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Macroscopic Limit for an Evaporation-Condensation Problem

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Abstract

We consider a rarefied gas mixture confined between two parallel walls consisting of vapor passing through the walls (evaporation, condensation), and a noncondensable which is totally reflected at the walls. Under a diffusive scaling we derive a macroscopic limit in which the noncondensable forms a well-defined boundary layer slowing down the vapor flow. The results differ substantially from others obtained with asymptotic analysis strategies. Our calculations are based on discrete velocity models.

Key words: Evaporation and condensation, discrete kinetic model, macroscopic limit.

MSC classification: 82B40, 76P05, 34B05

1 Introduction

Consider a gas mixture composed of two species confined between two parallel walls. Species A (”vapour”) is emitted according to a prescribed pressure and adsorbed at the boundaries. Species B (”noncondensable”) is totally reflected when hitting the walls. If there is a pressure decay from the wall at $x = a$ to the wall at $x = b$, then a flow of vapour is induced from a to b. At the same time one expects the noncondensable to follow the flow and form a boundary layer at b which slows down the vapor flow.

This problem has been studied in a couple of papers in recent years and in particular the fluid dynamic limit was of interest (see, e.g. [11, 1]). It turned out that the application of the standard asymptotic analysis for the fluid dynamic limit leads to a curious situation. In the limit both A and B are governed by the same Maxwellian with an infinitesimally small bulk velocity $\overline{v}$, (in fact, $\overline{v} = 0$) and a thin boundary layer of noncondensable is formed at b completely suppressing the vapour flow. This phenomenon contradicts physical intuition and is known in the literature as *ghost effect* [10].

In the present work we propose a different macroscopic limit, based on a scaling (“diffusive scaling”) which in the past has been applied in a variety of problems for the derivation of diffusion phenomena (see, e.g. [2, 5, 6]).It turns out that this kind of scaling leads in the limit to a boundary layer of well-defined thickness for the noncondensable which slows down but does not stop the vapor flow. The results are based on a careful investigation of the governing transport operator in the presence of a small drift. In case of zero drift, its nullspace has geometric dimension one (related to mass flow conservation) but algebraic multiplicity two. At the emergence of a drift, the two-dimensional nullspace splits up into two simple eigenspaces giving rise to a new nonzero eigenvalue.

We investigate the problem in the framework of Discrete Velocity Models (DVM). We consider the steady spatially one-dimensional problem in the simplest possible case of mechanically identical species A and B. This means that both are driven by the same Boltzmann collision operator. The only difference is the wall interaction. Denote by $\mathbf{g}$ the distribution of A and by $\mathbf{h}$ that of B. Then the sum $\mathbf{f} = \mathbf{g} + \mathbf{h}$ is governed by a nonlinear one-species Boltzmann collision operator J. If $\mathbf{f}$ is known, then $\mathbf{g}$ and $\mathbf{h}$ evolve according to a *linear* transport operator. We restrict to the case of $\mathbf{f}$ being a fixed global Maxwellian. In the case of zero flow between 0 and 1, $\mathbf{f}$ is a centered Maxwellian with zero bulk velocity. The corresponding transport operator L_0 exhibits a typical structure concerning the algebraic nullspace which in a similar situation has been observed in a couple of papers ([8] for the continuum case, [3, 7] for DVM).

For our investigation we require the DVM to satisfy four assumptions (see (2.3), (2.4), (3.2), (3.8) below), two of them being crucial. The first one is a symmetry condition and requires the velocity grid and the collision model to be invariant under a change of sign of the velocity components perpendicular to the walls. This leads to a linear ODE system with a matrix having a special antisymmetric block structure which is essential. (In the paper we exclude the case of zero normal velocities which would lead to a DAE rather than an ODE system. However, numerical experiments indicate that this condition can be weakened.) The second assumption concerns the existence of a maximal number of pairwise different nonzero eigenvectors. This in particular prohibits the existence of artificial invariants of the transport operator. (A discussion of this point may be found in [3, 7].)

2 The evaporation condensation problem

2.1 The model

Consider a gas mixture confined in the slab $[0, 1]$. The two components of the gas are species A ("vapour") with density function $\mathbf{g}(t, x, \mathbf{v})$ and species B ("noncondensable") with density function $\mathbf{h}(t, x, \mathbf{v})$. The two-dimensional velocities are represented in the form $\mathbf{v} = (v_x, v_\perp)$.

