C(\phi)^*)$ by [23, Lemma 3.21(d)] $\text{int}(C(\phi)^*) = C(\phi)^\#$. Thus $y^* \in C(\phi)^\#$ if and only if there is some $v \in Y^*$ with $\|v\|_* < 1$ and some $s > 0$ with

$$y^* = s(\phi + v) \neq 0_{Y^*} \ . \tag{5}$$

(i) By setting $\lambda = s/\alpha > 0$ we obtain from (5) $y^* = \alpha\lambda(\phi + v)$. For any $y \in C(\phi)$ we get by the definition of the augmented dual cones

$$\|y\| \leq \frac{1}{\alpha} y^*(y) = \lambda(\phi + v)(y) \leq \lambda\|\phi + v\|_*\|y\|$$

and thus $\lambda \geq 1/\|\phi + v\|_*$.

In case of $\text{int}(C(\phi)) \neq \emptyset$ and $(y^*, \alpha) \in C(\phi)^{a\circ}$ we obtain for all $y \in \text{int}(C(\phi))$ that $\|y\| < \frac{1}{\alpha} y^*(y)$ and this implies $\lambda > 1/\|\phi + v\|_*$.

(ii) According to (5) $y^* \in C(\phi)^\#$. For any $y \in C(\phi)$ we have according to the definition of the BP cone $\|y\| \leq \phi(y)$ and thus it holds

$$\begin{aligned} y^*(y) &= \alpha\lambda(\phi + v)(y) \\ &= \alpha\lambda\phi(y) + \alpha\lambda v(y) \\ &\geq \alpha\lambda\|y\| - \alpha\lambda\|v\|_*\|y\| \\ &= \alpha\lambda(1 - \|v\|_*)\|y\| \\ &\geq \alpha\|y\| \ . \end{aligned}$$

Hence, $(y^*, \alpha) \in C(\phi)^{a*}$.

If $\lambda > \frac{1}{1-\|v\|_*}$ then for any $y \in C(\phi)$, $y \neq 0$ it holds $y^*(y) > \alpha\|y\|$ and thus $(y^*, \alpha) \in C(\phi)^{a\#}$.

□

Remark 3.12. *Since $C(\phi) \neq \{0_Y\}$ only if $\|\phi\|_* \geq 1$ (see Proposition 3.8(ii)) we have for nontrivial BP cones $\|\phi + v\|_* \geq \|\phi\|_* - \|v\|_* \geq 1 - \|v\|_*$. Thus*

$$\frac{1}{\|\phi + v\|_*} \leq \frac{1}{1 - \|v\|_*}$$

and hence the necessary and the sufficient conditions of the above proposition only coincide if $\|\phi\|_* = 1$ and $v = s\phi$ with $s \in]-1, 0]$.

Next we examine under which conditions on cones the augmented dual cones have an element (y^*, α) with $\alpha \neq 0$.

Theorem 3.13. *Let $K \subset Y$ be a nontrivial convex cone of the normed space $(Y, \|\cdot\|)$. If there exists a bounded convex set B such that $K = cone(B)$ and $0_Y \notin cl(B)$ then there exists $\phi \in Y^*$ such that*

$$K \subset \{y \in Y \mid \|y\| \leq \phi(y)\} \ . \tag{6}$$

and $(\phi, 1) \in K^{a*}$, $(\phi, 0) \in K^{a\#}$. If additionally $int(K) \neq \emptyset$, then $(\phi, 1) \in K^{a\circ}$.

Proof. According to [18, Prop. 2.2.15], there exists $\phi \in Y^*$ such that (6) holds for any nontrivial convex cone K if and only if there exists a bounded convex set B such that $K = \text{cone}(B)$ and $0_Y \notin \text{cl}(B)$. The remaining follows by Proposition 3.10. □

The assumptions of Theorem 3.13 are satisfied for a convex cone $K \subset \mathbb{R}^m$ if and only if cl$K$ is pointed [18, Example 2.2.16]. Recall that a nontrivial convex cone $K \subset Y$ with a closed and bounded base is also representable as a BP cone [26].

In this paper we consider a new nonlinear scalarization functional which is based on elements (y^*, α) of the augmented dual cones K^{a*}. For these elements it holds

$$K \subset \left\{ y \in Y \;\middle|\; \|y\| \leq \underbrace{\left(\frac{1}{\alpha} y^*\right)}_{=:\phi}(y) \right\} \ .$$

This leads to some similarities to the scalarizations discussed in [10] where it was assumed that the considered cones K are BP cones $C(\phi)$ w.r.t. the norm of the space. In that case these special scalarization functionals coincide with the ones considered here for the case that the pair $(y^*, \alpha) = (\phi, 1)$ is chosen from the augmented dual cone.

Next we examine a necessary condition for the existence of elements $(y^*, \alpha) \in K^{a*}$ with $\alpha \neq 0$.

Proposition 3.14. *Let $(Y, \|\cdot\|)$ be a real normed space and $K \subset Y$ a nontrivial closed convex cone. If there exists some element $(y^*, \alpha) \in K^{a*}$ with $\alpha \neq 0$, then there exists a closed and bounded convex set B such that*

$$K = cone(B) \quad and \quad 0_Y \notin B \ .$$

Proof. If there exists $(\tilde{y}^*, \alpha) \in K^{a*}$ with $\alpha \neq 0$ then there also exists $(y^*, 1) \in K^{a*}$, compare Proposition 3.5. Thus $K \subset \{y \in Y \mid y^*(y) \geq \|y\|\}$ and by [18, Prop. 2.2.15] there exists a bounded convex set B such that $K = \mathrm{cone}(B)$ and $0_Y \notin \mathrm{cl}(B)$. As $K = \mathrm{cl}(K) = \mathrm{cone}(\mathrm{cl}(B))$ the set B can be chosen to be closed. □

4 Nonlinear scalarizations

Linear scalarizations have several drawbacks for characterizing nondominated elements w.r.t. a variable ordering [9]. Among others, as a necessary condition the convexity of the set has to be assumed. But also the sufficient conditions are in general too strong. In the literature, nonlinear scalarization functionals for characterizing nondominated elements of a vector optimization problem with an arbitrary variable ordering structure have hardly be examined so far. A possible approach is for instance a modification of the so-called Pascoletti-Serafini scalarization, see [11], which is a nonconvex functional in general (unless we have a constant cone-valued map $\mathcal{D}$). Another approach was proposed by Eichfelder and Ha [10] (see Example 4.1 below) which requires strong assumptions to be satisfied for deriving properties of the functionals.

In this section we use new nonlinear scalarization functionals. For that we assume that maps $(\ell^*, \alpha^*)\,(\ell^\circ, \alpha^\circ), (\ell^\#, \alpha^\#)\colon Y \to Y^* \times \mathbb{R}_+$ are given with the properties

$$(\ell^*, \alpha^*)(y) \in (\mathcal{D}(y))^{a*}, \quad (\ell^\circ, \alpha^\circ)(y) \in (\mathcal{D}(y))^{a\circ}, \text{ and } (\ell^\#, \alpha^\#)(y) \in (\mathcal{D}(y))^{a\#}$$

for all $y \in Y$. For the definition of $(\ell^\circ, \alpha^\circ)$ we assume the interior of the cones $\mathcal{D}(y)$ to be nonempty. We write (ℓ^N_y, α^N_y) for $(\ell^N, \alpha^N)(y)$ and any $N \in \{*, \circ, \#\}$ for shortness of the representation.

To any fixed $\bar{y} \in Y$ and $r \in Y$ we associate the following functionals:

$$\begin{aligned} \gamma^N_{\bar{y},r}(y) &:= \ell^N_{\bar{y}}(y - r) - \alpha^N_{\bar{y}}\|y - r\| \\ \xi^N_{\bar{y},r}(y) &:= \ell^N_{\bar{y}}(y - r) + \alpha^N_{\bar{y}}\|y - r\| \\ \eta^N_{\bar{y}}(y) &:= \ell^N_{y}(y - \bar{y}) - \alpha^N_{y}\|y - \bar{y}\| \\ \zeta^N_{\bar{y}}(y) &:= \ell^N_{y}(y - \bar{y}) + \alpha^N_{y}\|y - \bar{y}\| \end{aligned} \tag{7}$$

for each $y \in Y$ with $N \in \{*, \circ, \#\}$. Note that for $\gamma^N_{\bar{y},r}$ and for $\xi^N_{\bar{y},r}$ the pair $(\ell^N_{\bar{y}}, \alpha^N_{\bar{y}})$ is an element of the cone $(\mathcal{D}(\bar{y}))^{a*}$ and the pair does not change depending on the value of y for which the functionals $\gamma^N_{\bar{y},r}$ and $\xi^N_{\bar{y},r}$ are evaluated. In addition to that, $\gamma^N_{\bar{y},r}$ and $\xi^N_{\bar{y},r}$ are allowed to depend on some parameter r which is chosen as $r = \bar{y}$ for the definitions of $\eta^N_{\bar{y}}$ and $\zeta^N_{\bar{y}}$.

