Banach space-valued ergodic theorems and spectral approximation

seit 1558

DISSERTATION

zur Erlangung des akademischen Grades

doctor rerum naturalium

vorgelegt dem Rat
der Fakultät für Mathematik und Informatik
der Friedrich-Schiller-Universität Jena

von Felix Pogorzelski
geboren am 20.05.1984 in Erlenbach am Main

1. Gutachter: Prof. Dr. Daniel Lenz
   (Promotionsbetreuer, Friedrich-Schiller-Universiät Jena, Ernst-Abbe-Platz 2, 07743 Jena)
2. Gutatcher: Prof. Dr. Tatjana Eisner
   (Universität Leipzig, Augustusplatz 10, 04109 Leipzig)
3. Gutachter: Prof. Dr. Amos Nevo
   (Israel Institute of Technology (Technion), Technion Campus, 32000 Haifa, Israel)

Datum der Abgabe:	18.08.2014
Datum der Verteidigung:	17.10.2014

Zusammenfassung in deutscher Sprache

Die vorliegende Dissertationsschrift behandelt banachraumwertige, fast-additive Abbildungen. Diese sind von der Form

$$F : \mathcal{F}(G) \to Z,$$

wobei $\mathcal{F}(G)$ eine Menge von messbaren Teilmengen eines lokalkompakten Maßraumes G darstellt und Z für einen beliebigen Banachraum steht. Eine solche Funktion F nennt man fast-additiv, falls sie gewisse Beschränktheits- und Invarianzkriterien erfüllt und falls für jede endliche Vereinigung $Q = \bigsqcup_k Q_k$ paarweiser disjunkter Mengen $Q_k \in \mathcal{F}(G)$ die Differenz der Vektoren $F(Q)$ und $\sum_k F(Q_k)$ durch einen sogenannten 'Randterm' ('boundary term') kontrolliert werden kann. In gewissen solchen Räumen G mit Maß m findet man Ausschöpfungen $(U_j)_{j\in\mathbb{N}}$ von G mittels kompakter Mengen U_j, deren normalisierter Randwert asymptotisch verschwindet. In diesen Situationen lässt sich die Existenz der Grenzwertes

$$F^* := \lim_{j\to\infty} \frac{F(U_j)}{m(U_j)}$$

in der Topologie des Banachraumes Z beweisen. Der Hauptgegenstand der Arbeit ist der Nachweis solcher Konvergenzsätze (banachraumwertiger Ergodensätze) für eine große Klasse von Geometrien. Diese Resultate werden in einem zweiten Schritt auf Fragen spektraler Konvergenz angewendet. Dabei wird die gleichmäßige Existenz der integrierten Zustandsdichte von selbstadjungierten Operatoren mit endlicher Reichweite in verschiedenen diskreten Geometrien gezeigt. Zudem wird die Ihara Zetafunktion für sofische Graphings definiert und in der Topologie der gleichmäßigen Konvergenz auf kompakten Mengen durch normalisierte Zetafunktionen von endlichen Graphen approximiert. Die originalen Ergebnisse dieser Arbeit finden sich in den Aufsätzen [Pog13a, Pog13b, PS14, LPS14].

Der Text ist wie folgt gegliedert. Das Kapitel 2 behandelt eine Einführung in die Theorie von lokalkompakten, mittelbaren, hausdorffschen Gruppen, welche das zweite Abzählbarkeitsaxiom erfüllen. Für unimodulare Gruppen wird gezeigt, dass sich (bezüglich der Gruppenmultiplikation) genügend invariante Teilmengen gut durch Translate von bestimmten kompakten, kleineren Mengen überdecken lassen. Dies erweitert Ergebnisse aus [OW87]. Im folgenden Kapitel 3 werden Familien von Überdeckungen von genügend invarianten Teilmengen in G konstruiert, welche im Durchschnitt wünschenswerte Uniformitätseigenschaften aufweisen. Diese Resultate bilden die Grundlage zum Beweis zweier fast-additiver, banachraumwertiger Ergodensätze in Kapitel 4. Das Theorem für abzählbare mittelbare Gruppen (Theorem 4.4) verallgemeinert das Hauptresultat in [LSV11]. Für stetige Gruppen erhält man mit Theorem 4.15 einen fast-additiven, banachraumwertigen Mittelergodensatz, welcher den klassischen Fall [Gre73] erweitert. Das Kapitel 5 behandelt die Konvergenz von beschränkten, additiven Prozessen. Es wird gezeigt, dass Prozesse, welche sich durch

L^∞-Prozesse approximieren lassen, entlang von Tempelman-Følnerfolgen fast überall konvergieren, siehe Theorem 5.17. Dies ist eine fundamentale Erweiterung des geometrischen Rahmens der Ergebnisse in [Sat99, Sat03]. Die Anwendung der Konvergenzsätze für Gruppen auf Fragestellungen von spektraler Approximation erfolgt im Kapitel 6. Hier werden die Approximationsresultate für die integrierte Zustandsdichte aus den Arbeiten [LV09, LSV11] erweitert und ergänzt. Im Kapitel 7 werden Folgen von endlichen Graphen eingeführt, die gegen Graphings konvergieren. Diese Folgen heißen schwach konvergent oder auch Benjamini-Schramm-konvergent, cf. [BS01]. Besonderes Augenmerk liegt auf den hyperendlichen Graphenfolgen. In Theorem 7.10 wird gezeigt, dass schwach konvergente, hyperendliche Graphenfolgen eine Cauchyfolge bezüglich einer bestimmten Pseudometrik bilden. In diesem Fall spricht man von starker Konvergenz. Für stark konvergente Graphenfolgen lassen sich banachraumwertige fast-additive, sowie subadditive Konvergenzsätze beweisen. Dies ist Gegenstand der Theoreme 8.2 und 8.4 im Kapitel 8. Die Ergebnisse erweitern den geometrischen Rahmen bisheriger Resultate für endlich erzeugte Gruppen (siehe oben) und Halbgruppen [CSKC12]. Im Kapitel 9 wird die gleichmäßige Approximation der integrierten Zustandsdichte von selbstadjungierten, musterinvarianten Operatoren endlicher Reichweite auf Graphings entlang von hyperendlichen, schwach konvergenten Graphenfolgen bewiesen (Theorem 9.12). Ferner werden Verbindungen zur Lück-Approximation [Lüc94, ATV13] für Graphenfolgen gezogen. Das Kapitel 10 behandelt die Approximation der in [LPS14] eingeführten Ihara Zetafunktion für Graphings entlang normalisierter, endlicher Versionen zu schwach konvergenten Graphenfolgen. Für sofische Graphings wird in Theorem 10.5 die Konvergenz in der Topologie der gleichmäßigen Konvergenz auf kompakten Mengen gezeigt. Für periodische Graphen mit sofischer Gruppenwirkung werden entsprechende Folgen explizit konstruiert, siehe Theorem 10.8. Diese Ausführungen verallgemeinern die Konvergenzresultate in [CMS02, GŻ04, GIL08, GIL09]. Im Kapitel 11 werden zwei offene, sich direkt an diese Dissertation anschließende Fragen skizziert.

Table of Contents

Acknowledgements

First of all, I would like to thank my adviser Daniel Lenz for all his guidance and support during the last four years. He always had an open door for my questions and he generously shared with me both his mathematical knowledge and his ideas on research projects. Moreover, I am very grateful that he always provided me with all necessary means to travel within the scope of my research activities. I am also indebted to Daniel for helping me to improve my mathematical writing skills.
Thank you for supporting my academic career in every respect.

Next, I would like to thank my coworkers Siegfried Beckus, Matthias Keller, Daniel Lenz, Norbert Peyerimhoff, Marcel Schmidt, Christoph Schumacher and Fabian Schwarzenberger. Especially, many thanks go to Fabian Schwarzenberger. The cooperation with him laid a cornerstone for my first research successes. I will bear in mind all the great and intensive hours of discussions about mathematics and other important things. Moreover, I am particularly thankful to Matthias Keller and Norbert Peyerimhoff who introduced me to the world of polygonal complexes and who patiently explained to me links in metric geometry and spectral theory.

For their proofreading efforts, I am indebted to Siegfried Beckus, Alessandro Carderi, Daniel Lenz, Matthias Keller, Therese Mieth, Marcel Schmidt, Corinna Schreier and Fabian Schwarzenberger. Thank you for every valuable hint concerning language and structuring issues.

I also express my deepest thanks for the Analysis group around Daniel Lenz. It is a blessing to work at a place where supporting each other is a matter of course. Thank you Daniel, Marcel, Matthias, Siegfried and Xueping for creating a wonderful atmosphere of friendship and appreciation. I will not forget all the joyful hours we spent on both mathematical and non-mathematical activities.

I am very grateful for all invitations to research seminars and conferences. In this respect, I thank András Bátkai (Budapest), Michael Björklund (Göteborg), Phu Chung (Leipzig), Tatjana Eisner (Leipzig), Rainer Nagel (Tübingen), Tobias Oertel-Jäger (Erlangen at that time), Mathias Schacht (Hamburg) and Andreas Thom (Leipzig at that time) for giving me the opportunity to present my research results outside of Jena. Besides, I would like to express my thanks to Daniel Coronel and Michael Schraudner (both Santiago de Chile) for inviting me to Chile, thus making it possible for me to participate in two conferences and to give talks in Valparaíso and Santiago de Chile. At the same time I gratefully acknowlegde the financial travel support for Chile granted by the German National Academic Exchange Service (DAAD). Further, I thank Amos Nevo (Haifa) for two invitations to the Technion Institute. I experienced great hospitality there and I enjoyed the inspiring discussions about ergodic theorems of non-amenable groups. Special thanks go to Ivan Veselić who invited me to Chemnitz several times. It has always been a pleasure to work, to discuss, to give talks or just to spend time with the people from his group.

During the first year of my Ph.D. studies, I was supported by the Deutsche Forschungsgemeinschaft (DFG). Starting from the second year, I benefited from support through the German National Academic Foundation (Studienstiftung des deutschen Volkes). I am very beholden for the sponsorship from these two institutions as it offered me the freedom to focus on research and to travel in semester times.

I am very thankful to my dear parents for their irreplaceable support in three decades. The freedom they gave me, as well as the work attitude they taught be made this thesis possible.

Finally, I would like to thank my friends which have been there for me at all times. Thank you for all your solicitousness and for the joy you bring to my life.

1 Introduction

In this thesis, we consider Banach space-valued, almost-additive mappings. Abstractly, these functions are of the form

$$F : \mathcal{F}(G) \to Z,$$

where $\mathcal{F}(G)$ is a collection of finite measure subsets of a locally compact measure space (G, m) and Z is a Banach space with norm $\|\cdot\|_Z$. We will call such a mapping F almost-additive if it possesses certain boundedness and invariance conditions and if for a finite union $Q = \bigsqcup_k Q_k \in \mathcal{F}(G)$ of pairwise disjoint sets $Q_k \in \mathcal{F}(G)$, the difference of the vectors $F(Q)$ and $\sum_k F(Q_k)$ can be controlled with respect to $\|\cdot\|_Z$ by a boundary term. Inspired by elementary convergence theorems for subadditive sequences with values in $Z = \mathbb{R}$ (see e.g. [Gro99], [LW00], [Kri10]), the question arises whether one can find (partial) exhaustions of the space G by a sequence (U_j) of compact subsets such that the limit

$$F^* := \lim_{j\to\infty} \frac{F(U_j)}{m(U_j)}$$

exists in the topology of the Banach space Z. The goal of this dissertation is to give a positive answer to this question in a wide range of geometric situations. We will show results ranging from locally compact amenable groups to the geometrically very general situation of convergent graph sequences. All new assertions have appeared or will appear in one of the papers [Pog13a, Pog13b, PS14, LPS14].
The existence of the above limit provides interesting applications. One such example was given by Lenz in [Len02] for subshift dynamical systems. There, the author defines F on the set of associated words and characterizes unique ergodicity of the subshift by the normalized Banach space convergence of F along increasing boxes. It was observed in [LS05] that Banach space-valued, almost-additive mappings are a valuable tool to prove the uniform approximation of spectral quantities for large classes of self-adjoint operators via spectral data of their finite volume analogues. One such key quantity is given by the integrated density of states (IDS) which is of fundamental importance in the world of mathematical physics. In the context of amenable groups a considerable amount of uniform convergence results along Følner sequences can be found in the literature, see e.g. [KLS03, LS05, LMV08, LSV11]. In the two latter works [LMV08, LSV11], the authors use an abstract Banach space-valued ergodic theorem for their spectral approximation results along Følner sequences in countable amenable groups. Information about the coefficients of the operators under consideration are encoded in a colouring of the group by finitely many colours. As an ergodicity condition, it is assumed there that for all coloured patterns, their occurrence frequencies must exist along the Følner sequence. While the paper [LMV08] covers the case $G = \mathbb{Z}^d$, the convergence statement in [LSV11] holds true for all countable amenable groups containing a monotile Følner sequence with symmetric grid set. The situation of general amenable groups

remained open. One main result of this thesis closes the gap to the general case. Specifically, we show in Theorem 4.4 the Banach space convergence along Følner sequences in all countable, amenable groups. The key ingredient in the proof are ε-quasi tiling techniques developed by Ornstein and Weiss in [OW87] in order to 'approximate' large compact sets in the group by finite unions of nearly disjoint translates of 'smaller' prototile sets. We make use of constructions of Ornstein and Weiss, but we have to extend their results to obtain effective covering estimates. This is done in Theorem 3.2. We prove there the existence of families of ε-quasi tilings for all countable amenable groups such that on average, large compact sets are covered evenly by the translates of prototiles at our disposal. Parts of the covering theorems that we use here have been developed in cooperation with Schwarzenberger. In order to be clear with assigning originality we mention the cooperation at those points. The mentioned results appear in [PS14].
Almost-additive functions on subsets of locally compact, amenable groups can also been interpreted as a generalization of abstract averages appearing in classical ergodic theory. Hence, it is worth raising the question if under some compactness criteria on the Banach space Z (e.g. reflexivity), one can prove extensions of classical mean ergodic theorems. In Theorem 4.15, we give a positive answer to this problem. In fact, we prove the convergence along Følner sequences for almost-additive functions which are compatible with a weakly measurable action of the group on the Banach space. This result is a generalization of the classical ergodic theorems of von Neumann and Greenleaf [Gre73]. Like in the countable case, a major ingredient in the proof are uniform families of ε-quasi tilings of 'large' subsets of the group. Specifically, we show in Theorem 3.4 (which can be seen as an analogue of Proposition I.3.6. [OW87]) that in all amenable, locally compact, second countable unimodular Hausdorff groups, one can construct a family of ε-quasi tilings for 'highly invariant' compact sets T such that on average, the covering densities of the various prototile sets are constant in every part of T. In fact, for the proof of Theorem 4.15, we need more. We will have to work with two families of ε-quasi tilings referring to coverings of the same set and maintaining a certain independence from each other. This leads to the concept of uniform decomposition towers. We prove in Theorem 3.6 that those latter objects can always be constructed in all amenable groups under consideration.
Having the abstract mean ergodic theorem, Theorem 4.15, at hand, it is natural to ask whether there is a concept of a pointwise almost-everywhere convergence theorem for almost-additive functions defined on Bochner spaces Z. We give a positive answer to this question for additive functions F which will be referred to as bounded, additive processes. In Theorem 5.17, we prove the almost-everywhere convergence along increasing Tempelman Følner sequences for approximable processes taking their values in a Bochner space. The quotients in the ergodic theorem are more general than the integral averages appearing in the Lindenstrauss ergodic theorem, cf. [Lin01]. As for integral averages, one can prove pointwise convergence also along Shulman Følner sequences, our pointwise convergence theorem is not a direct generalization of the Lindenstrauss ergodic theorem. However, it extends the classical results in [Tem72, Eme74] and we show convergence for a larger class of ergodic averages, such that Delone point processes, cf. Example 5.7. Moreover, a Banach space-valued Lindenstrauss ergodic theorem can be shown to hold true, cf. Theorem 5.5. We omit the proof in the text since it has essentially been given in the diploma thesis of the author, cf. [Pog10]. The almost-sure convergence of bounded, additive processes has been investigated before in restricted geometric settings. In [Sat99, Sat03], Sato proves a

pointwise ergodic theorem for bounded, additive processes defined on $G = \mathbb{R}^{d+}$, where the semigroup action is dominated by a family of contraction majorants. We draw our attention to special contractions, thus being in a slightly more specific situation from the viewpoint of the underlying dynamics. Then, dealing with abstract amenable groups and with arbitrary increasing Tempelman Følner sequences, we can extend the results in [Sat99, Sat03] to a far more general geometric context. Note that the approximability condition is automatically satisfied in the convergence statements proven in the latter papers. As in the classical case, we prove the pointwise ergodic theorem by linking the mean ergodic theorem, Theorem 4.15, with a dominated ergodic theorem, given in Theorem 5.14. For the proof of the latter theorem, we adapt a proof given in [Kre85] in order to show a weak $(1,1)$-maximal inequality for associated dominating processes. These objects are inspired by classical differentiation theorems, see e.g. [AdJ81, Émi85, Sat98, Sat99, Sat03].

Most of the mentioned new results about uniform decomposition towers, the almost-additive mean ergodic theorem, as well as about the pointwise almost-everywhere convergence theorem for bounded, additive processes are published in [Pog13a].

Both of our Banach space-valued ergodic theorems (Theorems 4.4 and 4.15) can be applied in order to show the uniform existence of the integrated density of states for finite hopping range operators. For countable groups, we stick close to the setting of [LSV11], but we consider a random familiy of operators instead of one deterministic operator. Using Theorem 4.4, we show the uniform IDS approximation for all countable amenable groups in Theorem 6.5. While basic ingredients have already been known, in this form the result is new. For continuous groups we consider the geometric situation of [LV09]. In this work, the authors deal with discrete operators with their ground space being randomly chosen from a metric space with some quasi-isometry to some (not necessarily countable) amenable group. The coefficients of the operator are random as well. We use Corollary 4.16 to reproduce the IDS approximation result of the latter paper. In our text this is stated in Theorem 6.9.

Starting from Chapter 7 we turn to graphs and graphings. In particular, we deal with weakly convergent graph sequences of uniformly bounded vertex degree. This notion of convergence has been introduced by Benjamini and Schramm in the seminal work [BS01]. In the literature one calls these sequences also 'Benjamini-Schramm-convergent'. Precisely, this convergence means the asymptotic existence of the occurrence frequencies for all geometric patterns. Convergent graph sequences might have one deterministic (infinite) graph as a limit. This is for instance true in the case of sofic approximations of groups, see e.g. [Ele06b, Ele08a, ScSc12, AGV14]. However, in general, the limit is not just one countably infinite graph but rather a probability space of countable, bounded degree graphs with some additional structure. Namely, there are finitely many measure preserving involutions which induce a graph structure on the measure space of graphs ('graph of graphs'). This is captured in the notion of graphings, see Definition 7.3. Graphings have been constructed in various kinds and settings, see e.g. [Ele07b, Lov12, LPS14]. In this thesis, we will focus on hyperfinite graph sequences, cf. [Ele08a]. Roughly speaking, a family of graphs is hyperfinite if for all elements in the family, there is a way to delete a uniformly small portion of the edges in all elements in order to obtain (edge-)disjoint components with a uniformly small number of vertices. In [Ele08a], Elek introduced a pseudometric δ which quantifies geometric differences of finite graphs and defined strong convergence for graph sequences as being Cauchy in δ. Further, he conjectured in the same paper that weakly convergent and hyperfinite graph sequences are also strongly convergent. Using techniques of Elek in

[Ele12], we give a detailed proof in Theorem 7.10. A variant of this statement can already be found in Theorem 5 of [Ele12]. There, the author shows that two graphs with the same number of vertices in a hyperfinite family are almost isomorphic, whenever they are statistically close to each other. The main ingredient is the so-called Equipartition Theorem proven by Elek in Theorem 4 of the same paper. For the proof of Theorem 7.10, we base our argumentation on Elek's Equipartition Theorem as well. Another independent proof via algorithmic techniques has recently been given by Newman and Sohler in Theorem 3.1 of [NS13]. These considerations partially solve Conjecture 1 in [Ele08a]. The full conjecture refers to graphs with their vertices and edges coloured by finitely many colours. For its verification, one would have to prove a coloured version of the Equipartition Theorem. Hyperfiniteness is strongly linked with concepts of amenability, see e.g. [ES11].
Being inspired by Theorem 4.4, we then turn to the question whether one can prove Banach space-valued, almost-additive convergence theorems along weakly convergent, hyperfinite graph sequences. We give a positive answer in Theorem 8.2. Being the first assertion of its kind in a geometric situation without group or semigroup structures at hand, this result is a cornerstone in the theory of almost-additive convergence. In the setting of finitely generated amenable groups endowed with a trivial colouring (one colour), Theorem 8.2 is an analogue of Theorem 4.4. Further, we are able to derive a subadditive convergence assertion in Theorem 8.4. This result provides a generalization of a variant of the 'Ornstein-Weiss Lemma' which has been proven and used in the context of dynamical theory for countable amenable groups, cf. [Gro99, LW00, Kri10]. Moreover, Theorem 8.4 extends the geometric framework of Theorem 1.1 in [CSKC12], where the authors prove subadditive convergence along Følner nets in left-cancellative semigroups. Here, we have to pay the price of an additional monotonicity criterion. However, unlike in the latter work, we have to assume subadditivity only for disjoint decompositions of graphs.
As an application, we address the question of approximation of the IDS along weakly convergent graph sequences in Chapter 9. We do not claim originality for the results given in this chapter, but present them in a structured way according to the context of this thesis. In Theorem 9.9, we show the weak convergence of the normalized eigenvalue distributions of finite volume analogues towards the IDS of decomposable, pattern-invariant, finite hopping range operators on graphings. Here, weak convergence means convergence in the topology of weak convergence of measures. This result is not new and proofs of this fact can for instance be found in [Ele08a, Ele08b]. We give a direct and elementary proof which not only shows convergence but also gives an explicit description of the limit distribution. In analogy to Theorem 6.5, we also prove a uniform IDS approximation result for hyperfinite, weakly convergent graph sequences. In this context, Theorem 9.12 upgrades the weak convergence in Theorem 9.9 to uniform convergence of the spectral distribution functions. The fact that strong convergence implies uniform spectral convergence has already been proven in [Ele08a]. Using the Banach space-valued convergence theorem, Theorem 8.2, we provide a different and more structural approach to this question of spectral approximation. Another new ingredient is the fact that weakly convergent, hyperfinite graph sequences are strongly convergent (cf. Theorem 7.10) and hence, they allow for the application of the Banach space-valued convergence theorem. These results also imply the Lück conjecture for hyperfinite graph sequences, see e.g. [Lüc94, DLM$^+$03, Ele06b]. The validity of the conjecture for all (not necessarily hyperfinite) weakly convergent graph sequences is an open problem. Partial results for operators with algebraic integer coefficients can be found in [Tho08, ATV13].

The new main results of the Chapters 7, 8 and 9 are contained in the work [Pog13b].
Chapter 10 of this thesis is devoted to the Ihara Zeta function for graphings. In recent years, various methods have been used to define this function over infinite graphs via approximation. In cooperation with Lenz and Schmidt, the author of this thesis has developed a new approach in [LPS14] which unifies and extends the examples in the literature. More specifically, the authors define the Ihara Zeta function for so-called measure graphs. This latter class contains graphings as a special case. Further, it is proven there that this notion of Zeta function has a continuity property with respect to weak convergence of graphs. Precisely, the normalized versions of Zeta functions for the finite graphs in a weakly convergent graph sequence converge to the Zeta function of the corresponding limit graphing in the topology of uniform convergence on compact sets. As only local quantities need to be considered, hyperfiniteness is not needed. We give a direct and elementary proof of this continuity result for graphings in Theorem 10.5. This result extends all the approximation statements in [CMS02, GŻ04, GIL08, GIL09]. Moreover, our proof identifies the limit function as the Ihara Zeta function of the limit graphing. Further, we show in Theorem 10.8 that every countable graph coming along with a countable sofic subgroup of its automorphisms acting freely and co-finitely on the graph can be approximated by a weakly convergent graph sequence. Combined with Theorem 10.5, this implies the corresponding convergence theorem for the Ihara Zeta function, cf. Theorem 10.7. This generalizes the works of [GIL08] for amenable automorphism groups and of [CMS02] for residually finite groups acting on regular graphs via automorphisms.

The text is organized as follows. In Chapter 2, we start with a brief introduction on amenability of locally compact, second countable, unimodular Hausdorff groups (LCSCUH groups for short). For amenable LCSCUH groups, we extend the ε-quasi tiling techniques of [OW87] by precise covering estimates. Further, we show that 'large' sets in the group can be well approximated (ε-quasi tiled) by translates of compact sets taken from a Følner sequence, cf. Theorem 2.16. In the subsequent Chapter 3, we extend the previous covering results by showing the existence of families of ε-quasi tilings which on average have desirable uniformity properties. To do this, we will deal with countable and with continuous groups separately, cf. Theorems 3.2, 3.4 and 3.6. Using these uniform ε-quasi tilings, we prove almost-additive ergodic theorems in Chapter 4. Again, we proceed separately for countable groups and (possibly) continuous groups. In the countable situation, we prove the ergodic theorem for almost-additive functions along coloured Følner sequences, cf. Theorem 4.4. For continuous groups, we prove an abstract almost-additive mean ergodic theorem in Theorem 4.15. This statement generalizes classical mean ergodic theorems. Chapter 5 is devoted to Banach space-valued, bounded, additive processes on continuous amenable groups. We prove the almost-everywhere convergence for approximable processes along increasing Tempelman Følner sequences, cf. Theorem 5.17. In Chapter 6, we apply both ergodic theorems to prove the uniform approximation of the integrated density of states along finite volume analogues for a large class of random operators. This is done in the Theorems 6.5 and 6.9. Next, we turn to the world of graphs with uniformly bounded vertex degree. In this context, Chapter 7 deals with weak and strong convergence of graphs towards graphings. As an analogue of amenability, we describe the concept of hyperfinite families of graphs at this point. Moreover, we show that weakly convergent and hyperfinite graph sequences must also converge strongly, cf. Theorem 7.10. We use this result to prove almost-additive and subadditive, Banach space-valued convergence theorems for hyperfinite, weakly convergent graph sequences in the Theorems 8.2 and 8.4 of Chapter 8. As in the situation of finitely generated, amenable groups, we show the uniform approximation of the integrated density of states for finite hopping range operators on hyperfinite graphings in Theorem 9.12 of Chapter 9. Chapter 10 is devoted to an investigation of the Ihara Zeta function for graphings. We show the approximation of this function in the topology of uniform convergence on compact sets along weakly convergent graph sequences, cf. Theorem 10.5. Moreover, we construct graph sequences approximating graphs with periodicity induced by sofic automorphism groups in Theorem 10.8. This shows the compact approximation of the Ihara Zeta function for all those graphs, cf. Theorem 10.7. We conclude the thesis in Chapter 11 with a short outline of two open questions arising naturally from our elaborations.

2 Amenable groups

Throughout the whole thesis, we assume that Γ is a locally compact Hausdorff group. That is to say that Γ is a topological group (with its topology making the group multiplication and inversion continuous) with the property that each point possesses a compact neighbourhood and two different points can be separated by disjoint, open neighbourhoods. More specifically, we are interested in groups of this kind which are in addition second countable, i.e. there is a countable basis of the topology of Γ. The neutral element of Γ shall be denoted by e. We write $\mathcal{B}(\Gamma)$ for the Borel σ-algebra generated by the open subsets in Γ. By standard abstract harmonic analysis, one finds (up to constants) exactly one regular measure $m_L(\cdot)$ on $\mathcal{B}(\Gamma)$, called the left Haar measure which is invariant under group multiplication by elements from the left, i.e. $m_L(gA) = m_L(A)$ for every $g \in \Gamma$ and all $A \in \mathcal{B}(\Gamma)$. In most of our considerations, we restrict ourselves to *unimodular* groups, i.e. groups for which the unique Haar measure is both left- and right-invariant. In this case we simply write $|A|$ for the measure of some set $A \in \mathcal{B}(\Gamma)$. When integrating over sets in a unimodular group, we will use the notation $dg := dm_L(g)$. Note that for instance, all discrete and all abelian groups are unimodular. We shall write $\mathcal{F}(\Gamma) := \{A \in \mathcal{B}(\Gamma) \,|\, 0 < |A| < \infty\}$ for the collection of Borel sets in Γ with finite, positive measure. For the cardinality of some finite set $A \subseteq \Gamma$, we will write $\#(A)$. For countable groups, we use the counting measure as Haar measure, i.e. $\#(A) = |A|$ for all finite sets $A \subset \Gamma$. Most of the results in this thesis are stated and proven for locally compact, second countable, unimodular Hausdorff groups. As an abbreviation, we call those latter groups **LCSCUH groups**. We will always deal with groups of infinite Haar measure, i.e. $|\Gamma| = \infty$.

The following chapter is devoted to the presentation of general ε-quasi tiling results for amenable LCSCUH groups. In a first section, we define amenable groups by the existence of weak or strong Følner sequences. Next, we turn to the ε-quasi tiling techniques of Ornstein and Weiss. The goal of these tools is to cover 'large' (in the sense of invariant, see below) compact sets in Γ by 'smaller' tiling sets. Ideally, a large percentage of mass is covered by translates with few overlappings. We extend the results in [OW87] by precise covering and invariance estimates. Our essential idea to do so is to find upper bounds on covered volumes. This leads to the key Lemma 2.13. The main Theorem 2.16 of this chapter is proven in the last section. It is joint work with Schwarzenberger and appears in [PS14], Theorem 4.4. Generalizing the Theorems I.2.6 and I.3.3 in [OW87], the assertion shows that in every amenable LCSCUH group, one can construct ε-quasi tilings for sufficiently invariant compact sets. The finitely generated version of our ε-quasi tiling theory is also presented and used in the Ph.D. thesis of Schwarzenberger, cf. [Sch13].

2.1 Amenable groups

The following section is devoted to a brief introduction of amenable LCSCUH groups. In this context, we introduce a notion for a relative boundary (the K-boundary) of subsets in a group Γ with respect to compact sets and we use this concept to define so-called weak and strong Følner sequences. Those latter objects consist of non-empty, compact sets $T_n \subseteq \Gamma$ which are asymptotically invariant under left-translation by arbitrary compact sets. The existence of weak Følner sequences is commonly referred to as a characterization of amenability of the group. We will use their existence as the definition of amenability, cf. Definition 2.6. In Lemma 2.8 we state that each strong Følner sequence is also a weak Følner sequence and that the groups under consideration always possess a strong Følner sequence, cf. [PS14], Lemma 2.6. Further, we introduce growth conditions on the sequences in Definition 2.9. Assumptions of this kind will play a major role in the proofs of pointwise ergodic theorems, cf. Chapter 5. Most of the results in this section are contained in the papers [Pog13a, PS14].

Definition 2.1 (*K*-boundary).
Let Γ be a locally compact Hausdorff group. Assume that $\emptyset \neq K, T \subseteq \Gamma$ are subsets of Γ. We call the set $\partial_K(T)$, defined by

$$\partial_K(T) := \{g \in \Gamma \mid Kg \cap T \neq \emptyset \text{ and } Kg \cap (\Gamma \setminus T) \neq \emptyset\}$$

*the K-*boundary *of the set T.*

The next proposition is an immediate consequence of the above definition.

Proposition 2.2.
Let Γ be a locally compact Hausdorff group and assume $K, T \subseteq \Gamma$. Then

$$\partial_K(T) = K^{-1}T \cap K^{-1}(\Gamma \setminus T).$$

Proof.
Let $g \in \partial_K(T)$. Then by definition of the K-boundary, there are $k \in K$ and $t \in T$ such that $kg = t$. Hence $g \in K^{-1}T$. Again by definition of the K-boundary one finds $k' \in K$ and $t' \in \Gamma \setminus T$ with $k'g = t'$. Hence $g \in K^{-1}(\Gamma \setminus T)$ and the first inclusion follows. The converse inclusion is trivial. □

We collect some nice and useful properties of the relative boundary definition. For the proof of the following lemma we essentially stick to the presentation in [PS14], Lemma 2.4.

Lemma 2.3 (cf. [PS14], Lemma 2.4).
Let $T, S, K, L \subseteq \Gamma$ be given and assume $g \in \Gamma$. Then the following assertions hold true.

(i) $\partial_K(T) = \partial_K(\Gamma \setminus T)$,

(ii) $\partial_K(S \cup T) \subseteq \partial_K(S) \cup \partial_K(T)$,

(iii) $\partial_K(S \setminus T) \subseteq \partial_K(S) \cup \partial_K(T)$,

(iv) $\partial_K(T) \subseteq \partial_L(T)$ *if* $K \subseteq L$,

(v) $\partial_K(Tg) = \partial_K(T)g$,

(vi) $\partial_K(TS) \subseteq \partial_K(T)S$,

(vii) $\partial_K(T \setminus S) \subseteq \partial_K(T) \cup (\partial_K(S) \cap T)$ *if* $e \in K$,

(viii) $\partial_L(\partial_K(T)) \subseteq \partial_{KL}(T)$.

Proof.
The statements (i) to (iv) follow from Definition 2.1. To see (v), note that

$$Kh \cap S \neq \emptyset \Longleftrightarrow Khg \cap Sg \neq \emptyset$$

for every $h \in \Gamma$ and all sets $S \subseteq \Gamma$. By Definition 2.1, one has $h \in \partial_K(T)$ if and only if $hg \in \partial_K(Tg)$. We obtain $\partial_K(T) = \partial_K(Tg)g^{-1}$. To prove (vi), take $g \in \partial_K(TS)$. Thus $Kg \cap TS \neq \emptyset$ and thus, there is some $c \in S$ with $Kg \cap Tc \neq \emptyset$. Since $Kg \cap (\Gamma \setminus TS) \neq \emptyset$, it follows that $Kg \cap (\Gamma \setminus Tc) \neq \emptyset$. Hence, for every $g \in \partial_K(TS)$, we can find some $c \in S$ such that $g \in \partial_K(Tc)$. By (v), the latter set is equal to $\partial_K(T)c$ and since $c \in S$, we arrive at $g \in \partial_K(T)S$. We turn to the proof of the assertion (vii). So assume that $g \in \partial_K(T \setminus S)$, but $g \notin \partial_K(T)$. Then the fact that $Kg \cap (T \setminus S) \neq \emptyset$ leads to $Kg \cap T \neq \emptyset$, as well as to $Kg \cap (\Gamma \setminus S) \neq \emptyset$. So if $g \notin \partial_K(T)$, this is only possible if $Kg \subseteq T$. Since $e \in K$, it follows that $g \in T$. It remains to show that $Kg \cap S \neq \emptyset$, since then $g \in \partial_K(S)$. Indeed, since $g \in \partial_K(T \setminus S)$, we obtain $Kg \cap (\Gamma \setminus (T \setminus S)) \neq \emptyset$ and from $Kg \subseteq T$, it follows that $Kg \cap (S \cap T) \neq \emptyset$.
For the statement (viii), note that

$$\partial_L(\partial_K(T)) \subseteq \{g \in \Gamma \mid \text{ there is } l \in L : lg \in \partial_K(T)\} \subseteq \bigcup_{l \in L} \partial_{Kl}(T).$$

Now the property (iv) shows the claim. □

In our elaborations, we will have to compute the (Haar) measure of boundaries $\partial_K(T)$. Therefore, we have to know that these sets are in fact measurable. In most cases, we will have to deal with the situation where K is compact and T is an intersection of a closed set with an open set. In the following, we will refer to sets T of the latter kind as *locally closed* sets. The next proposition shows that for compact sets K and locally closed sets T in Γ, the set $\partial_K(T)$ indeed belongs to $\mathcal{B}(\Gamma)$.

Proposition 2.4.
Let Γ be a second countable, locally compact Hausdorff group. Then $\partial_K(S \cap O)$ belongs to $\mathcal{B}(\Gamma)$ whenever $K \subseteq \Gamma$ is compact, $S \subseteq \Gamma$ is closed and $O \subseteq \Gamma$ is open.

Proof.
Note that by Proposition 2.2, we have $\partial_K(S \cap O) = K^{-1}(S \cap O) \cap K^{-1}(\Gamma \setminus (S \cap O))$. Since the group multiplication is compatible with taking unions of sets, we can write

$$\begin{aligned} K^{-1}(\Gamma \setminus (S \cap O)) &= K^{-1}((\Gamma \setminus S) \cup (\Gamma \setminus O)) \\ &= K^{-1}(\Gamma \setminus S) \cup K^{-1}(\Gamma \setminus O). \end{aligned}$$

Since the inversion is continuous in topological groups, K^{-1} is a compact set as well. Hence the set $K^{-1}(\Gamma \setminus (S \cap O))$ is a union of an open and a closed set and therefore, it belongs to $\mathcal{B}(\Gamma)$. It remains to show that the set $K^{-1}(S \cap O)$ belongs to $\mathcal{B}(\Gamma)$. One can equivalently check that $K(S \cap O) \in \mathcal{B}(\Gamma)$ whenever K is compact, S is closed and O is open. Indeed, as Γ is a locally compact Hausdorff space, hence a T_3-space, which in addition is second countable, Urysohn's metrization theorem (cf. e.g. [How95], Theorem 1.4) tells us that Γ is metrizable. In particular, each open set O can be written as a countable union $\cup_n V_n$ of closed sets V_n. It follows from this that

$$K(S \cap O) = K \cdot \Big(\bigcup_{n=1}^{\infty} S \cap V_n \Big) = \bigcup_{n=1}^{\infty} K \cdot (S \cap V_n).$$

Thus, the latter set is also a countable union of closed sets, and hence is Borel measurable. □

For a positive number $0 < \delta < 1$ and a precompact $K \subseteq \Gamma$ (i.e. K has compact closure), we say that $T \subseteq \Gamma$ with $|T| > 0$ is (K, δ)-*invariant* if

$$\frac{|\partial_K(T)|}{|T|} < \delta.$$

We introduce the concept of Følner sequences in groups.

Definition 2.5.
Let (T_n) be a sequence of non-empty, compact subsets of a locally compact, second countable Hausdorff group Γ. Assume further that $|T_n| > 0$ for all $n \in \mathbb{N}$, i.e. $T_n \in \mathcal{F}(\Gamma)$. If

$$\lim_{n \to \infty} \frac{|T_n \triangle K T_n|}{|T_n|} = 0$$

for all non-empty, compact $K \subseteq \Gamma$, then (T_n) is called weak Følner sequence*. Here, $A \triangle B := (A \setminus B) \cup (B \setminus A)$ for sets $A, B \subseteq \Gamma$. If*

$$\lim_{n \to \infty} \frac{|\partial_K(T_n)|}{|T_n|} = 0$$

for all non-empty, compact $K \subseteq \Gamma$, then (T_n) is called strong Følner sequence*. We say that a (weak or strong) Følner sequence (T_n) is* nested *if $e \in T_1$ and $T_n \subseteq T_{n+1}$ for all $n \geq 1$. Each element T_n of a Følner sequence will be called* Følner set*.*

Having the notion of weak Følner sequences at hand, we can define amenability for groups.

Definition 2.6 (Amenability of groups).
Let Γ be a locally compact, second countable Hausdorff group. We say that Γ is amenable *if there is a weak Følner sequence in Γ.*

Let us give some examples for amenable groups.

Examples 2.7.

- Every compact group Γ is amenable and unimodular. A Følner sequence is given by $T_n = \Gamma$ for all $n \in \mathbb{N}$.
- Every abelian group is amenable. This can be shown using the Markov-Kakutani fixed point theorem (cf. [DS88], Theorem V.10.6). Note that abelian groups are also unimodular.
- Every locally compact, second countable, polynomially growing group is unimodular and amenable, cf. [Gui73], Theorem 1.1 and Lemma 1.3.
- There are groups of exponential growth which are amenable as well. One example is the *Lamplighter group*, which is defined as

  $$\Gamma := \mathbb{Z} \wr \mathbb{Z}_2 := \{(m,a) \,|\, m \in \mathbb{Z},\, a \in \bigoplus_{i\in\mathbb{Z}} \mathbb{Z}_2\},$$

  where $\mathbb{Z}_2$ is the cyclic group of order two. Thus, we have $\Gamma = \{(m,a) \,|\, m \in \mathbb{Z}, a \in \mathbb{Z}_2^{\mathbb{Z}}\}$ as a set. With σ as the left shift on the space of all 0-1-sequences over $\mathbb{Z}$ (i.e. $\sigma((x_n)_n) = (x_{n+1})_n$), the group operation is

  $$(m,a) \cdot (n,b) := (m+n, \sigma^n a + b).$$

  A Følner sequence for this group is given by

  $$T_n := \Big\{(m,a) \in \Gamma \,|\, |m| \le n,\, a = \sum_{k=-2n}^{2n} \alpha_k e_k,\, \alpha_k \in \{0,1\}\Big\}$$

  for $n \in \mathbb{N}$, where $e_k = (\delta_{lk})_{l\in\mathbb{Z}}$ is the Kronecker symbol, cf. [Pog10], Example 3.7 (3).
- The free group $\mathbb{F}_r$ of rank $r \in \mathbb{N}$, $r \ge 2$, is *not* amenable, cf. [Gre69], Example 1.2.3.
- Since the matrix groups $SL(2,\mathbb{R})$ and $GL(2,\mathbb{R})$, endowed with discrete topology, contain a subgroup which is isomorphic to the rank two group $\mathbb{F}_2$, they are *not* amenable, see e.g. [Run01], Theorem 1.2.7.

In [OW87], the authors proved that in each amenable group Γ the following holds: given a compact set $K \subseteq \Gamma$ and some positive number $\delta > 0$, there is a compact set T which is (K,δ)-invariant. Therefore, in each amenable, unimodular and second countable group, one always finds strong Følner sequences. Besides this, each strong Følner sequence is also a weak Følner sequence. We collect these observations in the following Lemma. For the proof we essentially stick to the presentation of Lemma 2.6 in [PS14].

Lemma 2.8 (cf. [PS14], Lemma 2.6).
Let Γ be an amenable LCSCUH group. Then the following statements hold true.

(i) There exists a strong Følner sequence in Γ.

(ii) Each strong Følner sequence is a weak Følner sequence.

(iii) If Γ is countable, then every weak Følner sequence is also a strong Følner sequence.

(iv) There exists a nested strong Følner sequence in Γ*.*

Proof.
Let Γ be an amenable LCSCUH group. To prove (i), we first denote by $\{V_n\}$ an enumeration of the countable base of the topology of Γ. Since Γ is locally compact, we can choose the V_n to have compact closure. We now set $K_n := \cup_{j=1}^n \overline{V}_j$. Then each K_n is compact and we have $K_n \subseteq K_{n+1}$ for $n \geq 1$, as well as $\cup_n K_n = \Gamma$. Let $K \subseteq \Gamma$ be compact. We claim that there is some $M \in \mathbb{N}$ such that $K \subseteq K_M$. For the proof, note first that for any $g \in \Gamma$, there is some $n(g) \in \mathbb{N}$ such that $g \in V_{n(g)}$. Hence the union $\cup_{g \in K} V_{n(g)}$ is an open cover of K. Since K is compact there must be a finite subcover $K \subseteq \cup_{j=1}^m V_{n(g_j)} \subseteq \cup_{j=1}^m \overline{V}_{n(g_j)}$. The latter union denotes a compact set and by construction of the K_n, we have $K \subseteq \cup_{j=1}^m \overline{V}_{n(g_j)} \subseteq K_M$, where $M := \max\{n(g_j) \mid j = 1, \dots, m\}$. Take a sequence (ε_n) of positive numbers converging to 0. By the above mentioned statement in [OW87], we find for each $n \in \mathbb{N}$ a compact set F_n such that

$$\frac{|\partial_{K_n}(F_n)|}{|F_n|} < \varepsilon_n.$$

Using property (iv) of Lemma 2.3, we conclude that for all $n \geq M$, one obtains with $K \subseteq K_M \subseteq K_n$ that

$$\frac{|\partial_K(F_n)|}{|F_n|} \leq \frac{|\partial_{K_M}(F_n)|}{|F_n|} \leq \frac{|\partial_{K_n}(F_n)|}{|F_n|} < \varepsilon_n.$$

So, clearly $\lim_{n\to\infty} |\partial_K(F_n)|/|F_n| = 0$. This shows the assertion (i).

To prove statement (ii), it is sufficient to verify the inclusion

$$T \triangle KT \subseteq \partial_{\left(K \cup K^{-1} \cup \{e\}\right)}(T)$$

for all compact sets $K, T \subseteq \Gamma$ of positive measure. Indeed, the set $L_K := K \cup K^{-1} \cup \{e\}$ is compact (it is a finite union of compact sets) and hence the convergence to zero follows from the convergence of strong Følner sequences to zero. To prove the inclusion, suppose first that $g \in KT \setminus T$. Then we can write $g = kt$ for some $k \in K$ and some $t \in T$, but $kt \notin T$. Since $k^{-1} \in L_K$ and $t \in T$, we obtain $L_K g \cap T \neq \emptyset$. Since $e \in L_K$ and $g \notin T$, we have $L_K g \cap (\Gamma \setminus T) \neq \emptyset$. Thus $g \in \partial_{L_K}(T)$. Now assume $g \in T \setminus KT$. Then for all $t \in T$ and every $k \in K$, $g \neq kt$ and therefore $k^{-1}g \notin T$ for all $k \in K$. Since $K^{-1} \subseteq L_K$, one observes $L_K g \cap (\Gamma \setminus T) \neq \emptyset$. As $e \in L_K$ and $g \in T$ we also have $L_K g \cap T \neq \emptyset$. Hence $g \in \partial_{L_K}(T)$.
Let us turn to assertion (iii). Assume that Γ is countable with counting measure $|\cdot|$. Let K, T be arbitrary non-empty, finite subsets of Γ. As above we set $L_K := K \cup K^{-1} \cup \{e\}$.

To show statement (iii), it is sufficient to prove

$$\partial_K(T) \subseteq \partial_{L_K}(T) \subseteq L_K(T \triangle L_K T),$$

since then $|\partial_K(T)| \leq |L_K| \cdot |T \triangle L_K T|$. Note that the first inclusion follows from Lemma 2.3 (iv). To see the second inclusion, take some $g \in \Gamma$ such that $L_K g$ intersects non-trivially both T and $\Gamma \setminus T$. By the symmetry of L_K, we have $g \in L_K^{-1} T = L_K T$. If $g \notin T$, then

$g \in L_K T \setminus T$ and since $e \in L_K$, we have proven the claim for this case. If $g \in T$, find some $k \in L_K$ such that $kg \in L_K T \setminus T$, which exists since $L_K g \cap (\Gamma \setminus T)$ is non-empty. Again by the symmetry of L_K, we have $g \in L_K(L_K T \setminus T)$. Now we have shown $g \in L_K(T \triangle L_K T)$ in both cases and this proves part (iii) of the Lemma.

Now we finally prove the claim (iv). Let (F_n) be a strong Følner sequence in Γ which exists by the assertion (i). We choose some $h \in F_1$ and set $T_1 := F_1 h^{-1}$, then we proceed inductively. If $T_1, \dots, T_k$ are chosen, then there is an $n \in \mathbb{N}$ such that F_n is $(T_k, 1/2)$-invariant. As $e \in T_k$ we have $F_n \setminus \partial_{T_k}(F_n) \subseteq \{g \in F_n \mid T_k g \subseteq F_n\} =: S$. Using $|F_n \setminus \partial_{T_k}(F_n)| \geq |F_n| - |\partial_{T_k}(F_n)| > 0$, this yields that S has positive Haar measure. Hence S is non-empty and we find some $g \in S$. Define $T_{k+1} := F_n g^{-1}$. Proceeding in this manner, we obtain a sequence (T_n) which is nested by construction. Further (T_n) is a strong Følner sequence as it is up to shifts a subsequence of (F_n). Note that here we used unimodularity of Γ and (v) of Lemma 2.3 stating that shifts via group multiplication do not change the measure of the K-boundary. □

Remark.
Lemma 2.8 shows that in an LCSCUH group, amenability is equivalent to both the existence of a weak Følner sequence, as well as to the existence of a strong Følner sequence. In the case of countable amenable groups we will only speak about Følner sequences as the specifications 'weak' and 'strong' are dispensable then.

It is a well-known fact that one cannot expect pointwise ergodic theorems for arbitrary Følner sequences, cf. [Eme74]. Hence, it is important for our purposes to impose some growth conditions on the Følner sequences under consideration.

Definition 2.9.
Let Γ be a second countable, unimodular, amenable group and assume that (T_n) is a weak or strong Følner sequence in Γ.

- *We say that (T_n) fulfills the* Tempelman *condition if there is a constant $C > 0$ such that*
$$\Big| \bigcup_{i \leq N} T_i^{-1} T_N \Big| \leq C\, |T_N|$$
for all $N \in \mathbb{N}$.
- *We say that (T_n) fulfills the* Shulman *condition if there is a constant $\tilde{C} > 0$ such that*
$$\Big| \bigcup_{i < N} T_i^{-1} T_N \Big| \leq \tilde{C}\, |T_N|$$
for all $N \in \mathbb{N}$. In this case, we say that (T_n) is a tempered *Følner sequence.*

Remark.
Note that the Tempelman condition is stronger than the Shulman condition. Moreover, tempered weak Følner sequences always exist in second countable amenable groups, cf. [Lin01]. However, there are second countable amenable groups without Tempelman Følner sequences. On the other hand, as the following Theorem shows, there are reasonable sufficient conditions on the group for the existence of Tempelman Følner sequences.

Theorem 2.10 ([Hoc07], Theorem 3.4).
If for a countable, abelian, amenable group Γ *we have*

$$r(\Gamma) := \sup\{n \in \mathbb{N} \mid \Gamma \text{ contains a subgroup isomorphic to } \mathbb{Z}^n\} < \infty,$$

then Γ *possesses at least one Tempelman Følner sequence.*

Remark.
The number $r(\Gamma)$ is called the *abelian rank* of Γ.

2.2 Ornstein Weiss tiling lemmas

In the following, we use ε-quasi tiling techniques for amenable groups developed by Ornstein and Weiss in [OW87]. The goal of the subsequent subsection is to cover 'highly invariant', compact sets $T \subseteq \Gamma$ (e.g. sets with large index in a Følner sequence) by Følner set translates $T_i c$, $1 \leq i \leq N$, $c \in C_i$ where N is an integer number, the C_i are finite subsets of Γ and the T_i are taken from a Følner sequence in Γ. Further, one wants those T_i-translates to have small overlaps and the union $\cup_i \cup_c T_i c$ shall cover most of T. The classical ε-quasi tiling theory shows how this can be done in unimodular amenable groups. More precisely, for small $\varepsilon > 0$ one obtains ε-disjoint (cf. Definition 2.12) translates that cover a portion of $(1 - 2\varepsilon)$ of the mass of T, cf. e.g. [OW87], Theorem I.3.3. For the construction, one first proves covering results for one single level $i \in \{1, \dots, N\}$. This section is devoted to an extension of the basic lemmas of the sections I.2 and I.3 in [OW87] to obtain precise covering and invariance estimates for the tiling sets $\{T_i c\}_{c \in C_i}$, where $1 \leq i \leq N$ is fixed.

We start with a lemma that can be proven along the lines of [OW87], Section I.2.

Lemma 2.11.
Let Γ *be a unimodular, amenable group with unit element* e*. Let some* $\delta > 0$ *be given and assume that* $K \subseteq \Gamma$ *is compact. Further, suppose that there is a* (K, δ)*-invariant set* $T \in \mathcal{F}(\Gamma)$ *such that the boundary* $\partial_K(T)$ *belongs to* $\mathcal{B}(\Gamma)$*. Define the set*

$$\hat{A}_K := \{h \in T \mid Kh \subseteq T\}.$$

Then the following assertions hold true.

(i) For all $z \in \Gamma$,

$$\int_\Gamma \mathbb{1}_{Kh}(z)\, dh = |K|. \tag{2.1}$$

(ii) If $e \in K$*, then* $\hat{A}_K \in \mathcal{B}(\Gamma)$ *and* $|\hat{A}_K| \geq (1 - \delta)|T|$.

(iii) If $e \in K$*, then for all* $S \in \mathcal{F}(\Gamma)$*, there is an element* $c \in \hat{A}_K$ *with*

$$|Kc \cap S| \leq \frac{|S|\,|K|}{(1 - \delta)|T|}. \tag{2.2}$$

Proof.
For the assertion (i), we compute using unimodularity

$$\int_\Gamma \mathbb{1}_{Kh}(z)\,dh = \int_\Gamma \mathbb{1}_{z^{-1}K}(h^{-1})\,dh = \int_\Gamma \mathbb{1}_{z^{-1}K}(h)\,dh = |z^{-1}K| = |K|.$$

To show (ii), note first that as $e \in K$, we have for all $h \in T$ that $Kh \cap T \neq \emptyset$. Thus, $\hat{A}_K = T \setminus (T \cap \partial_K(T)) \in \mathcal{B}(\Gamma)$ by the measurability assumption on $\partial_K(T)$. Since T is (K,δ)-invariant, we compute

$$\begin{aligned} |\hat{A}_K| &\geq |\{h \in T \,|\, Kh \cap T \neq \emptyset\}| - |\partial_K(T)| \\ &\geq |T| - \delta|T| = (1-\delta)|T|. \end{aligned}$$

Let us turn to the proof of the statement (iii). Assume that there exists a set $S \in \mathcal{F}(\Gamma)$ such that no $c \in \hat{A}_K$ satisfies the inequality (2.2). By (ii), $|\hat{A}_K| \geq (1-\delta)|T|$, and hence

$$\int_{\hat{A}_K} |Kc \cap S|\,dc > \int_{\hat{A}_K} \frac{|S||K|}{|T|(1-\delta)}\,dc = \frac{|S||K||\hat{A}_K|}{|T|(1-\delta)} \geq |S||K|. \tag{2.3}$$

However, on the other hand, it follows from Fubini's theorem and equality (2.1) that

$$\int_{\hat{A}_K} |Kc \cap S|\,dc = \int_{\hat{A}_K}\int_S \mathbb{1}_{Kc}(h)\,dh\,dc = \int_S\int_{\hat{A}_K} \mathbb{1}_{Kc}(h)\,dc\,dh \ \leq |S||K|,$$

which clearly is a contradiction to the strict inequality (2.3). □

Next, we introduce the notion of ε-disjoint sets, as well as of finite, ε-disjoint families of sets.

Definition 2.12 (ε-disjoint families).
Let $0 < \varepsilon < 1$ be a positive number and assume $N \in \mathbb{N}$. We say that a finite family $\{T_i\}_{i=1}^N$ of sets in $\mathcal{F}(\Gamma)$ is ε-disjoint if for all $1 \leq i \leq N$, there is a measurable set $\bar{T}_i \subseteq T_i$ such that $|\bar{T}_i| \geq (1-\varepsilon)|T_i|$ for all $1 \leq i \leq N$ and

$$\bigcup_{i=1}^N T_i = \bigsqcup_{i=1}^N \bar{T}_i,$$

where the latter union consists of pairwise disjoint sets. For $0 < \alpha < 1$, we say that a set $T \in \mathcal{F}(\Gamma)$ α-covers a set $S \in \mathcal{F}(\Gamma)$ if $|T \cap S| \geq \alpha|S|$.

Remark.
In the literature, one often finds a slightly weaker notion for ε-disjointness. For instance in [OW87], two finite measure subsets $T_1, T_2 \subseteq \Gamma$ are defined as ε-disjoint if there are subsets $\tilde{T}_i \subseteq T_i$ such that $\tilde{T}_1 \cap \tilde{T}_2 \neq \emptyset$ and $|\tilde{T}_i| \geq (1-\varepsilon)|T_i|$, $i \in \{1,2\}$. However, it is easy to construct sets from those $\tilde{T}_1$ and $\tilde{T}_2$ which satisfy Definition 2.12 with $N = 2$.

The following assertion is the main result of this section. It extends Lemma I.3.2. in [OW87] by two features. Firstly, we make sure that the ε-quasi tiles can even be made *disjoint* in a way such that they preserve certain invariance properties with respect to a fixed compact set. The techniques needed in the proof have been developed in private communication of the author with Weiss. Secondly, we do not only prove a lower bound for the mass proportion of some set T covered by ε-disjoint translates of the T_i, but we also show an *upper bound*. This is the essential step in order to compute precise covering densities in Theorem 2.16. For the proof of Lemma 2.13, we essentially follow the presentation of the proof of Lemma 3.2 in [PS14].

Lemma 2.13 (cf. [PS14], Lemma 3.2).
Let Γ be some unimodular, second countable Hausdorff group, $0 < \varepsilon, \delta < 1/2$ and $0 < \zeta < \delta/2$. Furthermore, let $T \in \mathcal{F}(\Gamma)$ and $K, B \subseteq \Gamma$ be compact sets such that $\partial_K(T), \partial_{KK^{-1}}(T) \in \mathcal{B}(\Gamma)$ and such that T is (KK^{-1}, δ)-invariant, K is (B, ζ^2)-invariant and let the sets K and B contain the unit element e. Then we can find finitely many elements $c_j, j = 1, \dots, n$ in T such that

(i) $Kc_j \subseteq T$, $j = 1, \dots, n$,

(ii) $\{Kc_j\}_{j=1}^n$ *is an ε-disjoint family of sets,*

(iii) for all $j = 1, \dots, n$, there is some measurable set $K_j \subseteq K$ with $|K_j| \geq (1 - \varepsilon)|K|$ such that

- *K_j is a locally closed set,*
- $\partial_B(K_j) \in \mathcal{B}(\Gamma)$,
- *K_j is $(B, 4\zeta)$-invariant,*
- $|\partial_B(K_j)| \leq |\partial_B(K)| + \zeta|K|$,
- *$\bigcup_{j=1}^l Kc_j = \bigsqcup_{j=1}^l K_j c_j$ for all $1 \leq l \leq n$ and the latter union consists of pairwise disjoint sets,*

(iv) $(\varepsilon - \delta)|T| \leq \left|\bigcup_{j=1}^n Kc_j\right| \leq (\varepsilon + \delta)|T|$.

Remark.
We could have stated the lemma without regarding the additional invariance conditions of K with respect to B. However, we will heavily use these properties in the main theorems of this and the next chapter.

Proof.
We start the proof with a simple observation to estimate the ratio $|K|/|T|$. For each $g \in \partial_K(T)$ and $t \in K$ we have $tg \in \partial_{KK^{-1}}(T)$, which immediately gives $|K| \leq |\partial_{KK^{-1}}(T)|$. This implies

$$\frac{|K|}{|T|} \leq \frac{|\partial_{KK^{-1}}(T)|}{|T|} < \delta, \tag{2.4}$$

as T is (KK^{-1}, δ)-invariant.

Now we formulate the following
Claim: if $c_j \in T$, $j = 1, \ldots, n$ are elements which fulfill conditions (i)-(iii) and

$$\Big|\bigcup_{j=1}^{n} Kc_j\Big| < \varepsilon(1-2\delta)|T|,$$

then there exists some $c_{n+1} \in T$ such that (i)-(iii) still hold true for c_j, $j = 1, \ldots, n+1$.

We postpone the proof of the claim and we assume for the moment that it holds. In this case, we start with some maximal disjoint family $\{Kc_j\}_{j=1}^{n}$ of translates of K contained in T with $n|K| \leq (\varepsilon + \delta)|T|$ and set $K_j := K$, $j = 1, \ldots, n$. Then obviously (i)-(iii) hold. It remains to check condition (iv). If

$$\Big|\bigcup_{j=1}^{n} Kc_j\Big| \geq \varepsilon(1-2\delta)|T|,$$

then we are done with the proof since $\varepsilon \leq 1/2$. Otherwise, we apply the claim and get some $c_{n+1} \in T$ such that conditions (i)-(iii) are still fulfilled for c_j, $j = 1, \ldots, n+1$. Moreover, it follows from the inequality (2.4) that

$$\Big|\bigcup_{j=1}^{n+1} Kc_j\Big| \leq \varepsilon(1-2\delta)|T| + \delta|T| \leq (\varepsilon+\delta)|T|.$$

If also the lower bound in condition (iv) is satisfied for $c_1, \ldots, c_{n+1}$, then we are done, if not, then we apply the claim again. This procedure will end after finitely many steps since T has finite measure and after each iteration we cover at least $(1-\varepsilon)|K|$ more than before. Thus, the only thing left to do is to prove the claim.

