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Locally definitizable operators: The local structure of the spectrum

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Abstract

We consider different types of spectral points of locally definitizable operators which can be defined with the help of approximate eigensequences. Their behavior allow a characterization in terms of the (local) spectral function. Moreover, we review some perturbation results for locally definitizable operators.

1 Introduction

As indicated in their name, locally definitizable (or more precise, operators definitizable over some domain in $\mathbb{C}$) are considered to be a class of operators which have locally the same spectral properties as definitizable operators in Krein spaces. Recall that a definitizable operator is a selfadjoint operator A in a Krein space $(\mathcal{H}, [\cdot,\cdot])$ with non-empty resolvent set and with a non-zero polynomial p such that $[p(A)x, x]$ is non-negative for all vectors x in the domain of $p(A)$, cf. [77]. Starting from this definition, a local version is not obvious. Therefore one proceeds in a different manner. The idea is to localize the "key properties" of definitizable operator. These "key properties" of definitizable operator are the following.

1. The non-real spectrum consists of finitely many points only which are poles of the resolvent.
2. Except for a finite set of exceptional points (critical points) the spectrum in $\mathbb{R}$ consists out of spectral points of positive and of negative type.
3. The growth of the resolvent close to $\mathbb{R}$ can be estimated by some power of $|\mathrm{Im}\,\lambda|^{-1}$.

Now a operator is called definitizable over some domain Ω (where Ω has to fulfill some additional assumptions, see Definition 2.5 below) if $\sigma(A) \cap (\Omega \setminus \overline{\mathbb{R}})$ consists of isolated points which are poles of the resolvent of A which do not accumulate to $\Omega \cap \overline{\mathbb{R}}$ such that 2. and 3. are fulfilled (with $\mathbb{R}$ replaced by Δ) for each closed subset Δ of $\Omega \cap \overline{\mathbb{R}}$.

This note is organized as follows. We start in Section 2 with the definition of spectral points of positive/negative type via approximative eigensequences. This approach has the advantage that it does not make use of a local spectral function. We also introduce various types of spectra and then we define locally definitizable operators. Often one obtains locally definitizable operators as a result of a perturbation of an operator with well-known spectral properties. Therefore we recall some of the perturbation results for finite rank perturbations/compact perturbations in Section 3 and for perturbations small in gap in Section 4.

Locally definitizable operators appear in many applications. For locally definitizable operators in the context of (indefinite) Sturm-Liouville problems we refer to [15, 20, 27, 64], for λ-dependent boundary value problems see [16, 19, 63] in the context of $\mathcal{PT}$-symmetric operators see [10, 11], in the study of partial differential equations [17, 29], for a special form of the Krein-Naimark formula see [13, 23] and in the study of problems of Klein-Gordon type see [56, 58].

2 Sign properties of spectral points of selfadjoint operators in Krein spaces

Let $(\mathcal{H}, [\cdot,\cdot])$ be a Krein space. We briefly recall that a complex linear space $\mathcal{H}$ with a Hermitian nondegenerate sesquilinear form $[\cdot,\cdot]$ is called a *Krein space* if there exists a so-called *fundamental decomposition*

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \tag{2.1}$$

with subspaces $\mathcal{H}_\pm$ being orthogonal to each other with respect to $[\cdot,\cdot]$ such that $(\mathcal{H}_\pm, \pm[\cdot,\cdot])$ are Hilbert spaces. To each decomposition (2.1) there corresponds a Hilbert space inner product $(\cdot,\cdot)$ and a selfadjoint operator J (the *fundamental symmetry*) with $J^2 = I, J = J^{-1}$ and $[x,y] = (Jx,y)$ for $x,y \in \mathcal{H}$. Recall that in a Krein space $(\mathcal{H}, [\cdot,\cdot])$ a vector $x \in \mathcal{H}$ is called *positive* (*negative*) if $[x,x] > 0$ ($[x,x] < 0$, respectively).

In the following, all topological notions are understood with respect to some Hilbert space norm $\|\cdot\|$ on $\mathcal{H}$ such that $[\cdot,\cdot]$ is $\|\cdot\|$-continuous. Any two such norms are equivalent (see, e.g., [77]). If $\mathcal{H}_-$ ($\mathcal{H}_+$) is finite dimensional, then $(\mathcal{H}, [\cdot,\cdot])$ is called a *Pontryagin space with finite rank of negativity* (resp. *positivity*). For basic properties of Krein spaces we refer to [67] and to the monographs [3, 35].

Let $(\mathcal{H}, [\cdot,\cdot])$ be a Krein space and let A be a bounded or unbounded selfadjoint linear operator in the Krein space $(\mathcal{H}, [\cdot,\cdot])$, i.e., A coincides with its adjoint A^+ with respect to the indefinite inner product $[\cdot,\cdot]$. If an operator is selfadjoint with respect to some Krein space inner product, then its spectral properties differ essentially from the spectral properties of selfadjoint operators in Hilbert spaces, e.g., the spectrum $\sigma(A)$ of A is in general not real and even $\sigma(A) = \mathbb{C}$ may occur.

The indefiniteness of the scalar product $[\cdot,\cdot]$ on $\mathcal{H}$ induces a natural classification of isolated real eigenvalues of a selfadjoint operator A in a Krein space $(\mathcal{H}, [\cdot,\cdot])$: A real isolated eigenvalue λ_0 of A is called of *positive* (*negative*) *type* if all corresponding eigenvectors are positive (negative, respectively). Observe that there is no Jordan chain of length greater than one which corresponds to a eigenvalue of A of positive type (or of negative type). This classification of real isolated eigenvalues is used frequently, we mention here as some example from theoretical physics [32, 33, 36, 47, 85].

