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A New Competitive Ratio of the Harmonic Algorithm for a k-Server Problem with Parallel Requests and Unit Distances

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1 Introduction

In this paper a generalized k-server problem with parallel requests where several servers can also be located on one point is discussed. The investigation of the generalized k-server problem was initiated by an operations research problem which consists of optimal conversions of machines or moulds (see [4] or [8]). It is sensible in the case of parallel requests to distinguish the surplus-situation where the request can be completely fulfilled by means of the k servers and and the scarcity-situation where the request cannot be completely met.

The k-server problem was introduced by Manasse, McGeoch and Sleator [11]. Meanwhile it is the most studied problem in the area of competitve online problems. Historical notes on k-server problems can be found in the book by A. Borodin and R. El-Yaniv [3] (sections 10.9 and 11.7) or also in the paper by Y. Bartal and E. Grove [2]. There the two important results are the competitiveness of the deterministic work-function algorithm (see E. Koutsoupias and C. Papadimitriou [9]) and of the randomized Harmonic k-server algorithm against an adaptive online adversary (see Y. Bartal and E. Grove [2]).

The work-function algorithm is an inefficient algorithm (with a good competitive ratio). In contrast the Harmonic k-server algorithm is memoryless and time-efficient. For this reason we first want to focus on a corresponding Harmonic k-server algorithm for the generalized k-server problem.

If one tries to generalize the proof by Y. Bartal and E. Grove [2] several subchains with different length must be considered and one will see that the computation of the weights $f(j)$ is not possible. In this paper we consider the general k-server problem in the case of unit distances. Using rough estimations of numbers of certain partitions we have shown in [7] that a corresponding Harmonic algorithm is competitive. The (usual) k-server problem with unit distances is known as the paging problem and the Harmonic k-server algorithm as RAND algorithm (see [3], chapters 3 and 4). Raghavan and Snir have shown that the RAND algorithm is k competitive against an adaptive online adversary. Although there can occur a lot more feasible requests in the case of the generalized k-server problem we will show in this paper (using detailed considerations related to sets of certain partitions) that the corresponding Harmonic k-server algorithm is $max\,\{k, R(k) - k + 1\}$ competitive (where $R(k)$ is a bound of the requests related to the scarcity-situation, see Theorem 1) and k competitive (just as RAND), if only the surplus-situation is allowed.

2 The formulation of the model

[1] Let $k \geq 1$ be an integer, and $M = (M, d)$ be a finite metric space where M is a set of points with $|M| = N$. An algorithm controls k mobile servers, which are located on points of M. Several servers can be located on one point. The algorithm is presented with a sequence $\sigma = r^1, r^2, \cdots, r^n$ of requests where a request r is defined as an N-ary vector of integers with $r_i \in \{0, 1, \cdots, k\}$ ("parallel requests"). The request means that r_i server are needed on point i $(i = 1, 2, \cdots, N)$. We say a request r is served if $\left\{\begin{array}{c}\text{at least}\\ \text{at most}\end{array}\right\}$ r_i servers lie on i $(i = 1, 2, \cdots, N)$ in case $\left\{\begin{array}{c}C[r,k]\\ C[k,r]\end{array}\right\}$. $C[r, k]$ denotes the case $\sum\limits_{i=1}^{N} r_i \leq k$ (surplus-situation, the request can be completely fulfilled) and $C[k, r]$ denotes the case $\sum\limits_{i=1}^{N} r_i \geq k$ (scarcity-situation, the request cannot be completely met, however it should be met as much as possible). By moving servers, the algorithm must serve the requests $r^1, r^2, \cdots, r^n$ sequentially. For any request sequence σ and any generalized k-server algorithm $ALG_{p(arallel)}$, $ALG_p(\sigma)$ is defined as the total distance (measured by the metric d) moved by the ALG_p's servers in servicing σ.

In this paper we will show that the corresponding Harmonic k-server algorithm attains a competitive ratio of $max\,\{k, R(k) - k + 1\}$ (see Theorem 1) against an adaptive online adversary in the case of unit distances (for the definitions of competitive ratio and adaptive online adversary see [2] or [3], sections 4.1 and 7.1). Analogous to [3], p. 152 working with lazy algorithms ALG_p is sufficient. For that reason we define the set of feasible servers positions with respect to s and r in the following way

$$\hat{A}_{N;k}(s, r) = \left\{ s' \in P_N(k) \;\middle|\; \begin{array}{l} r_i \leq s'_i \leq max\{s_i, r_i\},\ i = 1, \cdots, N,\ \text{in } C[r, k] \\ min\{s_i, r_i\} \leq s'_i \leq r_i,\ i = 1, \cdots, N,\ \text{in } C[k, r] \end{array} \right\} \tag{1}$$

[1]For basic knowledge of (usual) k-server problems see also [3], chapters 10 and 11 for example.

$$\text{where} \quad P_N(k) := \left\{ s \in \mathbb{Z}_+^n \mid \sum_{i=1}^{N} s_i = k \right\}. \tag{2}$$

The metric d implies that $(P_N(k), \hat{d})$ is also a finite metric space where $\hat{d}$ are the optimal values of the classical transportation problems with availabilities s and requirements s' of $P_N(k)$: $\sum_{i=1}^{N} \sum_{j=N}^{N} d(i,j)\, x_{ij} \;\rightarrow\; min$ subject to $\sum_{j=1}^{N} x_{ij} = s_i \;\forall i,\; \sum_{i=1}^{N} x_{ij} = s'_j \;\forall j,\; x \in \mathbb{Z}_+^n \times \mathbb{Z}_+^n$ (see [6], Lemma 3.6).