Example 4.1. *Let the variable ordering structure on Y be defined by the set-valued map $\mathcal{D}\colon Y \to 2^Y$ with $\mathcal{D}(y)$ being Bishop-Phelps cones w.r.t. the norm of the space, i.e. all BP cones are defined w.r.t. the same norm. Then there exists a map $\phi\colon Y \to Y^*$ with*

$$\mathcal{D}(y) = C(\phi(y)) = \{u \in Y \mid \|u\| \leq \phi(y)(u)\}. \tag{8}$$

Then, by Proposition 3.9, we can choose

$$(\ell^*_y, \alpha^*_y) = (\phi(y), 1) \quad \textit{for all } y \in Y$$

as well as

$$(\ell_y^{\#}, \alpha_y^{\#}) = (\phi(y), 0) \text{ for all } y \in Y \ .$$

If $\|\phi(y)\|_* > 1$ *for all* $y \in Y$ *also*

$$(\ell_y^{\circ}, \alpha_y^{\circ}) = (\phi(y), 1) \text{ for all } y \in Y \ .$$

By setting $r = \bar{y} \in Y$ *and* $N = *$ *or* $N = \circ$*, we obtain the nonlinear scalarization functionals*

$$\begin{aligned} \gamma_{\bar{y},r}^N(y) &= \phi(\bar{y})(y - \bar{y}) - \|y - \bar{y}\| \text{ for all } y \in Y, \\ \xi_{\bar{y},r}^N(y) &= \phi(\bar{y})(y - \bar{y}) + \|y - \bar{y}\| \text{ for all } y \in Y, \\ \eta_{\bar{y}}^N(y) &= \phi(y)(y - \bar{y}) - \|y - \bar{y}\| \text{ for all } y \in Y, \\ \zeta_{\bar{y}}^N(y) &= \phi(y)(y - \bar{y}) + \|y - \bar{y}\| \text{ for all } y \in Y, \end{aligned} \tag{9}$$

which are exactly those considered (under the above assumptions) in [10] in equation (12) and in the conclusions.

4.1 Properties of the scalarization functionals

Obviously, the functionals $\gamma_{\bar{y},r}^N$ and $\xi_{\bar{y},r}^N$ are Lipschitz continuous and $\xi_{\bar{y},r}^N$ is convex. In this subsection we show that the scalarization functionals $\eta_{\bar{y}}^N$ and $\zeta_{\bar{y}}^N$ inherit from the maps $\ell^*, \ell^\circ, \ell^{\#} \colon Y \to Y^*$ (assuming $\alpha_y^* = \alpha_y^\circ = \alpha_y^{\#} = 1$ for all $y \in Y$) such properties as continuity and Lipschitz continuity. We also show that the functional $\zeta_{\bar{y}}^N$ is convex under some assumptions.

Note that if there exists some map (ℓ^N, α^N) ($N \in \{*, \circ, \#\}$) with $(\ell_y^N, \alpha_y^N) \in (\mathcal{D}(y))^{aN}$ and $\alpha_y^N \neq 0$ for all $y \in Y$, then there also exists some map $(\tilde{\ell}^N, 1)$ with $(\tilde{\ell}_y^N, 1) \in (\mathcal{D}(y))^{aN}$ for all $y \in Y$ by setting $\tilde{\ell}_y^N := \frac{1}{\alpha_y^N} \ell_y^N$ (compare Proposition 3.5).

Lemma 4.2. *Let* $N \in \{*, \circ, \#\}$*. Assume there is a map* $\ell \colon Y \to Y^*$ *with* $(\ell(y), 1) \in (\mathcal{D}(y))^{aN}$ *for all* $y \in Y$*. Let the map* ℓ *be continuous. Then the functionals* $\eta_{\bar{y}}^N$ *and* $\zeta_{\bar{y}}^N$ *are continuous.*

If the map ℓ *is Lipschitz near* $y \in Y$*, then the functionals* $\eta_{\bar{y}}^N$ *and* $\zeta_{\bar{y}}^N$ *are also Lipschitz continuous near* $y \in Y$*.*

Proof. Using the same steps as in the proof given in [10, Prop. 3.13] one can show that the map $f \colon Y \to \mathbb{R}$, $f(y) := \ell(y)(y)$ is continuous. This implies the continuity of the scalarization functionals due to the continuity of the norm.

Assuming the Lipschitz continuity of ℓ near y, using the same steps as in the proof of [10, Prop. 3.15], one can show that f is Lipschitz continuous near y. As the norm is Lipschitz continuous, the result follows. □

Note that also results on the subdifferential of $\eta_{\bar{y}}^N$ and $\zeta_{\bar{y}}^N$ can be obtained in case ℓ is Lipschitz near $\bar{y}$ using again the arguments provided in [10, Prop. 3.15].

Lemma 4.3. *Let* $N \in \{*, \circ, \#\}$*. Assume there is a map* $\ell \colon Y \to Y^*$ *with* $(\ell(y), 1) \in (\mathcal{D}(y))^{aN}$ *for all* $y \in Y$*. Let the map* ℓ *be monotone, i.e.*

$$(\ell(y_1) - \ell(y_2))(y_1 - y_2) \geq 0 \text{ for all } y_1, y_2 \in Y \ ,$$

and linear. Then $\zeta_{\bar{y}}^N$ *is convex.*

Proof. Let $y_1, y_2 \in Y$ and $\lambda \in [0,1]$. Then

$$\begin{aligned}\zeta^N_{\bar{y}}(\lambda y_1 + (1-\lambda)y_2) &= \lambda\ell(y_1)(y_1 - \bar{y}) + (1-\lambda)\ell(y_2)(y_2 - \bar{y}) \\ &\quad -\lambda(1-\lambda)(\ell(y_1) - \ell(y_2))(y_1 - y_2) + \|\lambda y_1 + (1-\lambda)y_2 - \bar{y}\| \\ &\leq \lambda\ell(y_1)(y_1 - \bar{y}) + \lambda\|y_1 - \bar{y}\| \\ &\quad +(1-\lambda)\ell(y_2)(y_2 - \bar{y}) + (1-\lambda)\|y_2 - \bar{y}\| \\ &= \lambda\zeta^N_{\bar{y}}(y_1\bar{y}) + (1-\lambda)\zeta^N_{\bar{y}}(y_2).\end{aligned}$$

□

If we assume that $(\mathcal{D}(y))^\# \neq \emptyset$, then for any $\ell_y \in (\mathcal{D}(y))^\#$, it follows $(\ell_y, 0) \in (\mathcal{D}(y))^{aN}$ $(N \in \{*, \circ, \#\})$ for all $y \in Y$ and in this case the functionals in (7) are linear functionals.

4.2 Characterization results

Using the above functionals we can characterize (weakly/strongly) minimal and nondominated elements of a set A w.r.t. $\mathcal{D}$. Note that neither additional assumptions on $\mathcal{D}$ – besides that maps (ℓ^N, α^N) exist – nor convexity assumptions on the set A are presumed.

We start with the results for (weakly/strongly) minimal elements.

Theorem 4.4. *Let A be a nonempty subset of Y and $\bar{y} \in A$.*

(i) If the functional $\xi^*_{\bar{y},r}$ attains its strict minimum over A at $\bar{y}$, which means that

$$\xi^*_{\bar{y},r}(\bar{y}) < \xi^*_{\bar{y},r}(y), \ \forall y \in A \setminus \{\bar{y}\} \ ,$$

then $\bar{y} \in A$ is a minimal element of A w.r.t. the ordering map $\mathcal{D}$.