There is a corresponding notion for points from the approximate point spectrum $\sigma_{ap}(A)$[1]. Recall that for a selfadjoint operator A in a Krein space all real spectral points of A belong to $\sigma_{ap}(A)$ (see e.g. Corollary VI.6.2 in [35]). It is convenient to consider the point ∞ also either as a spectral point or as a point from the resolvent set. Hence, in the following, we will use the notion of the *extended spectrum* $\widetilde{\sigma}(A)$ of A which is defined by $\widetilde{\sigma}(A) := \sigma(A)$ if A is bounded and $\widetilde{\sigma}(A) := \sigma(A) \cup \{\infty\}$ if A is unbounded. Moreover, we set $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ and $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$.

The following definition was given in [70] and [79] for bounded selfadjoint operators.

Definition 2.1. *For a selfadjoint operator A in the Krein space $(\mathcal{H}, [\cdot,\cdot])$ a point $\lambda_0 \in \sigma(A)$ is called a spectral point of* positive (negative) type of A *if* $\lambda_0 \in \sigma_{ap}(A)$ *and for every sequence (x_n) in* $\operatorname{dom}(A)$ *with $\|x_n\| = 1$ and $\|(A - \lambda_0 I)x_n\| \to 0$ as $n \to \infty$ we have*

$$\liminf_{n\to\infty} [x_n, x_n] > 0 \quad \big(\textit{resp.}\ \limsup_{n\to\infty} [x_n, x_n] < 0\big).$$

[1]The *approximate point spectrum of* a closed operator A is denoted by $\sigma_{ap}(A)$ and consists of all $\lambda \in \mathbb{C}$ such that there exists a sequence (x_n) in dom A with $\|x_n\| = 1$, $n = 1, 2, \ldots,$

$$\|x_n\| = 1 \quad \text{and} \quad \lim_{n\to\infty} \|Ax_n - \lambda x_n\| = 0.$$

The sequence (x_n) is called an *approximative eigensequence*.

The point ∞ *is said to be of* positive (negative) type of A *if* A *is unbounded and for every sequence* (x_n) *in* $\operatorname{dom}(A)$ *with* $\lim_{n\to\infty}\|x_n\| = 0$ *and* $\|Ax_n\| = 1$ *we have*

$$\liminf_{n\to\infty}\,[Ax_n, Ax_n] > 0 \qquad \big(\textit{resp.}\ \limsup_{n\to\infty}\,[Ax_n, Ax_n] < 0\big).$$

We denote the set of all points of $\widetilde{\sigma}(A)$ *of positive (negative) type by* $\sigma_{++}(A)$ *(resp.* $\sigma_{--}(A)$*).*

The sets $\sigma_{++}(A)$ and $\sigma_{--}(A)$ are contained in $\overline{\mathbb{R}}$. Indeed, for $\lambda \in \sigma_{++}(A)\setminus\{\infty\}$ and (x_n) as in the first part of Definition 2.1 we have $-(\operatorname{Im}\lambda)[x_n, x_n] = \operatorname{Im}[(A-\lambda)x_n, x_n] \to 0$ for $n \to \infty$ which implies $\operatorname{Im}\lambda = 0$. In the following proposition we collect some properties. For a proof we refer to [7].

Proposition 2.2. *Let* λ_0 *be a point of* $\sigma_{++}(A)$ *(*$\sigma_{--}(A)$*, respectively). Then there exists an open neighborhood* $\mathcal{U}$ *in* $\overline{\mathbb{C}}$ *of* λ_0 *such that the following holds.*

(i) *We have*

$$\mathcal{U}\setminus\overline{\mathbb{R}} \subset \rho(A),$$

this is, the non-real spectrum of A *cannot accumulate to* $\sigma_{++}(A)\cup\sigma_{--}(A)$*.*

(ii) $\mathcal{U}\cap\widetilde{\sigma}(A)\cap\overline{\mathbb{R}} \subset \sigma_{++}(A)$ *(resp.* $\mathcal{U}\cap\widetilde{\sigma}(A)\cap\overline{\mathbb{R}} \subset \sigma_{--}(A)$*).*

(iii) *There exists a number* $M > 0$ *such that*

$$\|(A-\lambda)^{-1}\| \le \frac{M}{|\operatorname{Im}\lambda|} \quad \textit{for all } \lambda \in \mathcal{U}\setminus\overline{\mathbb{R}}.$$

It is shown in [79] for bounded selfadjoint operators A (and in [59] for unbounded selfadjoint operators) that if an open connected subset I in $\overline{\mathbb{R}}$ satisfies

$$I\cap\widetilde{\sigma}(A) \subset \sigma_{++}(A)\cup\sigma_{--}(A), \tag{2.2}$$

then there exists a local spectral function E of A of so-called positive type, i.e. for $\delta \subset I$ with $\delta\cap\widetilde{\sigma}(A) \subset \sigma_{\pm\pm}(A)$ the spectral subspace $(E(\delta)\mathcal{H}, \pm[\cdot,\cdot])$ is a Hilbert space. With the help of this (local) spectral function we obtain the following characterization.

Theorem 2.3. *Let* A *be a selfadjoint operator in the Krein space* $(\mathcal{H}, [\cdot,\cdot])$ *and let* I *be as in* (2.2) *with (local) spectral function* E*. A point* $\lambda \in I\cap\widetilde{\sigma}(A)$ *belongs to* $\sigma_{++}(A)$ *(*$\sigma_{--}(A)$*) if and only if there exists a connected set* $\delta \subset I$ *open in* $\overline{\mathbb{R}}$*,* $\lambda \in \delta$*, such that* $(E(\delta)\mathcal{H}, [\cdot,\cdot])$ *(resp.* $(E(\delta)\mathcal{H}, -[\cdot,\cdot])$*) is a Hilbert space.*

Roughly speaking, the spectral properties of the operator A are locally along I the same as of a selfadjoint operator in a Hilbert space.

Let, e.g., A be a $[\cdot,\cdot]$-non-negative selfadjoint operator in a Krein space $(\mathcal{H}, [\cdot,\cdot])$ with a non-empty resolvent set. Then $\sigma(A) \subset \mathbb{R}$ holds and the spectral points of A in $(0,\infty)$ and $(-\infty,0)$ are of *positive type* and *negative type*, respectively, which follows from Theorem 2.3 and the existence of a (unique) spectral function for non-negative operators in Krein spaces, see [77].