Let $c_j \in T$, $j = 1, \ldots, n$ be such that (i)-(iii) hold with sets K_j, $j = 1, \ldots, n$ and $|A| < \varepsilon(1-2\delta)|T|$, where $A := \bigcup_{j=1}^{n} Kc_j$. We set $S := \{g \in T \mid Kg \subseteq T\}$. If we had stated the lemma without the additional invariance properties of K with respect to B, we could apply Lemma 2.11 right away in order to finish the proof. In order to obtain the desired invariance conditions, we define the set

$$U := \Big\{ g \in S \,\Big|\, \frac{|Kg \cap \partial_B(A)|}{|K|} \leq \zeta \Big\}$$

which depends on ζ and the set B. Note that $K \subseteq KK^{-1}$. By Lemma 2.3 (iv) and by the fact that T is (KK^{-1}, δ)-invariant, we have that T is also (K, δ)-invariant. By Lemma 2.11 (ii), S is Borel measurable and $|S| \geq (1-\delta)|T|$. Since the group multiplication is continuous, it follows from the continuity of the Haar measure that U is Borel measurable as well. Further, by the definition of U, it follows with $T \setminus U \subseteq (T \setminus S) \cup (S \setminus U)$ that

$$\frac{|T \setminus U|}{|T|} \leq \frac{|T \setminus S|}{|T|} + \frac{|S \setminus U|}{|T|} \leq \delta + \int_S \frac{\mathbb{1}_{S\setminus U}(g)}{|T|}\, dg \leq \delta + \int_S \frac{|Kg \cap \partial_B(A)|}{\zeta|T|\,|K|}\, dg.$$

Here we used the definition of U which yields

$$\mathbb{1}_{S\setminus U}(g) \leq \frac{|Kg \cap \partial_B(A)|}{\zeta|K|}.$$

By Fubini's theorem and assertion (i) of Lemma 2.11, we arrive at

$$\begin{aligned}\int_S \frac{|Kg \cap \partial_B(A)|}{\zeta|T|\,|K|}\,dg &= \frac{1}{\zeta|T||K|}\int_S\int_{\partial_B(A)} \mathbb{1}_{Kg}(h)\,dh\,dg \\ &= \frac{1}{\zeta|T||K|}\int_{\partial_B(A)}\int_S \mathbb{1}_{Kg}(h)\,dg\,dh \\ \text{(Lemma 2.11 (i))} \quad &\leq \frac{|\partial_B(A)|}{\zeta|T|}.\end{aligned}$$

By ε-disjointness, the maximal number n of translates of K that can belong to A is bounded by $|T|/[(1-\varepsilon)|K|]$ such that with Lemma 2.3 (ii), we arrive at

$$\frac{|T \setminus U|}{|T|} \leq \frac{|\partial_B(\bigcup_{j=1}^n Kc_j)|}{\zeta|T|} \overset{\text{L. 2.3 (ii)}}{\leq} \delta + \frac{n\,|\partial_B(K)|}{\zeta|T|} \leq \delta + \frac{|\partial_B(K)|}{\zeta(1-\varepsilon)|K|} \leq \delta + \frac{\zeta}{(1-\varepsilon)} \leq 2\delta,$$

where the last inequality follows from $\varepsilon, \delta < 1/2$, $\zeta < \delta/2$ and the fact that K was chosen to be (B, ζ^2)-invariant. This yields $|U|/|T| \geq 1 - 2\delta$ which allows us to apply Lemma 2.11 (iii) to find some $c_{n+1} \in U$ such that

$$|Kc_{n+1} \cap A| \leq \frac{|A||K|}{|T|(1-2\delta)} < \varepsilon|K|$$

and hence condition (ii) holds for c_j, $j = 1, \ldots, n+1$. As $c_{n+1} \in S$, we have $Kc_{n+1} \subseteq T$, which gives (i). We set $K_{n+1} := (Kc_{n+1} \setminus A)\,c_{n+1}^{-1}$. Clearly, $K_{n+1} \in \mathcal{B}(\Gamma)$ and by the above inequality we get $|K_{n+1}| \geq (1-\varepsilon)|K|$. Also, K_{n+1} is a locally closed set, and by Proposition 2.4, the set $\partial_B(K_{n+1})$ is indeed measurable. Thus, with the statement (vii) of Lemma 2.3 and with $c_{n+1} \in U$, we have

$$|\partial_B(K_{n+1})| = |\partial_B(K \setminus Ac_{n+1}^{-1})| \leq |K \cap \partial_B(Ac_{n+1}^{-1})| + |\partial_B(K)| \leq \zeta|K| + |\partial_B(K)|$$

and using $0 < \varepsilon < 1/2$, one obtains

$$\frac{|\partial_B(K_{n+1})|}{|K_{n+1}|} \leq \frac{\zeta|K|}{(1-\varepsilon)|K|} + \frac{|\partial_B(K)|}{(1-\varepsilon)|K|} \leq 2\zeta + 2\zeta^2 \leq 4\zeta.$$

Thus (iii) holds as well and the claim is proven. □

In order to construct ε-quasi tilings for compact sets in the group Γ, we will use Lemma 2.13 inductively. The next lemma provides the necessary interfaces. This is joint work with Schwarzenberger and is contained in [PS14], Lemma 3.3.

Lemma 2.14 (cf. [PS14], Lemma 3.3).
Let Γ be some unimodular, second countable locally compact Hausdorff group, $0 < \varepsilon, \delta < 1/6$, $0 < \zeta < \delta/4$ and $\eta > 0$. Furthermore, let $T \in \mathcal{F}(\Gamma)$ be a locally closed set and assume that $K, L, B \subseteq \Gamma$ are compact sets with $e \in L \subseteq K$, $e \in B$. Moreover, let T be (KK^{-1}, δ)-invariant and K be (LL^{-1}, η)-invariant, as well as (B, ζ^2)-invariant. Then there is a finite set $C \in \mathcal{F}(\Gamma)$ such that $T \setminus KC$ is $(LL^{-1}, 2\delta + \eta)$-invariant and the properties (i) to (iv) of Lemma 2.13 are satisfied: $\{Kc\}_{c\in C}$ is an ε-disjoint family, $Kc \subseteq T$ for all $c \in C$ and

$$\varepsilon - \delta \leq \frac{|KC|}{|T|} \leq \varepsilon + \delta$$

holds. Furthermore, for each $c_j \in C$ *there is a measurable, locally closed set* $K_j \subseteq K$ *with* $\partial_B(K_j) \in \mathcal{B}(\Gamma)$ *which is* $(B, 4\zeta)$*-invariant, satisfies* $|K_j| \geq (1-\varepsilon)|K|$ *and* $|\partial_B(K_j)| \leq |\partial_B(K)| + \zeta|K|$ *and the sets* $K_j c_j$, $c_j \in C$ *are pairwise disjoint with* $KC = \bigsqcup_{j=1}^n K_{c_j} c_j$.

Proof.
Note that the sets $\partial_K(T)$ and $\partial_{KK^{-1}}(T)$ are measurable by Proposition 2.4. Thus, the assumptions of Lemma 2.13 are satisfied, and we get a set $C = \{c_1, \ldots, c_n\}$ such that the properties (i) to (iv) therein are fulfilled. It remains to prove that $T \setminus KC$ is $(LL^{-1}, 2\delta+\eta)$-invariant. To do so, note first that since T is a locally closed set, so is $T \setminus KC$ and it follows from Proposition 2.4 that the sets $\partial_{LL^{-1}}(T)$ and $\partial_{LL^{-1}}(T \setminus KC)$ must be measurable. We use the properties of the sets K_j, $1 \leq j \leq n$ to obtain

$$|T| \geq \frac{|KC|}{\varepsilon+\delta} = \frac{\sum_{j=1}^n |K_j c_j|}{\varepsilon+\delta} \geq \frac{1-\varepsilon}{\varepsilon+\delta}|K| \cdot n.$$

Therefore, using the upper bounds on ε and δ, we can compute

$$\frac{n}{|T|-|KC|} \leq \frac{n}{|T|-|K|n} \leq \frac{n}{(\frac{1-\varepsilon}{\varepsilon+\delta}-1)|K|n} = \frac{\varepsilon+\delta}{(1-2\varepsilon-\delta)|K|} \leq \frac{1}{|K|}. \tag{2.5}$$

Now apply the statements (iii) and (vi) of Lemma 2.3 to obtain

$$\frac{|\partial_{LL^{-1}}(T \setminus KC)|}{|T \setminus KC|} \leq \frac{|\partial_{LL^{-1}}(T)|}{|T \setminus KC|} + \frac{|\partial_{LL^{-1}}(KC)|}{|T \setminus KC|} \leq \frac{|\partial_{LL^{-1}}(T)|}{|T|-|KC|} + \frac{n \cdot |\partial_{LL^{-1}}(K)|}{|T|-|KC|}. \tag{2.6}$$

It follows from Lemma 2.13 (iv) that $|T| - |KC| \geq (1-(\varepsilon+\delta))|T|$. Combining this fact with inequality (2.5), we deduce from (2.6) that

$$\frac{|\partial_{LL^{-1}}(T \setminus KC)|}{|T \setminus KC|} \leq \frac{|\partial_{LL^{-1}}(T)|}{(1-(\varepsilon+\delta))\,|T|} + \frac{|\partial_{LL^{-1}}(K)|}{|K|} \leq 2\delta+\eta,$$

which shows our claim. Note that here, we used that T is (LL^{-1}, δ)-invariant since $L \subseteq K$ and since T is (KK^{-1}, δ)-invariant, cf. Lemma 2.3, (iv). □

2.3 The special tiling property

The results of the previous section put us in the position to obtain precise and effective ε-quasi tiling results for amenable LCSCUH groups. We prove that sufficiently invariant, compact sets $T \subseteq \Gamma$ can be arbitrarily well 'approximated' by finite, 'almost-disjoint' unions of Følner set translates $T_i c$. We proceed as follows. We say that an LCSCUH group has the *special tiling property* if highly invariant sets can be 'well approximated' by translates of Følner sets. This is made precise in Definition 2.15 which also enumerates all nice covering and invariance properties that we need for our later purposes. Then, in Theorem 2.16, we show that *all* amenable LCSCUH groups have the special tiling property. This generalizes in part the Theorems I.2.6 and I.3.3 in [OW87]. One new feature here is the fact that for each ε-prototile T_i we can precisely control the portion of mass of T covered by the collection of the corresponding T_i-translates. Moreover, we give an explicit construction for tilings of

T consisting of *pairwise disjoint* subsets of the T_i-translates preserving certain invariance conditions which respect to a fixed finite-measure set. The achievements of this section are joint work with SCHWARZENBERGER and will appear in [PS14] (cf. Theorem 4.4).

In the sequel we will use the following notation. For a positive, real number $w \in \mathbb{R}$, we write

$$\lceil w \rceil := \min\{l \in \mathbb{N} \,|\, l \geq w\}$$

for the smallest integer number $l \in \mathbb{N}$ greater or equal than w. We now define the special tiling property for amenable LCSCUH groups.

Definition 2.15 (Special tiling property (STP)).
Let Γ be an amenable LCSCUH group with unit element e. Then we say that Γ has the special tiling property *if for all given $\beta, \varepsilon > 0$, $0 < \beta < \varepsilon \leq 1/10$ and every nested Følner sequence $(S_n)_n$ in Γ, the following holds true: for $N(\varepsilon) := \lceil \log(\varepsilon)/\log(1-\varepsilon) \rceil \in \mathbb{N}$, there is a collection*

$$\{e\} \subseteq T_1 \subseteq \cdots \subseteq T_{N(\varepsilon)}, \qquad T_i \in \{S_n \,|\, n \geq i\} \quad (1 \leq i \leq N(\varepsilon))$$

of $N(\varepsilon)$ sets taken from (S_n), as well as some $\delta_0 = \delta(\varepsilon, \beta) > 0$ such that for all $0 < \delta < \delta_0$ and for every $(T_{N(\varepsilon)} T_{N(\varepsilon)}^{-1}, \delta)$-invariant compact set $T \subseteq \Gamma$, we can find finite center sets $C_i^T \subseteq T$, $1 \leq i \leq N(\varepsilon)$ such that

(i) $\bigcup_{c \in C_i^T} T_i c \subseteq T$ *for all* $1 \leq i \leq N(\varepsilon)$,

(ii) $\{T_i c\}_{c \in C_i^T}$ *is an ε-disjoint family of sets for all* $1 \leq i \leq N(\varepsilon)$,

(iii) the collection $\{T_i C_i^T\}_{i=1}^{N(\varepsilon)}$ *is a disjoint family of sets in* Γ,

(iv) $\left| \frac{|T_i C_i^T|}{|T|} - \eta_i(\varepsilon) \right| < \beta$ *for all* $1 \leq i \leq N(\varepsilon)$, *where* $\eta_i(\varepsilon) := \varepsilon(1-\varepsilon)^{N(\varepsilon)-i}$.

In this situation, we say that $\{T_i\}_{i=1}^{N(\varepsilon)}$ is a family of ε-prototiles for Γ and if for some compact $T \subseteq \Gamma$, the properties (i)-(iv) hold, we say that T has the special tiling property (STP) *with respect to $(\{T_i\}_{i=1}^{N(\varepsilon)}, (S_n)_n, \varepsilon, \beta)$ and that T is* ε-quasi tiled *(with parameter β) by the compact ε-prototiles T_i with finite center sets C_i^T.*

Roughly speaking, the special tiling property of a group means the following: one fixes arbitrary $0 < \varepsilon < 1/10$. Then one computes $N(\varepsilon) := \lceil \log(\varepsilon)/\log(1-\varepsilon) \rceil$ and one finds a finite sequence $\{T_i\}_{i=1}^{N(\varepsilon)}$ of ε-prototiles. For every $1 \leq i \leq N(\varepsilon)$ we get the densities $\eta_i(\varepsilon) := \varepsilon(1-\varepsilon)^{N(\varepsilon)-i}$. Now the special tiling property guarantees that for each $\beta << \varepsilon$ there is an invariance condition with respect to the $\{T_i\}$ such that whenever a compact set $T \in \mathcal{F}(\Gamma)$ satisfies this condition, it can be ε-quasi tiled by the ε-prototiles $\{T_i\}$ in such a way that for every $1 \leq i \leq N(\varepsilon)$, the mass proportion of T covered by the translates $T_i c$ is β-close to the value $\eta_i(\varepsilon)$.

A short computation shows that whenever some set $T \subseteq \Gamma$ is ε-quasi tiled by sets $\{T_i\}$ with center sets C_i^T according to Definition 2.15, then we can choose $\beta > 0$ in such a way that T

is $(1-2\varepsilon)$-covered by the T_i-translates, i.e.

$$\Big|\bigcup_{i=1}^{N(\varepsilon)} T_i C_i^T\Big| = \sum_{i=1}^{N(\varepsilon)} \Big|T_i C_i^T\Big| \geq (1-2\varepsilon)\,|T|. \tag{2.7}$$

Indeed, set $\beta := \varepsilon(2N(\varepsilon))^{-1}$ to obtain

$$\begin{aligned}\sum_{i=1}^{N(\varepsilon)} \Big|T_i C_i^T\Big| &\geq \sum_{i=1}^{N(\varepsilon)} \Big(\eta_i(\varepsilon)-\beta\Big)\,|T| \\ &\geq \Big(\sum_{i=1}^{N(\varepsilon)} \varepsilon(1-\varepsilon)^{N(\varepsilon)-i}\Big)|T| - \varepsilon\,|T| \geq (1-\varepsilon)|T| - \varepsilon\,|T|,\end{aligned}$$

cf. Remark 4.3 in [PS14].

In the main theorem of this section, we extend Theorem I.2.6 in [OW87]. It indicates that *all* amenable LCSCUH groups have the special tiling property. It is taken from [PS14], cf. Theorem 4.4. The proof contains ideas of boths authors. The idea to impose the upper bound $\varepsilon+\delta$ in Lemma 2.13 (see assertion (iv)) is due to the author of this thesis. This is the key ingredient for the precise computation of the covering densities being equal to $\eta_i(\varepsilon)$. For finitely generated groups, the proof of Theorem 2.16 has been depicted in Schwarzenberger's Ph.D. thesis, cf. [Sch13], proof of Theorem 5.20 in the appendix. Before stating and proving the theorem, we compare the techniques of Ornstein and Weiss with the present construction in two pictures. We start with a qualitative illustration of a 'typical' Ornstein/Weiss ε-quasi tiling.

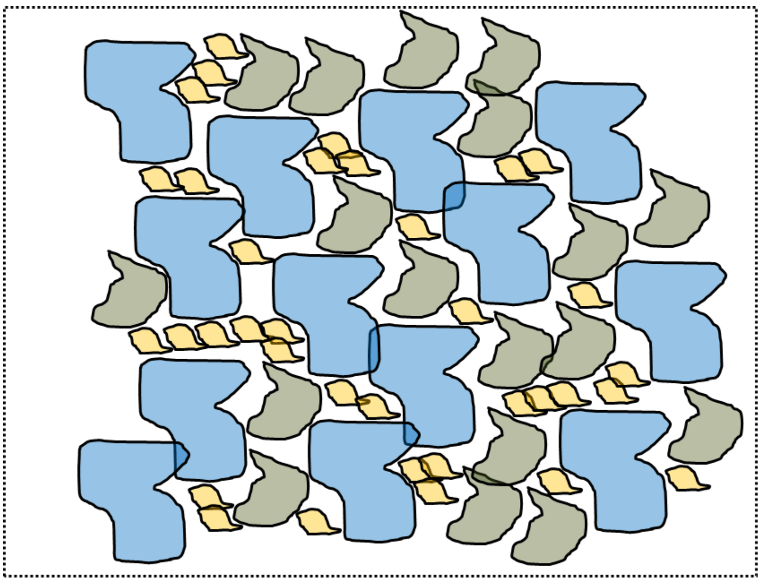

Figure 2.1: Ornstein/Weiss technique

Figure 2.1 shows a cutout of an ε-quasi tiling of some highly invariant compact set T. This cutout is indicated by the dashed line. For practical reasons, we have only drawn the translates of the three smallest ε-prototiles T_3 (blue), T_2 (green) and T_1 (yellow). (So one can imagine a very small cutout in the set which has much less mass than the next bigger ε-prototile T_4.) Note that translates of the same kind (i.e. of the same colour) may intersect each other in a small portion, while translates of different colours are pairwise disjoint. The construction in [OW87] is based on the following procedure. In all parts of the original set which have not yet been 'touched' by translates of ε-prototiles, one covers as much as possible with ε-disjoint translates of blue ε-prototiles. If it is not possible to include another blue translate in this way, one turns to translates of the next smaller ε-prototile which are marked green in the picture. Again, one finds a covering which is maximal in the sense that one cannot include another green translate with no forbidden intersections with the sets which have already been included. In the last step, one fills in the remaining gaps with translates of the yellow ε-prototile. It is just due to this construction that the total mass of all the translates of some bigger ε-prototile will in most cases be much bigger than the total mass of all the translates of some smaller ε-prototile. In the above figure, this can be seen clearly by comparing the total mass of the blue translates with the total mass covered by the yellow translates. By imposing upper bounds on the covered portions (as described above), we avoid this effect in ε-quasi tilings of the kind as described in Definition 2.15. In particular, the total mass covered by blue, green and yellow translates will be the same up to a very small factor of at most $(1 \pm 2\varepsilon)$. Consequently, one finds much more smaller than bigger translates in these ε-quasi tilings. The qualitative picture then looks quite different, see Figure 2.2.

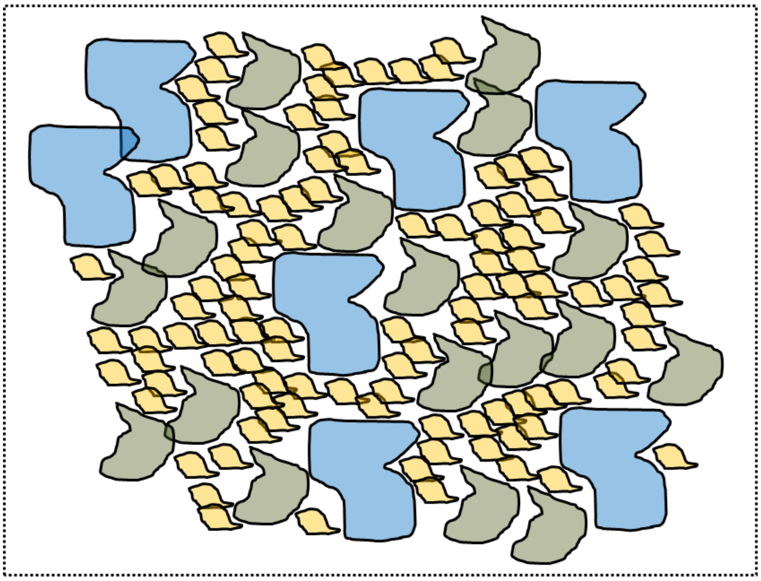

Figure 2.2: ε-quasi tiling as in Definition 2.15

Note that this heuristic reasoning with qualitative pictures presumes that the small cutouts are in fact representative ('typical') for the whole ε-quasi tiling. In general, this is not the case for every single cutout. However, there is a way to take care of this issue. Namely, in Chapter 3, we construct families of ε-quasi tilings which their 'average cutout' satisfying all the properties just explained through the above pictures.

We now turn to the announced theorem and show that every amenable LCSCUH group has the special tiling property. Thus, highly invariant compact sets in the group can indeed be ε-quasi tiled as illustrated above. Additionally, we show the validity of some technical features which will be essential later in the construction of uniform families of ε-quasi tilings, see Chapter 3. The theorem can also be found in [PS14], Theorem 4.4.

Theorem 2.16 (Amenable groups have the STP).
Let Γ be an amenable LCSCUH group. Let β, ε, $N := N(\varepsilon)$ and (S_n) be given as in Definition 2.15 and let $B \subseteq \Gamma$ be compact with $e \in B$ and $0 < \zeta < \varepsilon$. Then we can find (B, ζ^2)-invariant ε-prototiles T_i $(1 \leq i \leq N(\varepsilon))$ such that each locally closed, $(T_{N(\varepsilon)}T_{N(\varepsilon)}^{-1}, \delta)$-invariant $(0 < \delta < 6^{-N(\varepsilon)}\beta/4)$ set $T \in \mathcal{F}(\Gamma)$ can be ε-quasi tiled by translates $T_i c$, $(1 \leq i \leq N(\varepsilon)$, $c \in C_i^T)$ such that the tiling sets can be made disjoint in a way such that for all $1 \leq i \leq N(\varepsilon)$ and all $c \in C_i^T$, there is some measurable set $T_i^{(c)} \subseteq T_i$ with the following properties.

- *$T_i^{(c)}$ is a locally closed set.*
- $|T_i^{(c)}| \geq (1 - \varepsilon)|T_i|$.
- $\partial_B(T_i^{(c)}) \in \mathcal{B}(\Gamma)$.
- *$T_i^{(c)}$ is $(B, 4\zeta)$-invariant.*
- $|\partial_B(T_i^{(c)})| \leq |\partial_B(T_i)| + \zeta\, |T_i|$.
- *$T_i C_i^T = \bigsqcup_{c \in C_i^T} T_i^{(c)} c$, where the latter union consists of pairwise disjoint sets.*

Here we essentially stick to the presentation of the proof of Theorem 4.4 in [PS14].

Proof.
Let ε and β with $0 < \beta < \varepsilon \leq 1/10$ and $\zeta > 0$ be given and let (S_n) be some nested Følner sequence. Choose some $0 < \delta < \frac{\beta}{4} 6^{-N(\varepsilon)}$. Without loss of generality, we assume that S_n is (B, ζ^2)-invariant for all $n \in \mathbb{N}$ and that $\zeta < \delta/4$ (If ζ is not chosen to be smaller than $\delta/4$, then we can take some $\tilde{\zeta} < \delta/4$ and repeat all the steps of the proof again. Hence, all claimed statements will hold for the original ζ as well.) As usual, we use the notation $N := N(\varepsilon)$.

We start choosing the sets $T_i \in \{S_n \mid n \in \mathbb{N}\}$, $1 \leq i \leq N$ inductively in the following way: set $T_1 := S_1$ and if $T_i = S_k$ then take $T_{i+1} \in \{S_n \mid n \geq k+1\}$ which is $(T_i T_i^{-1}, \delta)$-invariant. Then obviously $e \in T_i \subseteq T_{i+1}$ and $T_i \in \{S_n \mid n \geq i\}$ for all $1 \leq i \leq N$. Define $D_0 := T$ and $\delta_0 := \delta$. Furthermore, set $\delta_l := (2^{l+1} - 1)\delta$ for $1 \leq l \leq N$. Note that for every $1 \leq l \leq N-1$, we have $\delta_l \leq \delta_{N-1} < \varepsilon < 1/10$. This allows us to apply Lemma 2.14 inductively. If for some $0 \leq l \leq N-1$, the set $D_l \in \mathcal{B}(\Gamma)$, which is an intersection of some open set with the closed set T is chosen as a $(T_{N-l}T_{N-l}^{-1}, \delta_l)$-invariant set, we apply the lemma with $T = D_l$, $K = T_{N-l}$, $L = T_{N-l-1}$,$B = B$, $\delta = \delta_l$, $\eta = \delta$ and $\zeta = \zeta$. We obtain a finite set $C_{N-l}^T \in \mathcal{F}(\Gamma)$ such that $D_{l+1} := D_l \setminus T_{N-l}C_{N-l}^T$ is $(T_{N-l-1}T_{N-l-1}^{-1}, \delta_{l+1})$-invariant, where

$\delta_{l+1} = 2\delta_l + \delta$. Furthermore, there are $(B, 4\zeta)$-invariant, measurable sets $T_{N-l}^{(c)} \subseteq T_{N-l}$ with $\bigcup_{c \in C_{N-l}^T} T_{N-l}c = \bigsqcup_{c \in C_{N-l}^T} T_{N-l}^{(c)}c$, as well as

$$|T_{N-l}^{(c)}| \geq (1-\varepsilon)|T_{N-l}| \quad \text{and} \quad |\partial_B(T_{N-l}^{(c)})| \leq |\partial_B(T_{N-l})| + \zeta\,|T_{N-l}| \quad \text{for all } c \in C_{N-l}^T.$$

This will give the additional properties listed above. It remains to verify the assertions (i)-(iv) of Definition 2.15. Note that Lemma 2.14 implies the inequalities

$$\varepsilon - \delta_l \leq \frac{|T_{N-l}C_{N-l}^T|}{|D_l|} \leq \varepsilon + \delta_l \tag{2.8}$$

for all $0 \leq l \leq N-1$.

Claim: for all $0 \leq l \leq N-1$, there is some constant κ_l *independent* of the parameters ε, β and δ such that

$$\left| \frac{|T_{N-l}C_{N-l}^T|}{|T|} - \varepsilon(1-\varepsilon)^l \right| \leq \kappa_l \cdot \delta_l. \tag{2.9}$$

In fact, we will see that $\kappa_l := 3^l$, $0 \leq l \leq N-1$ is a good choice. For the proof of the claim, we proceed by induction on l. Note that we have treated the case $l = 0$ in inequality (2.8) with $\kappa_0 = 1$. Now let $l \in \mathbb{N}_{\geq 1}$ and assume that (2.9) holds for all $0 \leq k \leq l-1$. By the induction hypothesis, we can sum up the resulting inequalities and arrive at

$$\varepsilon \cdot \sum_{k=0}^{l-1}(1-\varepsilon)^k - \sum_{k=0}^{l-1}\kappa_k\delta_k \leq \frac{|\bigcup_{k<l} T_{N-k}C_{N-k}^T|}{|T|} \leq \varepsilon \cdot \sum_{k=0}^{l-1}(1-\varepsilon)^k + \sum_{k=0}^{l-1}\kappa_k\delta_k.$$

Note that here we used the pairwise disjointness of the sets T_iC_i for $1 \leq i \leq N$. By the definition of D_l we obtain $T \setminus D_l = \bigcup_{k<l} T_{N-k}C_{N-k}^T$ and hence

$$1 - \varepsilon \cdot \sum_{k=0}^{l-1}(1-\varepsilon)^k - \sum_{k=0}^{l-1}\kappa_k\delta_k \leq \frac{|D_l|}{|T|} \leq 1 - \varepsilon \cdot \sum_{k=0}^{l-1}(1-\varepsilon)^k + \sum_{k=0}^{l-1}\kappa_k\delta_k. \tag{2.10}$$

Combining this inequality (2.10) with the estimate (2.8), we obtain

$$\begin{aligned}(\varepsilon - \delta_l)\left(1 - \varepsilon\sum_{k=0}^{l-1}(1-\varepsilon)^k - \sum_{k=0}^{l-1}\kappa_k\delta_k\right) \leq \frac{|T_{N-l}C_{N-l}^T|}{|T|} \\ \leq (\varepsilon + \delta_l)\left(1 - \varepsilon\sum_{k=0}^{l-1}(1-\varepsilon)^k + \sum_{k=0}^{l-1}\kappa_k\delta_k\right).\end{aligned}$$

Using $0 < \varepsilon < 1$ we obtain

$$\begin{aligned}\varepsilon\left(1 - \varepsilon\sum_{k=0}^{l-1}(1-\varepsilon)^k\right) &- \delta_l\left(1 + \sum_{k=0}^{l-1}\kappa_k\delta_k\right) - \sum_{k=0}^{l-1}\kappa_k\delta_k \leq \frac{|T_{N-l}C_{N-l}^T|}{|T|} \\ &\leq \varepsilon\left(1 - \varepsilon\sum_{k=0}^{l-1}(1-\varepsilon)^k\right) + \delta_l\left(1 + \sum_{k=0}^{l-1}\kappa_k\delta_k\right) + \sum_{k=0}^{l-1}\kappa_k\delta_k.\end{aligned}$$

By $\delta_k \le \delta_l \le 1$ and $1 - \varepsilon \sum_{k=0}^{l-1}(1-\varepsilon)^k = (1-\varepsilon)^l$, this implies

$$\left| \frac{|T_{N-l}C_{N-l}|}{|T|} - \varepsilon(1-\varepsilon)^l \right| \le \delta_l \left(1 + \sum_{k=0}^{l-1} \kappa_k \delta_k \right) + \sum_{k=0}^{l-1} \kappa_k \delta_k \le \left(1 + 2 \sum_{k=0}^{l-1} \kappa_k \right) \delta_l.$$

This shows the claim (2.9) with $\kappa_l := 1 + 2\sum_{k=0}^{l-1} \kappa_k$. Since $\kappa_0 = 1$ we can compute the κ_l recursively, namely $\kappa_l = 3^l$ for all $l \ge 0$. In particular, we have for all $1 \le i \le N$ that

$$\left| \frac{|T_i C_i^T|}{|T|} - \varepsilon(1-\varepsilon)^{N-i} \right| \le 3^N(2^{N+1} - 1)\delta \le 2 \cdot 6^N \delta < \beta,$$

by the choice of δ. This proves (iv) of Definition 2.15. Properties (i), (ii) and (iii) follow from the construction of the sets C_i^T, $1 \le i \le N$. The additional properties concerning the disjoint sets $T_i^{(c)}$ can immediately be deduced from Lemma 2.14. □

The following corollary is an immediate consequence of the preceding theorem.

Corollary 2.17.
Every amenable LCSCUH group satisfies the special tiling property.

PROOF.
This follows from Theorem 2.16. □

3 Uniform tiling results

The goal of this chapter is to use Theorem 2.16 in order to prove the existence of uniform ε-quasi tilings in amenable groups. Roughly speaking, we show that for highly invariant compact sets $T \subseteq \Gamma$, we can find a family of ε-quasi tilings as in Definition 2.15 such that *on average* (with respect to the family), almost all elements $u \in T$ occur in the center sets C_i^T with the same frequency. To do so, we significantly extend the ε-quasi tiling techniques in [OW87]. In fact, we give effective estimates on the covering densities and on the uniformity properties of ε-quasi tiling families. These results provide a major tool in the proofs of convergence theorems for almost-additive functions over Γ, cf. Chapter 4. We will deal with countable groups and general LCSCUH groups separately. In the latter situation, we do not only construct uniform ε-quasi tilings, but certain pairs of ε-quasi tiling families. Those pairs are called uniform decomposition towers, cf. Definition 3.5. They will become indispensable later in the proof of the abstract mean ergodic theorem for almost-additive functions over Γ, cf. Theorem 4.15. The decomposition theorem for countable amenable groups (cf. Theorem 3.2) appears in [PS14]. The analoguous Theorem 3.4 for general amenable LCSCUH groups, as well as the existence theorem for uniform decomposition towers (cf. Theorem 3.6) are published in [Pog13a].

3.1 Countable groups

In this section, we construct certain finite families of ε-quasi tilings in countable, amenable groups. We show that if a set T is ε-quasi tiled by the family, then on average, most of the elements $u \in T$ appear in the center sets C_i of the tiling with relative frequency $\varepsilon(1-\varepsilon)^{N-i}/|T_i|$. Since the latter number only depends on ε and the element $1 \leq i \leq N$, this is a remarkable result. The corresponding existence Theorem 3.2 is an analogue of Proposition I.2.7 in [OW87]. We also use constructions of the proof of that proposition. However, we compute different quantities. Moreover, we are able to give precise estimates for tiling densities and occurrence frequencies of elements in the center sets. The corresponding results can also be found in [PS14], Theorem 4.6.

Extending the special tiling property (cf. Definition 2.15), we now introduce the uniform special tiling property for countable groups.

Definition 3.1 (Uniform special tiling property (USTP)).
Let Γ *be a countable amenable group. Then we say that* Γ *satisfies the* uniform special tiling property *(USTP) if the following statements hold true.*

- *For arbitrary* $0 < \varepsilon \leq 1/10$, $N := N(\varepsilon) := \lceil \log(\varepsilon)/\log(1-\varepsilon) \rceil$, *for every* $0 < \beta < 2^{-N}\varepsilon$ *and for all nested Følner sequences* (S_n), *the group* Γ *satisfies the spectial tiling property according to Definition 2.15 with* ε*- prototiles*

$$\{e\} \subseteq T_1 \subseteq T_2 \subseteq \cdots \subseteq T_N,$$

  where $T_i \in \{S_n \mid n \geq i\}$ *for* $1 \leq i \leq N$.

- *For fixed numbers* ε *and* β *as above and for all Følner sequences* (U_n) *in* Γ, *there is a finite set* $Q(\varepsilon, \beta, T_i) =: Q \subset \Gamma$ *with* $e \in Q$ *depending on* ε, β *and the tiling sets* T_i, *as well as a number* $R \in \mathbb{N}$ *such that for each* $k \geq R$, *the set* U_k *is* (Q, β)*-invariant and there is a finite set* $\Lambda_k \subset \Gamma$, *along with a family* $\{C_i^\lambda \subseteq T \mid \lambda \in \Lambda_k, 1 \leq i \leq N\}$ *of finite center sets for the* T_i *such that the following holds: for all* $\lambda \in \Lambda_k$, *the set* U_k *is* ε*-quasi tiled with the properties (i) to (iii) of Definition 2.15 by the translates* $T_i c$, $c \in C_i^\lambda$ *and additionally,*

  (I) $\dfrac{\left|\bigcup_{i=1}^N T_i C_i^\lambda\right|}{|U_k|} \geq 1 - 4\varepsilon$ *for all* $\lambda \in \Lambda_k$,

  (II) $\left| |\Lambda_k|^{-1} \sum_{\lambda \in \Lambda_k} \mathbb{1}_{C_i^\lambda}(u) - \frac{\eta_i(\varepsilon)}{|T_i|} \right| < 3\,\frac{\beta}{|T_i|} + 2\,\varepsilon\gamma_i$ *for all* $1 \leq i \leq N$ *and every* $u \in U_k \setminus \partial_Q(U_k)$, *where* $\eta_i(\varepsilon) = \varepsilon(1-\varepsilon)^{N-i}$ *and the* $\gamma_i > 0$ *satisfy* $\sum_{i=1}^N \gamma_i |T_i| \leq 2$.

Remark.
The essential uniformity property is given by statement (II) of the second item. It states that for a percentage of at least $1 - \beta$ of the elements in the set U_k, a fixed element $u \in U_k$ serves as a center in the set C_i^λ in approximately $\varepsilon(1-\varepsilon)^{N-i}/|T_i|$ percent of the ε-quasi tilings in the family. In Proposition I.2.7 of [OW87], one finds a different uniformity result: it states that for most $u \in T$, the number of center sets C_i containing u compared to the total mass $\sum_\lambda |C_i^\lambda|$ is approximately proportional to $1/|T|$. However, it turns out that we cannot apply this fact for our purposes, but we need assertion (II) of the second item of Defintion 3.1.

The following theorem shows that uniform tilings as in Definition 3.1 exist in *all* countable, amenable groups. It is taken from [PS14], Theorem 4.6 and it is due to the author of this thesis. In agreement with the author of this thesis, Schwarzenberger presented the proof in his Ph.D. thesis, see the proof of Theorem 5.22 in the appendix. The result can also be found in cf. [PS14], Lemma 4.6.

Theorem 3.2 (Amenable groups have the USTP).
Every countable, amenable group Γ *satisfies the uniform special tiling property.*

Proof.
Let $0 < \varepsilon \leq 1/10$, set $N := N(\varepsilon) := \lceil \log(\varepsilon)/\log(1-\varepsilon) \rceil$ and choose $0 < \beta < 2^{-N}\varepsilon$. Further, let (U_n) be a Følner sequence of Γ and assume that $0 < \delta_0 < 6^{-N}\beta/20$.

Note that by Theorem 2.16, Γ is ε-quasi tiled by a finite sequence

$$\{\mathrm{e}\} \subseteq T_1 \cdots \subseteq T_N$$

of finite ε-prototiles sets taken from a nested Følner sequence (S_n).

Let $0 < \delta < \delta_0$. At various steps of the proof, we will have to make this parameter smaller. For the sake of the reader, we prefer doing this in a successive manner instead of imposing many technical conditions on δ right now. This is possible since the restrictions will only depend on ε, β and the basis sets T_i, $1 \leq i \leq N$, but not on things developed in the proof. One can then think of starting the proof all over again with a new condition on the parameter δ. We proceed in nine steps.

(1) We let $M := \lceil \log(\delta)/\log(1-\delta) \rceil$ and following Theorem 2.16, we find finite sets $\overline{T}_l \supseteq T_N$, $1 \leq l \leq M$, taken from the nested Følner sequence (S_n), such that $\{\overline{T}_l\}_{l=1}^M$ is a family of δ-prototiles in Γ. In addition to this, we make sure that $\overline{T}_l$ is $(T_N T_N^{-1}, \delta_0^2)$-invariant for all $1 \leq l \leq M$ (we can do so because further invariance properties can be chosen without changing the fact that the $\{\overline{T}_l\}$ δ-quasi tile Γ).

Then we find some integer $K \in \mathbb{N}$ such that for each $k \geq K$, the set $T := U_k$ is $(\overline{T}_l \overline{T}_l^{-1}, 2^{-l}\delta)$-invariant for all $1 \leq l \leq M$. Since δ will depend on ε, β and the basis sets T_i, so does the integer number K. Further, we choose $\hat{T}$ to be a (TT^{-1}, δ)-invariant finite set $\hat{T}$ which is also $(\overline{T}_l \overline{T}_l^{-1}, 2^{-l}\delta)$-invariant for all $1 \leq l \leq M$. Using Theorem 2.16, we can also make sure that $\hat{T}$ has the special tiling property with respect to $(\{\overline{T}_l\}_{l=1}^M, (S_n), \delta, \beta_1)$, where $0 < \beta_1 < 2^{-M}\delta$. For instance, take $\hat{T} := U_{\tilde{R}}$ for $\tilde{R} \in \mathbb{N}$ large enough.

Define $A := \{h \in \Gamma \,|\, Th \subseteq \hat{T}\}$. Since $\hat{T}$ is (TT^{-1}, δ)-invariant, we have by the Lemma 2.11 and the unimodularity of the counting measure on Γ that $|A| \geq (1-\delta)|\hat{T}|$.

(2) Since $\hat{T}$ has the special tiling property with respect to $(\{\overline{T}_l\}_{l=1}^M, (S_n), \delta, \beta_1)$ $(0 < \beta_1 < 2^{-M}\delta)$, we find a δ-quasi tiling of $\hat{T}$ with center sets $\overline{C}_l \subseteq \hat{T}$, $1 \leq l \leq M$ as in Theorem 2.16. Furthermore, we can make the $\overline{T}_l$-translates in this δ-quasi tiling actually disjoint such that the resulting disjoint translates $\overline{T}_l'(c)c \subseteq \overline{T}_l c$ $(c \in \overline{C}_l)$ are still $(T_N T_N^{-1}, 4\delta_0)$-invariant by applying Theorem 2.16 with $B = T_N T_N^{-1}$. Due to inequality (2.7), those sets maintain the covering properties of our tiling, i.e.

$$1 \geq \frac{|B(\delta, (\overline{T}_l), \hat{T})|}{|\hat{T}|} = \frac{\sum_{l=1}^M \sum_{c \in \overline{C}_l} |\overline{T}_l'(c)c|}{|\hat{T}|} \geq (1-2\delta), \tag{3.1}$$

where $B(\delta, (\overline{T}_l), \hat{T}) := \bigsqcup_{l=1}^M \bigsqcup_{c \in \overline{C}_l} \overline{T}_l'(c)c$.

(3) Since all sets $\overline{T}_l'(c)c$ are $(T_N T_N^{-1}, 4\delta_0)$-invariant for $1 \leq l \leq M$ and $c \in \overline{C}_l$, and since $\delta_0 < 6^{-N}\beta/4$, we can apply Theorem 2.16 with $\delta_0 = \delta_0$ to fix in each translate $\overline{T}_l'(c)c$, $(c \in \overline{C}_l)$ an ε-quasi tiling of $(T_i)_{i=1}^N$ with center sets $C_i(\overline{T}_l'(c))$ and

$$\left| \frac{\left| \bigcup_{\tilde{c} \in C_i(\overline{T}_l'(c))} T_i \tilde{c} \right|}{|\overline{T}_l'(c)|} - \eta_i(\varepsilon) \right| < \beta \tag{3.2}$$

for $1 \le i \le N$. Furthermore, we put

$$\hat{C}_i := \bigcup_{l=1}^{M} \bigcup_{c \in \overline{C}_l} C_i(\overline{T}'_l(c))$$

for $1 \le i \le N$. Then the elements in $\hat{C}_i$ can be considered as centers for the Følner sets T_i such that the $\{T_i c\}_{c \in \hat{C}_i}$ are ε-disjoint and such that for $1 \le i < j \le N$, the sets $T_i \hat{C}_i$ and $T_j \hat{C}_j$ are disjoint. Using the fact that $\beta < 2^{-N}\varepsilon$, we obtain by inequality (2.7)

$$\left| \bigcup_{i=1}^{N} \bigcup_{\tilde{c} \in C_i(\overline{T}'_l(c))} T_i \tilde{c} \right| \ge (1 - 2\varepsilon) \left| \overline{T}'_l(c) \right| \tag{3.3}$$

for each $1 \le l \le M$ and every $c \in \overline{C}_l$.

(4) We now would like to determine the portion of $\hat{T}$ that is covered by each set $T_i \hat{C}_i$. We will see that up to some small error (2β), this will be $\eta_i(\varepsilon) = \varepsilon(1-\varepsilon)^{N-i}$.

Let $1 \le i \le N$ be given. Using the disjointness of the $\overline{T}'_l(c)c$ for all $c \in \overline{C}_l$ and all $1 \le l \le M$, inequality (3.1) and (3.2), we obtain

$$\begin{aligned} \left| T_i \hat{C}_i \right| &= \sum_{l=1}^{M} \sum_{c \in \overline{C}_l} \left| \bigcup_{\tilde{c} \in C_i(\overline{T}'_l(c))} T_i \tilde{c} \right| \\ &\ge \sum_{l=1}^{M} \sum_{c \in \overline{C}_l} |\overline{T}'_l(c)|(\eta_i(\varepsilon) - \beta) \ge (1 - 2\delta)(\eta_i(\varepsilon) - \beta)|\hat{T}| \ge (\eta_i(\varepsilon) - 2\beta)|\hat{T}|, \end{aligned}$$

where the last step is true for sufficiently small δ. Note that this is the first of the announced conditions on the smallness of δ. Let us estimate in the other direction. Estimates (3.2) and (3.1) lead to

$$\left| T_i \hat{C}_i \right| \overset{(3.2)}{\le} \sum_{l=1}^{M} \sum_{c \in \overline{C}_l} |\overline{T}'_l(c)|(\eta_i(\varepsilon) + \beta) \overset{(3.1)}{=} (\eta_i(\varepsilon) + \beta)|B(\delta, (\overline{T}_l), \hat{T})| \le (\eta_i(\varepsilon) + \beta)|\hat{T}|.$$

The above bounds yield for all $1 \le i \le N$

$$\left| \frac{\left| T_i \hat{C}_i \right|}{|\hat{T}|} - \eta_i(\varepsilon) \right| < 2\beta. \tag{3.4}$$

(5) For each $1 \le i \le N$, define the ratio $\gamma_i := |\hat{C}_i| / |\hat{T}|$. We compare this expression with the ratio $\eta_i(\varepsilon)/|T_i|$. By exploiting ε-disjointness of the T_i-translates, as well as (3.4), we obtain for each $1 \le i \le N$

$$|T_i \hat{C}_i| \le |T_i||\hat{C}_i| \le (1-\varepsilon)^{-1} \sum_{c \in \hat{C}_i} |T'_i(c)c| = (1-\varepsilon)^{-1} \left| \bigcup_{c \in \hat{C}_i} T'_i(c)c \right| \le (1-\varepsilon)^{-1} |T_i \hat{C}_i|,$$

where the sets $T_i'(c)c, c \in \hat{C}_i$ are pairwise disjoint and $|T_i'(c)| \geq (1-\varepsilon)|T_i|$ for all $c \in \hat{C}_i$. This leads us to

$$\begin{aligned} \left|\gamma_i - \frac{\eta_i(\varepsilon)}{|T_i|}\right| &\leq \frac{1}{|T_i|}\left|\frac{|\hat{C}_i|\,|T_i|}{|\hat{T}|} - \frac{\left|T_i\hat{C}_i\right|}{|\hat{T}|}\right| + \frac{1}{|T_i|}\left|\frac{\left|T_i\hat{C}_i\right|}{|\hat{T}|} - \eta_i(\varepsilon)\right| \\ &\leq \frac{\varepsilon}{1-\varepsilon}\frac{|\hat{C}_i|}{|\hat{T}|} + \frac{2\beta}{|T_i|} \\ (\varepsilon < 1/2) \quad &\leq 2\gamma_i\,\varepsilon + \frac{2\beta}{|T_i|}. \end{aligned} \tag{3.5}$$

Using the ε-disjointness of the translates of T_i (and the rough bound $\varepsilon < 1/2$), it is straight forward to show that

$$\sum_{i=1}^{N} \gamma_i\,|T_i| \leq 2$$

holds true.

(6) In this step of the proof, it will be shown that most of the T-translates contained in $\hat{T}$ will be $(1-3\varepsilon)$-covered by the fixed pattern $\cup_{i=1}^{N} T_i\hat{C}_i$. Here, we will have to impose a second restriction on δ. We recall from step (1) that we chose the set A as the collection of elements $a \in \Gamma$ such that the translate Ta lies entirely in $\hat{T}$. For each $a \in A$, we set

$$X(a) := \frac{\left|Ta \cap \left(\hat{T} \setminus B(\delta, (\overline{T}_l), \hat{T})\right)\right|}{|T|} = \frac{\left|Ta \setminus B(\delta, (\overline{T}_l), \hat{T})\right|}{|T|}$$

and treat X as a function on A into $[0,1]$. Thus, it follows from the 'Chebyshev inequality' that

$$\sqrt{\delta}\left|\{a \in A \,|\, X(a) > \sqrt{\delta}\}\right| \leq \sum_{a\in A} X(a) = \frac{1}{|T|}\sum_{a\in A}\sum_{g\in\hat{T}\setminus B(\delta,(\overline{T}_l),\hat{T})} \mathbb{1}_{Ta}(g).$$

Using (3.1) and Lemma 2.11 (i), we continue estimating

$$\begin{aligned} \frac{1}{|T|}\sum_{a\in A}\sum_{g\in\hat{T}\setminus B(\delta,(\overline{T}_l),\hat{T})} \mathbb{1}_{Ta}(g) &= \frac{1}{|T|}\sum_{g\in\hat{T}\setminus B(\delta,(\overline{T}_l),\hat{T})}\sum_{a\in A} \mathbb{1}_{Ta}(g) \\ (\text{Lemma 2.11 (i)}) \quad &\leq |\hat{T} \setminus B(\delta, (\overline{T}_l), \hat{T})| \\ (\text{inequality (3.1)}) \quad &\leq 2\delta|\hat{T}|. \end{aligned}$$

This and $|A| \geq (1-\delta)|\hat{T}|$ (see step (1)), as well as $\delta \leq 1/2$ yield

$$\left|\{a \in A \,|\, X(a) > \sqrt{\delta}\}\right| \leq 2\sqrt{\delta}|\hat{T}| \leq \frac{2\sqrt{\delta}|A|}{1-\delta} \leq 4\sqrt{\delta}|A|,$$

or equivalently

$$|\Lambda| \geq (1-4\sqrt{\delta})|A|, \quad \text{where} \quad \Lambda := \{a \in A \mid X(a) \leq \sqrt{\delta}\}. \tag{3.6}$$

Thus, up to a portion of $4\sqrt{\delta}$, the translates of T which lie entirely in $\hat{T}$ are $(1-\sqrt{\delta})$-covered by the tiling with basis sets $(\overline{T}_l)$. However, it is convenient to work with quasi tilings of the sets Ta with their basis sets being *subsets* of Ta, where $a \in A$. Therefore, for each $a \in A$, we delete elements of the covering having a non-empty intersection with $\Gamma \setminus Ta$. To do so, define for $1 \leq l \leq M$ and $a \in A$ the sets

$$\begin{aligned} \partial(a,l) &:= \{c \in \overline{C}_l \,|\, \overline{T}'_l(c)c \cap Ta \neq \emptyset,\, \overline{T}'_l(c)c \cap (\Gamma \setminus Ta) \neq \emptyset\}, \quad \text{and} \\ I(a,l) &:= \{c \in \overline{C}_l \,|\, \overline{T}'_l(c)c \subseteq Ta\}. \end{aligned}$$

Then, $\overline{T}_l\partial(a,l) \subseteq \partial_{\overline{T}_l\overline{T}_l^{-1}}(Ta)$ for every $a \in A$ and each $1 \leq l \leq M$. Using the invariance properties of T assumed in step (1), for all $a \in A$, we arrive at

$$\begin{aligned} \frac{1}{|T|}\Big|\bigcup_{l=1}^{M}\bigcup_{c\in\partial(a,l)} \overline{T}'_l(c)c\Big| &\leq \frac{1}{|T|}\sum_{l=1}^{M}|\partial_{\overline{T}_l\overline{T}_l^{-1}}(T)| \\ &\leq \frac{1}{|T|}\sum_{l=1}^{M} 2^{-l}\delta|T| \\ &< \delta. \end{aligned}$$

It follows in particular, that if $\lambda \in \Lambda$, then the estimate

$$\begin{aligned} \Big|\bigcup_{l=1}^{M}\bigcup_{c\in I(\lambda,l)} \overline{T}'_l(c)c\Big| &\geq |T\lambda \cap B(\delta,(\overline{T}_l),\hat{T})| - \Big|\bigcup_{l=1}^{M}\bigcup_{c\in\partial(\lambda,l)} \overline{T}'_l(c)c\Big| \\ &\geq (1-\sqrt{\delta}-\delta)|T|. \end{aligned} \tag{3.7}$$

holds true. Now, for $a \in A$, we are able to estimate the portion of Ta which is covered by those $\hat{C}_i$-translates of $T_i, 1 \leq i \leq N$ which lie completely in Ta. To this end, we set

$$\tilde{C}_i(a) := \bigcup_{l=1}^{M}\bigcup_{c\in I(a,l)} C_i(\overline{T}'_l(c)), \quad \text{and} \quad D(a) := \bigcup_{i=1}^{N}\bigcup_{c\in\tilde{C}_i(a)} T_ic \subseteq Ta$$

for $1 \leq i \leq N$ and for $a \in A$. Using the disjointness of the translates $\overline{T}'_l(c)c$, $1 \leq l \leq M$, $c \in \overline{C}_l$ and the estimate (3.3), we obtain

$$|D(\lambda)| = \sum_{l=1}^{M}\sum_{c\in I(\lambda,l)}\Big|\bigcup_{i=1}^{N}\bigcup_{\tilde{c}\in C_i(\overline{T}'_l(c))} T_i\tilde{c}\Big| \geq (1-2\varepsilon)\sum_{l=1}^{M}\sum_{c\in I(\lambda,l)}|\overline{T}'_l(c)|$$

for all $\lambda \in \Lambda$. Thus, imposing another condition on the smallness of δ and using (3.7), we arrive at

$$|D(\lambda)| \geq (1-2\varepsilon)(1-\delta-\sqrt{\delta})|T| \geq (1-3\varepsilon)|T| \tag{3.8}$$

for every $\lambda \in \Lambda$.

(7) Next, we define the desired family of center sets and verify properties (i) - (iii) of Definition 2.15, as well as property (I) of Definition 3.1. In the previous step we have already defined the set Λ. Recall that Λ depends on $T = U_k$, which justifies the notion $\Lambda_k := \Lambda$. For each $k \in \mathbb{N}$, $\lambda \in \Lambda_k$ and $1 \le i \le N$, we set

$$C_i^\lambda := C_i^\lambda(T) := \tilde{C}_i(\lambda)\lambda^{-1}.$$

Then, we also have

$$\bigcup_{i=1}^{N} \bigcup_{c \in \tilde{C}_i^\lambda} T_i c = D(\lambda)\lambda^{-1} \subseteq T$$

for all $\lambda \in \Lambda$, which shows (i) of Definition 2.15. The properties (ii) and (iii) are fulfilled by construction. The bound in (I) of Definition 3.1 follows from (3.8).

(8) In this step, we prepare the proof of the uniform covering principle (II) of Definition 3.1. To do so, we define $Q(\delta, \varepsilon, \beta, T_i) := Q := \overline{T}_M \overline{T}_M^{-1}$. We now show that for each $1 \le i \le N$ and $u \in T \setminus \partial_Q(T)$, we have indeed

$$\Big| \frac{1}{|\Lambda|} \sum_{\lambda \in \Lambda} \mathbb{1}_{C_i^\lambda}(u) - \gamma_i \Big| \le \frac{\beta}{|T_i|}, \tag{3.9}$$

where the numbers γ_i and the sets C_i^λ are as explained in steps (5) and (7) respectively. Since $\tilde{C}_i(\lambda) \subseteq \hat{C}_i$, we obtain

$$\sum_{\lambda \in \Lambda} \mathbb{1}_{C_i^\lambda}(u) = \sum_{\lambda \in \Lambda} \mathbb{1}_{\tilde{C}_i(\lambda)}(u\lambda) \le \sum_{\lambda \in \Lambda} \mathbb{1}_{\hat{C}_i}(u\lambda) = |\hat{C}_i \cap u\Lambda| \le |\hat{C}_i|. \tag{3.10}$$

Next, use (3.6) and $|A| \ge (1-\delta)|\hat{T}|$ (cf. step (1)) to calculate

$$|\Lambda|^{-1} \sum_{\lambda \in \Lambda} \mathbb{1}_{C_i^\lambda}(u) \le \frac{|\hat{C}_i|}{(1-4\sqrt{\delta})|A|} \le \frac{|\hat{C}_i|}{(1-5\sqrt{\delta})|\hat{T}|} = \frac{\gamma_i}{1-5\sqrt{\delta}}$$

for all $1 \le i \le N$. With $\gamma_i \le 1$, we obtain

$$|\Lambda|^{-1} \sum_{\lambda \in \Lambda} \mathbb{1}_{C_i^\lambda}(u) - \gamma_i \le \left(\frac{1}{1-5\sqrt{\delta}} - 1 \right) \gamma_i \le \left(\frac{1}{1-5\sqrt{\delta}} - 1 \right) \le \frac{\beta}{|T_i|}, \tag{3.11}$$

where the last inequality holds true if δ is chosen small enough, which is the third assumption on the size of δ. Note here that the choice of the T_i only depends on ε but not other quantities developed in the proof. Let us verify the converse bound. At first, we claim that for all $\lambda \in \Lambda$ and every $1 \le i \le N$,

$$\hat{C}_i \lambda^{-1} \cap (T \setminus \partial_Q(T)) \subseteq C_i^\lambda. \tag{3.12}$$

To see this, let $u \in \hat{C}_i\lambda^{-1} \cap T \setminus \partial_Q(T)$ be given. By the definition of $\hat{C}_i$, we find $1 \le l \le M$, as well as $c \in \overline{C}_l$ such that $u \in C_i(\overline{T}'_l(c))\lambda^{-1}$. If $c \in I(\lambda, l)$, then $u \in \tilde{C}_i(\lambda)\lambda^{-1} = C_i^\lambda$

and there is nothing left to show. If $c \notin I(\lambda, l)$, then $\overline{T}'_l(c)c \cap (\Gamma \setminus T\lambda) \neq \emptyset$. On the other hand, as $u\lambda \in T\lambda$ and

$$u\lambda \in T_i u\lambda \subseteq \overline{T}'_l(c)c,$$

we also have $\overline{T}'_l(c)c \cap T\lambda \neq \emptyset$ and thus $c \in \partial(\lambda, l)$. It also follows from the latter inclusion (see step (6)) that $u\lambda \in \partial_{\overline{T}_l\overline{T}_l^{-1}}(T\lambda)$ which implies $u \in \partial_Q(T)$. But this is a contradiction to the choice of u. Thus, we must have $c \in I(\lambda, l)$ and the claimed inclusion (3.12) follows. Let now $u \in T \setminus \partial_Q(T)$ be given.
We use (3.12) to obtain

$$\mathbb{1}_{C_i^\lambda}(u) \geq \mathbb{1}_{\hat{C}_i\lambda^{-1}\cap(T\setminus\partial_Q(T))}(u) = \mathbb{1}_{\hat{C}_i\lambda^{-1}}(u) = \mathbb{1}_{u^{-1}\hat{C}_i}(\lambda),$$

which together with (3.10) implies

$$\sum_{\lambda\in\Lambda} \mathbb{1}_{C_i^\lambda}(u) \geq |u\Lambda \cap \hat{C}_i|.$$

With $\hat{C}_i \subseteq \hat{T}$ and $u\Lambda \subseteq \hat{T}$ (as $T\Lambda \subseteq \hat{T}$), we obtain $|\hat{T}| \geq |\Lambda|$ and hence, we calculate

$$\begin{aligned} \gamma_i - \frac{1}{|\Lambda|}\sum_{\lambda\in\Lambda} \mathbb{1}_{C_i^\lambda}(u) \quad &\leq \quad \frac{|\hat{C}_i|}{|\hat{T}|} - \frac{|u\Lambda \cap \hat{C}_i|}{|\hat{T}|} \\ (\hat{C}_i \subseteq \hat{T}) \quad &\leq \quad \frac{|\hat{T} \setminus u\Lambda|}{|\hat{T}|} = 1 - \frac{|\Lambda|}{|\hat{T}|}. \end{aligned}$$

Now, the inequalities (3.6) and $|A| \geq (1-\delta)|\hat{T}|$ yield

$$\gamma_i - \frac{1}{|\Lambda|}\sum_{\lambda\in\Lambda} \mathbb{1}_{C_i^\lambda}(u) \leq 1 - \frac{(1-4\sqrt{\delta})(1-\delta)|\hat{T}|}{|\hat{T}|} \leq 5\sqrt{\delta} \leq \frac{\beta}{|T_i|},$$

where the last inequality holds if δ is chosen small enough, which shall be the fourth and last additional condition on δ. This and inequality (3.11) show (3.9).

(9) Finally, we prove the property (II) of our theorem. To do so, we use the triangle inequality followed by the Inequalities (3.9) and (3.5). We obtain

$$\Big|\frac{1}{|\Lambda|}\sum_{\lambda\in\Lambda} \mathbb{1}_{C_i^\lambda}(u) - \frac{\eta_i(\varepsilon)}{|T_i|}\Big| \leq \Big|\frac{1}{|\Lambda|}\sum_{\lambda\in\Lambda} \mathbb{1}_{C_i^\lambda}(u) - \gamma_i\Big| + \Big|\gamma_i - \frac{\eta_i(\varepsilon)}{|T_i|}\Big| \leq 2\gamma_i\varepsilon + \frac{3\beta}{|T_i|}$$

for each $u \in T \setminus \partial_Q(T)$ and all $1 \leq i \leq N$, where $\sum_{i=1}^N \gamma_i|T_i| \leq 2$ as shown above.

Thus, we have finally finished the proof of the theorem. □

3.2 Continuous groups

We turn to the case of possibly continuous, LCSCUH groups. Again, we construct families of ε-quasi tilings with convenient uniformity properties. The corresponing existence Theorem 3.4 for amenable groups is an extension of Proposition I.3.6 of [OW87]. For the proof of the general mean ergodic theorem for almost-additive functions (cf. Chapter 4), we even need more sophisticated decompositions. This leads us to the concept of so-called uniform decomposition towers, cf. Definition 3.5. This latter object consists of a pair of uniform ε-quasi tiling famlies with index sets $\Upsilon, \Lambda \in \mathcal{F}(\Gamma)$. The crucial property is that for each $y \in \Upsilon$, we can compute the corresponding uniform ε-quasi tiling over Λ with center sets $\{C_i^{y,\lambda}(\cdot)\}_{\lambda\in\Lambda}$. For our later purposes, it will be essential that Λ is chosen independently of y. Thus, we need in fact decomposition pairs as described in Definition 3.5 and we must refrain from just 'glueing' two uniform families of ε-quasi tilings together. In Theorem 3.6, we show that in every amenable LCSCUH group, one can construct uniform decomposition towers. The results of this section have been published in [Pog13a].

Definition 3.3 (Uniform continuous decompositions).
Let Γ be an LCSCUH group. We say that Γ satisfies the uniform continuous decompositions condition *(UCDC) if the following statements holds true.*

- *For each $0 < \varepsilon \leq 1/10$, $N := N(\varepsilon) := \lceil \log(\varepsilon)/\log(1-\varepsilon)\rceil$, for arbitrary numbers $0 < \beta, \zeta < 2^{-N}\varepsilon$, for every nested Følner sequence (S_n), and for each compact set $e \in B \subseteq \Gamma$, the group Γ has the special tiling property according to Definition 2.15 with ε-prototiles*
$$\{e\} \subseteq T_1 \subseteq T_2 \subseteq \cdots \subseteq T_N,$$
where $T_i \in \{S_n \,|\, n \geq i\}$ for $1 \leq i \leq N(\varepsilon)$ and where the latter prototile sets are all (B, ζ^2)- invariant.