Concerning the gas particle interaction, both types are mechanically identical in the sense that both are governed by the same Boltzmann collision operator. The only difference lies in the gas-wall interaction. While species A may pass through the walls in both directions (condensation, evaporation), species B is totally reflected. As a consequence, there may be a total nonzero mass flux of A through the wall while the mass flux of B is zero.

We write $\mathbf{f}$ =$\mathbf{g}$+$\mathbf{h}$ and let the governing equations for $\mathbf{g}$ and $\mathbf{h}$ be the nonlinear two-species Boltzmann equation

$$(\partial_t + v_x\partial_x)\mathbf{g} = J[\mathbf{f}, \mathbf{g}] \tag{2.1}$$

$$(\partial_t + v_x\partial_x)\mathbf{h} = J[\mathbf{f}, \mathbf{h}] \tag{2.2}$$

with the collision operator $J[., .]$ to be specified below. Since $J[., .]$ is bilinear, a consequence of (2.1), (2.2) is that $\mathbf{f}$ solves the nonlinear Boltzmann equation

$$(\partial_t + v_x\partial_x)\mathbf{f} = J[\mathbf{f}, \mathbf{f}] \tag{2.3}$$

In most of the paper we restrict to the steady variant of the system,

$$v_x \partial_x \mathbf{g} = J[\mathbf{f}, \mathbf{g}] \tag{2.4}$$

$$v_x \partial_x \mathbf{h} = J[\mathbf{f}, \mathbf{h}] \tag{2.5}$$

In order to extend the equations to a well-posed boundary value problem, they have to be supplemented with boundary conditions either in the form of reflection laws or by prescribing the flows into the domain $[0, 1]$. For our purposes such a detailed description is not necessary.

(2.1) Definition: We call a pair $(\mathbf{g}, \mathbf{h})$ a *solution to the evaporation condensation problem*, if equations (2.4), (2.5) are satisfied and if $\mathbf{h}$ has zero mass flux, i.e.

$$\phi[\mathbf{h}](x) = \langle v_x \mathbf{h}(x) \rangle = 0. \tag{2.6}$$

In the rest of the paper we simplify the problem by considering only solutions of the evaporation condensation problem for which $\mathbf{f}$ is a known global equilibrium function of the Boltzmann collision operator. In this case, equations (2.4), (2.5) turn into a system of linear transport equations which can be solved by analyzing the corresponding transport operator. Furthermore, it is sufficient to construct $\mathbf{h}$, since $\mathbf{h}$ is a solution of the transport equation if and only if $\mathbf{g} = \mathbf{f} - \mathbf{h}$ is.

2.2 The discrete system

Let $\mathcal{V} = \{\mathbf{v}_1, \ldots, \mathbf{v}_N\} \subset \mathbb{R}^d$ $(d \geq 2)$ be a finite velocity set, $\mathbf{v}_i = (v_x^{(i)}, v_\perp^{(i)})$. ($v_x^{(i)}$ denotes the component in x-direction, and $v_\perp^{(i)}$ the orthogonal complement.) Let $\mathbf{f} = (f_i)_{i=1}^N$ be a distribution function on $\mathbb{R}^N$. A single collision event means a momentum exchange betwen pairs,

$$(\mathbf{v}_i, \mathbf{v}_l) \quad \leftrightarrow \quad (\mathbf{v}_j, \mathbf{v}_k)$$

For short we write $\alpha = (i, j, k, l)$ and $r_\alpha[\mathbf{f}, \mathbf{f}] = f_j f_k - f_i f_l$. The above collision is described by the *elementary collision operator*

$$(J_\alpha[\mathbf{f}, \mathbf{f}])_m = \begin{cases} r_\alpha[\mathbf{f}, \mathbf{f}] & \text{if} \quad m \in \{i, l\} \\ -r_\alpha[\mathbf{f}, \mathbf{f}] & \text{if} \quad m \in \{j, k\} \\ 0 & \text{else} \end{cases}$$