Not surprising, spectral points of positive and negative type are in general not stable under finite rank and compact perturbations. However, if the non-negative selfadjoint operator A from above is perturbed by a finite rank operator F such that the resulting operator $B = A + F$ is selfadjoint in $(\mathcal{H}, [\cdot,\cdot])$, then the Hermitian form $[B\cdot,\cdot]$ is still non-negative on the complement of a suitable finite dimensional subspace. Therefore, if (x_n) is an approximative eigensequence corresponding to $\lambda \in \sigma(B)\cap(0,\infty)$ $(\lambda \in \sigma(B)\cap(-\infty,0))$ and all x_n belong to a suitable linear manifold of finite codimension, then all accumulation points of the sequence $([x_n, x_n])$ are again positive (resp. negative). In [7] the latter property of approximative eigensequence serves as a definition of so-called spectral points of *type* π_+ and *type* π_-, respectively, for an arbitrary selfadjoint operator A in a Krein space which we recall here.

Definition 2.4. [7] *For a selfadjoint operator A in a Krein space $(\mathcal{H}, [\cdot,\cdot])$ a point $\lambda_0 \in \sigma_{ap}(A)$ is called a spectral point of* type π_+ (type π_-) of A *if there exists a linear manifold $\mathcal{H}_0 \subset \mathcal{H}$ with* $\operatorname{codim} \mathcal{H}_0 < \infty$ *such that for every sequence (x_n) in $\mathcal{H}_0 \cap \operatorname{dom} A$ with*

$$\|x_n\| = 1, \ n = 1, 2, \ldots, \ \text{and} \ \lim_{n\to\infty} \|(A - \lambda_0)x_n\| = 0 \tag{2.3}$$

we have

$$\liminf_{n\to\infty} \, [x_n, x_n] > 0 \quad (\text{resp. } \limsup_{n\to\infty} \, [x_n, x_n] < 0). \tag{2.4}$$

The point ∞ is said to be a point of type π_+ (type π_-) *if A is unbounded and if there exists a linear manifold $\mathcal{H}_0 \subset \mathcal{H}$ with* $\operatorname{codim} \mathcal{H}_0 < \infty$ *such that for every sequence (x_n) in $\mathcal{H}_0 \cap \operatorname{dom} A$ with*

$$\|Ax_n\| = 1, \ n = 1, 2, \ldots, \ \text{and} \ \lim_{n\to\infty} \|x_n\| = 0$$

we have

$$\liminf_{n\to\infty} \, [Ax_n, Ax_n] > 0 \quad (\text{resp. } \limsup_{n\to\infty} \, [Ax_n, Ax_n] < 0).$$

We denote the set of all points of type π_+ (type π_-) of A by $\sigma_{\pi_+}(A)$ (resp. $\sigma_{\pi_-}(A)$).

If in Definition 2.4 for all sequences (x_n) in $\mathcal{H} \cap \operatorname{dom} A$ with (2.3), property (2.4) follows (i.e. $\mathcal{H}_0 = \mathcal{H}$), then λ_0 is a spectral point of positive (resp. negative). An analogous statement holds for the point ∞. Hence,

$$\sigma_{++}(A) \subset \sigma_{\pi_+}(A) \quad \text{and} \quad \sigma_{--}(A) \subset \sigma_{\pi_-}(A).$$

The point ∞ plays a special role in the follwing sense (see [7]): $\infty \in \sigma_{\pi_+}(A)$ implies $\infty \in \sigma_{++}(A)$ and $\infty \in \sigma_{\pi_-}(A)$ implies $\infty \in \sigma_{--}(A)$.

In [7] Proposition 2.2 and Theorem 2.3 are generalized to spectral points of type π_+/π_-. It is proved that a real spectral point λ_0 of type π_+ of a selfadjoint operator A in a Krein space, which is not an interior point of the spectrum, has a deleted neighbourhood[2] consisting only of spectral points of positive type or of points from $\rho(A)$ and the growth of the resolvent $(A - \lambda)^{-1}$ can be estimated by some power of $|\operatorname{Im} \lambda|^{-1}$ for non-real λ in a neighborhood of λ_0. Such a behavior is also known for locally (in a neighborhood of λ_0) definitizable operators (see, e.g., [59]). Locally definitizable operators appeared first in a paper by H. Langer in 1967 (see [73]) without having a name at that time. Later, in a series of papers, P. Jonas studied these operators and introduced the notion of locally definitizable operators, cf. [52, 53, 55, 59, 60]. This class of operators will be of particular interest in the following, hence we recall here the definition of locally definitizable operators or, more precisely, operators definitizable over some subset of $\overline{\mathbb{C}}$.

Definition 2.5. *Let Ω be a domain in $\overline{\mathbb{C}}$ which is symmetric with respect to $\mathbb{R}$ such that $\Omega \cap \overline{\mathbb{R}} \neq \emptyset$ and the intersections with the open upper and lower half-plane are simply connected. Let A be a selfadjoint operator in the Krein space $(\mathcal{H}, [\cdot,\cdot])$ such that $\sigma(A) \cap (\Omega \setminus \overline{\mathbb{R}})$ consists of isolated points which are poles of the resolvent of A, and no point of $\Omega \cap \overline{\mathbb{R}}$ is an accumulation point of the non-real spectrum of A. The operator A is called* definitizable over Ω *if the following holds.*

(i) *For every closed subset Δ of $\Omega \cap \overline{\mathbb{R}}$ there exist an open neighborhood $\mathcal{U}$ of Δ in $\overline{\mathbb{C}}$ and numbers $m \geq 1$, $M > 0$ such that*

$$\|(A - \lambda)^{-1}\| \leq M(|\lambda| + 1)^{2m-2} |\operatorname{Im} \lambda|^{-m}$$

for all $\lambda \in \mathcal{U} \setminus \overline{\mathbb{R}}$.