The corresponding $HARMONIC_p$ k-server algorithm operates as follows: Serve a (not completely covered) request r with randomly chosen servers so that for the (new) server positions $s' \in \hat{A}_{N;k}(s,r)$ is valid with respect to the previous server positions s and the request r. More precisely, $HARMONIC_p$ leads to $s' \in \hat{A}_{N;k}(s,r)$ with probability

$$\frac{\frac{1}{\hat{d}(s,s')}}{\sum\limits_{s'': s'' \in \hat{A}_{N;k}(s,r)} \frac{1}{\hat{d}(s,s'')}}. \tag{3}$$

3 The competitiveness of $HARMONIC_p$ in case of unit distances

Unit distances means that $d(i,j) = 1 \;\forall i \neq j$. Thus, $\hat{d}(s,s') = \sum_{i=1}^{N} \frac{1}{2}|s_i - s'_i|$ for $s, s' \in P_N(k)$ follows and (1) yields $\hat{d}(s,s') = \begin{cases} \sum\limits_{i: r_i^t > s_i} (r_i^t - s_i) & \text{in } C[r,k] \\ \sum\limits_{i: r_i^t < s_i} (s_i - r_i^t) & \text{in } C[k,r] \end{cases}$ for every $s' \in \hat{A}_{N;k}(s,r)$. Then $s' \in \hat{A}_{N;k}(s,r)$ is chosen randomly and uniformly with probability $\frac{1}{|\hat{A}_{N;k}(s,r)|}$ among all elements of $\hat{A}_{N;k}(s,r)$ by $HARMONIC_p$.

In [7] can be found an example which shows that in order to prove the competitiveness an additional assumption (as $\sum_{i \in M} r_i^t \;\leq\; R(k)$ in the following theorem) in the case $C[k, r^t]$ is necessary.

Theorem 1. *The $HARMONIC_p$ k-server algorithm attains a competitive ratio of $C(k) = max\,\{k, R(k) - k + 1\}$ against an adaptive online adversary in case of unit distances if $\sum_{i \in M} r_i^t \;\leq\; R(k)\; \forall t$ for given $R(k) > k$.* [2]

[2] This condition is important for case $C[k, r^t]$. (According to the introduced model $\sum_{i \in M} r_i^t \leq k$ is true in case $C[r^t, k]$.) See also the above mentioned example.

Proof. We will use a potential function (see [2]) to prove the statement. In case of unit distances it is sufficient to use the following simple potential function

$$\Phi(s,s') := \hat{f} \sum_{i=1}^{N} \tfrac{1}{2}|s_i - s'_i| \; (= \hat{f}\, \hat{d}(s,s')),\;\; s,s' \in P_N(k). \tag{4}$$

At the beginning let $\hat{f} \geq 0$. We will solve for $\hat{f}$ later. More precisely and analogous to Bartal and Grove, let Φ_t denote the value of Φ at the end of the t-th step (corresponding to the t-th request r^t in the request sequence) and let $\Phi_t^{\sim}$ denote the value of Φ after the first stage of the t-th step (i.e., after the adversary's move and before the algorithm's move). In cases $C[r^t,k]$ and $C[k,r^t]$ we will show the following properties (see [2], pages 4 and 5)

$$\Phi \geq 0 \tag{5}$$

$$\Phi_t^{\sim} - \Phi_{t-1} \leq C(k) D_t, \tag{6}$$

where D_t denotes the distance moved by the offline servers (controlled by the adversary) to serve the request in the t-th step.

$$E(\Phi_t^{\sim} - \Phi_t) \geq E(Z_t), \tag{7}$$

where Z_t represents the cost which incurred by the online algorithm to serve the request in the t-th step.

If we can show that the potential function satisfies these properties then $HARMONIC_p$ is C(k) competitive.

In the following let

$\bar{s}$ ($\in P_N(k)$) denote the (offline) servers position controlled by the adversary at the end of the (t-1)-th step (i.e., at the beginning of the t-th step)

s ($\in P_N(k)$) denote the (online) servers position controlled by the algorithm at the beginning of the t-th step

s' ($\in \hat{A}_{N;k}(s,r^t)$) denote the (online) servers position at the end of the t-th step and

$\bar{s}'$ ($\in P_N(k)$) denote the (offline) servers position controlled by the adversary after the first stage of the t-th step.

Proof of (5) and (6):

(5) is straightforward if $\hat{f} \geq 0$. (6) follows by means of the triangle-equation of the metric $\hat{d}$:

$$\hat{f}\hat{d}(s,\bar{s}') - \hat{f}\hat{d}(s,\bar{s}) \leq \hat{f}\hat{d}(\bar{s},\bar{s}') = \hat{f} D_t \leq C(k) D_t \;\text{ if }\; C(k) \geq \hat{f}.$$

Proof of (7) in case $C[r^t, k]$:

In this case we will show that $\hat{f} = k$ (and hence each $\hat{f} \geq k$) satisfies the property (7).

For unit distances it follows that

$$\Phi_t^{\sim}(s, \bar{s}') = \hat{f} \sum_{i:\bar{s}'_i > s_i} (\bar{s}'_i - s_i) = \hat{f} \sum_{i:\bar{s}'_i < s_i} (s_i - \bar{s}'_i) \ \ \forall s' \in \hat{A}_{N;k}(s, r^t) \tag{8}$$

as well as

$$\begin{aligned} Z_t(s, s') = & \sum_{i:r_i^t > s_i} (r_i^t - s_i) \ \ \forall s' \in \hat{A}_{N;k}(s, r^t) \\ & =: Z_t(s, r^t) \end{aligned} \tag{9}$$

and (7) is equivalent to

$$\Phi_t^{\sim} - E(\Phi_t) \geq Z_t. \tag{7a}$$

We will reduce the problem in several steps and consider, firstly, certain cases for which the proof is simple. Finally, a remaining reduced problem will be investigated using properties of certain partitions of integers.