From physical considerations (momentum and energy conservation) we consider only elementary collisions for which $\overline{\mathbf{v}_i\mathbf{v}_l}$ and $\overline{\mathbf{v}_j\mathbf{v}_k}$ are the diagonals of a rectangle in $\mathbb{R}^d$. We denote by $R \subset \{1,\ldots,N\}^4$ all $\alpha = (i,j,k,l)$ representing a non-degenerate rectangle in the above sense. With this we can now choose collision frequencies $\gamma_\alpha \geq 0$ to define the *Boltzmann collision operator* on $\mathcal{V}$,

$$J[\mathbf{f},\mathbf{f}] = \sum_{\alpha\in R} \gamma_\alpha J_\alpha[\mathbf{f},\mathbf{f}] \tag{2.7}$$

A linear version of this is otained when considering the dynamics of a test particle (with distribution $\mathbf{g}$) in a given scattering field with distribution $\mathbf{f}$. The corresponding *linear transport operator* $J[\mathbf{f}]\mathbf{g}$ is given by the matrix

$$J[\mathbf{f}] = \sum_{\alpha\in R} \gamma_\alpha J_\alpha[\mathbf{f}]$$

with

$$(J_\alpha[\mathbf{f}]\mathbf{g})_m = \begin{cases} 0.5(f_j g_k + f_k g_j) - f_l g_i & \text{if} \quad m = i \\ 0.5(f_i g_l + f_l g_i) - f_k g_j & \text{if} \quad m = j \\ 0.5(f_i g_l + f_l g_i) - f_j g_k & \text{if} \quad m = k \\ 0.5(f_j g_k + f_k g_j) - f_i g_l & \text{if} \quad m = l \\ 0 & \text{else} \end{cases}$$

Its matrix representation is

$$J_\alpha[\mathbf{f}] = P_\alpha \begin{pmatrix} -f_l & 0.5f_k & 0.5f_j & 0 \\ 0.5f_l & -f_k & 0 & 0.5f_i \\ 0.5f_l & 0 & -f_j & 0.5f_i \\ 0 & 0.5f_k & 0.5f_j & -f_i \end{pmatrix} P_\alpha^T$$

with the $N \times 4$-Matrix P_α defined in column representation as

$$P_\alpha = (\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k, \mathbf{e}_l)$$

($\mathbf{e}_m$ the m-th canonical unit vector). It is well-known that Maxwellians, i.e. functions of the form

$$\mathbf{f}(\mathbf{v}) = \exp\left(-|\mathbf{v} - \overline{\mathbf{v}}|^2/2\Theta\right)$$

are equilibrium solutions of the nonlinear collision operator (and they are the only ones, if a sufficient number of rectangles appears in the sum (2.7), see [4]). In this case, a special situation arises. One easily checks that $\alpha = (i, j, k, l)$ describes a rectangle with $\overline{\mathbf{v}_i \mathbf{v}_l}$ and $\overline{\mathbf{v}_j \mathbf{v}_k}$ as diagonals if and only if

$$f_i f_l = f_j f_k \quad =: \phi_\alpha \tag{2.8}$$

Thus we can rewrite the terms of the transport operator, e.g.

$$0.5(f_j g_k + f_k g_j) - f_l g_i = \phi_\alpha \left(0.5(f_j^{-1} g_j + f_k^{-1} g_k) - f_i^{-1} g_i\right)$$

We end up with the compact formulation

$$J[\mathbf{f}] = C F^{-1} \tag{2.9}$$

with $F = \ \text{diag}(f_i, i = 1 \dots N)$,

$$C = \sum_{\alpha \in R} \pi_\alpha P_\alpha \underbrace{\begin{pmatrix} -1 & 0.5 & 0.5 & 0 \\ 0.5 & -1 & 0 & 0.5 \\ 0.5 & 0 & -1 & 0.5 \\ 0 & 0.5 & 0.5 & -1 \end{pmatrix}}_{=:\Gamma} P_\alpha^T \tag{2.10}$$

and $\pi_\alpha = \gamma_\alpha \phi_\alpha$.

Define

$$\mathbb{1} := (1 \dots 1)^T \in \mathbb{R}^N \quad \text{and} \quad \mathbf{f}^{-1} := (f_i^{-1}, i = 1 \dots N) = F^{-1} \mathbb{1}$$

The following result follows immediately from inspection of the matrix Γ.

(2.2) Lemma: (a) C conserves the total mass, i.e. $\mathbb{1}^T \cdot C = 0$.
(b) $\mathbb{1} \in \ker(C)$.

The first model assumption requires that the number of collisions is large enough to prohibit artificial invariants.

