Technische Universität Ilmenau
Institut für Mathematik

Preprint No. M 14/03

On limit point and limit circle classification for PT symmetric operators

Tomas Ya. Azizov and Carsten Trunk

2014

Impressum:
Hrsg.: Leiter des Instituts für Mathematik
Weimarer Straße 25
98693 Ilmenau
Tel.: +49 3677 69-3621
Fax: +49 3677 69-3270
http://www.tu-ilmenau.de/math/

On limit point and limit circle classification for $\mathcal{PT}$ symmetric operators

Tomas Ya. Azizov*and Carsten Trunk

Abstract

A prominent class of $\mathcal{PT}$-symmetric Hamiltonians is

$$H := \frac{1}{2}p^2 + x^2(ix)^N, \quad \text{for } x \in \Gamma$$

for some nonnegative number N. The associated eigenvalue problem is defined on a contour Γ in a specific area in the complex plane (Stokes wedges), see [3, 5]. In this short note we consider the case $N = 2$ only. Here we elaborate the relationship between Stokes lines and Stokes wedges and well-known limit point/limit circle criteria from [11, 6, 10].

Keywords: non-Hermitian Hamiltonian, Stokes wedges, limit point, limit circle, $\mathcal{PT}$ symmetric operator, spectrum, eigenvalues

1 Introduction

In this paper we consider the quantum system described by the Hamiltonian

$$H = \frac{1}{2m}p^2 - x^4, \tag{1.1}$$

where g is real and positive, see [4] (or [3] with $N = 4$). The Hamiltonian (1.1) is of particular interest because the corresponding $-\phi^4$ quantum field theory might be a good model for describing the dynamics of the Higgs sector of the standard model as the $-\phi^4$ theory is asymptotically free and thus nontrivial, cf. [4] and the references therein. Consider the one-dimensional Schrödinger eigenvalue problem (where we assume, for simplicity, all constants equal to one)

$$-y''(z) - z^4 y(z) = \lambda y(z), \quad z \in \Gamma, \tag{1.2}$$

*The research of T.Ya. Azizov was supported by grant 12-01-00102-a of the RFBR.

associated with the non-Hermitian Hamiltonian in (1.1). Here, $\lambda \in \mathbb{C}$ and the number z runs along a complex contour Γ which is within a Stokes wedge (for details we refer to Section 2). In the situation considered here, the Stokes wedge does not include the real-x axis. We will not use the same complex contour that Jones and Mateo employed in their operator analysis of the Hamiltonian (1.1) in [8]. Instead we use a more simple contour which is not as smooth as the one used in [4, 8]. In this short note, we associate with (1.1) an operator in a $L^2(\mathbb{R})$ space with some boundary conditions. Moreover, we determine the cases when the expression (1.1) is in limit point or limit circle case. This classification is due to [11]; for a more recent refinement see [6, 10].

2 Limit point and limit circle classification

Recall (see, e.g., [3, 4]) that the curve Γ is located in two Stokes wedges and tends to infinity in each of these wedges. A Stokes wedge is an open sector in the complex plane with vertex zero. In the situation considered here (N=4), the complex plane decomposes into six sectors, each with vertex zero, angle $\frac{\pi}{3}$, and with a boundary contained in the set of all complex numbers with

$$\arg z \in \{0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}\}.$$

To be more explicit: In the case considered here, we have six Stokes wedges S_j, $j = 1, \ldots, 6$, defined by

$$S_j = \left\{ z \in \mathbb{C} : (j-1)\frac{\pi}{3} < \arg\, z < j\frac{\pi}{3} \right\}.$$

According to the rules imposed by $\mathcal{PT}$-symmetry, the contour Γ has to satisfy some symmetry assumptions, i.e., Γ is assumed to be located in

$$S_1 \cup S_3 = \left\{ z \in \mathbb{C} : 0 < \arg\, z < \frac{\pi}{3} \;\text{ or }\; \frac{2\pi}{3} < \arg\, z < \pi \right\}. \qquad (2.1)$$

However, in this note we will also consider the case when Γ coincides with some Stokes line: $\Gamma \subset \{z \in \mathbb{C} : \arg\, z \in \{\frac{\pi}{3}, \frac{2\pi}{3}\}\}$.