(ii) If the functional $\xi^\#_{\bar{y},r}$ attains its minimum over A at $\bar{y}$, which means that

$$\xi^\#_{\bar{y},r}(\bar{y}) \leq \xi^\#_{\bar{y},r}(y), \ \forall y \in A \setminus \{\bar{y}\} \ ,$$

then $\bar{y} \in A$ is a minimal element of A w.r.t. the ordering map $\mathcal{D}$.

(iii) Suppose $int(\mathcal{D}(\bar{y}) \neq \emptyset$. If the functional $\xi^\circ_{\bar{y},r}$ attains its minimum over A at $\bar{y}$, which means that

$$\xi^\circ_{\bar{y},r}(\bar{y}) \leq \xi^\circ_{\bar{y},r}(y), \ \forall y \in A \ ,$$

then $\bar{y} \in A$ is a weakly minimal element of A w.r.t. the ordering map $\mathcal{D}$.

(iv) If $\bar{y} \in A$ is a strongly minimal element of A w.r.t. the ordering map $\mathcal{D}$ then the functional $\gamma^*_{\bar{y},r}$ attains its minimum over A at $\bar{y}$, which means that

$$\gamma^*_{\bar{y},r}(y) \geq \gamma^*_{\bar{y},r}(\bar{y}), \ \forall y \in A.$$

Proof. (i) Assume $\bar{y}$ is not a minimal element of A w.r.t. $\mathcal{D}$. Then for some $y \in A \setminus \{\bar{y}\}$, $\bar{y} - y \in \mathcal{D}(\bar{y}) \setminus \{0_Y\}$ and since $(\ell^*_{\bar{y}}, \alpha^*_{\bar{y}}) \in (\mathcal{D}(\bar{y}))^{a*}$

$$\ell^*_{\bar{y}}(\bar{y} - y) - \alpha^*_{\bar{y}}\|\bar{y} - y\| \geq 0.$$

This leads to $-\ell^*_{\bar{y}}(y - \bar{y}) - \alpha^*_{\bar{y}}\|y - \bar{y}\| \geq 0$, or

$$\ell^*_{\bar{y}}(y - r) + \alpha^*_{\bar{y}}\|y - r\| - \ell^*_{\bar{y}}(\bar{y} - r) - \alpha^*_{\bar{y}}\|\bar{y} - r\| \leq \ell^*_{\bar{y}}(y - \bar{y}) + \alpha^*_{\bar{y}}\|y - \bar{y}\| \leq 0$$

which contradicts the hypothesis.

(ii) Assume $\bar{y}$ is not a minimal element of A w.r.t. $\mathcal{D}$. Then for some $y \in A \setminus \{\bar{y}\}$ it holds $\bar{y} - y \in \mathcal{D}(\bar{y}) \setminus \{0_Y\}$ and since $(\ell^\#_{\bar{y}}, \alpha^\#_{\bar{y}}) \in (\mathcal{D}(\bar{y}))^{a\#}$

$$\ell^\#_{\bar{y}}(\bar{y} - y) - \alpha^\#_{\bar{y}}\|\bar{y} - y\| > 0.$$

This leads again to a contradiction of the hypothesis.

(iii) Assume $\bar{y}$ is not a weakly minimal element of A w.r.t. $\mathcal{D}$. Then there is some $y \in A$ with $\bar{y} - y \in \operatorname{int}(\mathcal{D}(\bar{y}))$ and thus $\ell^\circ_{\bar{y}}(\bar{y} - y) - \alpha^\circ_{\bar{y}}\|\bar{y} - y\| > 0$ and so

$$0 < \ell^\circ_{\bar{y}}(\bar{y} - y) - \alpha^\circ_{\bar{y}}\|\bar{y} - y\| \leq \ell^\circ_{\bar{y}}(\bar{y} - r) + \alpha^\circ_{\bar{y}}\|\bar{y} - r\| - \ell^\circ_{\bar{y}}(y - r) - \alpha^\circ_{\bar{y}}\|y - r\|,$$

or $\xi^\circ_{\bar{y},r}(\bar{y}) > \xi^\circ_{\bar{y},r}(y)$ in contradiction to $\bar{y}$ a minimizer of $\xi^\circ_{\bar{y},r}$ over A.

(iv) As $\bar{y}$ is a strongly minimal element of A w.r.t. $\mathcal{D}$ we have $y \in \{\bar{y}\} + \mathcal{D}(\bar{y})$ for all $y \in A$, or equivalently $y - \bar{y} \in \mathcal{D}(\bar{y})$ and thus for every $(\ell^*_{\bar{y}}, \alpha^*_{\bar{y}}) \in (\mathcal{D}(\bar{y}))^{a*}$ we have:

$$\begin{aligned}\gamma^*_{\bar{y},r}(y) - \gamma^*_{\bar{y},r}(\bar{y}) &= \ell^*_{\bar{y}}(y - r) - \alpha^*_{\bar{y}}\|y - r\| - \ell^*_{\bar{y}}(\bar{y} - r) + \alpha^*_{\bar{y}}\|\bar{y} - r\| \\ &\geq \ell^*_{\bar{y}}(y - \bar{y}) - \alpha^*_{\bar{y}}\|y - \bar{y}\| \\ &\geq 0\end{aligned}$$

for all $y \in A$.

□

Corollary 4.5. *Let A be a nonempty subset of Y and $\bar{y} \in A$. Additionally, let $\mathcal{D}(\bar{y})$ be a BP cone given by $\mathcal{D}(\bar{y}) = \{u \in Y \mid \phi(u) \geq \|u\|\}$ with $\phi \in Y^*$. Choose $(\ell^*_{\bar{y}}, \alpha^*_{\bar{y}}) := (\phi, 1)$ and if $\|\phi\| > 1$ also $(\ell^\circ_{\bar{y}}, \alpha^\circ_{\bar{y}}) := (\phi, 1)$ (see Proposition 3.9).*

(i) The functional $\xi^*_{\bar{y},\bar{y}}$ attains its strict minimum over A at $\bar{y}$ if and only if $\bar{y}$ is a minimal element of A w.r.t. $\mathcal{D}$.

(ii) Suppose int$(\mathcal{D}(\bar{y})) \neq \emptyset$. The functional $\xi^\circ_{\bar{y},\bar{y}}$ attains its minimum over A at $\bar{y}$ if and only if $\bar{y}$ is a weakly minimal element of A w.r.t. $\mathcal{D}$.

(iii) The functional $\gamma^*_{\bar{y},\bar{y}}$ attains its minimum over A at $\bar{y}$ if and only if $\bar{y}$ is a strongly minimal element of A w.r.t. $\mathcal{D}$.

Proof. The sufficient conditions of (i) and (ii) and the necessarity of (iii) are already proven in Theorem 4.4.

(i) Assume $\bar{y}$ is a minimal element of A w.r.t. $\mathcal{D}$ but there is some $y \in A \setminus \{\bar{y}\}$ with $\xi^*_{\bar{y},\bar{y}} \leq 0$. Then $\ell^*_{\bar{y}}(y-\bar{y}) + \alpha^*_y\|y-\bar{y}\| \leq 0$ and by the definition of (ℓ^*_y, α^*_y)

$$\phi(\bar{y} - y) - \|\bar{y} - y\| \geq 0.$$

The definition of $\mathcal{D}(\bar{y})$ implies $\bar{y} - y \in \mathcal{D}(\bar{y})$ in contradiction to $\bar{y}$ minimal.

(ii) The proof is similar to the proof of (i) but uses that $\operatorname{int}(\mathcal{D}(\bar{y})) = \{u \in Y \mid \phi(\bar{y})(u) \geq \|u\|\}$ by Proposition 3.8(iii).

(iii) If $\gamma^*_{\bar{y},\bar{y}}(y) \geq 0$ for all $y \in A$, then $\ell^*_{\bar{y}}(y-\bar{y}) - \alpha^*_{\bar{y}}\|y-\bar{y}\| \geq 0$ for all $y \in A$ and according to the definition of $\mathcal{D}(\bar{y})$ and of $(\ell^*_{\bar{y}}, \alpha^*_{\bar{y}})$ this implies $y - \bar{y} \in \mathcal{D}(\bar{y})$ for all $y \in A$.