(2.3) Model assumption: $C\mathbb{R}^N = \mathbb{1}^\perp$.

Equivalent to this assumption is that the restriction $C : \mathbb{1}^\perp \to \mathbb{1}^\perp$ is bijective. (When writing about the *inverse* C^{-1} *of* C we mean in the following the restriction $C^{-1} : \mathbb{1}^\perp \to \mathbb{1}^\perp$.)

A great part of the considerations to follow are symmetry arguments. Therefore we have to ensure that the velocity space and the collision model are symmetric with respect to reflections about the x-axes in the following sense.

(2.4) Model assumption: (i) If $\mathbf{v} = (v_x, v_\perp) \in \mathcal{V}$, then $v_x \neq 0$, and the reflected velocity $T_x\mathbf{v} := (-v_x, v_\perp) \in \mathcal{V}$.
(ii) The collision frequencies γ_α are T_x-invariant. This means: If $\alpha = (i, j, k, l)$ and $\alpha' = (i', j', k', l')$ are such that the corresponding velocities $\mathbf{v}_m$ and $\mathbf{v}_{m'}$ satisfy $\mathbf{v}_{m'} = T_x\mathbf{v}_m$ for $m \in \{i, j, k, l\}$, then $\gamma_{\alpha'} = \gamma_\alpha$.

From this follows that N is even, $N = 2n$. We choose a numering of $\mathcal{V}$ such that $v_x^{(i)} > 0$ for $i = 1 \dots n$, and $\mathbf{v}_{i+n} = T_x\mathbf{v}_i$. Notice that due to the model assumption (2.3) C is symmetric and has the block matrix stucture

$$C = \begin{pmatrix} A^* & B^* \\ B^* & A^* \end{pmatrix}$$

(see the discussion in [3]).

Since $\mathcal{V}$ contains no velocities $\mathbf{v}$ with $v_x = 0$, the matrix

$$V_x = \mathrm{diag}(v_x^{(i)}, i = 1 \dots N)$$

is regular, and the system (2.5) can be rewritten as the ODE system

$$\partial_x\mathbf{h} = L\mathbf{h} \tag{2.11}$$

with

$$L = V_x^{-1} C F^{-1} \tag{2.12}$$

We easily find the following properties, denoting by $\mathbf{v}_x^\perp$ the hyperplane perpendicular to $\mathbf{v}_x = (v_x^{(i)}, i = 1 \dots N)$.

(2.5) Lemma: (a) $\ker(L) = \mathrm{span}(\mathbf{f})$ and $L(\mathbb{R}^N) = \mathbf{v}_x^\perp$.
(b) The equation $L\mathbf{g} = \mathbf{h}$ is solvable if and only if $\mathbf{h} \in \mathbf{v}_x^\perp$.
(c) Any eigenvector $\mathbf{t}$ to an eigenvalue $\lambda \neq 0$ is orthogonal to $\mathbf{v}_x$.

Proof: (a) follows from the corresponding properties of C, see Lemma (2.2)(b) and assumption (2.3).
(b) is an application of Fredholm's alternative.
(c) follows from (b) and $L\mathbf{t} = \lambda\mathbf{t} \perp \mathbf{v}_x$. ◯

As a consequence, the restriction $L : (\mathbf{f}^{-1})^\perp = (F^{-1}\,\mathbb{1})^\perp \to \mathbf{v}_x^\perp$ is bijective. For short, when writing about the inverse L^{-1}, we mean the restriction

$$L^{-1} : \mathbf{v}_x^\perp \to (\mathbf{f}^{-1})^\perp, \quad L^{-1} = F C^{-1} V_x$$

If $\mathbf{h} \perp \mathbf{v}_x$, then $\mathbf{g} = L^{-1}\mathbf{h}$ is the unique solution of $L\mathbf{g} = \mathbf{h}$ orthogonal to $\mathbf{f}^{-1}$.