[2] A deleted neighbourhood of a point λ_0 is the set $\mathcal{U} \setminus \{\lambda_0\}$, where $\mathcal{U}$ is a neighbourhood of λ_0.

(ii) *Every point* $\lambda \in \Omega \cap \overline{\mathbb{R}}$ *has an open connected neighborhood* I_λ *in* $\overline{\mathbb{R}}$ *such that each of the two components of* $I_\lambda \setminus \{\lambda\}$ *is of positive or of negative type. That is, if* I_1, I_2 *are the two connected (disjoint) components of* $I_\lambda \setminus \{\lambda\} = I_1 \cup I_2$ *then* $I_1 \cap \widetilde{\sigma}(A) \subset \sigma_{++}(A)$ *or* $I_1 \cap \widetilde{\sigma}(A) \subset \sigma_{--}(A)$ *and a similar statement holds for* I_2*; either* $I_2 \cap \widetilde{\sigma}(A) \subset \sigma_{++}(A)$ *or* $I_2 \cap \widetilde{\sigma}(A) \subset \sigma_{--}(A)$*.*

It follows from [59, Theorem 4.7]) that A is definitizable if and only if A is locally definitizable over $\overline{\mathbb{C}}$. Definitizable operators are introduced and comprehensively studied by H. Langer in [72, 77] and appear in many applications, such as indefinite Sturm-Liouville problems, see e.g. [15, 21, 22, 25, 31, 34, 38, 39, 41, 66, 93], Krein-Feller operators [46], λ-dependent boundary value problems, see e.g. [24, 30, 42, 62, 63, 82], operator polynomials [68, 69, 71, 72, 74, 75, 76, 87], second order systems [50, 89, 90] and in the study of problems of Klein-Gordon type [83].

Using the notion of locally definitizable operators, the above mentioned result from [7] reads as follows.

Theorem 2.6. **[7]** *Let* A *be a selfadjoint operator in the Krein space* $(\mathcal{H}, [\cdot,\cdot])$*, and let* I *be a closed connected subset of* $\overline{\mathbb{R}}$ *such that*

$$I \cap \widetilde{\sigma}(A) \subset \sigma_{\pi_+}(A) \cup \sigma_{\pi_-}(A) \tag{2.5}$$

holds and that each point of I *is an accumulation point of* $\rho(A)$*. Then there exists a domain* Ω *in* $\overline{\mathbb{C}}$ *symmetric with respect to* $\mathbb{R}$ *with* $\Omega \cap \mathbb{C}^+$ *and* $\Omega \cap \mathbb{C}^-$ *being simply connected such that* $I \subset \Omega$ *and* A *is definitizable over* Ω*.*

It follows from [59, Section 3.4 and Remark 4.9] that in the situation of Definition 2.5 the operator A has a local spectral function $E(\delta)$ defined for all Borel subsets δ of $\Omega \cap \overline{\mathbb{R}}$ the endpoints of which belong to $\Omega \cap \overline{\mathbb{R}}$ (that is, $\overline{\delta} \subset \Omega \cap \overline{\mathbb{R}}$) and are, if the they belong to $\widetilde{\sigma}(A)$, spectral points of positive or negative type with respect to A. For such a set δ we collect in the following theorem some properties of $E(\delta)$, see [59, Section 3.4 and Remark 4.9].

Theorem 2.7. *The spectral projection* $E(\delta)$ *is a bounded* $[\cdot,\cdot]$*-selfadjoint projection with the following properties.*

(a) $E(\delta)$ *commutes with every bounded operator which commutes with the resolvent of* A*.*

(b) $\sigma(A|E(\delta)\mathcal{H}) \subset \sigma(A) \cap \overline{\delta}$*.*

(c) $\sigma(A|(I - E(\delta))\mathcal{H}) \subset \overline{\sigma(A) \setminus \operatorname{int}(\delta)}$*, where int*$(\delta)$ *is the interior of* δ *with respect to the topology of* $\overline{\mathbb{R}}$*.*

(d) *If, in addition,* δ *is a neighborhood of* ∞ *(with respect to the topology of* $\overline{\mathbb{R}}$*), then* $A|(I - E(\delta))\mathcal{H}$ *is a bounded operator.*

Contrary to the case of an interval satisfying (2.2), this local spectral function is no longer of positive type. Instead, we have the following.

Theorem 2.8. **[7]** *Let* A *be definitizable over* Ω *and let* E *be the spectral function of* A*. A real point* $\lambda \in \sigma(A) \cap \Omega$ *belongs to* $\sigma_{\pi_+}(A)$ $(\sigma_{\pi_-}(A))$ *if and only if there exists a bounded open interval* $\delta \subset \Omega$*,* $\lambda \in \delta$*, such that* $E(\delta)$ *is defined and* $(E(\delta)\mathcal{H}, [\cdot,\cdot])$ *is a Pontryagin space with finite rank of negativity* (*resp. positivity*)*.*

That is, the spectral properties of the operator A in a neighborhood of a point of type π_+ are the same as of a selfadjoint operator in a Pontryagin space.

Moreover, via the local spectral function, we obtain the following characterization of locally definitizable operators which also describe the relation between definitizable and locally definitizable operators.

Theorem 2.9. **[59, Theorem 4.8]** *Let A be a selfadjoint operator in the Krein space $(\mathcal{H}, [\cdot,\cdot])$ and let Ω be a domain as in Definition 2.5. The operator A is definitizable over Ω if and only if for every domain Ω' with the same properties as Ω, $\overline{\Omega'} \subset \Omega$, there exists a bounded selfadjoint projection E in $\mathcal{H}$ such that with respect to the decomposition*

$$\mathcal{H} = (I - E)\mathcal{H} \, [\dot{+}] \, E\mathcal{H} \tag{2.6}$$

the operator A can be written as a diagonal operator matrix

$$A = \begin{pmatrix} A_{I-E} & 0 \\ 0 & A_E \end{pmatrix}, \tag{2.7}$$

where one of the operators A_{I-E}, A_E is a bounded selfadjoint operator in the Krein space $((I-E)\mathcal{H}, [\cdot,\cdot])$ or $(E\mathcal{H}, [\cdot,\cdot])$, respectively, and the other one is either bounded or densely defined. Moreover, A_E is a definitizable operator in $(E\mathcal{H}, [\cdot,\cdot])$ and $\widetilde{\sigma}(A_{I-E}) \cap \Omega' = \emptyset$.