□

Next, the results for the (weakly/strongly) nondominated elements are given.

Theorem 4.6. *Let A be a nonempty subset of Y and $\bar{y} \in A$.*

(i) If the functional $\zeta^*_{\bar{y}}$ attains its strict minimum over A at $\bar{y}$, which means that

$$\zeta^*_{\bar{y}}(\bar{y}) = 0 < \zeta^*_{\bar{y}}(y), \ \forall y \in A \setminus \{\bar{y}\} \ ,$$

then $\bar{y} \in A$ is a nondominated element of A w.r.t. the ordering map $\mathcal{D}$.

(ii) If the functional $\zeta^{\#}_{\bar{y}}$ attains its minimum over A at $\bar{y}$, which means that

$$\zeta^{\#}_{\bar{y}}(\bar{y}) = 0 \leq \zeta^{\#}_{\bar{y}}(y), \ \forall y \in A \ ,$$

then $\bar{y} \in A$ is a nondominated element of A w.r.t. the ordering map $\mathcal{D}$.

(iii) Suppose int$(\mathcal{D}(y) \neq \emptyset$ for all $y \in Y$. If the functional $\zeta^{\circ}_{\bar{y}}$ attains its minimum over A at $\bar{y}$, which means that

$$\zeta^{\circ}_{\bar{y}}(\bar{y}) = 0 \leq \zeta^{\circ}_{\bar{y}}(y), \ \forall y \in A \ ,$$

then $\bar{y} \in A$ is a weakly nondominated element of A w.r.t. the ordering map $\mathcal{D}$.

(iv) If $\bar{y} \in A$ is a strongly nondominated element of A w.r.t. the ordering map $\mathcal{D}$ then the functional $\eta^*_{\bar{y}}$ attains its minimum over A at $\bar{y}$, which means that

$$\eta^*_{\bar{y}}(y) \geq \eta^*_{\bar{y}}(\bar{y}) = 0, \ \forall y \in A.$$

Proof. (i) Assume $\bar{y}$ is not a nondominated element of A w.r.t. $\mathcal{D}$. Then for some $y \in A$ it holds $\bar{y} - y \in \mathcal{D}(y) \setminus \{0_Y\}$ and since $(\ell_y^*, \alpha_y^*) \in (\mathcal{D}(y))^{a*}$ $\ell_y^*(\bar{y} - y) - \alpha_y^* \|\bar{y} - y\| \geq 0$, or

$$\ell_y^*(y - \bar{y}) + \alpha_y^* \|y - \bar{y}\| \leq 0 = \ell_{\bar{y}}^*(\bar{y} - \bar{y}) + \alpha_{\bar{y}}^* \|\bar{y} - \bar{y}\|$$

which contradicts the hypothesis.

(ii) Assume $\bar{y}$ is not a nondominated element of A w.r.t. $\mathcal{D}$. Then for some $y \in A$ it holds $\bar{y} - y \in \mathcal{D}(y) \setminus \{0_Y\}$ and since $(\ell_y^{\#}, \alpha_y^{\#}) \in (\mathcal{D}(y))^{a\#}$

$$\ell_y^{\#}(\bar{y} - y) - \alpha_y^{\#} \|\bar{y} - y\| > 0,$$

which again leads to a contradiction of the hypothesis.

(iii) Assume $\bar{y}$ is not a weakly nondominated element of A w.r.t. $\mathcal{D}$. Then there is some $y \in A$ with $\bar{y} - y \in \operatorname{int}(\mathcal{D}(y))$ and thus $\ell_y^{\circ}(\bar{y} - y) - \alpha_y^{\circ} \|\bar{y} - y\| > 0$ and so

$$0 > \ell_y^{\circ}(y - \bar{y}) + \alpha_y^{\circ} \|y - \bar{y}\|$$

in contradiction to $\bar{y}$ a minimizer of $\zeta_{\bar{y}}^{\circ}$ over A.

(iv) As $\bar{y}$ is a strongly nondominated element of A w.r.t. $\mathcal{D}$ we have $y - \bar{y} \in \mathcal{D}(y)$ and thus for every $(\ell_y^*, \alpha_y^*) \in (\mathcal{D}(y))^{a*}$ we have:

$$\eta_{\bar{y}}^*(y) = \ell_y^*(y - \bar{y}) - \alpha_y^* \|y - \bar{y}\| \geq 0$$

for all $y \in A$.

□

Corollary 4.7. *Let A be a nonempty subset of Y and $\bar{y} \in A$. Additionally, let $\mathcal{D}(y)$ be a BP cone for any $y \in Y$ given by $\mathcal{D}(y) = \{u \in Y \mid \phi(y)(u) \geq \|u\|\}$ with $\phi(y) \in Y^*$ for all $y \in Y$. Choose $(\ell_y^*, \alpha_y^*) := (\phi(y), 1)$ for all $y \in Y$ and if $\|\phi(y)\| > 1$ also $(\ell_y^{\circ}, \alpha_y^{\circ}) := (\phi(y), 1)$ for all $y \in Y$ (see Proposition 3.9).*

(i) The functional $\zeta_{\bar{y}}^*$ attains its strict minimum over A at $\bar{y}$ if and only if $\bar{y}$ is a nondominated element of A w.r.t. $\mathcal{D}$.

(ii) Suppose int$(\mathcal{D}(y)) \neq \emptyset$ for all $y \in Y$. The functional $\zeta_{\bar{y}}^{\circ}$ attains its minimum over A at $\bar{y}$ if and only if $\bar{y}$ is a weakly nondominated element of A w.r.t. $\mathcal{D}$.

(iii) The functional $\eta_{\bar{y}}^*$ attains its minimum over A at $\bar{y}$ if and only if $\bar{y}$ is a strongly nondominated element of A w.r.t. $\mathcal{D}$.

Proof. The sufficient conditions of (i) and (ii) and the necessarity of (iii) are already proven in Theorem 4.6.

(i) Assume $\bar{y}$ is a nondominated element of A w.r.t. $\mathcal{D}$ but there is some $y \in A\backslash\{\bar{y}\}$ with $\zeta^*_{\bar{y}}(y) \leq 0$. Then $\ell^*_y(y-\bar{y})+\alpha^*_y\|y-\bar{y}\| \leq 0$ and by the definition of (ℓ^*_y, α^*_y)

$$\phi(y)(\bar{y} - y) - \|\bar{y} - y\| \geq 0.$$

The definition of $\mathcal{D}(y)$ implies $\bar{y}-y \in \mathcal{D}(y)$ in contradiction to $\bar{y}$ nondominated.

(ii) The proof is similar to the proof of (i) but uses that $\text{int}(\mathcal{D}(y)) = \{u \in Y \mid \phi(y)(u) \geq \|u\|\}$ by Proposition 3.8(iii).

(iii) If $\eta^*_{\bar{y}}(y) \geq 0$ for all $y \in A$, then $\ell^*_y(y - \bar{y}) - \alpha^*_y\|y - \bar{y}\| \geq 0$ for all $y \in A$ and according to the definition of $\mathcal{D}(y)$ and of (ℓ^*_y, α^*_y) this implies $y - \bar{y} \in \mathcal{D}(y)$ for all $y \in A$.

□

The results of Corollary 4.7 coincide with the special scalarization results given in [10].

5 Properly optimal elements

Following the definitions for properly efficient elements given by Henig [20], Benson [2] and Borwein [4] in partially ordered space, we introduce the following generalizations for variable ordering structures. As before, let $(Y, \|\cdot\|)$ be a real normed space, A a nonempty subset of Y and and let $\mathcal{D}\colon Y \to 2^Y$ be the ordering map.