- *For fixed positive numbers $\varepsilon > 0, \beta > 0, \zeta > 0$ and for a fixed compact set $e \in B \subseteq G$, there is some integer $J \in \mathbb{N}$, as well as a positive number $0 < \delta_0 < \beta/4$ depending on ε, β and the basis sets T_i such that for each $0 < \delta_1 < \delta_0$, every locally closed, $(S_J S_J^{-1}, \delta_1)$-invariant set $T \subseteq \Gamma$ can be uniformly ε-quasi tiled in the following manner: we find a finite-measure set $\Lambda \in \mathcal{F}(\Gamma)$, along with a* family
$$\{C_i^\lambda(T) \,|\, \lambda \in \Lambda,\, 1 \leq i \leq N\}$$
of finite center sets for the basis sets T_i such that for each $\lambda \in \Lambda$, the set T is ε-quasi tiled by the translates $T_i c$, $1 \leq i \leq N$, $c \in C_i^\lambda(T)$ according to Definition 2.15 and moreover,

  (I) $\frac{|\bigcup_{i=1}^N T_i C_i^\lambda(T)|}{|T|} \geq 1 - 4\varepsilon$ *for all* $\lambda \in \Lambda$,

  (II) for all $1 \leq i \leq N$ and for every Borel set $S \subseteq T$,
$$\left| |\Lambda|^{-1} \int_\Lambda \frac{\#(C_i^\lambda(T) \cap S)}{|T|}\, d\lambda - \frac{\eta_i(\varepsilon)}{|T_i|} \cdot \frac{|S|}{|T|} \right| < 4\, \frac{\beta}{|T_i|} + 2\varepsilon \cdot \gamma_i,$$
  where $\eta_i(\varepsilon) := \varepsilon(1-\varepsilon)^{N-i}$ and the $\gamma_i > 0$ can be chosen such that $\sum_{i=1}^N \gamma_i |T_i| \leq 2$,

  (III) the translates $T_i c$ ($c \in C_i^\lambda(T)$, $1 \leq i \leq N$) can be made disjoint such that the resulting sets $T_i^{(c)} c$ have the properties listed in the statement of Theorem 2.16 for all $\lambda \in \Lambda$.

If for $T \in \mathcal{F}(\Gamma)$ the assertions (I)-(III) hold true we say that T is uniformly ε-quasi tiled by the translates $T_i c$.

Remark.
The essential property of Definition 3.3 is given by the inequality (II) in the second item. It shows that the center cets are uniformly distributed in T in the following sense. On average with respect to the measure $\nu(\cdot) := \frac{m_L(\cdot)}{|\Lambda|}$, the mean occurrence frequency of the center sets is nearly constant almost-everywhere of T. This statement is a refinement of Proposition I.3.6 in [OW87] and we use basic constructions of this assertion. Beyond that, we obtain quantitative estimates for the coverings in Definition 3.3. The uniform continuous decomposition condition is the continuous analogue of the uniform special tiling property in Definition 3.1. We also could have required property (III) already for the USTP. However, it was not needed at this point. We will make use of this feature of the UCD condition when we prove the existence of uniform decomposition towers in Theorem 3.6. Another difference is that in the uniformity estimate we consider an inequality normalized by the mass of T. The reason for this is that the number $\#(C_i^\lambda \cap S)$ might be large if S covers a large portion of T. This problem does not occur in the inequality in assertion (II) of Definition 3.1, where one deals with characteristic functions $\mathbb{1}_{C_i^\lambda}(u)$ for $u \in T$ and this latter function is clearly bounded by 1.

In the following theorem, we prove that each unimodular, amenable group is UCDC, i.e. it satisfies the uniform continuous decompositions condition. This is the continuous analogue of Theorem 3.2 for countable amenable groups and significant parts of the proof can be adapted. For the presentation of the proof we essentially follow the proof of Theorem 3.7 in [Pog13a].

Theorem 3.4 (Uniform continuous decompositions).
Each amenable LCSCUH group Γ satisfies the uniform continuous decompositions condition.

Proof.
Let $0 < \varepsilon \leq 1/10$ and $0 < \zeta, \beta < 2^{-N}\varepsilon$, as well as a compact set $e \in B \subseteq \Gamma$ be given, where as usual, $N := N(\varepsilon) := \lceil \log(\varepsilon)/\log(1-\varepsilon) \rceil$. Assume further that $0 < \delta_1 < 6^{-N}\beta/20$.

Note that by Theorem 2.16, we can find (B, ζ^2)-invariant basis sets

$$\{e\} \subseteq T_1 \subseteq \cdots \subseteq T_N,$$

taken from a nested Følner sequence (S_n) that ε-quasi tile the group such that each $(T_N T_N^{-1}, \delta_1)$-invariant set $T \subseteq \Gamma$ can be ε-quasi tiled by translates $T_i c$ that can be made disjoint in a way that they keep the claimed invariance properties with respect to the set B.

We choose $0 < \delta < 1/100$. As in the proof of Theorem 3.2, at various steps of the proof, we will have to make this parameter smaller. This is possible since the corresponding restrictions do not depend on objects developed in the following constructions, but only on ε, β and the basis sets T_i.

We stick close to the proof of Theorem 3.2 and again, we proceed in nine steps. In fact, the steps (1)-(7) can be proven in essentially the same manner as before. In step (8) we have to deal with a major difference in the calculations because for continuous groups, the Haar measure is not equal to the counting measure. Further, unlike in Theorem 3.2 we deal with quantities normalized by the mass of T.

(1) We set $M := \lceil \log(\delta)/\log(1-\delta) \rceil$ and we find $(T_N T_N^{-1}, \delta_1^2)$-invariant sets $\overline{T}_l \supseteq T_N$, $1 \le l \le M$, taken from (S_n) such that the $\overline{T}_l$ δ-quasi tile the group Γ (cf. Theorem 2.16). Define $J := \min\{n \in \mathbb{N} \,|\, S_n = \overline{T}_M\}$. Now let T be a locally closed set which is $(S_J S_J^{-1}, 2^{-M}\delta)$-invariant, hence $(\overline{T}_l \overline{T}_l^{-1}, 2^{-M}\delta)$-invariant for all $1 \le l \le M$ by Lemma 2.3 (iv). Further, we choose $\hat{T}$ to be a (TT^{-1}, δ)-invariant compact set inheriting all the mentioned invariance properties of T, i.e. it is also $(\overline{T}_l \overline{T}_l^{-1}, 2^{-M}\delta)$-invariant for all $1 \le l \le M$. Using Theorem 2.16, we can also make sure that $\hat{T}$ has the special tiling property with respect to $(\{\overline{T}_l\}_{l=1}^M, (S_n), \delta, \beta_1)$, where $0 < \beta_1 < 2^{-M}\delta$. We set $A := \{g \in \Gamma \,|\, Tg \subseteq \hat{T}\}$ and we note that $A \in \mathcal{F}(\Gamma)$ and

$$|A| \ge (1-\delta)|\hat{T}| \tag{3.13}$$

by Lemma 2.11 (ii) and the unimodularity of the group.

(2) We fix a δ-quasi tiling of $\hat{T}$ as in Theorem 2.16 with basis sets $\overline{T}_l$, $1 \le l \le M$, where we make the $\overline{T}_l$-translates actually disjoint such that the resulting disjoint translates $\overline{T}_l^{(c)}c$ are locally closed and still $(T_N T_N^{-1}, 4\delta_1)$-invariant. We note that these disjoint translates $(1-2\delta)$-cover the set $\hat{T}$, i.e.

$$\frac{\left|\bigcup_{l=1}^M \bigcup_{c \in \overline{C}_l} \overline{T}_l^{(c)} c\right|}{|\hat{T}|} = \frac{\sum_{l=1}^M \sum_{c \in \overline{C}_l} |\overline{T}_l^{(c)} c|}{|\hat{T}|} \ge 1 - 2\delta. \tag{3.14}$$

(3) Since all the sets $\overline{T}_l^{(c)}$ are still $(T_N T_N^{-1}, 4\delta_1)$-invariant for all $1 \le l \le M$ and every $c \in \overline{C}_l$ and $4\delta_1 < \delta_0$, it follows from Theorem 2.16 that we can fix in each translate $\overline{T}_l^{(c)}c$ an ε-quasi tiling with the basis sets T_i and finite center sets $C_i^{l,c}$ such that

$$\left| \frac{|T_i C_i^{l,c}|}{|\overline{T}_l^{(c)} c|} - \eta_i(\varepsilon) \right| < \beta \tag{3.15}$$

for every $1 \le i \le N$. Further, we set

$$\hat{C}_i := \bigcup_{l=1}^M \bigcup_{c \in \overline{C}_l} C_i^{l,c}$$

for $1 \le i \le N$ and we note that the $\hat{C}_i$ can be considered as center sets for the basis sets T_i such that the family $\{T_i c\}_{c \in \hat{C}_i}$ is ε-disjoint and such that for $1 \le i < j \le N$, the elements $T_i\hat{C}_i$ and $T_j\hat{C}_j$ are disjoint. For the covering properties of this ε-quasi tiling, a short computation using inequality (3.15) shows that

$$\left|\bigcup_{i=1}^{N} T_i\hat{C}_i\right| \geq (1-2\delta-2\varepsilon)|\hat{T}|,$$

cf. proof of Theorem 3.2, step (3).

(4) The step (4) of Theorem 3.2 shows that by imposing a first condition on δ depending on ε and β, we have

$$\left|\frac{|T_i\hat{C}_i|}{|\hat{T}|} - \eta_i(\varepsilon)\right| < 2\beta \tag{3.16}$$

for all $1 \leq i \leq N$.

(5) A short calculation using the ε-disjointness of the T_i-translates now shows with inequality (3.16) that

$$\left|\frac{\#(\hat{C}_i)}{|\hat{T}|} - \frac{\eta_i(\varepsilon)}{|T_i|}\right| < \frac{2\beta}{|T_i|} + \gamma_i\varepsilon, \tag{3.17}$$

where $\gamma_i := \#(\hat{C}_i)/|\hat{T}|$ for $1 \leq i \leq N$ and $\sum_{i=1}^{N}\gamma_i|T_i| \leq 2$, cf. proof of Theorem 3.2, step (5).

(6) We recall from step (1) that we chose A to be the collection of elements $a \in \Gamma$ such that the translate Ta lies entirely in $\hat{T}$. So for each $a \in A$, we define

$$X(a) := \frac{\left|Ta \cap \left(\hat{T} \setminus \bigcup_{l=1}^{M}\bigcup_{c\in\overline{C}_l}\overline{T}_l^{(c)}c\right)\right|}{|Ta|} = \frac{\left|Ta \setminus \bigcup_{l=1}^{M}\bigcup_{c\in\overline{C}_l}\overline{T}_l^{(c)}c\right|}{|T|}.$$

Note that X is a measurable function on the finite measure set A. It follows then from the Chebyshev inequality that

$$\frac{|\{a \in A \mid X(a) > \sqrt{\delta}\}|}{|A|} \leq \frac{1}{\sqrt{\delta}}\int_A \frac{\left|Ta \setminus \bigcup_{l=1}^{M}\bigcup_{c\in\overline{C}_l}\overline{T}_l^{c}c\right|}{|A|\cdot|T|}\,da.$$

Using the Inequalities (3.13) and (3.14), we obtain by interchanging integrals (Fubini's Theorem),

$$\begin{aligned}
\frac{|\{a \in A \mid X(a) > \sqrt{\delta}\}|}{|A|} &\leq \frac{1}{\sqrt{\delta}}\int_A |A|^{-1}|T|^{-1}\int_\Gamma \mathbb{1}_{Ta\setminus\left(\cup_{l=1}^{M}\cup_{c\in\overline{C}_l}\overline{T}_l^{(c)}c\right)}(g)\,dg\,da \\
&= \frac{1}{\sqrt{\delta}}|A|^{-1}|T|^{-1}\int_A\int_{\hat{T}\setminus\left(\cup_{l=1}^{M}\cup_{c\in\overline{C}_l}\overline{T}_l^{(c)}c\right)} \mathbb{1}_{Ta}(g)\,dg\,da \\
&= \frac{1}{\sqrt{\delta}}|A|^{-1}|T|^{-1}\int_{\hat{T}\setminus\left(\cup_{l=1}^{M}\cup_{c\in\overline{C}_l}\overline{T}_l^{(c)}c\right)}\left(\int_A \mathbb{1}_{Ta}(g)\,da\right)dg \\
&\leq \frac{1}{\sqrt{\delta}}\frac{|\hat{T}\setminus\cup_{l=l}^{M}\cup_{c\in\overline{C}_l}\overline{T}_l^{(c)}c|\cdot|T|}{(1-\varepsilon_1)|\hat{T}|\cdot|T|} \\
&\leq 6\sqrt{\delta}.
\end{aligned}$$

This shows that for most of the a's (up to a portion of $6\sqrt{\delta}$), the corresponding translates Ta are $(1-\sqrt{\delta})$-covered by the disjoint union

$$\bigsqcup_{l=1}^{M} \bigsqcup_{c\in\overline{C}_l,\, \overline{T}_l^{(c)}c\cap Ta\neq\emptyset} \overline{T}_l^{(c)}c.$$

It follows from this, as well as from the invariance properties of T that we can impose a second restriction on δ depending on ε to obtain that up to a portion of $6\sqrt{\delta}$ of the elements $a\in A$, the translates Ta are $(1-3\varepsilon)$-covered by the union $\bigcup_{i=1}^{N} T_i\hat{C}_i$, cf. step (6) in the proof of Theorem 3.2. However, for some elements $a\in A$ and some $c\in\hat{C}_i$, the translate T_ic will have non-trivial intersections with both Ta *and* its complement. In order to cope with this difficulty, we introduce the following notions. Define

$$\begin{aligned} I(a,l) &:= \{c\in\overline{C}_l \,|\, \overline{T}_l^{(c)}c\subseteq Ta\} \\ \partial(a,l) &:= \{c\in\overline{C}_l \,|\, \overline{T}_l^{(c)}c\cap Ta\neq\emptyset \wedge \overline{T}_l^{(c)}c\cap(G\setminus Ta)\neq\emptyset\} \end{aligned}$$

for $a\in A$ and $1\leq l\leq M$. Further, we set

$$C_i(a) := \bigcup_{l=1}^{M}\bigcup_{c\in I(a,l)} C_i^{l,c}$$

for $a\in A$ and $1\leq i\leq N(\varepsilon)$, where the sets $C_i^{l,c}$ are those defined in step (3).

(7) We are now in position to define the family Λ, as well as the corresponding center sets $C_i^{\lambda}(T):=C_i^{\lambda}$ for $1\leq i\leq N$ and $\lambda\in\Lambda$. Namely, we obtain Λ by erasing from the set A the 'bad' elements, i.e.

$$\Lambda := \{\lambda\in A \,|\, X(\lambda)\leq\sqrt{\delta}\}.$$

From the measurability of the map X we deduce that Λ is measurable as well. Note that by inequality (3.13), we have

$$|\Lambda| \geq (1-6\sqrt{\delta})(1-\delta)|\hat{T}|. \tag{3.18}$$

For $\lambda\in\Lambda$, we set

$$C_i^{\lambda} := \{d\in T \,|\, d\lambda\in C_i(\lambda)\}\subseteq \hat{C}_i\lambda^{-1}$$

for $1\leq i\leq N$. An easy computation using the previous step shows that for every $\lambda\in\Lambda$, the set T is $(1-4\varepsilon)$-covered by the union $\bigcup_{i=1}^{N} T_iC_i^{\lambda}$. By construction of the C_i^{λ}, we have indeed $T_iC_i^{\lambda}\subseteq T$ for all $1\leq i\leq N(\varepsilon)$ and every $\lambda\in\Lambda$. This shows property (I) of Definition 3.3.

(8) We still have to show the uniform covering property (II) of Definition 3.3. Set $\tilde{T} := T\setminus\partial_{\overline{T}_M\overline{T}_M^{-1}}(T)$. It follows from the invariance properties of T that with $\delta<\beta$, we have indeed $|\tilde{T}|\geq(1-\beta)|T|$.
Now, fix some Borel set $S\subseteq T$ and fix $1\leq i\leq N$. We show first that there is a constant $\kappa>0$ such that

$$\left|\int_\Lambda \#(C_i^{\lambda}\cap S)\,d\lambda - \#(\hat{C}_i)\cdot|S|\right| \leq \kappa\sqrt{\delta}\,\frac{|T|}{|T_i|}\,|\hat{T}| + \#(\hat{C}_i)\beta|T| \tag{3.19}$$

for all S and every $1 \leq i \leq N$. To do so, note first that

$$\int_\Lambda \#(C_i^\lambda \cap S)\, d\lambda \leq \sum_{c \in \hat{C}_i} \int_\Lambda \mathbb{1}_S(c\lambda^{-1})\, d\lambda.$$

One immediately obtains from this that

$$\int_\Lambda \#(C_i^\lambda \cap S)\, d\lambda \leq \#(\hat{C}_i) \cdot |S|. \tag{3.20}$$

This gives one part of the desired inequality.

On the other hand, we can prove the remaining in the following way. Since by construction, $T\Lambda \subset \hat{T}$, we have $\Lambda \subset T^{-1}\hat{T}$. Furthermore, in the same manner as in step (8) in the proof of Theorem 3.2, we obtain $\hat{C}_i\lambda^{-1} \cap \tilde{T} \subseteq C_i^\lambda$. This implies $C_i^\lambda \cap \tilde{T} = \hat{C}_i\lambda^{-1} \cap \tilde{T}$ and hence one computes

$$\begin{aligned}
\int_\Lambda \#(C_i^\lambda \cap S)\, d\lambda &\geq \int_\Lambda \#(C_i^\lambda \cap S \cap \tilde{T})\, d\lambda = \int_\Lambda \sum_{c \in \hat{C}_i} \mathbb{1}_{S \cap \tilde{T}}(c\lambda^{-1}) \\
&= \int_{T^{-1}\hat{T}} \sum_{c \in \hat{C}_i} \mathbb{1}_{S \cap \tilde{T}}(c\lambda^{-1})\, d\lambda - \int_{T^{-1}\hat{T} \setminus \Lambda} \sum_{c \in \hat{C}_i} \mathbb{1}_{S \cap \tilde{T}}(c\lambda^{-1})\, d\lambda \\
&\geq \int_{T^{-1}\hat{T}} \sum_{c \in \hat{C}_i} \mathbb{1}_{S \cap \tilde{T}}(c\lambda^{-1})\, d\lambda - \int_{T^{-1}\hat{T} \setminus \Lambda} \sum_{c \in \hat{C}_i} \mathbb{1}_{S}(c\lambda^{-1})\, d\lambda \\
&= \int_{T^{-1}\hat{T}} \sum_{c \in \hat{C}_i} \mathbb{1}_{(S \cap \tilde{T})^{-1}c}(\lambda)\, d\lambda - \int_{T^{-1}\hat{T} \setminus \Lambda} \#(\hat{C}_i \cap S\lambda)\, d\lambda.
\end{aligned}$$

It follows from the ε-disjointness of the translates $T_i c$ that the maximal number of elements in $\hat{C}_i$ which can belong to some translate $S\lambda$ must be bounded by

$$2\, \frac{|\partial_{T_i T_i^{-1}}(T\lambda) \cup T\lambda|}{(1-\varepsilon)|T_i|} \leq 2\, \frac{1+\delta}{1-\varepsilon}\, \frac{|T|}{|T_i|},$$

where we also used the $(T_i T_i^{-1}, \delta)$-invariance of the set T. Hence, we can estimate

$$\int_\Lambda \#(C_i^\lambda \cap S)\, d\lambda \geq \sum_{c \in \hat{C}_i} \int_{T^{-1}\hat{T}} \mathbb{1}_{(S \cap \tilde{T})^{-1}c}(\lambda)\, d\lambda - 2\, \frac{1+\delta}{1-\varepsilon}\, \frac{|T|}{|T_i|}\, |T^{-1}\hat{T} \setminus \Lambda|. \tag{3.21}$$

Moreover, with inequality (3.18), we obtain

$$\begin{aligned}
|T^{-1}\hat{T} \setminus \Lambda| &= |T^{-1}\hat{T}| - |\Lambda| \\
&\leq |\hat{T} \cup \partial_{TT^{-1}}(\hat{T})| - (1 - 6\sqrt{\delta})(1-\delta)|\hat{T}| \\
&\leq [1 + \delta - (1 - 6\sqrt{\delta})(1-\delta)]\, |\hat{T}| \\
&\leq 8\sqrt{\delta}\, |\hat{T}|
\end{aligned} \tag{3.22}$$

by the invariance properties of $\hat{T}$. Now making use of $\varepsilon, \delta < 1/2$ and the estimate in inequality (3.22), the inequality (3.21) yields

$$\begin{aligned}\int_\Lambda \#(C_i^\lambda \cap S)\, d\lambda &\geq \#(\hat{C}_i) \cdot |S \cap \tilde{T}| - 2 \cdot 4 \cdot \frac{|T|}{|T_i|} \cdot 8\sqrt{\delta}\, |\hat{T}| \\ &\geq \#(\hat{C}_i) \cdot |S \cap \tilde{T}| - 64\sqrt{\delta}\, \frac{|T|}{|T_i|}\, |\hat{T}| \\ &\geq \#(\hat{C}_i)|S| - \#(\hat{C}_i)\beta|T| - 64\sqrt{\delta}\, \frac{|T|}{|T_i|}\, |\hat{T}|.\end{aligned}$$

Note that we used here the fact that $|\tilde{T}| \geq (1-\beta)|T|$, as well as the inclusions $S \subseteq T$ and $\tilde{T} \subseteq T$. Together with inequality (3.20), this shows (3.19) with $\kappa = 64$.

(9) We infer from inequality (3.18) that for δ small enough,

$$\begin{aligned}\left| \frac{\#(\hat{C}_i)}{|\hat{T}|} - \frac{\#(\hat{C}_i)}{|\Lambda|} \right| &\leq \left(\frac{1}{(1-\delta)(1-6\sqrt{\delta})} - 1 \right) \frac{\#(\hat{C}_i)}{|\hat{T}|} \\ &\leq \varepsilon \cdot \gamma_i,\end{aligned}$$

where $\gamma_i = \#(\hat{C}_i)/|\hat{T}|$ as above.

From inequality (3.17), it follows that

$$\left| \frac{\#(\hat{C}_i)}{|\Lambda|} - \frac{\varepsilon(1-\varepsilon)^{N-i}}{|T_i|} \right| < 2\,\gamma_i \cdot \varepsilon + \frac{2\beta}{|T_i|}. \tag{3.23}$$

Hence, using the triangle inequality in inequality (3.19) and plugging in inequality (3.23), we arrive at

$$\begin{aligned}\left| |\Lambda|^{-1} \int_\Lambda \frac{\#(C_i^\lambda \cap S)}{|T|}\, d\lambda - \frac{\varepsilon(1-\varepsilon)^{N-i}}{|T_i|} \cdot \frac{|S|}{|T|} \right| &< 64 \frac{\sqrt{\delta}}{|T_i|} \frac{|\hat{T}|}{|\Lambda|} + 2\,\gamma_i \cdot \varepsilon + \frac{3\beta}{|T_i|} \\ &\overset{(3.18)}{\leq} \frac{64\sqrt{\delta}}{(1-\delta)(1-6\sqrt{\delta})|T_i|} + 2\,\gamma_i \cdot \varepsilon + \frac{3\beta}{|T_i|}.\end{aligned}$$

So it remains to choose δ small enough such that we have finally proven the theorem. □

For the general mean ergodic theorem which will be proven in Chapter 4, we will need the concept of a so-called *uniform decomposition tower*. In this context, we construct a uniform family of ε-quasi tilings of a highly TT^{-1}-invariant set $\hat{T}$ as in Theorem 3.4 (with index set $\Upsilon \subseteq \Gamma$) such that each ε-quasi tiling associated with a $y \in \Upsilon$ generates another uniform family of ε-quasi tilings for the set T with index set $\Lambda \subseteq \Gamma$. Moreover, this set Λ will not depend on the choice for $y \in \Upsilon$. In the following definition we make this concept precise.

Definition 3.5 (Uniform decomposition tower).
Let Γ be an amenable LCSCUH group. We say that Γ has the uniform decomposition tower condition *(UDTC) if for every strong Følner sequence (U_k) in Γ, the following statements hold true.*

- *For each* $0 < \varepsilon \leq 1/10$, $N := N(\varepsilon) := \lceil \log(\varepsilon)/\log(1-\varepsilon) \rceil$, *for arbitrary numbers* $0 < \beta, \zeta < 2^{-N}\varepsilon$, *for every nested Følner sequence* (S_n), *and for each compact set* $e \in B \subseteq \Gamma$, *the group* Γ *satisfies the special tiling property according to Definition 2.15 with* ε*-prototiles*
$$\{\mathrm{e}\} \subseteq T_1 \subseteq T_2 \subseteq \cdots \subseteq T_N,$$
where $T_i \in \{S_n \,|\, n \geq i\}$ *for* $1 \leq i \leq N(\varepsilon)$ *and where the latter prototile sets are also* (B, ζ^2)*-invariant.*

- *For fixed positive numbers* $\varepsilon > 0, \beta > 0, \zeta > 0$ *and for a fixed compact set* $e \in B \subseteq G$, *there are numbers* $K \in \mathbb{N}$ *and* $\eta_0 > 0$ *depending on* ε, β *and the basis sets* T_i *such that for each* $k \geq K$ *and for every* $0 < \eta < \eta_0$, *there is some* $(U_k U_k^{-1}, \eta)$*-invariant, compact set* $\hat{U}_k$, *as well as sets* $\Lambda_k, \Upsilon_k \in \mathcal{F}(\Gamma)$ *of finite measure, such that*

  (I) $U_k \Lambda_k \subseteq \hat{U}_k$, $(1-\beta)|\hat{U}_k| \leq |\Lambda_k| \leq |\hat{U}_k|$ *and* U_k *is* $(T_N T_N^{-1}, \beta)$*-invariant,*

  (II) there is a family
$$\{\hat{C}_i^y(\hat{U}_k) \,|\, y \in \Upsilon_k,\, 1 \leq i \leq N\}$$
  of finite center sets for the prototile sets T_i *such that the covering of* $\hat{U}_k$ *given by the translates* $T_i c, c \in \hat{C}_i^y(\hat{U}_k)$, *satisfies the assertions (I) and (II) of Definition 3.3,*

  (III) for each $y \in \Upsilon_k$, *the set* U_k *is uniformly* ε*-quasi tiled by the family*
$$\{C_i^{y,\lambda}(U_k) \,|\, \lambda \in \Lambda_k,\, 1 \leq i \leq N\}$$
  of finite center sets for ε*-quasi tiles* T_i *according to Definition 3.3, where*
$$\tilde{U}_k \cap \hat{C}_i^y(\hat{U}_k)\lambda^{-1} \subseteq C_i^{y,\lambda}(U_k) \subseteq U_k \cap \hat{C}_i^y(\hat{U}_k)\lambda^{-1}$$
  for all $1 \leq i \leq N$ *and every* $\lambda \in \Lambda_k$, *and* $\tilde{U}_k \subseteq U_k$ *is a measurable,* $(T_N T_N^{-1}, 4\beta)$*-invariant set with* $|\tilde{U}_k| \geq (1-\beta)|U_k|$.

  In this situation, we say that the pair (Υ_k, Λ_k) *is a* uniform decomposition tower *for the pair* $(U_k, \hat{U}_k)$ *with respect to* $(\{T_i\}_{i=1}^{N(\varepsilon)}, (S_n), \varepsilon, \beta)$.

In fact, the existence of uniform decomposition towers can be proven in full generality. This is shown in the following theorem, which is a strengthening of Theorem 3.4.

Theorem 3.6 (Uniform Decomposition Tower).
Each amenable LCSCUH group Γ *satisfies the uniform decomposition tower condition.*

Remark.
At a first sight, one might wonder whether the existence of uniform decomposition towers follows trivially from a repeated application of Theorem 3.4. However, we will have to use more involved arguments as we would like the sets Υ_k and Λ_k to be *independent* from each other.

PROOF.
Let $0 < \varepsilon \leq 1/10$ and $0 < \zeta, \beta < 2^{-N}\varepsilon$, as well as a compact set $e \in B \subseteq \Gamma$ be given, where $N := N(\varepsilon) := \lceil \log(\varepsilon)/\log(1-\varepsilon) \rceil$. Assume further that (U_k) is a strong Følner sequence.

By Theorem 2.16, we can find (B, ζ^2)-invariant basis sets

$$\{e\} \subseteq T_1 \subseteq \cdots \subseteq T_N,$$

taken from a nested Følner sequence (S_n) that ε-quasi tile the group according to Definition 2.15.

As before (proof of Theorem 3.4), we choose $0 < \delta < 1/100$, and at various steps of the proof, we reduce this parameter for our purposes (restrictions depending on ε, β and the T_i). As before, set $M := \lceil \log(\delta)/\log(1-\delta) \rceil$.

(1) Let $J \in \mathbb{N}$ and $0 < \delta_0 < \beta/4$ be the parameters from Definition 3.4 that can be found by Theorem 3.4. Then we can choose a finite sequence of $(S_J S_J^{-1}, \delta_0^2/64)$-invariant sets from (S_n), say $(\overline{T}_l)_l$, $1 \leq l \leq M$,

$$T_N \subseteq S_J \subseteq \overline{T}_1 \subseteq \overline{T}_2 \subseteq \cdots \subseteq \overline{T}_M,$$

which δ-quasi tile the group Γ according to Definition 2.15 for any parameter $0 < \beta_1 < 2^{-M}\delta$. Take some number $K \in \mathbb{N}$ depending on ε, β and the basis sets T_i such that for each $k \geq K$, the set $T := U_k$ is $(\overline{T}_l \overline{T}_l^{-1} T_N T_N^{-1}, 2^{-l}\delta)$-invariant for all $1 \leq l \leq M$. Moreover, we choose $\hat{T} := \hat{U}_k$, where $\hat{U}_k$ is a $(U_k U_k^{-1}, \delta)$-invariant set which has all the mentioned invariance properties of $T = U_k$ and which has the special tiling property with respect to $(\{\overline{T}\}_{l=1}^M, (S_n), \delta, \beta_1)$. This can for instance be done by setting $\hat{T} = \hat{U}_k = U_{\tilde{K}}$ for $\tilde{K} \in \mathbb{N}$ large enough. Defining $\eta_0 := \delta/2$, we will be able to show all assertions claimed for $\hat{T} = \hat{U}_k$.

(2) We use Theorem 2.16 to fix a δ-quasi tiling of $\hat{T}$ with basis sets $\overline{T}_l$ and finite center sets $\overline{C}_l$, $1 \leq l \leq M$. In fact, by Theorem 2.16, we can actually find *disjoint* translates $\overline{T}_l'(c)c$ which are still $(S_J S_J^{-1}, \delta_0/2)$-invariant for every $1 \leq l \leq M$ and all $c \in \overline{C}_l$.

(3) Next, we choose $\tilde{T}$ as a $(\overline{T}_M \overline{T}_M^{-1}, \tilde{\delta})$-invariant and $(\hat{T}\hat{T}^{-1}, \tilde{\delta})$-invariant, compact set, where

$$0 < \tilde{\delta} < \Big(\frac{\delta}{8 \sum_{l=1}^M \#(\overline{C}_l)} \Big)^2.$$

Now by Theorem 3.4, in every translate $\overline{T}_l'(c)c$, $1 \leq l \leq M, c \in \overline{C}_l$, we will find a uniform family of ε-quasi tilings with finite T_i-center sets $C_i^y(l,c)$, where the y are taken from a set $\Upsilon(l,c) \in \mathcal{F}(\Gamma)$ of finite measure. We have seen before that those uniform coverings can be induced by some background quasi tiling of a compact set $\widehat{\overline{T}_l'(c)c}$ which just has to be invariant enough with respect to $\overline{T}_l \overline{T}_l^{-1}$ for $1 \leq l \leq M$. Hence, without loss of generality, we can work with one single compact set $\tilde{T} \subseteq \Gamma$ replacing all the sets $\widehat{\overline{T}_l'(c)c}$ for $1 \leq l \leq M$ and every $c \in \overline{C}_l$.

(4) By the construction given in inequality (3.18) in the proof of Theorem 3.4 and with our choice for $\tilde{\delta}$ in the previous step, one obtains

$$\begin{aligned}
|\Upsilon(l,c)| &\geq (1-\tilde{\delta}-6\sqrt{\tilde{\delta}})|\tilde{T}| \\
&\geq (1-7\sqrt{\tilde{\delta}})\,|\tilde{T}| \\
&\geq \frac{1-7\sqrt{\tilde{\delta}}}{1+\tilde{\delta}}\,|\hat{T}^{-1}\tilde{T}| \\
&\overset{\tilde{\delta}<1/100}{\geq} (1-8\sqrt{\tilde{\delta}})\,|\hat{T}^{-1}\tilde{T}| \\
&\geq \left(1-\frac{\delta}{\sum_{l=1}^{M}\#(\overline{C}_l)}\right)|\hat{T}^{-1}\tilde{T}| \qquad (3.24)
\end{aligned}$$

for every $1 \leq l \leq M$ and each $c \in \overline{C}_l$. Note that in the third inequality we used the fact that $\tilde{T}$ was chosen to be $(\hat{T}\hat{T}^{-1}, \tilde{\delta})$-invariant.

Define

$$\Upsilon_k := \Upsilon := \bigcap_{l=1}^{M}\bigcap_{c\in\overline{C}_l}\Upsilon(l,c) \in \mathcal{F}(\Gamma).$$

Since $\Upsilon(l,c) \subset \hat{T}^{-1}\tilde{T}$ for all $1 \leq l \leq M$ and every $c \in \overline{C}_l$ (see proof of Theorem 3.4), it follows from elementary measure theory from (3.24) that for all $1 \leq l \leq M$ and all $c \in \overline{C}_l$,

$$|\Upsilon| \geq (1-\delta)\,|\hat{T}^{-1}\tilde{T}| \geq (1-\delta)\,|\Upsilon(l,c)|. \qquad (3.25)$$

(5) We now show the uniform covering property for the set $\hat{T} = \hat{U}_k$. To do so, fix a Borel set $\hat{S} \subset \hat{T}$, as well as some $1 \leq i \leq N$. As a preparative step, we use Theorem 3.4 to establish the uniform covering property for each translate $\overline{T}_l'(c)c$. Indeed, it follows from (II) in the second item of Definition 3.3 that

$$\begin{aligned}
&\left| |\Upsilon(l,c)|^{-1}\int_{\Upsilon(l,c)} \frac{\#[(\hat{S}\cap\overline{T}_l'(c)c)\cap C_i^y(l,c)]}{|\overline{T}_l'(c)|}\,dy - \frac{\varepsilon(1-\varepsilon)^{N-i}}{|T_i|}\,\frac{|\hat{S}\cap\overline{T}_l'(c)|}{|\overline{T}_l'(c)|}\right| \\
&\quad < 4\,\frac{\beta}{|T_i|} + 2\,\gamma_i(l,c)\,\varepsilon \qquad (3.26)
\end{aligned}$$

for all $1 \leq l \leq M$ and for every $c \in \overline{C}_l$, where $\sum_{i=1}^{N}\gamma_i(l,c)\,|T_i| \leq 2$. Furthermore, we compute

$$\begin{aligned}
&\left| |\Upsilon(l,c)|^{-1}\int_{\Upsilon(l,c)} \frac{\#[(\hat{S}\cap\overline{T}_l'(c)c)\cap C_i^y(l,c)]}{|\overline{T}_l'(c)|}\,dy - |\Upsilon|^{-1}\int_{\Upsilon}\frac{\#[(\hat{S}\cap\overline{T}_l'(c)c)\cap C_i^y(l,c)]}{|\overline{T}_l'(c)|}\,dy\right| \\
\leq\; & \left|\frac{|\Upsilon(l,c)|-|\Upsilon|}{|\Upsilon||\Upsilon(l,c)|}\right| \int_{\Upsilon}\frac{\#[(\hat{S}\cap\overline{T}_l'(c)c)\cap C_i^y(l,c)]}{|\overline{T}_l'(c)|}\,dy \\
&+ |\Upsilon(l,c)|^{-1}\int_{\Upsilon(l,c)\setminus\Upsilon}\frac{\#[(\hat{S}\cap\overline{T}_l'(c)c)\cap C_i^y(l,c)]}{|\overline{T}_l'(c)|}\,dy.
\end{aligned}$$

Note that due to δ-disjointness and due to the fact that all sets $\overline{T}_l'(c)$ are $(T_N T_N^{-1}, \delta_0/2)$-invariant, it is true that

$$\frac{\#[(\hat{S}\cap\overline{T}_l'(c)c)\cap C_i^y(l,c)]}{|\overline{T}_l'(c)|} \leq \frac{(1+\delta_0)|\overline{T}_l'(c)|}{(1-2\delta)|T_i|\cdot|\overline{T}_l'(c)|} = \frac{1+\delta_0}{(1-2\delta)|T_i|}.$$

Thus, with inequality (3.25) and $\delta < 1/100$, one arrives at

$$\left| |\Upsilon(l,c)|^{-1}\int_{\Upsilon(l,c)} \frac{\#[(\hat{S}\cap\overline{T}_l'(c)c)\cap C_i^y(l,c)]}{|\overline{T}_l'(c)|}\,dy - |\Upsilon|^{-1}\int_{\Upsilon} \frac{\#[(\hat{S}\cap\overline{T}_l'(c)c)\cap C_i^y(l,c)]}{|\overline{T}_l'(c)|}\,dy \right| \leq \frac{8\delta(1+\delta_0)}{|T_i|} \leq \frac{16\delta}{|T_i|}. \tag{3.27}$$

So combining the inequalities (3.26) and (3.27) with the triangle inequality yields

$$\left| |\Upsilon|^{-1}\int_{\Upsilon} \frac{\#[(\hat{S}\cap\overline{T}_l'(c)c)\cap C_i^y(l,c)]}{|\overline{T}_l'(c)|}\,dy - \frac{\varepsilon(1-\varepsilon)^{N-i}}{|T_i|}\,\frac{|\hat{S}\cap\overline{T}_l'(c)c|}{|\overline{T}_l'(c)|} \right| < 4\,\frac{\beta}{|T_i|} + 2\,\tilde{\gamma}_i\,\varepsilon + \frac{16\delta}{|T_i|} \tag{3.28}$$

for all $1 \leq l \leq M$ and every $c \in \overline{C}_l$, where $\tilde{\gamma}_i := \max_{l,c}\, \gamma_i(l,c)$.

We are now in position to prove the uniform covering property of the family Υ for the whole set $\hat{T}$. Firstly, we set

$$\hat{C}_i^y := \hat{C}_i^y(\hat{T}) := \hat{C}_i^y(\hat{U}_k) := \bigcup_{l=1}^{M}\bigcup_{c\in\overline{C}_l} C_i^y(l,c).$$

Then, for each $y \in \Upsilon$, the finite set $\hat{C}_i^y$ contains the T_i-centers of an ε-quasi tiling of the set $\hat{T}$, see the steps (3) to (5) of the proof of Theorem 3.4. Next, by disjointness of the $\overline{T}_l'(c)c$, we compute with

$$\begin{aligned}
&\left| |\Upsilon|^{-1}\int_{\Upsilon} \frac{\#(\hat{S}\cap\hat{C}_i^y)}{|\hat{T}|}\,dy - \frac{|\hat{S}|}{|\hat{T}|}\cdot\frac{\eta_i(\varepsilon)}{|T_i|} \right| \\
\leq\; &\left| \sum_{l=1}^{M}\sum_{c\in\overline{C}_l} \frac{|\overline{T}_l'(c)c|}{|\hat{T}|}\,|\Upsilon|^{-1}\int_{\Upsilon}\frac{\#[(\hat{S}\cap\overline{T}_l'(c)c\cap C_i^y(l,c)]}{|\overline{T}_l'(c)c|}\,dy - \frac{\left|\hat{S}\cap\bigcup_{l=1}^{M}\bigcup_{c\in\overline{C}_l}\overline{T}_l'(c)c\right|}{|\hat{T}|}\cdot\frac{\eta_i(\varepsilon)}{|T_i|} \right| \\
&+ \frac{\left|\hat{T}\setminus\left(\bigcup_{l=1}^{M}\bigcup_{c\in\overline{C}_l}\overline{T}_l'(c)c\right)\right|}{|\hat{T}|}\cdot\frac{\eta_i(\varepsilon)}{|T_i|} + |\Upsilon|^{-1}\int_{\Upsilon}\frac{\#\left((\hat{T}\setminus\bigcup_{l=1}^{m}\bigcup_{c\in\overline{C}_l}\overline{T}_l'(c)c)\cap\hat{C}_i^y\right)}{|\hat{T}|}\,dy.
\end{aligned}$$

Due to ε-disjointness and Lemma 2.3 (ii),(iii) we obtain

$$\begin{aligned}
\frac{\#\Big((\hat{T}\setminus\bigcup_{l=1}^{M}\bigcup_{c\in\overline{C}_l}\overline{T}_l'(c)c)\cap\hat{C}_i^y\Big)}{|\hat{T}|} &\le \frac{\Big|\hat{T}\setminus\Big(\bigcup_{l=1}^{M}\bigcup_{c\in\overline{C}_l}\overline{T}_l'(c)c\Big)\Big|}{(1-2\varepsilon)|T_i||\hat{T}|}+\frac{|\partial_{T_NT_N^{-1}}(\hat{T})|}{(1-2\varepsilon)|T_i||\hat{T}|}\\
&\quad+\frac{\sum_{l=1}^{M}\sum_{c\in\overline{C}_l}|\partial_{T_NT_N^{-1}}(\overline{T}_l'(c)c)|}{(1-2\varepsilon)|T_i||\hat{T}|}\\
&\le \frac{2\delta+\delta+\delta_0}{(1-2\varepsilon)|T_i|}=\frac{3\delta+\delta_0}{(1-2\varepsilon)|T_i|}
\end{aligned}$$

for all $y\in\Upsilon$. We continue with

$$\begin{aligned}
&\left||\Upsilon|^{-1}\int_\Upsilon\frac{\#(\hat{S}\cap\hat{C}_i^y)}{|\hat{T}|}\,dy-\frac{|\hat{S}|}{|\hat{T}|}\cdot\frac{\eta_i(\varepsilon)}{|T_i|}\right|\\
\le&\ \Bigg|\sum_{l=1}^{M}\sum_{c\in\overline{C}_l}\frac{|\overline{T}_l'(c)c|}{|\hat{T}|}\,|\Upsilon|^{-1}\int_\Upsilon\frac{\#[(\hat{S}\cap\overline{T}_l'(c)c)\cap C_i^y(l,c)]}{|\overline{T}_l'(c)c|}\,dy\\
&-\sum_{l=1}^{M}\sum_{c\in\overline{C}_l}\frac{|\overline{T}_l'(c)c|}{|\hat{T}|}\,\frac{|\hat{S}\cap\overline{T}_l'(c)c|}{|\overline{T}_l'(c)c|}\,\frac{\eta_i(\varepsilon)}{|T_i|}\Bigg|\ +\ 2\delta\,\frac{\eta_i(\varepsilon)}{|T_i|}\ +\ \frac{3\delta+\delta_0}{(1-2\varepsilon)|T_i|}\\
\le&\ \sum_{l=1}^{M}\sum_{c\in\overline{C}_l}\frac{|\overline{T}_l'(c)c|}{|\hat{T}|}\cdot\left||\Upsilon|^{-1}\int_\Upsilon\frac{\#[(\hat{S}\cap\overline{T}_l'(c)c)\cap C_i^y(l,c)]}{|\overline{T}_l'(c)|}\,dy-\frac{|\hat{S}\cap\overline{T}_l'(c)c|}{|\overline{T}_l'(c)|}\,\frac{\eta_i(\varepsilon)}{|T_i|}\right|\\
&+2\delta\,\frac{\eta_i(\varepsilon)}{|T_i|}+\frac{3\delta+\delta_0}{(1-2\varepsilon)|T_i|}\\
\overset{(3.28)}{\le}&\ 4\,\frac{\beta}{|T_i|}+2\,\tilde{\gamma}_i\,\varepsilon+\frac{22\delta}{|T_i|}+\frac{2\delta_0}{|T_i|}.
\end{aligned}$$

By noting that $\delta_0<\beta/4$ and by making δ small enough (depending on ε and β), this shows the uniformity estimate claimed in the second item of (II) of Definition 3.5. The remaining assertion in the item (I) of Definition 3.3 is satisfied by construction.

(6) Next, we verify the statement (III) of the second item of Definition 3.5. We choose $\Lambda=\Lambda_k$ in exactly the same manner as in the proof of Theorem 3.4, steps (6) and (7). Note that Λ results from considerations concerning the sets $T,\hat{T}$, and the $\overline{T}_l'(c)c$ for $1\le l\le M$ and $c\in\overline{C}_l$, but not from the tilings constructed above. In this sense, Λ is indeed independent of $\Upsilon=\Upsilon_k$.

As demonstrated above, for every $y\in\Upsilon$, the set $\hat{T}$ is ε-quasi tiled by the basis sets T_i with corresponding finite center sets $\hat{C}_i^y$ for $1\le i\le N$ and all the translates T_ic ($1\le i\le N$, $c\in\hat{C}_i^y$) are contained in some translate $\overline{T}_l^d d$, where $1\le l\le M$ and $d\in\overline{C}_l$. Thus, if we fix $y\in\Upsilon$, we can argue as in the proof of Theorem 3.4, steps (6) and (7), to define

$$\begin{aligned}
C_i^{y,\lambda}(U_k):=C_i^{y,\lambda}(T) &:= \{d\in T\,|\,d\lambda\in C_i^y(\lambda)\}\\
&= T\cap C_i^y(\lambda)\lambda^{-1}
\end{aligned}$$

for all $1 \leq i \leq N$ and all $\lambda \in \Lambda$, where

$$C_i^y(\lambda) = \bigcup_{l=1}^{M} \bigcup_{c \in I(\lambda,l)} C_i^y(l,c) \subseteq \hat{C}_i^y \cap U_k.$$

Note that as in step (6) of the proof of Theorem 3.6, we have set $I(\lambda, l) := \{c \in \overline{C}_l \,|\, \overline{T}_l'(c)c \subseteq T\lambda\}$. Continueing with the steps (8) and (9) in the proof of Theorem 3.4, we obtain the desired uniform covering. Further, set $\tilde{U}_k := U_k \setminus \partial_{\overline{T}_M \overline{T}_M^{-1}}(U_k)$. Then indeed, $|\tilde{U}_k| \geq (1-\beta)|U_k|$ and $\hat{C}_i^y \lambda^{-1} \cap \tilde{U}_k \subseteq C_i^{y,\lambda}$ for all $\lambda \in \Lambda$. Further, we infer from Lemma 2.3 (iv) and (viii) and the fact that U_k is $(\overline{T}_M \overline{T}_M^{-1} T_N T_N^{-1}, \beta)$-invariant (for δ small enough) that

$$\frac{|\partial_{T_N T_N^{-1}}(\tilde{U}_k)|}{|\tilde{U}_k|} \leq 2\,\frac{|\partial_{T_N T_N^{-1}}(U_k)|}{|U_k|} + 2\,\frac{|\partial_{\overline{T}_M \overline{T}_M^{-1} T_N T_N^{-1}}(U_k)|}{|U_k|} \leq 4\beta.$$

Since these considerations must hold true for all $y \in \Upsilon$, we have proven the statement (III) in the second item of Definition 3.5.

(7) The validity of statement (I) in the second item of Definition 3.5 follows by construction for δ small enough depending on β.

This finishes the proof of the theorem. □

4 Banach space-valued ergodic theorems

The following elaborations deal with mappings

$$F : \mathcal{F}(\Gamma) \to (Z, \|\cdot\|_Z),$$

where Γ is an amenable LCSCUH group and where Z is a Banach space with norm $\|\cdot\|_Z$. We will assume that the values $\|F(Q)\|_Z/|Q|$ are uniformly bounded. Further, we suppose that for unions $Q = \bigsqcup_{k=1}^m Q_k$ of pairwise disjoint sets $Q_k \in \mathcal{F}(\Gamma)$, the difference of the expressions $F(Q)$ and $\sum_{k=1}^m F(Q_k)$ in norm is controlled by a boundary term function $b : \mathcal{F}(\Gamma) \to [0, \infty)$. Those latter mappings have the crucial property that they asymptotically relatively vanish for strong Følner sequences (U_j), i.e. $\lim_{j\to\infty} b(U_j)/|U_j| = 0$. We will F call an almost-additive, Z-valued function on $\mathcal{F}(\Gamma)$. Canonical examples for almost-additive functions are ergodic integral averages: let Γ act on a probability space (Ω, μ) by measure preserving transformations, set $Z = L^2(\Omega, \mu)$ and take $f \in L^2(\Omega, \mu)$. In this situation, one can define an almost-additive (in fact additive) mapping via $F(Q)(\omega) := \int_Q f(g\omega)\, dg$. By von Neumann's ergodic theorem (cf. e.g. [Gre73], Corollary 3.4.), there is some $f^* \in L^2(\Omega, \mu)$ such that

$$\lim_{j\to\infty} \left\| |U_j|^{-1} \int_{U_j} f(g\cdot)\, dg - f^* \right\|_{L^2(\Omega,\mu)} = \lim_{j\to\infty} \left\| \frac{F(U_j)}{|U_j|} - f^* \right\|_{L^2(\Omega,\mu)} = 0$$

for every (strong or weak) Følner sequence (U_j). Thus, it is a natural question to find an abstract setting in which one can expect norm convergence of the expressions $F(U_j)/|U_j|$ for general almost-additive mappings F. In this chapter, we show that this holds true under natural ergodicity assumptions. We discuss two different approaches, one for countable groups and another one for general amenable (possibly continuous) LCSCUH groups. Firstly, we endow countable amenable groups with a colouring map $\mathcal{C} : \Gamma \to \mathcal{A}$, where $\mathcal{A}$ is a finite set. For suitable Følner sequences, $\mathcal{C}$ induces some kind of measure on the set of all possible finite coloured patterns in the coloured group. In Theorem 4.4, we prove the Banach space convergence for almost-additive functions F which take their values according to the shape of the patterns. The latter assertion is valid for all countable, amenable groups. Hence, it is a major extension of the corresponding theorems in [LMV08, LSV11]. This result can be found in [PS14]. Secondly, one might work with classical ergodic theory for general amenable LCSCUH groups. By imposing a mild compactness condition on the Banach space under consideration, we use the uniform tiling results of Chapter 3 to generalize the abstract mean ergodic theorem for integral averages given in Theorem 3.2 of [Gre73]. Precisely, we show in Theorem 4.15 that there is some element $F^* \in Z$ such that for *all* strong Følner sequences, the ratios $F(U_j)/|U_j|$ converge to F^* in the topology of Z. This result has appeared in [Pog13a], Theorem 5.7.

4.1 Countable amenable groups

In this section, we assume that Γ is a countable amenable group. As before, we write $\mathcal{F}(\Gamma)$ for the set of all non-empty, finite subsets of Γ. We consider Γ to be coloured by finitely many colours. As an ergodicity assumption, we consider Følner sequences along which the occurrence frequencies of all possible coloured patterns must exist as a limit. The first model of this kind has been outlined by Lenz in [Len02], where the author characterizes the existence of pattern frequencies in a subshift dynamical system by the validity of an almost-additive convergence theorem. In Theorem 1 of [LMV08] the authors prove the Banach space convergence along Følner sequences for almost-additive functions over $\Gamma = \mathbb{Z}^d$. This latter assertion was generalized in [LSV11], cf. Theorem 3.1, to all countable, amenable groups that possess a monotile Følner sequence with symmetric grid system. In Theorem 4.4, we get rid of this assumption and prove the convergence of almost-additive functions over *arbitrary* countable, amenable groups. The major tool for this undertaking is the uniform special tiling property of all countable amenable groups, see Theorem 3.2. This achievement appears (among other things) in a common work of the author with Schwarzenberger, cf. Theorem 5.5 in [PS14]. In his Ph.D. thesis [Sch13], Schwarzenberger adapts our proof of Theorem 4.4 to the case of finitely generated, amenable groups.

We assume in the following that $\mathcal{A}$ is a finite set that colours the group Γ via some mapping $\mathcal{C} : \Gamma \to \mathcal{A}$. For $Q \in \mathcal{F}(\Gamma)$, we call $P := \mathcal{C}_{|Q}$ the *pattern of* Q. In this situation, we say that Q is the *domain* $D(P)$ of P. The set of all finite patterns is denoted by $\mathcal{P}$ and for a fixed $Q \in \mathcal{F}(\Gamma)$, we write $\mathcal{P}(Q) \subseteq \mathcal{P}$ for the set of possible patterns with domain Q. Further, if P is a pattern and $Q \subseteq D(P)$, the *restriction* of P to Q is given by

$$P_{|Q} : Q \to \mathcal{A} : h \mapsto P(h).$$

Moreover, for a pattern $P \in \mathcal{P}$ and $g \in \Gamma$, we define the *translation* Pg of P by

$$Pg : D(P)g \to \mathcal{A} : hg \mapsto P(h).$$

Translations and restrictions of whole colourings are defined analogously. Note that translations by group elements attach equivalence classes $\tilde{P}$ to patterns $P \in \mathcal{P}$. Precisely, two patterns are equivalent if and only if one pattern is some translation of the other. The set of all possible induced equivalence classes shall be denoted by $\tilde{\mathcal{P}}$.

Given two patterns $P, P' \in \mathcal{P}$, the *number of occurrences* of the pattern P in P' is defined as

$$\#_P(P') := \# \left(\{ g \in \Gamma \mid D(P)g \subseteq D(P'),\ P'_{|D(P)g} = Pg \} \right).$$

Counting occurrencies of patterns along a Følner sequence $(U_j)_{j\in\mathbb{N}}$ leads to the definition of *frequencies*. If for a pattern P and a Følner sequence $(U_j)_{j\in\mathbb{N}}$, the limit

$$\nu_P := \lim_{j\to\infty} \frac{\#_P(\mathcal{C}_{|U_j})}{|U_j|}$$

exists, we call ν_P the *frequency of* P *in the coloring* $\mathcal{C}$ *along* $(U_j)_{j\in\mathbb{N}}$.

In order to define the notion of an almost-additive set function, we need an object to measure the degree of additivity of a mapping. We do this via a so-called *boundary term*.

Definition 4.1 (Boundary term).
Let Γ be a countable, amenable group. A mapping

$$b : \mathcal{F}(\Gamma) \to [0, \infty)$$

is called boundary term *for Γ if*

(i) it is bounded, *i.e. there exists some constant $D > 0$ such that $b(Q) \leq D\,|Q|$ for all $Q \in \mathcal{F}(\Gamma)$,*

(ii) $\lim_{j\to\infty} \frac{b(U_j)}{|U_j|} = 0$ for every Følner sequence (U_j) in Γ,

(iii) it is translation invariant, *i.e. $b(Q) = b(Qg)$ for all $Q \in \mathcal{F}(\Gamma)$ and every $g \in \Gamma$,*

(iv) it satisfies the inequalities

$$b(Q \cap Q') \leq b(Q) + b(Q'), \quad b(Q \cup Q') \leq b(Q) + b(Q'), \quad b(Q \setminus Q') \leq b(Q) + b(Q')$$

for all $Q, Q' \in \mathcal{F}(\Gamma)$.

For each coloured pattern P, we then define $b(P) := b(D(P))$. (Note that due to the invariance property (iii), the value $b(P)$ will only depend on the equivalence class of a pattern.)

With this definition at hand, we are in position to introduce the notion of an almost-additive, Banach space-valued set function.

Definition 4.2.
Let $(Z, \|\cdot\|)$ be a Banach space and assume that $\mathcal{A}$ is a finite set. Suppose that Γ is a countable group with a colouring $\mathcal{C} : \Gamma \to \mathcal{A}$. A function

$$F : \mathcal{F}(\Gamma) \to (Z, \|\cdot\|)$$

is called almost-additive *if*

(i) it is $\mathcal{C}$-invariant, i.e. $F(Q) = F(Q')$ whenever $Q, Q' \in \mathcal{F}(\Gamma)$ are such that $\mathcal{C}_{|Q} = \mathcal{C}_{|Q'}$,

(ii) there is a boundary term b such that for every finite collection $\{Q_i\}_{i=1}^k$ of pairwise disjoint sets in $\mathcal{F}(\Gamma)$

$$\left\| F(Q) - \sum_{i=1}^{k} F(Q_i) \right\| \leq \sum_{i=1}^{k} b(Q_i),$$

where $Q := \bigsqcup_{i=1}^k Q_i$.

Every almost-additive function $F : \mathcal{F}(\Gamma) \to (Z, \|\cdot\|)$, gives rise to a function $\tilde{F}$ on $\mathcal{P}$ instead of on $\mathcal{F}(\Gamma)$:

$$\tilde{F}(P) = \begin{cases} F(Q) & \text{if } Q \in \mathcal{F}(\Gamma) \text{ such that } \tilde{\mathcal{C}_{|Q}} = \tilde{P}, \\ 0 & \text{else.} \end{cases} \tag{4.1}$$

This is well-defined by the $\mathcal{C}$-invariance of F. The next result yields properties of the functions F and $\tilde{F}$. It is joint work with SCHWARZENBERGER. It shows that even for ε-disjoint families $\{Q_k\}_{k=1}^m$, $Q_k \in \mathcal{F}(\Gamma)$, an almost-additive mapping F still satisfies a certain almost-additivity property.

Lemma 4.3 (cf. [PS14], Lemma 5.3).
Let Γ be a countable amenable group. Let a Banach space $(Z, \|\cdot\|)$, as well as some finite set $\mathcal{A}$ and a colouring $\mathcal{C} : \Gamma \to \mathcal{A}$ be given. Furthermore let $F : \mathcal{F}(\Gamma) \to Z$ be $\mathcal{C}$-invariant and almost-additive with boundary term b.

(i) Then F and $\tilde{F}$ are bounded, i.e. there exists a constant $C > 0$ such that

$$\|F(Q)\| \leq C|Q| \quad \text{and} \quad \|\tilde{F}(P)\| \leq C|D(P)|,$$

for all $Q \in \mathcal{F}(\Gamma)$ and $P \in \mathcal{P}$, where $\tilde{F}$ is given by (4.1).

(ii) If furthermore $0 < \varepsilon < 1/2$ is given and Q_i, $1 \leq i \leq k$ are ε-disjoint sets and $Q = \bigcup_{i=1}^k Q_i$, then

$$\left\| F(Q) - \sum_{i=1}^k F(Q_i) \right\| \leq (3C + 9D)\varepsilon|Q| + 3\sum_{i=1}^k b(Q_i),$$

where C is the constant from (i) and D is given by property (i) of the boundary term, cf. Definition 4.1.

PROOF.
Firstly note that as $\mathcal{A}$ is a finite set and F is a $\mathcal{C}$-invariant function, the maximum $m := \max_{x\in\Gamma} \|F(\{x\})\|$ exists. Therefore, we have for $Q \in \mathcal{F}(\Gamma)$

$$\|F(Q)\| \leq \left\| F(Q) - \sum_{x\in Q} F(\{x\}) \right\| + \left\| \sum_{x\in Q} F(\{x\}) \right\| \leq \sum_{x\in Q} (b(\{x\}) + \|F(\{x\})\|) \leq C|Q|,$$

where $C := b(\{e\}) + m$ and e is the unit element in Γ. Hence, F is indeed bounded with that choice for C. By definition of $\tilde{F}$, this proves the second estimate in (i) as well.
We now turn to the proof of assertion (ii). Now let $\varepsilon \in (0, 1/2)$ and ε-disjoint sets Q_i, $1 \leq i \leq k$ be given and set $Q = \bigcup_{i=1}^k Q_i$. Thus, there are pairwise disjoint sets $\bar{Q}_i \subseteq Q_i$ such that $|\bar{Q}_i| \geq (1-\varepsilon)|Q_i|$ for all $1 \leq i \leq k$. Using the triangle inequality, we compute

$$\begin{aligned} \left\| F(Q) - \sum_{i=1}^k F(Q_i) \right\| \leq & \left\| F(Q) - F\left(\bigcup_{i=1}^k \bar{Q}_i\right) \right\| + \left\| F\left(\bigcup_{i=1}^k \bar{Q}_i\right) - \sum_{i=1}^k F(\bar{Q}_i) \right\| \\ & + \sum_{i=1}^k \left\| F(Q_i) - F(\bar{Q}_i) \right\|. \end{aligned} \tag{4.2}$$

Exploiting the fact that F is almost-additive with boundary term b, we obtain

$$\left\| F\left(\bigcup_{i=1}^k \bar{Q}_i\right) - \sum_{i=1}^k F(\bar{Q}_i) \right\| \leq \sum_{i=1}^k b(\bar{Q}_i),$$

as well as

$$\begin{aligned}\left\|F(Q_i)-F(\bar{Q}_i)\right\| &\le \left\|F(Q_i)-F(\bar{Q}_i)-F(Q_i\setminus\bar{Q}_i)\right\|+\left\|F(Q_i\setminus\bar{Q}_i)\right\|\\ &\le b(\bar{Q}_i)+b(Q_i\setminus\bar{Q}_i)+C\left|Q_i\setminus\overline{Q}_i\right|\\ &\le b(\bar{Q}_i)+\varepsilon(C+D)\left|Q_i\right|.\end{aligned}$$

In the same manner, we derive

$$\begin{aligned}\left\|F(Q)-F\left(\bigcup_{i=1}^k\bar{Q}_i\right)\right\| &\le b\left(\bigcup_{i=1}^k\bar{Q}_i\right)+b\left(Q\setminus\bigcup_{i=1}^k\bar{Q}_i\right)+C\left|Q\setminus\bigcup_{i=1}^k\bar{Q}_i\right|\\ &\le b\left(\bigcup_{i=1}^k\bar{Q}_i\right)+\varepsilon(C+D)\left|Q\right|,\end{aligned}$$

where we used $|Q\setminus\bigcup_{i=1}^k\bar{Q}_k|\le\varepsilon|Q|$ (which is due to ε-disjointness). Putting the last estimates together and using property (iv) of the boundary term b, inequality (4.2) leads to

$$\left\|F(Q)-\sum_{i=1}^k F(Q_i)\right\|\le 3\sum_{i=1}^k b(\bar{Q}_i)+\varepsilon(C+D)\left(|Q|+\sum_{i=1}^k|Q_i|\right).$$

If we furthermore apply

$$b(\bar{Q}_i)\le b(Q_i)+b(Q_i\setminus\bar{Q}_i)\le b(Q_i)+D|Q_i\setminus\bar{Q}_i|\le b(Q_i)+\varepsilon D|Q_i|,$$

which holds for all $1\le i\le k$ again by property (iv) of the boundary term, we obtain

$$\begin{aligned}\left\|F(Q)-\sum_{i=1}^k F(Q_i)\right\| &\le 3\sum_{i=1}^k b(Q_i)+3\varepsilon D\sum_{i=1}^k|Q_i|+\varepsilon(C+D)\left(|Q|+\sum_{i=1}^k|Q_i|\right)\\ &\le 3\sum_{i=1}^k b(Q_i)+\varepsilon(C+4D)\sum_{i=1}^k|Q_i|+\varepsilon(C+D)|Q|.\end{aligned}$$

Using $\sum_{i=1}^k|Q_i|\le(1-\varepsilon)^{-1}|Q|\le 2|Q|$, we arrive at the desired estimate. □

We now prove our main theorem. This result appears in [PS14], Theorem 5.5.