Next the set $M = \{i = 1, \cdots, N\}$ of points is partitioned in relation to s, $\bar{s}'_i$, r^t, s'_i in case $C[r^t, k]$ where $r_i^t \leq s'_i \leq max\{r_i^t, s_i\}$ for $i = 1, \cdots, N$:

$$\begin{aligned} M_I &= \{i \in M \mid s_i > s'_i \geq \bar{s}'_i = r_i^t \text{ or } s_i \geq s'_i > \bar{s}'_i = r_i^t\}, \\ M_{IIa} &= \{i \in M \mid s_i > s'_i \geq \bar{s}'_i > r_i^t \text{ or } s_i \geq s'_i > \bar{s}'_i > r_i^t\}, \\ M_{IIb} &= \{i \in M \mid s_i > \bar{s}'_i > s'_i \geq r_i^t\}, \\ M_{III} &= \{i \in M \mid r_i^t \leq s'_i < s_i \leq \bar{s}'_i \text{ or } r_i^t < s'_i = s_i \leq \bar{s}'_i\}, \\ M_{IV} &= \{i \in M \mid s_i \leq r_i^t = s'_i \leq \bar{s}'_i\} = \{i \in M \mid s_i \leq r_i^t\}. \end{aligned}$$

A first reduced model (reduction in 3 steps):

The quantities of the property (7a) will not change by the following manipulations. Particularly, the essential structure of $\hat{A}_{N;k}$ (see (1)) and $|\hat{A}_{N;k}|$ will also not change. k must be reduced in corresponding way.

1. $\Delta_i := \min\{s_i, r_i^t, \bar{s}'_i\}$ for $i \in M$, $k := k - \sum_{i \in M} \Delta_i$,
   $s_i := s_i - \Delta_i, \bar{s}'_i := \bar{s}'_i - \Delta_i, s'_i := s'_i - \Delta_i, r_i^t := r_i^t - \Delta_i$ for $i \in M$
2. $\bar{s} := \sum_{i \in M_{III}} (\bar{s}'_i - s_i) + \sum_{i \in M_{IV}} (\bar{s}'_i - r_i^t)$,
   $\bar{s}'_i := s_i$ for $i \in M_{III}$, $\bar{s}'_i := r_i^t (= s'_i)$ for $i \in M_{IV}$. Temporarily, we set $s_{i'} = s'_{i'} = r_{i'}^t := 0, \bar{s}'_{i'} := \bar{s}$ for an additional $i' \in \{i'\} =: M_V$.
3. We replace the elements of M_{IV} by one element i where
   $r^* := \sum_{i \in M_{IV}} r_i^t =: s'_i =: \bar{s}'_i$.

Then with regard to $s, s', \bar{s}', r^t$ the following possibilities remain for the reduced model:

$$s_i > s'_i \geq \bar{s}'_i = 0 = r_i^t \text{ or } s_i \geq s'_i > \bar{s}'_i = 0 = r_i^t \ \text{ for } i \in M_I,$$

$s_i > s'_i \geq \bar{s}'_i > 0 = r^t_i$ or $s_i \geq s'_i > \bar{s}'_i > 0 = r^t_i$ for $i \in M_{IIa}$,
$s_i > \bar{s}'_i > s'_i \geq r^t_i = 0$ for $i \in M_{IIb}$,
$r^t_i = 0 \leq s'_i < s_i = \bar{s}'_i$ or $r^t_i = 0 < s'_i = s_i = \bar{s}'_i$ for $i \in M_{III}$,
$s_i = 0 < r^* = s'_i = \bar{s}'_i$ for $i \in M_{IV} = \{i\}$,
$s_{i'} = s'_{i'} = r^t_{i'} = 0,\ \bar{s}'_{i'} = \bar{s}$ for $i' \in M_V = \{i'\}$
where

$$r^* = \sum_{i \in M_I \cup M_{IIa} \cup M_{IIb} \cup M_{III}} (s_i - s'_i). \tag{10}$$

Furthermore,

$Z_t(s, s') = r^* \ \forall s' \in \hat{A}_{N;k}(s, r)$ (see (9)),

$$\Phi^{\sim}_t(s, \bar{s}') = \hat{f}\ (r^* + \bar{s}) \text{ (see (8))} \tag{11}$$

and

$$\begin{aligned}(0 \leq)\Phi_t(s', \bar{s}') &= \hat{f}\ \left(\sum_{i \in M_{IIb} \cup M_{III}} (\bar{s}'_i - s'_i) + \bar{s} \right) \\ &= \hat{f}\ \left(\sum_{i \in M_{IIb}} (s_i - s'_i) - \sum_{i \in M_{IIb}} (s_i - \bar{s}'_i) + \sum_{i \in M_{III}} (s_i - s'_i) + \bar{s} \right) \\ &= \hat{f}\ \left(r^* - \sum_{i \in M_I \cup M_{IIa}} (s_i - s'_i) - \sum_{i \in M_{IIb}} (s_i - \bar{s}'_i) + \bar{s} \right)\ (\leq \hat{f}\ (r^* + \bar{s}))\end{aligned} \tag{12}$$

follow (the last equation by means of (10)).