Definition 5.1. *Let $\bar{y} \in A$. We say that*

(a) $\bar{y}$ *is a* properly nondominated *element of A w.r.t. the ordering map $\mathcal{D}$ (in the sense of Henig [20]) if it is a nondominated element of A w.r.t. the ordering map $\mathcal{D}$ and if there is a cone-valued map $\mathcal{K}\colon Y \to 2^Y$ with $\mathcal{D}(y) \setminus \{0_Y\} \subset$ int$(\mathcal{K}(y))$ for all $y \in Y$ such that $\bar{y}$ is a nondominated element of A w.r.t. $\mathcal{K}$, i.e.*

$$\bar{y} \notin \{y\} + \mathcal{K}(y)\ \forall\ y \in A \setminus \{\bar{y}\}.$$

(b) $\bar{y}$ *is a* properly nondominated *element of A w.r.t. the ordering map $\mathcal{D}$ (in the sense of Benson [2]) if it is a nondominated element of A w.r.t. the ordering map $\mathcal{D}$ and if $\bar{y}$ is a nondominated element of the set*

$$\{\bar{y}\} + cl(cone(\bigcup_{a\in A}(\{a\} + \mathcal{D}(a)) - \{\bar{y}\})).$$

(c) $\bar{y}$ *is a* properly nondominated *element of A w.r.t. the ordering map $\mathcal{D}$ (in the sense of Borwein [4]) if it is a nondominated element of A w.r.t. the ordering map $\mathcal{D}$ and if $\bar{y}$ is a nondominated element of the set*

$$\{\bar{y}\} + T(\bigcup_{a\in A}(\{a\} + \mathcal{D}(a)), \bar{y}).$$

(d) $\bar{y}$ *is a properly minimal element of A w.r.t. the ordering map* $\mathcal{D}$ *(in the sense of Henig [20]) if it is a minimal element of A w.r.t. the ordering map* $\mathcal{D}$ *and if there is a cone-valued map* $\mathcal{K}\colon Y \to 2^Y$ *with* $\mathcal{D}(y) \setminus \{0_Y\} \subset int(\mathcal{K}(y))$ *for all* $y \in Y$ *such that* $\bar{y}$ *is a minimal element of A w.r.t.* $\mathcal{K}$*, i.e.*

$$y \notin \{\bar{y}\} - \mathcal{K}(\bar{y}) \;\forall\; y \in A \setminus \{\bar{y}\}.$$

(e) $\bar{y}$ *is a properly minimal element of A w.r.t. the ordering map* $\mathcal{D}$ *(in the sense of Benson [2]) if it is a minimal element of A w.r.t. the ordering map* $\mathcal{D}$ *and if* $\bar{y}$ *is a minimal element of the set*

$$\{\bar{y}\} + cl(cone(A + \mathcal{D}(\bar{y}) - \{\bar{y}\})).$$

(f) $\bar{y}$ *is a properly minimal element of A w.r.t. the ordering map* $\mathcal{D}$ *(in the sense of Borwein [4]) if it is a minimal element of A w.r.t. the ordering map* $\mathcal{D}$ *and if* $\bar{y}$ *is a minimal element of the set*

$$\{\bar{y}\} + T(A + \mathcal{D}(\bar{y}), \bar{y}).$$

Thereby, $T(\Omega, \bar{y})$ with $\Omega \subset Y$, $\Omega \neq \emptyset$ and $\bar{y} \in \mathrm{cl}(\Omega)$ denotes the *contingent cone* (or the *Bouligand tangent cone*) to S at $\bar{y}$ as defined for instance in [23, Def. 3.41]:

$$\begin{aligned} T(\Omega, \bar{y}) := \{h \in Y \mid\; & \exists (\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{R}_{++},\; \exists (y_n)_{n\in\mathbb{N}} \subset \Omega \\ & \text{such that } \lim_{n\to\infty} y_n \to \bar{y} \text{ and } h = \lim_{n\to\infty} \lambda_n (y_n - \bar{y})\}. \end{aligned}$$

If $\mathcal{D}(y) = K$ for all $y \in Y$ then the definitions of a properly nondominated element w.r.t. $\mathcal{D}$ and of a properly minimal element w.r.t. $\mathcal{D}$ coincide with the concepts of a properly efficient element in a partially ordered space ordered by the convex cone K.

The generalizations of Henig proper efficiency to Henig proper nondominatedness and Henig proper minimality are straightforward. However, for instance for the generalization of the notion of Borwein efficiency to Borwein proper nondominatedness, one needs to reformulate the condition that 0_Y is an efficient element of the set $M := T(A + K, \{\bar{y}\})$ with K the ordering cone. Possible approaches are to consider the set

$$M_1 := \{\bar{y}\} + T(\bigcup_{a\in A} (\{a\} + \mathcal{D}(a)), \bar{y})$$

and check if $\bar{y}$ is a nondominated element as done above, but also for instance

$$M_2 := \{\bar{y}\} + T(A + \bigcup_{a\in A} \mathcal{D}(a), \bar{y})$$

and checking whether $\bar{y}$ is a nondominated element or

$$M_3 := T(\bigcup_{a\in A} (\{a\} + \mathcal{D}(a)), \bar{y})$$

and checking whether the zero is a nondominated element would be possible approaches. The following example demonstrates that the approach proposed in Definition 5.1(c) is a meaningful generalization.

Example 5.2. *Let $A := \{y \in \mathbb{R}^2 \mid \|y\| \leq 1\}$ be the unit ball in the Euclidean space $Y = \mathbb{R}^2$ and let an ordering map $\mathcal{D}$ be defined by*

$$\mathcal{D}(y) = \begin{cases} \{z \in \mathbb{R}^2 \mid z_1 \leq 0,\ z_2 \geq 0\} & \text{if } y \in \{(0,1),(2,0)\}, \\ \mathbb{R}^2_+ & \text{else.} \end{cases}$$

Without doubt, the point $\bar{y} = (-1/\sqrt{2}, -1/\sqrt{2})$ should be a properly nondominated point in the sense of Borwein and it is in fact if we use Definition 5.1. As $\bigcup_{a\in A} \mathcal{D}(a) = \{z \in \mathbb{R}^2 \mid z_2 \geq 0\}$ we obtain however $M_2 = \mathbb{R}^2$ and $\bar{y}$ is dominated for instance by $(-1,-1)$. The element $(2,0)$ is in M_3 and thus 0_Y is not a nondominated element of M_3.

Remark 5.3. *(i) By [23, Theorem 3.44], any Benson properly nondominated/ minimal element of A w.r.t. $\mathcal{D}$ is also a Borwein properly nondominated/ minimal element of A w.r.t. $\mathcal{D}$.*

(ii) If the set $M := \bigcup_{a\in A} (\{a\} + \mathcal{D}(a))$ is a convex set, then Benson's proper nondominatedness is equivalent to Borwein's proper nondominatedness by [28, Lemma 3.1.1]. If M is starshaped w.r.t. some element $\bar{y} \in A$, then by [23, Cor. 3.46] the element $\bar{y}$ is a Benson properly nondominated element if and only if it is a Borwein properly nondominated element.

(iii) Also by [28, Lemma 3.1.1] and [23, Cor. 3.46], if $A+\mathcal{D}(\bar{y})$ is starshaped w.r.t. $\bar{y} \in A$, then the element $\bar{y}$ is a Benson properly minimal element if and only if it is a Borwein properly minimal element.

(iv) If the cones $\mathcal{D}(y)$ are closed, pointed, nontrivial and convex cones, then by [28, Theorem 3.1.2] Henig's proper minimality is equivalent to Benson's proper minimality.

Lemma 5.4. *$\bar{y}$ is a properly minimal element of A w.r.t. the ordering map $\mathcal{D}$ (in the sense of Henig) if and only if it is a minimal element of A w.r.t. the ordering map $\mathcal{D}$ and if there is a convex cone K with $\mathcal{D}(\bar{y}) \setminus \{0_Y\} \subset int(K)$ such that $y \notin \{\bar{y}\} - K$ for all $y \in A \setminus \{\bar{y}\}$.*

Proof. The only-if-part is immediate from Definition 5.1(b) by setting $K := \mathcal{K}(\bar{y})$. For the if-part, the set-valued map $\mathcal{K}$ can be defined by

$$\mathcal{K}(y) := \begin{cases} Y & \text{if } y \neq \bar{y}, \\ K & \text{if } y = \bar{y}. \end{cases}$$

Then $\mathcal{D}(y) \setminus \{0_Y\} \subset \text{int}(\mathcal{K}(y))$ for all $y \in Y$ and we are done. □

The next lemma follows directly from the definitions.