Theorem 4.4 (Ergodic theorem for countable amenable groups).
Let Γ be a countable, amenable group along with a colouring $\mathcal{C}:\Gamma\to\mathcal{A}$, where $\mathcal{A}$ is a finite set. Further, assume that $(S_n)_{n\in\mathbb{N}}$ is a nested Følner sequence and suppose that (U_j) is a Følner sequence in Γ such that for all patterns $P\in\mathcal{P}$, the frequencies ν_P exist along (U_j). If $F:\mathcal{F}(\Gamma)\to Z$ is an almost-additive (and $\mathcal{C}$-invariant) map and if $\tilde{F}$ is given as in (4.1), then the following statements hold true.

(i) There exists an element $F^*\in Z$ such that

$$\lim_{j\to\infty}\left\|\frac{F(U_j)}{|U_j|}-F^*\right\|=0.$$

(ii) The element F^* can be expressed as the limit

$$F^* = \lim_{\varepsilon \searrow 0} \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \left(\sum_{P \in \mathcal{P}(T_i^\varepsilon)} \nu_P \frac{\tilde{F}(P)}{|T_i^\varepsilon|} \right),$$

where for each $0 < \varepsilon < 1/10$, we set $N(\varepsilon) := \lceil \log(\varepsilon)/\log(1-\varepsilon)] \rceil$ and $\eta_i(\varepsilon) := \varepsilon(1-\varepsilon)^{N(\varepsilon)-i}$ for $1 \le i \le N(\varepsilon)$ and where the finite sequence $(T_i^\varepsilon)_{i=1}^{N(\varepsilon)}$ with the T_i^ε taken from (S_n) is given as in Definition 2.15 with parameters $\beta = 2^{-N(\varepsilon)-1}\varepsilon$ and $\delta_0(\beta) < 6^{-N(\varepsilon)}\beta/20$.

(iii) For every $0 < \varepsilon < 1/10$, there is some $j_0 := j_0(\varepsilon, \beta) \in \mathbb{N}$ and some finite set $Q \in \mathcal{F}(\Gamma)$ such that for every $j \ge j_0$, U_j is (Q, β)-invariant and the difference

$$\Delta(j,\varepsilon) := \left\| \frac{F(U_j)}{|U_j|} - \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \sum_{P \in \mathcal{P}(T_i^\varepsilon)} \nu_P \frac{\tilde{F}(P)}{|T_i^\varepsilon|} \right\|$$

satisfies the estimate

$$\Delta(j,\varepsilon) \le (12C + 33D)\varepsilon + C \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \sum_{P \in \mathcal{P}(T_i^\varepsilon)} \left| \frac{\#_P(\mathcal{C}_{|U_j})}{|U_j|} - \nu_P \right|$$
$$+ 4 \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \frac{b(T_i^\varepsilon)}{|T_i^\varepsilon|} + (2C + 4D) \frac{|\partial_Q(U_j)|}{|U_j|} \sum_{i=1}^{N(\varepsilon)} |T_i^\varepsilon|. \tag{4.3}$$

Proof.
Fix $0 < \varepsilon < 1/10$. We first show the estimate (4.3). To do so, choose $j_0 = j_0(\varepsilon, \beta, T_i^\varepsilon) \in \mathbb{N}$ such that for each $j \ge j_0$, the set U_j is sufficiently invariant to apply Theorem 3.2. Hence, for each $j \ge j_0$, we find a finite family Λ_j^ε of ε-quasi tilings for the set $T = U_j$ satisfying the uniform special tiling property (USTP), cf. Definition 3.1. With no loss of generality, we may assume that all the $T_i = T_i^\varepsilon$ are taken from a subsequence $\{S_{n_k}\}_{k=1}^\infty$ such that the expressions $b(S_{n_k})/|S_{n_k}|$ converge to zero monotonically as $k \to \infty$. Additionally, we make sure that $T_i^\varepsilon \in \{S_{n_l} \,|\, l \ge i\}$ for all $1 \le i \le N$. Then, for fixed $j \ge j_0$ we estimate with the triangle inequality

$$\Delta(j,\varepsilon) \le \left\| \frac{F(U_j)}{|U_j|} - \frac{1}{|\Lambda_j^\varepsilon|} \sum_{\lambda \in \Lambda_j^\varepsilon} \sum_{i=1}^{N(\varepsilon)} \sum_{c \in C_i^\lambda(U_j)} \frac{F(T_i^\varepsilon c)}{|U_j|} \right\|$$
$$+ \left\| \frac{1}{|\Lambda_j^\varepsilon|} \sum_{\lambda \in \Lambda_j^\varepsilon} \sum_{i=1}^{N(\varepsilon)} \sum_{c \in C_i^\lambda(U_j)} \frac{F(T_i^\varepsilon c)}{|U_j|} - \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \sum_{P \in \mathcal{P}(T_i^\varepsilon)} \frac{\#_P(\mathcal{C}_{|U_j})}{|U_j|} \frac{\tilde{F}(P)}{|T_i^\varepsilon|} \right\|$$
$$+ \left\| \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \sum_{P \in \mathcal{P}(T_i^\varepsilon)} \Big(\frac{\#_P(\mathcal{C}_{|U_j})}{|U_j|} - \nu_P \Big) \frac{\tilde{F}(P)}{|T_i^\varepsilon|} \right\|.$$

Again by the triangle inequality, we then obtain

$$\Delta(j,\varepsilon) \le D_1(j,\varepsilon) + D_2(j,\varepsilon) + D_3(j,\varepsilon),$$

where

$$D_1(j,\varepsilon) := \frac{1}{|U_j||\Lambda_j^\varepsilon|} \sum_{\lambda\in\Lambda_j^\varepsilon} \left\| F(U_j) - \sum_{i=1}^{N(\varepsilon)} \sum_{c\in C_i^\lambda(U_j)} F(T_i^\varepsilon c) \right\|,$$

$$D_2(j,\varepsilon) := \frac{1}{|U_j|} \left\| \frac{1}{|\Lambda_j^\varepsilon|} \sum_{\lambda\in\Lambda_j^\varepsilon} \sum_{i=1}^{N(\varepsilon)} \sum_{c\in C_i^\lambda(U_j)} F(T_i^\varepsilon c) - \sum_{i=1}^{N(\varepsilon)} \frac{\eta_i(\varepsilon)}{|T_i^\varepsilon|} \sum_{P\in\mathcal{P}(T_i^\varepsilon)} \#_P(\mathcal{C}_{|U_j}) \tilde{F}(P) \right\|,$$

$$D_3(j,\varepsilon) := \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \sum_{P\in\mathcal{P}(T_i^\varepsilon)} \left| \frac{\#_P(\mathcal{C}_{|U_j})}{|U_j|} - \nu_P \right| \frac{\|\tilde{F}(P)\|}{|T_i^\varepsilon|}.$$

We will now separately estimate from above the expressions $D_3(j,\varepsilon)$, $D_1(j,\varepsilon)$ and $D_2(j,\varepsilon)$ (in this order). The boundedness of $\tilde{F}$, see Lemma 4.3, yields

$$D_3(j,\varepsilon) \leq C \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \sum_{P\in\mathcal{P}(T_i^\varepsilon)} \left| \frac{\#_P(\mathcal{C}_{|U_j})}{|U_j|} - \nu_P \right|. \tag{4.4}$$

In order to estimate $D_1(j,\varepsilon)$, we make use of the almost-additivity of the function F and we use part (ii) of Lemma 4.3. This gives for each $j \geq j_0$ and for every $\lambda \in \Lambda_j^\varepsilon$

$$\left\| F(U_j) - \sum_{i=1}^{N(\varepsilon)} \sum_{c\in C_i^\lambda(U_j)} F(T_i^\varepsilon c) \right\| \leq \left\| F(U_j) - F\left(A_{j,\lambda}^\varepsilon\right) \right\| + \left\| F\left(A_{j,\lambda}^\varepsilon\right) - \sum_{i=1}^{N(\varepsilon)} \sum_{c\in C_i^\lambda(U_j)} F(T_i^\varepsilon c) \right\|$$
$$\leq b(A_{j,\lambda}^\varepsilon) + b(U_j \setminus A_{j,\lambda}^\varepsilon) + \|F(U_j \setminus A_{j,\lambda}^\varepsilon)\| + (3C+9D)\varepsilon|U_j| + 3\sum_{i=1}^{N(\varepsilon)} \sum_{c\in C_i^\lambda(U_j)} b(T_i^\varepsilon c),$$

where

$$A_{j,\lambda}^\varepsilon = \bigcup_{i=1}^{N(\varepsilon)} \bigcup_{c\in C_i^\lambda(U_j)} T_i^\varepsilon c.$$

It also follows from the properties listed in Definition 3.1 that for each $\lambda \in \Lambda_j^\varepsilon$, the set U_j is $(1-4\varepsilon)$-covered by the translates $T_i^\varepsilon c$, $1 \leq i \leq N$, $c \in C_i^\lambda(U_j)$ that are entirely contained in U_j. Thus, we have $|U_j \setminus A_{j,\lambda}^\varepsilon| \leq 4\varepsilon|U_j|$. Using this and the properties of the boundary term b, we obtain after a short calculation

$$D_1(j,\varepsilon) \leq \frac{1}{|U_j||\Lambda_j^\varepsilon|} \sum_{\lambda\in\Lambda_j^\varepsilon} \left((7C+13D)\varepsilon|U_j| + 4\sum_{i=1}^{N(\varepsilon)} \sum_{c\in C_i^\lambda(U_j)} b(T_i^\varepsilon c) \right)$$
$$(b \text{ translation invariant}) \leq (7C+13D)\varepsilon + \frac{4}{|U_j||\Lambda_j^\varepsilon|} \sum_{\lambda\in\Lambda_j^\varepsilon} \left(\sum_{i=1}^{N(\varepsilon)} |C_i^\lambda(U_j)| b(T_i^\varepsilon) \right)$$
$$= (7C+13D)\varepsilon + 4\sum_{i=1}^{N(\varepsilon)} \frac{b(T_i^\varepsilon)}{|\Lambda_j^\varepsilon|} \sum_{\lambda\in\Lambda_j^\varepsilon} \frac{|C_i^\lambda(U_j)|}{|U_j|}.$$

Next, we make use of property (II) of the second item in Definition 3.1 to find some $Q \in \mathcal{F}(\Gamma)$ such that for all $u \in U_j \setminus \partial_Q(U_j)$, the uniformity inequality in (II) is satisfied. Note that additionally, U_j is (Q,β)-invariant. We will use this fact later. Splitting the sum over U_j into sums over $U_j \setminus \partial_Q(U_j)$ and over $\partial_Q(U_j)$, one obtains

$$\begin{aligned}
\frac{1}{|\Lambda_j^\varepsilon|}\sum_{\lambda\in\Lambda_j^\varepsilon}\frac{|C_i^\lambda(U_j)|}{|U_j|} &= \frac{1}{|\Lambda_j^\varepsilon||U_j|}\sum_{u\in U_j\setminus\partial_Q(U_j)}\sum_{\lambda\in\Lambda_j^\varepsilon}\mathbb{1}_{C_i^\lambda(U_j)}(u) + \frac{1}{|\Lambda_j^\varepsilon||U_j|}\sum_{u\in\partial_Q(U_j)}\sum_{\lambda\in\Lambda_j^\varepsilon}\mathbb{1}_{C_i^\lambda(U_j)}(u)\\
&\le \frac{|U_j\setminus\partial_Q(U_j)|}{|U_j|}\left(\frac{\eta_i(\varepsilon)}{|T_i^\varepsilon|}+\frac{3\beta}{|T_i^\varepsilon|}+\varepsilon\gamma_i^{\varepsilon,j}\right) + \frac{1}{|\Lambda_j^\varepsilon||U_j|}\sum_{u\in\partial_Q(U_j)}\sum_{\lambda\in\Lambda_j^\varepsilon}1\\
&\le \frac{\eta_i(\varepsilon)}{|T_i^\varepsilon|}+\frac{3\beta}{|T_i^\varepsilon|}+\varepsilon\gamma_i^{\varepsilon,j}+\frac{|\partial_Q(U_j)|}{|U_j|}.
\end{aligned}$$

Inserting this in the last estimate for $D_1(j,\varepsilon)$ and exploiting the properties of the boundary term b, as well as $\sum_{i=1}^{N(\varepsilon)}\gamma_i^{\varepsilon,j}|T_i^\varepsilon| \le 2$, we arrive at

$$\begin{aligned}
D_1(j,\varepsilon) &\le (7C+13D)\varepsilon + 4\sum_{i=1}^{N(\varepsilon)} b(T_i^\varepsilon)\left(\frac{\eta_i(\varepsilon)}{|T_i^\varepsilon|}+\frac{3\beta}{|T_i^\varepsilon|}+\varepsilon\gamma_i^{\varepsilon,j}+\frac{|\partial_Q(U_j)|}{|U_j|}\right)\\
&\le (7C+13D)\varepsilon + \left(4\sum_{i=1}^{N(\varepsilon)}\eta_i(\varepsilon)\frac{b(T_i^\varepsilon)}{|T_i^\varepsilon|}\right) + 12\beta D N(\varepsilon) + 8\varepsilon D + \left(4\frac{|\partial_Q(U_j)|}{|U_j|}\sum_{i=1}^{N(\varepsilon)} b(T_i^\varepsilon)\right)\\
&\le (7C+33D)\varepsilon + \left(4\sum_{i=1}^{N(\varepsilon)}\eta_i(\varepsilon)\frac{b(T_i^\varepsilon)}{|T_i^\varepsilon|}\right) + 4D\frac{|\partial_Q(U_j)|}{|U_j|}\sum_{i=1}^{N(\varepsilon)}|T_i^\varepsilon|, \qquad (4.5)
\end{aligned}$$

where the last step uses $\beta N(\varepsilon) \le \varepsilon$.

We finally estimate $D_2(j,\varepsilon)$, where again, we use the property (II) of Definition 3.1. To do so, note that

$$\sum_{P\in\mathcal{P}(T_i^\varepsilon)} \#_P(\mathcal{C}_{|U_j})\tilde{F}(P) = \sum_{\substack{u\in U_j\\ T_i^\varepsilon u\subseteq U_j}} F(T_i^\varepsilon u).$$

Therefore, we have

$$\begin{aligned}
D_2(j,\varepsilon) &= \frac{1}{|U_j|}\left\|\sum_{i=1}^{N(\varepsilon)}\sum_{u\in U_j}\frac{1}{|\Lambda_j^\varepsilon|}\sum_{\lambda\in\Lambda_j^\varepsilon}\mathbb{1}_{C_i^\lambda(U_j)}(u)F(T_i^\varepsilon u) - \sum_{i=1}^{N(\varepsilon)}\frac{\eta_i(\varepsilon)}{|T_i^\varepsilon|}\sum_{\substack{u\in U_j\\ T_i^\varepsilon u\subseteq U_j}} F(T_i^\varepsilon u)\right\|\\
&\le \frac{1}{|U_j|}\sum_{i=1}^{N(\varepsilon)}\sum_{u\in U_j}\left|\frac{1}{|\Lambda_j^\varepsilon|}\sum_{\lambda\in\Lambda_j^\varepsilon}\mathbb{1}_{C_i^\lambda(U_j)}(u) - \frac{\eta_i(\varepsilon)}{|T_i^\varepsilon|}\right|\left\|F(T_i^\varepsilon u)\right\|.
\end{aligned}$$

Again, we split the sum over U_j into two parts: one where we are able to apply property (II) and a “small” one. Besides this, we use the boundedness of F and $\beta N(\varepsilon) \le \varepsilon$ to arrive at

$$D_2(j,\varepsilon) \le \frac{C}{|U_j|}\sum_{i=1}^{N(\varepsilon)} |T_i^\varepsilon| \Bigg(\sum_{\substack{u\in U_j \\ u\notin \partial_Q(U_j)}} \bigg| \sum_{\lambda\in\Lambda_j^\varepsilon} \frac{\mathbb{1}_{C_i^\lambda(U_j)}(u)}{|\Lambda_j^\varepsilon|} - \frac{\eta_i(\varepsilon)}{|T_i^\varepsilon|}\bigg|$$

$$+ \sum_{u\in\partial_Q(U_j)} \bigg| \sum_{\lambda\in\Lambda_j^\varepsilon} \frac{\mathbb{1}_{C_i^\lambda(U_j)}(u)}{|\Lambda_j^\varepsilon|} - \frac{\eta_i(\varepsilon)}{|T_i^\varepsilon|}\bigg|\Bigg)$$

$$(\text{ Def. 3.1, ineq. (II) }) \quad \le \frac{C}{|U_j|}\sum_{i=1}^{N(\varepsilon)} |T_i^\varepsilon| \left(|U_j| \Big(\frac{3\beta}{|T_i^\varepsilon|} + \varepsilon\gamma_i^{\varepsilon,j}\Big) + 2\,|\partial_Q(U_j)|\right)$$

$$= C\sum_{i=1}^{N(\varepsilon)} \big(3\beta + \varepsilon\gamma_i^{\varepsilon,j}|T_i^\varepsilon|\big) + 2\,\frac{C|\partial_Q(U_j)|}{|U_j|}\sum_{i=1}^{N(\varepsilon)} |T_i^\varepsilon|$$

$$(\beta N(\varepsilon)\le\varepsilon) \quad \le 5\varepsilon C + 2\,\frac{C|\partial_Q(U_j)|}{|U_j|}\sum_{i=1}^{N(\varepsilon)} |T_i^\varepsilon|. \tag{4.6}$$

To finish the proof of (iii), we combine the Inequalities (4.5), (4.6) and (4.4) and obtain

$$\begin{aligned}\Delta(j,\varepsilon) &\le D_1(j,\varepsilon) + D_2(j,\varepsilon) + D_3(j,\varepsilon)\\ &\le (12C+33D)\varepsilon + C\sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \sum_{P\in\mathcal{P}(T_i^\varepsilon)} \left| \frac{\#_P(\mathcal{C}_{|U_j})}{|U_j|} - \nu_P\right|\\ &\quad + 4\sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon)\frac{b(T_i^\varepsilon)}{|T_i^\varepsilon|} + (2C+4D)\frac{|\partial_Q(U_j)|}{|U_j|}\sum_{i=1}^{N(\varepsilon)}|T_i^\varepsilon|\end{aligned}$$

for all $j \ge j_0(\varepsilon,\beta,T_i^\varepsilon)$. Since $0<\varepsilon<1/10$ (and therefore also β) was arbitrarily chosen, this shows the desired estimate (4.3) for $j\ge j_0(\varepsilon,\beta,T_i^\varepsilon)$. Having this result at our disposal, it is not hard to prove the remaining statements (i) and (ii) of the theorem.
The choice of the T_i^ε and the monotonicity assumption on the sequence $b(S_{n_k})/|S_{n_k}|$ yield that

$$\lim_{\varepsilon\searrow 0}\sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon)|T_i^\varepsilon|^{-1}b(T_i^\varepsilon) \le \lim_{\varepsilon\to 0}\sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon)|S_{n_i}|^{-1}b(S_{n_i}) = 0.$$

By the assumption that the frequencies ν_P exist along $(U_j)_j$, we obtain with (4.3) and the fact that U_j is (Q,β)-invariant that

$$\lim_{\varepsilon\to 0}\lim_{j\to\infty}\Delta(j,\varepsilon) = 0. \tag{4.7}$$

Now the triangle inequality shows that

$$\left\| \frac{F(U_j)}{|U_j|} - \frac{F(U_m)}{|U_m|}\right\| \quad\le\quad \Delta(j,\varepsilon) + \Delta(m,\varepsilon)$$

for all $0<\varepsilon<1/10$. By (4.7), the sequence $(|U_j|^{-1}F(U_j))_{j\in\mathbb{N}}$ must be Cauchy and hence, it converges in the Banach space Z to some element F^*. Hence, statement (i) of the theorem is proven. The limit in (4.7) also shows that

$$F^* = \lim_{\varepsilon \searrow 0} \sum_{i=1}^{N(\varepsilon)} \varepsilon (1-\varepsilon)^{N(\varepsilon)-i} \sum_{P \in \mathcal{P}(T_i)} \nu_P \frac{\tilde{F}(P)}{|T_i|}$$

and thus, statement (ii) holds as well. □

Remark.
Note that in general, the limit element F^* will depend on the choice of the Følner sequence (U_j), see Chapter 6.

Being valid for all countable amenable groups, Theorem 4.4 generalizes the works [LMV08, LSV11]. Another possible extension is to consider convergence of real-valued, *subadditive* functions defined over amenable groups. In this context, Krieger uses ε-quasi tiling techniques in [Kri07, Kri10] to show the convergence along Følner nets. An analogous semigroup result for cancellative amenable semigroups can be found in [CSKC12]. However, the authors need a periodicity condition on f, i.e. the (semi-)group is coloured by one colour only. This assumption is too restrictive for the spectral applications that we have in mind, cf. Chapter 6. We will get back to convergence issues for subadditive functions in Chapter 8.

4.2 Continuous amenable groups

If not stated otherwise, we now consider Γ to be an amenable (possibly continuous) LCSCUH group. The main result of this section is Theorem 4.15, which is a mean convergence theorem for almost-additive Banach space-valued mappings defined on $\mathcal{F}(\Gamma)$. Here, we will not deal with coloured groups, but we will emanate from classical ergodic theory. Precisely, we make use of a mean ergodic theorem for integral averages proven by Greenleaf in [Gre73]. The major ingredient for the proof will be the concept of uniform decomposition towers (Definition 3.5). The elaborations of this section can also be found in [Pog13a].

Abstract mean ergodicity

At first, we summarize well-known results about mean ergodic theorems for amenable groups, see [Gre73]. This includes the definition of abstract integral averages in Definition 4.6 and the corresponding mean ergodic theorem in Theorem 4.7. We will use this latter result in order to prove convergence along Følner sequences for general almost-additive functions.

Definition 4.5.
Let Γ be a second countable, locally compact Hausdorff group and assume that Z is a Banach space with corresponding dual space Z^*. We then say that Γ acts weakly measurably on Z via uniformly bounded operators $\{T_g\}_{g\in\Gamma}$ if there is a constant $A > 0$ and a map

$$T : \Gamma \times Z \to Z : (g, f) \mapsto T_g f$$

with the following properties.

(i) $T_g : Z \to Z$ *is a linear operator for each* $g \in \Gamma$*,*

(ii) $\|T_g\| \leq A$ *for all* $g \in \Gamma$*,*

(iii) $T_e f = f$ *for each* $f \in Z$*, where* e *is the unit element in* Γ*,*

(iv) $T_{g_1}(T_{g_2} f) = T_{g_1 g_2} f$ *for each* $f \in Z$ *and all* $g_1, g_2 \in \Gamma$*,*

(v) For each $f \in Z$ *and every* $h \in Z^*$*, the map*

$$\Phi_{f,h} : \Gamma \to \mathbb{C} : g \mapsto \langle T_g f, h \rangle_{Z,Z^*}$$

is measurable with respect to the Borel σ*-algebras on* Γ *and* $\mathbb{C}$ *respectively, where* $\langle \cdot, \cdot \rangle_{Z,Z^*}$ *denotes the dual pairing of elements in* Z *and* Z^**.*

Moreover, we define $\mathrm{Fix}(T_\Gamma) := \{f \in Z \,|\, T_g f = f \text{ for all } g \in \Gamma\}$ *as the space of elements in* Z *which are fixed under the action of all* $g \in \Gamma$*.*

With the notion of weakly measurable actions at hand, we define the following abstract ergodic integral averages.

Definition 4.6 (Integral averages).
Let Γ *be a second countable, locally compact Hausdorff group acting weakly measurably on a Banach space* Z *via a family of linear, uniformly bounded operators* $\{T_g\}_{g \in \Gamma}$ *as in Definition 4.5. Then, if* (U_j) *is a sequence of compact sets with positive measure in* Γ*, we denote for* $f \in Z$ *the* j*-th abstract ergodic average* $A_j f$ *as*

$$A_j f := |U_j|^{-1} \int_{U_j} T_{g^{-1}} f \, dm_L(g), \quad j \in \mathbb{N}.$$

Remark.
Note that in the first instance, the abstract ergodic averages are only defined in a weak sense, i.e. $A_j f \in Z^{**}$ for $f \in Z$, where Z^{**} is the bidual space of Z. However, under mild compactness assumptions on the T_g-orbit of f, one can show $A_j f \in Z$, see the remark following Theorem 4.3 in [Pog10]. One sufficient condition is that for $f \in Z$, the closure of the convex hull of the set $\{T_g f \,|\, g \in \Gamma\}$ is compact in the weak topology of Z. This condition will be satisfied in all our subsequent considerations.

The following mean ergodic theorem is well-known, see e.g. Theorem 3.3 in [Gre73] or Theorem 4.3 in [Pog10].

Theorem 4.7 (Mean ergodic theorem for integral averages).
Let Γ *be a second countable, locally compact Hausdorff group acting weakly measurably on a Banach space* $(Z, \|\cdot\|_Z)$ *via a family of linear, uniformly bounded operators* $\{T_g\}_{g \in \Gamma}$ *as in Definition 4.5. If for each* $f \in Z$*, the convex hull* $\mathrm{co}\{T_g f \,|\, g \in \Gamma\}$ *has compact closure in the weak topology of* Z*, then there is a bounded projection* P *on* Z *such that given a weak Følner sequence* (U_j) *in* Γ *along with the corresponding ergodic averages* $(A_j)_j$*,*

- $\lim_{j \to \infty} \|A_j f - Pf\|_Z = 0$ *for all* $f \in Z$*.*

- $Z = \mathrm{ran}(P) \oplus \ker(P)$, *where* $\mathrm{ran}(P)$ *denotes the range and* $\ker(P)$ *denotes the kernel of* P *respectively.*
- $\mathrm{ran}(P) = \mathrm{Fix}(T_\Gamma)$.
- $\ker(P)$ *is equal to strong closure of the set of those* $f \in Z$ *spanned by finite linear combinations of elements* $h - T_g h$, *where* $h \in Z, g \in \Gamma$.

PROOF.
See [Gre73], Theorems 3.1 and 3.2, as well as [Pog10], Theorems 4.2 and 4.3. □

In fact, there is no need to assume that the compactness condition holds true for all $f \in Z$. This leads to the next corollary. It will serve as a major ingredient in the proof of the abstract mean ergodic theorem, cf. Theorem 4.15.

Corollary 4.8.
Assume the structure of the previous Theorem 4.7. If $f \in Z$ *is such that the convex hull* $\mathrm{co}\{T_g f \mid g \in \Gamma\}$ *has compact closure in the weak topology on* Z, *then there is some* $f^* \in Z$ *such that*

$$\lim_{j\to\infty} \|A_j f - f^*\|_Z = 0.$$

PROOF.
See [Gre73], Theorem 3.3. □

A mean ergodic theorem for almost-additive functions

In the following, we elaborate the concept of almost-additive functions on $\mathcal{F}(\Gamma)$, where Γ is a possibly continuous group. This notion is the analogue of Definition 4.2 in the countable case with a minor difference: we have to drop the assumption on the boundary term b to be bounded. The reason for this is that one cannot expect a reasonable boundedness condition for the measure of K-boundary sets in continuous groups. For instance, consider $\Gamma := (\mathbb{R}, +)$, $K := [-\delta, \delta]$ for an arbitrary $\delta > 0$ and $T_n := [0, 1/n] \cup (\mathbb{Q} \cap [0, 1])$ for $n \in \mathbb{N}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, one has that $[1/n, 1] \subseteq \partial_K(T_n)$, hence $|\partial_K(T_n)| \geq 1 - 1/n$, but $|T_n| = 1/n$, where $|\cdot|$ stands for the Lebesgue measure on $\mathbb{R}$. Sending n to infinity, one observes that $\sup_n \frac{|\partial_K(T_n)|}{|T_n|} = \infty$. To overcome this difficulty, we introduce tiling-admissible, weak boundary terms in the Definitions 4.9 and 4.11. This leads to the notion of admissibly almost-additive mappings $F : \mathcal{F}(\Gamma) \to Z$. In Theorem 4.15, we show that under mild compactness conditions, there is an element $F^* \in Z$ such that

$$\lim_{j\to\infty} \left\| \frac{F(U_j)}{|U_j|} - F^* \right\|_Z = 0$$

for every strong Følner sequence (U_j) in Γ. This extends Theorem 3.2 in [Gre73]. We start with the concept of weak boundary terms.

Definition 4.9 (Weak boundary term).
Let Γ be an amenable LCSCUH group. A mapping

$$b : \mathcal{F}(\Gamma) \to [0, \infty)$$

is called weak boundary term *if*

(i) $\lim_{j\to\infty} \frac{b(U_j)}{|U_j|} = 0$ *for every strong Følner sequence* (U_j) *in* Γ,

(ii) it is translation invariant, i.e. $b(Qg) = b(Q)$ *for all* $Q \in \mathcal{F}(\Gamma)$ *and every* $g \in \Gamma$,

(iii) one has the inequalities

$$b(Q \cap Q') \leq b(Q) + b(Q'), \quad b(Q \cup Q') \leq b(Q) + b(Q'), \quad b(Q \setminus Q') \leq b(Q) + b(Q')$$

for all $Q, Q' \in \mathcal{F}(\Gamma)$.

As in the proof of Theorem 4.4, we use uniform families of ε-quasi tilings in order to prove our mean ergodic Theorem 4.15. For the ε-prototiles T_i appearing in the proof, some estimates on the measure of their K-boundaries will be indispensable. However, our notion of weak boundary terms will not guarantee boundedness in general. We cope with this difficulty by introducing the concept of so-called *tiling-admissible*, weak boundary terms. Those mappings possess certain boundedness properties for the sets arising from ε-quasi tilings.

Definition 4.10.
Let $0 < \varepsilon < 1/10$ *be a positive number and assume that* b *is a weak boundary term. We call a set* $\mathcal{C}$ *consisting of finite,* ε*-disjoint families of sets in* $\mathcal{F}(\Gamma)$ *an* ε-admissible collection for b *if there is a constant* $\tilde{D} > 0$ *such that for each such family* $\{Q_k\}_{k=1}^m$ *in* $\mathcal{C}$ *we have*

$$b(Q_k) \leq \tilde{D}\, |Q_k|$$

for all $1 \leq k \leq m$ *and one can find a family* $\{\bar{Q}_k\}_{k=1}^m$ *of measurable, pairwise disjoint sets with*

- $|\bar{Q}_k| \geq (1 - \varepsilon)|Q_k|$,
- $b(\bar{Q}_k) \leq \tilde{D}\,(b(Q_k) + \varepsilon|Q_k|)$

for all $1 \leq k \leq m$.

In our later considerations, we will have to work with weak boundary terms which are compatible with ε-quasi tilings coming from uniform decomposition towers, see Definition 3.5. This leads to the following definition.

Definition 4.11.
Let Γ *be an amenable LCSCUH group and assume that* b *is a weak boundary term for* Γ*. We call* b tiling-admissible *if for every nested, strong Følner sequence* (S_n) *in* Γ*, there is a constant* $\tilde{D} > 0$ *such that for all* $0 < \varepsilon < 1/10$ *and for each choice of* ε*-prototiles*

$$\{e\} \subseteq T_1^\varepsilon \subseteq \cdots \subseteq T_{N(\varepsilon)}^\varepsilon$$

taken from (S_n) according to Definition 2.15, every ε-quasi tiling as constructed in Theorem 3.6 is an ε-admissible family of translates $T_i^\varepsilon c$ for b with constant $\tilde{D}$.

The following proposition shows that the natural choice $b(Q) = D\,|\partial_L(Q)|$ for some constant $D > 0$ and for some compact $L \subseteq \Gamma$ defines a weak, tiling-admissible boundary term.

Proposition 4.12.
Suppose that Γ is an amenable LCSCUH group. Let $D > 0$ be arbitrary and assume that $\{e\} \subseteq L$ is a compact set in $\mathcal{F}(\Gamma)$. Then, the mapping

$$b : \mathcal{F}(\Gamma) \to [0,\infty) : b(Q) := D\,|\partial_L(Q)|$$

is a weak, tiling-admissible boundary term.

Proof.
The fact that b is a weak boundary term follows from the Følner condition and from Lemma 2.3.
Let (S_n) be a nested, strong Følner sequence in Γ and set

$$\overline{D} := \sup_{n\in\mathbb{N}} \frac{b(S_n)}{|S_n|}.$$

By the Følner condition, we have $\overline{D} < \infty$. Now let $\mathcal{C}$ be an ε-disjoint family consisting of translates $T_i^\varepsilon c$ appearing in some ε-quasi tiling as constructed in Theorem 3.6, where T_i^ε is an element of (S_n). Clearly, $b(T_i^\varepsilon c) \leq \overline{D}\,|T_i^\varepsilon c|$ for all those translates. By construction, the translates also satisfy the condition given in part (III) of the second item in Definition 3.3, which guarantees the existence of measurable sets $T_i^\varepsilon(c)$ such that $\partial_L(T_i^\varepsilon(c))$ is measurable as well, $|T_i^\varepsilon(c)| \geq (1-\varepsilon)|T_i^\varepsilon|$ and

$$|\partial_L(T_i^\varepsilon(c)c)| \leq |\partial_L(T_i^\varepsilon)| + \varepsilon\,|T_i^\varepsilon c|,$$

see also Definition 2.15. Thus, we have proven that b is indeed tiling-admissible with the constant $\tilde{D} := \overline{D} + 1$. □

Analogously to the case of countable groups, we can define Banach space-valued, almost-additive functions on $\mathcal{F}(\Gamma)$, where Γ is a possibly continuous group. In order to avoid confusions with Definition 4.2, we call those mappings F *admissibly almost-additive.*

Definition 4.13 (Admissibly almost-additive function).
Let Γ be an amenable LCSCUH group and suppose that $(Z, \|\cdot\|_Z)$ is a Banach space. A map

$$F : \mathcal{F}(\Gamma) \to Z$$

is called admissibly almost-additive *if*

(i) *F is* bounded, *i.e. there exists some constant $C > 0$ such that*

$$C = \sup_{Q\in\mathcal{F}(\Gamma)} \frac{\|F(Q)\|_Z}{|Q|} < \infty,$$

(ii) there is some tiling-admissible, weak boundary term $b : \mathcal{F}(\Gamma) \to [0, \infty)$ such that F is almost-additive with respect to b, *i.e.*

$$\left\| F(Q) - \sum_{k=1}^{m} F(Q_k) \right\|_Z \leq \sum_{k=1}^{m} b(Q_k)$$

for any union $Q = \bigsqcup_k Q_k$ of pairwise disjoint sets in $\mathcal{F}(\Gamma)$.

The following proposition is the analogue to Lemma 4.3, part (ii). It appears in [Pog13a], Proposition 5.6.

Proposition 4.14.
Let Γ be an amenable LCSCUH group and let $(Z, \|\cdot\|_Z)$ be a Banach space. Assume that $F : \mathcal{F}(\Gamma) \to Z$ is admissibly almost-additive with boundary term $b : \mathcal{F}(\Gamma) \to [0, \infty)$. Further, let $0 < \varepsilon < 1/10$ and denote by $\mathcal{C}$ an ε-admissible collection for b with constant $\tilde{D}$. Then if $\{Q_k\}_{k=1}^m$ is an element in $\mathcal{C}$ such that $\cup_k Q_k \subseteq Q$ and $|\cup_k Q_k| \geq \alpha\, |Q|$ for some parameter $0 < \alpha \leq 1$, the following error estimate holds true.

$$\left\| F(Q) - \sum_{k=1}^{m} F(Q_k) \right\|_Z \leq C\,(2\varepsilon + 1 - (1-\varepsilon)\alpha)\,|Q| + 10\tilde{D}\varepsilon\,|Q| + b(Q) + (5\tilde{D} + 1) \sum_{k=1}^{m} b(Q_k),$$

where C is the boundedness constant for F.

Proof.
Since $\mathcal{C}$ is ε-admissible for b with constant $\tilde{D}$, for each $1 \leq k \leq m$, one finds a measurable set $\bar{Q}_k \subseteq Q_k$ with $|\bar{Q}_k| \geq (1-\varepsilon)|Q_k|$ such that the $\bar{Q}_k$ are pairwise disjoint and $b(\bar{Q}_k) \leq \tilde{D}\, b(Q_k) + \tilde{D}\,\varepsilon\,|Q_k|$ for all $1 \leq k \leq m$. Since Q is α-covered by the Q_k, one obtains

$$\left| \bigcup_{k=1}^{m} \bar{Q}_k \right| \geq (1-\varepsilon)\,\alpha\,|Q|. \tag{4.8}$$

By the triangle inequality, we get

$$\begin{aligned} \left\| F(Q) - \sum_{k=1}^{m} F(Q_k) \right\|_Z \quad \leq \quad & \left\| F(Q) - F\left(\bigcup_{k=1}^{m} \bar{Q}_k \right) \right\|_Z + \left\| F\left(\bigcup_{k=1}^{m} \bar{Q}_k \right) - \sum_{k=1}^{m} F(\bar{Q}_k) \right\|_Z + \\ & \sum_{k=1}^{m} \left\| F(\bar{Q}_k) - F(Q_k) \right\|_Z . \end{aligned}$$

For the first expression, we obtain from the almost-additivity of F that

$$\left\| F(Q) - F\left(\bigcup_{k=1}^{m} \bar{Q}_k \right) \right\|_Z \quad \leq \quad b\left(\bigcup_{k=1}^{m} \bar{Q}_k \right) + b\left(Q \setminus \bigcup_{k=1}^{m} \bar{Q}_k \right) + \left\| F\left(Q \setminus \bigcup_{k=1}^{m} \bar{Q}_k \right) \right\|_Z .$$

Since $|Q \setminus \cup_k \bar{Q}_k| \leq (1 - (1-\varepsilon)\alpha)|Q|$ and by Definition 4.9 (iii), we obtain by using boundedness and inequality (4.8)

$$\left\| F(Q) - F\left(\bigcup_{k=1}^{m} \bar{Q}_k \right) \right\|_Z \quad \leq \quad 2 \sum_{k=1}^{m} b(\bar{Q}_k) + b(Q) + C\,(1 - (1-\varepsilon)\alpha)\,|Q|.$$

Moreover, by the fact that $b(\bar{Q}_k) \leq \tilde{D}\,(b(Q_k) + \varepsilon|Q_k|)$ for $1 \leq k \leq m$ and as $\varepsilon < 1/2$,

$$\left\| F(Q) - F\left(\bigcup_{k=1}^{m} \bar{Q}_k\right)\right\|_Z \leq 2\tilde{D}\sum_{k=1}^{m} b(Q_k) + 4\tilde{D}\varepsilon\,|Q| + b(Q) + C\,(1-(1-\varepsilon)\alpha)\,|Q|.$$

For the second expression, we use the disjointness of the $\bar{Q}_k$ to get

$$\left\| F\left(\bigcup_{k=1}^{m} \bar{Q}_k\right) - \sum_{k=1}^{m} F(\bar{Q}_k)\right\|_Z \leq \sum_{k=1}^{m} b(\bar{Q}_k).$$

By the considerations above and by the ε-disjointness of the Q_k (with $\varepsilon < 1/2$), we arrive at

$$\left\| F\left(\bigcup_{k=1}^{m} \bar{Q}_k\right) - \sum_{k=1}^{m} F(\bar{Q}_k)\right\|_Z \leq \tilde{D}\sum_{k=1}^{m} b(Q_k) + 2\tilde{D}\varepsilon\,|Q|.$$

For the third expression, we compute similarly as before,

$$\begin{aligned} \|F(Q_k) - F(\bar{Q}_k)\|_Z &\leq b(\bar{Q}_k) + b(Q_k \setminus \bar{Q}_k) + \|F(Q_k \setminus \bar{Q}_k)\|_Z \\ &\leq b(Q_k) + 2\,b(\bar{Q}_k) + C\varepsilon\,|Q_k| \end{aligned}$$

for $1 \leq k \leq m$. Taking sums, one obtains with the previous considerations that

$$\sum_{k=1}^{m} \|F(Q_k) - F(\bar{Q}_k)\|_Z \leq (2\tilde{D}+1)\sum_{k=1}^{m} b(Q_k) + 2C\,\varepsilon|Q| + 4\tilde{D}\varepsilon\,|Q|.$$

Summing the partial results up, this proves the claim. □

We are finally in position to prove the abstract mean ergodic theorem for admissibly almost-additive mappings on amenable groups. It has been published in [Pog13a], Theorem 5.7.

Theorem 4.15 (Mean ergodic theorem for set functions).
Let Γ be an amenable LCSCUH group, (U_j) a strong Følner sequence in Γ, $(Z, \|\cdot\|_Z)$ a Banach space and $\{T_g\}_{g\in\Gamma}$ a family of linear, uniformly bounded operators acting weakly measurably on Z. Further, denote by

$$F : \mathcal{F}(\Gamma) \to (Z, \|\cdot\|_Z)$$

some bounded (with constant C), admissibly almost-additive mapping with tiling-admissible, weak boundary term b defined on $\mathcal{F}(\Gamma)$ with the additional property that

$$T_g F(Q) = F(Qg^{-1})$$

for all $Q \in \mathcal{F}(\Gamma)$ and every $g \in \Gamma$.
If in addition to this, for every $Q \in \mathcal{F}(\Gamma)$, the convex hull $C_{F,Q} := \mathrm{co}\{F(Qg) \,|\, g \in \Gamma\}$ has compact closure in the weak topology on Z, then the following assertions hold true.

(A) For each $Q \in \mathcal{F}(\Gamma)$, the limit

$$S(Q) := \lim_{j\to\infty} |U_j|^{-1} \int_{U_j} F(Qg)\, dm_L(g)$$

exists in the topology of Z and $S(Q)$ does not depend on the choice for (U_j).

(B) For each $\varepsilon > 0$ and $N(\varepsilon) := \lceil \log(\varepsilon)/\log(1-\varepsilon)\rceil$, consider ε-prototiles $\{T_i^\varepsilon\}_{i=1}^{N(\varepsilon)}$ in the group Γ as in Definition 2.15 with $0 < \beta < 2^{-N(\varepsilon)}\varepsilon$. Then the following limits exist in Z and are equal:

$$F^* := \lim_{j\to\infty} \frac{F(U_j)}{|U_j|} = \lim_{\varepsilon\to 0} \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon)\, \frac{S(T_i^\varepsilon)}{|T_i^\varepsilon|},$$

where we have $\eta_i(\varepsilon) := \varepsilon(1-\varepsilon)^{N(\varepsilon)-i}$ for $1 \le i \le N(\varepsilon)$. Further, F^* is independent of the choice of (U_j).

(C) The limit F^* is a T_g-fixed point, i.e. for all $g \in \Gamma$, we have

$$T_g F^* = F^*.$$

Proof.
For the proof of statement (A), let $Q \in \mathcal{F}(\Gamma)$. By Corollary 4.8, the claim now follows from the relative weak compactness of $C_{F,Q}$ together with the invariance $T_{g^{-1}}F(Q) = F(Qg)$ for all $g \in \Gamma$. Note that it follows also from Corollary 4.8 that $S(Q)$ must be independent of (U_j).

For the proof of (B), fix $0 < \varepsilon < 1/10$ and $\beta := 2^{-N(\varepsilon)}\varepsilon$. By Theorem 3.6, we find $K = K(\varepsilon, \beta, T_i^\varepsilon) \in \mathbb{N}$ such that for each $j \ge K$, one can construct a uniform decomposition tower for U_j. Further, we set

$$\Delta(j,\varepsilon) := \left\| \frac{F(U_j)}{|U_j|} - \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon)\, \frac{S(T_i^\varepsilon)}{|T_i^\varepsilon|} \right\|_Z$$

for $\varepsilon > 0$ and $j \in \mathbb{N}$. In the following, we fix $j \ge K$. With $0 < \eta < \eta_0$, where η_0 is chosen as in Definition 3.5, we can find

- some $(U_j U_j^{-1}, \eta)$-invariant set $\hat{U}_j$ along with
- a uniform decomposition tower (Υ, Λ) with prototile sets T_i^ε, $(1 \le i \le N(\varepsilon))$,
- a family of finite center sets $\hat{C}_i^y (y \in \Upsilon)$ for the ε-quasi tiling of $\hat{U}_j$,
- and for each $y \in \Upsilon$, a family of finite center sets $C_i^{y,\lambda} (\lambda \in \Lambda)$ for the ε- quasi tiling of U_j.

With no loss of generality, we assume that all the T_i^ε are taken from a subsequence $(S_{n_k})_{k=1}^\infty$ of a strong Følner sequence such that the expressions $b(S_{n_k})/|S_{n_k}|$ converge to zero monotonically as $k \to \infty$. Additionally, we make sure that $T_i^\varepsilon \in \{S_{n_l} \mid l \ge i\}$ for all $1 \le i \le N(\varepsilon)$.

It is our goal to show that $\lim_{\varepsilon\to 0}\lim_{j\to\infty}\Delta(j,\varepsilon)=0$. To do so, we combine the construction of the uniform decomposition tower for $(U_j,\hat{U}_j)$, cf. Theorem 3.6, statement (III), with the triangle inequality and we arrive at

$$\Delta(j,\varepsilon)\le D_1(j,\varepsilon)+D_2(j,\varepsilon)+D_3(j,\varepsilon)+D_4(j,\varepsilon)+D_5(j,\varepsilon)$$

with

$$D_1(j,\varepsilon):=\left\|\frac{F(U_j)}{|U_j|}-\sum_{i=1}^{N(\varepsilon)}|\Upsilon|^{-1}|\Lambda|^{-1}\int_\Upsilon\int_\Lambda\sum_{c\in C_i^{y,\lambda}}\frac{F(T_i^\varepsilon c)}{|U_j|}\,d\lambda\,dy\right\|_Z,$$

$$D_2(j,\varepsilon):=\left\|\sum_{i=1}^{N(\varepsilon)}|\Upsilon|^{-1}|\Lambda|^{-1}\int_\Upsilon\int_\Lambda\left(\sum_{c\in C_i^{y,\lambda}}\frac{F(T_i^\varepsilon c)}{|U_j|}-\sum_{c\in\hat{C}_i^y\lambda^{-1}\cap U_j}\frac{F(T_i^\varepsilon c)}{|U_j|}\right)\right\|\,d\lambda\,dy,$$

$$D_3(j,\varepsilon):=\left\|\sum_{i=1}^{N(\varepsilon)}|\Upsilon|^{-1}|\Lambda|^{-1}\int_\Upsilon\left(\sum_{c\in\hat{C}_i^y}\int_{U_j\setminus c\Lambda^{-1}}\frac{T_{\lambda^{-1}}F(T_i^\varepsilon)}{|U_j|}\,d\lambda\right)\,dy\right\|_Z,$$

$$D_4(j,\varepsilon):=\left\|\sum_{i=1}^{N(\varepsilon)}\left(|\Upsilon|^{-1}\int_\Upsilon\frac{\#(\hat{C}_i^y)}{|\Lambda|}\,dy\right)\left(\int_{U_j}\frac{T_{\lambda^{-1}}F(T_i^\varepsilon)}{|U_j|}\,d\lambda-S(T_i^\varepsilon)\right)\right\|_Z$$

and

$$D_5(j,\varepsilon):=\left\|\sum_{i=1}^{N(\varepsilon)}\left(|\Upsilon|^{-1}\int_\Upsilon\frac{\#(\hat{C}_i^y)}{|\Lambda|}\,dy\right)S(T_i^\varepsilon)-\sum_{i=1}^{N(\varepsilon)}\eta_i(\varepsilon)\,\frac{S(T_i^\varepsilon)}{|T_i^\varepsilon|}\right\|_Z.$$

Here, to obtain $D_3(j,\varepsilon)$ and $D_4(j,\varepsilon)$, we used

$$\begin{aligned}\int_\Lambda\sum_{c\in\hat{C}_i^y\lambda^{-1}\cap U_j}F(T_i^\varepsilon c)\,d\lambda &= \int_\Lambda\sum_{c\in\hat{C}_i^y}\mathbb{1}_{U_j}(c\lambda^{-1})\cdot F(T_i^\varepsilon c\lambda^{-1})\,d\lambda\\ &= \sum_{c\in\hat{C}_i^y}\int_{c\Lambda^{-1}}\mathbb{1}_{U_j}(\lambda)\cdot F(T_i^\varepsilon\lambda)\,d\lambda\\ &= \sum_{c\in\hat{C}_i^y}\int_{U_j\cap c\Lambda^{-1}}T_{\lambda^{-1}}F(T_i^\varepsilon)\,d\lambda\end{aligned}$$

for all $1\le i\le N(\varepsilon)$ and $y\in\Upsilon$. We also used here that the group Γ is unimodular.
We will now give estimates for these expressions.

(1) We start with $D_1(j,\varepsilon)$. Since U_j is $\alpha:=(1-4\varepsilon)$-covered by ε-disjoint translates $\{T_ic\}$, $1\le i\le N$, $c\in C_i^{y,\lambda}$ for each $y\in\Upsilon$ and each $\lambda\in\Lambda$ and as the weak boundary term b

is tiling-admissible for some constant $\tilde{D} \geq 1$, it follows from Proposition 4.14 that for every $j \geq K$,

$$\begin{aligned} D_1(j,\varepsilon) &\leq (7C + 10\tilde{D})\,\varepsilon + \frac{b(U_j)}{|U_j|} \\ &\quad + (5\tilde{D}+1) \sum_{i=1}^{N(\varepsilon)} \left(|\Upsilon|^{-1}|\Lambda|^{-1} \int_\Upsilon \int_\Lambda \frac{\#(C_i^{y,\lambda})}{|U_j|}\, d\lambda\, dy \right) \cdot b(T_i^\varepsilon). \end{aligned}$$

By Theorem 3.6, we can use the inequality in (II) of Definition 3.3 with $S = T = U_j$ and by the boundedness property of Definition 4.10 for the sets T_i^ε, this yields

$$\begin{aligned} D_1(j,\varepsilon) &\leq (7C + 10\tilde{D})\,\varepsilon + \frac{b(U_j)}{|U_j|} \\ &\quad + (5\tilde{D}+1) \sum_{i=1}^{N(\varepsilon)} \left(\eta_i(\varepsilon)\, b(T_i^\varepsilon) + 2\tilde{D}\tilde{\gamma}_i |T_i^\varepsilon|\,\varepsilon + 4\tilde{D}\,\beta \right). \end{aligned}$$

By the triangle inequality, $\sum_i \tilde{\gamma}_i |T_i^\varepsilon| \leq 2$ and $\beta N(\varepsilon) < 2\varepsilon$, one obtains

$$\begin{aligned} D_1(j,\varepsilon) &\leq (7C + 10\tilde{D})\,\varepsilon + \frac{b(U_j)}{|U_j|} \\ &\quad + (5\tilde{D}+1) \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon)\, b(T_i^\varepsilon) + 4\tilde{D}(5\tilde{D}+1)\,\varepsilon + 8\tilde{D}(5\tilde{D}+1)\,\varepsilon \\ &\leq (7C + 10\tilde{D} + 12\tilde{D}(5\tilde{D}+1))\,\varepsilon + \frac{b(U_j)}{|U_j|} \\ &\quad + (5\tilde{D}+1) \sum_{i=1}^{N(\varepsilon)} \varepsilon(1-\varepsilon)^{N(\varepsilon)-i} \cdot \frac{b(T_i^\varepsilon)}{|T_i^\varepsilon|} \\ &\leq (7C + 10\tilde{D} + 12\tilde{D}(5\tilde{D}+1))\,\varepsilon + \frac{b(U_j)}{|U_j|} \\ &\quad + (5\tilde{D}+1) \sum_{i=1}^{N(\varepsilon)} \varepsilon(1-\varepsilon)^{N(\varepsilon)-i} \cdot \frac{b(S_{n_i})}{|S_{n_i}|}. \end{aligned}$$

for every $j \geq K$. As $\lim_{j\to\infty} b(U_j)/|U_j| = 0$ we arrive at

$$\begin{aligned} \limsup_{j\to\infty} D_1(j,\varepsilon) &\leq (7C + 10\tilde{D} + 12\tilde{D}(5\tilde{D}+1))\,\varepsilon \\ &\quad + (5\tilde{D}+1) \sum_{i=1}^{N(\varepsilon)} \varepsilon(1-\varepsilon)^{N(\varepsilon)-i} \cdot \frac{b(S_{n_i})}{|S_{n_i}|}. \end{aligned} \tag{4.9}$$

(2) We continue with the estimate for $D_2(j,\varepsilon)$. It follows from the property (III) of the Definition 3.5 of the uniform decomposition tower that there is a set $\tilde{U}_j \subseteq U_j$ with $|\tilde{U}_j| \geq (1-\beta)|U_j|$ and

$$\tilde{U}_j \cap \hat{C}_i^y \lambda^{-1} \subseteq C_i^{y,\lambda} \subseteq U_j \cap \hat{C}_i^y \lambda^{-1}$$

for $1 \le i \le N(\varepsilon)$, $y \in \Upsilon$ and $\lambda \in \Lambda$. Further, one can choose $\tilde{U}_j$ such that it is a $(T^\varepsilon_{N(\varepsilon)} T^{\varepsilon\,-1}_{N(\varepsilon)}, 4\beta)$-invariant set.

By the triangle inequality and the boundedness of F, we then obtain

$$\begin{aligned} D_2(j,\varepsilon) &\le \sum_{i=1}^{N(\varepsilon)} |\Upsilon|^{-1}|\Lambda|^{-1} \int_\Upsilon \int_\Lambda \sum_{c \in \hat{C}_i^y \lambda^{-1} \cap (U_j \setminus \tilde{U}_j)} \frac{\|F(T_i^\varepsilon c)\|}{|U_j|} \, d\lambda \, dy \\ &\le C \sum_{i=1}^{N(\varepsilon)} |\Upsilon|^{-1}|\Lambda|^{-1} \int_\Upsilon \int_\Lambda \frac{\#(\hat{C}_i^y \cap (U_j \setminus \tilde{U}_j)\lambda)}{|U_j|} |T_i^\varepsilon| \, d\lambda \, dy. \end{aligned}$$

Since the set U_j is $(T^\varepsilon_{N(\varepsilon)} T^{\varepsilon\,-1}_{N(\varepsilon)}, \beta)$-invariant, we have

$$\frac{|\partial_{T_i^\varepsilon T_i^{\varepsilon\,-1}}(U_j \setminus \tilde{U}_j)|}{|U_j|} < 5\beta$$

for all $1 \le i \le N(\varepsilon)$. The ε-disjointness of the T_i^ε-translates then implies the bound

$$\begin{aligned} \frac{\#(\hat{C}_i^y \cap (U_j \setminus \tilde{U}_j)\lambda)}{|U_j|} |T_i^\varepsilon| &\le \frac{|(U_j \setminus \tilde{U}_j) \cup \partial_{T_i^\varepsilon T_i^{\varepsilon\,-1}}(U_j \setminus \tilde{U}_j)|}{|U_j||T_i^\varepsilon|(1-2\varepsilon)} |T_i^\varepsilon| \\ &\le 2\,(\beta + 5\beta) = 12\beta \end{aligned}$$

for every $1 \le i \le N(\varepsilon)$, for each $y \in \Upsilon$ and all $\lambda \in \Lambda$. Putting these estimates together, we arrive at

$$D_2(j,\varepsilon) \le C \sum_{i=1}^{N(\varepsilon)} 12\,\beta \overset{\beta N < 2\varepsilon}{\le} 24C\,\varepsilon.$$

Thus,

$$\limsup_{j\to\infty} D_2(j,\varepsilon) \le 24C\varepsilon. \tag{4.10}$$

(3) For a good estimate for $D_3(j,\varepsilon)$, the concept of a uniform decomposition tower is crucial. At first, we observe that $c \in \hat{U}_j \setminus u\Lambda$ whenever $u \in U_j$ and $c \in \hat{C}_i^y \subseteq \hat{U}_j$ are such that $u \notin c\Lambda^{-1}$. Hence, due to the boundedness of F, we have for $j \ge K$ that

$$\begin{aligned} D_3(j,\varepsilon) &\le \sum_{i=1}^{N(\varepsilon)} |\Upsilon|^{-1}|\Lambda|^{-1} \int_\Upsilon \left(\sum_{c \in \hat{C}_i^y} |U_j|^{-1} \int_{U_j \setminus c\Lambda^{-1}} \|T_{u^{-1}} F(T_i^\varepsilon)\|_Z \, du \right) dy \\ &\le C \sum_{i=1}^{N(\varepsilon)} |\Upsilon|^{-1}|\Lambda|^{-1} \int_\Upsilon \left(\sum_{c \in \hat{C}_i^y} |U_j|^{-1} \int_{U_j} \mathbb{1}_{\hat{U}_j \setminus u\Lambda}(c) \cdot |T_i^\varepsilon| \, du \right) dy \\ &\le C \sum_{i=1}^{N(\varepsilon)} |U_j|^{-1} \int_{U_j} |\Upsilon|^{-1} \left(\int_\Upsilon \frac{\#((\hat{U}_j \setminus u\Lambda) \cap \hat{C}_i^y)}{|\Lambda|} \, dy \right) du \cdot |T_i^\varepsilon| \\ &\overset{\beta<1/2}{\le} 2C \sum_{i=1}^{N(\varepsilon)} |U_j|^{-1} \int_{U_j} |\Upsilon|^{-1} \left(\int_\Upsilon \frac{\#((\hat{U}_j \setminus u\Lambda) \cap \hat{C}_i^y)}{|\hat{U}_j|} \, dy \right) du \cdot |T_i^\varepsilon|, \end{aligned}$$

where the last inequality is due to Definition 3.5, statement (I) and $\beta < 1/2$.
The independence of Λ and Υ comes into play now. As the set Λ is the same for every chosen $y \in \Upsilon$, we can exploit the properties (I) and (II) of Definition 3.5. Note that the property (I) gives $|\hat{U}_j \setminus u\Lambda| = |\hat{U}_j| - |\Lambda| \leq \beta\,|\hat{U}_j|$ for all $u \in U_j$. Next, we apply the inequality in (II) of Definition 3.3 for each $u \in U_j$ with $\hat{T} = \hat{U}_j$, $\hat{S} = \hat{U}_j \setminus u\Lambda$. Now using the boundedness of F, $\beta < 2^{-N(\varepsilon)}\varepsilon$, $\sum_i \tilde{\gamma}_i |T_i^\varepsilon| \leq 2$ and $\beta < \varepsilon$ we arrive at

$$\begin{aligned} D_3(j,\varepsilon) &\leq 2C \sum_{i=1}^{N(\varepsilon)} \left(\eta_i(\varepsilon) \cdot \frac{\beta}{|T_i^\varepsilon|} + 4\frac{\beta}{|T_i^\varepsilon|} + 2\tilde{\gamma}_i\,\varepsilon \right) |T_i^\varepsilon| \\ &\leq 2\,C\,(\beta + 8\varepsilon + 4\varepsilon) \\ &\leq 26\,C\,\varepsilon. \end{aligned}$$

Consequently,

$$\limsup_{j\to\infty} D_3(j,\varepsilon) \leq 26C\,\varepsilon. \tag{4.11}$$

(4) For $D_4(j,\varepsilon)$, it is a direct consequence of assertion (II) of Definition 3.3 with $\hat{S} = \hat{T}$ and with $|\Lambda| \geq (1-\beta)|\hat{U}_j|$ that

$$\begin{aligned} D_4(j,\varepsilon) &\leq \sum_{i=1}^{N(\varepsilon)} (1-\beta)^{-1} \left(\frac{\eta_i(\varepsilon)}{|T_i^\varepsilon|} + \frac{4\beta}{|T_i^\varepsilon|} + 2\,\tilde{\gamma}_i\,\varepsilon \right) \left\| \int_{U_j} \frac{T_{\lambda^{-1}} F(T_i^\varepsilon)}{|U_j|}\, d\lambda - S(T_i^\varepsilon) \right\|_Z \\ &\overset{\beta<1/2}{\leq} 2 \sum_{i=1}^{N(\varepsilon)} \left[\frac{\eta_i(\varepsilon)}{|T_i^\varepsilon|} \left\| |U_j|^{-1} \int_{U_j} T_{\lambda^{-1}} F(T_i^\varepsilon)\, d\lambda - S(T_i^\varepsilon) \right\|_Z + \right. \\ &\qquad \left. \left(4\frac{\beta}{|T_i^\varepsilon|} + 2\,\tilde{\gamma}_i\,\varepsilon \right) C \cdot 2 \cdot |T_i^\varepsilon| \right] \\ &\leq 2 \sum_{i=1}^{N(\varepsilon)} \frac{\eta_i(\varepsilon)}{|T_i^\varepsilon|} \left\| |U_j|^{-1} \int_{U_j} T_{\lambda^{-1}} F(T_i^\varepsilon)\, d\lambda - S(T_i^\varepsilon) \right\|_Z + 48\varepsilon\, C \end{aligned}$$

for every $j \geq K$. It now follows from the claim (A) that

$$\limsup_{j\to\infty} D_4(j,\varepsilon) \leq 48C\,\varepsilon. \tag{4.12}$$

(5) Finally, again by using Theorem 3.6 (i.e. property (II) of Definition 3.5 with $\hat{S} = \hat{T} = \hat{U}_j$), we also get an estimate for $D_5(j,\varepsilon)$. By the uniform distribution of the $\hat{C}_i^y$ and since $|\Lambda| \geq (1-\beta)|\hat{U}_j|$ by the statement (I) of Definition 3.5, we obtain

$$\begin{aligned} D_5(j,\varepsilon) &\leq \sum_{i=1}^{N(\varepsilon)} \left| |\Upsilon|^{-1} \int_\Upsilon \frac{\#(\hat{C}_i^y)}{|\Lambda|}\, dy - \frac{\eta_i(\varepsilon)}{|T_i^\varepsilon|} \right| \|S(T_i^\varepsilon)\|_Z \\ &\leq C\,[(1-\beta)^{-1} - 1] \cdot |\Upsilon|^{-1} \int_\Upsilon \sum_{i=1}^{N(\varepsilon)} \frac{\#(\hat{C}_i^y)\,|T_i^\varepsilon|}{|\hat{U}_j|}\, dy\, + \\ &\quad C \sum_{i=1}^{N(\varepsilon)} \left(\frac{4\beta}{|T_i^\varepsilon|} + 2\tilde{\gamma}_i\varepsilon \right) |T_i^\varepsilon| \end{aligned}$$

for $j \geq K$. Since the translates $\{T_i^\varepsilon c\}$, $c \in \hat{C}_i^y$ are ε-disjoint and as $\sum_{i=1}^{N(\varepsilon)} \tilde{\gamma}_i |T_i^\varepsilon| \leq 2$, we arrive at

$$\begin{aligned} D_5(j,\varepsilon) &\leq C\,\frac{(1-\beta)^{-1}-1}{1-\varepsilon} + C \sum_{i=1}^{N(\varepsilon)} (4\beta + 2\tilde{\gamma}_i |T_i^\varepsilon|\,\varepsilon) \\ &\overset{\beta<2^{-N(\varepsilon)}\varepsilon}{\leq} 16C\,\varepsilon \end{aligned}$$

for $j \geq K$ and thus,

$$\limsup_{j\to\infty} D_5(j,\varepsilon) \leq 16C\,\varepsilon. \tag{4.13}$$

To conclude the proof of the theorem, we derive from the Inequalities (4.9), (4.10) (4.11), (4.12), as well as (4.13) that indeed,

$$\lim_{\varepsilon\to 0} \limsup_{j\to\infty} \Delta(j,\varepsilon) = 0.$$

Here, we use in the estimate (4.9) the monotonicity of $(b(S_{n_i})/|S_{n_i}|)$ to obtain

$$\lim_{\varepsilon\to 0} \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon) \frac{b(S_{n_i})}{|S_{n_i}|} = 0.$$

The triangle inequality now yields

$$\begin{aligned} \limsup_{k,l\to\infty} \left\| \frac{F(U_l)}{|U_l|} - \frac{F(U_k)}{|U_k|} \right\|_Z &\leq \lim_{\varepsilon\to 0} \limsup_{k\to\infty} \Delta(k,\varepsilon) + \lim_{\varepsilon\to 0} \limsup_{l\to\infty} \Delta(l,\varepsilon) \\ &= 0. \end{aligned}$$

Hence, $F(U_j)/|U_j|$ is a Cauchy sequence and thus, it converges in the Banach space Z. The representation as the second limit is now an easy consequence of the triangle inequality. To see the independence of the Følner sequence (U_j) under consideration, note first that by claim (A), all the $S(T_i^\varepsilon)$ do not depend on (U_j) and hence the above ε-limit only depends of the choice of the tiling sets T_i^ε. However, we have chosen an arbitrary collection of ε-prototiles in the group Γ which does not depend on (U_j). Thus, the expressions $F(U_j)/|U_j|$ must converge to the same limit for *every* strong Følner sequence (U_j) in Γ.