Let ϕ with $0 < \phi \leq \frac{\pi}{3}$. Here (for simplicity) we assume that Γ is given by

$$\Gamma := \{x e^{i\phi \operatorname{sgn} x} : x \in \mathbb{R}\}.$$

Note that $0 < \phi < \frac{\pi}{3}$ corresponds to the case that Γ is contained in a Stokes wedge. This case is usually assumed, cf. [3, 4, 5, 8, 9] whereas $\phi = \frac{\pi}{3}$ corresponds to the case that Γ coincides with some Stokes lines.

Our approach starts with the idea of Mostafazadeh in [9] to map the problem (1.2) back onto the real axis using a real parametrization. Here (contrary to [9]) we use the following parametrization $z : \mathbb{R} \to \mathbb{C}$,

$$z(x) := x e^{i\phi \operatorname{sgn} x}$$

Then y solves (1.2) for $z \neq 0$ if and only if w, $w(x) := y(z(x))$, solves

$$-e^{-2i\phi}w''(x) - e^{4i\phi}x^4 w(x) = \lambda w(x) \qquad \text{if } x > 0, \tag{2.2}$$
$$-e^{2i\phi}w''(x) - e^{-4i\phi}x^4 w(x) = \lambda w(x) \qquad \text{if } x < 0. \tag{2.3}$$

We define for a complex number α the operator A_α with domain $\operatorname{dom} A_\alpha$ in $L^2(\mathbb{R})$. The domain $\operatorname{dom} A_\alpha$ consists of all $w \in L^2(\mathbb{R})$ which are locally absolutely continuous on $\mathbb{R}$ such that w' is locally absolutely continuous on $\mathbb{R} \setminus \{0\}$ with

$$A_\alpha w \in L^2(\mathbb{R}) \quad \text{and} \quad w'(0+) = \alpha w'(0-).$$

For $w \in \operatorname{dom} A_\alpha$ we define $A_\alpha w$ in the following way:

$$A_\alpha w := \begin{cases} -e^{-2i\phi}w''(x) - e^{4i\phi}x^4 w(x) & \text{if } x > 0, \\ -e^{2i\phi}w''(x) - e^{-4i\phi}x^4 w(x) & \text{if } x < 0. \end{cases}$$

The two (linearly independent) solutions $y^\pm$ of (2.2) satisfy as $x \to \infty$ (see, e.g., [7, pg. 58])

$$y^\pm(x) \sim \left[e^{-4i\phi}s(x)\right]^{-1/4} exp\left(\pm \int_0^\infty \operatorname{Re} s(t)^{1/2} dt\right)$$

with $s(x) := -e^{6i\phi}x^4 - e^{2i\phi}\lambda$. We use the notation $f(x) \sim g(x)$ to mean that $f(x)/g(x) \to 1$ as $x \to \infty$ The same holds for the two solutions of (2.3) (as $x \to -\infty$) which is easily seen by replacing x by $-x$. We have

$$\operatorname{Re} s(t)^{1/2} \sim -t^2 \sin 3\phi.$$

The following theorem is the main result of this note. It is a consequence of the above observations and follows from the classification given in [11] (see also [6, 10]).

Theorem 2.1. (i) *If $0 < \phi < \frac{\pi}{3}$, then (2.2) and (2.3) are in limit point case. In particular this implies that one solution of (2.2) is not in $L^2(\mathbb{R}^+)$ and that one solution of (2.3) is not in $L^2(\mathbb{R}^-)$.*

(ii) *If $\phi = \frac{\pi}{3}$, then (2.2) and (2.3) in limit circle case. In particular this implies that both solutions of (2.2) are in $L^2(\mathbb{R}^+)$ and that both solution of (2.3) are in $L^2(\mathbb{R}^-)$.*

Theorem 2.1 allows the following mathematical interpretation: If Γ coincides with a Stokes line, then (2.2) and (2.3) are in limit circle case. If Γ is contained in a Stokes wedge, then (2.2) and (2.3) are in limit circle case.