Let A be definitizable over Ω. Points from $\Omega \cap \widetilde{\sigma}(A) \setminus (\sigma_{++}(A) \cup \sigma_{--}(A))$ are sometimes called *critical points*. Now Theorems 2.3 and 2.8 allow the following classification of the spectral points of an operator A definitizable over Ω: Each spectral point $\lambda_0 \in \Omega \cap \overline{\mathbb{R}}$ is

- either a point of $\sigma_{++}(A) \cup \sigma_{--}(A)$. This is the set of spectral points of definite type where the spectral properties of the operator are locally the same as of a selfadjoint operator in a Hilbert space, see Theorem 2.3.
- Or a point of $\sigma_{\pi_+}(A) \cup \sigma_{\pi_-}(A)$ but no point of $\sigma_{++}(A) \cup \sigma_{--}(A)$. These points are a subset of the critical points and the spectral properties of the operator are locally the same as of a selfadjoint operator in a Pontryagin space, see Theorem 2.8.
- Or the point λ_0 belongs to $\widetilde{\sigma}(A) \setminus \big(\sigma_{\pi_+}(A) \cup \sigma_{\pi_-}(A)\big)$. The points are called the *essentially critical points* (see, e.g. [62]) and they have the property that for every open connected set $\delta \subset \Omega$, $\lambda \in \delta$, such that $E(\delta)$ is defined the space $(E(\delta)\mathcal{H}, [\cdot,\cdot])$ is not a Pontryagin space. That is, in every fundamental decomposition of the Krein subspace $(E(\delta)\mathcal{H}, [\cdot,\cdot])$ the two components are infinite dimensional.

Remark 2.10. *A definitizable operator has only finitely many critical points, cf. [77]. There exists operators definitizable over some subset Ω of $\mathbb{C}$ with infinitely many critical or infinitely many essentially critical points. Critical points and essential critical points of operators definitizable over Ω may only accumulate to the boundary of Ω.*

The following theorem illustrate the properties of sign type spectrum from a different point of view. For a proof we refer to [90]. Here we denote by $\sigma_{ess}(A)$ the *essential spectrum*[3] of A.

Theorem 2.11. *Let A be a self-adjoint operator in $(\mathcal{H}, [\cdot,\cdot])$.*

[3]A closed, densely defined operator A in some Banach space is called *Fredholm* if the dimension of the kernel of A and the codimension of the range of A are finite. The set

$$\sigma_{ess}(A) := \{\lambda \in \mathbb{C} \mid A - \lambda I \text{ is not Fredholm}\}$$

is called the *essential spectrum* of A.

(i) *If A satisfies*

$$\widetilde{\sigma}(A) = \sigma_{++}(A) \quad (\textit{resp. } \widetilde{\sigma}(A) = \sigma_{--}(A)),$$

then $(\mathcal{H}, [\cdot,\cdot])$ is a Hilbert space (anti-Hilbert space, respectively).

(ii) *If A satisfies*

$$\widetilde{\sigma}(A) = \sigma_{++}(A) \cup \sigma_{--}(A),$$

then A is similar to a self-adjoint operator in a Hilbert space.

(iii) *If A with $\rho(A) \neq \emptyset$ satisfies*

$$\sigma_{ess}(A) \subset \mathbb{R} \quad \textit{and} \quad \widetilde{\sigma}(A) = \sigma_{\pi_+}(A) \quad (\textit{resp. } \widetilde{\sigma}(A) = \sigma_{\pi_-}(A)),$$

then $(\mathcal{H}, [\cdot,\cdot])$ is a Pontryagin space with finite rank of negativity (resp. positivity). Moreover, the non-real spectrum of A consists of at most finitely many points which belong to $\sigma_p(A) \setminus \sigma_{ess}(A)$.

(iv) *If A with $\rho(A) \neq \emptyset$ satisfies*

$$\sigma_{ess}(A) \subset \mathbb{R} \quad \textit{and} \quad \widetilde{\sigma}(A) = \sigma_{\pi_+}(A) \cup \sigma_{\pi_-}(A), \tag{2.8}$$

then the non-real spectrum of A consists of at most finitely many points which belong to $\sigma_p(A) \setminus \sigma_{ess}(A)$. Moreover, the operator A is definitizable.

3 Compact and finite rank perturbations of definitizable and locally definitizable operators

Roughly speaking, the property of an operator to be definitizable or to be locally definitizable is stable under finite rank perturbations. However, this property is not stable under compact perturbation unless the unperturbed operator has no essential critical points. It is the purpose of the following section the make these statements more precise.

We start with a result on finite rank perturbations from J. Behrndt [14].

Theorem 3.1. [14] *Let A_0 and A_1 be selfadjoint operators in a Krein space $(\mathcal{H}, [\cdot,\cdot])$ with $\rho(A_0) \cap \rho(A_1) \neq \emptyset$ and assume that for some $\lambda_0 \in \rho(A_0) \cap \rho(A_1)$ the difference*

$$(A_0 - \lambda_0)^{-1} - (A_1 - \lambda_0)^{-1} \tag{3.1}$$

is a finite rank operator. Then A_0 is definitizable over Ω if and only if A_1 is definitizable over Ω.

Moreover, if A_0 is definitizable over Ω and $\delta \subset \Omega \cap \overline{\mathbb{R}}$ is an open interval with endpoint $\mu \in \Omega \cap \overline{\mathbb{R}}$ and the spectral points of A_0 in δ are only of positive type (negative type), then there exists an open interval δ', $\delta' \subset \delta$, with endpoint μ such that the spectral points of A_1 in δ' are only of positive type (resp. negative type).