To show claim (C), we take an arbitrary $g \in \Gamma$. Note that for all $0 < \varepsilon < 1/10$ and every $1 \leq i \leq N(\varepsilon)$, we have $T_g S(T_i^\varepsilon) = S(T_i^\varepsilon)$ by the Følner property of the sequence (U_j). By the boundedness (continuity) of the operator T_g and by the convergence result in claim (B), the following computation finishes our proof.

$$\begin{aligned} T_g F^* &= T_g \left(\lim_{\varepsilon\to 0} \sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon)\, \frac{S(T_i^\varepsilon)}{|T_i^\varepsilon|} \right) = \lim_{\varepsilon\to 0} \left(\sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon)\, \frac{T_g S(T_i^\varepsilon)}{|T_i^\varepsilon|} \right) \\ &= \lim_{\varepsilon\to 0} \left(\sum_{i=1}^{N(\varepsilon)} \eta_i(\varepsilon)\, \frac{S(T_i^\varepsilon)}{|T_i^\varepsilon|} \right) = F^*. \end{aligned}$$

□

For spectral applications, we will also have to deal with situations where we do not have an action of the group Γ on the Banach space Z at our disposal. However, assuming the existence of certain abstract limits, we are still able to derive the abstract mean ergodic theorem.

Corollary 4.16.
Let Γ be an amenable LCSCUH group, (U_j) a strong Følner sequence in Γ, $(Z, \|\cdot\|_Z)$ a Banach space and $\{T_g\}_{g\in\Gamma}$ a family of linear, uniformly bounded operators acting weakly measurably on Z. Further, denote by

$$F : \mathcal{F}(\Gamma) \to (Z, \|\cdot\|_Z)$$

some bounded (constant C), admissibly almost-additive mapping with tiling-admissible, weak boundary term b defined on $\mathcal{F}(\Gamma)$. Assume that for each $Q \in \mathcal{F}(G)$ and all elements h in the dual space Z^* of Z, the mappings

$$\psi_{Q,h} : G \to \mathbb{C} : g \mapsto \langle F(Qg), h\rangle_{Z,Z^*}$$

are measurable, where $\langle\cdot,\cdot\rangle_{Z,Z*}$ denotes the dual pairing of Z with Z^*. For a positive sequence $\varepsilon_k \to 0$ and $N(\varepsilon_k) := \lceil \log(\varepsilon_k)/\log(1-\varepsilon_k)\rceil$, take ε_k-prototiles $\{T_i^{\varepsilon_k}\}_{i=1}^{N(\varepsilon_k)}$ in Γ as in Definition 2.15 with $0 < \beta < 2^{-N(\varepsilon_k)}\varepsilon_k$.
Then, if for each $k \in \mathbb{N}$ and every $1 \leq i \leq N(\varepsilon_k)$, the expression

$$S(T_i^{\varepsilon_k}) := \lim_{j\to\infty} |U_j|^{-1} \int_{U_j} F(T_i^{\varepsilon_k} g)\, dg$$

exists in Z, the following limits exist in Z and are equal:

$$F^* := \lim_{j\to\infty} \frac{F(U_j)}{|U_j|} = \lim_{k\to\infty} \sum_{i=1}^{N(\varepsilon_k)} \eta_i(\varepsilon_k)\, \frac{S(T_i^{\varepsilon_k})}{|T_i^{\varepsilon_k}|},$$

where $\eta_i(\varepsilon_k) := \varepsilon(1-\varepsilon_k)^{N(\varepsilon_k)-i}$ for $1 \leq i \leq N(\varepsilon_k)$.
In addition to this, F^* does not depend on the choice of the sequence (U_j).

Proof.
This follows by a simple modification of the proof of (B) of Theorem 4.15. □

5 Bounded, additive processes on groups

This chapter is devoted to the investigation of a specific class of almost-additive functions. Precisely, we focus on so-called *bounded, additive processes* F which can be interpreted as admissibly almost-additive functions in the sense of Definition 4.13 with boundary term $b = 0$ and with $Z = L^1(\Omega, Y)$ being a Bochner space with reflexive Banach space Y. The main theorem of the present chapter is Theorem 5.17. It is shown therein that for approximable bounded, additive processes over probability spaces, there is some $F^* \in L^1(\Omega, Y)$ such that for increasing Tempelman Følner sequences (U_j),

$$\lim_{j\to\infty} \left\| \frac{F(U_j)(\omega)}{|U_j|} - F^*(\omega) \right\|_Y = 0$$

for almost-all $\omega \in \Omega$. As far as the underlying geometry is concerned, this significantly extends the setting of the considerations of Sato in [Sat99, Sat03], where he proves convergence along d-dimensional cubes for an $\mathbb{R}^d$-semigroup action. In the situation of the latter work, the approximability is automatically satisfied. In the general setting of the present thesis, we need to assume that L^1-processes can be approximated in a suitable way by a certain class of L^∞-processes. Our result provides interesting applications and implications to results from the literature. For instance, if F is absolutely continuous, i.e. if it is an integral average as in Definition 4.6, then we obtain a classical pointwise ergodic theorem, see e.g. [Tem72, Eme74, Lin01]. By slightly extending the techniques in [Lin01], one can even prove the convergence for Shulman sequences and for σ-finite measure spaces in this special situation. This has been done by the author of this thesis in Theorem 6.8 of [Pog13a]. We will state and discuss this result without proof in Theorem 5.5. More interestingly, our main Theorem 5.17 applies to a larger class of bounded, additive processes. The present chapter is divided into two parts. At first, we introduce Bochner spaces and bounded, additive processes. In the second part, we present and discuss the new pointwise almost-everywhere assertions and draw some links to the literature. The corresponding results are partially taken from Chapter 6 in [Pog13a]. During the preparation of the present chapter, the author of this thesis realized that in [Pog13a], there is a gap in the proof of the pointwise ergodic theorem. To deal with this, we add the assumption of approximability in the present version, cf. Theorem 5.17. An erratum on the pointwise almost-everywhere convergence statements of [Pog13a] is in preparation.

5.1 Bounded, additive processes

In the following, we introduce bounded, additive processes on LCSCUH groups with values in a Bochner space. Moreover, we give some illustrative examples. Considering abstract

Poisson point processes in Example 5.6, we demonstrate that there are bounded, additive processes which are not of the integral form in Definition 4.6. In Theorem 5.5, we state the almost-everywhere convergence result for absolutely continuous processes. This latter assertion can be shown by using classical techniques. Therefore, we do not give a proof and refer the reader to Theorem 6.8 in [Pog13a].

Definition 5.1.
Let Y be a Banach space and $(\Omega, \mathcal{F}, \mu)$ be a σ-finite measure space. For $1 \leq p < \infty$, we denote by $L^p(\Omega, Y)$ the (Bochner) space of all equivalence classes $f : \Omega \to Y$ such that each representative f is strongly measurable with respect to $\mathcal{F}$ and

$$\|f\|_{L^p(\Omega,Y)} := \left(\int_\Omega \|f(\omega)\|_Y^p \, d\mu(\omega) \right)^{1/p} < \infty,$$

i.e. $\|f(\cdot)\|_Y \in L^p(\Omega, \mathbb{R}) := L^p(\Omega, \mathcal{F}, \mu)$. (Two functions f and h are considered as equivalent if $f(\omega) = h(\omega)$ for μ-almost every $\omega \in \Omega$.)
For $p = \infty$, we set $L^\infty(\Omega, Y)$ as the space of strongly measurable equivalence classes f such that

$$\|f\|_{L^\infty(\Omega,Y)} := \operatorname*{ess\,sup}_{\omega \in \Omega} \|f(\omega)\|_Y < \infty,$$

i.e. $\|f(\cdot)\|_Y \in L^\infty(\Omega, \mathbb{R}) := L^\infty(\Omega, \mathcal{F}, \mu)$.

Remark.
The strong measurability condition mentioned in the above definition is the common notion for measurability in Bochner spaces, cf. [Boc33]. Precisely, a mapping $f : \Omega \to Y$ is strongly measurable if it can be obtained as a μ-almost-everywhere limit of simple classes (f_n), i.e. for each $n \in \mathbb{N}$, there are $L \in \mathbb{N}$, $y_i \in Y$ and $A_i \in \mathcal{F}$, $(1 \leq i \leq L)$ such that with

$$f_n := \sum_{i=1}^{L} y_i \mathbb{1}_{A_i},$$

we have $\lim_{n\to\infty} \|f_n(\omega) - f(\omega)\|_Y = 0$ for μ-almost every $\omega \in \Omega$.

In the following, we are interested in the case where $Z := L^p(\Omega, Y)$ for some $1 \leq p < \infty$ and Y is a reflexive Banach space. Further, we suppose that the group Γ acts measurably on $(\Omega, \mathcal{F}, \mu)$ by measure preserving transformations. This is made precise in the following definition.

Definition 5.2.
Let Γ be an LCSCUH group with unity e and Borel σ-algebra $\mathcal{B}(\Gamma)$. Suppose that $(\Omega, \mathcal{F}, \mu)$ is a probability space. We say that Γ acts on Ω measure preservingly *or* by measure preserving transformations *if there is a $(\mathcal{B}(\Gamma) \otimes \mathcal{F})$-$\mathcal{F}$-measurable map*

$$\pi : \Gamma \times \Omega \to \Omega$$

such that

- *$\pi(e, \omega) = \omega$ for all $\omega \in \Omega$,*

- $\pi(g, \pi(h, \omega)) = \pi(gh, \omega)$ *for all* $g, h \in \Gamma$ *and every* $\omega \in \Omega$,
- $\mu(A) = \mu(\pi(g, A))$ *for all* $g \in \Gamma$ *and every* $A \in \mathcal{F}$.

If in addition, a set $A \in \mathcal{F}$ *can only be fixed under the* Γ*-action (i.e.* $\pi(g, A) = A$ *for all* $g \in \Gamma$*) if* $\mu(A) \in \{0, 1\}$*, then we say that the action of* Γ *on* Ω *is* ergodic*. In this situation, we say that* μ *is an* ergodic measure *for the action of* Γ *on* Ω.

For the sake of clarity, we write in the following simply $g\omega := g \cdot \omega := \pi(g, \omega)$ for $g \in \Gamma$ and $\omega \in \Omega$.

We assume that the action of Γ on the probability space Ω induces a weakly measurable action of Γ on Z according to Definition 4.5. The connection of the corresponding operators $\{T_g\}$ with the group action on Ω shall be given by a measurable group homomorphism $\varphi : \Gamma \to \Gamma$. The latter must satisfy the following regularity condition: there is some constant $\kappa > 0$ such that for every $g \in \Gamma$ and for each $f \in Z$

$$\|T_g f(\omega)\|_Y \leq \kappa \, \|f(\varphi(g)^{-1}\omega)\|_Y$$

for μ-almost every $\omega \in \Omega$. Since the action of Γ on Ω preserves the measure μ, it follows that $\sup_{g\in\Gamma} \|T_g\|_{L^p(\Omega,Y)} \leq \kappa$. As φ is a group homomorphism, the above inequality implies

$$\begin{aligned} \|f(\varphi(g)^{-1}\omega)\|_Y &= \|[T_{g^{-1}}(T_g f)](\varphi(g)^{-1}\omega)\|_Y \\ &\leq \kappa \, \|(T_g f)(\varphi(g^{-1})^{-1}\varphi(g)^{-1}\omega)\|_Y = \kappa \, \|(T_g f)(\omega)\|_Y. \end{aligned}$$

This shows that the above regularity condition is equivalent to the assumption that there is a number $\kappa > 0$ such that

$$\kappa^{-1} \, \|f(\varphi(g)^{-1}\omega)\|_Y \leq \|T_g f(\omega)\|_Y \leq \kappa \, \|f(\varphi(g)^{-1}\omega)\|_Y \tag{5.1}$$

for all $g \in \Gamma$ and every $f \in Z$. This also implies that $\kappa \geq 1$. Note that this setting includes the 'standard situation' of the usual ground space transformation, where one has $(T_g f)(\omega) = f(g^{-1}\omega)$ for $f \in L^p(\Omega, Y)$, $g \in \Gamma$ and $\omega \in \Omega$.

In the following, we denote by (U_j) a tempered, strong Følner sequence in Γ, cf. Definitions 2.5 and 2.9. Recall that in Definition 4.6, we defined the j-th ($j \in \mathbb{N}$) abstract ergodic average with respect to (U_j) as

$$A_j f := |U_j|^{-1} \int_{U_j} T_{g^{-1}} f \, dm_L(g)$$

for $f \in Z$.

We are now in position to define the notion of bounded, additive processes on groups with values in a Bochner space.
In this context, we restrict ourselves to the set $\mathcal{F}^0(\Gamma)$ consisting of all elements in $\mathcal{F}(\Gamma)$ which additionally are locally closed and precompact (i.e. they have compact closure).

Definition 5.3 (Bounded, additive processes).
Let Γ be an LCSCUH group and denote by Y some reflexive Banach space. For a σ-finite measure space $(\Omega, \mathcal{F}, \mu)$ with μ-invariant Γ-action, as well as for $1 \leq p < \infty$, we assume that there is a family $\{T_g\}_{g\in\Gamma}$ of uniformly bounded, linear operators acting weakly measurably on $L^p(\Omega, Y)$ such that inequality (5.1) is satisfied for some $\kappa \geq 1$ and for some measurable group homomorphism $\varphi : \Gamma \to \Gamma$. In this situation, we call the map

$$F : \mathcal{F}^0(\Gamma) \to L^p(\Omega, Y)$$

a bounded, additive process *on Γ if the following statements hold.*

(i) F *is* bounded*, i.e.* $K := \sup\{\|F(Q)\|_{L^p(\Omega,Y)}/|Q| \,|\, Q \in \mathcal{F}^0(\Gamma), |Q| > 0\} < \infty$,

(ii) F *is* additive*, i.e.* $F(Q) = \sum_{k=1}^m F(Q_k)$ *if* $Q \in \mathcal{F}^0(\Gamma)$ *is a disjoint union of the* $Q_k \in \mathcal{F}^0(\Gamma)$ *for* $1 \leq k \leq m$,

(iii) F *is* equivariant*, i.e.* $T_g F(Q) = F(Qg^{-1})$ *for all* $Q \in \mathcal{F}^0(\Gamma)$ *and every* $g \in \Gamma$.

Before we approach the issues of norm and pointwise convergence of these mappings F, let us first give a couple of examples.

Examples 5.4.

- Assume that Γ acts on a σ-finite measure space $(\Omega, \mathcal{F}, \mu)$ by measure preserving transformations. Then, for every $f \in L^p(\Omega, \mathbb{R})$ $(1 \leq p < \infty)$, the map

$$\mathcal{F}^0(\Gamma) \to L^p(\Omega, \mathbb{R}) : F(Q)(\cdot) := \int_Q T_{g^{-1}} f \, dg \, (\cdot)$$

  defines a bounded, additive process for the canonical action $T_g h(\omega) = h(g^{-1}\omega)$ on $L^p(\Omega, \mathbb{R})$. Note that the regularity condition, inequality (5.1) is satisfied with $\kappa = 1$ and the identity $\varphi = \mathrm{id}_\Gamma$. In this case, we say that the process F is *absolutely continuous with density* f with respect to Γ. The boundedness constant is given by $C := \|f\|_{L^p(\Omega,\mathbb{R})}$.

- Let $\Gamma = \mathbb{R}^d$ $(d \geq 1)$ and assume that $F : \mathcal{F}^0(\Gamma) \to L^1_+(\Omega, \mathbb{R})$ is a bounded, additive process for a measure preserving action $T_g h(\omega) = h(g^{-1}\omega)$ on the canonical non-negative cone in $L^1(\Omega, \mathbb{R})$. It is shown in [AdJ81] that in this situation, we can write $F = F_1 + F_2$, where F_1 is some absolutely continuous process with a non-negative density and where F_2 is a *singular process*, i.e. a bounded, additive process which does not dominate any absolutely continuous, non-zero, non-negative process.

- In the previous example, set $(\Omega, \mathcal{F}, \mu) = (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mathcal{L}_d)$, where $\mathcal{L}_d$ is the usual d-dimensional Lebesgue measure in the euclidean space. Assume further that the action of Γ on $\mathbb{R}^d$ is given by translation, i.e. $T_g f(x) = f(x - g)$ for all $g, x \in \mathbb{R}^d$ and where $f \in L^1_+(\mathbb{R}^d)$. Then every singular bounded, additive process with respect to $\{T_g\}$ is of the form $F(Q)(x) = \nu(x + Q)$ $(x \in \mathbb{R}^d)$, where ν is a Borel measure which is singular with respect to $\mathcal{L}_d$ (cf. [AdJ81], Example 4.12).

- The same result as in the previous example holds true if we consider $(\Omega, \mathcal{F}, \mu) = (\mathbb{T}^d, \mathcal{B}(\mathbb{T}^d), \mathcal{L}_d)$, where $\mathcal{L}_d$ is the d-dimensional Lebesgue measure on the d-dimensional torus $\mathbb{T}^d$ and $\Gamma = \mathbb{R}^d$ acts by rotations.

In [Pog13a], the author of this thesis has extended the Lindenstrauss pointwise ergodic theorem from $L^1(\tilde{\Omega}, \mathbb{R})$ (with $\tilde{\Omega}$ being a probability space) to general Bochner spaces $L^p(\Omega, Y)$ as defined above. Since most parts of the proof are minor adaptions of LINDENSTRAUSS' techniques, we state the assertion here without proof. We have seen in the first example above that on Bochner spaces, abstract ergodic averages of the form as in Definition 4.6 can be interpreted as absolutely continuous, bounded additive processes F. In this context, Theorem 5.5 solves the question of almost-everywhere convergence of $F(U_j)(\omega)/|U_j|$ in case that F has an $L^p(\Omega, Y)$-density f, i.e. if F is absolutely continuous.

Theorem 5.5 (see [Pog13a], Theorem 6.8).
Let Γ be an amenable LCSCUH group which acts on a σ-finite measure space (Ω, μ) by measure preserving transformations. Assume further that for $1 \leq p < \infty$ and some reflexive Banach space Y, the group acts weakly measurably on $L^p(\Omega, Y)$ via a family $\{T_g\}_{g \in \Gamma}$ of uniformly bounded operators such that the regularity condition in inequality (5.1) is satisfied. Then, for each $f \in L^p(\Omega, Y)$, there is a unique element $f^* \in L^p(\Omega, Y)$ such that for all tempered, weak Følner sequences (U_j) in Γ,

$$\lim_{j \to \infty} \left\| |U_j|^{-1} \int_{U_j} (T_{g^{-1}} f)(\omega)\, dg - f^*(\omega) \right\|_Y = 0$$

for μ-almost every $\omega \in \Omega$.
Moreover, we have $T_g f^* = f^*$ in $L^p(\Omega, Y)$ for all $g \in \Gamma$.

Note that the Theorem 5.5 can also be interpreted as a convergence theorem for absolutely continuous bounded, additive processes. Thus, it is natural to come up with the question of pointwise almost-everywhere convergence of the averages $F(U_j)(\omega)/|U_j|$ for general bounded, additive processes F. Before approaching this question, one has to make sure that the Definition 5.3 is in fact more universal, i.e. that there are bounded, additive processes which are not absolutely continuous. In order to show that processes of this latter kind do in fact exist, we give the following example.

Example 5.6 (Poisson point processes).
Let Γ be a locally compact, second countable, unimodular Hausdorff group with left measure $m = |\cdot|$. Moreover, let $(\Omega, \mathcal{F}, \mu)$ be a probability space such that $\alpha\, m$ $(\alpha > 0)$ is the intensity measure of some Γ-set valued, homogeneous Poisson point process X defined on (Ω, μ). Concerning their existence and further information on point processes on locally compact groups, the reader may refer to [Kin93]. Note that there is a μ-preserving action of Γ from the right on Ω and we have $X(g\omega) := X(\omega(g)) = X(\omega)g^{-1}$. In this case, the Poisson random variables $F(Q)(\omega) := \#(X(\omega) \cap Q)$ $(\omega \in \Omega, Q \in \mathcal{F}^0(\Gamma))$ define a bounded, additive process for the canonical translations (see above) $\{T_g\}$ on $L^1_+(\Omega, \mathbb{R})$. This can readily be checked.

- For the boundedness, note that by the distributional properties of the Poisson point process, we have
$$\|F(Q)\|_{L^1(\Omega)} = \mathbb{E}_\mu \left(\#(X(\cdot) \cap Q)\right) = \alpha\, |Q|, \quad (Q \in \mathcal{F}^0(\Gamma)),$$
where $\mathbb{E}_\mu$ denotes the expected value with respect to the measure μ. Thus, F is bounded with constant $\alpha > 0$.

- For the additivity, assume that $Q = \bigsqcup_{k=1}^{m} Q_k$ is a disjoint union of elements in $\mathcal{F}^0(\Gamma)$. Then indeed,

$$F(Q)(\omega) = \#(X(\omega) \cap Q) = \sum_{k=1}^{m} \#(X(\omega) \cap Q_k) = \sum_{k=1}^{m} F(Q_k)(\omega)$$

  for all $\omega \in \Omega$.

- For the equivariance, we compute

$$\begin{aligned} F(Qg)(\omega) &= \#(X(\omega) \cap Qg) = \#(X(\omega)g^{-1} \cap Q) \\ &= \#(X(\omega(g)) \cap Q) = F(Q)(g\omega) \end{aligned}$$

  for $Q \in \mathcal{F}^0(\Gamma)$, $\omega \in \Omega$, $g \in \Gamma$.

A further interesting fact is the following: if Γ is not discrete, then the process is not absolutely continuous. This might be known but we have not found a proof in the literature. In the following, we give a brief justification which was delevoped by XUEPING HUANG and reported to the author in private communication. So assume that there is some measurable function f on Ω such that

$$\#(X(\omega) \cap Q) = \int_Q f(g\omega)\, dg$$

for every $Q \in \mathcal{F}^0(\Gamma)$. Now take an arbitrary open set $Q \in \mathcal{F}^0(\Gamma)$. Integration yields that $f \in L^1(\Omega, \mu)$ and that there is a set $\Omega(Q) \subseteq \Omega$ of full measure such that the integral $\int_Q f(g\omega)\, dg$ is finite for all $\omega \in \Omega(Q)$. Taking a countable cover Q_j of the (second countable) group consisting of open sets of finite measure, we find a subset $\overline{\Omega} \subseteq \Omega$ of full measure such that for all $\omega \in \overline{\Omega}$ and every $j \in \mathbb{N}$, the expression $\int_{Q_j} f(g\omega)\, dg$ is finite. This also demonstrates that for almost all $\omega \in \Omega$ and every compact set $A \subseteq \Gamma$,

$$\#(X(\omega) \cap A) = \int_A f(g\omega)\, dg < \infty. \tag{5.2}$$

Now fix $\omega_0 \in \overline{\Omega}$, along with an arbitrary $x \in X(\omega_0)$. Further, take a decreasing sequence A_n of compact sets such that $\cap_n A_n = \{x\}$. Since Γ does not possess atoms with respect to the Haar measure, $\lim_{n\to\infty} m(A_n) = 0$. Also, $\liminf_{n\to\infty} \#(X(\omega_0) \cap A_n) \geq 1$. On the other hand, it follows from Vitali's theorem that $\lim_{n\to\infty} \int_{A_n} f(g\omega_0)\, dg = 0$. This clearly is a contradiction to the equality (5.2). Hence, there is no such $f \in L^1(\Omega, \mu)$. We conclude that those Poisson point processes do not belong to the class of absolutely continuous processes.

We give another example which has some similarities to the previous one.

Example 5.7.
Let $\Gamma = (\mathbb{R}^d, +)$. Clearly, this is a non-discrete, abelian LCSCUH group. A set $\omega \subseteq \mathbb{R}^d$ is called *r-uniformly discrete* for $r > 0$ if $\#(\omega \cap B_r(g)) \leq 1$ for all $g \in \mathbb{R}^d$, where $B_r(g)$ is the standard ball of radius r around g. A set $\omega \subseteq \mathbb{R}^d$ is *R-relatively dense* for $R > 0$ if $\omega \cap B_R(g) \neq \emptyset$ for all $g \in \mathbb{R}^d$. We say that a set $\omega \subseteq \mathbb{R}^d$ is a *Delone set* if there exist $r, R > 0$ such that ω is r-uniformly discrete and R-relatively dense. Note that this definition

only makes sense if $2R \geq r$. For a fixed pair (r, R) of positive numbers with $2R \geq r$, we denote by $\mathcal{D}(r, R)$ the collection of all Delone sets in $\mathbb{R}^d$ which are r-uniformly discrete and R-relatively dense. Then, there is a canonical way to come up with a topology that turns $\mathcal{D}(r, R)$ into a compact space, cf. e.g. [Sol98] or [LS03] as well. Further, the canonical action of $\mathbb{R}^d$ on $\mathcal{D}(r, R)$ by translations is continuous with respect to this topology. Since $\mathbb{R}^d$ is abelian (and hence amenable), there exists an invariant probability measure μ on $\mathcal{D}(r, R)$. Similarly, as for the Poisson point processes above, we define

$$F(Q)(\omega) := \#(Q \cap \omega)$$

for $Q \in \mathcal{F}^0(\mathbb{R}^d)$ and $\omega \in \mathcal{D}(r, R)$. By uniform discreteness, there is a constant C_r depending on r such that $F(Q)(\omega) \leq C_r |Q|$ for all $Q \in \mathcal{F}^0(\mathbb{R}^d)$ and every $\omega \in \mathcal{D}(r, R)$. Thus, the process F maps to $L^\infty(\mathcal{D}(r, R), \mathbb{R})$ and since $\mu(\mathcal{D}(r, R)) = 1$, F is bounded in $L^1(\mathcal{D}(r, R), \mathbb{R})$. Clearly, F is additive as well and satisfies the equivariance condition that for all $g \in \mathbb{R}^d$, every $Q \in \mathcal{F}^0(\mathbb{R}^d)$,

$$T_{g^{-1}} F(Q)(\omega) = F(Q)(g\omega) = F(Qg)(\omega)$$

for (μ-almost-)every $\omega \in \mathcal{D}(r, R)$. The same argument as in the previous example shows that there cannot be an L^1-class f such that $F(Q)(\cdot) = \int_Q f(g \cdot)\, dg$. Therefore, as in the previous example, F is not absolutely continuous.

5.2 Pointwise convergence

In the following, we prove a pointwise almost-everywhere ergodic theorem for approximable bounded, additive processes, cf. Theorem 5.17. If not stated otherwise, we always assume that the underlying processes take their values in $L^1(\Omega, Y)$, where Ω is a probability space with measure μ and Y is some reflexive Banach space. The strategy will be in analogy to classical proofs. Precisely, we use our abstract mean ergodic Theorem 4.15 in combination with an L^1-maximal inequality. The latter inequality is proven in a so-called *dominated ergodic theorem*, cf. Theorem 5.14. Unlike in the absolutely continuous situation, we do not have an integral representation of the process under consideration. Consequently, we will have to assume that the Følner sequence satisfies the Tempelman condition. We conclude this chapter with a short discussion, where we compare our pointwise convergence Theorem 5.17 with the results in the literature. The achievements mentioned in this section are partially taken from [Pog13a].

We will see below that with the notion of some bounded, additive process F at hand, it is worth investigating this process on the level of the $\| \cdot \|_Y$-norm of the elements $F(Q)(\omega)$ with $Q \in \mathcal{F}^0(\Gamma)$ and $\omega \in \Omega$. For this purpose, we define the following $\mathbb{R}$-valued (in fact non-negative) expressions. The analogue for the semigroup case $\Gamma = \mathbb{R}^{d+}$ can e.g. be found in [Sat99, Sat03].

Definition 5.8 (Associated dominating process).
Let Γ be an LCSCUH group. For a bounded, additive process F on Γ, we define the associ-

ated dominating process F^0 *as*

$$F^0 : \mathcal{F}^0(\Gamma) \to L^0(\Omega, \mathbb{R}) :$$
$$F^0(Q)(\omega) := \text{ess sup} \left\{ \sum_{k=1}^{L} \|F(Q_k)\|_Y(\omega) \,\Big|\, Q = \bigsqcup_{k=1}^{L} Q_k \text{ disj.}, \, Q_k \in \mathcal{F}^0(\Gamma), \, 1 \le k \le L, \, L \in \mathbb{N} \right\}$$

for $Q \in \mathcal{F}^0(\Gamma)$ and for $\omega \in \Omega$, where $L^0(\Omega, \mathbb{R})$ is the set of all real-valued, measurable functions on Ω.

We have to justify the measurability of the $F^0(Q)$ first. Note that in the case of discrete groups, we simply have $F^0(Q)(\omega) := \sum_{g \in Q} \|F(\{g\})\|_Y(\omega)$ for $\omega \in \Omega$ and a finite set $Q \subseteq \Gamma$. If Γ is non-discrete, the measurability is guaranteed by the following lemma. For processes on $\mathbb{R}^d$-intervals, a variant of this assertion has been used in the proof of Lemma 3.7 in [Émi85].

Lemma 5.9.
Let Γ be a continuous, amenable, LCSCUH group. Further, suppose that F is a bounded, additive process on Γ with values in $L^1(\Omega, Y)$ as indicated above. Then, we can find a sequence of partitions $\{P_m\}_{m \in \mathbb{N}}$ of Γ consisting of countably many sets in $\mathcal{F}^0(\Gamma)$ such that the following properties hold.

- $\bigsqcup_{A \in P_m} A = \Gamma$ *for all $m \in \mathbb{N}$.*
- *For each $A \in P_m$, we have $|A| < 2^{-m}$ ($m \in \mathbb{N}$).*
- *P_{m+1} is a refinement of P_m for each $m \in \mathbb{N}$, i.e. for each $A \in P_{m+1}$, there exists a unique $B \in P_m$ such that $A \subseteq B$.*
- *For each $Q \in \mathcal{F}^0(\Gamma)$, we have the representation $F^0(Q)(\omega) = \lim_{m \to \infty} F^0_{P_m}(Q)(\omega)$ for μ-almost every $\omega \in \Omega$, where $F^0_{P_m}(Q)(\cdot) := \sum_{A \in P_m} \|F(A \cap Q)(\cdot)\|_Y$ for $m \in \mathbb{N}$.*

Proof.
Since Γ is second countable and Hausdorff and by the outer regularity of the Haar measure, for every $n \in \mathbb{N}$, one finds a precompact and open neighbourhood V_n of the unity e in Γ such that $\cap_{n \in \mathbb{N}} V_n = \{e\}$, $V_{n+1} \subseteq V_n$ and $|V_n| < 2^{-n}$ for all $n \in \mathbb{N}$. So let $Q \in \mathcal{F}^0(\Gamma)$. We will define a sequence of successively refined partitions of Q in the following manner. For each $n \in \mathbb{N}$, cover the closure of Q by left-translates of V_n. Due to the precompactness of Q, we can extract a finite subcover $\cup_{i=1}^{K(n)} g_i^{(n)} V_n$ of Q. Next, we make the translates of this union disjoint such that

$$\bigcup_{i=1}^{K(n)} g_i^{(n)} V_n = \bigsqcup_{i=1}^{K(n)} g_i^{(n)} \tilde{V}_n^i$$

where $\bigsqcup$ stands for the disjoint union and the sets $\tilde{V}_n^i \subseteq V_n$ belong to $\mathcal{F}^0(\Gamma)$ for all $1 \le i \le K(n)$. Hence, putting $Q_i^{(n)} := Q \cap g_i^{(n)} \tilde{V}_n^i$ for $1 \le i \le K(n)$, we have $Q = \bigsqcup_{i=1}^{K(n)} Q_i^{(n)}$, as well as $|Q_i^{(n)}| < 2^{-n}$ for every $1 \le i \le K(n)$. We define the partitions

$$\begin{aligned} P_1(Q) &:= \{Q_i^{(1)} \,|\, 1 \le i \le K(1)\}, \\ P_m(Q) &:= \{Q_i^{(m)} \cap \tilde{Q}^{(m-1)} \,|\, 1 \le i \le K(m), \, \tilde{Q}^{(m-1)} \in P_{m-1}\} \quad \text{for } m \ge 1, \end{aligned}$$

for the set Q. It is obvious that $P_{m+1}(Q)$ is finer than $P_m(Q)$ for every $m \geq 1$, and we write $P_{m+1}(Q) \geq P_m(Q)$. Further, by construction, $|A| \leq 2^{-m}$ for $A \in P_m$ ($m \geq 1$). Since Γ is locally compact and second countable, it can be exhausted by a countable sequence (Γ_n) of increasing, compact sets. We set $\tilde{\Gamma}_n := \Gamma_n \setminus \Gamma_{n-1}$ with the convention that $\Gamma_0 := \{e\}$. Then, the sets $\tilde{\Gamma}_n$ are locally closed and precompact, i.e. they belong to $\mathcal{F}^0(\Gamma)$ for every $n \in \mathbb{N}$. and we repeat the above construction for each precompact set $Q = \tilde{\Gamma}_n$, $n \geq 1$. Then clearly, for each $m \geq 1$, the expression

$$P_m := \bigcup_{n \in \mathbb{N}} P_m(\tilde{\Gamma}_n)$$

is a partition of the group satisfying the first three items of the Lemma 5.9.

To show the approximation result for F^0, let $Q \in \mathcal{F}^0(\Gamma)$ and define

$$F^0_{P_m}(Q)(\omega) := \sum_{A \in P_m} \|F(A \cap Q)\|_Y(\omega)$$

for $m \geq 1$, where the partitions P_m have been defined above. By the triangle inequality, we have $F^0_{P_m}(Q) \leq F^0_{P_{m+1}}(Q)$ μ-almost-surely for all $m \geq 1$. Further, it is clear from the boundedness of F that $\|F^0_{P_m}(Q)\|_{L^1(\mathbb{R},Y)} \leq C\,|Q|$ for every $m \geq 1$. Hence, we can define

$$\overline{F}^0(Q)(\omega) := \lim_{m \to \infty} F^0_{P_m}(Q)(\omega)$$

for μ-almost every $\omega \in \Omega$ and we set $\overline{F}^0(Q)(\omega) = 0$ for the remaining $\omega \in \Omega$. We will now show that in fact $F^0(Q) = \overline{F}^0(Q)$ μ-almost-everywhere. This shows that the equivalence class $F^0(Q)$ is well-defined and in particular measurable. Note that it follows from the definition of F^0 that for each $m \in \mathbb{N}$, we have

$$F^0_{P_m}(Q)(\omega) \leq F^0(Q)(\omega)$$

for μ-almost every $\omega \in \Omega$. This implies $F^0(Q) \geq \overline{F}^0(Q)$ almost-everywhere. For the converse inequality, we choose a finite, disjoint union $Q = \bigsqcup_{l=1}^{L} Q_l$, where $Q_l \in \mathcal{F}^0(\Gamma)$ for $1 \leq l \leq L$. For $\omega \in \Omega$, we obtain with the triangle inequality that

$$\begin{aligned}
\sum_{l=1}^{L} \|F(Q_l)\|_Y(\omega) &\leq \sum_{l=1}^{L} \sum_{A \in P_m} \|F(Q_l \cap A)\|_Y(\omega) \\
&\leq \sum_{\substack{A \in P_m, \\ \exists\, l:\, A \subseteq Q_l}} \|F(A \cap Q)\|_Y(\omega) + \sum_{l=1}^{L} \sum_{\substack{A \in P_m,\, \exists\, l_1 \neq l: \\ A \cap Q_l,\, A \cap Q_{l_1} \neq \emptyset}} \|F(A \cap Q_l)\|_Y(\omega) \\
&\leq F^0_{P_m}(Q)(\omega) + \sum_{l=1}^{L} \sum_{\substack{A \in P_m,\, \exists\, l_1 \neq l: \\ A \cap Q_l,\, A \cap Q_{l_1} \neq \emptyset}} \|F(A \cap Q_l)\|_Y(\omega) \qquad (5.3)
\end{aligned}$$

for all $m \in \mathbb{N}$. Further, we have by the boundedness of F that

$$\left\| \sum_{l=1}^{L} \sum_{\substack{A \in P_m,\, \exists\, l_1 \neq l: \\ A \cap Q_l,\, A \cap Q_{l_1} \neq \emptyset}} \|F(A \cap Q_l)\|_Y(\cdot) \right\|_{L^1(\Omega,\mathbb{R})} \leq C \sum_{l=1}^{L} |\partial_{V_m V_m^{-1}}(Q_l)|.$$

Note that since $\cap_{m\in\mathbb{N}} V_m V_m^{-1} = \{\mathrm{id}\}$ and as Q_l has compact closure, it follows from the continuity of the (Haar) measure from above that

$$|\partial_{V_m V_m^{-1}}(Q_l)| = |V_m V_m^{-1} Q_l \cap V_m V_m^{-1}(\Gamma \setminus Q_l)| \overset{m\to\infty}{\to} 0$$

for each $1 \leq l \leq L$. Therefore, we can find a subsequence (m_k) such that the second sum in inequality (5.3) converges to zero almost-everywhere. Hence, we deduce from that same inequality that

$$\sum_{l=1}^{L} \|F(Q_l)\|_Y(\omega) \quad \leq \quad \limsup_{k\to\infty} F^0_{P_{m_k}}(Q)(\omega) = \overline{F}^0(Q)(\omega)$$

for almost-every $\omega \in \Omega$. Hence $F^0(Q) \leq \overline{F}^0(Q)$ and since $Q \in \mathcal{F}^0(\Gamma)$ was arbitrarily chosen, this shows in fact that $F^0 = \overline{F}^0$ on $\mathcal{F}^0(\Gamma)$. □

In the following proposition, we collect useful properties of associated dominating processes.

Proposition 5.10.
Assume that Γ is an amenable LCSCUH group. Let F be some bounded, additive process on Γ with values in $L^1(\Omega, Y)$. Then the associated dominating process F^0 takes values in L^1, i.e. $F^0(Q) \in L^1(\Omega,\mathbb{R})$ for every $Q \in \mathcal{F}^0(\Gamma)$. Moreover, the following assertions hold true.

(i) $\sup_{Q\in\mathcal{F}^0(\Gamma)} \frac{\|F^0(Q)\|_{L^1(\Omega,\mathbb{R})}}{|Q|} \leq \sup_{Q\in\mathcal{F}^0(\Gamma)} \frac{\|F(Q)\|_{L^1(\Omega,Y)}}{|Q|}$.

(ii) $F^0(Q) = \sum_{l=1}^{L} F^0(Q_l)$ *for every disjoint union* $Q = \bigsqcup_{l=1}^{L} Q_l$ *in* $\mathcal{F}^0(\Gamma)$.

Proof.
Set $\tilde{\gamma} := \sup_{Q\in\mathcal{F}^0(\Gamma)} \|F^0(Q)\|_{L^1(\Omega,\mathbb{R})}/|Q|$ and $\gamma := \sup_{Q\in\mathcal{F}^0(\Gamma)} \|F(Q)\|_{L^1(\Omega,Y)}/|Q|$. It follows from Proposition 5.9 and the monotone convergence theorem that $\|F^0(Q)\|_{L^1(\Omega,\mathbb{R})} \leq \gamma\,|Q|$ for every $Q \in \mathcal{F}^0(\Gamma)$. This shows $\tilde{\gamma} \leq \gamma$.
We turn to the proof of the fact that $F^0(Q) = \sum_{k=1}^{m} F^0(Q_k)$ for disjoint unions $Q = \bigsqcup_{k=1}^{m} Q_k$. The $'\leq'$-inequality follows from the triangle inequality and the approximation statement in Lemma 5.9. For the converse $'\geq'$-inequality we compute with inequality (5.3) and with the approximation statement in the previous lemma that

$$\begin{aligned}
\sum_{k=1}^{m} F^0(Q_k)(\omega) \quad &= \quad \sum_{k=1}^{m} \lim_{l\to\infty} \sum_{A\in P_l} \|F(Q_k\cap A)\|_Y(\omega) \\
&= \quad \lim_{l\to\infty} \sum_{k=1}^{m} \sum_{A\in P_l} \|F(Q_k\cap A)\|_Y(\omega) \\
&\leq \quad \lim_{l\to\infty} F^0_{P_l}(Q)(\omega) \\
&= \quad F^0(Q)(\omega)
\end{aligned}$$

almost-surely. This concludes the proof. □

The following lemma describes how the dominating process F^0 for some F is related to the T_g-action on $L^p(\Omega, Y)$.

Lemma 5.11.
Let F be some bounded, additive process according to Definition 5.3 with its associated dominating process F^0. Then, if $\kappa \geq 1$ is the constant and if $\varphi : \Gamma \to \Gamma$ is the measurable group homomorphism in inequality (5.1), we obtain for all $g \in \Gamma$ and for every $Q \in \mathcal{F}^0(\Gamma)$,

$$\kappa^{-1} F^0(Q)(\varphi(g)\omega) \leq F^0(Qg)(\omega) \leq \kappa F^0(Q)(\varphi(g)\omega)$$

for μ-almost every $\omega \in \Omega$.

Proof.
The claim follows from the above representation of F^0 in Lemma 5.9 (fourth item) and from the inequality (5.1). □

Note that Proposition 5.10 and Lemma 5.11 show that F^0 is a bounded, additive process on $\mathcal{F}^0(\Gamma)$. We are now in position to introduce the concept of L^1-maximal inequalities for bounded, additive processes.

Definition 5.12.
Suppose that Γ is an amenable LCSCUH group. Assume that (U_j) is a weak Følner sequence. Further, let $F : \mathcal{F}^0(\Gamma) \to L^1(\Omega, Y)$ be some bounded, additive process along with its associated dominating process F^0 on $\mathcal{F}^0(\Gamma)$. We say that F satisfies an L^1-maximal inequality (or the L^1-maximality condition) for the sequence (U_j) if there is a constant $\gamma > 0$ such that for all $\lambda > 0$ and for every $M \in \mathbb{N}$, one has

$$\mu\left(\left\{\omega \in \Omega \,\Big|\, \sup_{j \geq M} \frac{F^0(U_j)(\omega)}{|U_j|} > \lambda\right\}\right) \leq \frac{\gamma}{\lambda} \sup_{j \geq M} \frac{\|F^0(U_j)\|_{L^1(\Omega,\mathbb{R})}}{|U_j|}.$$

A similar concept can be found in the book of Krengel, see § 6.4.2. in [Kre85]. For the proof of the dominated ergodic theorem, we need the following combinatorial lemma.

Lemma 5.13.
Let Γ be an amenable LCSCUH group, along with a weak Følner sequence (U_j) with $U_j \subseteq U_{j+1}$ for all $n \in \mathbb{N}$. Further, let $M \leq N \in \mathbb{N}$ be integer numbers and assume that $B, F \in \mathcal{F}^0(\Gamma)$ are given such that $U_N B \subseteq F$. Then, for every map $\theta : B \to \{M, \dots, N\}$, there exists a finite subset $\tilde{B} \subseteq B$ for which the sets $U_{\theta(b)} b$ $(b \in \tilde{B})$ are disjoint and such that $B \subseteq \bigcup_{b \in \tilde{B}} U_{\theta(b)}^{-1} U_{\theta(b)} b$.

Proof.
See [Kre85], Lemma 6.4.3. □

We are now in position to show that bounded, additive processes satisfy the L^1-maximality condition for every increasing, weak Tempelman Følner sequence in some unimodular group Γ. It is an open problem whether this result can be extended to Shulman Følner sequences.

The method of LINDENSTRAUSS for absolutely continuous processes, see Theorem 5.5, relies strongly on the integral structure of the ergodic averages (absolutely continuous case) and we cannot make use of this fact here. We adapt a proof given in [Kre85]. There, the originality of the dominated ergodic theorem is ascribed to TEMPEL'MAN.

Theorem 5.14 (Dominated ergodic theorem, cf. [Pog13a], Theorem 7.9).
Let Γ be an amenable LCSCUH group, along with a weak Følner sequence (U_j) such that $U_j \subseteq U_{j+1}$ for all $j \in \mathbb{N}$ and which satisfies the Tempelman condition. Then, every bounded, additive process $F : \mathcal{F}^0(\Gamma) \to L^1(\Omega, Y)$ respecting the regularity condition in inequality (5.1) *satisfies an L^1-maximal inequality.*

PROOF.
The proof is a modification of the proof of Theorem 6.4.2 in [Kre85]. At first, we fix an integer $M \in \mathbb{N}$ and $\lambda > 0$. Further, let $N \geq M$ be an integer and denote by $\delta > 0$ an arbitrary positive number. Define the compact set $K := \cup_{l=M}^{N} U_l$ and since (U_j) is a weak Følner sequence, we find a compact set U_{k_N} $(k_N \geq N)$ such that $|U_{k_N} \triangle \overline{U}_N| < \delta\, |U_{k_N}|$, where $\overline{U}_N := KU_{k_N}$. Moreover, define the sets

$$D_{\lambda,M,N} := \{\omega \in \Omega \,|\, \sup_{M \leq l \leq N} F^0(U_l)(\omega)/|U_l| > \lambda\},$$

where F^0 is the associated dominating process for F. For $\omega \in \Omega$, set

$$B := B(\omega, \lambda, M, N) := \{g \in U_{k_N} \,|\, \varphi(g)\omega \in D_{\lambda,M,N}\},$$

where $\varphi : \Gamma \to \Gamma$ is the homomorphism taken from inequality (5.1). Then, for any element $b \in B$, there must be some number $\ell \in \{M, \dots, N\}$ such that $F^0(U_\ell)(\varphi(b)\omega) > \lambda\, |U_\ell|$. Picking for each b such an element ℓ gives rise to a map $\theta : B \to \{M, \dots, N\} : \theta(b) := \ell$. By Lemma 5.13, we find some finite subset $\tilde{B} \subseteq B$ such that the sets $U_{\theta(b)}b$ are disjoint for $b \in \tilde{B}$ and

$$B(\omega, \lambda, M, N) \subseteq \bigcup_{b \in \tilde{B}} U_{\theta(b)}^{-1} U_{\theta(b)} b.$$

Since the sequence (U_j) satisfies the Tempelman condition for some constant $\tilde{\kappa} > 0$, it follows that

$$|B(\omega, \lambda, M, N)| \leq \tilde{\kappa} \left| \bigsqcup_{b \in \tilde{B}} U_{\theta(b)} b \right|. \tag{5.4}$$

By construction of B, we have $U_{\theta(b)}b \subseteq \overline{U}_N$ for $b \in \tilde{B}$. We compute with the additivity statement (ii) of Proposition 5.10 and with Lemma 5.11 that

$$\begin{aligned} F^0(\overline{U}_N)(\omega) &\geq \sum_{b \in \tilde{B}} F^0(U_{\theta(b)}b)(\omega) \geq \kappa^{-1} \sum_{b \in \tilde{B}} F^0(U_{\theta(b)})(\varphi(b)\omega) \\ &\overset{b \in \tilde{B}}{\geq} \frac{\lambda}{\kappa} \left| \bigsqcup_{b \in \tilde{B}} U_{\theta(b)} b \right|, \end{aligned}$$

where $\kappa \geq 1$ is the constant and $\varphi : \Gamma \to \Gamma$ is the homomorphism taken from inequality (5.1). It follows from inequality (5.4) that

$$\begin{aligned} |B(\omega, \lambda, M, N)| &\leq \frac{\kappa\tilde{\kappa}}{\lambda} F^0(\overline{U}_N)(\omega) \\ &\leq \frac{\kappa\tilde{\kappa}}{\lambda} \left(F^0(\overline{U}_N \setminus U_{k_N})(\omega) + F^0(U_{k_N})(\omega) \right). \end{aligned} \tag{5.5}$$

Note that the latter inequality holds true μ-almost-everywhere. Integration of the left hand side yields

$$\int_\Omega |B(\omega, \lambda, M, N)| \, d\mu(\omega) = |U_{k_N}| \cdot \mu(D_{\lambda,M,N}), \tag{5.6}$$

since the action of Γ on Ω is μ-preserving. Note further that by the choice of the set $\overline{U}_N$, it is true that $|\overline{U}_N \setminus U_{k_N}| < \delta \, |U_{k_N}|$ and therefore, integrating the right hand side of inequality (5.5), we obtain with $\gamma := \sup_{Q \in \mathcal{F}^0(\Gamma)} F^0(Q)/|Q| < \infty$ (cf. Proposition 5.10) that

$$\frac{\kappa\tilde{\kappa}}{\lambda} \int_\Omega \left(F^0(\overline{U}_N \setminus U_{k_N})(\omega) + F^0(U_{k_N})(\omega) \right) d\mu(\omega) \leq |U_{k_N}| \cdot \frac{\kappa\tilde{\kappa}}{\lambda} \left(\gamma \cdot \delta + \frac{\|F^0(U_{k_N})\|_{L^1(\Omega,\mathbb{R})}}{|U_{k_N}|} \right).$$

Combining this fact with the inequality (5.6), we get with $k_N \geq M$

$$\mu(D_{\lambda,M,N}) \leq \frac{\kappa\tilde{\kappa}}{\lambda} \left(\gamma \cdot \delta + \sup_{l \geq M} \frac{\|F^0(U_l)\|_{L^1(\Omega,\mathbb{R})}}{|U_l|} \right)$$

and with $\delta \to 0$, we arrive at

$$\mu(D_{\lambda,M,N}) \leq \frac{\kappa\tilde{\kappa}}{\lambda} \sup_{l \geq M} \frac{\|F^0(U_l)\|_{L^1(\Omega,\mathbb{R})}}{|U_l|}.$$

Since the right hand side of the latter inequality does not depend on N, we can exploit the continuity of the measure μ as $N \to \infty$ to finish the proof. □

We now set the preparations for a pointwise ergodic theorem for bounded, additive processes. To do so, we introduce the notion of *approximable*, bounded, additive processes.

Definition 5.15 (Approximable processes).
Let Γ be an amenable LCSCUH Hausdorff group. Let some bounded, additive process $F : \mathcal{F}^0(\Gamma) \to L^1(\Omega, Y)$ be given which satisfies the regularity condition (5.1)*. In this situation, we call F* approximable *if there is a sequence (F_n) of bounded, additive processes on $\mathcal{F}^0(\Gamma)$ with the following properties.*

- *For each $n \in \mathbb{N}$, the process F_n takes values in $L^\infty(\Omega, Y) \subseteq L^1(\Omega, Y)$.*
- *For every $n \in \mathbb{N}$, the process $F - F_n$ satisfies the regularity condition given by inequality* (5.1)*.*

- *For every weak Tempelman Følner sequence* (U_j) *in* Γ *with* $U_j \subseteq U_{j+1}$*, the following boundedness condition holds true. For every* $n \in \mathbb{N}$*, there is a* $j_0 \in \mathbb{N}$ *such that for all* $j \geq j_0$
$$F_n^0(U_j)(\omega) \leq n\,|U_j|$$
for almost-every $\omega \in \Omega$*, where* F_n^0 *is the associated dominating process for* F_n*.*
- *For every weak Tempelman Følner sequence* (U_j) *in* Γ *with* $U_j \subseteq U_{j+1}$*, the sequence* (F_n) *approximates* F *along* (U_j) *in the sense that*
$$\lim_{n\to\infty} \limsup_{j\to\infty} \frac{\|H_n^0(U_j)\|_{L^1(\Omega,\mathbb{R})}}{|U_j|} = 0,$$
where H_n^0 *is the dominating process associated with the process* $F - F_n$*.*

Remark.
The classical integral averages give rise to approximable processes. Let $f \in L^1(\Omega, Y)$ be fixed and consider the bounded, additive process
$$F : \mathcal{F}^0(\Gamma) \to L^1(\Omega, Y) : F(Q)(\omega) := \int_Q f(g\omega)\, dg.$$
Now, let $n \in \mathbb{N}$. Then, there is a set B_n of measure less than $\|f\|_{L^1}/n$ such that $\|f(\omega)\|_Y \leq n$ for all $\omega \notin B_n$. Define
$$F_n(Q)(\omega) := \int_Q \tilde{f}_n(g\omega)\, dg$$
for $Q \in \mathcal{F}^0(\Gamma)$ and $\omega \in \Omega$, where we set $\tilde{f}_n(\omega) = f(\omega)$ for $\omega \notin B_n$ and $\tilde{f}_n(\omega) = 0$ for $\omega \in B_n$. It is easy to check that the F_n are bounded, additive processes and that the sequence (F_n) approximates F in the sense of the previous definition.

The following proposition shows that the elements of an approximating sequence (F_n) for some bounded, additive process F converge almost-surely along increasing Tempelman Følner sequences.

Proposition 5.16.
Let Γ *be an amenable LCSCUH Hausdorff group and suppose that* $F : \mathcal{F}^0(\Gamma) \to L^\infty(\Omega, Y) \subseteq L^1(\Omega, Y)$ *is a bounded, additive process satisfying the regularity condition* (5.1)*. Further, assume that* (U_j) *is an increasing Tempelman Følner sequence and that there are numbers* $n, j_0 \in \mathbb{N}$ *such that for every* $j \geq j_0$*, we have*
$$F^0(U_j)(\omega) \leq n\,|U_j| \tag{5.7}$$
for almost-all $\omega \in \Omega$*, where* F^0 *is the dominating process associated to* F*.*
Then, there is a set $\tilde{\Omega} \subseteq \Omega$ *of full measure such that for all* $\omega \in \tilde{\Omega}$*, the sequence* $F(U_j)(\omega)/|U_j|$ *converges in the topology of* Y *as* $j \to \infty$*.*

Proof.
Let (U_j) be an increasing Tempelman Følner sequence. Without loss of generality, we will work with the sequence obtained by deleting the first j_0 elements in (U_j). For the sake of

simplicity, we will call this sequence (U_j) again. Then, $F^0(U_j)(\cdot) \leq n\,|U_j|$ almost-surely for all $j \in \mathbb{N}$. We define the sequence (S_n) of sets in Γ via $S_n := U_n u^{-1}$, $n \geq 1$, where $u \in U_1$ is an arbitrary element. Since (U_j) is increasing, the sequence (S_n) is a nested Følner sequence.

Now, take a sequence (ε_k) of positive numbers converging to zero. For every $k \in \mathbb{N}$, we set $N(\varepsilon_k) := \lceil \log(\varepsilon_k)/\log(1-\varepsilon_k)\rceil$ and we choose ε_k-prototiles $\{T_i^{\varepsilon_k}\}_{i=1}^{N(\varepsilon_k)}$ taken from the sequence S_n according to Definition 2.15 with $0 < \beta_k < 2^{-N(\varepsilon_k)}\varepsilon_k$. By Theorem 5.5, we can find a set $\hat{\Omega} \subseteq \Omega$ with $\mu(\hat{\Omega}) = 1$ such that for each $k \in \mathbb{N}$, for every $1 \leq i \leq N(\varepsilon_k)$, and for all $\omega \in \hat{\Omega}$, the limits

$$S(T_i^{\varepsilon_k})(\omega) := \lim_{j\to\infty} |U_j|^{-1} \int_{U_j} F(T_i^{\varepsilon_k} g)(\omega)\, dg \tag{5.8}$$

exist in the topology of the Banach space Y. Now, fix $k \in \mathbb{N}$. By Theorem 3.6, we find $K = K(\varepsilon_k, \beta_k, T_i^{\varepsilon_k}) \in \mathbb{N}$ such that for every $j \geq K$, we find a decomposition tower emanating from the set U_j. Define

$$\Delta(j, \varepsilon_k, \omega) := \left\| \frac{F(U_j)(\omega)}{|U_j|} - \sum_{i=1}^{N(\varepsilon_k)} \eta_i(\varepsilon_k)\, \frac{S(T_i^{\varepsilon_k})(\omega)}{|T_i^{\varepsilon_k}|} \right\|_Y$$

for $j \geq K$. In the following, we fix $j \geq K$, choose η_0 as in Definition 3.5 and fix $0 < \eta < \eta_0$. Then, we find

- some $(U_j U_j^{-1}, \eta)$-invariant set $\hat{U}_j$ along with
- an associated uniform decomposition tower (Υ, Λ) with prototile sets $T_i^{\varepsilon_k}$, $(1 \leq i \leq N(\varepsilon_k))$,
- a family of finite center sets $\hat{C}_i^y (y \in \Upsilon)$ for the ε_k-quasi tilings of $\hat{U}_j$,
- and for each $y \in \Upsilon$, a family of finite center sets $C_i^{y,\lambda} (\lambda \in \Lambda)$ for the ε_k-quasi tilings of U_j.