$\hat{f}\bar{s}$ vanishes in the difference $\Phi^{\sim}_t(s, \bar{s}') - \Phi_t(s', \bar{s}')$. That's why we consider an unbalanced reduced model without M_V and with the difference of $\bar{s}$ between $\sum_{i \in M} s_i$ and $\sum_{i \in M} \bar{s}'_i$. $Z_t(s, s') = r^* \ \forall s' \in \hat{A}_{N;k}(s, r)$ remains valid. We set $\Phi^{\sim}_t(s, \bar{s}') = \hat{f}\ r^*$ and $\Phi_t(s', \bar{s}') = \hat{f}\ \hat{d}'(s', \bar{s}')$ where

$$\hat{d}'(s', \bar{s}') = \sum_{i \in M_{IIb} \cup M_{III}} (\bar{s}'_i - s'_i) \text{ for } s' \in \hat{A}_{N;k}(s, r). \tag{13}$$

In this way $\Phi^{\sim}_t(s, \bar{s}') - \Phi_t(s', \bar{s}')$ does not change.

Clearly that

$$k = \sum_{i \in M_I \cup M_{IIa} \cup M_{IIb} \cup M_{III}} s_i \geq r^* + \sum_{i \in M_{IIa} \cup M_{IIb} \cup M_{III}} \bar{s}'_i \tag{14}$$

is right for the unbalanced model.

We can then conclude from (12) that

$$\Phi_t(s', \bar{s}') = r^* \ \Leftrightarrow\ s_i - s'_i = 0 \text{ for } i \in M_I \cup M_{IIa},\ M_{IIb} = \varnothing, \tag{15}$$

thus

$$\Phi_t(s', \bar{s}') = r^* \ \Leftrightarrow\ r^* = \sum_{i \in M_{III}} (s_i - s'_i) \text{ (see (10))} \ \Rightarrow \sum_{i \in M_{III}} s_i \geq r^*. \tag{16}$$

With regard to the reduced model (7a) is equivalent to

$\hat{f}\; r^* - \hat{f} \underset{s' \in \hat{A}_{N;k}(s,r)}{E} (\hat{d}'(s', \bar{s}')) \geq r^*$ and hence also to $\frac{r^*}{r^* - \underset{s' \in \hat{A}_{N;k}(s,r)}{E} (\hat{d}'(s', \bar{s}'))} \leq \hat{f}$.

Finally the conjecture $\hat{f} = k = C(k)$ [3] in case $C[r^t, k]$ leads to the following representation of (7a):

$$\underset{s' \in \hat{A}_{N;k}(s,r)}{E} (\hat{d}'(s', \bar{s}')) \leq r^* \tfrac{k-1}{k}. \tag{17}$$

The inequality

$$\underset{s' \in \hat{A}_{N;k}(s,r)}{E} (\hat{d}'(s', \bar{s}')) \leq r^* - 1 \tag{18}$$

is sufficient for the validity of (17) since $k \geq r^*$, see (14).

Case: $\exists\, s' \in \hat{A}_{N;k}(s,r) :\; M_{IIb} \neq \varnothing$

Let $\hat{A}^I_{N;k}(s,r) = \left\{ s' \in \hat{A}_{N;k}(s,r) \mid M_{IIb} = \varnothing \right\}$ and $\hat{A}^{II}_{N;k}(s,r) = \hat{A}_{N;k}(s,r) \setminus \hat{A}^I_{N;k}(s,r)$.

Then (15) implies that $\underset{s' \in \hat{A}^{II}_{N;k}(s,r)}{E} (\hat{d}'(s', \bar{s}')) \leq r^* - 1$.

Using the relation

$$\underset{s' \in \hat{A}_{N;k}(s,r)}{E} (\hat{d}'(s', \bar{s}')) = \underset{s' \in \hat{A}^I_{N;k}(s,r)}{E} (\hat{d}'(s', \bar{s}')) \frac{|\hat{A}^I_{N;k}(s,r)|}{|\hat{A}_{N;k}(s,r)|} + \underset{s' \in \hat{A}^{II}_{N;k}(s,r)}{E} (\hat{d}'(s', \bar{s}')) \frac{|\hat{A}^{II}_{N;k}(s,r)|}{|\hat{A}_{N;k}(s,r)|}$$

we get

a) $\underset{s' \in \hat{A}_{N;k}(s,r)}{E} (\hat{d}'(s', \bar{s}')) \leq r^* - 1$ if $\underset{s' \in \hat{A}^I_{N;k}(s,r)}{E} (\hat{d}'(s', \bar{s}')) \leq r^* - 1$ and the conjecture (17) is true according to (18),

b) $\underset{s' \in \hat{A}_{N;k}(s,r)}{E} (\hat{d}'(s', \bar{s}')) < \underset{s' \in \hat{A}^I_{N;k}(s,r)}{E} (\hat{d}'(s', \bar{s}'))$ if $\underset{s' \in \hat{A}^I_{N;k}(s,r)}{E} (\hat{d}'(s', \bar{s}')) > r^* - 1$ and it is sufficient to consider a reduced model with $M_{IIb} = \varnothing \;\forall\, s' \in \hat{A}_{N;k}(s,r)$.

Case: $M_{IIb} = \varnothing \;\; \forall\, s' \in \hat{A}_{N;k}(s,r)$

In this case we can use a further reduction step:

$s_i := s_i - \bar{s}'_i,\; {s'}_i := {s'}_i - \bar{s}'_i,\; \bar{s}'_i := 0$ for $i \in M_{IIa},\; k := \sum_{i \in M_I \cup M_{IIa} \cup M_{III}} s_i$.

Then the set M_{IIa} can be integrated into the set M_I and a reduced model with the following possibilities in relation to $s, s', \bar{s}', r^t$ remains:

$s_i \;>\; s'_i \;\geq\; \bar{s}'_i = 0 \;=\; r^t_i$ or $s_i \;\geq\; s'_i \;>\; \bar{s}'_i = 0 \;=\; r^t_i$ for $i \in M_I$,

[3]It is unproblematic to use the reduced k since the original k is greater or equal than the reduced.

