Technische Universität Ilmenau
Institut für Mathematik

Preprint No. M **13/15**

Two-dimensional Hamiltonian systems

Henrik Winkler

November 2013

Impressum:
Hrsg.: Leiter des Instituts für Mathematik
Weimarer Straße 25
98693 Ilmenau
Tel.: +49 3677 69-3621
Fax: +49 3677 69-3270
http://www.tu-ilmenau.de/math/

TWO-DIMENSIONAL HAMILTONIAN SYSTEMS

HENRIK WINKLER

Abstract. This survey article contains various aspects of the direct and inverse spectral problem for two–dimensional Hamiltonian systems, that is, two–dimensional canonical systems of homogeneous differential equations of the form

$$Jy'(x) = -zH(x)y(x),\ x \in [0, L),\ 0 < L \leq \infty,\ z \in \mathbb{C},$$

with a real non-negative definite matrix function $H \geq 0$ and a signature matrix J, and with a standard boundary condition of the form $y_1(0+) = 0$. Additionally it is assumed that Weyl's limit point case prevails at L. In this case the spectrum of the canonical system is determined by its Titchmarsh-Weyl coefficient Q which is a Nevanlinna function, that is, a function which maps the upper complex half-plane analytically into itself. In this article an outline of the Titchmarsh-Weyl theory for Hamiltonian systems is given and the solution of the direct spectral problem is shown. Moreover, Hamiltonian systems comprehend the class of differential equations of vibrating strings with a non-homogenous mass-distribution function as considered by M.G. Krein. The inverse spectral problem for two–dimensional Hamiltonian systems was solved by L. de Branges by use of his theory of Hilbert spaces of entire functions, showing that each Nevanlinna function is the Titchmarsh-Weyl coefficient of a uniquely determined normed Hamiltonian. More detailed results of this connection for e.g. systems with a semibounded or discrete or finite spectrum are presented, and also some results concerning spectral perturbation, which allow an explicit solution of the inverse spectral problem in many cases.

1. Introduction

In this survey article direct and inverse spectral problems for two–dimensional canonical systems of homogeneous differential equations of the form

$$Jy'(x) = -zH(x)y(x),\ x \in [0, L),\ 0 < L \leq \infty,\ z \in \mathbb{C},$$

with a real non-negative definite matrix function $H \geq 0$ and a signature matrix J are considered. There is a standard boundary condition of the form $y_1(0+) = 0$, and it is assumed that Weyl's limit point case prevails at L. Then the spectral properties of the canonical system are determined by an unique Titchmarsh-Weyl coefficient Q, which belongs to the class $\mathbf{N}$ of Nevanlinna functions. Without loss of generality one can suppose that H is trace normed, that is $\operatorname{trace} H \equiv 1$ on $[0, \infty)$. If the canonical system corresponds to an operator, see Theorem 2.6 below, its spectral measure σ is given by means of the Titchmarsh-Weyl coefficient Q via the common representation formula for Nevanlinna functions (2.14).

The study of two or higher-dimensional canonical systems of differential equations has its roots in the Hamilton-Jacobi formalism in theoretical mechanics, see [K1] for details. Canonical systems were investigated by M.G. Krein and I.S.

Gohberg under operator-theoretic aspects, see e.g. [K1], [GK], by F.V. Atkinson, see e.g. [At], and by L.A. Sakhnovich, see e.g. [S2]–[S6]. Further contributions were made by V.I. Potapov, A.L. Sakhnovich, H. Dym and A. Iacob, see e.g. [P, S1, DI, GKS], and many other authors. A function-theoretic approach to canonical systems can be found in the works of D.B. Hinton and J.K. Shaw [HSh1, HSh2], V.I. Kogan and F.S. Rofe-Beketov [KoR], and A.M. Krall [Kr].

Titchmarsh-Weyl coefficients were originally introduced in the context of Sturm-Liouville problems, see [T, We]; for the inclusion of these problems in the theory of canonical systems see [At]. A spectral theory for two-dimensional canonical systems was presented by L. de Branges in his theory about Hilbert spaces of entire functions, see [dB1]–[dB5]. The methods of L. de Branges led to the solution of the inverse spectral problem for two–dimensional systems, that is each function $Q \in \mathbf{N}$ is the Titchmarsh-Weyl coefficient of a canonical system with a trace normed Hamiltonian H which is uniquely determined by Q, see Theorem 2.4 below.

The approach to general canonical systems via the extension theory of linear relations goes back to B.C. Orcutt [O] and I.S. Kac [Ka1]–[Ka3], and is extended in [LeM] and [BHSW]. Canonical systems on $\mathbb{R}$ such that the limit point case prevails at both ends where considered in [Ka4], and with interface conditions in e.g. [HSW, SW1, SW2] and [BH]. For systems with the spectral parameter in the boundary condition see [DLS1, DLS2]. Direct and inverse spectral results for higher dimensional canonical systems where investigated by D. Arov and H. Dym, see [AD1]–[AD7].

For spaces with an indefinite metric, so-called Pontryagin spaces of entire functions, a generalization of the theory of L. de Branges is presented by M. Kaltenbäck and H. Woracek in [KaW1]–[KaW7], where a generalization of the inverse spectral result of L. de Branges is contained. Spectral problems for higher dimensional canonical systems in an indefinite situation where considered by J. Rovnyak and L.A. Sakhnovich, see [RS1]–[RS4].

M.G. Krein and H. Langer, see [KL1, KL2], considered canonical systems in connection with the continuation problem for positive definite functions. For applications to moment problems see [At].

The class of canonical systems of differential equations contains large classes of linear ordinary differential equations studied in the literature. The theory of vibrating strings with non-homogeneous mass distributions developed by M.G. Krein, see e.g. [KaK2], [K3]–[K5], [Ka5], is included in the theory of canonical systems, see e.g. [L, LW, KaWW]. A presentation of the theory of strings with applications in the extrapolation problem for stationary stochastic processes going back to M.G. Krein [K5] is given by H. Dym and H.P. McKean in [DMcK], where the theory of de Branges is connected with operator-theoretic methods.

There has been an extension of canonical systems to so-called S-hermitian systems, studied by H.D. Niessen, F.W. Schaeffke and A. Schneider, see [HS1]–[HS3] for current results, and H. Langer and R. Mennicken [LM] have shown how S-hermitian systems can be reduced to canonical systems. In particular, A. Schneider [Sch] has shown how large classes of differential expressions can be written in terms of canonical and S-hermitian systems; this includes ordinary differential operators, see [C1, C2], and pairs of ordinary differential operators.

2. Canonical Systems

Let H be a real, symmetric and non-negative definite matrix function on $[0, L)$ with $0 < L \leq \infty$:

$$H(x) = \begin{pmatrix} h_1(x) & h_3(x) \\ h_3(x) & h_2(x) \end{pmatrix}, \quad x \in [0, L), \tag{2.1}$$

with locally integrable functions $h_1,\ h_2$ and h_3, and $H(x) \neq 0$ for $x \in [0, L)$ Two matrix functions H_1 and H_2 are considered to be equivalent if $H_1(x) = H_2(x)$ a.e. on $[0, L)$ with respect to the Lebesgue measure. Let J be the following matrix:

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

A two dimensional *canonical system with Hamiltonian* H, shortly *Hamiltonian system*, is a homogeneous differential equation of the form

$$J\frac{dy(x)}{dx} = -zH(x)y(x),\ x \in [0, L), \tag{2.2}$$

which is considered to hold almost everywhere on $[0, L)$. Here $z \in \mathbb{C}$ is a parameter and y is a vector-valued function, $y(x) = (y_1(x)\ y_2(x))^T$, which satisfies the boundary condition

$$y_1(0+) = 0. \tag{2.3}$$

If not stated otherwise, it is also assumed that for the Hamiltonian (2.1) the condition

$$\int_0^L \operatorname{trace} H(x)dx = \infty \tag{2.4}$$

holds. Note that the condition (2.4) is equivalent to the fact that for the canonical system (2.2) Weyl's limit point case prevails at L.