We will show

$$\lim_{l\to\infty} \lim_{j\to\infty} \Delta(j, \varepsilon_{k_l}, \omega) = 0$$

almost-surely for a subsequence (ε_{k_l}). To do so, we follow the lines of the proof of Theorem 4.15. Fixing $\omega \in \tilde{\Omega}$, we obtain by means of the triangle inequality

$$\Delta(j, \varepsilon_k, \omega) \leq \sum_{\ell=1}^{5} D_\ell(j, \varepsilon_k, \omega),$$

where the expressions $D_\ell(j, \varepsilon_k, \omega)$ are defined as in the proof of the mean ergodic theorem. Using the boundedness in inequality (5.7) and the limit relations (5.8), we obtain

$$\limsup_{j\to\infty} \sum_{\ell=2}^{5} D_\ell(j, \varepsilon_k, \omega) \leq (24 + 26 + 48 + 16) n\, \varepsilon_k = 114 n\, \varepsilon_k \tag{5.9}$$

as in the steps (2) to (5) of the mentioned proof with $C = n$. For $D_1(j, \varepsilon_k, \omega)$, we will have to argue in a slightly different way. To do so, note first that due to the additivity of the process, there is no boundary term present, i.e. $b \equiv 0$. For $\lambda \in \Lambda$ and $y \in \Upsilon$, we define

$$A_{y,\lambda}^{\varepsilon_k} := \bigcup_{i=1}^{N(\varepsilon_k)} \bigcup_{c \in C_i^{y,\lambda}} T_i^{\varepsilon_k} c.$$

Further, for $c \in C_i^{y,\lambda}$, we denote by $T_i^{\varepsilon_k}(c)$ a subset of $T_i^{\varepsilon_k}$ with $|T_i^{\varepsilon_k}(c)| \geq (1-\varepsilon)|T_i^{\varepsilon_k}|$ with the property that the sets $T_i^{\varepsilon_k}(c)c$ are pairwise disjoint for all $1 \leq i \leq N(\varepsilon_k)$, $c \in C_i^{y,\lambda}$ and

$$A_{y,\lambda}^{\varepsilon_k} = \bigsqcup_{i=1}^{N(\varepsilon_k)} \bigsqcup_{c \in C_i^{y,\lambda}} T_i^{\varepsilon_k}(c)c.$$

Using the additivity of the process F, we compute

$$\begin{aligned} D_1(j, \varepsilon_k, \omega) \quad \leq \quad & |\Upsilon|^{-1} \, |\Lambda|^{-1} \int_\Upsilon \int_\Lambda \Bigg(\frac{\|F(U_j \setminus A_{y,\lambda}^{\varepsilon_k})(\omega)\|_Y}{|U_j|} \\ & + \quad \frac{\sum_{i=1}^{N(\varepsilon_k)} \sum_{c \in C_i^{y,\lambda}} \|F(T_i^{\varepsilon_k} c \setminus T_i^{\varepsilon_k}(c)c)(\omega)\|_Y}{|U_j|} \Bigg) \, d\lambda \, dy \end{aligned}$$

for every $\omega \in \tilde{\Omega}$. With inequality (5.7), one obtains that the function

$$D_1(\varepsilon_k, \omega) := \limsup_{j \to \infty} D_1(j, \varepsilon_k, \omega)$$

is bounded by the constant $3n$ for all $\omega \in \tilde{\Omega}$. Moreover, the dominated ergodic theorem combined with the boundedness of the process F in $L^1(\Omega, Y)$ yield

$$\|D_1(\varepsilon_k, \cdot)\|_{L^1(\Omega,\mathbb{R})} \leq 4\varepsilon_k + 2\varepsilon_k = 6\varepsilon_k.$$

Note that we used here that the sets U_j are $(1 - 4\varepsilon_k)$-coverd by the sets $A_{y,\lambda}^{\varepsilon_k}$ and that the ε_k-disjoint translates $T_i^{\varepsilon_k} c$ are $(1 - \varepsilon_k)$ covered by the disjoint translates $T_i^{\varepsilon_k}(c)c$. Finally, take a subsequence ε_{k_l} such that

$$\lim_{l \to \infty} D_1(\varepsilon_{k_l}, \omega) = 0$$

for all $\omega \in \tilde{\Omega} \cap \hat{\Omega}$, where $\hat{\Omega}$ is a set of full measure as well. With inequality (5.9), we arrive at

$$\lim_{l \to \infty} \limsup_{j \to \infty} \Delta(j, \varepsilon_{k_l}, \omega) = 0$$

almost-surely. In the same manner as in the proof of Theorem 4.15, we conclude that $(F(U_j)(\omega)/|U_j|)_j$ is convergent in Y for almost-every $\omega \in \Omega$. □

We are finally in position to prove the main theorem of this chapter. Precisely, we verify the pointwise almost-everywhere convergence for approximable bounded, additive processes along increasing Tempelman Følner sequences. The method of the proof follows classical concepts. The major steps towards pointwise convergence are the following.

- Firstly, one proves the almost-everywhere convergence for essentially bounded processes. We have done this in Proposition 5.16.
- For a given bounded, additive process, one finds an approximating sequence of L^∞-processes, cf. Definition 5.15. We will assume this in the pointwise ergodic theorem.
- One makes sure that one has an L^1-maximal inequality at disposal. We have guaranteed this in the dominated ergodic theorem, cf. Theorem 5.14.
- Finally, one combines the mentioned concepts to obtain a pointwise almost-everywhere ergodic theorem. Precisely, we approximate the underlying process in L^1 by essentially bounded processes. The pointwise convergence holds true for all elements in the approximating sequence. Now, we can use the L^1-maximal inequality to conclude that the almost-everywhere convergence must also hold true for the original process.

Theorem 5.17 (Pointwise convergence of bounded, additive processes).
Assume that Γ is an amenable LCSCUH group and denote by (U_j) a strong Tempelman Følner sequence such that $U_j \subseteq U_{j+1}$ $(j \in \mathbb{N})$. Let $F : \mathcal{F}^0(\Gamma) \to L^1(\Omega, Y)$ be a bounded, additive process, and let Y be a reflexive Banach space. Further, suppose that F is compatible with a family $\{T_g\}_{g\in\Gamma}$ of uniformly bounded operators acting weakly measurably on $L^1(\Omega, Y)$ (i.e. $T_g F(Q) = F(Qg^{-1})$ and the regularity condition given in inequality (5.1) is satisfied for all $g \in \Gamma$ and every $Q \in \mathcal{F}^0(\Gamma)$).
If, in addition, the process F is approximable, then we obtain a unique $F^* \in L^1(\Omega, Y)$ such that

$$\lim_{j\to\infty} \left\| \frac{F(U_j)}{|U_j|}(\omega) - F^*(\omega) \right\|_Y = 0$$

for μ-almost every $\omega \in \Omega$.
Further, for every $g \in \Gamma$, we have $T_g F^* = F^*$ μ-almost-everywhere.

Proof.
Since F is approximable, for every $n \in \mathbb{N}$, we find an approximating process

$$F_n : \mathcal{F}^0(\Gamma) \to L^\infty(\Omega, Y)$$

as described in Definition 5.15. Now, define

$$H_n : \mathcal{F}^0(\Gamma) \to L^1(\Omega, Y) : H_n(Q)(\omega) := F(Q)(\omega) - F_n(Q)(\omega)$$

for $n \in \mathbb{N}$. By Definition 5.15, the H_n are bounded, additive processes satisfying the regularity condition given in equality (5.1). Lemma 5.11 shows that the same holds true for the processes H_n^0. Since the processes (F_n) approximate F along (U_j), we have

$$\lim_{n\to\infty} \limsup_{j\to\infty} \|H_n^0(U_j)/|U_j|\|_{L^1(\Omega,\mathbb{R})} = 0.$$

With this, we derive from the dominated ergodic theorem, Theorem 5.14, that for every $\varepsilon > 0$, there is some integer $n(\varepsilon) \in \mathbb{N}$ such that for all $\lambda > 0$ and for every $n \geq n(\varepsilon)$, we have

$$\begin{aligned} \mu\left(\left\{\omega \in \Omega \,\middle|\, \limsup_{j\to\infty} \frac{H_n^0(U_j)(\omega)}{|U_j|} > \lambda\right\}\right) &\leq \frac{\gamma}{\lambda} \limsup_{j\to\infty} \frac{\|H_n^0(U_j)\|_{L^1(\Omega,\mathbb{R})}}{|U_j|} \\ &\leq \frac{\gamma}{\lambda}\varepsilon, \end{aligned}$$

where $\gamma > 0$ is a constant independent of ε, n and λ. This shows that

$$\lim_{n\to\infty} \limsup_{j\to\infty} \left\| \frac{F(U_j)(\omega)}{|U_j|} - \frac{F_n(U_j)(\omega)}{|U_j|} \right\|_Y = 0 \tag{5.10}$$

almost-surely. Further, it follows from Proposition 5.16 that for all $n \in \mathbb{N}$, there is an element $F_n^* \in L^1(\Omega, Y)$ such that

$$\lim_{j\to\infty} \left\| \frac{F_n(U_j)(\omega)}{|U_j|} - F_n^*(\omega) \right\|_Y = 0$$

almost-surely. Inserting this into the limit relation 5.10, we arrive at

$$\lim_{n\to\infty} \limsup_{j\to\infty} \left\| \frac{F(U_j)(\omega)}{|U_j|} - F_n^*(\omega) \right\|_Y = 0.$$

This shows that for almost-all $\omega \in \Omega$, the sequence $\big(F(U_j)(\omega)/|U_j|\big)_j$ is Cauchy in the Banach space Y. Thus, it converges to some element $\bar{F}^*$ almost-surely. By Theorem 4.15, the ratios $F(U_j)/|U_j|$ converge in $L^1(\Omega, Y)$ to some element $F^* \in L^1(\Omega, Y)$ with the property that $T_g F^*(\omega) = F^*(\omega)$ for almost-all $\omega \in \Omega$. (To check that the compactness criterion required in the abstract mean ergodic theorem holds true, the reader may e.g. refer to [DU77], Theorem IV.2.1.) Thus, $\bar{F}^* = F^*$ almost-surely and we have finished the proof of the theorem. □

Let us briefly discuss Theorem 5.17. We have seen above that for $f \in L^p(\Omega, \mathbb{R})$ and some strong Følner sequence (U_j), the classical ergodic averages

$$A_j f(\omega) := |U_j|^{-1} \int_{U_j} f(g\omega)\, dg$$

defined on $L^p(\Omega, \mathbb{R})$, $1 \leq p < \infty$ can be interpreted as values of an absolutely continuous, bounded, additive process with density f. Therefore, in the situation of measure preserving actions of unimodular, amenable groups with an increasing strong Tempelman Følner sequence (U_j), Theorem 5.17 generealizes the pointwise ergodic theorems of TEMPEL'MAN [Tem72], EMERSON [Eme74] and LINDENSTRAUSS [Lin01]. Using convenient features of integral averages in the proof of the L^1-maximal inequality for absolutely continuous processes, cf. Theorem 6.6 in [Pog13a], convergence can even be obtained in the setting of tempered weak Følner sequences. In fact, if one restricts oneself to absolutely continuous processes (and this has been done in [Tem72, Eme74, Lin01]), Theorem 5.5 is more general than Theorem 5.17. The question is unsolved whether our theorem also holds true in the full generality of Shulman Følner sequences. Further, in order to guarantee the mild compactness criteria of the almost-additive mean ergodic Theorem 4.15, we have to accept the condition on the measure space to be finite. On the other hand, Theorem 5.17 also includes processes which are not absolutely continuous. One class of these instances is given by point processes arising from Delone sets in non-discrete, abelian LCSCUH groups, cf. e.g. Example 5.7. In this sense, it extends the assertions in [Tem72, Eme74, Lin01].
In [Sat99, Sat03], SATO has proven a semigroup result for $\mathbb{R}^d$-semigroup actions on the

Bochner space $L^1(\Omega, Y)$, where (Ω, μ) is a σ-finite measure space and Y is a reflexive Banach space. The convergence is shown along d-dimensional cubes exhausting the semigroup $\Gamma = \mathbb{R}^{d+}$. For the dynamics under consideration, it is assumed in the latter papers that the weakly measurable transformations $\{T_g\}$ have contraction majorants, i.e. for all $g \in \Gamma$, there is a contraction P_g on $L^1(\Omega, \mathbb{R})$ such that

$$\|T_g f(\omega)\|_Y \leq (P_g \|f(\cdot)\|_Y)(\omega)$$

μ-almost-everywhere for all $f \in L^1(\Omega, Y)$, where $\|f(\cdot)\|_Y$ is the norm function for f on $L^1(\Omega, \mathbb{R})$. In the present thesis, we have fixed the operators P_g by inequality (5.1) via $(P_g f)(\omega) = \kappa f(\varphi(g)^{-1}\omega)$ for $f \in L^1(\Omega, \mathbb{R})$. In this sense, the present situation is more special. However, we obtain in Theorem 5.17 a significant geometric extension, as we are able to deal with arbitrary increasing strong Tempelman Følner sequences in arbitrary unimodular, amenable groups. (The approximability condition is automatically satisfied in the settings considered in [Sat98, Sat99].)

An interesting problem is also the issue of pointwise convergence for *subadditive*, bounded processes. In some special euclidean situations and for certain random processes, mulitiparameter ergodic theorems with subadditivity properties have been proven, see e.g. [Ngu79, KP87, Sch88]. For certain countable amenable groups, a Kingman type ergodic theorem can be found in the recent work [DGZ13]. The ingredients of the proof of Theorem 5.17 might provide essential tools for future investigations about abstract, subadditive pointwise ergodic theorems for continuous amenable groups.

6 Spectral approximation for amenable groups

This chapter is devoted to uniform approximation results concerning the integrated density of states (IDS) for random operators on discrete structures. We discuss two models in this context. To do so, we make use of the ergodic theorems developed in the Chapters 4 and 5. The first application is inspired by [LSV11]. Using Theorem 4.4, we prove a statement for all countable, amenable groups. Secondly, we reproduce major results in [LV09]. More specifically, we show how the abstract ergodic Theorems 4.15 and 5.5 can be applied to obtain the uniform existence of the IDS for operators resulting from certain point processes in abstract metric spaces. The corresponding results can also be found in [Pog13a, PS14].

6.1 IDS approximation for countable groups

In this section, we give a major application of Theorem 4.4. In fact, we show the uniform approximation of the integrated density of states (IDS) for self-adjoint, finite hopping range operators on Cayley graphs of amenable groups. The approach to work with almost-additive, Banach space-valued set functions has been used before, see [Len02, LS05, LMV08]. In [LSV11], the authors prove a Banach space-valued ergodic theorem for amenable groups in which a certain kind of Følner sequences can be found. Since Theorem 4.4 is valid for *all* countable, amenable groups, we are now able to verify spectral approximation results in a more general geometric situation. The corresponding results are joint work with Schwarzenberger and can also be found in [PS14]. In the main Theorem 6.5 of this section, we prove the almost-sure uniform IDS convergence for an ergodic family of bounded, self-adjoint, finite hopping range operators defined on a randomly coloured Cayley graph. Having the abstract convergence Theorem 4.4 at our disposal, we do not need to develop additional spectral theoretic tools. Therefore, our proof will be an adaption of the Theorem 4.5 in [LSV11]. Besides our elaborations, there are more possible applications of almost-additive convergence theorems. One example is to show the almost-sure convergence of cluster densities in an amenable bound percolation model. In [PS14], the authors use Theorem 4.4 to extend previous results of Grimmett, [Gri76, Gri99]. In order not to go beyond the scope of this thesis, we refer the reader to the literature for more details. This section is divided into three parts. At first, endowing countable amenable groups with a random colouring, we draw a link to classical ergodic theory. Using the Lindenstrauss ergodic theorem, we show that in this situation, the frequencies of the coloured patterns exist almost-surely. Moreover, the limit attained is independent of the (tempered) Følner sequence under consideration. Secondly, we define an ergodic family $\{H_\omega\}$ of bounded, self-adjoint operators on randomly coloured Cayley graphs and we show the almost-sure uniform convergence of the integrated density of states along restrictions of the operators on Følner sequences. As in the classical situation, the limit will not depend on the choice of the sequence. We conclude this section by citing a model in [LSV11], Example 4.8., which shows that the IDS can serve as an example for an almost-additive set function for which the limit F^* in Theorem 4.4 does in fact depend on the chosen Følner sequence.

Random colourings and existence of frequencies

We assume that Γ is a countable, amenable group. Let $\mathcal{A}$ be a finite set of colours. We define

$$\Omega := \{\omega = (\omega(g))_{g\in\Gamma} \,|\, \omega(g) \in \mathcal{A}\}$$

and endow this latter set with the canonical σ-algebra $\mathcal{F}$ generated by the finite-cylinder sets. Then Γ acts naturally on the space Ω via the maps $g\omega = (\omega(gh))_{h\in\Gamma}$ for $g \in \Gamma$. Further, we suppose that μ is a probability measure on $(\Omega, \mathcal{F})$ which is invariant under the translation by Γ, i.e. $\mu(gA) = \mu(A)$ for all $g \in \Gamma$ and every $A \in \mathcal{F}$. In addition to this, we assume μ to be ergodic with respect to the Γ-action.
In this situation, we call the collection $(\Omega, \mathcal{F}, \Gamma, \mathcal{A}, \mu)$ an *ergodic random colouring* of the group, where the colours are chosen from the finite set $\mathcal{A}$. Using the Lindenstrauss ergodic theorem, we show that for μ-almost every $\omega \in \Omega$, the frequencies ν_P^ω exist along weak and tempered Følner sequences for all possible coloured patterns $P \in \mathcal{P}$, where $\mathcal{P}$ is the collection of all coloured patterns, cf. Chapter 4.
Let $P \in \mathcal{P}$ be a finite coloured pattern with domain $D(P) \in \mathcal{F}(\Gamma)$ containing the unity e. For this P, set $A_P := \{\omega \in \Omega \,|\, \omega_{|D(P)} = P\}$, which is the set of all colourings of Γ that coincide with P on $D(P)$. The indicator function on A_P shall be denoted by $\mathbb{1}_P$. Let (U_j) be a weak, tempered Følner sequence in Γ. With these notions at hand, we obtain

$$\sum_{g\in U_j \setminus \partial_{D(P)D(P)^{-1}}(U_j)} \mathbb{1}_P(g\omega) \leq \#_P(\omega_{|U_j}) \leq \sum_{g\in U_j} \mathbb{1}_P(g\omega) \tag{6.1}$$

for all $j \in \mathbb{N}$, see [PS14], Theorem 6.2.

The following theorem is an immediate consequence of the Lindenstrauss ergodic theorem, cf. [Lin01].

Theorem 6.1.
Let Γ be a countable, amenable group along with a random colouring as described above. Then for every weak, tempered Følner sequence (U_j), there is a set $\Omega_0 \subseteq \Omega$ of full measure, $\mu(\Omega_0) = 1$, such that the limit

$$\nu_P^\omega := \lim_{j\to\infty} \frac{\#_P(\omega_{|U_j})}{|U_j|}$$

exists for all $\omega \in \Omega_0$ and for all finite coloured patterns $P \in \mathcal{P}$ and it is equal to $\mu(A_P)$.

Proof.
For one single pattern $P \in \mathcal{P}$, it follows from the Inequalities (6.1) and from the Lindenstrauss ergodic theorem that there is a set Ω_P with $\mu(\Omega_P) = 1$ such that the above limit

expression exists. As the set $\mathcal{A}$ contains only finitely many colours, there are at most countably many different patterns that might occur in the random colouring. Intersecting yields that the set $\Omega_0 := \cap_P \Omega_P$ is a set of full measure as well. By ergodicity of the action, the convergence must be towards the measure $\mu(A_P)$. □

Uniform convergence of the IDS

We turn to the proof of the uniform approximation of the integrated density of states for Følner subgraphs of Cayley graphs induced by finitely generated, amenable groups. We start by making this model precise.

Definition 6.2.
A group Γ *is said to be* finitely generated *if there is a finite set* $S \subset \Gamma$ *such that each element* $g \in \Gamma$ *can be written as a finite product of elements in* S*. In this situation, we call* S *a* generating system for Γ*.*

Every finitely generated group Γ with generating system S determines a canonical graph structure. Namely, we call $G := \mathrm{Cay}(\Gamma, S)$ the *Cayley graph for* Γ *with repect to* S if G is a graph with vertex set Γ and two elements $g, h \in \Gamma$ are linked by an edge if and only if there is some $s \in S$ such that $sg = h$. At first hand, this definition forces us to work with directed edges. Assuming in the following that the set S is symmetric (i.e. $s \in S$ implies that $s^{-1} \in S$), we can forget about the directions since then either both or no directions exist between two vertices. For two vertices $g, h \in \Gamma$, we can define the canonical minimal path distance $d_{\Gamma,S}$ in $\mathrm{Cay}(\Gamma, S)$ by

$$d_{\Gamma,S}(g,h) := \min\{L \in \mathbb{N} \,|\, \exists\, s_i \in S, 1 \leq i \leq L, gh^{-1} = s_1 s_2 \dots s_L\}.$$

For some integer number $R \in \mathbb{N}$ and $g \in \Gamma$, we denote by $B_R(g) := \{h \in \Gamma \,|\, d_{\Gamma,S}(g,h) \leq R\}$ the ball of radius R around g. For balls around the unit element $e \in \Gamma$, we simply write B_R instead of $B_R(e)$.

Let us turn to operators on amenable Cayley graphs. At first, we have to define the underlying spaces. Assume that $\mathcal{H}$ is a finite-dimensional Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and induced norm $\| \cdot \|_{\mathcal{H}}$. Further, set

$$\ell^2(\Gamma, \mathcal{H}) := \Big\{ u : \Gamma \to \mathcal{H} \,\Big|\, \sum_{g \in \Gamma} \|u(g)\|_{\mathcal{H}}^2 < \infty \Big\},$$

which again will be a Hilbert space with canonical ℓ^2-scalar product $\langle \cdot, \cdot \rangle_{\ell^2}$. For some subset $Q \subseteq \Gamma$, we identify the subspace $\ell^2(Q, \mathcal{H}) := \{u : Q \to \mathcal{H} \,|\, \sum_{g \in Q} \|u(g)\|_{\mathcal{H}}^2 < \infty\}$ as the collection of all elements in $\ell^2(\Gamma, \mathcal{H})$ which are supported on Q. Thus, we get a canonical projection p_Q, as well as a canonical inclusion map i_Q. Precisely,

$$p_Q : \ell^2(\Gamma, \mathcal{H}) \to \ell^2(Q, \mathcal{H}) : (p_Q\, u)(g) := u(g),$$

for $g \in Q$, and

$$i_Q : \ell^2(Q, \mathcal{H}) \to \ell^2(\Gamma, \mathcal{H}) : (i_Q\, u)(g) := \begin{cases} u(g), & \text{if } g \in Q, \\ 0, & \text{else.} \end{cases}$$

In order to deal with random operators, we introduce measurable families of operators on $\ell^2(\Gamma, \mathcal{H})$.

Definition 6.3.
Let $(\Omega, \mathcal{F})$ be a measurable space. Then a family $\{H_\omega\}_{\omega\in\Omega}$ of bounded, measurable operators on $\ell^2(\Gamma, \mathcal{H})$ is called weakly measurable *if for all $u, v \in \ell^2(\Gamma, \mathcal{H})$, the mapping*

$$\rho_{u,v} : \Omega \to \mathbb{C} : \rho_{u,v}(\omega) := \langle u, H_\omega v\rangle_{\ell^2}$$

is $\mathcal{F}$-$\mathcal{B}(\mathbb{C})$-measurable, where $\mathcal{B}(\mathbb{C})$ is the natural Borel σ-algebra on $\mathbb{C}$. Further, we say that the family is self-adjoint *if for all $u, v \in \ell^2(\Gamma, \mathcal{H})$ and every $\omega \in \Omega$,*

$$\langle u, H_\omega v\rangle_{\ell^2} = \langle H_\omega u, v\rangle_{\ell^2}.$$

We are now in position to define the family of random operators $\{H_\omega\}$ on Cayley graphs for which we will prove the uniform spectral approximation.

Definition 6.4.
Let Γ be an amenable group with finite generating system S and denote by $\mathcal{H}$ some finite-dimensional Hilbert space. Assume that $\mathcal{A}$ is a finite set. Further, suppose that $(\Omega, \mathcal{F}, \Gamma, \mathcal{A}, \mu)$ is an ergodic, random colouring of the group. Then, we call a family $\{H_\omega\}_{\omega\in\Omega}$ of bounded, self-adjoint operators on $\ell^2(\Gamma, \mathcal{H})$ admissible *if*

- *$\{H_\omega\}_{\omega\in\Omega}$ is weakly measurable in the sense of Definition 6.3,*
- *$\{H_\omega\}_{\omega\in\Omega}$ is of* finite hopping range*, i.e. there is a constant $M \in \mathbb{N}$ such that $p_{\{g\}} H_\omega i_{\{h\}} = 0$ for all $g, h \in \Gamma$ with $d_{\Gamma,S}(g, h) > M$ and for every $\omega \in \Omega$,*
- *$\{H_\omega\}_{\omega\in\Omega}$ is* equivariant*, i.e. for every $\omega \in \Omega$ and for all $g, h, t \in \Gamma$, one has*

  $$p_{\{tg\}} H_\omega i_{\{th\}} = p_{\{g\}} H_{t\omega} i_{\{h\}}.$$

- *$\{H_\omega\}_{\omega\in\Omega}$ is* colouring invariant*, i.e. there is a constant $N \in \mathbb{N}$ such that for every $\omega \in \Omega$ and for all $t, g \in \Gamma$, the fact $\omega_{|B_{2N}(g)} \sim \omega_{|B_{2N}(tg)}$ implies that*

  $$p_{\{h\}} H_\omega i_{\{\tilde{h}\}} = p_{\{th\}} H_\omega i_{\{t\tilde{h}\}}$$

  for all $h, \tilde{h} \in B_N(g)$.

Remark.
Note that the operators H_ω strongly take into account the underlying graph structure. Thus, they are not only dependent on the group Γ, but also on the generating system S.

For every element H_ω of an admissible family of bounded, self-adjoint operators and for each finite set $U \in \mathcal{F}(\Gamma)$, we define the restriction of H_ω to U as

$$H_\omega[U] : \ell^2(U, \mathcal{H}) \to \ell^2(U, \mathcal{H}) : H_\omega[U] := p_U\, H_\omega\, i_U.$$

Note in case of a finite set U, the operator $H_\omega[U]$ is a finite-dimensional matrix possessing $\dim(\mathcal{H})|U|$ real eigenvalues $\lambda_i \in \mathbb{R}$, where $\dim(\mathcal{H})$ denotes the dimension of $\mathcal{H}$. Therefore, we can define the cumulative eigenvalue function on $\mathbb{R}$ by

$$n_\omega[U] : \mathbb{R} \to \mathbb{N} : n_\omega[U](E) := |\{i \in \mathbb{N} \mid \exists\, \lambda_i \leq E\}|,$$

where the λ_i, $1 \leq i \leq \dim(\mathcal{H})|U|$ are the eigenvalues of the matrix $H_\omega[U]$, counted with their multiplicities. Note that for each finite set $U \in \mathcal{F}(\Gamma)$ and for all $\omega \in \Omega$, the function $N_\omega[U](\cdot) := \frac{n_\omega[U]}{\dim(\mathcal{H})|U|}(\cdot)$ is bounded and continuous from the right. Thus, those mappings belong to the Banach space $\mathcal{C}_{br}(\mathbb{R}) = \mathcal{C}_{br}(\mathbb{R}, \|\cdot\|_\infty)$ of all bounded and right-continuous functions on $\mathbb{R}$ equipped with the sup-norm.

We are now able to state and prove the announced Banach space spectral approximation result for admissible families of bounded, self-adjoint operators. To keep notation simple, we will write $U_R := U \setminus \partial_{B_{2R}}(U)$ for $U \in \mathcal{F}(\Gamma)$ and $R \in \mathbb{N}$.

Theorem 6.5.
Let Γ be an amenable group with finite generating system S. Suppose that $\mathcal{A}$ is a finite set of colours and that $\mathcal{H}$ is a finite-dimensional Hilbert space. Further, let $(\Omega, \mathcal{F}, \Gamma, \mathcal{A}, \mu)$ be an ergodic random colouring of Γ and assume that $\{H_\omega\}_{\omega\in\Omega}$ is an admissible family of bounded, self-adjoint operators on $\ell^2(\Gamma, \mathcal{H})$. Set $R := \max\{M, N\}$, where the constants $M, N \in \mathbb{N}$ are those of Definition 6.4. Then, there is a unique element $N^* \in C_{br}(\mathbb{R}, \|\cdot\|_\infty)$ such that for every weak, tempered Følner sequence (U_j) in Γ, one obtains the Banach space convergence

$$\lim_{j\to\infty} \|N_\omega[U_{j,R}](\cdot) - N^*(\cdot)\|_\infty = 0$$

for μ-almost every $\omega \in \Omega$.

Remark.
The element $N^* \in C_{br}(\mathbb{R}, \|\cdot\|_\infty)$ is called *integrated density of states (IDS)* of $\{H_\omega\}$. Theorem 6.5 shows that we have almost-everywhere convergence to some abstract limit. In fact, this limit can be identified by the so-called *Pastur-Shubin trace formula*, i.e.

$$N^*(E) := |Q|^{-1} \int_\Omega \operatorname{tr}\left(\mathbb{1}_Q\, \mathbb{1}_{]-\infty,E]}(H_\omega)\right) d\mu(\omega) \tag{6.2}$$

for $E \in \mathbb{R}$, where $Q \in \mathcal{F}(\Gamma)$ is an *arbitrary* non-empty, finite set. Here, the characteristic function $\mathbb{1}_Q$ is identified with the corresponding multiplication operator and $\mathbb{1}_{]-\infty,E]}(H_\omega)$ denotes the spectral projection of the operator H_ω in the interval $]-\infty, E]$. We will not prove the validity of equality (6.2) as the techniques can be found in the literature, see e.g. [LPV07, LV09, LSV11, PSS13].

We now turn to the proof of Theorem 6.5. The main task is to check that the assumptions of Theorem 4.4 are satisfied. We will not carry out the standard calculations in detail, but refer to the literature at those points.

Proof (of Theorem 6.5).
By Theorem 6.1, there is a set Ω_0 of full measure such that for all $\omega \in \Omega_0$, the frequencies

$$\nu_P^\omega := \lim_{j\to\infty} \frac{\#_P(\omega_{|U_j})}{|U_j|}$$

exist for all possible finite patterns $P \in \mathcal{P}$ and they are independent of the choice of the Følner sequence (U_j). So fix $\omega \in \Omega_0$ and consider the group Γ with colouring ω. Define the map

$$F_\omega : \mathcal{F}(\Gamma) \to (C_{br}(\mathbb{R}), \|\cdot\|_\infty) : F_\omega(Q)(\cdot) := \frac{1}{\dim(\mathcal{H})} n_\omega[Q_R](\cdot).$$

In order to show the ω-invariance, let two finite sets $Q, \tilde{Q} \in \mathcal{F}(\Gamma)$ be given such that $\omega_{|Q} \sim \omega_{|\tilde{Q}}$. Since the operator H_ω is colouring-invariant, the eigenvalues of the operators $H_\omega[Q_R]$ and $H_\omega[\tilde{Q}_R]$ coincide and this implies $F_\omega(Q) = F_\omega(\tilde{Q})$. It remains to find a boundary term b on $\mathcal{F}(\Gamma)$ with respect to which the map F_ω is almost-additive. We claim that the function

$$b : \mathcal{F}(\Gamma) \to [0, \infty) : b(Q) := 4 \dim(\mathcal{H}) |\partial_{B_{2R}}(Q)|$$

is in fact appropriate. Clearly, b is a boundary term. To check the almost-additivity, we proceed as in the proof of Proposition 4.6 in [LSV11]. Let $Q = \bigsqcup_{k=1}^m Q_k$ a union of pairwise disjoint subsets in $\mathcal{F}(\Gamma)$. By definition of R, the operators $H_\omega[Q_{k,R}]$ decouple, i.e.

$$H_\omega\left[\bigcup_{k=1}^m Q_{k,R}\right] = \bigoplus_{k=1}^m H_\omega[Q_{k,R}]$$

and we can count the eigenvalues of $H_\omega[Q_{k,R}]$ separately for $1 \le k \le m$. This yields

$$n_\omega\left[\bigcup_{k=1}^m Q_{k,R}\right] = \sum_{k=1}^m n_\omega[Q_{k,R}]. \tag{6.3}$$

The key step is a rank estimate that also can be found in [LSV11], see Proposition 7.2. We arrive at

$$\left\| n_\omega[Q_R] - n_\omega\left[\bigcup_{k=1}^m Q_{k,R}\right]\right\|_\infty \le 4 \dim(\mathcal{H}) \sum_{k=1}^m |\partial_{B_{2R}}(Q_k)|.$$

Thus, with equality (6.3), we obtain that F_ω is almost-additive with boundary term b.
To conclude the proof of the theorem, note that we have verified the assumptions of Theorem 4.4. Therefore, we find indeed some $N^* \in (C_{br}(\mathbb{R}), \|\cdot\|_\infty)$ such that

$$\lim_{j\to\infty} \left\| \frac{n_\omega[U_{j,R}]}{\dim(\mathcal{H})\, |U_j|} - N^* \right\|_\infty = 0$$

in the $\|\cdot\|_\infty$-topology. Moreover, since the frequencies ν_P^ω do not depend on the choice of the Følner sequence (U_j), we can deduce from the ε-limit expression in Theorem 4.4 that the same must hold true for N^*. We finish the proof with the observation that $\lim_{j\to\infty} |U_{j,R}|/|U_j| = 1$. □

Theorem 6.5 is a random version of Theorem 7.5 in [PS14]. In the latter assertion, the authors work in a deterministic setting and they *assume* to have a Følner sequence at hand such that all pattern frequencies exist along this sequence. In the present situation, this condition is guaranteed by the Lindenstrauss ergodic theorem. In [PSS13], the authors discuss a model which is similar to the present one. Precisely, they show the almost-sure uniform IDS approximation for random Schrödinger operators on amenable *quantum graphs* with random boundary conditions at the vertices. Again, the key ingredients for the proof are the Lindenstrauss ergodic theorem and our almost-additive ergodic Theorem 4.4. A previous version for the case $\Gamma = \mathbb{Z}^d$ can be found in [GLV07].

Non-uniqueness of the limit

In the above model, we strongly made use of the Lindenstrauss pointwise ergodic theorem which assures that the values of the frequencies do not depend on the tempered Følner sequence. However, even in situations in which no classical theorem can hold true, Theorem 4.4 might be applicable. Due to lack of intrinsic ergodicity in those cases, the limit expression F^* can be different for different Følner sequences (U_j) and (V_j). We give a concrete spectral theoretic example which is taken from [LSV11], cf. Example 4.8.

Example 6.6.
Let $\Gamma = \mathbb{Z}$ along with generating system $S = \{-1, +1\}$ and consider the Cayley graph $\mathrm{Cay}(\Gamma, S)$. Further, assume that Γ is labeled by two colours 0 and 1 ($\mathcal{A} := \{0, 1\}$) by the colouring $\mathcal{C}$, given by

$$\mathcal{C} : \mathbb{Z} \to \mathcal{A} : \mathcal{C}(m) := \begin{cases} 1, & \text{if } m \geq 0 \text{ or } m = 3k \text{ for some } k \in \mathbb{Z}, \\ 0, & \text{else.} \end{cases}$$

Now delete all edges in $\mathrm{Cay}(\Gamma, S)$ which are incident to a 1-vertex to obtain a new graph G. Let H be the adjacency operator on G, i.e.

$$H : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) : (Hu)(g) := \sum_{z \in \mathbb{Z}} h(g, z)\, u(z)$$

with $h(g, z) = 1$ if g and z share an edge in G and $h(g, z) = 0$ otherwise. For a finite set $Q \subset \mathbb{Z}$, we set

$$H[Q] : \ell^2(Q) \to \ell^2(Q) : H[Q] := p_Q\, H\, i_Q,$$

where, as above, p_Q and i_Q are the canonical projection and inclusion respectively. By the same reasoning as above, the mapping $Q \mapsto n[Q](\cdot)$, where

$$n[Q](E) := |\{i \in \mathbb{N} \mid \lambda_i \leq E \text{ is eigenvalue of } H[Q]\}|,$$

is almost-additive with boundary term $b(Q) := 4\,|\partial_{B_R}(Q)|$. We define two (weak) Følner sequences (U_j) and (V_j) in $\mathbb{Z}$ via

$$U_j := \{1, 2, \ldots, 3j-1, 3j\} \quad \text{and} \quad V_j := \{-3j, -3j+1, \ldots, -2, -1\}.$$

for $j \in \mathbb{N}$. Obviously, the frequencies along (U_j) and (V_j) do exist for all coloured patterns. Thus, we can apply Theorem 4.4. However, we obtain different limits for (U_j) and (V_j)

respectively. Since for all $j \in \mathbb{N}$, all entries of the matrix $H[U_j]$ must be equal to zero, the IDS with respect to the sequence (U_j) reads as

$$N^*_{(U_j)}(E) := \begin{cases} 0, & \text{if } E < 0, \\ 1, & \text{otherwise.} \end{cases}$$

On the other hand, basic linear algebra yields that the eigenvalues of the matrices $H[V_j]$ are $-1, 0$ and 1 and each of those occurs with multiplicity j. Thus, the corresponding IDS can be computed as

$$N^*_{(V_j)}(E) := \begin{cases} 0, & \text{if } E < -1 \\ 1/3, & \text{if } -1 \leq E < 0 \\ 2/3, & \text{if } 0 \leq E < 1 \\ 1, & \text{otherwise,} \end{cases}$$

which obviously is a different function from $N^*_{(U_j)}$.

6.2 Continuous groups

In this section, we give an application of our ergodic theorems, Theorems 4.15 and 5.5. More precisely, we prove pointwise convergence of the normalized eigenspace dimensions for a class of random operators on randomly chosen discrete structures, cf. Theorem 6.8. The underlying space possesses a quasi isometry to an amenable group. By standard arguments, this also leads to the almost-sure uniform convergence of the integrated density of states. Those latter results are due to Lenz and Veselić and have been published in [LV09]. We do not claim originality but show that almost-additive ergodic theorems can be used to solve problems arising naturally in mathematical physics. With our tools at hand, we will able to deal with operators which are randomly chosen as point processes over general LCSCUH groups. We start by explaining the model in [LV09], see also [Pog13a].

Suppose that (X, d_X) is a locally compact metric space with a countable basis of the topology. Let Γ be an amenable LCSCUH group equipped with an invariant metric d_Γ such that every bounded ball in Γ has compact closure. Moreover, we assume that Γ acts continuously from the right by isometries on X such that the following two properties hold:

- There exists a right fundamental domain J' with compact closure J, which is a countable union of compact sets,
- The map $\Phi : X \to \Gamma : x \mapsto g$, whenever $x \in J'g$, is a *quasi isometry*, i.e. there exist $a \geq 1$ and $b \geq 0$ with

  $$\frac{1}{a}\, d_\Gamma(\Phi(x), \Phi(y)) - b \leq d_X(x, y) \leq a\, d_\Gamma(\Phi(x), \Phi(y)) + b$$

  for all $x, y \in X$.

For a set $A \subseteq \Gamma$ and $r > 0$, we write $A_r := \{g \in \Gamma \,|\, d_\Gamma(g, \Gamma \setminus A) > r\}$, as well as $A^r := \{g \in \Gamma \,|\, d_\Gamma(g, A) < r\}$ and $\partial^r(A) := A^r \setminus A_r$. Analogously, with the metric d_X at hand, we introduce this notation for subsets of the space X.

For some fixed parameter $\eta > 0$, we set $\mathcal{D}$ as the family of η-uniformly discrete subsets of X, i.e.

$$\mathcal{D} := \{A \subset X \,|\, d_X(x, y) \geq \eta, \quad \text{for } x, y \in A \text{ with } x \neq y\}.$$

We define the set $\tilde{\mathcal{D}}$ as

$$\tilde{\mathcal{D}} := \{(A, h) \,|\, A \in \mathcal{D}, h : A \times A \to \mathbb{C}^*\},$$

where $\mathbb{C}^*$ is an arbitrary compactification of $\mathbb{C}$.

This space can be naturally equipped with the vague topology. It is then compact, cf. [LV09]. The action of Γ on X induces an action from the right on $\tilde{\mathcal{D}}$ by $g \cdot (A, h) = (Ag^{-1}, h(xg, yg))$ for $g \in \Gamma$ and $(A, h) \in \tilde{\mathcal{D}}$. In this situation, there exists a Γ-invariant ergodic probability measure on $\tilde{\mathcal{D}}$, whose topological support will be denoted by Ω, cf. [LV09]. Then, Ω is a compact subset of $\tilde{\mathcal{D}}$. Note that each element $\omega \in \Omega$ can be written as $\omega := (X(\omega), h_\omega) \in \tilde{\mathcal{D}}$, where $X(\omega)$ is η-uniformly discrete and $h_\omega : X(\omega) \times X(\omega) \to \mathbb{C}^*$ is a map. Each $X(\omega)$ gives rise to a Hilbert space $\ell^2(X(\omega))$, endowed with the canonical counting measure $\delta_{X(\omega)} := \sum_{x \in X(\omega)} \delta_x$.

We will draw our attention to the bounded operators H_ω on $\ell^2(X(\omega))$, defined as

$$(H_\omega u)(x) := \sum_{y \in X(\omega)} h_\omega(x, y)\, u(y)$$

for each $x \in X(\omega)$. Moreover, we assume that the H_ω are *of finite hopping range*, i.e. there exists some number $R > b$ such that for all $\omega \in \Omega$, we have $h_\omega(x, y) = 0$ whenever $d_X(x, y) \geq R$. For $g \in \Gamma$, we let

$$U_g : \ell^2(X(\omega)) \to \ell^2(X(g\omega)) : (U_g u)(x) := u(xg)$$

with adjoint $U_g^* = U_{g^{-1}}$. With that notion, we assume that the operators H_ω are *equivariant*, i.e.

$$U_g^* \, H_{g\omega} \, U_g = H_\omega$$

for all $g \in \Gamma$ and every $\omega \in \Omega$. Also, we need that the H_ω are *self-adjoint*.

As in the previous section, we need to restrict and to expand the operators H_ω. In light of that, for $Q \in \mathcal{F}(\Gamma)$, we denote by $i_Q : \ell^2(X(\omega) \cap (JQ)_R) \to \ell^2(X(\omega))$ the canonical inclusion operator and by $p_Q : \ell^2(X(\omega)) \to \ell^2(X(\omega) \cap (JQ)_R)$ the canonical projection operator for $\omega \in \Omega$, where $(JQ)_R$ stands for the R-interior of JQ, i.e.

$$(JQ)_R = \{x \in JQ \,|\, d_X(x, X \setminus JQ) \geq R\}.$$

For every $\omega \in \Omega$, we consider the restricted operators $H_\omega^R[Q] : \ell^2(X(\omega) \cap (JQ)_R) \to \ell^2(X(\omega) \cap (JQ)_R)$, where

$$H_\omega^R[Q] := p_Q H_\omega i_Q$$

for $Q \in \mathcal{F}(\Gamma)$. Since $X(\omega)$ is η-uniformly discrete, each such $H_\omega^R[Q]$ can be described by a finite, quadratic matrix. Now fix some energy level $E \in \mathbb{R}$. We define the function

$$\begin{aligned} F_\cdot^E : \mathcal{F}(\Gamma) \to L^1(\Omega, \mathbb{R}) : F_\omega^E(Q) &:= \#\{i \in \mathbb{N} \mid \lambda_i \text{ is eigenvalue of } H_\omega^R[Q] \text{ and } \lambda_i = E\} \\ &= \operatorname{tr}\Big(\mathbb{1}_{\{E\}} H_\omega^R[Q]\Big). \end{aligned}$$

Note that $F_\omega^E[Q]$ simply gives the multiplicity of the eigenvalue E for the operator $H_\omega^R[Q]$. We consider the action of Γ on Ω as a weakly measurable action via operators $\{T_g\}_{g\in\Gamma}$ on $L^1(\Omega, \mathbb{R})$. By the same methods as in the case of countable groups, we obtain the following proposition.

Proposition 6.7.
In the above model, the following holds true: for every $E \in \mathbb{R}$ and for all $\omega \in \Omega$, the mapping $F_\omega^E : \mathcal{F}(\Gamma) \to \mathbb{R}$ is admissibly almost-additive with tiling-admissible, weak boundary term

$$b : \mathcal{F}(\Gamma) \to [0, \infty) : b(Q) := D\,|\partial_{B_{2\overline{R}}}(Q)|,$$

where $D > 0$ and $\overline{R} > 0$ are constants depending on a, b, η and R and $B_{2\overline{R}}$ is the open ball of radius $2\overline{R}$ around the unity e in the group Γ. Further, the equivariance condition

$$T_{g^{-1}} F_\omega^E(Q) := F_{g\omega}^E(Q) = F_\omega^E(Qg)$$

is satisfied for all $E \in \mathbb{R}$, every $\omega \in \Omega$, each $Q \in \mathcal{F}(\Gamma)$ and every $g \in \Gamma$.

Proof.
This follows by standard arguments, see e.g. the Propositions 8.1, 8.2 and 8.5 in [Pog13a]. For the almost-additivity (b is in fact tiling-admissible, see Proposition 4.12), the main ingredient is the rank estimate given by Proposition 7.2 in [LSV11]. □

We now prove the main theorem of this section.

Theorem 6.8.
In the above model, the following spectral approximation result holds true: For every $E \in \mathbb{R}$, there is a number $0 \leq F^{E\,*} \leq 1$ such that for every tempered strong Følner sequence (U_j), one can find a measurable set $\tilde{\Omega} \subseteq \Omega$ of full measure such that for each $\omega \in \tilde{\Omega}$,

$$\lim_{j\to\infty} \frac{F_\omega^E(U_j)}{|U_j|} = F^{E\,*}.$$

Proof.
With Γ acting measure preservingly and ergodically on Ω, we obtain a canonical weakly measurable and ergodic action of operators $\{T_g\}_{g\in\Gamma}$ on $L^1(\Omega, \mu)$ via $T_g f(\omega) := f(g^{-1}\omega)$. Fix $E \in \mathbb{R}$. By Proposition 6.7, we have $T_g F_\omega^E(Q) = F_\omega^E(Qg^{-1})$ for all $g \in \Gamma$, each $Q \in \mathcal{F}(\Gamma)$

and every $\omega \in \Omega$. It now follows from the ergodic Theorem 5.5 (resp. from the Lindenstrauss ergodic theorem) that for all $Q \in \mathcal{F}(\Gamma)$, the limit

$$S(Q) := \lim_{j\to\infty} |U_j|^{-1} \int_{U_j} F_\omega^E(Qg)\, dg$$

exists for all $\omega \in \hat{\Omega}$, where $\hat{\Omega}$ is a set of full measure. Note that $S(Q)$ does not depend on the choice of (U_j). Moreover, for each $Q \in \mathcal{F}(\Gamma)$ and for every $g \in \Gamma$, we can find a set $\bar{\Omega} \subseteq \hat{\Omega}$ of full measure such that $S(Qg)(\omega) = S(Q)(\omega)$ for all $\omega \in \bar{\Omega}$. Take a sequence (ε_k) of positive numbers converging to zero. For each ε_k, we have an ε_k-quasi tiling sequence $\{T_i^{\varepsilon_k}\}_{i=1}^{N(\varepsilon_k)}$, $N(\varepsilon_k) := \lceil \log \varepsilon_k / \log(1-\varepsilon_k)\rceil$ as in Definition 2.15. By the above considerations, we find a set $\tilde{\Omega} \subseteq \Omega$ of full measure such that for every $\omega \in \tilde{\Omega}$, the limits $S(T_i^{\varepsilon_k})(\omega)$ exist for all $k \in \mathbb{N}$ and all $1 \leq i \leq N(\varepsilon_k)$. We have seen in the above Proposition 6.7 that for all $\omega \in \Omega$, the map F_ω^E is admissibly almost-additive with tiling-admissible, weak boundary term $b(Q) := D\,|\partial_{B_{2\overline{R}}}(Q)|$, where the constants $\overline{R}, D > 0$ depend on the parameters a, b, η, R of our model. This puts us in the position to apply Corollary 4.16 to obtain the convergence to some number $0 \leq F_\omega^{E*} \leq 1$ for every $\omega \in \tilde{\Omega}$. It follows from the representation

$$F_\omega^{E*} := \lim_{k\to\infty} \sum_{i=1}^{N(\varepsilon_k)} \varepsilon_k (1-\varepsilon_k)^{N(\varepsilon_k)-i} \frac{S(T_i^{\varepsilon_k})(\omega)}{|T_i^{\varepsilon_k}|}$$

and the T_g-invariance of the averages $S(T_i^{\varepsilon_k})$ that for all $g \in \Gamma$, one obtains $T_g F_\omega^{E*} = F_\omega^{E*}$ μ-almost-surely. The ergodicity of the action of Γ on Ω yields that F_ω^{E*} must be constant almost-surely. This finishes the proof. □

If one considers the empirical eigenvalue distributions of the operators $H_\omega^R[U_j]$ as measures ν_ω^j, then a standard calculation shows that almost-surely, the sequence (ν_ω^j) converges weakly to a deterministic measure ν ('density of states'). Moreover, ν is given by a trace representation on the group von Neumann algebra, see Theorem 2.1. in [LV09]. In fact, we have $F^{E*} = \nu(\{E\})$ for all $E \in \mathbb{R}$. For detailed discussions, the reader may e.g. refer to Lemma 6.1 in [LV09] or Chapter 9 of the present thesis. Theorem 6.8 shows the almost-everywhere convergence of the normalized eigenspace dimensions for all energies $E \in \mathbb{R}$. For continuity points E of the IDS, it follows from standard measure theory that the convergence follows already from the weak convergence, cf. e.g. Satz 4.12 in [Els05]. As the IDS allows only for countably many discontinuity points, there is a set $\tilde{\Omega} \subseteq \Omega$ of full measure such that the convergence statement in Theorem 6.8 holds true simultaneously for all $E \in \mathbb{R}$, whenever $\omega \in \tilde{\Omega}$. Now, another tool from measure theory shows that weak convergence and the just mentioned considerations about pointwise convergence even yield uniform convergence of the distribution functions, cf. Lemma 6.3 in [LV09]. Further, the Pastur Shubin trace formula holds true. This leads to the following uniform convergence result.

Theorem 6.9 (Uniform approximation of the IDS).
Let the conditions of Theorem 6.8 be given. Then, for every tempered strong Følner sequence (U_j) in Γ, there is a measurable set $\tilde{\Omega} \subseteq \Omega$ of full measure such that for each $\omega \in \tilde{\Omega}$,

$$\lim_{j\to\infty} \sup_{E\in\mathbb{R}} \left| \frac{N_\omega^E(U_j)}{|U_j|} - N^{E*} \right| = 0,$$

where $N_\omega^E(U_j) := \#\{i \in \mathbb{N} \mid \lambda_i \leq E \text{ is eigenvalue of } H_\omega^R[U_j]\}$ *and* $N^{E\,*} := \nu((-\infty, E])$.

Note that this theorem has already been proven before. It can be found in [LV09], Theorems 2.1 and 2.4.

7 Graphs and graphings

In this chapter, we deal with concepts of convergence for graph sequences. Firstly, we discuss weak convergence [BS01] of finite graphs towards *graphings*. The latter objects encode certain measurable equivalence relations which can be interpreted as a probability distribution on the set of all isomorphism classes of countable graphs of bounded vertex degree. Explicit constructions for graphings can e.g. be found in [Ele07b, Lov12, LPS14]. In a second part we define hyperfinite graph sequences as in [Ele07a] and show that weakly convergent graph sequences of this kind converge in a stronger sense. More precisely, this means that the graph sequence is Cauchy in a certain pseudometric δ, cf. Theorem 7.10. This partially confirms Conjecture 1 of Elek in [Ele08a]. Using the Equipartition Theorem of Elek (cf. Theorem 4 in [Ele12]), we give a detailed proof for this assertion. The fundamental insight in this context that two graphs in a hyperfinite family having the same number of vertices are geometrically alike (i.e. close in δ) whenever they have similar local graph statistics, has already been obtained in Theorem 3.1 of [NS13] and Theorem 5 in [Ele12]. Theorem 7.10 has far reaching consequences. In the following chapters, we are able to prove a Banach space-valued ergodic theorem for almost-additive functions defined on graphs (Chapter 8), as well as the uniform approximation of the integrated density of states for pattern-invariant, finite hopping range operators on the graphs (Chapter 9). The results of this chapter are taken from [Pog13b].

7.1 Weak convergence and graphings

This section is devoted to a brief introduction of weakly convergent graph sequences. The concept of weak convergence for graph sequences with uniform vertex degree bound has been introduced by Benjamini and Schramm in the influential work [BS01]. Further, we define limit graphings and sofic graphings as natural limit objects of convergent graph sequences.

We start with some basic notation.

Throughout the remaining parts of this thesis, we deal with graphs $G = (V, E)$, where V is some set called the *vertices* of G and $E \subseteq (V \times V) \setminus U$ is a symmetric set of *edges* of G, where $U := \{(v, v) \,|\, v \in V\}$ is the diagonal set. We say that two vertices $v, w \in V$ are linked by an edge in G if $(v, w) \in E$ (and by symmetry also $(w, v) \in E$). For $v \in V$, we denote by $\deg(v)$ the *vertex degree* of v, i.e. the number of $w \in V$ such that $(v, w) \in E$. Throughout the whole thesis, we will deal with graphs (V, E) for which there is some $D \in \mathbb{N}$ such that $\sup_{v \in V} \deg(v) \leq D$. Further, a graph G is *connected* if for any two vertices $v, w \in V$, one can find a finite sequence $v_0, \dots, v_L \in L$ with $v_0 = v$ and $v_L = w$ such that $(v_i, v_{i+1}) \in E$ for all $0 \le i \le L-1$. In this situation, we call $(v_0, \dots, v_L)$ a *path* connecting v with w. A component in a graph G is a subgraph of G, where all vertices in the subgraph are connected to each other via at least one finite path. Note that there are at most countably many vertices in each such component. We say that two components are *edge-disjoint* in G if the vertex sets of both components are disjoint and there is no edge in E linking a vertex of one component with a vertex of the other component. For graphs $G = (V, E)$, there is a canonincal *path metric* on V, given by

$$d_G : V \times V \to [0, \infty) : d_G(v, w) := \min\{L \in \mathbb{N} \,|\, \exists \text{ path } (v_0, \dots, v_L) \text{ connecting } v \text{ and } w\}.$$

If for $v, w \in V$, no connecting path can be found, we set $d_G(v, w) = \infty$ as a convention. A graph $G = (V, E, o)$ together with some distinguished vertex $o \in V$ is called *rooted* and we say that $o \in V$ is the *root* of G. For a finite, connected, rooted graph, we define the *radius* $\rho(G)$ of G as

$$\rho(G) := \max\{d_G(v, o) \,|\, v \in V\}.$$

Given two finite, rooted graphs $G = (V, E, o)$ and $\tilde{G} = (\tilde{V}, \tilde{E}, \tilde{o})$, we say that G and $\tilde{G}$ are *rooted isomorphic* if there is a bijective mapping $\varphi : V \to \tilde{V}$ such that $\varphi(o) = \tilde{o}$ and $(v, w) \in E$ if and only if $(\varphi(v), \varphi(w)) \in \tilde{E}$. In this situation we write $G \simeq \tilde{G}$. Fixing $D \in \mathbb{N}$, as well as some $r \in \mathbb{N}$, we write $\mathcal{A}_r^D$ for the finite set of all rooted isomorphism classes (graphs identified by rooted graph isomorphisms) of rooted graphs with radius at most r and vertex degree bounded by D. The collection of all isomorphism classes of rooted graphs is denoted by the countable set $\mathcal{A}^D := \bigcup_{r=1}^{\infty} \mathcal{A}_r^D$.

Further, if $G = (V, E)$ and $T \subseteq V$, we say that $G_T := (T, E_T)$ is the *subgraph of G induced by T* if $E_T := \{(v, w) \,|\, (v, w) \in E, v, w \in T\}$. In the following elaborations, $\mathcal{S}^D$ denotes a set consisting of finite, unrooted graphs with vertex degree bound $D \in \mathbb{N}$. For some $G = (V, E) \in \mathcal{S}^D$ and $\alpha \in \mathcal{A}^D$, we set

$$p(G, \alpha) := \frac{|\{v \in V \,|\, B^G_{\rho(\alpha)}(v) \simeq \alpha\}|}{|V|},$$

where $\rho(\alpha)$ is the radius of a (all) representative(s) of α. Here, we mean that for each $v \in V$, we consider the subgraph $B^G_{\rho(\alpha)}(v)$ of G induced by the $\rho(\alpha)$-ball in G around v as rooted graph with root v and we check whether this element is rooted isomorphic to the class α. Therefore, the expression $p(G, \alpha)$ measures the empirical occurrence frequency of the class α in the graph G. We say that a sequence (G_n) in $\mathcal{G}^D$ converges weakly if asymptotically, the limit frequencies exist for all classes $\alpha \in \mathcal{A}^D$. This is made precise in the following definition.

Definition 7.1 (Weak convergence of graphs).
Let (G_n) be a sequence of finite, connected graphs with uniform vertex degree bound $D \in \mathbb{N}$ and such that $\lim_{n\to\infty} |V_n| = \infty$. Then we say that (G_n) is weakly convergent *if for all $\alpha \in \mathcal{A}^D$, the limit*

$$p(\alpha) := \lim_{n\to\infty} p(G_n, \alpha)$$

exists.

This concept of convergence has been introduced by Benjamini and Schramm in [BS01]. Therefore, weakly convergent graph sequences are also called *Benjamini-Schramm convergent* in the literature. A natural question arising immediately is to attach some limit element to weakly convergent graph sequences. Before approaching this issue, let us give some examples first.

Examples 7.2.

- Let Γ be a finitely generated (generating system S, unit element e), amenable group along with its Cayley graph $G = \mathrm{Cay}(\Gamma, S)$. Let (T_n) be a Følner sequence in Γ. For each $n \in \mathbb{N}$, denote by G_n the subgraph of G induced by T_n. Then (G_n) is a weakly convergent graph sequence with limit probabilities

$$p(\alpha) := \begin{cases} 1, & B^G_{\rho(\alpha)}(e) \simeq \alpha \\ 0, & \text{otherwise} \end{cases}$$

  for all $\alpha \in \mathcal{A}^D$.

- The above concept can be extended to the class of all *sofic* groups Γ with finite generating system $S \subset \Gamma$. Sofic groups have been invented by Gromov in [Gro99]. The name 'sofic' was given by Weiss in [Wei00] who derived it from the Hebrew word סוף [sof] for 'finite'. In fact, a group is sofic if it can be approximated by certain weakly convergent graph sequences, i.e. if there is a (particularly labeled) weakly convergent graph sequence (G_n) with uniform vertex degree bound $D = |S| \in \mathbb{N}$ such that

$$p(\alpha) := \begin{cases} 1, & B^G_{\rho(\alpha)}(e) \simeq \alpha \\ 0, & \text{otherwise} \end{cases}$$

  for all $\alpha \in \mathcal{A}^D$, where again $G = \mathrm{Cay}(\Gamma, S)$. Note that the graphs G_n are not necessarily induced subgraphs of G. Subgraphs of this kind can only be found in amenable groups. However, the class of sofic groups is much larger; in fact, it is not known whether there are groups which do not have this property. In particular, all amenable and all residually finite groups are sofic, cf. [Wei00]. A precise definition for sofic groups will be given in Chapter 10.

- Assume that G is a countably infinite graph (with vertex degree bound $D \in \mathbb{N}$) along with a sofic group Γ acting freely and co-finitely on G by graph automorphisms. Suppose that F is a finite fundamental domain for G/Γ. For $\alpha \in \mathcal{A}^D$, we set

$$F_\alpha := \{f \in F \mid B^G_{\rho(\alpha)}(f) \simeq \alpha\}.$$

  We show in Chapter 10 that in this situation, there is a weakly convergent graph sequence (G_n) such that

$$\lim_{n\to\infty} p(G_n, \alpha) = \frac{|F_\alpha|}{|F|}$$

  for all $\alpha \in \mathcal{A}^D$. Note that this is an extension of the group case since every sofic group acts freely and co-finitely on its own Cayley graph by group automorphisms.

In all the above examples, the countably infinite graph G can be interpreted as a limit object of some weakly convergent graph sequence. However, this cannot be done in general. We will see in the following that weakly convergent graph sequences give rise to probability distributions of countably infinite graphs. This leads us to the concept of graphings and their induced measure graphs.

Definition 7.3 (Graphing).
Let X be a compact topological Hausdorff space and μ be a Borel probability measure on X. Let furthermore finitely many measure preserving, continuous involutions I_k, $1 \leq k \leq D$, $D \in \mathbb{N}$ on (X, μ) be given. Then we call the collection

$$\mathcal{G} := (X, \mu, I_1, \ldots, I_D)$$

a graphing *over (X, μ).*

Note that the involutions I_k of a graphing $\mathcal{G}$ give rise to a finitely generated group

$$\Gamma := \langle I_k \,|\, 1 \leq k \leq D \rangle,$$

where group multiplication is just composition of bijective mappings from X into itself. Obviously, Γ acts on (X, μ) by measure preserving transformations. In the following, we refer to Γ as the group associated with $\mathcal{G}$. Given a graphing, one obtains a concept for so-called measure graphs as given in [LPS14]. These objects are induced from certain measurable equivalence relations with countable classes, see e.g. [FM77]. Let us make this concept more precise.

Every graphing $\mathcal{G} := (X, \mu, I_k)$, $1 \leq k \leq D$ comes along with a canonical graph structure. Precisely, we can define a graph G_X with vertex set X and two distinct elements $x, y \in X$ being by an edge coloured by the label k if and only if $y = I_k(x)$. Then G_X is a (possibly uncountable) collection of pairwise disjoint, countable, coloured graphs with vertex sets Γx, where $x \in X$. Unfortunately, this graph does not fiber over its connected components and it turns out that the quotient space X/Γ has very unpleasant measurability properties, cf. [LPV07]. To overcome this problem, we will work with a particular measurable equivalence relation over X. We then obtain a measurable coloured graph, cf. [LPS14]. Define

$$V := \{(x, y) \in X \times X \,|\, \Gamma x = \Gamma y\}.$$

Note that the set $V \subseteq X \times X$ is an eqivalence relation, where two elements $x, y \in X$ shall be equivalent if and only if there is some $\gamma \in \Gamma$ such that $y = \gamma x$. The set V is a measurable subset of $X \times X$ when endowing the latter space with the product Borel σ-algebra. This follows from the observation that Γ is countable and

$$V = \bigcap_{\gamma \in \Gamma} (\mathrm{id}_X \times I_\gamma)^{-1} \operatorname{diag}(X),$$

where $I_\gamma(x) := \gamma\, x$ and $\operatorname{diag}(X) := \{(x, x) \,|\, x \in X\}$ is the diagonal set in $X \times X$. We also used here that the mappings I_γ are measurable (in fact they are also continuous). We now define an edge relation on V. To do so, we set

$$E := \big\{\big((w, x), (y, z)\big) \in V \times V \,|\, w = y, x \neq z,\, \exists\, 1 \leq k \leq D : x = I_k z\big\}.$$

In this way, we obtain a graph $G := (V, E)$ with vertex set V and edge set E. It is even true that G is a measurable graph as defined in Definition 1.2 of [LPS14]. (For a rigorous justification of this fact, see Section 7 of this latter paper.) We mention the most important features which will be used in the following. For $\alpha \in \mathcal{A}^D$, we set

$$X_\alpha := \{x \in X \mid B^G_{\rho(\alpha)}((x,x)) \simeq \alpha\}.$$

It is not hard to see that all sets X_α are measurable, cf. [LPS14]. Further, set

$$V_\alpha := \{v \in V \mid B^G_{\rho(\alpha)}(v) \simeq \alpha\}.$$

Due to the representation

$$V_\alpha = \bigcap_{\gamma\in\Gamma} (I_\gamma \times \mathrm{id}_X)^{-1} \operatorname{diag}(X_\alpha), \quad \operatorname{diag}(X_\alpha) := \{(x,x) \mid x \in X_\alpha\},$$

the sets V_α are measurable subsets of V for all $\alpha \in \mathcal{A}^D$. It follows also from the considerations in [LPS14] that there is a finite measure M on the set V with the property that $\mu(X_\alpha) = M(V_\alpha)$ for every $\alpha \in \mathcal{A}^D$. Furthermore, this measure can be desintegrated as $M = \mu \circ \eta$, where for every $x \in X$, η^x is the counting measure on the set $V^x := \{x\} \times \Gamma x \subseteq V$. Then for every non-negative, measurable function f on V, we have

$$\int_V f(v)u(v)\, dM(v) = \int_X f((x,x))\, d\mu(x),$$

where $u : V \to \mathbb{R}$ is a certain measurable, non-negative averaging function. This equality follows from Lemma 1.16 (b) in [LPS14] which is strongly based on the non-commutative integration theory of Connes (see [Con79]).