$r_i^t = 0 \leq s'_i < s_i = \bar{s}'_i$ or $r_i^t = 0 < s'_i = s_i = \bar{s}'_i$ for $i \in M_{III}$,
$s_i = 0 < r^* = s'_i = \bar{s}'_i$ for $i \in M_{IV}$.

The union of sets in the formulas (14) and (10) must then be replaced by $M_I \cup M_{III}$ or M_{III}. Furthermore (10) leads to

$$k \geq r^* \tag{19}$$

and

$$\sum_{i \in M_I} s_i \geq r^*. \tag{20}$$

If $\sum\limits_{i \in M_{III}} s_i \leq r^* - 1$ then $\underset{s' \in \hat{A}_{N;k}(s,r)}{E} (\hat{d}'(s', \bar{s}')) \leq r^* - 1$ follows from (16) and the conjecture (17) is true according to (18). Hence it remains to investigate the reduced problem with $M_{IIb} = \varnothing$ where (20) and

$$\sum_{i \in M_{III}} s_i \geq r^* \tag{21}$$

are assumed.

Let us set: $M_I = \{1, \cdots, m\}$, $M_{III} = \{m+1, \cdots, m+n\}$, $M_{IV} = \{m+n+1\}$ and $s^I = (s_{m+1}, \cdots, s_{m+n})$, $s^{II} = (s_1, \cdots, s_m)$.

In order to prove the conjecture for the reduced problem we use ordered restricted partitions of integers x (compositions) into exactly n non-negative parts (written as vectors):

$P_s^n(x) := \left\{ \bar{x} \in \mathbb{Z}_+^n \mid 0 \leq \bar{x}_i \leq s_i \text{ for } i = 1, \cdots, n, \ \sum\limits_{i=1}^{n} \bar{x}_i = x \right\}$, $p_s^n(x) := |P_s^n(x)|$ for $x \in \mathbb{Z}_+$ and given $n \in \mathbb{N}, s \in \mathbb{Z}_+^n$.

A one-to-one correspondence between the elements of the sets $\hat{A}_{N;k}(s,r)$ and $P^{n+m}_{(s^{II}, s^I)}(r^*)$ is defined by:

$$\{s' \leftrightarrow \bar{x}\} \Leftrightarrow \{s_i - s'_i = \bar{x}_i \text{ for } \ i = 1, \cdots, n+m\}. \tag{22}$$

$|\hat{A}_{N;k}(s,r)| = p^{n+m}_{(s^{II}, s^I)}(r^*) = \sum\limits_{x=0}^{r^*} p^n_{s^I}(x)\, p^m_{s^{II}}(r^* - x)$ (pay attention to (21) and (20)) and $\hat{d}'(s', \bar{s}') = \sum\limits_{i \in M_{III}} (\bar{s}'_i - s'_i) = \sum\limits_{i=m+1}^{m+n} (s_i - s'_i) = \sum\limits_{i=m+1}^{m+n} \bar{x}_i$ (see (13)) imply the following representation of the conjecture (17):

$$\underset{s' \in \hat{A}_{N;k}(s,r)}{E} (\hat{d}'(s', \bar{s}')) = \frac{\sum\limits_{x=0}^{r^*} x\, p^n_{s^I}(x)\, p^m_{s^{II}}(r^* - x)}{\sum\limits_{x=0}^{r^*} p^n_{s^I}(x)\, p^m_{s^{II}}(r^* - x)} \leq r^* \frac{k-1}{k}, \tag{23}$$

where (20) and (21) are assumed.

In order to prove (23) we use the following properties of $p_s^n(x)$ and the following relationship (29):

- $$p_s^n(x) = p_s^n(\sum_{i=1}^{n} s_i - x) \text{ (symmetry)} \tag{24}$$

- $p_s^n(x)$ is monotonically increasing for
$$x \in \{0, 1, \cdots, \max\{s_n, \lceil \tfrac{1}{2} \sum_{i=1}^{n} s_i \rceil\}\} \tag{25}$$

- $$p_s^n(x) \text{ is strictly increasing for } x \in \{0, 1, \cdots, \min\{\sum_{i=1}^{n-1} s_i;\ \lfloor \frac{1}{2} \sum_{i=1}^{n} s_i \rfloor\}\} \tag{26}$$

- $$(r^*{+}1) p_s^n(r^*{+}1) \leq n \sum_{x=0}^{r^*} p_s^n(x), r^* \in \mathbb{Z}_+ \tag{27}$$

- $$\tfrac{(x+1)p_s^n(x+1)+yp_s^n(y)}{p_s^n(x+1)+p_s^n(y)} \leq \tfrac{(y+1)p_s^n(y+1)+xp_s^n(x)}{p_s^n(y+1)+p_s^n(x)} \ (\leq y+2)$$
$$\text{for } \ x < y \text{ and } y+1 \ (y+2) \leq \max\{s_n, \lceil \tfrac{1}{2} \sum_{i=1}^{n} s_i \rceil\} \tag{28}$$

- Let $a_i > 0$ for $i = 0, \cdots, l,\ b_0 \geq b_1 \geq \cdots \geq b_l > 0$ and $A_0 \leq A_1 \leq \cdots \leq A_l$. Then $\frac{A_0 a_0 b_0 + \cdots + A_l a_l b_l}{a_0 b_0 + \cdots + a_l b_l} \leq \frac{A_0 a_0 + \cdots + A_l a_l}{a_0 + \cdots + a_l}$ is valid. (29)

(See the following Remarks 1 for (24), $\cdots$, (27) and (29). Simple computations yield (28).)