For $\phi \in [0, \pi)$, denote

$$\xi_\phi = \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}. \tag{2.5}$$

An open subinterval I of $[0, L)$ is called *H-indivisible of type ϕ,* $0 \leq \phi < \pi$, if the relation

$$H(x) = k(x)\xi_\phi\xi_\phi^T,\ \text{a.e. on } I, \tag{2.6}$$

holds with some positive function k on I (see [Ka3]). An H-indivisible interval is called *maximal* if it is not a proper subset of another H-indivisible interval. Note that $\det H = 0$ on H-indivisible intervals. According to L. de Branges, a point $x \in [0, L)$ is called *singular* if x belongs to an H-indivisible interval, otherwise x is called *regular*.

With the Hamiltonian H are associated the following linear spaces, see [Ka1, Ka3]: The Hilbert space $L^2(H)$ is the set of all (equivalence classes of) 2-vector functions $f(x) = (f_1(x)\ f_2(x))^T$ on $[0, L)$ with the property that

$$\int_0^L f(x)^*H(x)f(x)dx < +\infty,$$

equipped with the inner product

$$[f,g]_{L^2(H)} := \int_0^L g(x)^* H(x) f(x) dx.$$

Let $\widehat{L}^2(H)$ be the linear subspace of $L^2(H)$ which consists of all (equivalence classes of) functions $f \in L^2(H)$ with the property that for each H-indivisible interval I of type ϕ there exists a constant $c_{I,\phi,f} \in \mathbb{C}$ such that

$$\xi_\phi^T f(x) = c_{I,\phi,f}, \quad x \in I.$$

It can be shown that $\widehat{L}^2(H)$ is a closed linear subspace of $L^2(H)$, in particular, $\widehat{L}^2(H)$ is a Hilbert space.

For the Hamiltonian H the matrix initial value problem

$$\frac{dW(x,z)}{dx} J = zW(x,z)H(x), \; W(0,z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.7}$$

is considered. Its solution, the 2×2 matrix function

$$W(x,z) = \begin{pmatrix} w_{11}(x,z) & w_{12}(x,z) \\ w_{21}(x,z) & w_{22}(x,z) \end{pmatrix},$$

is said to be the *fundamental matrix function* of the canonical system (2.2). The relation (2.7) implies that the matrix function $W(\cdot, \bar{z})^*$ is the solution of the initial value problem

$$J \frac{dW(x,\bar{z})^*}{dx} = -zH(x)W(x,\bar{z})^*, \quad x \in [0,L), \quad W(0+,\bar{z})^* = I. \tag{2.8}$$

It follows that for $z, \lambda \in \mathbb{C}$

$$W(x,z)JW(x,\lambda)^* - J = (z - \bar{\lambda}) \int_0^x W(t,z)H(t)W(t,\lambda)^* \, dt, \quad x \in [0,L), \tag{2.9}$$

and in particular, for $z \in \mathbb{C}$

$$W(x,z)JW(x,\bar{z})^* = J, \quad W(x,\bar{z})^* JW(x,z) = J, \quad x \in [0,L). \tag{2.10}$$

The matrix function $W(\cdot, z)$ is entire in $z \in \mathbb{C}$ and real, i.e. $\overline{W(\cdot, \bar{z})} = W(\cdot, z)$. Moreover,

$$\det W = 1, \tag{2.11}$$

which follows from (2.10).

For each $x \in [0,L)$ and z with $\operatorname{Im} z > 0$, the linear fractional transformation

$$\omega \to \frac{w_{11}(x,z)\omega + w_{12}(x,z)}{w_{21}(x,z)\omega + w_{22}(x,z)}, \; \omega \in \mathbb{C}^+,$$

maps the upper halplane $\mathbb{C}^+$ onto a disk $D(x,z) \subset \mathbb{C}^+$. Moreover, see [dB2], if $x_0 < x_1$ then $D(x_1,z) \subset D(x_0,z)$. Let $\mathbf{N}$ be the set of Nevanlinna functions, i.e. the set of all functions Q which are analytic on $\mathbb{C} \backslash \mathbb{R}$, satisfy the symmetry condition $Q(z) = \widehat{Q}(\bar{z})$ for $z \in \mathbb{C} \setminus \mathbb{R}$, and map $\mathbb{C}^+$ into $\mathbb{C}^+ \cup \mathbb{R}$.

The following theorem goes back to [dB2], an alternative proof is contained in [HSW].

Theorem 2.1. *Let $W(\cdot, z)$ be the solution of* (2.7). *Then for each $t(z) \in \mathbf{N} \cup \{\infty\}$ the limit*

$$Q(z) = \lim_{x \to L} \frac{w_{11}(x,z)t(z) + w_{12}(x,z)}{w_{21}(x,z)t(z) + w_{22}(x,z)}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \tag{2.12}$$

is independent of $t(z)$ and belongs to $\mathbf{N} \cup \{\infty\}$. Moreover, for each $z \in \mathbb{C} \setminus \mathbb{R}$

$$\chi(z) = \chi(\cdot, z) = W(\cdot, \bar{z})^* \begin{pmatrix} 1 \\ -Q(z) \end{pmatrix} \in \widehat{L}^2(H). \tag{2.13}$$

If Q is a real constant or ∞, the only solution of (2.2) *which belongs to $L^2(H)$ is equivalent to the trivial solution. If Q is not a real constant, the function χ in* (2.13) *is the only nontrivial solution of* (2.2) *which belongs to $L^2(H)$.*

The function Q is called the *Titchmarsh-Weyl coefficient* of the canonical system (2.2) or of the Hamiltonian H. Since Q is a Nevanlinna function, it has the unique spectral representation, the so-called Riesz-Herglotz representation, see [AG, Do, KaK1],

$$Q(z) = a + bz + \int_{-\infty}^{+\infty} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) d\sigma(\lambda) \tag{2.14}$$

with $a \in \mathbb{R},\ b \geq 0$ and

$$\int_{-\infty}^{+\infty} \frac{d\sigma(\lambda)}{1 + \lambda^2} < \infty. \tag{2.15}$$

The non-negative measure σ is called the *spectral measure* of the canonical system (2.2) or of the Hamitonian H.

Example 2.2. Let the Hamiltonian H be given by

$$H(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad x \in [0, \infty). \tag{2.16}$$

The solution $W(\cdot, z)$ of the corresponding equation (2.7) is given by

$$W(x,z) = \begin{pmatrix} \cos(xz) & \sin(xz) \\ -\sin(xz) & \cos(xz) \end{pmatrix}.$$

Hence the Titchmarsh-Weyl coefficient $Q(z)$ is given by

$$Q(z) = \lim_{x \to \infty} \frac{\sin(xz)}{\cos(xz)} = -i \lim_{x \to \infty} \frac{e^{ixz} - e^{-ixz}}{e^{ixz} + e^{-ixz}} = i, \quad \text{Im } z > 0,$$

since $\lim_{x \to \infty} e^{ixz} = 0$ and $\lim_{x \to \infty} e^{-ixz} = \infty$ if Im $z > 0$. The Stieltjes Inversion formula, see [dB5, Do, KaK1], now implies that

$$\sigma'(x) = \frac{1}{\pi} \lim_{\epsilon \to 0+} \text{Im } Q(x + i\epsilon) = 1,$$

and thus

$$d\sigma(\lambda) = \frac{1}{\pi} d\lambda$$

Example 2.3. Let the Hamiltonian H be given by

$$H(x) = \xi_\phi \xi_\phi^T, \quad x \in [0, \infty). \tag{2.17}$$

Then $H(x)$ is trace-normed and has rank 1 on $\mathbb{R}^+$. The solution $W(\cdot, z)$ of the corresponding equation (2.7) is given by

$$W(x,z) = \begin{pmatrix} 1 - zx\cos\phi\sin\phi & zx\cos^2\phi \\ -zx\sin^2\phi & 1 + zx\cos\phi\sin\phi \end{pmatrix}.$$

Hence the Titchmarsh-Weyl coefficient $Q(z)$ is given by

$$Q(z) = \cot\phi.$$

The only solution of (2.2) which belongs to $L^2(H)$ is given by

$$\chi(z) = \chi(\cdot, z) = W(\cdot, \bar{z})^* \begin{pmatrix} 1 \\ -\cot\phi \end{pmatrix} = \begin{pmatrix} 1 \\ -\cot\phi \end{pmatrix}.$$

Clearly, $H(x)\chi(z) = 0$, i.e. the solution $\chi(z)$ is equivalent to the trivial one.