Lemma 5.5. *$\bar{y}$ is a properly minimal element of A in the sense of Henig/Benson/ Borwein w.r.t. the ordering map $\mathcal{D}$ if and only if it is a properly efficient element in the sense of Henig/Benson/Borwein of A in the space Y partially ordered by the convex cone $K := \mathcal{D}(\bar{y})$.*

Remark 5.6. *As a consequence of the above Lemma and [19, Theorem 4.2], if $\bar{y}$ is a properly minimal element in the sense of Henig, then $\bar{y}$ is also a properly minimal element in the sense of Benson. As a consequence of the above Lemma and [24, Theorems 5.2], if $\bar{y}$ is a properly minimal element in the sense of Benson and if $\mathcal{D}(\bar{y})$ has a weakly compact base, then $\bar{y}$ is also a properly minimal element in the sense of Henig w.r.t. $\mathcal{D}$.*

5.1 Characterizing properly minimal elements

In the following we show that the newly introduced scalarization functionals are also useful for characterizing proper optimal elements. For Benson proper minimality, sufficient conditions are given in Theorem 5.7 and 5.8(iii) while necessary conditions are given in Theorem 5.8(ii). Necessary and sufficient conditions for Henig properly minimal elements are proposed in Theorem 5.8(i) and (iii). In Theorem 5.7 we provide sufficient conditions for Borwein properly minimal elements. Necessary conditions for Borwein properly minimal elements will be presented in the next subsection in Theorem 5.13.

Recall that the functional $\xi^{\#}_{\bar{y},\bar{y}}$ is defined by

$$\xi^{\#}_{\bar{y},\bar{y}}(y) = \ell^{\#}_{\bar{y}}(y-\bar{y}) + \alpha^{\#}_{\bar{y}}\|y-\bar{y}\|$$

for all $y \in Y$ with $(\ell^{\#}_{\bar{y}}, \alpha^{\#}_{\bar{y}}) \in (\mathcal{D}(\bar{y}))^{a\#}$.

Theorem 5.7. *Let $\bar{y} \in A$ be a minimal solution of the problem*

$$\min_{y\in A} \xi^{\#}_{\bar{y},\bar{y}}(y),$$

i.e.

$$\ell^{\#}_{\bar{y}}(y-\bar{y}) + \alpha^{\#}_{\bar{y}}\|y-\bar{y}\| \geq 0 \quad \textit{for all} \quad y \in A.$$

Then $\bar{y}$ is a Benson properly minimal element of A w.r.t. $\mathcal{D}$ and also a Borwein properly minimal element of A w.r.t. $\mathcal{D}$.

Proof. By Theorem 4.4(ii), $\bar{y}$ is a minimal element of A w.r.t. $\mathcal{D}$. Next, let

$$y \in \{\bar{y}\} + \mathrm{cl}(\mathrm{cone}(A + \mathcal{D}(\bar{y}) - \{\bar{y}\}))$$

be arbitrarily chosen. Then there exists a sequence of nonnegative real numbers $(\lambda_n)_{n\in\mathbb{N}}$ and sequences $(y_n)_{n\in\mathbb{N}} \subset A$, $(d_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\bar{y})$ with $y = \bar{y} + h$ and $h := \lim_{n\to\infty} \lambda_n(y_n + d_n - \bar{y})$. By the continuity and linearity of $\ell^{\#}_{\bar{y}}$, as $d_n \in \mathcal{D}(\bar{y})$ for all $n \in \mathbb{N}$ and $(\ell^{\#}_{\bar{y}}, \alpha^{\#}_{\bar{y}}) \in (\mathcal{D}(\bar{y}))^{a\#}$ and by the assumption as $y_n \in A$ for all $n \in \mathbb{N}$

$$\begin{aligned}
\xi^{\#}_{\bar{y},\bar{y}}(y) &= \ell^{\#}_{\bar{y}}(h) + \alpha^{\#}_{\bar{y}}\|h\| \\
&\geq \lim\nolimits_{n\to\infty} \lambda_n \Big(\ell^{\#}_{\bar{y}}(y_n-\bar{y}) + \alpha^{\#}_{\bar{y}}\|y_n-\bar{y}\| + \ell^{\#}_{\bar{y}}(d_n) - \alpha^{\#}_{\bar{y}}\|d_n\|\Big) \\
&\geq \lim\nolimits_{n\to\infty} \lambda_n \Big(\ell^{\#}_{\bar{y}}(y_n-\bar{y}) + \alpha^{\#}_{\bar{y}}\|y_n-\bar{y}\|\Big) \\
&\geq 0.
\end{aligned}$$

Again, by Theorem 4.4(ii), $\bar{y}$ is a minimal element of $\{\bar{y}\} + \mathrm{cl}(\mathrm{cone}(A + \mathcal{D}(\bar{y}) - \{\bar{y}\}))$ w.r.t. $\mathcal{D}$ and hence a Benson properly minimal element of A w.r.t. $\mathcal{D}$. By Remark 5.3(i), $\bar{y}$ is also a Borwein properly minimal element of A w.r.t. $\mathcal{D}$. □

The following theorem is a direct consequence of Lemma 5.5, Remark 5.3(i), Remark 5.6 and [24, Theorems 5.7 and 5.8].

Theorem 5.8. *Let $(Y, \|\cdot\|)$ be a reflexive Banach space and let $\bar{y}$ be some element of A. Let $\mathcal{D}_\varepsilon(\bar{y})$ be an ε-conic neighborhood of $\mathcal{D}(\bar{y})$ for any positive real number $\varepsilon \in (0,1)$ such that $\mathcal{D}_\varepsilon(\bar{y})$ and $\mathcal{D}(\bar{y})$ satisfy the separation property for all $\varepsilon \in (0,1)$.*

(i) If $\bar{y}$ is a properly minimal element of A w.r.t. $\mathcal{D}$ in the sense of Henig, then there exists $(\ell^{\#}_{\bar{y}}, \alpha^{\#}_{\bar{y}}) \in (\mathcal{D}(\bar{y}))^{a\#}$ such that $\xi^{\#}_{\bar{y},\bar{y}}$ attains its minimum over A at $\bar{y}$.

(ii) If the cone $\mathcal{D}(\bar{y})$ has additionally a weakly compact base, then the necessary condition (i) also holds for a properly minimal element of A w.r.t. $\mathcal{D}$ in the sense of Benson.

(iii) If the cone $\mathcal{D}(\bar{y})$ has additionally a weakly compact base and there exists $(\ell^{\#}_{\bar{y}}, \alpha^{\#}_{\bar{y}}) \in (\mathcal{D}(\bar{y}))^{a\#}$ such that $\xi^{\#}_{\bar{y},\bar{y}}$ attains its minimum over A at $\bar{y}$ then $\bar{y}$ is a properly minimal element of A w.r.t. $\mathcal{D}$ in the sense of Henig and Benson and therefore in the sense of Borwein.

The scalar characterization theorem for Borwein properly minimal elements will be given in the next subsection in Theorem 5.13.

5.2 Characterizing properly nondominated elements

We start by giving sufficient conditions for the properly nondominated elements in the sense of Henig, Benson and Borwein by using the proposed scalarization functionals. After that, we also provide necessary conditions for all proper optimality notions. Recall that the functional $\zeta^{\#}_{\bar{y}}$ is defined by

$$\zeta^{\#}_{\bar{y}}(y) = \ell^{\#}_{y}(y - \bar{y}) + \alpha^{\#}_{y}\|y - \bar{y}\|$$

with some $(\ell^{\#}_{y}, \alpha^{\#}_{y}) \in (\mathcal{D}(y))^{a\#}$ for all $y \in Y$.

Theorem 5.9. *Let $\bar{y} \in A$ be a unique minimal solution of*

$$\min_{y \in A} \zeta^{\#}_{\bar{y}}(y),$$

i.e.

$$\ell^{\#}_{y}(y - \bar{y}) + \alpha^{\#}_{y}\|y - \bar{y}\| > 0 \quad \text{for all} \quad y \in A \setminus \{\bar{y}\}.$$

Then $\bar{y}$ is a Henig properly nondominated element of A w.r.t. $\mathcal{D}$.