Theorem 3.1 also holds for definitizable operators as the class of definitizable operators over $\overline{\mathbb{C}}$ coincides with the class of definitizable operators ([59, Theorem 4.7]). For definitizable operators this fact is already contained in [61].

Moreover, it is shown in [26] that the finiteness of the number of eigenvalues in a spectral gap of a definitizable or locally definitizable operator is preserved under finite rank perturbations.

If the difference in (3.1) is no longer a finite rank operator but a compact operator, it is well-known that, in general, the assertions of Theorem 3.1 will not hold, see e.g. [61, Proposition 3]. The notion of points of type π_+ and π_- is particularly convenient when compact perturbations are considered. Under a compact perturbation a spectral point of type π_+ remains a spectral point of type π_+ or becomes a point from the resolvent set:

Theorem 3.2. **[7]** *Let* A_0 *and* A_1 *be selfadjoint operators in the Krein space* $(\mathcal{H}, [\cdot,\cdot])$. *Assume that* $\rho(A_0) \cap \rho(A_1) \neq \emptyset$ *and that for some* $\mu \in \rho(A_0) \cap \rho(A_1)$ *the difference*

$$(A_0 - \mu)^{-1} - (A_1 - \mu)^{-1} \text{ is compact.} \tag{3.2}$$

Then

$$(\sigma_{\pi_+}(A_0) \cup \rho(A_0)) \cap \mathbb{R} = (\sigma_{\pi_+}(A_1) \cup \rho(A_1)) \cap \mathbb{R},$$
$$(\sigma_{\pi_-}(A_0) \cup \rho(A_0)) \cap \mathbb{R} = (\sigma_{\pi_-}(A_1) \cup \rho(A_1)) \cap \mathbb{R}.$$

Moreover, $\infty \in \sigma_{++}(A_0)$ $(\infty \in \sigma_{--}(A_0))$ *if and only if* $\infty \in \sigma_{++}(A_1)$ (*resp.* $\infty \in \sigma_{--}(A_1)$).

Theorem 3.2 together with the results presented in the preceding Section 2 give the following perturbation result for locally definitizable operators in Krein spaces from [7] (which is presented here in a slightly different form).

Theorem 3.3. **[7]** *Let* A_0, A_1 *be selfadjoint operators in the Krein space* $(\mathcal{H}, [\cdot,\cdot])$ *with* $\sigma_{ess}(A_0) \subset \mathbb{R}$. *Let* A_0 *be definitizable over a domain* $\Omega \subset \overline{\mathbb{C}}$ *with* $\Omega \cap \overline{\mathbb{R}} = I$. *Assume that* $\rho(A_1) \cap \Omega \neq \emptyset$ *and for some* $\mu \in \rho(A_0) \cap \rho(A_1)$ (3.2) *holds. If*

$$I \cap \widetilde{\sigma}(A_0) \subset \sigma_{\pi_+}(A_0) \cup \sigma_{\pi_-}(A_0),$$

then A_1 *is definitizable over* Ω *and*

$$I \cap \widetilde{\sigma}(A_1) \subset \sigma_{\pi_+}(A_1) \cup \sigma_{\pi_-}(A_1).$$

Theorem 3.3 has a long list of well-known precursors: H. Langer proved in [73] 1967 the assertion of Theorem 3.3 in the case of a bounded selfadjoint fundamentally reducible[4] operator A_0 such that the difference of the resolvents (3.2) belongs to the so-called Matsaev-class. Recall that the Matsaev-class consists of all compact operators with s-numbers (s_j) satisfying $\sum_{j=1}^{\infty}(2j-1)^{-1}s_j < \infty$. P. Jonas extended this result in [53, 54] to unbounded selfadjoint fundamentally reducible operators A_0 such that (3.2) belongs again to the Matsaev-class. In the paper [79] of H. Langer, A. Markus and V. Matsaev in 1997 these assumptions are relaxed: The assertions of Theorem 3.3 are proved for the case of a bounded selfadjoint (no more fundamentally reducible) operator such that (3.2) is compact (no more of Matsaev-class). We mention that the proof of this result from [79] is based upon the existence of maximal spectral subspaces (cf. [86]). Moreover, it is formulated in terms of the so-called eigenvalues of finite index of negativity[5] which are precisely the spectral points of type π_+ being not of positive type, cf. [7]. Finally, J. Behrndt and P. Jonas succeeded to prove the assertions of Theorem 3.3 in 2005 (cf. [18]). We mention that the proofs given in [18] and [7] use completely different methods. Both papers were published in 2005 but [18] was submitted more than one year earlier as [7].

Note that Theorem 3.3 is not suitable for operators A_0 being non-negative in a neighborhood of ∞, or, more precisely, for operators A_0 with ∞ being an essential critical point. However, this case is intensively studied and we refer to [18, 40, 51, 53, 55, 57, 91, 92].

[4] Operators which are selfadjoint in a Krein space and at the same time selfadjoint with respect to some Hilbert space inner product $(\cdot,\cdot)$ such that $[\cdot,\cdot]$ is $\|\cdot\|$-continuous are called *fundamental reducible*.

[5] In [79] an eigenvalue of finite index of negativity of a bounded selfadjoint operator A is defined in the following way: Assume $\lambda_0 \in \sigma_p(A)$ and assume that there exists an open interval (α, β) with $\lambda_0 \in (\alpha, \beta)$ and $(\alpha,\beta)\backslash\{\lambda_0\} \subset \sigma_{++}(A) \cup \rho(A)$. Using the local spectral function we find a restriction of A to some spectral subset with resolvent set in (α, α') and (β', β), $\alpha < \alpha\prime < \lambda_0 < \beta' < \beta$. The interval $(\alpha\prime, \beta')$ is a spectral set of the restriction in the sense of Dunford, hence we have a spectral projection. If this spectral projection projects onto a Pontryagin space with finite rank of negativity, then the point λ_0 is called in [79] an eigenvalue of finite index of negativity.