With no loss of generality one can suppose that the Hamiltonian H is *trace normed,* that is $h_1(x) + h_2(x) = 1$ a.e. on $[0, \infty)$. To justify this, let H be any Hamiltonian on $[0, L)$ and let y be a solution of the corresponding problem (2.2). By $\hat{x} := \int\limits_0^x \operatorname{trace} H(t)dt$ and $\hat{H}(\hat{x}) := H(x)(\operatorname{trace} H(x))^{-1}$ a trace normed Hamiltonian $\hat{H}$ on $[0, \infty)$ is defined. Clearly, the relation (2.4) implies that $\hat{x} \to \infty$ if $x \to L$. It follows easily that with $\hat{y}(\hat{x}) := y(x)$ the equation

$$J\frac{d\hat{y}(\hat{x})}{d\hat{x}} = -z\hat{H}(\hat{x})\hat{y}(\hat{x}), \ \hat{y}_1(0+) = 0$$

is satisfied. Taking into account (2.12), the Hamiltonians H and $\hat{H}$ correspond to the same Titchmarsh-Weyl coefficient. For a detailed investigation concerning the reparametrization of non-traced Hamiltonians see [WW2].

For the class of trace normed Hamiltonians a basic result in [dB4] can be formulated as follows (see [W1]):

Theorem 2.4. *(The inverse spectral Theorem) Each function $Q \in \mathbf{N}$ is the Titchmarsh-Weyl coefficient of a canonical system with a trace normed Hamiltonian H on $[0, \infty)$ which is not identically equal to $H(x) = \operatorname{diag}\ (1\ 0)$, $x \in [0, \infty)$; this correspondence is bijective if two Hamiltonians which coincide almost everywhere are identified.*

This result holds only for two-dimensional canonical systems; for a inverse results for canonical systems of higher dimension see [AD1]–[AD7]. The Titchmarsh-Weyl coefficient corresponding to the trace normed Hamiltonian H is denoted by Q. Note the following result from [dB2] (see also [W1]):

Lemma 2.5. *For the number b in the representation* (2.14) *of Q holds the relation*

$$b = \sup\{x :\ (0, x) \text{ is } H\text{-indivisible of type } 0\} \cup \{0\}.$$

Hence, $H(x) = \operatorname{diag}\ (1, 0)$ on $(0, b)$.

Let A be the linear relation which consists of all pairs $\{f, g\} \in (\widehat{L}^2(H))^2$ where f is absolutely continuous, $f_1(0+) = 0$, and

$$J\frac{df(x)}{dx} = -H(x)g(x), \ x \in [0, L), \tag{2.18}$$

holds. The domain of A is denoted by $\operatorname{dom} A$. The next theorem goes back to I.S. Kac, see [Ka3]:

Theorem 2.6. *If the Hamiltonian H of a canonical system satisfies the following two conditions:*

a) $\int_0^\epsilon h_2(x)dx > 0$ *for each* $\epsilon > 0$,

b) $(0, L)$ *is not an H-indivisible interval,*

then $\operatorname{dom} A$ *is dense in* $\widehat{L}^2(H)$ *and A is a self-adjoint operator.*

We say that a Hamiltonian H corresponds to an operator A if it satisfies the conditions a) and b) of Theorem 2.6. According to Lemma 2.5, the condition a) excludes that 0 is the left end point of an H-indivisible interval of type 0. In the terminology of [Ka3] this means that the "first exceptional case" is excluded.

Under the assumptions a) and b), the spectral measure σ of the canonical system is a spectral measure of the operator A in the following sense: There exists a linear and isometric mapping $\mathbf{F}$ from $\widehat{L}^2(H)$ into L^2_σ with the property $\mathbf{F}A\mathbf{F}^{-1} \subset M_\lambda$, where M_λ is the operator of multiplication by the independent variable in L^2_σ: $M_\lambda(f)(\lambda) := \lambda f(\lambda),\ f \in L^2_\sigma$. Indeed, $\mathbf{F}$ can be chosen to be the following "Fourier transformation": Denote by $\widehat{L}^2_0(H)$ the subset of $\widehat{L}^2(H)$ of elements which vanish identically near L, and define

$$\hat{f}(z) := (\mathbf{F}f)(z) := \int_0^L (w_{21}(x,z)\ w_{22}(x,z))H(x)f(x)dx. \tag{2.19}$$

Then the mapping $f \mapsto \mathbf{F}f$ is an isometry from $\widehat{L}^2_0(H)$ onto a dense subset of L^2_σ. Hence it can be extended by continuity to all of $\widehat{L}^2(H)$. The inverse transformation, mapping L^2_σ onto $\widehat{L}^2(H)$, is given by

$$f(x) = \text{l.i.m.}_{N\to+\infty} \int_{-N}^{+N} (w_{21}(x,\lambda)\ w_{22}(x,\lambda))^T \hat{f}(\lambda)d\sigma(\lambda),\ x \in [0, L), \tag{2.20}$$

where l.i.m. denotes the limit in the norm of $\widehat{L}^2(H)$. For $f, g \in \widehat{L}^2(H)$, the relation $[f,g]_{\widehat{L}^2(H)} = [\hat{f},\hat{g}]_{L^2_\sigma}$ is also called Parseval's identity, see [dB3, Ka3].

3. Hilbert spaces of entire functions

Now some aspects of the theory of L. de Branges [dB1]-[dB5] on Hilbert spaces of entire functions (see also [DMcK]) are considered. A de Branges space $\mathbf{K}$ is defined to be a Hilbert space whose elements are entire functions such that the following axioms are satisfied (see [dB1]):

(H1) Whenever the function f belongs to $\mathbf{K}$ and has a non-real zero ω, the function g defined by $g(z) := f(z)(z-\bar{\omega})/(z-\omega)$ belongs to $\mathbf{K}$ and has the same norm as f.

(H2) For each non-real ω, the functional F_ω defined on $\mathbf{K}$ by $F_\omega f := f(\omega)$ is continuous and linear.

(H3) The function f^* given by $f^*(z) := \overline{f(\bar{z})}$ belongs to $\mathbf{K}$ whenever f belongs to $\mathbf{K}$ and has the same norm as f.

An entire function E with the property

$$|E(z)| > |E(\bar{z})| \quad \text{for} \quad \operatorname{Im} z > 0 \tag{3.1}$$

will be called a de Branges-function (as in [DMcK]). For a given de Branges-function $E(z)$ the set $\mathbf{K}(E)$ of entire functions $f(z)$, which satisfy the conditions

$$\|f\|^2 := \int_{-\infty}^{+\infty} |f(t)|^2 |E(t)|^{-2} dt < \infty \tag{3.2}$$

and

$$|f(z)|^2 \leq \|f\|^2 \frac{|E(z)|^2 - |E(\bar{z})|^2}{2\pi i(\bar{z} - z)}, \ \operatorname{Im} z \neq 0, \tag{3.3}$$

is a de Branges space satisfying (H1), (H2) and (H3) with respect to the scalar product

$$\langle f, g\rangle := \int_{-\infty}^{+\infty} f(t)\overline{g(t)}|E(t)|^{-2} dt \text{ for } f, g \in \mathbf{K}(E). \tag{3.4}$$