Case $p_{s^I}^n(r^* - 1) > p_{s^I}^n(r^*)$:

Together with (24) and (26) the relations $r^* > 1$ and $p_{s^I}^n(r^*{-}2) \geq p_{s^I}^n(r^*)$ follow. The last inequality implies $p_{s^I}^n(r^*{-}2)\ p_{s^{II}}^m(2) \geq p_{s^I}^n(r^*) = p_{s^I}^n(r^*)\ p_{s^{II}}^m(0)$ and $\frac{(r^*-2)\ p_{s^I}^n(r^*-2)\ p_{s^{II}}^m(2) + r^*\ p_{s^I}^n(r^*)\ p_{s^{II}}^m(0)}{p_{s^I}^n(r^*-2)\ p_{s^{II}}^m(2) + p_{s^I}^n(r^*)\ p_{s^{II}}^m(0)} \leq r^* - 1$.

Hence $\underset{s' \in \hat{A}_{N;k}(s,r)}{E}(\hat{d}'(s', \bar{s}')) = \frac{\sum_{x=0}^{r^*} x\ p_{s^I}^n(x)\ p_{s^{II}}^m(r^*-x)}{\sum_{x=0}^{r^*} p_{s^I}^n(x)\ p_{s^{II}}^m(r^*-x)} \leq r^* - 1$, and (23) is valid in this case (see (18)).

Case $p_{s^I}^n(r^* - 1) \leq p_{s^I}^n(r^*)$:

We show:

I
$$\frac{\sum_{x=0}^{r^*} x\ p_{s^I}^n(x)\ p_{s^{II}}^m(r^*-x)}{\sum_{x=0}^{r^*} p_{s^I}^n(x)\ p_{s^{II}}^m(r^*-x)} \leq \frac{\sum_{x=0}^{r^*} x\ p_{s^I}^n(x)}{\sum_{x=0}^{r^*} p_{s^I}^n(x)} \tag{30}$$

(this means $|M_I| = m = 1$ yields an upper bound)

II

$$\frac{\sum\limits_{x=0}^{r^*} x\, p^n_{s^I}(x)}{\sum\limits_{x=0}^{r^*} p^n_{s^I}(x)} \;\leq\; r^* \frac{k-1}{k}. \tag{31}$$

To I: Let us set $l = \min\{r^*, \left\lfloor \frac{1}{2}\sum\limits_{i=1}^{m} s_i^{II} \right\rfloor\}$, $L = \max\{0, 2r^* - \sum\limits_{i=1}^{m} s_i^{II}\}$,

$\bar{L} = \lfloor \frac{L}{2} \rfloor = \max\{0, r^* - \left\lceil \frac{1}{2}\sum\limits_{i=1}^{m} s_i^{II} \right\rceil\}$, $\bar{i} = i + \lceil \frac{L}{2} \rceil$ for $i = 0, \ldots, l$,

$b_i = p^m_{s^{II}}(r^* - \bar{i})$ for $i = 0, \ldots, l$ and

$$\left.\begin{array}{l} A_i = \frac{\bar{i}p^n_{s^I}(\bar{i}) + (L-\bar{i})p^n_{s^I}(L-\bar{i})}{p^n_{s^I}(\bar{i}) + p^n_{s^I}(L-\bar{i})} \text{ for } i = 0, \cdots, \bar{L}, \\ a_i = p^n_{s^I}(\bar{i}) + p^n_{s^I}(L-\bar{i}) \text{ for } i = 0, \cdots, \bar{L} \text{ if } L \text{ is odd or} \\ a_0 = p^n_{s^I}(\frac{L}{2}),\ a_i = p^n_{s^I}(\bar{i}) + p^n_{s^I}(L-\bar{i}) \text{ for } i = 1, \cdots, \bar{L} \text{ if } L \text{ is} \\ \hfill \text{even} \end{array}\right\} \text{ if } L > 0,$$

$$\left.\begin{array}{r} A_i = i + (r^* - l) \text{ and } a_i = p^n_{s^I}(i + (r^* - l)) \\ \text{for } i = 0, \ldots, l \text{ if } L = 0 \text{ or} \\ \text{for } i = \bar{L}+1, \ldots, l \text{ if } L > 0 \end{array}\right\} \text{ if } \bar{L} < l.$$

It is clear that $A_{\bar{L}} = \frac{Lp^n_{s^I}(L) + 0p^n_{s^I}(0)}{p^n_{s^I}(L) + p^n_{s^I}(0)}$ and

$$A_{\bar{L}+1} = \bar{L} + 1 + r^* - l = r^* - \left\lceil \frac{1}{2}\sum_{i=1}^{m} s_i^{II} \right\rceil + 1 + r^* - \left\lfloor \frac{1}{2}\sum_{i=1}^{m} s_i^{II} \right\rfloor = L + 1$$

if $L > 0$ ($\Rightarrow\ l = \left\lfloor \frac{1}{2}\sum\limits_{i=1}^{m} s_i^{II} \right\rfloor$) and $\bar{L} < l$.

The symmetry $b_i = p^m_{s^{II}}(r^* - \bar{i}) = p^m_{s^{II}}(\sum\limits_{i=1}^{m} s_i^{II} - r^* + \bar{i}) = p^m_{s^{II}}(r^* - (L - \bar{i}))$

for $i = 0, \cdots, \bar{L}$ if $L > 0$ (see (24)) leads to $\frac{\sum\limits_{x=0}^{r^*} x\, p^n_{s^I}(x)\, p^m_{s^{II}}(r^*-x)}{\sum\limits_{x=0}^{r^*} p^n_{s^I}(x)\, p^m_{s^{II}}(r^*-x)} = \frac{A_0a_0b_0 + \cdots + A_la_lb_l}{a_0b_0 + \cdots + a_lb_l}$.

Using $r^* - \bar{i} \leq r^* - \lceil \frac{L}{2} \rceil \leq \left\lfloor \frac{1}{2}\sum\limits_{i=1}^{m} s_i^{II} \right\rfloor$ for $i = 0, \ldots, l$ together with with (24) implies that $\{b_i\}$ is monotonically decreasing.