3 The steady transport operator

3.1 Zero bulk velocity

We start with the case of a centered Maxwellian,

$$\mathbf{f}_0(\mathbf{v}) = \exp(-|\mathbf{v}|^2/2\Theta)$$

Due to assumption (2.3) and the chosen numbering, the corresponding diagonal matrix F_0 takes the block diagonal structure

$$\begin{aligned} F_0 &= \operatorname{diag}(\mathbf{f}_0(\mathbf{v}_i), i = 1\dots N) = \begin{pmatrix} \operatorname{diag}(\mathbf{f}_0(\mathbf{v}_i), i = 1\dots n) & 0 \\ 0 & \operatorname{diag}(\mathbf{f}_0(\mathbf{v}_i), i = 1\dots n) \end{pmatrix} \\ &=: \begin{pmatrix} F_0^{(1/2)} & 0 \\ 0 & F_0^{(1/2)} \end{pmatrix} \end{aligned}$$

For the same reasons we can decompose V_x into

$$V_x = \operatorname{diag}(V_x^{(1/2)}, -V_x^{(1/2)})$$

This equips the operator $L_0 = V_x^{-1} C F_0^{-1}$ with the block structure

$$L_0 = \begin{pmatrix} (V_x^{(1/2)})^{-1} A^* (F_0^{(1/2)})^{-1} & (V_x^{(1/2)})^{-1} B^* (F_0^{(1/2)})^{-1} \\ -(V_x^{(1/2)})^{-1} B^* (F_0^{(1/2)})^{-1} & -(V_x^{(1/2)})^{-1} A^* (F_0^{(1/2)})^{-1} \end{pmatrix} =: \begin{pmatrix} A & B \\ -B & -A \end{pmatrix} \quad (3.1)$$

The spectrum of matrices of this form was studied in [3, 7]. The following results are of interest for our purposes.

(3.1) Lemma: $\lambda > 0$ is an eigenvalue of L_0 if and only if $-\lambda$ is eigenvalue.

Proof: Define $\mathbf{t}^+ := (\mathbf{p}, \mathbf{q})^T$ and $\mathbf{t}^- := (\mathbf{q}, \mathbf{p})^T$. Then we find easily

$$L_0 \mathbf{t}^+ = \lambda \mathbf{t}^+ \quad \Leftrightarrow \quad L_0 \mathbf{t}^- = -\lambda \mathbf{t}^- \qquad \bigcirc$$

The following model assumption is generic in the class of DVM we are considering (see the discussions in [3, 7]).

(3.2) Model assumption: L_0 has $n-1$ pairwise different strictly positive eigenvalues λ_i, $i = 1 \dots n-1$.

We collect the results concerning the spectrum of L_0. A central property if the solvability of the equation

$$L_0 \mathbf{r}_0 = \mathbf{f}_0 \quad (3.2)$$

which follows from the Lemma below. We call a vector *even* if it is of the form $(\mathbf{p},\mathbf{p})^T$, and *odd* if it is $(\mathbf{p},-\mathbf{p})^T$. The subset of even resp. odd vectors is denoted by $\mathbb{R}^N_{even}$ resp. $\mathbb{R}^N_{odd}$. Furthermore we call an operator M *even* if it maps *even* into *even* and *odd* into *odd*; we call M *odd* if it maps *even* into *odd* and *odd* into *even*.

(3.3) Lemma: (a) L_0 is odd.
(b) $L_0(\mathbb{R}^N_{odd}) = \mathbb{R}^N_{even}$.
(c) There exists a unique solution $\mathbf{r}_0 = L_0^{-1}\mathbf{f}_0 \in \mathbb{R}^N_{odd}$ to equation (3.2).
(d) $L_0(\mathbb{R}^N_{even}) \subsetneq L_0(\mathbb{R}^N_{even}) \oplus \text{span}\{\mathbf{r}_0\} = \mathbb{R}^N_{odd}$

Proof: (a) By model assumption (2.3) and the numbering of the velocity space $\mathcal{V}$, C and F_0 are even and V_x^{-1} is odd.
(b) follows from (a) and

$$\mathbf{v}_x^\perp = \mathbb{R}^N_{even} \cup \left(\mathbb{R}_{odd} \cap \mathbf{v}_x^\perp\right) = L(\mathbb{R}^N_{odd}) \cup L(\mathbb{R}^N_{even})$$