For the components w_{21} and w_{22} of the fundamental matrix W of a canonical system the function

$$E(x, z) := w_{22}(x, z) + iw_{21}(x, z) \tag{3.5}$$

is a de Branges–function for any fixed $x \in [0, L)$. If x is a regular point, the space $\mathbf{K}(E(x, \cdot))$ is isometrically imbedded in L^2_σ. L. de Branges has shown in [dB1]-[dB5] that for each non-negative Borel measure σ which satisfies (2.15) there exists a canonical system such that σ is its spectral measure (Theorem 12, [dB2]). Moreover, for each spectral measure σ there exists exactly one family of in L^2_σ isometrically imbedded de Branges spaces which is is completely ordered with respect to inclusion (Theorem 7, [dB4]). This result is essential for the proof of Theorem 2.4 (see [W1]). The function

$$K_x(\omega, z) = \frac{w_{22}(x, z)\overline{w_{21}(x, \omega)} - w_{21}(x, z)\overline{w_{22}(x, \omega)}}{z - \bar{\omega}}. \tag{3.6}$$

is a reproducing kernel for the de Branges space $\mathbf{K}(E(x, \cdot))$, that is

$$f(z) = \langle f, K_x(z, .)\rangle = \int_{-\infty}^{+\infty} f(\lambda)\overline{K_x(z, \lambda)} d\sigma(\lambda). \tag{3.7}$$

for each $f \in \mathbf{K}(E(x, \cdot))$, and the set of functions $\{K_x(\omega, \cdot)\}$ is dense in $\mathbf{K}(E(x, \cdot))$. Let

$$u(x, z) := \begin{pmatrix} w_{21}(x, z) \\ w_{22}(x, z) \end{pmatrix}. \tag{3.8}$$

Note that u satisfies the equation

$$J\frac{du(x, z)}{dx} = -zH(x)u(x, z), \ x \in [0, L), \ u(0, z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.9}$$

and that

$$K_x(\lambda, z) = \int_0^x u(t, \lambda)^* H(t)u(t, z) dt = -\frac{u(x, \lambda)^* J u(x, z)}{z - \bar{\lambda}} \tag{3.10}$$

An entire function F belongs to the Cartwright class of the exponential type T if

$$\int_{-\infty}^{+\infty} \frac{|\ln|F(x)||}{1+x^2} dx < \infty, \text{ and } T = \limsup_{|z|\to\infty} |z|^{-1} \ln|F(z)|.$$

In [dB2] it is shown that the components $w_{ij}(x,\cdot)$, $\quad i,j = 1,2$, of the fundamental matrix $W(x,\cdot)$ belong to the Cartwright class of the same exponential type $T(x)$, given by

$$T(x) = \int_0^x \sqrt{\det H(t)} dt.$$

Moreover, the de Branges space $\mathbf{K}(E(x,\cdot))$ consists of functions of exponential type not exceeding $T(x)$.

4. The spectrum of canonical systems

In this section some explicit relations between the Hamiltonian H and the corresponding Titchmarsh-Weyl coefficient Q or the spectral measure σ are listed.

4.1. **Relations between H and Q.** This subsection contains some results from [W1]-[W3] about eigenvalues at 0, the meaning of the real constant a in (2.14), and the shift of the spectrum.

Theorem 4.1. *(see* [W3]*) For a canonical system with Hamiltonian H defined on $[0, L)$ and spectral measure σ the relation*

$$\sigma(\{0\}) = \left(\int_0^L h_2(t) dt \right)^{-1} \tag{4.1}$$

holds.

Lemma 4.2. *(see* [W1]*) Let a canonical system with a trace normed Hamiltonian H, fundamental matrix W and Titchmarsh-Weyl coefficient Q be given. For $s \in \mathbb{R}$ and $S = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$, let*

$$\tilde{x}(x) := \operatorname{trace} \left(S \int_0^x H(t) dt S^T \right)$$

and

$$\tilde{H}(\tilde{x}(x)) := SH(x)S^T \left(\operatorname{trace}\left(SH(x)S^T\right)\right)^{-1}.$$

Then $\lim_{x\to\infty} \tilde{x}(x) = \infty$ and $\tilde{H}$ (with respect to the scale $\tilde{x}$) is the trace normed Hamiltonian corresponding to the fundamental matrix $\tilde{W}$ given by

$$\tilde{W}(\tilde{x}, z) = SW(x,z)S^{-1}$$

and the Titchmarsh-Weyl coefficient

$$\tilde{Q} = Q + s.$$

The following result (see [W2]) shows how the Hamiltonian changes if the corresponding spectral measure is "shifted" along the real line.

Lemma 4.3. *Let Q be the Titchmarsh-Weyl coefficient of a canonical system with trace normed Hamiltonian H and fundamental matrix W. For $l \in \mathbb{R}$, by*

$$\begin{aligned} \widetilde{x}(x) &= \operatorname{trace}\left(\int_0^x W(t,-l)H(t)W(t,-l)^T\,dt\right) \\ \widetilde{H}(\widetilde{x}) &= W(x,-l)H(x)W(x,-l)^T dx \left(\frac{d\widetilde{x}}{dx}\right)^{-1}, \end{aligned} \tag{4.2}$$

a Hamiltonian $\widetilde{H}$ is determined on $[0,\infty)$ whose Titchmarsh-Weyl coefficient $\widetilde{Q}$ has the property

$$\widetilde{Q}(z) = Q(z-l),\ d\widetilde{\sigma}(\lambda) = d\sigma(\lambda - l).$$

4.2. **Canonical Systems with a semibounded spectrum.** Recall that a subset of $\mathbb{R}$ is called semibounded if its infimum or its supremum is finite. A canonical system is called *semibounded* if its spectrum, that is the support of its spectral measure σ, is semibounded. The following result is from [W3].

Theorem 4.4. *If $\operatorname{supp}\sigma$ is semibounded and the corresponding Hamiltonian is trace normed, then $\det H = 0$ a.e. on $[0,+\infty)$ and the components h_1, h_2 and h_3 of H are functions of locally bounded variation.*

Corollary 4.5. *If the spectral measure σ of the canonical system with Hamiltonian H has the property $\operatorname{supp}\sigma \subset [0,\infty)$, then the set $\mathcal{D}(v) := \{x : x \geq 0,\ h_2(x) > 0\}$ is connected and the function*

$$v(x) := \frac{h_3(x)}{h_2(x)} \left(= \frac{h_1(x)}{h_3(x)}\right),\ x \in \mathcal{D}(v),$$

is non-decreasing. If $\operatorname{supp}\sigma \subset (0,\infty)$, then $\mathcal{D}(v) = (b,\infty)$ with the constant b from the representation (2.14) of the Titchmarsh-Weyl coeffcient Q.

At the points where $h_2 > 0$ the Hamiltonian of a semibounded canonical system is characterized by the function v. It turns out that it is sometimes more convenient to consider Hamiltonians which are normalized as follows:

$$H(x) = \begin{cases} \begin{pmatrix} v^2(x) & v(x) \\ v(x) & 1 \end{pmatrix} & \text{if } h_2(x) \neq 0, \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \text{if } h_2(x) = 0. \end{cases} \quad x \in [0,L),\ L \leq +\infty, \tag{4.3}$$

Note that the function v^2 is locally integrable on $[0,L)$ as $\int_0^x \operatorname{trace} H(t)dt < +\infty$ if $x < L$.

Theorem 4.6. *(see [W3]) A canonical system which is semibounded from below has a Hamiltonian of the form (4.3) with the following properties: Let $I_k,\ k = 1,2,...$ be the ordered sequence of the maximal H-indivisible intervals of type 0, and let $E := (0,L) \setminus \bigcup_k \overline{I_k}$. Then the intervals I_k can only accumulate at L. There is an at most countable number of exceptional points $x_i \in E$, $i = 1,2,\ldots$, whose only possible accumulation point is L such that on each interval of $E \setminus \{x_1, x_2, \ldots\}$ the function v is non-decreasing and right-continuous. At an exceptional point $x_i \in E$ the function v has either a (finite) negative jump, that is $v(x_i+) - v(x_i-) < 0$, or it becomes singular with $v(x_i-) = +\infty$ or $v(x_i+) = -\infty$.*

A corresponding result for canonical systems which are semibounded from above exists, see [W3].