Theorem 3.3 also applies to definitizable operators in Krein spaces. Based on the Theorems 3.2, 3.3 and Theorem 2.11 we obtain the following perturbation result for definitizable operators, which follows already from the results in the frequently cited paper [61] of P. Jonas and H. Langer from 1979.

Theorem 3.4. *Let A_0 and A_1 be selfadjoint operators in the Krein space $(\mathcal{H}, [\cdot,\cdot])$. Assume that $\rho(A_0) \cap \rho(A_1) \neq \emptyset$ and that for some (and hence for all) $\mu \in \rho(A_0) \cap \rho(A_1)$ the difference* (3.2) *is compact. If A_0 is a definitizable operator with*

$$\sigma_{ess}(A_0) \subset \mathbb{R} \quad \text{and} \quad \widetilde{\sigma}(A_0) = \sigma_{\pi_+}(A_0) \cup \sigma_{\pi_-}(A_0), \tag{3.3}$$

then A_1 is a definitizable operator and (3.3) *holds for A_0 replaced by A_1.*

Theorem 3.4 is in the following sense optimal (cf. [61, Proposition 3]): To every bounded definitizable selfadjoint operator in a Krein space $(\mathcal{H}, [\cdot,\cdot])$ with an non-empty set of essentially critical points there exists a compact selfadjoint operator K in $(\mathcal{H}, [\cdot,\cdot])$ such that the operator $A + K$ is not definitizable.

In [28] the investigation of spectral points of type π_+ and type π_- of selfadjoint operators started in [7] is continued. A sharp lower bound for the codimension of the linear manifold $\mathcal{H}_0$ occurring in Definition 2.4 is given in [28] and this number is smaller or equal to the negativity (resp. positivity) index of the spectral subspaces corresponding to small intervals containing λ_0. Moreover, in [28], a special finite dimensional perturbation is constructed which turns a real point of type π_+ (type π_-) into a point of positive (resp. negative) type.

The above notions and results are also valid for linear relations, see [6]. For the notion of definitizable and locally definitizable linear relations we refer to [44] and [59].

Finaly, we note that the concept of spectral points of positive/negative type is also used as a standard tool in the analysis of selfadjoint operator functions. For further details on the sign type properties of an associated linear operator (i.e. the linearization) in a Krein space and the local spectral functions for selfadjoint operator functions we refer to [1, 78, 80, 81].

4 Compact perturbations and perturbations small in gap of linear relations

The perturbation results presented in this section hold for locally definitizable operators. However, they even hold for arbitrary closed operators and linear relations in Krein spaces and, hence, we present them for such a general class of operators.

The notions of spectral points of positive/negative type and of type π_+/π_- extend naturally to non-selfadjoint operators and to closed linear relations. Recall that closed linear relations in a Hilbert or Krein space $\mathcal{H}$ are closed linear subspaces of the Cartesian product $\mathcal{H} \times \mathcal{H}$. Linear operators are always identified with linear relations via their graphs. For the definitions of the usual operations with relations like the inverse, the spectrum etc. we refer to [2, 43], and to the monographs [37, 48]. Here the (extended) set of regular type $\widetilde{r}(A)$ of a closed linear relation A is defined by $\widetilde{r}(A) := \mathbb{C} \setminus \sigma_{ap}(A)$ if $0 \in \sigma_{ap}(A^{-1})$ and $\widetilde{r}(A) := \overline{\mathbb{C}} \setminus \sigma_{ap}(A)$ otherwise, where $\sigma_{ap}(A)$ is the approximate point spectrum[6] of A.

[6]We say that $\lambda \in \mathbb{C}$ belongs to the *approximate point spectrum* $\sigma_{ap}(A)$ of a closed linear relation A if there exists a sequence $\left(\binom{x_n}{\tilde{x}_n}\right)$ with $\binom{x_n}{\tilde{x}_n} \in A$, $n = 1, 2, \ldots$, such that

$$\|x_n\| = 1 \quad \text{and} \quad \lim_{n\to\infty} \|\tilde{x}_n - \lambda x_n\| = 0.$$

Definition 4.1. **[6]** *Let A be a closed linear relation in a Krein space $(\mathcal{H}, [\cdot,\cdot])$. A point $\lambda_0 \in \sigma_{ap}(A)$ is said to be of* type π_+ (type π_-) *with respect to A, if there exists a linear relation $S \subset A$ with* $\operatorname{codim}_A S < \infty$ *such that for every sequence* $\left(\binom{x_n}{\tilde{x}_n}\right)$ *with* $\binom{x_n}{\tilde{x}_n} \in S$, $n = 1, 2 \ldots$, $\|x_n\| = 1$ *and* $\lim_{n\to\infty} \|\tilde{x}_n - \lambda_0 x_n\| = 0$ *we have*

$$\liminf_{n\to\infty} \, [x_n, x_n] > 0 \quad (\textit{resp.} \ \limsup_{n\to\infty} \, [x_n, x_n] < 0).$$

A similar definition is given for the point ∞, see [6]. If for $\lambda \in \sigma_{\pi_+}(A)$ (resp. $\lambda \in \sigma_{\pi_-}(A)$) it is possible to choose in Definition 4.1 $S = A$, then we call λ a point of *positive type* (resp. *negative type*) of A. As in Section 2 we denote the set of all points of positive, of negative type, of type π_+ and of type π_- by $\sigma_{++}(A)$, $\sigma_{--}(A)$, $\sigma_{\pi_+}(A)$ and $\sigma_{\pi_-}(A)$, respectively.

As a first result we obtain (see [6] and, for selfadjoint operators, [7]) some sign-type properties of eigenvalues from the set $\sigma_{\pi_+}(A)\backslash\sigma_{++}(A)$.