(c) follows from $\mathbf{f}_0 \in \mathbb{R}_{even} \perp \mathbf{v}_x$ and Lemma (2.4).
(d) Let be $\mathbf{t}_i^+ = (\mathbf{p}_i,\mathbf{q}_i)^T$ be eigenvectors for the positive eigenvalues λ_i and denote $\mathbf{t}_i^- = (\mathbf{q}_i,\mathbf{p}_i)^T$ as the corresponding eigenvectors vor $-\lambda_i$, $i = 1\ldots n-1$. Then $\mathbf{s}_i^+ = \mathbf{t}_i^+ + \mathbf{t}_i^- \in \mathbb{R}^N_{even}$ and $\mathbf{s}_i^- = \mathbf{t}_i^+ - \mathbf{t}_i^- \in \mathbb{R}^N_{odd}$ span $(n-1)$-dimensional subspaces of $\mathbb{R}^N_{even}$ resp. $\mathbb{R}^N_{odd}$, and $L_0^2\mathbf{s}_i^- = \lambda_i^2\mathbf{s}_i^-$ and $L_0^2\mathbf{r}_0 = 0$. Thus

$$\mathbf{r}_0 \notin \text{span}(\mathbf{s}_i^-, i = 1\ldots n-1) = L_0(\mathbb{R}^N_{even}) \qquad \bigcirc$$

This result yields a complete description of the eigenspace structure of L_0 and proves the following theorem.

(3.4) Theorem: (a) L_0 is similar to the Jordan normal form

$$N = \text{diag}(\Lambda, -\Lambda, N_0) \quad \text{with} \quad \Lambda = \text{diag}(\lambda_i, i = 1\ldots n-1), \quad N_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

(b) A corresponding transformation matrix is

$$T(0) = \{\mathbf{t}_1^+, \ldots \mathbf{t}_{n-1}^+, \mathbf{t}_1^-, \ldots \mathbf{t}_{n-1}^-, \mathbf{f}_0, \mathbf{r}_0\}$$

with $\mathbf{t}_i^\pm = \mathbf{t}_i^\pm(0)$ as in the proof of Lemma (3.3)(d), and $\mathbf{r}_0 = L_0^{-1}\mathbf{f}_0$.

(3.5) Corollary: The general solution $\mathbf{h}$ of $\partial_x \mathbf{h} = L_0 \mathbf{h}$ in the slab $[0,1]$ takes the form

$$\mathbf{h}(x) = \sum_{i=1}^{n-1} \gamma_i^- \exp(-\lambda_i x)\mathbf{t}_i^- + \sum_{i=1}^{n-1} \gamma_i^+ \exp(-\lambda_i(1-x))\mathbf{t}_i^+ + (\gamma_0 + \gamma_r \cdot x)\mathbf{f}_0 + \gamma_r \mathbf{r}_0$$

The first two sums represent boundary layers at $x=0$ and $x=1$ which are used to model prescribed inflow conditions at the boundaries.

(3.6) Remark: Model assumption (2.3) is quite restrictive since it prohibits zero x-components of the velocities. It turns out that this can be weakened. In all numerical experiments we performed as far, the system (2.5) turned out to represent an index-1 differential algebraic system which could be transformed into an ODE system with a matrix which has precisely the same Jordan structure as that given in the Theorem.

3.2 Shifted Maxwellian

We replace the centered Maxwellian $\mathbf{f}_0$ with a new one shifted in x-direction,

$$\mathbf{f}_{\overline{v}}(\mathbf{v}) = \exp\left(-\frac{1}{2\Theta}|\mathbf{v} - \overline{\mathbf{v}}|^2\right) = \mathbf{f}_0 + \overline{v} \cdot \Delta\mathbf{f}(\overline{v})$$

$\overline{\mathbf{v}} = (\overline{v}, 0)^T$. $\Delta\mathbf{f}$ is continuous with

$$\mathbf{f}' := \Delta\mathbf{f}(0) = \frac{1}{\Theta} \cdot V_x \mathbf{f}_0$$

This introduces an analytic change of all operators introduced above. These are in particular continuously differentiable with respect to $\overline{v}$. The prime indicates in the following the derivative at $\overline{v} = 0$. The transport operator is affected by the change in two ways. The operator C_0 changes into $C_{\overline{v}} = C_0 + \overline{v} \cdot \Delta C(\overline{v})$ with ΔC continuous,

$$\begin{aligned} \Delta C(0) = C' &= \sum_{\alpha \in R} \gamma_\alpha \phi'_\alpha P_\alpha \Gamma P_\alpha^T, \\ \phi'_\alpha &= \frac{v_x^{(i)} + v_x^{(l)}}{\Theta} \cdot \mathbf{f}_0(\mathbf{v}_i)\mathbf{f}_0(\mathbf{v}_l), \quad \alpha = (i,j,k,l) \end{aligned}$$