4.3. **Canonical systems with a finite number of negative eigenvalues.** A canonical system has κ ($< \infty$) negative eigenvalues if its spectral measure σ has the property that $\operatorname{supp}\sigma \cap (-\infty, 0)$ is a set of κ points. In this case, if the Hamiltonian H corresponds to an operator A, the number of negative squares of the form $\mathbf{a}[f,f] := [Af, f]_{\widehat{L}^2(H)}$ is equal to κ and A has κ negative eigenvalues. By Theorem 4.4, the corresponding Hamiltonian H has the property that $\det H \equiv 0$. The following theorem from [W3] characterizes the Hamiltonian H of the form (4.3) corresponding to a canonical system with a finite number of negative eigenvalues.

Theorem 4.7. *Suppose that the canonical system has a finite number κ of negative eigenvalues. Then its Hamiltonian H of the form* (4.3) *has the following properties:*

(1) *The number κ_1 of all bounded and maximal H-indivisible intervals $I_1, \dots, I_{\kappa_1}$ of type* 0 *is finite.*
(2) *There is a finite number κ_2 of exceptional points $x_i \in E := (0, L) \setminus \bigcup_k \overline{I_k}$ such that on each interval of $E \setminus \{x_1, \dots, x_{\kappa_2}\}$ the function v is nondecreasing and right-continuous. At an exceptional point the function v has either a negative jump or it becomes singular.*
(3) *If $I_1 = (0, b)$, $b > 0$, then $\kappa = \kappa_1 + \kappa_2 - 1$, otherwise, if $0 \in \overline{E}$, the relation $\kappa = \kappa_1 + \kappa_2$ holds.*

Conversely, if the Hamiltonian H of the form (4.3) *has the properties* (1) *and* (2), *then the canonical system has a finite number of negative eigenvalues.*

If the canonical system has a finite number κ of positive eigenvalues, a corresponding result exists, see [W3]. For Hamiltonians which are partially of the form (4.3) the following results hold, see [WW1].

Theorem 4.8. *Let H be a Hamiltonian defined on $[0, L)$ and assume that for some $\epsilon \in (0, L)$ one has*

$$H(x) = \begin{pmatrix} v(x)^2 & v(x) \\ v(x) & 1 \end{pmatrix}, \ x \in (0, \epsilon),$$

with a nondecreasing function $v : (0, \epsilon) \to \mathbb{R}$. Then the limit $\lim_{y\to+\infty} Q(iy)$ exists in $\mathbb{R} \cup \{-\infty\}$ and in fact

$$\lim_{y\to+\infty} Q(iy) = \lim_{x\searrow 0} v(x)\,. \tag{4.4}$$

Theorem 4.9. *Let H be a Hamiltonian defined on $[0, L)$ and assume that for some $l \in (0, L)$ one has*

$$H(x) = \begin{pmatrix} v(x)^2 & v(x) \\ v(x) & 1 \end{pmatrix}, \ x \in (l, L),$$

with a nondecreasing function $v : (l, L) \to \mathbb{R}$. Then the Tichmarsh-Weyl coefficient Q is meromorphic in $\mathbb{C} \setminus [0, +\infty)$, the negative real poles of Q cannot accumulate at 0, *and the limit $\lim_{z\nearrow 0} Q(z)$ exists in $\mathbb{R} \cup \{+\infty\}$. In fact,*

$$\lim_{z\nearrow 0} Q(z) = \lim_{x\nearrow L} v(x). \tag{4.5}$$

4.4. **Canonical Systems with a discrete spectrum.** The following result from I.S. Kac, see [Ka6],[Ka7], contains a sufficent condition about the discreteness of the spectrum of a canonical system, meaning that $\operatorname{supp}\sigma$ has no accumulation point in $\mathbb{R}$.

Theorem 4.10. *Let H be a Hamiltonian of the form (2.1) defined on $[0,\infty)$, and assume that the function $B(x) := \int_0^x h_3(t)dt$ is bounded on $[0,\infty)$. Then the spectral measure σ is discrete if one of the following two relations hold:*

$$\lim_{x\to+\infty}\left(\int_0^x h_2(t)dt\int_x^{+\infty} h_1(t)dt\right)=0,\qquad \lim_{x\to+\infty}\left(\int_0^x h_1(t)dt\int_x^{+\infty} h_2(t)dt\right)=0. \tag{4.6}$$

Moreover, if $h_3 = 0$ on $[0,\infty)$, then this condition is necessary.

4.5. **Canonical systems with a finite spectral measure.** In this subsection canonical systems whose Titchmarsh-Weyl coefficients have the property that the corresponding spectral measure is finite are considered. It turns out that the Hamiltonian starts then with an indivisible interval even in case that $b = 0$, see Lemma 2.5 if $b > 0$. The following result is from [W4].

Theorem 4.11. *The Titchmarsh-Weyl coefficient Q of a canonical system with Hamiltonian H has the property that $b = 0$ and $\int_{-\infty}^{+\infty} d\sigma(\lambda) < \infty$ if and only if 0 is the left end point of an H-indivisible interval of type different from zero. If H is trace normed and Q has a representation of the form*

$$Q(z) = a + \int_{-\infty}^{+\infty}\frac{d\sigma(\lambda)}{\lambda - z} \tag{4.7}$$

and $(0,l)$ is the maximal H-indivisible interval of type $\phi \neq 0$, then the relations $a = \cot\phi$ and $\int_{-\infty}^{+\infty} d\sigma(\lambda) = (l\sin^2\phi)^{-1}$ hold.

If all moments of the spectral measure σ of a canonical system exist, M.G. Krein and H. Langer gave in [KL2] explicit representations of the Hamiltonian by means of corresponding orthogonal polynomials (see also [A]).

4.6. **Canonical systems with a diagonal Hamiltonian and strings.** Canonical systems of differential equations are generalizations of the so-called differential equations of vibrating strings with a non-homogenous mass-distribution function as considered by M.G. Krein, see e.g. [KaK2]. The particular case of strings with a discrete mass distribution (like a pearl necklace) was already investigated by T. Stieltjes, see [St, K3]. If the Hamiltonian H is a diagonal matrix, the entries h_1 and h_2 are related to the length and the mass of a string. The following result is from [LW].

Lemma 4.12. *The Hamiltonian H in (2.1) is of diagonal form,*

$$H(x) = \begin{pmatrix} h_1(x) & 0 \\ 0 & h_2(x)\end{pmatrix}, \quad x \in [0, L)$$

if and only if the corresponding Titchmarch-Weyl coefficient Q satisfies the relation

$$Q(z) = -Q(-z), \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

By defining

$$x(t) := \int_0^t h_1(s)ds, \quad m(x) := \int_0^{\max\{t:x(t)=x\}} h_2(s)ds, \quad l := \int_0^L h_1(s)ds \tag{4.8}$$

and

$$\phi(x, z^2) := w_{22}(t, z), \quad z\psi(x, z^2) := w_{12}(t, z),$$

the functions $\varphi(x, z)$ and $\psi(x, z)$ are solutions of the following differential equation of a string as considered by M.G. Krein (see e.g. [KaK2])

$$dy'(x) + zy(x)dm(x) = 0, \; y'(0-) = 0, \quad x \in [0, l). \tag{4.9}$$

This problem arises if Fourier's method is applied to the partial differential equation which describes the vibrations of a string with free left endpoint 0 on the interval $[0, l)$, where $m(x)$ is the mass of the string on the interval $[0, x]$

Between the principal Titchmarsh-Weyl coefficient Q_S (see [KaK2]) of the string and the corresponding Titchmarsh-Weyl coefficient Q of the canonical system the following relation holds:

$$zQ_S(z^2) = Q(z) \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{4.10}$$

For more connections between strings and canonical systems see, also in an indefinite situation, see e.g. [KaK2, Ka5, KL1, L, LW, KaWW].