Obviously, $\{A_i\}$ is monotonically increasing if $L = 0$. In case $L > 0$ the monotonicity follows from (28) since $p^n_{s^I}(r^* - 1) \leq p^n_{s^I}(r^*)$ implies $r^* \leq \max\{s_n, \lceil \frac{1}{2}\sum\limits_{i=1}^{n} s_i \rceil\}$ (see (25)).

We can now apply (29) and

$$\frac{\sum_{x=0}^{r^*} x\, p^n_{s^I}(x)\, p^m_{s^{II}}(r^*-x)}{\sum_{x=0}^{r^*} p^n_{s^I}(x)\, p^m_{s^{II}}(r^*-x)} = \frac{A_0a_0b_0+\cdots+A_la_lb_l}{a_0b_0+\cdots+a_lb_l} \leq \frac{A_0a_0+\cdots+A_la_l}{a_0+\cdots+a_l} = \frac{\sum_{x=0}^{r^*} x\, p^n_{s^I}(x)}{\sum_{x=0}^{r^*} p^n_{s^I}(x)}$$ is proved.

To II: Firstly, $k = s_1 + \sum_{i=2}^{n+1} s_i \geq n+1$ follows from $|M_I| = m = 1$ and (14).

Then it is sufficient to show that $\frac{\sum_{x=0}^{r^*} x\, p^n_{s^I}(x)}{\sum_{x=0}^{r^*} p^n_{s^I}(x)} \leq r^* \frac{n}{n+1}$. This inequality can be proved by a simple mathematical induction using (27).

Proof of (7) in case $C[k, r^t]$:

In this case it must be showed that $\hat{f} = max\,\{k, R(k) - k + 1\} =: C(k)$ satisfies the property (7).

We can use many ideas from case $C[r^t, k]$ in a similar way.

For this we replace $\left\{ \begin{array}{c} \overset{<}{(-)} \\ \overset{>}{(-)} \end{array} \right\}$ by $\left\{ \begin{array}{c} \overset{>}{(-)} \\ \overset{<}{(-)} \end{array} \right\}$ in (9), (14) and in the definitions of the sets $M_I, \cdots, M_{IV}$ and furthermore $\bullet - o$ by $o - \bullet$ in (9), (10), (12), (13), (15) and (22) (and in corresponding formulas without numbers) where $\bullet = 0$ is possible and the corresponding terms are shorter.
These manipulations implicate that r and s change roles in a way.

Moreover we substitute k by $C(k)$ in the conjectures.
In case $C[k, r^t]$ we must also keep $R(k)$ and $R(k) - k$ in mind.

The above mentioned manipulations lead to

$$Z_t(s, s') = \sum_{i: r^t_i < s_i} (s_i - r^t_i) \;\; \forall s' \in \hat{A}_{N;k}(s, r^t).$$

The considered subsets of M are

$$\begin{aligned}
M_I &= \{i \in M \mid s_i < s'_i \leq \bar{s}'_i = r^t_i \text{ or } s_i \leq s'_i < \bar{s}'_i = r^t_i\},\\
M_{IIa} &= \{i \in M \mid s_i < s'_i \leq \bar{s}'_i < r^t_i \text{ or } s_i \leq s'_i < \bar{s}'_i < r^t_i\},\\
M_{IIb} &= \{i \in M \mid s_i < \bar{s}'_i < s'_i \leq r^t_i\},\\
M_{III} &= \{i \in M \mid r^t_i \geq s'_i > s_i \geq \bar{s}'_i \text{ or } r^t_i > s'_i = s_i \geq \bar{s}'_i\},\\
M_{IV} &= \{i \in M \mid s_i \geq r^t_i = s'_i \geq \bar{s}'_i\} = \{i \in M \mid s_i \geq r^t_i\}.
\end{aligned}$$

in case $C[k, r^t]$.

The first reduction steps also include the reduction of $R(k)$:
$R(k) = R(k) - \sum_{i \in M} \Delta_i$. (Note that the difference $R(k) - k$ remains the original one.)

Since $\Delta_i := \min\left\{s_i, r_i^t, \bar{s}_i'\right\}$ are others in case $C[k, r^t]$ the steps 2 and 3 are realized in the following way

2. $\bar{s} := \sum\limits_{i \in M_{III}} (s_i - \bar{s}_i') + \sum\limits_{i \in M_{IV}} (r_i^t - \bar{s}_i') = \sum\limits_{i \in M_{III}} s_i + \sum\limits_{i \in M_{IV}} r_i^t,$
   $s_i := 0 (= \bar{s}_i'), r_i^t := r_i^t - s_i, s_i' := s_i' - s_i$ for $i \in M_{III}$
   $s_i' = r_i^t := 0 (= \bar{s}_i'), s_i := s_i - r_i^t$ for $i \in M_{IV}$. Temporarily,
   we set $s_{i'} = s_{i'}' = r_{i'}^t := \bar{s}, \bar{s}_{i'}' := 0$ for an additional $i' \in \{i'\} =: M_V$.
3. We replace the elements of M_{IV} by one element i where
   $s^* := \sum\limits_{i \in M_{IV}} s_i,\ r_i^t := s_i' := \bar{s}_i' := 0.$

Then s^* takes the place of r^*.

An equation related to $R(k)$ must be added to (14):

$$k = s^* \le \sum_{i \in M_I \cup M_{IIa} \cup M_{IIb}} \bar{s}_i' \le \sum_{i \in M_I \cup M_{IIa} \cup M_{IIb} \cup M_{III}} r_i^t = R(k). \tag{32}$$

Furthermore $\sum\limits_{i \in M_{III}} s_i \ge r^*$ (see (16)) is replaced by $\sum\limits_{i \in M_{III}} r_i^t \ge s^*$.