Proof. By Theorem 4.6(ii), $\bar{y}$ is a nondominated element of A w.r.t. $\mathcal{D}$. For each $y \in A$ we define the set

$$S(\ell^{\#}_{y}, \alpha^{\#}_{y}) := \{w \in Y \mid \ell^{\#}_{y}(w) + \alpha^{\#}_{y}\|w\| \leq 0\}$$

which is obviously a closed, convex and pointed cone. Next, let $y \in A \setminus \{\bar{y}\}$ be arbitrarily chosen. Then $y - \bar{y} \notin S(\ell^{\#}_{y}, \alpha^{\#}_{y})$ being equivalent to

$$\bar{y} \notin \{y\} - S(\ell^{\#}_{y}, \alpha^{\#}_{y}).$$

We define the cone $\mathcal{K}(y) := -S(\ell_y^{\#}, \alpha_y^{\#})$. Then

$$\bar{y} \notin \{y\} + \mathcal{K}(y).$$

By Lemma 3.4,

$$\mathcal{D}(y) \setminus \{0_Y\} \subset -\mathrm{int}(S(\ell_y^{\#}, \alpha_y^{\#})) = \mathrm{int}(\mathcal{K}(y)).$$

As $y \in A \setminus \{\bar{y}\}$ was chosen arbitrarily, this completes the proof. □

In the following, let $\bar{y} \in A$ and let

$$M := \bigcup_{y \in A} (\{y\} + \mathcal{D}(y)) \text{ and } \bar{D} := \bigcup_{y \in \widetilde{M}_{\bar{y}}} \mathcal{D}(y)$$

with $\widetilde{M}_{\bar{y}} := \{\bar{y}\} + \mathrm{cl}(\mathrm{cone}(M - \{\bar{y}\}))$. By Proposition 3.6,

$$\bar{D}^{a\#} = \bigcap_{y \in \widetilde{M}_{\bar{y}}} (\mathcal{D}(y)^{a\#}).$$

Theorem 5.10. *Let for some* $\bar{y} \in A$ *and some* $(\ell^{\#}, \alpha^{\#}) \in \bar{D}^{a\#}$

$$\ell^{\#}(y - \bar{y}) + \alpha^{\#}\|y - \bar{y}\| \geq 0 \quad \textit{for all } y \in A.$$

Then $\bar{y}$ *is a Benson properly nondominated element of* A *w.r.t.* $\mathcal{D}$ *and also a Borwein properly nondominated element of* A *w.r.t.* $\mathcal{D}$.

Proof. Let $(\ell^{\#}, \alpha^{\#}) \in \bar{D}^{a\#}$. By the definition of $\bar{D}$ we have $(\ell^{\#}, \alpha^{\#}) \in (\mathcal{D}(y))^{a\#}$ for all $y \in \widetilde{M}_{\bar{y}} = \{\bar{y}\} + \mathrm{cl}(\mathrm{cone}(M - \{\bar{y}\}))$, and since $A \subset \widetilde{M}_{\bar{y}}$, we also have $(\ell^{\#}, \alpha^{\#}) \in (\mathcal{D}(y))^{a\#}$ for all $y \in A$. Then, noting that the same functional $(\ell^{\#}, \alpha^{\#})$ is used for all $y \in A$ we can say that the functional $\zeta_{\bar{y}}^{\#}$ coincides with the functional of this theorem, i.e.

$$\zeta_{\bar{y}}^{\#}(y) = \ell^{\#}(y - \bar{y}) + \alpha^{\#}\|y - \bar{y}\| \text{ for all } y \in A.$$

Then, it follows from Theorem 4.6(ii) that $\bar{y}$ is a nondominated element of A w.r.t. $\mathcal{D}$.

Next, let

$$y \in \{\bar{y}\} + \mathrm{cl}(\mathrm{cone}(M - \{\bar{y}\}))$$

be arbitrarily chosen. Then there exists a sequence of nonnegative real numbers $(\lambda_n)_{n \in \mathbb{N}}$ and sequences $(y_n, d_n)_{n \in \mathbb{N}}$ with $y_n \in A$ and $d_n \in \mathcal{D}(y_n)$ for all $n \in \mathbb{N}$ such that $y = \bar{y} + h$ and $h := \lim_{n \to \infty} \lambda_n(y_n + d_n - \bar{y})$. By the continuity and linearity of $\ell^{\#}$, as $d_n \in \mathcal{D}(y_n) \subset \bar{D}$ for all $n \in \mathbb{N}$ and $(\ell^{\#}, \alpha^{\#}) \in \bar{D}^{a\#}$ and by the assumption as $y_n \in A$ for all $n \in \mathbb{N}$ we have

$$\begin{aligned} \zeta_{\bar{y}}^{\#}(y) &= \ell^{\#}(h) + \alpha^{\#}\|h\| \\ &\geq \lim_{n \to \infty} \lambda_n \left(\ell^{\#}(y_n - \bar{y}) + \alpha^{\#}\|y_n - \bar{y}\| + \ell^{\#}(d_n) - \alpha^{\#}\|d_n\|\right) \\ &\geq \lim_{n \to \infty} \lambda_n \left(\ell^{\#}(y_n - \bar{y}) + \alpha^{\#}\|y_n - \bar{y}\|\right) \\ &\geq 0. \end{aligned}$$

Again, by Theorem 4.6(ii), $\bar{y}$ is a nondominated element of $\{\bar{y}\} + \mathrm{cl}(\mathrm{cone}(M - \{\bar{y}\}))$ and thus a Benson properly nondominated element of A w.r.t. $\mathcal{D}$. By Remark 5.3(i), $\bar{y}$ is also a Borwein properly nondominated element of A w.r.t. $\mathcal{D}$. □

Theorem 5.11. *Let $(Y, \|\cdot\|)$ be a reflexive Banach space. Let $\mathcal{D}_\varepsilon(y)$ be an ε-conic neighborhood of $\mathcal{D}(y)$ for any positive real number $\varepsilon \in (0,1)$ such that $\mathcal{D}_\varepsilon(y)$ and $\mathcal{D}(y)$ satisfy the separation property for all $\varepsilon \in (0,1)$ and for all $y \in A$. Let $\bar{y} \in A$ be a Henig properly nondominated element of A w.r.t. $\mathcal{D}$. Then for all $y \in A$ there exists a pair $(l_y^{\#}, \alpha_y^{\#}) \in D(y)^{a\#}$ such that*

$$l_y^{\#}(y - \bar{y}) + \alpha_y^{\#}\|y - \bar{y}\| \geq 0. \tag{10}$$

Proof. Since $\bar{y}$ is a properly nondominated element of A w.r.t. the ordering map $\mathcal{D}$ in the sense of Henig there is a cone-valued map $\mathcal{C}\colon Y \to 2^Y$ with $\mathcal{D}(y) \setminus \{0_Y\} \subset \operatorname{int}(\mathcal{C}(y))$ for all $y \in Y$ such that $\bar{y}$ is a nondominated element of A w.r.t. $\mathcal{C}$, i.e.

$$\bar{y} \notin \{y\} + \mathcal{C}(y) \;\forall\; y \in A \setminus \{\bar{y}\}.$$

For every $y \in A$, we can choose a sufficiently small positive ε, such that $\mathcal{D}(y)\setminus\{0_Y\} \subset \operatorname{int}(\mathcal{D}_\varepsilon(y))$ and $\mathcal{D}_\varepsilon(y) \setminus \{0_Y\} \subset \operatorname{int}(\mathcal{C}(y))$. Then we have

$$\bar{y} \notin \{y\} + \mathcal{D}_\varepsilon(y),$$

hence

$$y - \bar{y} \notin -\mathcal{D}_\varepsilon(y). \tag{11}$$

By the hypothesis, $\mathcal{D}(y)$ and $\mathcal{D}_\varepsilon(y)$ satisfy the separation property. Then, by (Separation) Theorem 3.3, there exists a pair $(l_y^{\#}, \alpha_y^{\#}) \in D(y)^{a\#}$ such that

$$l_y^{\#}(d) + \alpha_y^{\#}\|d\| < 0 \leq l_y^{\#}(z) + \alpha_y^{\#}\|z\| \tag{12}$$

for all $d \in -\mathcal{D}(y) \setminus \{0_Y\}$, and $z \in \partial(-\mathcal{D}_\varepsilon(y))$.