5. Small Perturbations of Canonical Systems

The proof of Theorem 2.4 that there is a bijective correspondence between trace-normed canonical systems and their Titchmarsh-Weyl coefficients is not constructive and does not allow the determination of the Hamiltonian for a given Nevanlinna function Q. Therefore it seems to be of interest, to give some general rules, how the Hamiltonian changes if the Titchmarsh-Weyl coefficient (or its spectral measure) undergoes certain transformations. They can be considered as generalizations of corresponding results of M.G. Krein for strings and their spectral measures, see [K3, K4, DMcK, DK1, DK2]. Some of the transformation formulas in [W2] are special cases of Theorem 5.1 below.

If only the constants b and a in the representation (2.14) of Q are changed, the corresponding transformations for the Hamiltonian are given by Lemma 2.5 and Lemma 4.2, see also [W1]. In this section transformations concerning the spectral measure σ are considered. Let $\sigma^\bullet$ be the spectral measure of a Titchmarsh-Weyl coefficient $Q^\bullet$ with corresponding Hamiltonian $H^\bullet$. If the spectral measure $\sigma^\bullet$ is a small perturbation of σ, that is if the signed measure $\tilde{\sigma} := \sigma^\bullet - \sigma$ is sufficiently small (see below), the scale $x^\bullet$ of the Hamiltonian $H^\bullet$ can be parametrized in terms of the scale x of the Hamiltonian H corresponding to the spectral measure $\sigma : x^\bullet = x^\bullet(x)$, such that the corresponding de Branges spaces $\mathbf{K}(E^\bullet(x^\bullet(x), \cdot))$ and $\mathbf{K}(E(x, \cdot))$ contain the same set of functions. We mention that these de Branges spaces need not coincide as Hilbert spaces. Under the assumption that the spectral measure $\sigma^\bullet$ is a small perturbation of σ, in this chapter the corresponding Hamiltonian $H^\bullet$ in terms of the canonical system corresponding to σ is presented.

The spectral measure $\sigma^\bullet$ is a *small perturbation of* σ if

(i) there is a bounded interval $E \subset \mathbb{R}$ and a constant $c > 0$ such that on $\mathbb{R} \setminus E$ the measure $\sigma^\bullet$ is absolutely continuous with respect to σ and for $\lambda \in \operatorname{supp} \sigma \cap \mathbb{R} \setminus E$ the relation $\frac{d|\tilde{\sigma}|}{d\sigma}(\lambda) \le \frac{c}{1+\lambda^2}$ holds

(ii) the measure σ is absolutely continuous with respect to $\sigma^\bullet$.

In the sequel the expression $\int^x f(t)dt$ denotes some primitive of the function f.

Theorem 5.1. *Let a canonical system with Hamiltonian H on $[0,l)$ and spectral measure σ and functions u and K_x (see (3.9), (3.10)) be given. Assume that the spectral measure $\sigma^\bullet := \sigma + \tilde{\sigma}$ is a small perturbation of σ. Let $\mathrm{a}(x,z)$ be the solution of the integral equation*

$$u(x,z) = \mathrm{a}(x,z) + \int_{-\infty}^{+\infty} \mathrm{a}(x,\lambda) K_x(\lambda,z) d\tilde{\sigma}(\lambda), \tag{5.1}$$

let a and b denote the vector functions defined by

$$a(x) := \mathrm{a}(x,0),\ b(x) := \frac{d\mathrm{a}(x,z)}{dz}|_{z=0}, \tag{5.2}$$

and let P be the following matrix function:

$$P(x) := \begin{pmatrix} q(x)^T \\ a(x)^T \end{pmatrix} = \begin{pmatrix} q_1(x) & q_2(x) \\ a_1(x) & a_2(x) \end{pmatrix}, \tag{5.3}$$

where

$$q(x) := \frac{b(x)}{a(x)^T J b(x)} - a(x) \int^x \frac{a(t)^T H(t) b(t)}{\left(a(t)^T J b(t)\right)^2} dt. \tag{5.4}$$

Then $\det P = 1$ *holds. Let*

$$x^\bullet(x) := \int_0^x \operatorname{trace} P(t) H(t) P(t)^T dt,\ l^\bullet := \lim_{x \to l} x^\bullet(x). \tag{5.5}$$

Then by the relation

$$H^\bullet(x^\bullet) := P(x) H(x) P(x)^T \left(\frac{dx^\bullet}{dx} \right)^{-1},\ x^\bullet \in [0, l^\bullet], \tag{5.6}$$

on the interval $[0, l^\bullet]$ a Hamiltonian $H^\bullet$ (with independent variable $x^\bullet$) corresponding to the spectral measure $\sigma^\bullet := \sigma + \tilde{\sigma}$ is given.

Explanation: The definition of q shows that it is only given up to multiples of the function a. Clearly, for each $s \in \mathbb{R}$, the Hamiltonian $H_s^\bullet$ defined by means of the matrix function

$$P_s(x) := \begin{pmatrix} (q(x) + s a(x))^T \\ a(x)^T \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} P(x) \tag{5.7}$$

corresponds also to the spectral measure $\sigma^\bullet$, and for the corresponding Titchmarsh-Weyl coefficient $Q_s^\bullet$ holds the relation $Q_s^\bullet - Q^\bullet = s$. Conversely, to each (trace normed) Hamiltonian $\widetilde{H^\bullet}$ corresponding to the spectral measure $\sigma^\bullet$ there exists $s \in \mathbb{R}$ such that $\widetilde{H^\bullet} = H_s^\bullet$ holds, see Lemma 4.2 above.

Example 5.2. Assume that $m \in \mathbb{R}$ is such that $m + \sigma([0]) \geq 0$. Let

$$\begin{aligned} S(x) &= 1 + m \int_0^x h_2(t)dt, \\ A(x) &= 2 \int_0^x S(t)h_3(t)dt, \\ P(x) &= \begin{pmatrix} S(x) & -mS(x)A(x), \\ 0 & S(x)^{-1} \end{pmatrix} \end{aligned}$$

and define $H^\bullet(x^\bullet)$ by the relation (5.6), then the following holds:

$$\begin{aligned} Q^\bullet = &\quad Q(z) - mz^{-1}, \\ \sigma^\bullet = &\qquad \sigma + m\delta_0. \end{aligned}$$

References

[A] N.I. Achiezer: *The Classical Moment Problem.* Oliver & Boyd, Edinburgh, 1965.

[AG] N.I. Achieser and I.M. Glasmann: *Theorie der linearen Operatoren im Hilbert Raum.* Akademie-Verlag, Berlin, 1954.

[AD1] D.Z. Arov and H. Dym: J-inner matrix functions, interpolation and inverse problems for canonical systems, I: Foundations. Integral Equations Oper. Theory **29** (1997), No. 4, 373–454.

[AD2] D.Z. Arov and H. Dym: J-inner matrix functions, interpolation and inverse problems for canonical systems, II: The inverse monodromy problem. Integral Equations Oper. Theory **36** (2000), No. 1, 11–70.

[AD3] D.Z. Arov and H. Dym: J-inner matrix functions, interpolation and inverse problems for canonical systems, III: More on the inverse monodromy problem. Integral Equations Oper. Theory **36** (2000), No. 2, 127–181.

[AD4] D.Z. Arov and H. Dym: J-inner matrix functions, interpolation and inverse problems for canonical systems, IV: Direct and inverse bitangential input scattering problems. Integral Equations Oper. Theory **43** (2002), No. 1, 1–67.

[AD5] D.Z. Arov and H. Dym: J-inner matrix functions, interpolation and inverse problems for canonical systems, V: The inverse input scattering problem for Wiener class and rational $p \times q$ input scattering matrices. Integral Equations Oper. Theory **43** (2002), No. 1, 68–129.

[AD6] D.Z. Arov and H. Dym: The bitangential inverse input impedance problem for canonical systems, I: Weyl-Titchmarsh classification, existence and uniqueness. Integral Equations Oper. Theory **47** 2003, No. 1, 3–49.