and F_0^{-1} is to be replaced with $F_{\overline{v}}^{-1} = F_0^{-1} + \overline{v}\Delta F^{-1}(\overline{v})$ with

$$(F^{-1})' := \Delta F^{-1}(0) = -\frac{1}{\Theta} \cdot F_0^{-2} V_x$$

Finally, $L_0 = V_x^{-1} C F_0^{-1}$ changes into $L_0 + \overline{v}\Delta L(\overline{v})$ with

$$\Delta L(0) = L' = V_x^{-1} C' F_0^{-1} - \frac{1}{\Theta} \cdot L_0 \cdot F_0^{-1} V_x$$

It is important to mention that L' is an *even* operator. We list some of the main properties.

(3.7) Lemma: For small $\overline{v}$ there exist positive eigenvalues $\lambda_i^+(\overline{v})$ and negative eigenvalues $-\lambda_i^-(\overline{v})$ depending analytically on $\overline{v}$ with $\lambda_i^\pm \to \lambda_i$ for $\overline{v} \to 0$.

Proof: This is a standard result from perturbation theory since all $\pm\lambda_i$ are simple eigenvalues of L_0 (see Kato). ○

Crucial for the following is how the algebraic nullspace is affected by the change. If there were a solution of $L_{\overline{v}}\mathbf{r}_{\overline{v}} = \mathbf{f}_{\overline{v}}$ then its Jordan normal form remained the same as before. The following (generic) model assumption prohibits this. Recall that $\mathbb{R}^N_{odd}$ is spanned by $\mathbf{s}_i^- = \mathbf{t}_i^+ - \mathbf{t}_i^-$, $i = 1 \dots n-1$, and $\mathbf{r}_0$ (Lemma (3.3)(d)).

(3.8) Model assumption: $\mathbf{f}' + L'\mathbf{r}_0 \notin \{\mathbf{s}_i^-, i = 1 \dots n-1\}$.

Under this assumption the two-dimensional nullspace splits up into two simple eigenspaces as is shown now.

(3.9) Lemma: In a neighborhood $U_0 = (-\overline{v}_0, \overline{v}_0)$ of $\overline{v} = 0$ there exists a continuous mapping $\overline{v} \to (\Delta\lambda(\overline{v}), \Delta\mathbf{r}(\overline{v}))$ such that the pair

$$(\lambda(\overline{v}), \mathbf{r}_{\overline{v}}) = (\overline{v} \cdot \Delta\lambda(\overline{v}), \mathbf{r}_0 + \overline{v} \cdot \Delta\mathbf{r}(\overline{v})) \tag{3.3}$$

solves

$$L_{\overline{v}}\mathbf{r}_{\overline{v}} = \mathbf{f}_{\overline{v}} + \lambda_{\overline{v}} \cdot \mathbf{r}_{\overline{v}} \tag{3.4}$$

$\lambda_{\overline{v}}^{-1}\mathbf{f}_{\overline{v}} + \mathbf{r}_{\overline{v}}$ is eigenvector with eigenvalue $\lambda_{\overline{v}}$. Furthermore,

$$\lambda' := \Delta\lambda(0) \quad = \quad \frac{\mathbf{v}_x^T(L'\mathbf{r}_0 - \mathbf{f}')}{\mathbf{v}_x^T\mathbf{r}_0} \tag{3.5}$$

and $\mathbf{r}' := \Delta\mathbf{r}(0)$ is solution of

$$L_0\mathbf{r}' = \mathbf{f}' - L'\mathbf{r}_0 + \lambda' \cdot \mathbf{r}_0 \tag{3.6}$$

Proof: A necessary condition for the continuity of $\Delta\lambda$ and $\Delta\mathbf{r}$ at $\overline{v} = 0$ is obtained inserting the ansatz (3.3) into equation (3.4) and taking the limit $\overline{v} \to 0$. This leads to equation (3.6). From Fredholm's alternative, this equation is solvable if and only if the right hand side is orthogonal to $\mathbf{v}_x$. Thus $\mathbf{f}' - L'\mathbf{r}_0 + \lambda' \cdot \mathbf{r}_0$ has to be the projection of