Definition 7.4 (Measure graph induced by a graphing).
Let $\mathcal{G} = (X, \mu, I_k)$ be a graphing. Let G be the graph with vertex set V and edge set E as above and let M be the finite measure on V constructed from μ as above. Then, we call (G, M) the measure graph induced by the graphing *$\mathcal{G}$.*

As said before, graphings can be constructed in a canonical way from weakly convergent graph sequences. This is stated in the following theorem. For a proof, see [LPS14], Proposition 9.3. The crucial observation that is needed here is the fact that one can attach to every finite graph a canonical probability measure which is invariant under transformations shifting a fixed root to another vertex. The existence of an invariant limit measure follows from a compactness condition for probability measures. Similar constructions for graphings have been obtained before, see e.g. [Ele07b, AL07].

Theorem 7.5 (Existence of graphings, cf. [LPS14], Proposition 9.3).
Let (G_n) be a weakly convergent sequence of graphs with uniform vertex degree bound $D \in \mathbb{N}$. Then, there is a graphing $\mathcal{G} = (X, \mu, I_k)$ along with a measure graph (G, M) induced by $\mathcal{G}$ such that for all $\alpha \in \mathcal{A}^D$

$$\lim_{n\to\infty} p(G_n, \alpha) = \mu(X_\alpha),$$

where X_α is the collection of x such that the subgraph in $G = (V, E)$ induced by the ball of radius $\rho(\alpha)$ around $(x,x) \in V$ is rooted isomorphic to α.

We will refer to the graphings arising from convergent graph sequences as so-called *limit graphings*. The converse question is an open problem, i.e. it is not known whether all graphings (with suitable topological and measurable assumptions) are a limit graphing for some weakly convergent sequence (G_n) of finite graphs). This is a question of Aldous and Lyons [AL07, Sch08]. A positive answer to this question would imply that all finitely generated groups are sofic. However, we can define the large class of *sofic graphings* as those graphings which are attained as a limit of Benjamini-Schramm convergent sequences.

Definition 7.6 (Sofic graphings).
Let $\mathcal{G} = (X, \mu, I_k)$ be a graphing along with its induced measure graph (G, M). Then $\mathcal{G}$ is called sofic *if there is a weakly covergent sequence (G_n) of finite graphs such that for all $\alpha \in \mathcal{A}^D$*

$$\lim_{n\to\infty} p(G_n, \alpha) = \mu(X_\alpha),$$

where X_α is the collection of x such that the subgraph in $G = (V, E)$ induced by the ball of radius $\rho(\alpha)$ around $(x, x) \in V$ is rooted isomorphic to α.

Sofic graphings have been considered before. This concept has been introduced in [EL10] as sofic measurable equivalence relations. Further, some examples are given in the latter paper. For instance, it is shown there that every treeable equivalence relation is sofic.

7.2 Hyperfiniteness and Banach space convergence

In this section, we consider a different notion of convergence for graphs with uniform vertex degree bound $D \in \mathbb{N}$. Precisely, a sequence (G_n) of graphs taken from a set $\mathcal{S}^D$ will be called strongly convergent if it is Cauchy in a particular pseudometric δ on $\mathcal{S}^D$ introduced by Elek in [Ele08a]. The goal of the next chapter is to prove a Banach space convergence result for almost-additive mappings on $\mathcal{S}^D$. Here, we provide the necessary preparations. In this context, we will have to work with hyperfinite graph sequences. This latter condition roughly says that all elements in the graph sequence can be cut in small edge-disjoint components by deleting only a small portion of the edges in each graph. There is a significant link to concepts of amenability for graphs. For instance, a statement of Elek and Szabó [ES11] shows that sofic approximations of a group are hyperfinite if and only if the group is amenable. It is well-known that strong convergence implies weak convergence and hyperfiniteness. One can conclude from the results in [NS13] or in [Ele12] that the converse assertion holds also true. This is a partial positive answer to a conjecture of Elek in [Ele08a]. The full conjecture refers to graph sequences with edge- and vertex colourings by finitely many colours. In Theorem 7.10, we give a clear and detailed proof which is based on the so-called Equipartition Theorem, cf. Theorem 4 in [Ele12]. It is an open problem whether the latter statement also holds true for edge- and vertex coloured graphs. Private communication of the author of this thesis with Elek gives rise to evidence that the Equipartition Theorem holds also true in the coloured case. A nice consequence would be the validity of Elek's conjecture in full generality. This in turn would allow for the proof of a Banach space-valued, almost-additive convergence theorem along hyperfinite, *coloured* weakly convergent graph sequences.

As usual, we assume that $\mathcal{S}^D$ is a set consisting of finite graphs with uniform vertex degree bound $D \in \mathbb{N}$. Suppose that $G, \tilde{G} \in \mathcal{S}^D$ are defined on the same vertex set V and that the vertices in both graphs are labeled with the numbers $\{1, \dots, |V|\}$. We set

$$\delta_V(G, \tilde{G}) := \frac{|\{v \in V \mid S^G(v) \neq S^{\tilde{G}}(v)\}|}{|V|},$$

where $S^G(v)$ denotes the labeled 1-ball (the *star*) around $v \in V$ in the graph G. Thus, the $\neq$-sign must be understood not only in the sense of rooted isomorphism classes but it also decodes the differences in the vertex numberings of both stars. As shown in Lemma 2.1 of [Ele08a], δ_V defines a metric on the graphs in $\mathcal{S}^D$ modulo isomorphism which are defined on the common vertex set V. For the empty graph $\mathcal{E}$, we set $\delta_V(G, \mathcal{E}) = 1$ for all $G \in \mathcal{S}^D$. Next, we define a metric which is invariant under permutations of the vertex numbers. Referring to Lemma 2.2 in [Ele08a], we define the metric

$$\delta_S(G, \tilde{G}) := \min_{\sigma \in \mathrm{Sym}(|V|)} \delta_V(G, \tilde{G}^\sigma)$$

on all graphs in $G, \tilde{G} \in \mathcal{S}^D$ modulo isomorphism defined on V, where $\mathrm{Sym}(|V|)$ is the symmetric group over the set $\{1, \dots, |V|\}$ and $\tilde{G}^\sigma$ is the graph $\tilde{G}$ with its vertex numbering translated by σ. Finally, we get rid of the assumption that graphs need to be defined over the same vertex set. This leads us to the so-called *geometric distance*

$$\delta(G, \tilde{G}) := \inf_{\{p, q \in \mathbb{N} \mid p\,|V| = q\,|\tilde{V}|\}} \delta_S(p\,G, q\,\tilde{G}),$$

where the graph $p\,G$ consists of p edge-disjoint copies of the graph G. This is a pseudometric over the set of all unrooted isomorphism classes in $\mathcal{S}^D$, cf. [Ele08a], Proposition 2.1. In fact, we have $\delta(G, \tilde{G}) = 0$ if and only if there is some graph $\bar{G} \in \mathcal{S}^D$ such that both graphs $G, \tilde{G}$ consist of edge-disjoint copies of $\bar{G}$. For $G \in \mathcal{S}^D$, we denote by $\alpha(G)$ the corresponding unrooted isomorphism class for G. Now, we are finally in the position to define strong convergence for graph sequences (G_n).

Definition 7.7 (Strong convergence of graphs).
Let $(G_n) = (V_n, E_n)$ *be a sequence of finite, connected, graphs in* $\mathcal{S}^D$ *with* $\lim_{n\to\infty} |V_n| = \infty$. *Then* (G_n) converges strongly *if* $(\alpha(G_n))_n$ *is Cauchy in the pseudometric* δ.

It is well-known that strongly convergent graph sequences are also weakly convergent, see e.g. [Ele08a], Proposition 2.2. We now turn to the concept of hyperfiniteness for graphs.

Definition 7.8 (Hyperfinite families of graphs).
A family $\mathcal{P} \subseteq \mathcal{S}^D$ *is called* hyperfinite *if for every* $\varepsilon > 0$, *there is an integer* $K_\varepsilon \in \mathbb{N}$ *such that there is a way to remove from each* $G \in \mathcal{P}$ *a portion of at most* ε *of the edges such that the remaining graph* G'_ε *consists of edge-disjoint components consisting of at most* K_ε *vertices.*
We say that a graph sequence (G_n) *is hyperfinite if the set* $\mathcal{P} := \{G_n \mid n \in \mathbb{N}\} \subseteq \mathcal{S}^D$ *is a hyperfinite family.*

Here are some examples for hyperfinite graph sequences.

Examples 7.9.

- Let $\mathcal{P} := \{P_n\}_{n=1}^{\infty}$, where P_n is a path of length n. Then $\mathcal{P}$ is hyperfinite. One may choose $K_\varepsilon := 2/\varepsilon$ for $\varepsilon > 0$.
- Let $\mathcal{P} := \{G_n\}_{n=1}^{\infty}$, where each G_n is the subgraph of $\mathrm{Cay}(\Gamma, S)$ induced by the set T_n, where (T_n) is a Følner sequence in an amenable group Γ generated by some finite set $S \subset \Gamma$. Using the ε-quasi tiling result of Theorem 2.16, it can be seen that $\mathcal{P}$ is hyperfinite.
- Let $\mathcal{P} := \{G_n = (V_n, E_n)\}_{n=1}^{\infty}$ be a graph sequence of subexponential growth, i.e. there is a function $f : \mathbb{N} \to \mathbb{N}$ of subexponential growth such that for every $n \in \mathbb{N}$, for each $x \in V_n$ and for all $r \in \mathbb{N}$, one has $|B_r^G(x)| \leq f(r)$. Note that f is said to have subexponential growth if for each $\beta > 0$, there is a number $r_\beta > 0$ such that $f(r) \leq \exp(\beta\, r)$ for every $r \geq r_\beta$, cf. [Ele08a]. Then, $\mathcal{P}$ is a hyperfinite family.
- A large class of non-examples is given by sofic approximations for non-amenable groups, see the second item in the example list 7.2. In fact, sofic approximations for a group Γ are hyperfinite if and only if Γ is amenable, cf. Proposition 4.1 in [ES11]. In particular, if (G_n) is a sofic approximation for the free group $\mathbb{F}_r$ of rank $r \geq 2$, then the set $\mathcal{P} := \{G_n\}_{n=1}^{\infty}$ is *not* a hyperfinite family of graphs.

Hyperfiniteness is implied by strong convergence, cf. Proposition 2.3 in [Ele08a]. We now turn to the converse statement and we show that weakly convergent, hyperfinite graph sequences are in fact strongly convergent.

Theorem 7.10 ([Pog13b], Theorem 3.1).
Let (G_n) be a weakly convergent sequence of graphs with uniform vertex degree bound $D \in \mathbb{N}$. If in addition, (G_n) is hyperfinite, then (G_n) is strongly convergent.

Using algorithmic techniques, this statement has essentially been proven in [NS13]. In Theorem 3.1 of this latter work, the authors show that two graphs on the same vertex set taken from a hyperfinite family have small δ-distance from each other if they look alike statistically. We follow a different approach and use the *Equipartition Theorem* which is due to Elek. Roughly speaking, it says that one may delete a small portion of the edges in the graphs of some hyperfinite Benjamini-Schramm convergent sequence in order to cut them into small egde-disjoint components in such a way that asymptotically, the number of the various remaining components stabilizes. It is shown in [Ele08a] that every hyperfinite, weakly convergent graph sequence contains a strongly convergent subsequence. Using a part of the proof of this latter statement, as well as the Equipartition Theorem, we obtain convergence for all such sequences. We still need some notational preparation.

At first, we introduce another distance function d_π which measures statistical differences of two graphs. Take an arbitrary enumeration $(\alpha_i)_{i \in \mathbb{N}}$ of the elements in $\mathcal{A}^D$. Now consider the map

$$\mathcal{L} : \mathcal{S}^D \to [0,1]^{\mathbb{N}} : \mathcal{L}(G) := (p(G, \alpha_i))_{i \in \mathbb{N}}.$$

Then $\mathcal{L}$ is almost-injective in the sense that $\mathcal{L}(G) = \mathcal{L}(\tilde{G})$ implies that there is a graph $\bar{G} \in \mathcal{S}^D$ such that both G and $\tilde{G}$ are edge-disjoint unions of $\bar{G}$-copies, cf. [Ele12]. With this notion at hand, we set

$$d_\pi(\mathcal{L}(G), \mathcal{L}(\tilde{G})) := \sum_{i=1}^{\infty} 2^{-i} \frac{|p(G,\alpha_i) - p(\tilde{G},\alpha_i)|}{1 + |p(G,\alpha_i) - p(\tilde{G},\alpha_i)|}$$

for $G, \tilde{G} \in \mathcal{S}^D$. Using this distance function, we can characterize weak convergence of some graph sequence (G_n) by saying that $(\mathcal{L}(G_n))_n$ is Cauchy with respect to d_π.
Moreover, for $D, r \in \mathbb{N}$ we denote by $\mathring{\mathcal{A}}_r^D$ the set of all unrooted isomorphism classes of finite graphs with vertex degree bound D of diameter at most r. Note that the diameter of a finite graph is the longest possible distance between two distinct points. Precisely, a graph $G \in \mathcal{S}^D$ is unrooted isomorphic to a class $\mathring{\alpha} \in \mathring{\mathcal{A}}_r^D$ if there is a bijection between both vertex sets preserving the edge relations in G and $\mathring{\alpha}$. Set $\mathring{\mathcal{A}}^D := \bigcup_{r=1}^{\infty} \mathring{\mathcal{A}}_r^D$. Then, the Equipartition Theorem reads as follows.

Theorem 7.11 (Equipartition Theorem).
Let $\mathcal{P} \subseteq \mathcal{S}^D$ be a hyperfinite family. Then, for all $\varepsilon > 0$, one finds a number $K_\varepsilon \in \mathbb{N}$ such that the following holds true: for each $\beta > 0$, there exists $\delta > 0$ such that if $G = (V_G, E_G) \in \mathcal{P}$ and $\tilde{G} = (V_{\tilde{G}}, E_{\tilde{G}}) \in \mathcal{S}^D$ are such that $d_\pi(\mathcal{L}(G), \mathcal{L}(\tilde{G})) \leq \delta$, then there is a way to delete less than $2\varepsilon\,|E_G|$ edges in G, as well as less than $2\varepsilon\,|E_{\tilde{G}}|$ edges in $\tilde{G}$ such that

- *in the remaining graphs G'_ε and $\tilde{G}'_\varepsilon$, all connected (edge-disjoint) components have vertex size at most K_ε,*
- $\sum_{\mathring{\alpha} \in \mathring{\mathcal{A}}_{K_\varepsilon}^D} |c_{\mathring{\alpha}}^{G'_\varepsilon} - c_{\mathring{\alpha}}^{\tilde{G}'_\varepsilon}| < \beta$,

where $c_{\mathring{\alpha}}^{G'_\varepsilon} := |C_{\mathring{\alpha}}^{G'_\varepsilon}|/|V_G|$ and $C_{\mathring{\alpha}}^{G'_\varepsilon}$ is the set of vertices V_G which lie in a component of G'_ε which is unrooted isomorphic to $\mathring{\alpha}$.

PROOF.
See [Ele12], Theorem 4. □

Remark.
We would like to point out that ELEK shows more in Theorem 4 of [Ele12]. Namely, it is assumed there that one, but not necessarily both of the graphs G and $\tilde{G}$ must belong to the hyperfinite family $\mathcal{P}$. This shows testability of important properties for bounded degree graphs such as planarity. For our purposes, the weaker version given above is sufficient.

Having this result at hand, we are now able to prove Theorem 7.10.

PROOF (OF THEOREM 7.10).
Assume that $(G_n) = (V_{G_n}, E_{G_n})$ is weakly convergent and hyperfinite as a set in $\mathcal{S}^D$. Let $\varepsilon > 0$, set $\varepsilon_1 := \varepsilon/(6D)$ and for this ε_1, choose K_{ε_1} according to the Equipartition Theorem, Theorem 7.11. For every $n \in \mathbb{N}$, we remove at most $2\varepsilon_1\,|E_{G_n}|$ edges of G_n such that in the remaining graphs G'_{n,ε_1}, all connected components consist of at most K_{ε_1} vertices. As in [Ele08a], we call a vertex $v \in V_{G_n}$ *exceptional* for ε_1 if at least one of the edges in E_{G_n} incident to v has been removed. Hence, for every graph G_n, there are at most $2D\varepsilon_1|V_{G_n}|$ exceptional vertices in V_{G_n}. Denote the various unrooted isomorphism classes of diameter

at most K_{ε_1} by $\mathring{\alpha}_1, \mathring{\alpha}_2, \ldots, \mathring{\alpha}_{M_{\varepsilon_1}}$ ($M_{\varepsilon_1} \in \mathbb{N}$). For $n \in \mathbb{N}$ and for $1 \leq i \leq M_{\varepsilon_1}$, we write $\kappa_i^{n,\varepsilon_1} \in \mathbb{N}$ for the number of connected components in G'_{n,ε_1} which are unrooted isomorphic to $\mathring{\alpha}_i$. Set $\gamma_i^{n,\varepsilon_1} := \kappa_i^{n,\varepsilon_1}/|V_{G_n}|$ for $n \in \mathbb{N}$ and $1 \leq i \leq M_{\varepsilon_1}$. With $C_{\mathring{\alpha}_i}^{n,\varepsilon_1}$ being defined as the collection of vertices that lie in a component of G'_{n,ε_1} unrooted isomorphic to $\mathring{\alpha}_i$, we have $\kappa_i^{n,\varepsilon_1} = |C_{\mathring{\alpha}_i}^{n,\varepsilon_1}|/|V(\alpha_i)|$. Since (G_n) is weakly convergent, the sequence $(\alpha(G_n))$ is Cauchy in the distance function d_π and Theorem 7.11 is applicable. Hence for $\beta := \varepsilon_1/(K_{\varepsilon_1} M_{\varepsilon_1})$, we find some number $L \in \mathbb{N}$ such that

$$|\gamma_i^{n,\varepsilon_1} - \gamma_i^{m,\varepsilon_1}| < \beta = \frac{\varepsilon}{6DM_{\varepsilon_1}K_{\varepsilon_1}} \tag{7.1}$$

for all $n, m \geq L$ and every $1 \leq i \leq M_{\varepsilon_1}$. It remains to show that $\delta(G_n, G_m) \leq \varepsilon$ for $n, m \geq L$. To do so, we follow the lines of the proof of Lemma 2.3 in [Ele08a]. Take $n, m \geq L$. We denote by H_n the graph given by $|V_{G_m}|$ disjoint copies of G_n and by H_m the graph given by $|V_{G_n}|$ disjoint copies of G_m. Then there is no loss in generality to assume that both graphs H_n and H_m are defined on a common vertex set V and that they both have a vertex labeling with numbers $\{1, 2, \ldots, |V|\}$. Analogously, we let H'_{n,ε_1} and H'_{m,ε_1} be the subgraphs consisting of $|V_{G_m}|$ respectively of $|V_{G_n}|$ edge-disjoint copies of G'_{n,ε_1} respectively of G'_{m,ε_1}. Then, there are $|V_{G_m}|\kappa_i^{n,\varepsilon_1}$ components in H'_{n,ε_1} which are unrooted isomorphic to $\mathring{\alpha}_i$ and $|V_{G_n}|\kappa_i^{m,\varepsilon_1}$ components in H'_{m,ε_1} which are unrooted isomorphic to $\mathring{\alpha}_i$. Set $q_{i,\varepsilon_1} := \min\{|V_{G_m}|\kappa_i^{n,\varepsilon_1}; |V_{G_n}|\kappa_i^{m,\varepsilon_1}\}$ and $Q_{i,\varepsilon_1} := \max\{|V_{G_m}|\kappa_i^{n,\varepsilon_1}; |V_{G_n}|\kappa_i^{m,\varepsilon_1}\}$. Further, for each i, choose q_{i,ε_1} edge-disjoint components in H'_{n,ε_1} which are unrooted isomorphic to $\mathring{\alpha}_i$. If for $v \in V$, there is no $1 \leq i \leq M_{\varepsilon_1}$, such that v is contained in one of the chosen components of H'_{n,ε_1} isomorphic to $\mathring{\alpha}_i$, then call v a *non-matching* (cf. [Ele08a], proof of Lemma 2.3) vertex. Now, find a bijective map $\sigma \in \mathrm{Sym}(|V|)$ such that there is a subset V' of V of vertex size $\sum_i q_{i,\varepsilon_1} |V(\alpha_i)|$ such that σ restricted to V' preserves the star relations between H'_{n,ε_1} and H'_{m,ε_1}, i.e. the edge relations including the number labeling. On the numbers associated with vertices in V which are non-matching for H'_{n,ε_1} or for H'_{m,ε_1}, we may define σ arbitrarily. We show that

$$\delta_V(H_n, H_m^\sigma) \leq \varepsilon.$$

To see this, observe that if v is neither non-matching nor exceptional for ε_1 in H'_{n,ε_1} and neither non-matching nor exceptional for ε_1 in $H^{\sigma\prime}_{m,\varepsilon_1}$, then its star including the vertex numbering is the same in H_n and as in H_m^σ respectively. The number of non-matching vertices is bounded by

$$\sum_{i=1}^{M_{\varepsilon_1}} (Q_{i,\varepsilon_1} - q_{i,\varepsilon_1})\, K_{\varepsilon_1}.$$

By multiplication with $|V|$, we deduce from inequality (7.1) that

$$|Q_{i,\varepsilon_1} - q_{i,\varepsilon_1}| < \frac{\varepsilon}{6D\, M_{\varepsilon_1} K_{\varepsilon_1}}\, |V|$$

for all $1 \leq i \leq M_{\varepsilon_1}$. By construction, the union of the exceptional vertices in H_n and in H_m^σ has cardinality less than $4\varepsilon_1\, D\, |V|$. With our estimate for the non-matching vertices, we conclude that

$$\delta_V(H_n, H_m^\sigma) \leq \frac{4\varepsilon_1\, D\, |V|}{|V|} + \frac{\varepsilon}{6D} < \varepsilon.$$

Finally, passing to infima, we arrive at

$$\delta(G_n, G_m) \leq \delta_V(H_n, H_m^\sigma) < \varepsilon.$$

This finishes the proof of the theorem. □

In [Ele08a], the author considers graphs with their edges and their vertices being labeled by finitely many colours. In this context, weak convergence means that for all *coloured* isomorphism classes, the occurrence frequencies exist in the limit. Thus, if one proved the Equipartition Theorem for coloured graph sequences, one could use the same method as above to solve the full conjecture of Elek claiming that weakly convergent, hyperfinite *coloured* graph sequences are strongly convergent (note that in this situation, the stars in the graphs to measure the δ-distance take the vertex und edge colourings into account in the canonical manner). We have not pursued this goal in this thesis. A positive answer to this issue can then also be considered as an extension of Theorem 4.4 in the finitely generated situation. Precisely, one would be able to prove a Banach space convergence theorem for almost-additive functions along hyperfinite, weakly convergent coloured graph sequences. With the methods at our disposal, we can prove the convergence for uncoloured graph sequences. This is the topic of the next chapter.

8 Convergence theorems for graphs

In this chapter, we prove two convergence theorems along weakly convergent, hyperfinite (i.e. strongly convergent) graph sequences. The first result refers to almost-additive, Banach space-valued functions on some set $\mathcal{S}^D$ consisting of finite graphs with uniform vertex degree bound $D \in \mathbb{N}$. In this context, we discover that a natural notion for almost-additivity is given by a particular continuity property with respect to the distance function δ introduced in the previous chapter. It turns out that there is a nice analogy to our Banach space convergence theorem for countable amenable groups, see Chapter 4. Namely, considering finitely generated, amenable groups and sticking to the uncoloured situation, the convergence theorem, Theorem 8.2, developed in this chapter can be interpreted as an extension of Theorem 4.4. Being the first assertion of its kind for graph sequences, Theorem 8.2 is a breakthrough in combinatorial approximation theory.
Secondly, we turn to the issue of subadditive convergence for graphs. Subadditive convergence theorems play a major role in proving the existence of invariants in topological dynamical systems, cf. e.g. [Gro99, LW00, CSKC12]. Using Theorem 7.10, we prove normalized convergence along weakly convergent, hyperfinite graph sequences for a large class of subadditive functions on $\mathcal{S}^D$, cf. Theorem 8.4. As far as the underlying geometry is concerned, this provides a significant extension of variants of the 'Ornstein-Weiss Lemma' for amenable groups [Gro99, LW00, Kri10] and semigroups [CSKC12]. All results of this chapter are contained in [Pog13b].

8.1 Almost-additive, Banach space convergence

In this section, we provide an almost-additive Banach space-valued convergence theorem for almost-additive functions. They will be defined on some set $\mathcal{S}^D$ consisting of finite graphs with uniform vertex degree bound $D \in \mathbb{N}$. This assertion is analogous to Theorem 4.4 for uncoloured, finitely generated groups.

Definition 8.1 (Almost-additive functions on graphs).
Let $(Z, \|\cdot\|)$ be an arbitrary Banach space and denote by $\mathcal{P}$ some subset of $\mathcal{S}^D$. A mapping

$$F : \mathcal{S}^D \to Z$$

is called almost-additive *on $\mathcal{P}$ if $F(\tilde{\emptyset}) = 0$ for the empty graph $\tilde{\emptyset}$ (in case $\tilde{\emptyset} \in \mathcal{S}^D$) and if there is a constant C depending on D such that*

$$\|p\, F(G) - q\, F(\tilde{G})\| \leq C\, \delta_S(pG, q\tilde{G})\, p\, |V_G|,$$

whenever $G, \tilde{G} \in \mathcal{P}$ and $p, q \in \mathbb{N}$ are such that $p\,|V_G| = q\,|V_{\tilde{G}}|$. (The distance function δ_S is defined as in the previous chapter.)

We are now in position to prove the convergence result.

Theorem 8.2 (Almost-additive convergence theorem for graph sequences). *Suppose that F is a mapping on $\mathcal{S}^D$ with values in some Banach space $(Z, \|\cdot\|)$. If $(G_n) = (V_n, E_n)$ is a weakly convergent, hyperfinite graph sequence in $\mathcal{S}^D$, and if F is almost-additive on $\mathcal{P} := \{G_n \,|\, n \in \mathbb{N}\}$, then there is some $F^* \in Z$ such that*

$$\lim_{n\to\infty} \left\| \frac{F(G_n)}{|V_n|} - F^* \right\| = 0.$$

Proof.
Take a sequence $(G_n) = (V_n, E_n)$ in $\mathcal{S}^D$ which is hyperfinite and weakly convergent. By Theorem 7.10, (G_n) is strongly convergent as well. Fix $\varepsilon > 0$ and find some integer $L \in \mathbb{N}$ such that $\delta(G_n, G_m) < \varepsilon$ for $n, m \geq L$. By definition, we find $p, q \in \mathbb{N}$ with $p|V_n| = q|V_m|$ such that $\delta_S(pG_n, qG_m) \leq 2\varepsilon$. Since F is almost-additive on (G_n) for some constant $C > 0$, we obtain

$$\left\| \frac{F(G_n)}{|V_n|} - \frac{F(G_m)}{|V_m|} \right\| = \left\| \frac{p\,F(G_n)}{p\,|V_n|} - \frac{q\,F(G_m)}{q\,|V_m|} \right\| \leq 2C\,\varepsilon$$

for all $n, m \geq L$. Therefore, the sequence $(F(G_n)/|V_n|)_n$ is Cauchy in the Banach space Z and thus, it converges to some limit $F^* \in Z$. □

Let us briefly explain the links of the above result to Theorem 4.4. Consider the Cayley graph $G := \mathrm{Cay}(\Gamma, S)$ induced by an amenable group Γ with finite generating system $S \subset \Gamma$. Further, suppose that Γ is coloured trivially by a constant map $\mathcal{C} : \Gamma \to \{a\}$, where a is the only colour at disposal. Then, the setting described in Chapter 4 fits into the context of Theorem 8.2. To see this, note first that every mapping

$$F : \mathcal{F}(\Gamma) \to Z$$

can be rewritten as

$$F' : \mathcal{S}^{|S|}(\Gamma) \to Z$$

by identifying all finite sets in $Q \in \mathcal{F}(\Gamma)$ with their induced subraphs $G(Q)$ in G such that

$$\mathcal{S}^{|S|}(\Gamma) := \{G(Q) \,|\, Q \in \mathcal{F}(\Gamma)\}$$

and $F'(G(Q)) = F(Q)$ for all $Q \in \mathcal{F}(\Gamma)$. Let (U_j) be a Følner sequence in Γ. Then $(G(U_j))_j$ is weakly convergent and hyperfinite. Now, Theorem 8.2 tells us that it is sufficient to show that F' is almost-additive on $\{G(U_j) \,|\, j \in \mathbb{N}\}$. This is proven in part (iii) of Theorem 4.4 (even for the situation where Γ is endowed with an ergodic colouring). There we use the ε-quasi tiling theory to cut the U_j into ε-disjoint translates of T_i^ε and asymptotically (with j getting large), the occurrence frequencies of the different ε-quasi tiles stabilize. This implies that there is some $j_0 \in \mathbb{N}$ such that for $j, \tilde{j} \geq j_0$, the graphs $G(U_j)$ and $G(U_{\tilde{j}})$ are close in the δ-distance.

The extension of Theorem 7.10 to the case of coloured graph sequences is an open problem. Note that the proof essentially relies on the fact that (G_n) is Cauchy in δ. Therefore, if the theorem holds true also for coloured graphs, we would immediately obtain a Banach space convergence theorem that gives an extension of Theorem 4.4 (for finitely generated groups). As explained before, this would require an Equipartition Theorem for hyperfinite, coloured graph sequences.

8.2 Subadditive convergence

In classical analysis, Fekete's Lemma says that for every sequence (a_n) of real numbers with $a_{n+m} \leq a_n + a_m$ $(n, m \in \mathbb{N})$, the sequence (a_n/n) converges to its infimum (which may be $-\infty$). For the sake of applications, it is an important question whether one can replace the index set of the sequence by more complicated structures such as sets or graphs. In Theorem 1.1 of [CSKC12], the authors prove convergence along Følner nets for functions on left-cancellative amenable semigroups. Using this result, they obtain the existence of topological entropy and topological mean dimension for continuous dynamical systems. We prove a corresponding subadditive convergence assertion in the context of weakly convergent, hyperfinite graph sequences. To do so, we have to accept an additional monotonicity condition on the subadditive function F under consideration. More precisely, we need F to be non-decreasing with respect to the subgraph relation. However, unlike in [CSKC12], we have to assume subadditivity only for disjoint decompositions of graphs. In this context, Theorem 8.4 extends a variant of the 'Ornstein-Weiss Lemma' which was used in the framework of amenable group dynamical systems theory, cf. [OW87, Gro99, LW00].

We start with the definition of a subadditive function on $\mathcal{S}^D$. Note that $G = (V, E)$ is a subgraph of $\tilde{G} = (\tilde{V}, \tilde{E})$ if $V \subseteq \tilde{V}$ and if there is an injective map $\varphi : V \to \tilde{V}$ with the property that $(x, y) \in E$ implies $(\varphi(x), \varphi(y)) \in \tilde{E}$.

Definition 8.3 (Subadditive functions on graphs).
Let $\mathcal{P} \subseteq \mathcal{S}^D$. A mapping $h : \mathcal{S}^D \to \mathbb{R}$ is subadditive *if it satisfies the following properties.*

- *There is a constant $C > 0$ such that $h(G) \leq C\,|V|$ for all $G = (V, E) \in \mathcal{S}^D$* (boundedness).
- *If G is a subgraph of $\tilde{G} \in \mathcal{S}^D$, then $h(G) \leq h(\tilde{G})$* (monotonicity).
- *If $G = (V, E)$ and $G' = (V', E')$ are both subgraphs of some $\tilde{G} = (\tilde{V}, \tilde{E}) \in \mathcal{S}^D$ such that $\tilde{V}$ is the disjoint union of V and V', then*

  $$h(\tilde{G}) \leq h(G) + h(G') \quad \text{(subadditivity)}.$$

  If in addition, we are in the edge-disjoint situation, i.e. $e \in \tilde{E}$ if and only if either $e \in E$ or $e \in E'$, *then we have in fact*

  $$h(\tilde{G}) = h(G) + h(G') \quad \text{(special additivity)}.$$

- *If $G \in \mathcal{P}$ is (unrooted) isomorphic to $G' \in \mathcal{P}$, then $h(G) = h(G')$* (pattern-invariance).

At a first glance, the special additivity assumption may look a bit strong. However, in most settings it appears just as a technicality, e.g. if one considers induced subgraphs of connected graphs. In the proof of the convergence theorem, we have to make use of this special criterion at one technical point. Namely, we will have to work with the distance function δ which forces us to consider q-fold ($q \in \mathbb{N}$) vertex- and edge-disjoint copies of graphs $G \in \mathcal{S}^D$. For those objects, the special additivity condition in combination with the pattern-invariance condition makes sure that $h(qG) = qh(G)$.

Theorem 8.4 (Subadditive convergence theorem for graph sequences).
Let $(G_n) = (V_n, E_n)$ be a weakly convergent, hyperfinite graph sequence in $\mathcal{S}^D$ and assume that $h : \mathcal{S}^D \to \mathbb{R}$ is subadditive. Then, there is an element $\lambda \in \mathbb{R} \cup \{-\infty\}$ such that

$$\lim_{n\to\infty} \frac{h(G_n)}{|V_n|} = \lambda.$$

Proof.
Let $(G_n) = (V_n, E_n)$ be a weakly convergent, hyperfinite graph sequence in $\mathcal{S}^D$. By Theorem 7.10, (G_n) is in fact strongly convergent. Define

$$\lambda := \liminf_{n\to\infty} \frac{h(G_n)}{|V_n|}.$$

Due to the boundedness of h, we have $\lambda \in [-\infty, C]$, where C is the boundedness constant for h. Note that $\lambda = -\infty$ is possible as well. Denote by $K \subseteq \mathbb{N}$ an infinite set containing the indices of some subsequence of $(h(G_n)/|V_n|)$ that converges to λ (or diverges to $-\infty$). Now fix an arbitrary $\varepsilon > 0$. By strong convergence, we find some $k_0 \in K$ such that for all $n, k \geq k_0$, we have $\delta(G_n, G_m) < \varepsilon$. We fix such integers n and k and we also make sure that $k \in K$. Take a pair of integers $q_n, q_k \in \mathbb{N}$ such that $q_n|V_n| = q_k|V_k|$ and

$$\delta_S(q_n\, G_n, q_k\, G_k) \leq 2\varepsilon. \tag{8.1}$$

With no loss of generality, we may assume that the graphs $q_n\, G_n$ and $q_k\, G_k$ are defined on a common vertex set $V_{n,k}$. Then we find a subset $V'_{n,k} \subseteq V_{n,k}$ containing those vertices such that the labeled stars (i.e. 1-balls including vertex numberings in both graphs) coincide in both graphs. By inequality (8.1), we obtain $|V'_{n,k}| \geq (1-2\varepsilon)|V_{n,k}|$. For this set $V'_{n,k}$, we denote the corresponding induced subgraph in $q_n\, G_n$ (respectively in $q_k\, G_k$) by $G'_{n,k}$. Using the boundedness, subadditivity, special additivity, as well as the pattern invariance property of h, we get

$$\frac{h(G_n)}{|V_n|} = \frac{h(q_n\, G_n)}{q_n\, |V_n|} \leq \frac{h(G'_{n,k})}{|V_{n,k}|} + 2C\,\varepsilon. \tag{8.2}$$

As h is monotone, it follows that $h(G'_{n,k}) \leq h(q_k\, G_k)$. Now the special additivity property and the pattern-invariance of h yield $h(G'_{n,k}) \leq q_k\, h(G_k)$. Thus, we deduce from inequality (8.2) that

$$\frac{h(G_n)}{|V_n|} \leq \frac{q_k\, h(G_k)}{|V_{n,k}|} + 2C\,\varepsilon = \frac{h(G_k)}{|V_k|} + 2C\,\varepsilon.$$

Since $k \in K, n \in \mathbb{N}, n, k \geq k_0$ were chosen arbitrarily, the latter inequality holds true for all large enough $k \in K$ and every large enough $n \in \mathbb{N}$. We conclude that

$$\limsup_{n\to\infty} \frac{h(G_n)}{|V_n|} \leq \liminf_{k\to\infty} \frac{h(G_k)}{|V_k|} + 2C\,\varepsilon = \lambda + 2C\,\varepsilon.$$

Thus, sending $\varepsilon \to 0$ finishes the proof. □

9 Spectral approximation for graphs

In this chapter, we present some known results about spectral approximation in a structured way fitting in the context of the previous chapters. We do not come up with new results, but give direct proofs via calculations with explicit expressions for all involved quantities. The chapter is divided into two parts.

Firstly, we associate to each graphing some canonical von Neumann algebra. The latter space consists of those operators which act fiberwise on the connected components of the graphing. This will lead us to the concept of so-called *Carleman operators*. Moreover, we are able to define a notion of a trace on the von Neumann algebra. To do so, we stick to the construction given in [LPS14] for measure graphs over a groupoid. The corresponding operator algebraic considerations in this latter article rely on the non-commutative integration theory developed by Connes in the form discussed in [LPV07]. For similar results on operator algebras on measurable equivalence relations, see e.g. the papers [FM77, Ele08a, Ele08b].

In the second part, we turn to the question of the IDS approximation for operators on sofic graphings along their finite analogues on weakly convergent graph sequences. We start by proving weak convergence of the empirical spectral distributions towards the trace of the spectral family. This result has been proven before, see e.g. [Ele08a, Ele08b]. Having an explicit formula for the IDS of Carleman operators on graphings at hand, we are able to give a slightly more structural proof below which is based on measuring the geometric differences between the finite graphs in the graph sequence and the random geometric patterns in the graphing. Briefly describing the Lück conjecture for weakly convergent graph sequences (see e.g. [ATV13]), we give a short outline concerning the issue of uniform convergence in possibly non-hyperfinite situations. While partial results on this matter have been obtained through algebraic tools (cf. [Tho08, ATV13]), the full conjecture, as well as a geometric approach to a possible solution remain open. The main result of this chapter is Theorem 9.12, where we show that for finite range Carleman operators on sofic *hyperfinite* graphings, we can approximate the IDS uniformly. This is a variant of the convergence result in [Ele08a] for strongly convergent graph sequences. Our proof relies on a different method, namely the Banach space-valued convergence theorem from the previous chapter, cf. Theorem 8.2. Being valid for all hyperfinite, weakly convergent graph sequences, it provides a direct approach describing the combinatorial structure behind spectral approximation results. Moreover, Theorem 9.12 generalizes previous assertions, see e.g. [Ele06a, DLM$^+$03, Ele06b]. In the context of uncoloured Cayley graphs of amenable groups, it also extends the spectral approximation results in [LSV11, PS14].

9.1 Operators on graphings

The von Neumann algebra of a graphing

In the following, we obtain a canonical von Neumann algebra arising naturally from graphings. It turns out that this von Neumann algebra comes along with a finite and faithful trace. The results are a special instance of the more general concept of measure graphs over some measurable groupoid, cf. [LPS14].

We start with a graphing $\mathcal{G} = (X, \mu, I_k)$ (see Definition 7.3) with the group $\Gamma := \langle I_k \rangle$ acting measure preservingly on X. As discussed after Definition 7.3, every graphing induces a measure graph $G = (V, E)$ with M being the corresponding finite measure on V. For every $x \in X$, we define

$$V^x := \{x\} \times \Gamma x,$$

which clearly is a subset of V. Now for $f \in L^2(V, M)$, there is a bundle $(f_x)_{x \in X}$ of mappings $f_x \in \ell^2(V^x)$ such that $f((x, \gamma x)) = f_x((x, \gamma x))$ for μ-almost every $x \in X$ and each $\gamma \in \Gamma$. There is a canonical way to write $L^2(V, M)$ as a direct integral over the bundle spaces, i.e.

$$L^2(V, M) \simeq \int_X^{\oplus} \ell^2(V^x)\, d\mu(x).$$

For a more detailed discussion of this latter fact, see e.g. [Con79, LPV07]. More background material in direct integral theory can e.g. be found in [Dix81].

We will make use of this decomposition in order to consider bounded, linear operators on $L^2(V, M)$ acting as a graph operator on the fibers V^x. This leads to the notion of *decomposable operators*.

Definition 9.1.
Let $\mathcal{G} = (X, \mu, I_1, \ldots, I_D)$ be a graphing with induced measure graph (G, M), where $G = (V, E)$. Then, we call a bounded, linear operator $H : L^2(V, M) \to L^2(V, M)$ decomposable *if for μ-almost every $x \in X$, there exists a bounded, linear operator $H^x : \ell^2(V^x) \to \ell^2(V^x)$ such that for $v \in V^x$, we have $(Hf)(v) = H^x f_x(v)$. In this situation, we also use the direct integral notation*

$$H := \int_X^{\oplus} H^x\, d\mu(x).$$

Further, we call (H^x) a decomposition for H and we write $H \simeq (H^x)$.

We still need an equivariance condition for the operators under consideration. So for $x \in X$ and $\gamma \in \Gamma$, define the unitary operator $U_{x,\gamma}$ as

$$U_{x,\gamma} : \ell^2(V^x) \to \ell^2(V^{\gamma x}) : (U_{x,\gamma} h)((y, \gamma' y)) := h((\gamma^{-1} y, \gamma^{-1}\gamma' y)).$$

Note that we could also write x instead of $\gamma^{-1} y$. Now, we are in the position to define a class of operators which is suitable for our purposes.

Definition 9.2 (Bounded random operators).
Let $\mathcal{G} = (X, \mu, I_1, \dots, I_D)$ be a graphing with induced measure graph (G, M), where $G = (V, E)$. Suppose that $H : L^2(V, M) \to L^2(V, M)$ is decomposable. Then, we say that H is a bounded random operator *on the graphing $\mathcal{G}$ if there is a decomposition $H \simeq (H^x)_{x\in X}$, $H^x : \ell^2(V^x) \to \ell^2(V^x)$ such that*

- *the mapping $x \mapsto \langle f_x, H^x g_x\rangle_{\ell^2(V^x)}$ is measurable for all $f, g \in L^2(V, M)$,*
- *there exists a constant $C > 0$ such that $\|H^x\| \leq C$ for μ-almost every $x \in X$,*
- *the equivariance condition is satisfied, i.e.*

$$H^{\gamma x} = U_{x,\gamma} H^x U^*_{x,\gamma}$$

for all $x \in X$ and each $\gamma \in \Gamma$.

In this context, we say that two decompositions (H^x) and $(\tilde{H}^x)$ are equivalent if $H^x = \tilde{H}^x$ holds true μ-almost-surely. Therefore, we will from this point on identify bounded random operators with the equivalence class of its decompositions, i.e. $H = [H^x]$. For the operator norm, we define

$$\|H\| := \inf \{C \geq 0 \,|\, [H^x] = H \text{ and } \|H^x\| \leq C \ \mu\text{-a.e.} \}.$$

Considering the usual addition and multiplication on the fibers, one notes that the set $\mathcal{N}(V, \mathcal{G})$ of all bounded, random operators on the graphing $(V, \mathcal{G}, M)$ is an algebra. Even more can be said.

Theorem 9.3.
The set $\mathcal{N}(V, \mathcal{G})$ is a von Neumann algebra.

Proof.
See e.g. [Con79], Theorem V.2. □

Denote by $\mathcal{N}^+(V, \mathcal{G})$ the set of all non-negative, self-adjoint operators in $\mathcal{N}(V, \mathcal{G})$. Here, we say that $H \in \mathcal{N}(V, \mathcal{G})$ is non-negative if the spectrum of H is contained in $[0, \infty)$. Then, for each $H \in \mathcal{N}^+(V, \mathcal{G})$, there is a canonical notion of a trace.

Definition 9.4 (Trace on $\mathcal{N}^+(V, \mathcal{G})$).
Let $\mathcal{G} = (X, \mu, I_1, \dots, I_D)$ be a graphing with induced measure graph (G, M), where $G = (V, E)$. Then, for $H \in \mathcal{N}^+(V, \mathcal{G})$, we define the trace *$\tau(H)$ of H as*

$$\tau(H) := \int_X \langle \delta_{(x,x)}, H^x \delta_{(x,x)}\rangle_{\ell^2(V^x)} \, d\mu(x),$$

where $(H^x)_{x\in X}$ is an arbitrary decomposition with $H \simeq (H^x)$ and $\delta_{(x,x)} : V^x \to \{0, 1\}$ is the usual delta function giving weight 1 only to the element (x, x).

Note that the mapping $\tau : \mathcal{N}^+(V, \mathcal{G}) \to [0, \infty] : H \mapsto \tau(H)$ is a *weight* on $\mathcal{N}(V, \mathcal{G})$, i.e. a (positive) linear functional on $\mathcal{N}^+(V, \mathcal{G})$. Following the lines of [LPS14], it can be readily checked that τ gives in fact rise to a trace on the von Neumann algebra $\mathcal{N}(V,\mathcal{G})$ in the classical sense. Moreover, this trace is *finite* (i.e. it attaches a finite number to the identity operator) and *faithful* (i.e. $\tau(H) = 0$ implies that $H = 0$), see Theorem 7.6 in [LPS14]. Also, there is a unique extension of τ to a continuous map on the whole algebra $\mathcal{N}(V,\mathcal{G})$.

Carleman operators

It turns out that the structure of the elements in $\mathcal{N}(V,\mathcal{G})$ can be described more explicitly. More precisely, they can be represented as operators on $L^2(V,M)$ with a decomposable kernel function. Those elements are called Carleman operators, see e.g. [Wei80, LPS14]. In the following, denote by π the measurable projection of the elements in V to its first coordinate, i.e.

$$\pi : V \to X : v = (x, \gamma x) \mapsto x.$$

Definition 9.5 (Carleman operators).
Let $\mathcal{G} = (X, \mu, I_1, \dots, I_D)$ be a graphing with induced measure graph (G,M), where $G = (V,E)$. An operator H on $L^2(V,M)$ is called Carleman operator *if there is a measurable function on the space $V \times V$ with*

$$h(v,\cdot) \in \ell^2(V^{\pi(v)})$$

for all $v \in V$ such that for every $f \in L^2(V,M)$, one obtains

$$Hf(v) = \sum_{w \in V^{\pi(v)}} h(v,w)\, f(w) =: (H^{\pi(v)} f_{\pi(v)})(v)$$

in the L^2-sense. The function h is called the kernel function *for the Carleman operator H.*

Note that in the above definition, one can use an arbitrary decomposition $(H^x)_{x\in X}$ of H. We denote by $\mathcal{K}$ the set of all Carleman operators in the von Neumann algebra $\mathcal{N}(V,\mathcal{G})$ such that the kernel function h is compatible with the action of Γ on V in the sense that

$$h(\gamma \cdot v, \gamma \cdot w) = h(v,w),$$

where $\gamma \cdot v := (\gamma x, \gamma\gamma' x)$ for $v = (x, \gamma' x) \in V$.

It is known that $\mathcal{K}$ is a right ideal in $\mathcal{N}(V,\mathcal{G})$, see Proposition 4.4 in [LPV07]. Since the identity operator also belongs to $\mathcal{K}$ with kernel function $h(v,w) = 1$ if $v = w$ and $h(v,w) = 0$ otherwise, one directly obtains the following.

Proposition 9.6.
Let $\mathcal{G} = (X, \mu, I_1, \dots, I_D)$ be a graphing with induced measure graph (G,M), where $G = (V,E)$. Then, every element in the von Neumann algebra $\mathcal{N}(V,\mathcal{G})$ is a Carleman operator in $\mathcal{K}$.

Note that the converse statement of the above proposition is trivial. Hence, one arrives at $\mathcal{N}(V,\mathcal{G}) = \mathcal{K}$. For this reason, there is no loss in generality to consider Carleman operators as the 'right' class of decomposable operators on graphings.

9.2 Convergence of the IDS

The goal is to prove the IDS approximation for pattern-invariant, finite hopping range Carleman operators on sofic graphings along weakly convergent graph sequences. The convergence will be in the sense of weak convergence of (spectral) measures. In this context, our Theorem 9.9 unifies the results in the literature in a concise and explicit way. This will be discussed next.
The issue of spectral convergence along Benjamini-Schramm convergent graph sequences has already been covered before, see for instance the work of Elek. In [Ele08a], the author proves weak convergence on finite analogues towards the integrated density of states of an element in a von Neumann algebra obtained from a GNS construction on graph sequences. In a second paper [Ele08b], the author obtains weak convergence for the spectral distributions of discrete Laplace operators towards an element in the abstract von Neumann algebra constructed from measurable equivalence relations, see e.g. [FM77]. In both cases, the limit is given by a trace expression which coincides with the trace τ on the graphing von Neumann algebra $\mathcal{N}(V, \mathcal{G})$ as introduced above. Using Carleman operators on graphings, we give an explicit description for the arising limits. From the geometric point of view, Theorem 9.9 also extends the deterministic considerations of [ScSc12]. In the latter work, the authors deal with more general, unbounded random operators. We will not pursue this goal in this thesis. However, this seems to be an interesting project for future investigations.
As in the situation of (finitely generated) amenable groups, we will have our main focus on the convergence of spectral distribution functions *uniformly* in all energies $E \in \mathbb{R}$. Precisely, we show as an application of Theorem 8.2 that for self-adjoint, finite range operators on *hyperfinite* graphings, one obtains indeed uniform convergence along weakly convergent, hyperfinite graph sequences.
Passing to operators with matrix valued kernel, with only few steps we could now obtain the Lück approximation in its original formulation (cf. [Lüc94]) for hyperfinite sequences. This also generalizes the convergence statements along Følner sequences in finitely generated groups, see [DLM$^+$03, Ele06b]. For length issues, we refrain from giving detailed descriptions of those latter results, but we refer the interested reader to the mentioned literature.

Weak convergence along Benjamini-Schramm sequences

We start with the definition of finite hopping range Carleman operators. Let $\mathcal{G} = (X, \mu, I_k)$ be a graphing. As usual, we denote the corresponding induced measure graph by $G = (V, E)$ and the finite measure on V is denoted by M. For $x \in X$, we write G^x for the subgraph of G induced by $V^x \subset V$. We say that G^x is the *leaf graph* for x.

Definition 9.7.
Let $\mathcal{G} = (X, \mu, I_1, \dots, I_D)$ be a graphing with induced measure graph (G, M), and $G = (V, E)$. Assume that H is a Carleman operator on $L^2(V, M)$. We say that H is of finite hopping range *if there is a constant $\tilde{R} \in \mathbb{N}$ such that for the kernel function h of H, one has $h(v, w) = 0$ whenever $\pi(v) = \pi(w)$ and $d^{\pi(v)}(v, w) > \tilde{R}$ for $v, w \in V$, where $d^{\pi(v)}$ is the canonical path metric in $G^{\pi(v)}$.*

In the following, we assume that $\mathcal{G}$ is a sofic graphing with approximating sequence $(G_n) := (V_n, E_n)$. Further, denote by H a Carleman operator on $L^2(V, M)$ with finite hopping range $R \in \mathbb{N}$. Further, we require H to satisfy the following invariance condition. There is a constant $R > 0$ such that for all pairs $v, w \in V$ satisfying $B_{2R}^{G^{\pi(v)}}(v) \simeq B_{2R}^{G^{\pi(w)}}(w)$ via some rooted graph isomorphism φ, we have

$$h(z, z') = h(\varphi(z), \varphi(z'))$$

for all choices of z, z' in the vertex set of $B_R^{G^{\pi(v)}}(v)$. This essentially means that the coefficients of the operator depend only on the local geometric patterns occurring in the graphing. In the following, we will refer to those elements as *pattern-invariant*, finite hopping range Carleman operators. Then for large enough $n \in \mathbb{N}$, there is a canonical way to define finite analogues of H on $\ell^2(V_n)$. Let us describe this construction in detail.

Fix an arbitrary $\varepsilon > 0$. Since (G_n) is weakly convergent, we find some number $N(\varepsilon) \in \mathbb{N}$ large enough such that for all $\alpha \in \mathcal{A}^D$ with $\rho(\alpha) \leq 4R$, we have that

$$\left|p(G_n, \alpha) - \mu(X_\alpha)\right| < \varepsilon / D^{4R+1}$$

whenever $n \geq N(\varepsilon)$, where $X_\alpha := \{x \in X \mid B_{\rho(\alpha)}^G((x, x)) \simeq \alpha\}$. Decreasing ε if necessary (thus increasing $N(\varepsilon)$), we have that for $n \geq N(\varepsilon)$ that the sets

$$T(G_n, \alpha) := \{a \in V_n \mid B_{\rho(\alpha)}^{G_n}(a) \simeq \alpha\}$$

are non-empty for all $\alpha \in \bar{\mathcal{A}}_{4R}^D$, where the latter set denotes the collection of those $\alpha \in \mathcal{A}^D$ with $\rho(\alpha) \leq 4R$ and $\mu(X_\alpha) > 0$. We define the following operators on the graphs G_n. Namely,

$$H_n : \ell^2(V_n) \to \ell^2(V_n) : (H_n u)(a) := \sum_{b \in V_n} h^n(a, b)\, u(b),$$

where the kernel function $h^n(a, b)$ is given by

$$h^n(a, b) := \begin{cases} h_x(\varphi(a), \varphi(b)), & \exists\, v \in V_n, x \in X : a, b \in B_R^{G_n}(v), \\ & B_{4R}^{G_n}(v) \simeq^{\varphi} B_{4R}^{G_x}((x, x)) \simeq \alpha,\ \mu(X_\alpha) > 0 \\ 0, & \text{else.} \end{cases}$$

We need to show that the kernel functions are well-defined, i.e. that they do not depend on the choices for $v \in V_n$ and $x \in X$.

Lemma 9.8.
Suppose that the operators (H_n) ($n \in \mathbb{N}$) are as described above. Then, for every $n \in \mathbb{N}$ and for all vertices $a, b \in V_n$, the value $h^n(a, b)$ does not depend on the choice of the $v \in V_n$ given in the above definition.

Proof.
For simplicity, we use in this proof the notation $B_R^G(v)$ both for the subgraph in G induced by the set of vertices with distance at most R from v, as well as for the underlying vertex set itself.

Fix $n \in \mathbb{N}$ and take $a, b \in V_n$. Let v and v' be elements in V_n with associated rooted graph isomorphisms φ and φ', as well as with elements $x, x' \in X$ as given in the definition of the value $h^n(a,b)$. Due to the triangle inequality, we can compute for $w \in B_{2R}^{G_n}(v)$

$$d_{G_n}(w, v') \leq d_{G_n}(w, v) + d_{G_n}(v, a) + d_{G_n}(a, v') \leq 4R$$

and thus, $B_{2R}^{G_n}(v) \subseteq B_{4R}^{G_n}(v')$. Now, set

$$\Phi : \varphi'(B_{2R}^{G_n}(v)) \to \varphi(B_{2R}^{G_n}(v)) : \Phi := \varphi \circ (\varphi'^{-1}_{|\varphi'(B_{2R}^{G_n}(v))}).$$

This is a rooted graph isomorphism of induced subgraphs in the measure graph induced by the graphing. By definition of φ', we have

$$\varphi'(a), \varphi'(b) \in \varphi'(B_R^{G_n}(v)) = B_R^{G^{x'}}(\varphi'(v)).$$

By the invariance property of the kernel function h of the graphing, we obtain

$$h(\varphi'(a), \varphi'(b)) = h(\Phi \circ \varphi'(a), \Phi \circ \varphi'(b)) = h(\varphi(a), \varphi(b)).$$

This shows our claim. □

The previous lemma shows that if H is a pattern-invariant, finite hopping range Carleman operator, then the operator H_n is a well-defined, finite dimensional version of H on $\ell^2(V_n)$. Even more can be deduced. Namely, if H is self-adjoint (symmetric), so is H_n as well for large enough n. So assuming that H is a finite hopping range, self-adjoint Carleman operator and if (H_n) are the approximating operators along some graph sequence converging to the graphing, we can define the empirical eigenvalue distributions of the H_n as follows. Set

$$N_n : \mathbb{R} \to [0,1] : N_n(E) := \frac{|\{\lambda \leq E \,|\, \lambda \text{ is eigenvalue of } H_n|}{|V_n|}$$

and

$$N^* : \mathbb{R} \to [0,1] : N^*(E) := \tau(\mathbb{1}_{]-\infty,E]}(H)),$$

where τ is the trace on $\mathcal{N}^+(V, \mathcal{G})$ introduced above and $\mathbb{1}_{]-\infty,E]}(H)$ denotes the spectral projection of the operator H to the real set $]-\infty, E]$, see also [Wei80]. As in Chapter 7, we call N^* the integrated density of states (IDS) for the operator H. Note that for each $n \in \mathbb{N}$, the functions N_n are elements of the Banach space $\mathcal{C}_{br}(\mathbb{R})$ of all bounded, right-continuous functions endowed with the sup-norm $\|\cdot\|_\infty$.

Theorem 9.9 (Weak IDS convergence).
Let $\mathcal{G} = (X, \mu, I_1, \ldots, I_D)$ be a sofic graphing with induced measure graph (G, M), where $G = (V, E)$. Further, let $(G_n) = (V_n, E_n)$ be a weakly convergent approximating sequence. Now, assume that $H = [H^x]_{x \in X}$ is a self-adjoint, pattern-invariant Carleman operator on $L^2(V, M)$ which is of finite hopping range with parameter $\tilde{R} \in \mathbb{N}$. Then, for all bounded, continuous, real-valued functions f defined on the spectrum $\sigma(H)$ of H, one has

$$\lim_{n\to\infty} |V_n|^{-1} \sum_{a \in V_n} \langle \delta_a, f(H_n)\delta_a \rangle_{\ell^2(V_n)} = \int_X \langle \delta_{(x,x)}, f(H^x)\delta_{(x,x)} \rangle_{\ell^2(V^x)} \, d\mu(x),$$

where all expressions are defined according to the continuous spectral calculus for bounded, self-adjoint operators.

The above theorem shows weak convergence of the empirical spectral measures to the density of states measure of the limit operator H. It follows from a basic result in measure theory that this implies the convergence of the spectral distribution functions in all continuity points E of N^*.

Corollary 9.10.
In the situation of the previous theorem, we obtain

$$\lim_{n\to\infty} N_n(E) = N^*(E)$$

for all continuity points E of N^*, where the $N_n(\cdot)$ are the empirical eigenvalue distribution functions for the operators H_n and $N^*(\cdot)$ denotes the IDS of the operator H.

Proof.
This follows from Theorem 9.9 in combination with Satz 4.12 in [Els05]. □

We now give the proof of Theorem 9.9.