[AD7] D.Z. Arov and H. Dym: The bitangential inverse input impedance problem for canonical systems II. Formulas and examples. Integral Equations Oper. Theory **51** (2005), No. 2, 155–213.

[At] F.V. Atkinson: *Discrete and Continuous Boundary Problems.* Academic Press, New York, 1964.

[BH] H. Behncke and D. Hinton: Two singular point linear Hamiltonian systems with an interface condition. Math. Nachr. **283**, (3), (2010) 365–378.

[BHSW] J. Behrndt, S. Hassi, H. de Snoo and R. Wietsma: Square-integrable solutions and Weyl functions for singular canonical systems. Math. Nachr. **284**, (11/12), (2011) 1334–1384.

[dB1] L. de Branges: Some Hilbert spaces of entire functions. Trans. Amer. Math. Soc. **96** (1960), 259–295.

[dB2] L. de Branges: Some Hilbert spaces of entire functions II. Trans. Amer. Math. Soc. **99** (1961), 118–152.

[dB3] L. de Branges: Some Hilbert spaces of entire functions III. Trans. Amer. Math. Soc. **100** (1961), 73–115.

[dB4] L. de Branges: Some Hilbert spaces of entire functions IV. Trans. Amer. Math. Soc. **105** (1962), 43–83.

[dB5] L. de Branges: *Hilbert Spaces of Entire Functions.* Prentice Hall, Englewood Cliffs, N.J., 1968.

[CL] E.A. Coddington and N. Levinson: On the nature of the spectrum of singular second order linear differential equations. Canadian J. Math. **3** (1951), 335–338.

[C1] E.A. Coddington: The spectral representation of ordinary self-adjoint differential operators. Ann. of Math., 60 (1954), 192–211.

[C2] E.A. Coddington: Generalized resolutions of the identity for symmetric ordinary differential operators. Ann. of Math., 68 (1958), 378–392.

[DLS1] A. Dijksma, H. Langer, and H.S.V. de Snoo: Hamiltonian systems with eigenvalue depending boundary conditions. Operator Theory: Adv. Appl. **35** (1988), 37–83.

[DLS2] A. Dijksma, H. Langer, and H.S.V. de Snoo: Eigenvalues and pole functions of Hamiltonian systems with eigenvalue depending boundary conditions. Math. Nachr. **161** (1993), 107–154.

[Do] W.F. Donoghue: *Monotone matrix functions and analytic continuation.* Springer-Verlag, New York-Heidelberg, 1974.

[DI] H. Dym and A. Iacob: Positive definite extensions, canonical equations and inverse problems. Operator Theory: Adv. Appl. **12** (1984), 141–240.

[DK1] H. Dym and N. Kravitsky: On Recovering the Mass Distribution of a String from its Spectral Function. Topics in Functional Analysis, Advances in Mathematics Supplementary Studies **3** (1978), 45–90.

[DK2] H. Dym and N. Kravitsky: On the Inverse Spectral Problem for the String Equation. Integral Equations Operator Theory **1/2** (1978), 270–277.

[DMcK] H. Dym and H.P. McKean: *Gaussian Processes, Function Theory, and the Inverse Spectral Problem.* Academic Press, New York, 1976.

[GL] I.M. Gelfand and B.M. Levitan: On the determination of a differential equation from its spectral function. AMS Transl. (2), **1** (1955), 253–304.

[GKS] I. Gohberg, M.A. Kaashoek and A.L.Sakhnovich: Canonical systems with rational spectral densities: explicit formulas and applications. Math. Nachr. **194** (1998), 93–125.

[GK] I. Gohberg and M.G. Krein: *Theory and Applications of Volterra Operators in Hilbert Space.* Transl. Math. Monographs **24**, Amer. Math. Soc., Providence, Rhode Island, 1970.

[HSW] S. Hassi, H.S.V. de Snoo, and H. Winkler: Boundary-value problems for two-dimensional canonical systems, Integral Equations Oper. Theory **36** (2000), 445 - 479.

[HS1] D.B. Hinton and A. Schneider: On the Titchmarsh-Weyl coefficients for singular S-Hermitian systems I. Math. Nachr. **163** (1993), 323–342.

[HS2] D.B. Hinton and A. Schneider: On the Titchmarsh-Weyl coefficients for singular S-Hermitian systems II. Math. Nachr. **185** (1) (1997), 67–84.

[HS3] D.B. Hinton and A. Schneider: Titchmarsh-Weyl coefficients for odd-order linear Hamiltonian systems. J. Spectral Math. Appl. **1** (2006) 1–36.

[HSh1] D.B. Hinton and J.K. Shaw: On Titchmarsh-Weyl $M(\lambda)$-functions for linear Hamiltonian systems. J. Differential Equations **40** (1981), 316–342.

[HSh2] D.B. Hinton and J.K. Shaw: Hamiltonian systems of limit point or limit circle type with both endpoints singular. J. Differential Equations **50** (1983), 444–464.

[Ka1] I.S. Kac: "On the Hilbert spaces generated by monotone Hermitian matrix functions", Zap. Mat. Otd. Fiz.-Mat. Fak. i Har'kov. Mat. Obsc, **22** (1951), 95–113.

[Ka2] I.S. Kac: On the multiplicity of the spectrum of a second-order differential operator. Dokl. Akad. Nauk. SSSR Ser. Mat. **145** (1962), 510–514 (Russian) (English translation: Sov. Math. **3** (1962), 1035–1039).

[Ka3] I.S. Kac: Linear relations, generated by a canonical differential equation on an interval with a regular endpoint, and expansibility in eigenfunctions (Russian). Deposited paper 517.9, Odessa, 1984.

[Ka4] I.S. Kac: Expansibility in eigenfunctions of a canonical differential equation on an interval with singular endpoints and associated linear relations (Russian). Deposited paper 517.984, Odessa, 1986.

[Ka5] I.S. Kac: Spectral theory of strings. Ukrainian mathematical Journal **46**, No.3, (1994), 155–176. (Russian)

[Ka6] I.S. Kac: A criterion for the discreteness of a singular canonical system. (Russian), Funktsional. Anal. i Prilozhen. **29** (1995), No. 3, 75–78, English translation: Funct. Anal. Appl. **29** (1995), No. 3, 207–210.

[Ka7] I.S. Kac: On the nature of the de Branges Hamiltonian. Ukrainian mathematical Journal **59**, No.5, (2007), 718–743.

[KaK1] I.S. Kac and M.G. Krein: R-functions–Analytic functions mapping the upper halfplane into itself. Amer. Math. Soc. Transl. (2), **103** (1974), 1–18.

[KaK2] I.S. Kac and M.G. Krein: On the spectral functions of the string. Amer. Math. Soc. Transl. (2), **103** (1974), 19–102.

[KaW1] M. Kaltenbäck and H. Woracek: Pontryagin spaces of entire functions I. Integral Equations Oper. Theory **33** (1999), 34–97.

[KaW2] M. Kaltenbäck and H. Woracek: Pontryagin spaces of entire functions II. Integral Equations Oper. Theory, **33** (1999), 305–380.

[KaW3] M. Kaltenbäck and H. Woracek: Pontryagin spaces of entire functions III. Acta Sci. Math. (Szeged) **69** (2003), 241–310.

[KaW4] M. Kaltenbäck and H. Woracek: Pontryagin spaces of entire functions IV. Acta Sci. Math. (Szeged) **72** (3/4) (2006), 709–835.

[KaW5] M. Kaltenbäck and H. Woracek: Pontryagin spaces of entire functions VI. Acta Sci. Math. (Szeged) **76** (2010), 511–560

[KaW6] M. Kaltenbäck and H. Woracek: Pontryagin spaces of entire functions V. Acta Sci. Math. (Szeged) **77** (2011), 223–336.