$\underset{s' \in \hat{A}_{N;k}(s,r)}{E} (\hat{d}'(s', \bar{s}')) \le s^* - 1$ is sufficient (see (18)) since $s^* \le k \le C(k)$ according to (32).

Finally the reduced model with

$0 = s_i < s_i' \le \bar{s}_i' = r_i^t$ or $0 = s_i \le s_i' < \bar{s}_i' = r_i^t$ for $i \in M_I$,
$r_i^t \ge s_i' > s_i = \bar{s}_i' = 0$ or $r_i^t > s_i' = s_i = \bar{s}_i' = 0$ for $i \in M_{III}$,
$s^* > r_i = s_i' = \bar{s}_i' = 0$ for $i \in M_{IV}$.

is considered where the difference $R(k) - k$ calculated by the reduced $R(k)$ and k is less or equal to the original difference.

Analogous to (19) and (20)

$$k = s^*$$

and

$$s^* \le \sum_{i \in M_I} \bar{s}_i' = \sum_{i \in M_I} r_i^t$$

are valid in case $C[k, r^t]$. In addition

$$\sum_{i \in M_{III}} r_i^t \le R(k) - k. \tag{33}$$

A one-to-one correspondence between the elements of the sets $\hat{A}_{N;k}(s, r)$ and $P^{n+m}_{(r^{II}, r^I)}(s^*)$ is defined by:

$$\{s' \leftrightarrow \bar{x}\} \Leftrightarrow \{s_i' - s_i = s_i' = \bar{x}_i \text{ for } i = 1, \cdots, n+m\},$$

where $r^I = (r^t_{m+1}, \cdots, r^t_{m+n})$, $r^{II} = (r^t_1, \cdots, r^t_m)$.

(23) is replaced by

$$\underset{s' \in \hat{A}_{N;k}(s,r)}{E}(\hat{d}'(s',\bar{s}')) = \frac{\sum\limits_{x=0}^{s^*} x\, p^n_{r^I}(x)\, p^m_{r^{II}}(s^*-x)}{\sum\limits_{x=0}^{s^*} p^n_{r^I}(x)\, p^m_{r^{II}}(s^*-x)} \leq s^* \frac{C(k)-1}{C(k)}.$$

Finally we use $C(k) \geq R(k) - k + 1 \geq |M_{III}| + |M_I| = n + 1$ which follows from (33) in order to prove II. ■

Remarks 1 *There is lot of theory about ordered restricted partitions of integers into* ***positive*** *parts (see [1] for example). In contrary for the case of ordered restricted partitions into* ***non-negative*** *parts not much results are known.* [4] *Therefore we give some basic ideas for the proofs concerning the above used properties.*

a) The one-to-one correspondence $\bar{x} \leftrightarrow s - \bar{x}$ between the elements of the sets $P^n_s(x)$ and $P^n_s(\sum\limits_{i=1}^{n} s_i - x)$ leads to the symmetry-property (24).

b) The monotonicity (25) (and based on that the strict monotonicity (26)) can be proved by means of mathematical induction using (24) and the following simple recursive formula

$$p^{n+1}_{(s_1,\cdots,s_{n+1})}(x+1) = p^{n+1}_{(s_1,\cdots,s_{n+1})}(x) + p^n_{(s_1,\cdots,s_n)}(x+1) - p^n_{(s_1,\cdots,s_n)}(x - s_{n+1}).$$

c) We now want to sketch the main idea for the proof of (27). Based on $P^n_s(x)$ the r^*+1 sets $M^{r^*+1}_P(x)$ $(x = 0, \cdots, r^*)$ of partitions of r^*+1 are generated in the following way:

$M^{r^*+1}_P(x) := \{\, \bar{x} + (r^*+1-x)e_i |\, \bar{x} \in P^n_s(x), \bar{x}_i + (r^*+1-x) \leq s_i, i = 1, \cdots, n\}$, *where e_i is the i-th unit vector. (Each set $M^{r^*+1}_P(x)$ has at most $np^n_s(x)$ elements.) With that it remains to show that every partition of $P^n_s(r^*+1)$ is thereby generated exactly r^*+1 times.*

d) Finally, we want to state the proof of (29):

$$0 \leq \sum_{i=0}^{l} \sum_{j:j<i} a_i a_j (b_j - b_i)(A_i - A_j)$$

$$0 \leq \sum_{i=0}^{l} \sum_{j:j<i} [a_i a_j A_i (b_j - b_i) + a_j a_i A_j (b_i - b_j)]$$

$$\sum_{i=0}^{l} \sum_{j:j<i} (a_i a_j A_i b_i + a_j a_i A_j b_j) \leq \sum_{i=0}^{l} \sum_{j:j<i} (a_i a_j A_i b_j + a_j a_i A_j b_i)$$

[4] Although the method of the generating function and others can also be applied to ordered partitions into *non-negative* parts (s. Andrews).

$$\sum_{i=0}^{l}\sum_{\substack{j=0\\j\neq i}}^{l} a_i a_j A_i b_i \leq \sum_{i=0}^{l}\sum_{\substack{j=0\\j\neq i}}^{l} a_i a_j A_i b_j$$

$$\sum_{i=0}^{l}\sum_{j=0}^{l} a_i a_j A_i b_i \leq \sum_{i=0}^{l}\sum_{j=0}^{l} a_i a_j A_i b_j$$

$$\frac{A_0 a_0 b_0 + \cdots + A_l a_l b_l}{a_0 b_0 + \cdots + a_l b_l} \leq \frac{A_0 a_0 + \cdots + A_l a_l}{a_0 + \cdots + a_l}. \qquad \blacksquare$$

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