3 Point spectrum of A_α in the limit circle case

In the case Γ coincides with a Stokes line, both solutions of (2.2) are in $L^2(\mathbb{R}^+)$ and that both solution of (2.3) are in $L^2(\mathbb{R}^-)$. It is easily seen, that there exist a linear combination of these solutions which is in $\operatorname{dom} A_\alpha$ and the following theorem follows.

Theorem 3.1. *Assume that Γ coincides with a Stokes line. Then the point spectrum $\sigma_p(A_\alpha)$ of A_α coincides with the complex plane,*

$$\sigma_p(A_\alpha) = \mathbb{C}.$$

In the situation of Theorem 3.1 a boundary condition is missing. In order to avoid the situation in Theorem 3.1, one has to impose so-called boundary conditions at $\pm\infty$, see e.g., [1, 2].

References

[1] T.Ya. Azizov and C. Trunk, *On domains of $\mathcal{PT}$ symmetric operators related to $-y''(x) + (-1)^n x^{2n} y(x)$*, J. Phys. A: Math. Theor. **43** (2010), 175303.

[2] T.Ya. Azizov und C. Trunk, *$\mathcal{PT}$ symmetric, Hermitian and $\mathcal{P}$-self-adjoint operators related to potentials in $\mathcal{PT}$ quantum mechanic*, J. Math. Phys. **53** (2012), 012109.

[3] C.M. Bender and S. Boettcher, *Real spectra in non-Hermitian Hamiltonians having $\mathcal{PT}$ symmetry*, Phys. Rev. Lett. **80** (1998), 5243–5246.

[4] C.M. Bender, D.C. Brody, J.-H. Chen, H.F. Jones, K.A. Milton and M.C. Ogilvie, *Equivalence of a complex $\mathcal{PT}$-symmetric quartic Hamiltonian and a Hermitian quartic Hamiltonian with an anomaly*, Phys. Rev. D **74** (2006), 025016.

[5] C.M. Bender, D.C. Brody and H.F. Jones, *Complex extension of quantum mechanics* Phys. Rev. Lett. **89** (2002) 270401.

[6] B.M. Brown, D.K.R. McCormack, W.D. Evans and M. Plum, *On the spectrum of second-order differential operators with complex coefficients*, Proc. R. Soc. A **455** (1999), 1235–1257.

[7] M.S.P. Eastham, *The asymptotic solution of linear differential systems*, London Mathematical Society, Monograph **4**, 1989.

[8] H.F. Jones and J. Mateo, *Equivalent Hermitian Hamiltonian for the non-Hermitian $-x^4$ potential*, Phys. Rev. D **73** (2006), 085002.

[9] A. Mostafazadeh, *Pseudo-Hermitian description of $\mathcal{PT}$-symmetric systems defined on a complex contour*, J. Phys. A **38** (2005), 3213.

[10] J. Qi, H. Sun and Z. Zheng, *Classification of Sturm-Liouville differential equations with complex coefficients and operator realizations*, Proc. R. Soc. A **467** (2011), 1835–1850.

[11] A.R. Sims, *Secondary conditions for linear differential operators of the second order*, J. Math. Mech. **6** (1957), 247–285.

Contact information

Tomas Ya. Azizov

Voronezh State University, Faculty of Mathematics
Universitetskaya pl. 1, 394006 Voronezh, Russia
azizov@math.vsu.ru

Carsten Trunk

Institut für Mathematik, Technische Universität Ilmenau
Postfach 100565, D-98684 Ilmenau, Germany
carsten.trunk@tu-ilmenau.de