[KaW7] M. Kaltenbäck and H. Woracek: Canonical differential equations of Hilbert-Schmidt type. Operator Theory Adv.Appl. 175 (2007), 159-168.

[KaWW] M. Kaltenbäck, H. Winkler and H. Woracek: Strings, dual strings and related canonical systems. Math. Nachr. **280** (13/14) (2007), 1518–1536.

[K] T. Kato: *Perturbation Theory for Linear Operators.* Springer-Verlag, Berlin, Heidelberg, New York, 1980.

[KoR] V.I. Kogan and F.S. Rofe-Beketov: On square-integrable solutions of symmetric systems of differential equations of arbitrary order. Proc. Roy. Soc. Edinburgh Sect. A **74** (1976), 5–40.

[Kr] A.M. Krall: Hilbert Space, Boundary Value Problems and Orthogonal Polynomials. Oper. Theory Adv. Appl. Vol. 133 (Birkhäuser Verlag, Basel, 2002).

[K1] M.G. Krein: *Topics in differential and integral equations and operator theory* Ed. by I. Gohberg. Transl. from the Russian by A. Iacob. Oper. Theory Adv. Appl. Vol.7 Birkhäuser Verlag, Basel, 1983.

[K2] M.G. Krein: On the theory of entire matrix functions of exponential type (Russian). Ukrain. Mat. Z. **3**, 2 (1951), 164–173.

[K3] M.G. Krein: On a generalization of investigations of Stieltjes (Russian). Dokl. Akad. Nauk. SSSR **87** (1952), 881–884.

[K4] M.G. Krein: On some cases of the effective determination of the density of a nonhomogeneous string from its spectral funktion (Russian). Dokl. Akad. Nauk. SSSR **93** (1953), 617–620.

[K5] M.G. Krein: On a fundamental approximation problem in the theory of extrapolation and filtration of stationary random processes (Russian). Dokl. Akad. Nauk. SSSR **94** (1954), 13–16 (English translation: Select. Transl. Math. Stat. Probab. **4** (1963) 127–131).

[KL1] M.G. Krein and H. Langer: On some extension problems which are closely connected with the theory of hermitian operators in a space Π_κ. III. Indefinite analogues of the Hamburger and Stieltjes moment problems. Part (1): Beiträge zur Analysis **14** (1979), 25–40. Part (2): Beiträge zur Analysis **15** (1981), 27–45.

[KL2] M.G. Krein and H. Langer: On some continuation problems which are closely connected with the theory of operators in spaces Π_κ. IV. Continuous analogues of orthogonal polynomials on the unit circle with respect to an indefinite weight and related continuation problems for some classes of functions. J. Oper. Theory, **13** (1985), 299–417.

[L] H. Langer: Spektralfunktionen einer Klasse von Differentialoperatoren zweiter Ordnung mit nichtlinearem Eigenwertparameter. Ann. Acad. Sci. Fenn. Ser. A I Math. **2** (1976), 269-301.

[LM] H. Langer and R. Mennicken: A transformation of right-definite S-hermitian systems to canonical systems. Differ. Integral Equ. **3**, (1990) 901–908.

[LW] H. Langer and H. Winkler: Direct and inverse spectral problems for generalized strings. Integral Equations Operator Theory **30** (1998) 409–431.

[LeM] M. Lesch and M. M. Malamud: On the deficiency indices and self-adjointness of symmetric Hamiltonian systems. J. Differ. Equations **189** (2003) 556–615.

[O] B.C. Orcutt *Canonical differential equations*, Doctoral dissertation, University of Virginia, 1969.

[P] V.P. Potapov: The multiplicative structure of J–contractive matrix functions. Amer. Math. Soc. Transl. (2), **15** (1960), 131–243.

[RS1] J. Rovnyak and L.A. Sakhnovich: Some indefinite cases of spectral problems for canonical systems of difference equations. Linear Algebra Appl. **343/344** (2002), 267–289.

[RS2] J. Rovnyak and L.A. Sakhnovich: Spectral problems for some indefinite cases of canonical differential equations. J Operator Theory **51** (1) (2004), 115–139.

[RS3] J. Rovnyak and L.A. Sakhnovich: Inverse problems for canonical differential equations with singularities, IWOTA 2005 proceedings, Oper. Theory Adv. Appl. **179** (2007), 257–288.

[RS4] J. Rovnyak and L.A. Sakhnovich: Pseudospectral functions for canonical differential systems. Modern Analysis and Applications: The Mark Krein Centenary Conference, Vol. 2, Oper. Theory Adv. Appl. **191** (2009), 187–219.

[S1] A.L. Sakhnovich: Spectral functions of a canonical system of order $2n$. Math. USSR Sbornik **71** (1992), 355–369.

[S2] L.A. Sakhnovich: Factorization problems and operator identities. Russian Math. Surveys **41** (1986), 1–64.

[S3] L.A. Sakhnovich: Nonlinear equations and inverse problems on a semi–axis. preprint, Inst. of Math., Acad. of Sciences of the UkrSSr, Kiev (1987), 1–55 (Russian).

[S4] L.A. Sakhnovich: The method of operator identities and problems of analysis. Algebra and Analysis **5** (1993), 4–80.

[S5] L.A. Sakhnovich: *Spectral Theory of Canonical Differential Systems. Method of Operator Identities.* Birkhäuser, Basel, 1999.

[S6] L.A. Sakhnovich: Dirac type and canonical systems: spectral and Weyl-Titchmarsh matrix functions, direct and inverse problems. Inverse Problems **18** (2) (2002), 331–348.

[Sch] A. Schneider: Zur Einordnung selbstadjungierter Rand-Eigenwertprobleme bei gewöhnlichen Differentialgleichungen in die Theorie S-hermitescher Rand-Eigenwertprobleme. Math. Ann., **178** (1968), 277–294.

[SW1] H.S.V. de Snoo, H. Winkler: Two-dimensional trace-normed canonical systems of differential equations and selfadjoint interface conditions. Integral Equations Oper. Theory **51** (2005), 73 - 108.

[SW2] H.S.V. de Snoo, H. Winkler: Canonical systems of differential equations with selfadjoint interface conditions on graphs. Proc. Roy. Soc. Edinburgh **135 A** (2005), 297 - 315.

[St] T.J.Stieltjes: *Œuvres completes/Collected papers. Vol. I, II*, reprint of the 1914–1918 edition. Springer-Verlag, Berlin 1993.

[T] E.C. Titchmarsh: *Eigenfunction expansions associated with second-order differential equations, Part One.* second edition, Oxford University Press, Oxford, 1962.

[We] H. Weyl: Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Funktionen. Math. Ann. **68** (1910), 220–269.

[W] H. Winkler: *Zum inversen Spektralproblem für zweidimensionale kanonische Systeme.* Doctoral dissertation, Technische Universität Wien, 1993.

[W1] H. Winkler: The inverse spectral problem for canonical systems. Integral Equations Operator Theory. **22** (1995), 360–374.

[W2] H. Winkler: On transformations of canonical systems. Operator Theory: Adv. Appl. **80**, (1995), 276–288.

[W3] H. Winkler: Canonical systems with a semibounded spectrum. Operator Theory: Adv. Appl. **106** (1998), 397–417.

[W4] H. Winkler: Spectral estimations for canonical systems, Math. Nachr. **220** (2000), 115–141.

[W5] H. Winkler: Small perturbations of canonical systems, Integral Equations Oper. Theory **38** (2000), 222–250.

[W6] H. Winkler: On generalized Friedrichs and Kreĭn-von Neumann extensions and canonical systems. Math. Nachr. **236** (2002), 175–191.

[WW1] H. Winkler, H. Woracek: On semibounded canonical systems. Linear Algebra Appl. **429** (2008) 1082–1092.

[WW2] H. Winkler, H. Woracek: Reparameterizations of non trace-normed Hamiltonians. Oper.Theory Adv.Appl. **221** (2012), 667–690.

Institute of Mathematics, Ilmenau University of Technology, Weimarer Str. 25, 98693 Ilmenau, Germany