Let $K(y) := cl(Y \setminus (-\mathcal{D}_\varepsilon(y)))$. Then, it is easy to show that the right hand side relation in (12) holds true for all $z \in K(y)$. Indeed, assume to the contrary that $l_y^{\#}(k) + \alpha_y^{\#}\|k\| < 0$ for some $k \in K(y) \setminus \partial(K(y))$, and let $d \in -\mathcal{D}(y) \setminus \{0_Y\}$. Since $-\mathcal{D}(y) \subset -\mathcal{D}_\varepsilon(y)$ and $\partial(K(y)) = \partial(-\mathcal{D}_\varepsilon(y))$ by the construction, there exists some $\lambda \in (0,1)$ such that $z = \lambda d + (1-\lambda)k \in \partial(K(y))$. Then, by the convexity of the function $l_y^{\#}(\cdot) + \alpha_y^{\#}\|\cdot\|$ we have

$$\begin{aligned} l_y^{\#}(z) + \alpha_y^{\#}\|z\| &= l_y^{\#}(\lambda d + (1-\lambda)k) + \alpha_y^{\#}\|\lambda d + (1-\lambda)k\| \\ &\leq \lambda(l_y^{\#}(d) + \alpha_y^{\#}\|d\|) + (1-\lambda)(l_y^{\#}(k) + \alpha_y^{\#}\|k\|) \\ &< 0, \end{aligned}$$

which contradicts (12).

Now, by (11), we have $y - \bar{y} \in K(y)$. Therefore it is clear that

$$l_y^{\#}(d) + \alpha_y^{\#}\|d\| < 0 \leq l_y^{\#}(z) + \alpha_y^{\#}\|z\| \tag{13}$$

for all $d \in -\mathcal{D}(y) \setminus \{0_Y\}$ and for $z = y - \bar{y}$, or

$$l_y^{\#}(y - \bar{y}) + \alpha_y^{\#}\|y - \bar{y}\| \geq 0. \tag{14}$$

As $y \in A$ was chosen arbitrarily this proves the theorem. □

Theorem 5.12. *Let $(Y, \|\cdot\|)$ be a reflexive Banach space. Let $\mathcal{D}_\varepsilon(y)$ be an ε-conic neighborhood of $\mathcal{D}(y)$ for any positive real number $\varepsilon \in (0,1)$ such that $\mathcal{D}_\varepsilon(y)$ and $\mathcal{D}(y)$ satisfy the separation property for all $\varepsilon \in (0,1)$ and for all $y \in Y$. Let $\bar{y} \in A$ be a Borwein properly nondominated element of A w.r.t. $\mathcal{D}$ and let $M := \bigcup_{a\in A}(\{a\} + \mathcal{D}(a))$ be starshaped w.r.t. $\bar{y}$. Then for all $y \in A$ there exists a pair $(l_y^{\#}, \alpha_y^{\#}) \in D(y)^{a\#}$ such that (10) is satisfied for all $y \in M$.*

Proof. Since $\bar{y}$ is a Borwein proper nondominated element, we have that $\bar{y}$ is a nondominated element of

$$\{\bar{y}\} + T(\bigcup_{a\in A}(\{a\} + \mathcal{D}(a)), \bar{y}) = \{\bar{y}\} + T(M, \bar{y}).$$

Then

$$\bar{y} \notin \{\bar{y} + h\} + \mathcal{D}(\bar{y} + h) \text{ for all } h \in T(M, \bar{y}) \setminus \{0_Y\}$$

or

$$h \notin -\mathcal{D}(\bar{y} + h) \text{ for all } h \in T(M, \bar{y}) \setminus \{0_Y\}.$$

Now let $h \in T(M, \bar{y}) \setminus \{0_Y\}$ be arbitrarily chosen. Since $-\mathcal{D}(\bar{y}+h)$ is a closed convex cone by assumption, there exists $\varepsilon > 0$ such that

$$h \notin -\mathcal{D}_\varepsilon(\bar{y} + h).$$

By the hypothesis, $\mathcal{D}(\bar{y} + h)$ and $\mathcal{D}_\varepsilon(\bar{y} + h)$ satisfy the separation property.

Then by the (Separation) Theorem 3.3 there exists $(\ell_{\bar{y}+h}^{\#}, \alpha_{\bar{y}+h}^{\#}) \in \mathcal{D}(\bar{y} + h)^{a\#}$ with

$$\ell_{\bar{y}+h}^{\#}(d)+\alpha_{\bar{y}+h}^{\#}\|d\| < 0 \leq \ell_{\bar{y}+h}^{\#}(u)+\alpha_{\bar{y}+h}^{\#}\|u\| \text{ for all } d \in -\mathcal{D}(\bar{y}+h),\ u \in \partial(-\mathcal{D}_\varepsilon(\bar{y}+h)).$$

Let $K(\bar{y} + h) := \mathrm{cl}(Y \setminus (-\mathcal{D}_\varepsilon(\bar{y} + h)))$. Then it is clear that $h \in K(\bar{y} + h)$ and that the separation property is also true for $u = h$ (see the proof of the previous theorem). Thus,

$$0 \leq \ell_{\bar{y}+h}^{\#}(h) + \alpha_{\bar{y}+h}^{\#}\|h\|.$$

As $h \in T(M, \bar{y}) \setminus \{0_Y\}$ was chosen arbitrarily, this implies

$$0 \leq \ell_{\bar{y}+h}^{\#}(h) + \alpha_{\bar{y}+h}^{\#}\|h\| \text{ for all } h \in T(M, \bar{y}).$$

As M is starshaped w.r.t. $\bar{y}$, by [23, Theorem 3.43], $M - \{\bar{y}\} \subset T(M, \bar{y})$. This implies for $h = y - \bar{y}$ with $y \in M$

$$0 \leq \ell_y^{\#}(y - \bar{y}) + \alpha_y^{\#}\|y - \bar{y}\| \text{ for all } y \in M.$$

□

Theorem 5.13. *Let $(Y, \|\cdot\|)$ be a reflexive Banach space and let $\bar{y} \in A$ be a Borwein properly minimal element of A w.r.t. the ordering map $\mathcal{D}$. Let $\mathcal{D}_\varepsilon(\bar{y})$ be an ε-conic neighborhood of $\mathcal{D}(\bar{y})$ for any positive real number $\varepsilon \in (0,1)$ such that $\mathcal{D}_\varepsilon(\bar{y})$ and $\mathcal{D}(\bar{y})$ satisfy the separation property for all $\varepsilon \in (0,1)$. Then there exists $(\ell_{\bar{y}}^{\#}, \alpha_{\bar{y}}^{\#}) \in (\mathcal{D}(\bar{y}))^{a\#}$ such that $\xi_{\bar{y},\bar{y}}^{\#}$ attains its minimum over $\{\bar{y}\} + T(A + \mathcal{D}(\bar{y}), \bar{y})$ at $\bar{y}$.*

Proof. The proof is similar to the proof of Theorem 5.12. □

Theorem 5.14. *Let $(Y, \|\cdot\|)$ be a reflexive Banach space. Let $\mathcal{D}_\varepsilon(y)$ be an ε-conic neighborhood of $\mathcal{D}(y)$ for any positive real number $\varepsilon \in (0,1)$ such that $\mathcal{D}_\varepsilon(y)$ and $\mathcal{D}(y)$ satisfy the separation property for all $\varepsilon \in (0,1)$ and for all $y \in A$. Let $\bar{y} \in A$ be a Benson properly nondominated element of A. Then there exists a pair $(l_y^{\#}, \alpha_y^{\#}) \in D(y)^{a\#}$ such that (10) is satisfied for all $y \in M$, where $M := \bigcup_{a \in A} (\{a\} + \mathcal{D}(a))$.*

Proof. The proof is similar to the proof of Theorem 5.12 but no starshapness of the set $M := \bigcup_{a \in A} (\{a\} + \mathcal{D}(a))$ is needed. □

6 Conclusions

The paper introduces notions for properly minimal and properly nondominated elements in vector optimization with variable ordering structures for the first time. These notions are generalizations of known definitions of proper optimality given by Henig, Borwein and Benson in partially ordered vector spaces. The paper also presents characterization results for properly optimal elements via scalarization. For this purpose new nonlinear scalarization functionals are introduced and their properties are studied.

Other scalar characterizations of the proper optimality notions as well as optimality conditions based on the derivative or the subdifferential of the new scalar functionals are subject to future research.

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