Proof (of Theorem 9.9).
We first prove the theorem for the function f with $f(x) = x^k$ for all $x \in \sigma(H)$, where $k \in \mathbb{N}$ is an integer number. Then note that

$$f(H) = H^k := \underbrace{H \circ \cdots \circ H}_{k}.$$

As $\mathcal{N}(V, \mathcal{G})$ is an algebra, $f(H)$ is also a Carleman operator. Further, computing the kernel by matrix multiplication yields that $f(H)$ is of finite hopping range with parameter $\tilde{R}' := k \cdot \tilde{R}$. For the weakly convergent graph sequence (G_n), construct the operators $f(H_n)$ from $f(H)$ as described above. Then for all large enough $n \in \mathbb{N}$ and $a \in V_n$, the value $\theta_n(a) := \langle \delta_a, H_n^k \delta_a \rangle_{\ell^2(V_n)}$ only depends on the local geometry around a in G_n. Indeed, since H is pattern-invariant, there is a constant $R \in \mathbb{N}$ depending on $\tilde{R}'$ such that if $\alpha \in \mathcal{A}^D$ has radius $\rho(\alpha) = 4R$, then all $a \in T(G_n, \alpha)$ give rise to the same value $\theta_n(a)$. The analogous assertion holds true for all $x \in X$, i.e. $\theta(x) := \langle \delta_{(x,x)}, H^k \delta_{(x,x)} \rangle_{\ell^2(V^x)}$ is constant on each set X_α of positive measure, where $\alpha \in \mathcal{A}^D$ has radius $\rho(\alpha) = 4R$. Moreover, it follows from the construction of the operators $f(H_n)$ that $\theta_n(a) = \theta(x) =: \theta_\alpha$ if there is some $\alpha \in \mathcal{A}^D$ of radius $\rho(\alpha) = 4R$ such that $a \in T(G_n, \alpha)$, $x \in X_\alpha$ and $\mu(X_\alpha) > 0$. Due to the weak convergence of the sequence (G_n) and the connectedness of the G_n, we obtain

$$\begin{aligned}
\lim_{n\to\infty} |V_n|^{-1} \sum_{a \in V_n} \langle \delta_a, f(H_n)\delta_a \rangle_{\ell^2(V_n)} &= \lim_{n\to\infty} \sum_{\substack{\alpha \in \mathcal{A}^D \\ \rho(\alpha)=4R}} \frac{|T(G_n,\alpha)|}{|V_n|}\, \theta_\alpha \\
&= \sum_{\substack{\alpha \in \mathcal{A}^D, \rho(\alpha)=4R \\ \mu(X_\alpha)>0}} \theta_\alpha\, \mu(X_\alpha) \\
&= \int_X \langle \delta_{(x,x)}, f(H^x)\delta_{(x,x)} \rangle_{\ell^2(V^x)}\, d\mu(x). \qquad (9.1)
\end{aligned}$$

This shows the claim for $f(x) = x^k$. By linearity, we can extend the statement to the space $\mathcal{P}(H)$ consisting of all real coefficient polynomials defined on $\sigma(H)$. Since the latter set is compact, we obtain from the Stone-Weierstraß Theorem (cf. [Wer00], Satz VIII.4.7) that $\mathcal{P}(H)$ is dense in the set of all bounded, continuous functions on $\sigma(H)$. We conclude that the limit relation (9.1) also holds true for general bounded, continuous functions on $\sigma(H)$. $\square$

As mentioned before, we are interested in the question whether we can obtain *uniform* convergence of the eigenvalue distribution functions $N_n(E)$. It is well-known that the distribution functions of a sequence of finite measures converges uniformly whenever the measures converge weakly and the distribution functions converge pointwise, cf. e.g. Lemma 6.3 in [LV09]. So knowing that weak convergence holds true, we obtain even uniform convergence of $N_n(\cdot)$ to $N^*(\cdot)$ *if* we can show

$$\lim_{n\to\infty} |V_n|^{-1} \sum_{a\in V_n} \langle \delta_a, \mathbb{1}_{\{E\}}(H_n)\, \delta_a\rangle_{\ell^2(V_n)} = \int_X \langle \delta_{(x,x)}, \mathbb{1}_{\{E\}}(H^x)\delta_{(x,x)}\rangle_{\ell^2(V^x)}\, d\mu(x) \qquad (9.2)$$

for all energies $E \in \mathbb{R}$, where for $n \in \mathbb{N}$ and $x \in X$, the operators $\mathbb{1}_{\{E\}}(H_n)$ and $\mathbb{1}_{\{E\}}(H^x)$ are defined according to the measurable spectral calculus. Note that here,

$$\begin{aligned} \sum_{a\in V_n} \langle \delta_a, \mathbb{1}_{\{E\}}(H_n)\, \delta_a\rangle_{\ell^2(V_n)} &= \operatorname{tr}\big(\mathbb{1}_{\{E\}}(H_n)\big) \\ &= \dim\big(\ker(H_n - E)\big) \end{aligned}$$

for all $n \in \mathbb{N}$ and for every $E \in \mathbb{R}$, where $\operatorname{tr}(\cdot)$ denotes the standard trace for matrices. The question whether the limit relation (9.2) holds true for arbitrary weakly convergent graph sequences is known as the Lück conjecture for Benjamini-Schramm sequences. Using the Banach space almost-additive convergence theorem from the previous section (Theorem 8.2), we solve this question for *hyperfinite* graph sequences in the next subsection. The issue of uniform convergence for possibly non-hyperfinite, weakly convergent graph sequences (Lück approximation for non-hyperfinite graph sequences) is an open problem. A partial answer is given by Thom in [Tho08] (Theorem 4.3), where it is shown that uniform convergence holds true along sofic approximations of groups for operators with algebraic integers as matrix coefficients. The extension of this latter statement to general weakly convergent graph sequences was realized in [ATV13], Theorem 4. The method of the corresponding proofs is given by diophantine approximation techniques. However, this algebraic approach cannot be used for operators with arbitrary complex coefficients. Thus, it is an interesting (and seemingly hard) problem to use geometric tools to prove the Lück conjecture for graph sequences in its full generality.

Uniform convergence in the hyperfinite case

The goal of the following subsection is to prove the uniform approximation of the integrated density of states (IDS) of pattern-invariant, finite hopping range Carleman operators on *hyperfinite* sofic graphings. We do so by applying our almost-additive Banach space-valued ergodic theorem, Theorem 8.2. The corresponding Theorem 9.12 has essentially been stated before as a convergence result for strongly convergent graph sequences, cf. [Ele08a], Proposition 3.2. However, our method of proof detects the underlying combinatorial structure of spectral convergence assertions of this kind. Moreover, we obtain uniform approximation in the more natural formulation of weakly convergent, hyperfinite graph sequences. To show almost-additivity, we stick to the arguments given in [Ele08a]. For the sake of completeness, we sketch the major steps in the proof and emphasize the use of our Banach space-valued convergence theorem. We start with the definition of hyperfinite sofic graphings.

Definition 9.11 (Sofic hyperfinite graphings).
Let $\mathcal{G}$ be a sofic graphing. Then, we say that $\mathcal{G}$ is hyperfinite *if it is a limit graphing of some hyperfinite, weakly convergent graph sequence (G_n).*

Note that one may also define hyperfiniteness for general graphings without referring to approximating sequences, see e.g. [KM04, Ele12]. However, the context of sofic graphings is absolutely sufficient for our purposes. In this situation the graphing will be hyperfinite if and only if one/all approximating sequences are hyperfinite, cf. Theorem 1 in [Ele12] or Theorem 1.1 in [Sch08].

The uniform IDS approximation theorem now reads as follows.

Theorem 9.12 (Uniform IDS approximation for hyperfinite graphings).
Let $\mathcal{G} = (X, \mu, I_1, \dots, I_D)$ be a sofic graphing with induced measure graph (G, M), where $G = (V, E)$ and τ is the canonical trace on the von Neumann algebra $\mathcal{N}(V, \mathcal{G})$. Suppose that $(G_n) := (V_n, E_n)$ is an approximating sequence for $\mathcal{G}$. Then, for every self-adjoint, pattern-invariant Carleman operator $H = [H^x]_{x \in X}$ on $L^2(V, M)$ which is of finite hopping range, one obtains

$$\lim_{n \to \infty} \|N_n - N^*\|_\infty = 0,$$

where the $N_n \in \mathcal{C}_{br}(\mathbb{R})$ are the empirical normalized spectral distribution functions corresponding to the finite analogues (H_n) and $N^*(E) := \tau(\mathbb{1}_{]-\infty,E]}(E))$.

Proof.
For $n \in \mathbb{N}$ and every subset $Q \subseteq V_n$, we denote by H_n^Q the induced operator

$$\ell^2(Q) \to \ell^2(Q) : H_n^Q := p_Q \, H_n \, i_Q,$$

where i_Q and p_Q are the canonical injection and projection respectively, see also Chapter 6. Further, denote by $G(n, Q)$ the subgraph of G_n induced by the set $Q \subseteq V_n$. We set

$$\mathcal{S}^D := \{G(n, Q) \,|\, n \in \mathbb{N}, Q \subseteq V_n\}$$

and define the mapping

$$F : \mathcal{S}^D \to (\mathcal{C}_{br}(\mathbb{R}), \|\cdot\|_\infty) : \; F(G(n, Q))(E) := \big|\{\lambda \le E \,|\, \lambda \text{ eigenvalue of } H_n^Q\}\big|.$$

We claim that F restricted to $\mathcal{S}' := \{G_n \,|\, n \in \mathbb{N}\}$ is almost-additive in the sense of Definition 8.1. Indeed, for $m, n \in \mathbb{N}$, we find $p_m, q_n \in \mathbb{N}$ such that $p_m|V_m| = q_n|V_n|$ and we can assume that the two graphs $p_m \, G_m$ and $q_n \, G_n$ are defined on the same vertex set V labeled with numbers $\{1, \dots, |V|\}$. Further, define in a canonical way $H_m' := \bigoplus_{p_m} H_m$ and

$H'_n := \bigoplus_{q_n} H_n$. It follows from the uniform rank estimate (cf. e.g. [LSV11], Proposition 7.2 or [Ele08a], Lemma 3.6) that there is a constant $C > 0$ depending on R and on D such that

$$\operatorname{rank}\left(H'_m - H'_n\right) < C\,\delta_V(p_m\,G_m, q_n\,G_n)\,|V|.$$

Hence (cf. e.g. [Ele08a], Lemma 3.5), one arrives at

$$\|F'[H'_n] - F'[H'_m]\|_\infty \le C\,\delta_V(p_m\,G_m, q_n\,G_n)\,|V|,$$

where $F'[H'_n](E) := |\{\lambda \le E \,|\, \lambda \text{ eigenvalue of } H'_n\}|$ for $n \in \mathbb{N}$. By passing to infima, we conclude that the same holds true when replacing the distance function δ_V by the distance function δ_S. By the definition of F, we also have $F'[H'_n] = p_n F(G_n)$ for all $n \in \mathbb{N}$. Therefore, F is almost-additive on $\mathcal{S}'$. Now, by Theorem 8.2, there must be some $F^* \in \mathcal{C}_{br}(\mathbb{R})$ such that

$$\lim_{n\to\infty} \|N_n - F^*\|_\infty = \lim_{n\to\infty} \left\| \frac{F(G_n)}{|V_n|} - F^* \right\|_\infty = 0.$$

This shows the claimed convergence. We still need to identify the limit F^* as N^*. Note that by Theorem 9.9, the N_n converge to N^* weakly, i.e. $\lim_n N_n(E) = N^*(E)$ in all continuity points E of N^*. Thus $F^*(E) = N^*(E)$ for all those points $E \in \mathbb{R}$. Due to monotonicity, there are at most countably many points of discontinuity of N^* and for each such point E_0, there is a sequence E_n of continuity points converging to E_0 from the right. Since F^* is right-continuous, we arrive at

$$N^*(E_0) = \lim_{n\to\infty} N^*(E_n) = \lim_{n\to\infty} F^*(E_n) = F^*(E_0).$$

Thus, we have finished the proof of the theorem. □

10 The Ihara Zeta function for graphings

In this chapter, we define the Ihara Zeta function for graphings. In its original form, this Zeta function has been introduced by Ihara in order to count prime elements in certain p-adic groups, see [Iha66b, Iha66a]. Some years later it was discovered by Sunada that there is a natural extension of this concept to finite, regular graphs. Further results concerning the Ihara Zeta function for finite graphs can be found in [Has89, Has90, Has92, Has93, KS00]. In the past decade, various attempts have been made to define the Ihara Zeta function for infinite graphs. For certain periodic graphs, see for instance the works [CMS01, GIL08]. Another approach for weighted graphs can be found in [Dei14]. Recently, it has been shown by Lenz, Schmidt and the author of this thesis that there is a canonical way to define the Ihara Zeta function for the very general class of measure graphs over a groupoid [LPS14]. This gives in particular a natural way to define a Zeta function for graphings and this includes the notions of [CMS01, GŻ04, GIL08, GIL09] as special cases. Further, the article [LPS14] extends some classical results to a very general setting, amongst them determinant formulae and convergence statements.
Here, we will prove the approximation of the Ihara Zeta function for sofic graphings by normalized versions for elements in a weakly convergent graph sequence. Since the geometric quantities under considerations depend only on local patterns, we obtain uniform convergence on compact sets for *all* (and in particular, for possibly non-hyperfinite) graph sequences. The corresponding Theorem 10.5 significantly generalizes the earlier results. In fact, it contains all the approximation statements in [CMS02, GŻ04, GIL08, GIL09] as special cases. Moreover, we give an explicit construction of an approximation sequence for countable graphs endowed with a free, co-finite action by a countable sofic group of automorphisms. This includes and unifies important cases given in the literature, i.e. if the subgroup Γ of the automorphisms is amenable, cf. [GIL08], or if Γ is residually finite and acts on a regular graph, cf. [CMS02].
Theorem 5.3 of [LPS14] expresses the Ihara Zeta function in terms of a determinant formula. Thus, approximation results for the Zeta function can also be interpreted as convergence statements for an abstract notion of determinant. Hence, the question arises if one can prove approximating statements for other kinds of determinants as well. An example is given in Theorem 1.4 of [LT14]. There, the authors prove a corresponding result for the Fuglede-Kadison determinant on the von Neumann algebra associated with a countable, amenable group Γ. The approximation is attained along normalized finite analogues induced by a Følner sequence in Γ. For this notion of determinant, one has to cope with singularity issues in the case of non-invertible operators. Thus, the extension of this result to general (even hyperfinite) weakly convergent sequences seems to be a non-trivial problem.
The results of this chapter are contained in [LPS14].

Let $G = (V, E)$ be a graph, where as before, $E \subseteq V \times V$ is a symmmetric set. If $e =$

$(v_1, v_2) \in E$, then we say that $o(e) := v_1$ is the *origin* of e and $t(e) := v_2$ is the *terminus* of e. A closed path of length $l \in \mathbb{N}$ in G is a sequence $(e_1, e_2, \ldots, e_l)$ of edges $e_j \in E$ such that $t(e_j) = o(e_{j+1})$ for all $1 \leq j \leq l-1$ and $t(e_l) = o(e_1)$. In order to fix the rules for counting cycles in a graph, we need to define some properties for closed paths.

Definition 10.1 (Properties of closed paths).
Let $G = (V, E)$ be a graph and assume that $P := (e_1, e_2, \ldots, e_l)$ is a closed path of length $l \in \mathbb{N}$ in G. Then

- *we say that P is* backtracking *if there is some $1 \leq j \leq l-1$ such that $e_{j+1} = \overline{e}_j$, where $\overline{e}_j$ is the edge linking the vertices of e_j with interchanged roles for its origin and its terminus. A closed path with no backtracking is said to be* proper*;*
- *we say that P has a* tail *if there is a number $k \in \mathbb{N}$ such that $\overline{e}_j = e_{l-j+1}$ for every $1 \leq j \leq k$;*
- *we say that P is* primitive *if it is not obtained by going $k \geq 2$ times around a shorter closed path;*
- *we say that P is* reduced *if it is neither backtracking nor has a tail;*
- *we say that P is a* prime cycle *if it is reduced and primitive.*

In the following, we denote the set of prime cycles of finite length in a graph by $\mathcal{P}$. Note that by 'forgetting' the starting point of some element $P \in \mathcal{P}$, one may consider equivalence classes $[P]$ of prime cycles P. With a slight abuse of notation, we will then write $[P] \in \mathcal{P}$ in order to emphasize that we are interested in prime cycles 'modulo' their starting points.

We now have all necessary tools at our disposal in order to define the Ihara Zeta function for finite graphs. The original definition was given by the following Euler product formula.

Definition 10.2 (Ihara Zeta function for finite graphs).
Let $G = (V, E)$ be a finite graph with vertex degree bound $D \in \mathbb{N}$. Then, the Ihara Zeta function for G is defined as

$$Z_G(u) := \prod_{[P] \in \mathcal{P}} \left(1 - u^{l(P)}\right)^{-1},$$

where $l(P)$ is the length of the cycles represented by $[P]$ and $u \in \mathbb{C}$ with $|u| < (D-1)^{-1}$.

Obviously, one has to verify that the above function is well-defined, i.e. that for $u \in \mathbb{C}$ with $|u| < (D-1)^{-1}$ the above product exists. This can be seen by an alternative representation of $Z_G(u)$. Namely, $Z_G(u)$ can equivalently be written as an exponential function involving the number of finite closed paths in G. For a fixed $l \in \mathbb{N}$ and $p \in V$, we denote by $N_l(p)$ the number of reduced (not necessarily primitive) closed paths of length l in G which start and end at p. We further set

$$N_l := \sum_{p \in V} N_l(p)$$

for $l \in \mathbb{N}$. Then, the following holds true.

Proposition 10.3 (Exp-representation of Z_G).
Let $G = (V, E)$ be a finite graph with vertex degree bound $D \in \mathbb{N}$. Then, the Ihara Zeta function for G can be written as

$$Z_G(u) = \exp\left(\sum_{l=1}^{\infty} \frac{N_l}{l}\, u^l\right)$$

for $u \in \mathbb{C}$ with $|u| < (D-1)^{-1}$.

Proof.
We define the function

$$\zeta_G(u) := \exp\left(\sum_{l\geq 1} \frac{N_l}{l} u^l\right), \quad |u| < (D-1)^{-1},$$

where N_l is the number of closed paths of length l in G without tail and without back-tracking. Note that for $|u| < (D-1)^{-1}$, there exists some $D^* \in \mathbb{R}$ with $D^* > D-1$ and $|u| \leq {D^*}^{-1}$. Since $N_l \leq D(D-1)^{l-1}$ we get

$$N_l\,|u|^l \leq \frac{D}{D^*} \cdot \left(\frac{D-1}{D^*}\right)^{l-1}$$

for all $l \geq 1$. Estimating with the geometric series, we observe that the series occurring in the definition of $\zeta_G(u)$ exists. Further, we compute

$$\begin{aligned}
\log \zeta_G(u) &= \sum_{l\geq 1} \frac{N_l}{l} u^l = \sum_{\textit{reduced } C} \frac{u^{l(C)}}{l(C)} \\
\text{(passing to primitive loops)} &= \sum_{P\in\mathcal{P}} \sum_{j\geq 1} \frac{u^{l(P^j)}}{l(P^j)} = \sum_{P\in\mathcal{P}} \sum_{j\geq 1} \frac{u^{jl(P)}}{jl(P)} \\
\text{(passing to primitive classes)} &= \sum_{[P]\in\mathcal{P}} \sum_{j\geq 1} \frac{u^{jl(P)}}{j} \\
\text{(logarithmic series)} &= -\sum_{[P]\in\mathcal{P}} \log\left(1 - u^{l(P)}\right) \\
&= \log\left(\prod_{[P]\in\mathcal{P}} \left(1 - u^{l(P)}\right)^{-1}\right),
\end{aligned}$$

where $l(C)$ and $l(P)$ denote the lengths of the closed paths C and P respectively. Taking exponentials yields $\zeta_G(u) = Z_G(u)$. This finishes the proof. □

Note that for infinite graphs, there might exist lengths $l \in \mathbb{N}$ such that $N_l = \infty$. Hence, the corresponding Ihara Zeta function cannot be just defined in the same manner as in the finite case. We will deal with this issue by normalization. Precisely, for a finite graph $G = (V, E)$, we define the normalized version of $Z_G(u)$ as

$$Z_{G,norm}(u) := \exp\left(\sum_{l=1}^{\infty} \frac{N_l}{|V|}\, \frac{u^l}{l}\right).$$

We immediately observe that $Z_{G,norm}(u)^{|V|} = Z_G(u)$. Hence, the values $Z_{G,norm}(u)$ could be interpreted as some $|V|$-th root of the values $Z_G(u)$. However, taking roots in the field of complex numbers, we might find several solutions. Therefore, we refrain from giving a definition involving rational (non-integer) powers of $Z_G(u)$.

Now, one idea to define the Ihara Zeta function for a (countably) infinite graph $G = (V, E)$ (with vertex degree bound $D \in \mathbb{N}$) is to exhaust G by induced subgraphs (G_n) of finite volume and to verify that the limit

$$Z_G(u) := \lim_{n\to\infty} Z_{G_n,norm}(u)$$

exists in a suitable topology. In [GIL09], the authors prove the existence of this limit along some Følner type sequence $(G_n) = (V_n, E_n)$ of finite, self-similar graphs. To do so, the authors show that there is a notion of a trace τ defined for certain finite hopping range operators on $\ell^2(V)$ such that for each such operator H, one obtains τ from the limit process

$$\tau(H) := \lim_{n\to\infty} \frac{\operatorname{tr}\left(P(V_n)H\right)}{|V_n|},$$

where $P(V_n)H = p_{V_n} H i_{V_n}$ denotes the canonical projection of H to $\ell^2(V_n)$ and $\operatorname{tr}(\cdot)$ stands for the natural finite dimensional trace. It is not hard to see that the latter limit relation implies that the sequence (G_n) is in fact weakly convergent. These examples raise the question whether it is possible to extend this definition to a Zeta function for limit graphings. It turns out that this is true. We can even follow the more elegant way to define the Ihara Zeta function for arbitrary graphings and we show afterwards that in the sofic situation, these functions are approximated in uniform convergence on compact sets by the corresponding normalized versions of the elements of the approximating sequence (G_n).

Definition 10.4 (Ihara Zeta function for graphings).
Let $\mathcal{G} = (X, \mu, I_1, \dots, I_D)$ be a graphing with induced measure graph (G, M), where $G = (V, E)$. Then we define the Ihara Zeta function $Z_{\mathcal{G}}$ for this graphing as

$$Z_{\mathcal{G}}(u) := \exp\left(\sum_{l=1}^{\infty} \frac{\overline{N}_l}{l} u^l\right), \quad u \in \mathbb{C}, \quad |u| < (D-1)^{-1},$$

where

$$\overline{N}_l := \int_X N_l(x)\, d\mu(x)$$

and $N_l(x)$ denotes the number of reduced closed paths of length l in G starting and ending at $(x, x) \in V$.

Remark.
Note that the function $N_l(x)$ is measurable: as the connected components of the graphing are infinite, the value $N_l(x)$ just depends on the geometry of the ball $B^G_{l+1}((x, x))$ as induced subgraph of the graph G induced by the graphing $\mathcal{G}$. Thus, for all $\alpha \in \mathcal{A}^D$ with $\rho(\alpha) \geq l+1$, the mapping $N_l(x)$ is constant on the set

$$X_\alpha = \{x \in X \mid B^G_{\rho(\alpha)}((x, x)) \simeq \alpha\},$$

which has already been defined in Chapter 7. Recall that those latter sets X_α are measurable by the construction of graphings, see e.g. [LPS14].
Furthermore, we have $\overline{N}_l < \infty$ for all $l \in \mathbb{N}$, as $N_l(x) \leq D(D-1)^{l-1}$ for all $x \in X$ and μ is a probability measure.

10.1 Approximation of sofic graphings

We can immediately state and prove the approximation theorem for the Ihara Zeta functions associated with a sofic graphing. As only local quantities need to be considered, we do not have to impose amenability/hyperfiniteness conditions on the graphing. We have seen above that Ihara Zeta functions for infinite graphs can be defined via convergence of finite, normalized analogues. Moreover, there are infinite graphs with a natural definition for its Ihara Zeta function. For many of these functions, one can find sequences of finite graphs such that the associated normalized Ihara Zeta functions converge to a notion of Zeta function for the original graph in the topology of uniform convergence on compact sets. For instance in [CMS02], the authors show compact convergence along residually finite approximations of groups acting freely on regular graphs. A similar approach involving the integrated density of states of Markov operators on regular graphs can be found in [GŻ04]. An approximation theorem for graphs with a free action by countable amenable groups has been shown in [GIL08].
It turns out that all those limit functions can be interpreted as the Ihara Zeta function for some sofic graphing and the underlying approximation is via weak convergence of graph sequences, see [LPS14]. Hence, considering the graphs in [CMS02, GŻ04, GIL08, GIL09] as sofic graphings, we can interpret our approximation theorem, Theorem 10.5 as a major extension of the mentioned convergence results. The identification of the corresponding Zeta functions with a Zeta function of a limit graphing is straight forward in all cases where the Ihara Zeta function is defined via a limit relation. For other notions, this problem will in general be more difficult. We will show how to do this for periodic graphs in the next section.

Let us state and prove the main theorem of this section. It shows that the Ihara Zeta function satisfies a continuity property with respect to weak convergence of graphs.

Theorem 10.5 (Approximation of the Ihara Zeta function).
Let $\mathcal{G} = (X, \mu, I_1, \dots, I_D)$ be a sofic graphing with induced measure graph (G, M), where $G = (V, E)$. Moreover, let $(G_n) = (V_n, E_n)$ be an approximating sequence of finite, connected graphs with uniform vertex degree bound $D \in \mathbb{N}$. If $Z_{\mathcal{G}}$ denotes the Ihara Zeta function of the graphing, then

$$\lim_{n\to\infty} Z_{G_n,norm}(u) = Z_{\mathcal{G}}(u)$$

in the topology of uniform convergence on compact sets in the set of numbers $u \in \mathbb{C}$ with $|u| < (D-1)^{-1}$.

Proof.
Note that by dominated convergence and by the continuity of the exponential function, it

is sufficient to check that

$$\lim_{n\to\infty} \frac{N_l^n}{|V_n|} = \overline{N}_l$$

for every length $l \in \mathbb{N}$. Here, $\overline{N}_l$ is defined as in the definition of $Z_{\mathcal{G}}$ (see above) and

$$N_l^n := \sum_{v\in V_n} N_l^n(v),$$

where $N_l^n(v)$ denotes the number of reduced closed paths of length l in G_n which start and finish in v. So fix $l \in \mathbb{N}$ and an arbitrary $l' \geq l/2 + 1$. Since the connected components of the graphing $\mathcal{G}$ are infinite and as

$$\overline{N}_l = \int_X N_l(x)\, d\mu(x)$$

with $N_l(x)$ being constant on all sets X_α for $\alpha \in \mathcal{A}^D$ with $\rho(\alpha) = l'$, we obtain

$$\overline{N}_l = \sum_{\substack{\alpha\in\mathcal{A}^D \\ \rho(\alpha)=l'}} N_l(\alpha)\, \mu(X_\alpha).$$

Here, $N_l(\alpha)$ denotes the number of reduced closed paths of length l starting and ending at the root in the class α. Since N_l^n is a local quantity and as the G_n are connected, we have

$$\frac{N_l^n}{|V_n|} = \sum_{\substack{\alpha\in\mathcal{A}^D \\ \rho(\alpha)=l'}} N_l(\alpha)\, p(G_n, \alpha)$$

for all but finitely many $n \in \mathbb{N}$. Now (G_n) is an approximating sequence and hence,

$$\lim_{n\to\infty} p(G_n, \alpha) = \mu(X_\alpha)$$

for all $\alpha \in \mathcal{A}^D$. Hence, the convergence follows from the above representations of $\overline{N}_l$ and $N_l^n/|V_n|$ as finite sums. □

Remark.
The above theorem is also given in Theorem 9.5 of [LPS14]. There, the main ingredients for the proof are the compactness of the space of invariant, normalized measures on X, as well as the continuity of the Ihara Zeta function with respect to the weak topology of finite measures. For details, we refer the reader to the Theorems 3.2 and 9.5 in [LPS14].

10.2 Approximation for periodic graphs

In this section, we consider the class of countably infinite, connected graphs $G = (V, E)$ with a countable subgroup Γ of its automorphisms $\mathrm{Aut}(G)$ acting freely and co-finitely on G. Precisely, this means that the Γ-action does not have non-trivial fixed points and that there is a finite fundamental domain $F \subseteq V$ for the action of Γ on G. We show here that if Γ is sofic, then the Ihara Zeta function $Z_G(u)$ for G is equal to the Ihara Zeta function for a sofic graphing. To do so, we explicitely construct weakly convergent sequences (G_n) of finite graphs recovering the graph statistics of G in Theorem 10.8. Then, Theorem 10.5 implies the approximation of $Z_G(u)$ by the normalized Zeta functions associated with the graphs G_n, cf. Theorem 10.7. As mentioned before, the class of sofic groups is very large and it is not known whether there is a non-sofic group. A nice survey on the topic can be e.g. found in [Pes08]. In non-amenable situations, subgraph exhaustions of G are no sofic approximations. Consequently, a continuity result for those sequences cannot be expected. On the other hand, Følner exhaustions in amenable (hyperfinite) graphs are indeed sofic approximations. Therefore, Theorem 10.7 is a significant generalization of the convergence theorem for graphs endowed with an automorphism action through amenable groups in [GIL08]. Moreover, Schreier graph approximations in residually finite groups are sofic approximations as well. Hence, our Theorem 10.7 substantially extends the convergence statement of [CMS02] for regular graphs endowed with a free and co-finite action by some countable residually finite automorphism group.

For $\alpha \in \mathcal{A}^D$ and some fundamental domain $F \subseteq V$ for the action of Γ on G, we define

$$F_\alpha := \{f \in F \mid B^X_{\rho(\alpha)}(f) \simeq \alpha\}.$$

Then, it is natural to denote the Ihara Zeta function for the graph G by

$$Z_G(u) := \exp\left(\sum_{l=1}^{\infty} \frac{\overline{N}_l}{l}\, u^l\right)$$

for $u \in \mathbb{C}$ with $|u| < (D-1)^{-1}$, where

$$\overline{N}_l := \sum_{\substack{\alpha \in \mathcal{A}^D \\ \rho(\alpha)=l'}} |F_\alpha|\, N_l(\alpha)$$

for some (every) $l' \geq l/2 + 1$, see also [LPS14]. Here again, $N_l(\alpha)$ is the number of reduced closed paths of length l which start and end at the root of α. It is easy to see that the definition of Z_G is independent of the choice of F. Moreover, we would like to point out that it is not hard to define the Ihara Zeta function also in the case of possibly non-free actions with finite stabilizer sets for all points $f \in F$. Moreover, it was shown by Schmidt that this notion corresponds to the usual definition via the Euler product representation, cf. Proposition 2.10 in [LPS14]. Thus, we work indeed with the 'right' notion of Zeta function. However, since our approximation result (and likewise all mentioned results in the literature) hold only true for free actions, we prefer sticking to this situation.

For a precise definition of sofic groups, we need a slight piece of preparation. For $N \in \mathbb{N}$, we denote by $\mathrm{Sym}(N)$ the symmetric group over $\{1, \ldots, N\}$ with unit element Id_N. This group is naturally endowed with the *normalized Hamming distance* d_H, defined as

$$d_H(\sigma, \tau) := \frac{\#\Big\{a \in \{1, \ldots, N\} \mid \sigma(a) \neq \tau(a)\Big\}}{N}$$

for $\sigma, \tau \in \mathrm{Sym}(N)$. One can check that d_H is a metric on $\mathrm{Sym}(N)$, see e.g. [Pes08]. We now define sofic groups via almost-homomorphisms with respect to the Hamming distance d_H.

Definition 10.6 (Sofic groups).
A group Γ with unit element e is called sofic *if for every finite set $T \subseteq \Gamma$ and for each $\varepsilon > 0$, there exist $N \in \mathbb{N}$, along with a mapping*

$$\sigma : T \to \mathrm{Sym}(N) : s \mapsto \sigma_s$$

such that

(i) if $s, t, st \in T$, then $d_H(\sigma_s \sigma_t, \sigma_{st}) < \varepsilon$,

(ii) if $e \in T$, then $d_H(\sigma_e, \mathrm{Id}_n) < \varepsilon$,

(iii) if $s, t \in T$ with $s \neq t$, then $d_H(\sigma_s, \sigma_t) \geq 1 - \varepsilon$.

If for T and ε, there is some map σ satisfying (i) and (ii), then we say that σ is an almost homomorphism *for (T, ε).*

Our goal is to prove the following theorem.

Theorem 10.7 (Approximation of the Ihara Zeta function for periodic graphs).
Let G be a countably infinite graph with vertex degree bound $D \in \mathbb{N}$. Further, assume that $\Gamma \leq \mathrm{Aut}(G)$ is a countable sofic group acting freely and co-finitely on G. Denote by F some finite fundamental domain. Then, there is a weakly convergent graph sequence $G_n = (V_n, E_n)$ such that

$$\lim_{n\to\infty} Z^{|F|}_{G_n,norm}(u) = Z_G(u)$$

in the topology of uniform convergence on compact subsets in the set of $u \in \mathbb{C}$ with $|u| < (D-1)^{-1}$.

The main task in the proof of the above theorem is to construct a weakly convergent graph sequence $(G_n) = (V_n, E_n)$ such that

$$\lim_{n\to\infty} p(G_n, \alpha) = \frac{|F_\alpha|}{|F|}$$

for all $\alpha \in \mathcal{A}^D$. It will follow then from Theorem 10.5 that the functions $Z_{G_n,norm}(\cdot)$ converge to the Ihara Zeta function $Z_{\mathcal{G}}(\cdot)$ of the limit graphing $\mathcal{G}$ associated to the sequence (G_n). Since $Z_G = Z^{|F|}_{\mathcal{G}}$ (which is due to the finite sum representation for the values $\overline{N}_l$), Theorem 10.7 will be a consequence of the following theorem.

Theorem 10.8.
Let G be a countably infinite graph with vertex degree bound $D \in \mathbb{N}$. Further, assume that $\Gamma \leq \mathrm{Aut}(G)$ is a countable sofic group acting freely and co-finitely on $G = (V, E)$. Let $F \subseteq V$ be a finite fundamental domain for the Γ-action on G. Then, there is a weakly convergent graph sequence $G_n = (V_n, E_n)$ such that

$$\lim_{n\to\infty} p(G_n, \alpha) = \frac{|F_\alpha|}{|F|}$$

for every $\alpha \in \mathcal{A}^D$, where $F_\alpha := \{f \in F \mid B^G_{\rho(\alpha)}(f) \simeq \alpha\}$.

Proof.
Take $r \in \mathbb{N}$, as well as $\delta > 0$ and pick a fundamental domain F. It is our goal to construct a finite graph $G_{r,\delta}$ such that

$$\left|p(G_{r,\delta}, \alpha) - |F_\alpha|/|F|\right| < \delta \tag{10.1}$$

holds true for every $\alpha \in \mathcal{A}_r^D$. Now for every $v \in V$, there are unique elements $\gamma_v \in \Gamma$ and $f \in F$ such that $v = \gamma_v f$. We write $\pi : V \to F$ for the corresponding covering map with $\pi(v) := f$. Set

$$T := \Big\{\gamma_v \,|\, v \in \bigcup_{f \in F} B_r^G(f)\Big\}.$$

Since the action of Γ on G is free, this latter set is finite and we have $e \in T$. Next, we define

$$\tilde{T} := TT \cup (TT)^{-1} \cup T^{-1}T \cup TT^{-1}.$$

Note that $T \cup T^{-1} \subseteq \tilde{T}$, which is due to $e \in T$. Set $\varepsilon := \delta/(2|\tilde{T}|^2)$. Since Γ is sofic, we find an $N \in \mathbb{N}$ depending on $\tilde{T}$ and ε, along with a map $\sigma : \tilde{T} \to \mathrm{Sym}(N)$ fulfilling the properties (i), (ii) and (iii) of Definition 10.6. This puts us in the position to define the graph $G_{r,\delta} = (V_{r,\delta}, E_{r,\delta})$ through an algebraic equality involving σ. To do so, define first $V_{r,\delta} := F \times \{1, 2, \dots, N\}$. Further, two vertices (f, i) and (g, j) with $f, g \in F$ and $i, j \in \{1, 2, \dots, N\}$ shall be linked by an edge if and only if there are elements $\gamma_f, \gamma_g \in \Gamma$ such that $\gamma_f f \sim \gamma_g g$ in G and such that the equation

$$\sigma_{\gamma_f}(i) = \sigma_{\gamma_g}(j)$$

holds. Now, for each $1 \leq i \leq N$, we define

$$\varphi_i : B_r^G(F) \to V_{r,\delta} : v \mapsto (\pi(v), \sigma_{\gamma_v^{-1}}(i)),$$

where we have set $B_r^G(F) := \bigcup_f B_r^G(f)$. The key observation for the proof of inequality (10.1) is the following.

Claim: For at least $(1-\delta)N$ of the numbers in $\{1, 2, \dots, N\}$, the map φ_i is a graph isomorphism onto its image in $V_{r,\delta}$.

Let us prove the claim. We denote by $\tilde{N}$ the set of those numbers i such that the following properties are fulfilled at the same time.

(a) We have $\sigma_\gamma(\sigma_{\gamma'}(i)) = \sigma_{\gamma\gamma'}(i)$ whenever $\gamma, \gamma' \in T \cup T^{-1}$. We remind the reader at this point that $T \cup T^{-1} \subseteq \tilde{T}$ and by the definition of $\tilde{T}$, it is also true that $\left(T \cup T^{-1}\right)^2 \subseteq \tilde{T}$.

(b) For all $\gamma, \gamma' \in \tilde{T}$, the equation $\sigma_\gamma(i) = \sigma_{\gamma'}(i)$ implies $\gamma = \gamma'$.

Now, we use the fact that Γ is a sofic group. We will show that up to a portion of δ, the properties (a) and (b) are satisfied for the numbers i. Indeed, by item (i) of Definition 10.6, there are at most $\varepsilon|\tilde{T}|^2 N$ indices violating property (a) and by item (iii) of Definition 10.6, there are not more than $\varepsilon|\tilde{T}|^2 N$ indices violating property (b). Hence, we arrive at

$$|\tilde{N}| \geq (1 - 2\varepsilon|\tilde{T}|^2)N = (1-\delta)N,$$

where the latter equality is due to the choice of ε. We now show that for every $i \in \tilde{N}$, the map φ_i as defined above is indeed a graph isomorphism onto the image $\varphi_i(B_r(F))$. So, fix $i \in \tilde{N}$. Since F is a fundamental domain and as the assertion (b) is satisfied, φ_i is injective, hence bijective onto its image. We still need to show that φ_i and its inverse preserve the edge relations given by E and defined in $E_{r,\delta}$ above, respectively. To do so, suppose that $(\pi(v), \sigma_{\gamma_v^{-1}}(i)) \sim (\pi(w), \sigma_{\gamma_w^{-1}}(i))$ in $G_{r,\delta}$ for $v, w \in B_r^G(F)$. We show that this is true if and only if $v \sim w$ in G. By the definition of $E_{r,\delta}$ the above edge relation is equivalent to the existence of $\gamma, \gamma' \in T$ satisfying

$$\gamma\pi(v) \sim \gamma'\pi(w) \text{ in } G \text{ and } \sigma_\gamma(\sigma_{\gamma_v^{-1}}(i)) = \sigma_{\gamma'}(\sigma_{\gamma_w^{-1}}(i)).$$

It follows from property (a) that this holds true if and only if there exist $\gamma, \gamma' \in T$ such that

$$\gamma\pi(v) \sim \gamma'\pi(w) \text{ in } G \text{ and } \sigma_{\gamma\gamma_v^{-1}}(i) = \sigma_{\gamma'\gamma_w^{-1}}(i).$$

By property (b), this latter statement is equivalent to the existence of $\gamma, \gamma' \in T$ with

$$\gamma\pi(v) \sim \gamma'\pi(w) \text{ in } G \text{ and } \gamma\gamma_v^{-1} = \gamma'\gamma_w^{-1}.$$

Recall that $\pi(v) = \gamma_v^{-1}v$ and $\pi(w) = \gamma_w^{-1}w$. Since the element $\gamma\gamma_v^{-1} = \gamma'\gamma_w^{-1}$ is a graph automorphism by assumption, the previous statement is equivalent to $v \sim w$ in G. This proves the claim.

We still need to finish the proof of the theorem. To do so, fix an arbitrary $\alpha \in \mathcal{A}_r^D$. The previous claim shows that

$$(1-\delta)|F_\alpha|N \le |\{(f,i) \mid B_{\rho(\alpha)}^{G_{r,\delta}}((f,i)) \simeq \alpha\}| \le |F_\alpha|N + \delta|F|N,$$

which in turn implies inequality (10.1). This finishes the proof. □

With this, Theorem 10.7 can be proven very quickly.

Proof (of Theorem 10.7).
By Theorem 10.8, we find a weakly convergent graph sequence $(G_n) = (V_n, E_n)$ with limit probabilities

$$\lim_{n\to\infty} p(G_n, \alpha) = \frac{|F_\alpha|}{|F|}$$

for every $\alpha \in \mathcal{A}^D$. It follows from Theorem 10.5 that the normalized Zeta functions $Z_{G_n,norm}$ converge uniformly on compact sets in the complex $(D-1)^{-1}$-neighbourhood around 0 towards the Zeta function of the limit graphing $\mathcal{G}$ with counting functions

$$\overline{N}_l = \int_X N_l(x)\, d\mu(x)$$

for $l \in \mathbb{N}$, cf. Definition 10.4. Since the G_n are connected, the functions $N_l(\cdot)$ are constant on the sets X_α for all $\alpha \in \mathcal{A}^D$ with $\rho(\alpha) = l+1$. Thus, $\overline{N}_l = \sum_{\alpha:\rho(\alpha)=l+1} N_l(\alpha)\frac{|F_\alpha|}{|F|}$. Note that these expressions coincide with the counting functions for periodic graphs up to the normalization by the power $|F|$. Hence, we arrive at $Z_G = Z_{\mathcal{G}}^{|F|}$, and this proves the desired result. □

11 Open questions

In this chapter, we briefly summarize two open questions emanating from the elaborations of this thesis. The first question refers to the Equipartition Theorem of Elek, cf. Theorem 7.11. It is open whether there is a coloured version of this assertion.

Question 1.
Does the Equipartition Theorem also hold true for hyperfinite families of graphs with their vertices and edges labeled by finitely many colours?

Private communication with Elek indicates that there is considerable evidence for a positive answer to this question. As a consequence, one could prove a Banach space-valued convergence theorem for coloured, hyperfinite Benjamini-Schramm graph sequences. Moreover, one would obtain a coloured version of Theorem 8.2 standing in one line with the ergodic theorem for coloured amenable groups, cf. Theorem 4.4. This in turn would imply the uniform approximation of the integrated density of states of a larger class of Carleman operators, cf. Theorem 9.12.

The attempt to prove Banach space-valued convergence theorems for non-hyperfinite graph sequences will fail in general. The reason for this is that almost-additivity essentially means continuity with respect to the pseudometric δ which will satisfy the Cauchy criterion if and only if the graph sequence is hyperfinite, cf. Theorem 7.10. However, it is known that for operators with algebraic integer coefficients, the integrated density of states does converge uniformly along (possibly non-hyperfinite) weakly convergent approximations, cf. [Tho08, ATV13]. The strategy of the proof is via diophantine approximation and cannot be applied to operators with arbitrary real or complex coefficients. Therefore, it is natural to raise the following question.

Question 2.
Let H be a self-adjoint, pattern-invariant, finite hopping range Carleman operator on a (possibly non-hyperfinite) sofic graphing $\mathcal{G}$. Let (G_n) be a weakly convergent graph sequence for $\mathcal{G}$ and suppose that the operators H_n are the finite approximations of H on the G_n. Is it true that

$$\lim_{n\to\infty} \|N_n - N^*\|_\infty = 0,$$

where the N_n are the empirical spectral distribution functions for the operators H_n and where N^* is the IDS of H?

Note that the emphasize in Question 2 is on the uniform convergence of the spectral distribution functions. For the issue of weak convergence, we refer the reader at this point to the discussions in Chapter 9. In the literature, this question also appears under the name

Lück approximation. In [Lüc94], Lück showed the spectral convergence for combinatorial Laplacians on Schreier graphs of residually finite groups for $E = 0$. This result was a starting point for a considerable amount of investigations which finally led to the general issue concerned with arbitrary coefficients. It would be nice to find a geometric approach to this question which does not depend on the algebraic properties of the coefficients. In particular, it is an interesting task to investigate spectral convergence in non-amenable groups. One approach might be to use the tools developed by Bowen and Nevo in their proofs of ergodic theorems for non-amenable groups, cf. [BN13a, BN13b].

Ehrenwörtliche Erklärung

Hiermit erkläre ich,

- dass mir die Promotionsordnung der Fakultät bekannt ist,
- dass ich die Dissertation selbst angefertigt habe, keine Textabschnitte oder Ergebnisse eines Dritten oder eigene Prüfungsarbeiten ohne Kennzeichnung übernommen und alle von mir benutzten Hilfsmittel, persönliche Mitteilungen und Quellen in meiner Arbeit angegeben habe,
- dass ich die Hilfe eines Promotionsberaters nicht in Anspruch genommen habe und dass Dritte weder unmittelbar, noch mittelbar geldwerte Leistungen von mir für Arbeiten erhalten haben, die im Zusammenhang mit dem Inhalt der vorgelegten Dissertation stehen,
- dass ich die Dissertation noch nicht als Prüfungsarbeit für eine staatliche oder andere wissenschaftliche Prüfung eingereicht habe,
- dass ich weder die gleiche, noch eine in wesentlichen Teilen ähnliche, noch eine andere Abhandlung bereits bei einer anderen Hochschule als Dissertation eingereicht habe.

Jena, den 20. Oktober 2014

Ort, Datum

Felix Pogorzelski

Bibliography

[AdJ81] Mustafa A. Akcoglu and Andrés del Junco. Differentiation of n-dimensional additive processes. *Canad. J. Math.*, 33(3):749–768, 1981.

[AGV14] Miklós Abért, Yair Glasner, and Bálint Virág. Kesten's theorem for invariant random subgroups. *Duke Math. J.*, 163(3):465–488, 2014.

[AL07] David Aldous and Russell Lyons. Processes on unimodular random networks. *Electron. J. Probab.*, 12(54):1454–1508, 2007.

[ATV13] Miklós Abért, Andreas Thom, and Bálint Virág. Benjamini-Schramm convergence and pointwise convergence of the spectral measure. *Preprint*, 2013.

[BN13a] Lewis Bowen and Amos Nevo. Geometric covering arguments and ergodic theorems for free groups. *L'Enseignement Mathématique*, 59:133–164, 2013.

[BN13b] Lewis Bowen and Amos Nevo. Pointwise ergodic theorems beyond amenable groups. *Ergodic Theory Dynam. Systems*, 33(3):777–820, 2013.

[Boc33] Salomon Bochner. Integration von Funktionen, deren Werte die Elemente eines Vektorraumes sind. *Fundamenta Math.*, 20(1):262–276, 1933.

[BP13] Siegfried Beckus and Felix Pogorzelski. Spectrum of Lebesgue measure zero for Jacobi matrices of quasicrystals. *Math. Phys. Anal. Geom.*, 16(3):289–308, 2013.

[BS01] Itai Benjamini and Oded Schramm. Recurrence of distributional limits of finite planar graphs. *Electron. J. Probab.*, 6(23), 2001.

[CMS01] Bryan Clair and Shahriar Mokhtari-Sharghi. Zeta functions of discrete groups acting on trees. *J. Algebra*, 237(2):591–620, 2001.

[CMS02] Bryan Clair and Shahriar Mokhtari-Sharghi. Convergence of zeta functions of graphs. *Proc. Amer. Math. Soc.*, 130(7):1881–1886 (electronic), 2002.

[Con79] Alain Connes. Sur la théorie non commutative de l'intégration. In *Algèbres d'opérateurs (Sém., Les Plans-sur-Bex, 1978)*, volume 725 of *Lecture Notes in Math.*, pages 19–143. Springer, Berlin, 1979.

[CSKC12] Tullio Ceccherini-Silberstein, Fabrice Krieger, and Michel Coornaert. An analogue of Fekete's lemma for subadditive functions on cancellative amenable semigroups. *To appear in J. Anal. Math.*, 2012. arXiv: `http://arxiv.org/pdf/1209.6179`.

[Dei14] Anton Deitmar. Ihara Zeta functions for infinite weighted graphs. *Preprint*, 2014. arXiv: `http://arxiv.org/abs/1402.4945`.

[DGZ13] Anthony Dooley, Valentyn Golodets, and Guohua Zhang. Sub-additive ergodic theorems for countable amenable groups. *Preprint*, 2013. arXiv: `http://arxiv.org/abs/1310.1198`.

[Dix81] Jacques Dixmier. *Von Neumann algebras*, volume 27 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam-New York, 1981. With a preface by E. C. Lance, Translated from the second French edition by F. Jellett.

[DLM+03] Józef Dodziuk, Peter Linnell, Varghese Mathai, Thomas Schick, and Stuart Yates. Approximating L^2-invariants and the Atiyah conjecture. *Comm. Pure Appl. Math.*, 56(7):839–873, 2003.

[DS88] Nelson Dunford and Jacob T. Schwartz. *Linear operators. Part I.* John Wiley & Sons Inc., New York, 1988.

[DU77] Joseph Diestel and Jerry J. Jr. Uhl. *Vector measures.* American Mathematical Society, Providence, R.I., 1977. With a foreword by B. J. Pettis, Mathematical Surveys, No. 15.

[EL10] Gábor Elek and Gábor Lippner. Sofic equivalence relations. *J. Funct. Anal.*, 258(5):1692–1708, 2010.

[Ele06a] Gábor Elek. L^2-spectral invariants and quasi-crystal graphs. arXiv: `http://arxiv.org/abs/math/0607198`, 2006.

[Ele06b] Gábor Elek. The strong approximation conjecture holds for amenable groups. *J. Funct. Anal.*, 239(1):345–355, 2006.

[Ele07a] Gábor Elek. The combinatorial cost. *Enseign. Math. (2)*, 53(3-4):225–235, 2007.

[Ele07b] Gábor Elek. On limits of finite graphs. *Combinatorica*, 27(4):503–507, 2007.

[Ele08a] Gábor Elek. L^2-spectral invariants and convergent sequences of finite graphs. *J. Funct. Anal.*, 254(10):2667–2689, 2008.

[Ele08b] Gábor Elek. Weak convergence of finite graphs, integrated density of states and a Cheeger type inequality. *J. Combin. Theory Ser. B*, 98(1):62–68, 2008.

[Ele12] Gábor Elek. Finite graphs and amenability. *J. Funct. Anal.*, 263(9):2593–2614, 2012.

[Els05] Jürgen Elstrodt. *Maß- und Integrationstheorie*. Springer-Lehrbuch. [Springer Textbook]. Springer-Verlag, Berlin, fourth edition, 2005. Grundwissen Mathematik. [Basic Knowledge in Mathematics].

[Eme74] William R. Emerson. The pointwise ergodic theorem for amenable groups. *Amer. J. Math.*, 96:472–487, 1974.

[Émi85] Richard Émilion. A general differentiation theorem for n-dimensional additive processes. *Math. Scand.*, 57(1):206–214, 1985.

[ES11] Gábor Elek and Endre Szabó. Sofic representations of amenable groups. *Proc. Amer. Math. Soc.*, 139(12):4285–4291, 2011.

[FM77] Jacob Feldman and Calvin C. Moore. Ergodic equivalence relations, cohomology, and von Neumann algebras. II. *Trans. Amer. Math. Soc.*, 234(2):325–359, 1977.

[GIL08] Daniele Guido, Tommaso Isola, and Michel L. Lapidus. Ihara's zeta function for periodic graphs and its approximation in the amenable case. *J. Funct. Anal.*, 255(6):1339–1361, 2008.

[GIL09] Daniele Guido, Tommaso Isola, and Michel L. Lapidus. A trace on fractal graphs and the Ihara zeta function. *Trans. Amer. Math. Soc.*, 361(6):3041–3070, 2009.

[GLV07] Michael J. Gruber, Daniel H. Lenz, and Ivan Veselić. Uniform existence of the integrated density of states for random Schrödinger operators on metric graphs over $\mathbb{Z}^d$. *J. Funct. Anal.*, 253(2):515–533, 2007.

[Gre69] Frederick P. Greenleaf. *Invariant means on topological groups and their applications.* Van Nostrand Mathematical Studies, No. 16. Van Nostrand Reinhold Co., New York, 1969.

[Gre73] Frederick P. Greenleaf. Ergodic theorems and the construction of summing sequences in amenable locally compact groups. *Comm. Pure Appl. Math.*, 26:29–46, 1973.

[Gri76] Geoffrey R. Grimmett. On the number of clusters in the percolation model. *J. London Math. Soc. (2)*, 13(2):346–350, 1976.

[Gri99] Geoffrey Grimmett. *Percolation*, volume 321 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1999.

[Gro99] Misha Gromov. Topological invariants of dynamical systems and spaces of holomorphic maps. I. *Math. Phys. Anal. Geom.*, 2(4):323–415, 1999.

[Gui73] Yves Guivarc'h. Croissance polynomiale et périodes des fonctions harmoniques. *Bull. Soc. Math. France*, 101:333–379, 1973.

[GŻ04] Rostislav I. Grigorchuk and Andrzej Żuk. The Ihara zeta function of infinite graphs, the KNS spectral measure and integrable maps. In *Random walks and geometry*, pages 141–180. Walter de Gruyter GmbH & Co. KG, Berlin, 2004.

[Has89] Ki-ichiro Hashimoto. Zeta functions of finite graphs and representations of p-adic groups. In *Automorphic forms and geometry of arithmetic varieties*, volume 15 of *Adv. Stud. Pure Math.*, pages 211–280. Academic Press, Boston, MA, 1989.

[Has90] Ki-ichiro Hashimoto. On zeta and L-functions of finite graphs. *Internat. J. Math.*, 1(4):381–396, 1990.

[Has92] Ki-ichiro Hashimoto. Artin type L-functions and the density theorem for prime cycles on finite graphs. *Internat. J. Math.*, 3(6):809–826, 1992.

[Has93] Ki-ichiro Hashimoto. Artin L-functions of finite graphs and their applications. *Sūrikaisekikenkyūsho Kōkyūroku*, 840:70–81, 1993. Algebraic combinatorics (Japanese) (Kyoto, 1992).

[Hoc07] Michael Hochman. Averaging sequences and abelian rank in amenable groups. *Israel J. Math.*, 158:119–128, 2007.

[How95] Norman R. Howes. *Modern analysis and topology.* Springer-Verlag, New York, 1995.

[Iha66a] Yasutaka Ihara. Discrete subgroups of PL(2, $k_{\wp}$). In *Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965)*, pages 272–278. Amer. Math. Soc., Providence, R.I., 1966.

[Iha66b] Yasutaka Ihara. On discrete subgroups of the two by two projective linear group over p-adic fields. *J. Math. Soc. Japan*, 18:219–235, 1966.

[Kin93] John F. C. Kingman. *Poisson processes*, volume 3 of *Oxford Studies in Probability.* The Clarendon Press Oxford University Press, New York, 1993. Oxford Science Publications.

[KLS03] Steffen Klassert, Daniel Lenz, and Peter Stollmann. Discontinuities of the integrated density of states for random operators on Delone sets. *Comm. Math. Phys.*, 241(2-3):235–243, 2003.

[KM04] Alexander S. Kechris and Benjamin D. Miller. *Topics in orbit equivalence*, volume 1852 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 2004.

[KP87] Ulrich Krengel and Ronald Pyke. Uniform pointwise ergodic theorems for classes of averaging sets and multiparameter subadditive processes. *Stochastic Process. Appl.*, 26(2):289–296, 1987.

[KPP14] Matthias Keller, Norbert Peyerimhoff, and Felix Pogorzelski. Sectional curvature of polygonal complexes with planar substructures. *Preprint*, 2014. arXiv: `http://arxiv.org/abs/1407.4024`.

[Kre85] Ulrich Krengel. *Ergodic theorems.* Walter de Gruyter, 1985.

[Kri07] Fabrice Krieger. Le lemme d'Ornstein-Weiss d'après Gromov. In *Dynamics, ergodic theory, and geometry*, volume 54 of *Math. Sci. Res. Inst. Publ.*, pages 99–111. Cambridge Univ. Press, Cambridge, 2007.

[Kri10] Fabrice Krieger. The Ornstein-Weiss lemma for discrete amenable groups. Max Planck Institute for Mathematics Bonn, MPIM Preprint 2010-48, 2010.

[KS00] Motoko Kotani and Toshikazu Sunada. Zeta functions of finite graphs. *J. Math. Sci. Univ. Tokyo*, 7(1):7–25, 2000.

[Len02] Daniel Lenz. Uniform ergodic theorems on subshifts over a finite alphabet. *Ergodic Theory Dynam. Systems*, 22(1):245–255, 2002.

[Lin01] Elon Lindenstrauss. Pointwise theorems for amenable groups. *Invent. Math.*, 146(2):259–295, 2001.

[LMV08] Daniel Lenz, Peter Müller, and Ivan Veselić. Uniform existence of the integrated density of states for models on $\mathbb{Z}^d$. *Positivity*, 12(4):571–589, 2008.

[Lov12] László Lovász. *Large networks and graph limits*, volume 60 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2012.

[LPS14] Daniel Lenz, Felix Pogorzelski, and Marcel Schmidt. The Ihara zeta function for infinite graphs. *Preprint*, 2014.
arXiv: `http://arxiv.org/abs/1408.3522`.

[LPV07] Daniel Lenz, Norbert Peyerimhoff, and Ivan Veselić. Groupoids, von Neumann algebras and the integrated density of states. *Math. Phys. Anal. Geom.*, 10(1):1–41, 2007.

[LS03] Daniel Lenz and Peter Stollmann. Delone dynamical systems and associated random operators. In *Operator algebras and mathematical physics (Constanţa, 2001)*, pages 267–285. Theta, Bucharest, 2003.

[LS05] Daniel Lenz and Peter Stollmann. An ergodic theorem for Delone dynamical systems and existence of the integrated density of states. *J. Anal. Math.*, 97:1–24, 2005.

[LSV11] Daniel Lenz, Fabian Schwarzenberger, and Ivan Veselić. A Banach space-valued ergodic theorem and the uniform approximation of the integrated density of states. *Geom. Dedicata*, 150:1–34, 2011. Erratum 159:411–413, 2012.

[LT14] Hanfeng Li and Andreas Thom. Entropy, determinants, and L^2-torsion. *J. Amer. Math. Soc.*, 27(1):239–292, 2014.

[Lüc94] Wolfgang Lück. Approximating L^2-invariants by their finite-dimensional analogues. *Geom. Funct. Anal.*, 4(4):455–481, 1994.

[LV09] Daniel Lenz and Ivan Veselić. Hamiltonians on discrete structures: jumps of the integrated density of states and uniform convergence. *Math. Z.*, 263(4):813–835, 2009.

[LW00] Elon Lindenstrauss and Benjamin Weiss. Mean topological dimension. *Israel J. Math.*, 115:1–24, 2000.

[Ngu79] Xuan-Xanh Nguyen. Ergodic theorems for subadditive spatial processes. *Z. Wahrsch. Verw. Gebiete*, 48(2):159–176, 1979.

[NS13] Ilan Newman and Christian Sohler. Every property of hyperfinite graphs is testable. *SIAM J. Comput.*, 42(3):1095–1112, 2013.

[OW87] Donald S. Ornstein and Benjamin Weiss. Entropy and isomorphism theorems for actions of amenable groups. *J. Anal. Math.*, 48:1–141, 1987.

[Pes08] Vladimir G. Pestov. Hyperlinear and sofic groups: a brief guide. *Bull. Symbolic Logic*, 14(4):449–480, 2008.

[Pog10] Felix Pogorzelski. Ergodic theorems on amenable groups. Diplomarbeit, Eberhard-Karls-Universität Tübingen, 2010.
`http://www.analysis-lenz.uni-jena.de/ls_analysis_multimedia/Felix/Pogorzelski10_diplomarbeit.pdf`.

[Pog13a] Felix Pogorzelski. Almost-additive ergodic theorems for amenable groups. *J. Funct. Anal.*, 265(8):1615–1666, 2013.

[Pog13b] Felix Pogorzelski. Convergence theorems for graph sequences. *Preprint*, 2013. arXiv: `http://arxiv.org/abs/1304.0945`.

[PS14] Felix Pogorzelski and Fabian Schwarzenberger. A Banach space-valued ergodic theorem for amenable groups and applications. *To appear in J. Anal. Math.*, 2014. arXiv: `http://arxiv.org/abs/1205.3649`.

[PSS13] Felix Pogorzelski, Fabian Schwarzenberger, and Christian Seifert. Uniform existence of the integrated density of states on metric Cayley graphs. *Lett. Math. Phys.*, 103(9):1009–1028, 2013.

[Run01] Volker Runde. *Lectures on Amenability.* Springer Verlag Berlin Heidelberg New York, 2001.

[Sat98] Ryotaro Sato. Vector valued differentiation theorems for multiparameter additive processes in L_p spaces. *Positivity*, 2(1):1–18, 1998.

[Sat99] Ryotaro Sato. Vector-valued ergodic theorems for multiparameter additive processes. *Colloq. Math.*, 79(2):193–202, 1999.

[Sat03] Ryotaro Sato. Vector-valued ergodic theorems for multiparameter additive processes. II. *Colloq. Math.*, 97(1):117–129, 2003.

[Sch88] Klaus Schürger. Almost subadditive multiparameter ergodic theorems. *Stochastic Process. Appl.*, 29(2):171–193, 1988.

[Sch08] Oded Schramm. Hyperfinite graph limits. *Electron. Res. Announc. Math. Sci.*, 15:17–23, 2008.

[Sch13] Fabian Schwarzenberger. *The integrated density of states for operators on groups.* PhD thesis, Technische Universität Chemnitz, 2013.

[ScSc12] Christoph Schumacher and Fabian Schwarzenberger. Approximation of the integrated density of states on sofic groups. *To appear in Ann. Henri Poincaré*, 2012. arXiv: `http://arxiv.org/abs/1212.5207`.

[Sol98] Boris Solomyak. Spectrum of dynamical systems arising from Delone sets. In *Quasicrystals and discrete geometry (Toronto, ON, 1995)*, volume 10 of *Fields Inst. Monogr.*, pages 265–275. Amer. Math. Soc., Providence, RI, 1998.

[Tem72] Arkady A. Tempel'man. Ergodic theorems for general dynamical systems. *Trudy Moskov. Mat. Obšč.*, 26:95–132, 1972.

[Tho08] Andreas Thom. Sofic groups and Diophantine approximation. *Comm. Pure Appl. Math.*, 61(8):1155–1171, 2008.

[Wei80] Joachim Weidmann. *Linear operators in Hilbert spaces*, volume 68 of *Graduate Texts in Mathematics.* Springer-Verlag, New York-Berlin, 1980. Translated from the German by Joseph Szücs.

[Wei00] Benjamin Weiss. Sofic groups and dynamical systems. *Sankhyā Ser. A*, 62(3):350–359, 2000. Ergodic theory and harmonic analysis (Mumbai, 1999).

[Wer00] Dirk Werner. *Funktionalanalysis*. Springer-Verlag, Berlin, extended edition, 2000.