Theorem 4.2. **[6]** *Let A be a closed linear relation in a Krein space $(\mathcal{H}, [\cdot,\cdot])$. If $\lambda_0 \in \sigma_{\pi_+}(A)\backslash\sigma_{++}(A)$ ($\lambda_0 \in \sigma_{\pi_-}(A)\backslash\sigma_{--}(A)$), then λ_0 is an eigenvalue of A with a corresponding non-positive* (*resp. non-negative*) *eigenvector. If $\infty \in \sigma_{\pi_+}(A)\backslash\sigma_{++}(A)$ ($\infty \in \sigma_{\pi_-}(A)\backslash\sigma_{--}(A)$), then the multivalued part of A contains a non-positive* (*resp. non-negative*) *vector.*

In order to investigate the behavior of spectral points of type π_+ and of type π_- under compact perturbations and perturbations small in norm we use the orthogonal projections P_A and P_B in $\mathcal{H}\oplus\mathcal{H}$ onto two closed subspaces A and B of $\mathcal{H}\oplus\mathcal{H}$. Recall that the quantity $\hat{\delta}(A, B) := \|P_A - P_B\|$ is called the *gap* between A and B, cf. [65]. We shall say that A is a compact (finite rank) perturbation of B if $P_A - P_B$ is a compact (resp. finite dimensional) operator. The following description of compact perturbations of closed linear relations is obtained in [5].

Theorem 4.3. *Let A and B be closed linear relations. Then the following assertions are equivalent:*

(i) *$P_A - P_B$ is a compact operator,*

(ii) *for every $\varepsilon > 0$ there exists a closed linear relation F such that $P_B - P_F$ is a finite rank operator and*

$$\hat{\delta}(A, F) = \|P_A - P_F\| < \varepsilon.$$

If, in addition, $\rho(A) \cap \rho(B) \neq \emptyset$, then A is a compact perturbation of B if and only if $(A - \lambda)^{-1} - (B - \lambda)^{-1}$ is a compact operator for some (and hence for all) $\lambda \in \rho(A) \cap \rho(B)$.

Moreover, it is shown in [5] that A is a finite rank perturbation of B if and only if A and B are both finite dimensional extensions of their common part $A \cap B$.

In [6] the following perturbation result for arbitrary non-selfadjoint operators (and relations) in Krein spaces is obtained. We mention that usually perturbation problems are only considered for special subclasses of closed operators, e.g., selfadjoint (see above), normal or dissipative operators in Krein spaces, see [4, 8, 9, 12, 45, 84, 88].

Theorem 4.4. **[6]** *Let A and B be closed linear relations in a Krein space $(\mathcal{H}, [\cdot,\cdot])$ and suppose that A is a compact perturbation of B. Then we have*

$$\sigma_{\pi_+}(A) \cup \widetilde{r}(A) = \sigma_{\pi_+}(B) \cup \widetilde{r}(B) \quad \textit{and} \quad \sigma_{\pi_-}(A) \cup \widetilde{r}(A) = \sigma_{\pi_-}(B) \cup \widetilde{r}(B).$$

The main result in [6] is devoted to perturbations which are small in the gap metric. Roughly speaking, it is shown that spectral points of type π_+ and type π_- type are stable under perturbations small in the gap metric. A similar result holds for spectral points of positive and negative type, see [6].

Theorem 4.5. **[6]** *Let A be a closed linear relation in a Krein space $(\mathcal{H}, [\cdot,\cdot])$ and let $\mathfrak{F} \subset \overline{\mathbb{C}}$ be a compact set with $\mathfrak{F} \subset \sigma_{\pi_+}(A) \cup \widetilde{r}(A)$ ($\mathfrak{F} \subset \sigma_{\pi_-}(A) \cup \widetilde{r}(A)$). Then there exists a constant $\gamma \in (0,1)$ such that for all closed linear relations B with $\hat{\delta}(A,B) < \gamma$ we have*

$$\mathfrak{F} \subset \sigma_{\pi_+}(B) \cup \widetilde{r}(B) \quad (\textit{resp. } \mathfrak{F} \subset \sigma_{\pi_-}(B) \cup \widetilde{r}(B)).$$

The above introduced notions of spectral points of positive and negative type are very convenient in the study of fundamentally reducible closed linear relations under perturbations small in gap, see [6]. A relation A is said to be *fundamentally reducible* if there exists a fundamental decomposition of the Krein space of the form (2.1) and A can be written as

$$A = A_+ \dotplus A_-, \qquad \text{direct sum}, \tag{4.1}$$

where $A_+ := A \cap \mathcal{H}_+^2$ and $A_- := A \cap \mathcal{H}_-^2$ are closed linear relations in the Hilbert spaces $(\mathcal{H}_+, [\cdot,\cdot])$ and $(\mathcal{H}_-, -[\cdot,\cdot])$, respectively. If λ belongs to $\mathbb{C} \setminus \sigma_{ap}(A_-)$ the estimate

$$\|\tilde{y}^- - \lambda y^-\| \geq k_{\lambda,-}\|y^-\| \tag{4.2}$$

holds for some $k_{\lambda,-} > 0$ and all $\binom{y^-}{\tilde{y}^-} \in A_-$.

The following result from [6] can be viewed as a natural generalization of a result for bounded selfadjoint operators in [79, Theorem 4.1]. For simplicity we formulate it here only for spectral points of positive type (for spectral points of negative type, type π_+ or type π_- we refer to [6]).

Theorem 4.6. **[6]** *Let A be a fundamentally reducible closed linear relation in $\mathcal{H}$ as in (4.1) and let B be a closed linear relation in $\mathcal{H}$. If for some $\lambda \in \mathbb{C} \setminus \sigma_{ap}(A_-)$, $k_{\lambda,-} > 0$ as in (4.2) and $\gamma > 0$*

$$\hat{\delta}(A-\lambda, B-\lambda) < \gamma \quad \textit{and} \quad \gamma^2\left(1 + \frac{1}{k_{\lambda,-}^2}\right) < \frac{1}{4}$$

hold, then

$$\lambda \in \sigma_{++}(B) \cup \widetilde{r}(B).$$

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