$\mathbf{f}' - L'\mathbf{r}_0$ along $\mathbf{r}_0$ onto $\mathbf{v}_x^T$. From this follows (3.5).
Given $\mathbf{r} \neq 0$, define its corresponding normalized vector $\hat{\mathbf{r}} = \|\mathbf{r}\|^{-1}\mathbf{r}$ and the projection $P_{\hat{\mathbf{r}}}$ along $\mathbf{r}$ onto $\mathbf{v}_x^T$,

$$P_{\hat{\mathbf{r}}}\mathbf{g} = \mathbf{g} - \mathbf{v}_x^T\mathbf{g} \cdot \frac{\hat{\mathbf{r}}}{\mathbf{v}_x^T\hat{\mathbf{r}}}$$

The solution of (3.4) is equivalent to finding a fixed point of the mapping

$$\hat{\mathbf{r}} \to c \cdot L^{-1} P_{\hat{\mathbf{r}}}\mathbf{f}_{\overline{v}} \tag{3.7}$$

(with c normalizing constant). Since $\mathbf{v}_x^T\mathbf{f}_{\overline{v}} = \mathcal{O}(\overline{v})$ and since $\mathbf{r}_0$ is fixed point for $\overline{v} = 0$, it follows that for $\overline{v}$ small the mapping (3.7) is a contraction with a unique fixed point. The continuity of the mapping follows from the continuity of simple eigenvectors of analytically perturbed operators (see [9]). ○

This leads to the proof of the main result of this section.

(3.10) Main Theorem:
(a) For $|\overline{v}| \neq 0$ sufficiently small there exists a new eigenvalue $\lambda(\overline{v}) = \overline{v} \cdot \Delta\lambda(\overline{v})$ depending analytically on $\overline{v}$ with $\Delta\lambda(0) = \lambda' \neq 0$ and a corresponding eigenvector of the form $\mathbf{t} = \mathbf{r}_{\overline{v}} + \lambda(\overline{v})^{-1} \cdot \mathbf{f}_{\overline{v}}$. $\mathbf{r}_{\overline{v}} \perp \mathbf{v}_x$ is the unique solution of $L_{\overline{v}}\mathbf{r}_{\overline{v}} = \mathbf{f}_{\overline{v}} + \lambda(\overline{v})\mathbf{r}_{\overline{v}}$.
(b) $L_{\overline{v}}$ is similar to the diagonal matrix

$$N = \mathrm{diag}(\lambda_1^+ \dots \lambda_{n-1}^+, -\lambda_1^- \dots - \lambda_{n-1}^-, 0, \lambda(\overline{v}))$$

Proof: In order to prove (a) we remark that the Jordan block J_0 of the nullspace of L_0 changes into

$$J_{\mathbf{v}} = \begin{pmatrix} 0 & 1 \\ 0 & \lambda(\overline{v}) \end{pmatrix}$$

which is similar to $\mathrm{diag}(0, \lambda(\overline{v}))$. (b) follows then immediately. ○

4 A macroscopic limit

In order to derive a meaningful macroscopic limit we introduce the diffusive scaling (see, e.g. [2, 5, 6]) for the equation

$$(\partial_t + v_x\partial_x)\mathbf{g} = J[\mathbf{f}]\mathbf{g}$$

It consists in replacing the macroscopic variables t and x with $\epsilon^{-2}t$ and $\epsilon^{-1}x$ and leads to the rescaled equation

$$(\partial_t + \epsilon^{-1} v_x \partial_x)\mathbf{g} = \epsilon^{-2} J[\mathbf{f}]\mathbf{g} \tag{4.1}$$

Formally this is equivalent to replacing the space $\mathcal{V}$ of microscopic velocities with $\epsilon^{-1}\mathcal{V}$ and scaling up the collision frequency by a factor ϵ^{-2}. This is the approach which we take here.

Replacing $\mathbf{v}_i$ with $\mathbf{w}_i = \epsilon^{-1}\mathbf{v}_i$ requires to change the Maxwellians $\mathbf{f}_{\overline{v}} = (\exp(-|\mathbf{v}_i - \overline{\mathbf{v}}|^2/2\Theta), i = 1 \dots N)$ to $\mathbf{f}_{\overline{v}}^{(\epsilon)} = (\exp(-|\mathbf{w}_i - \overline{\mathbf{v}}|^2/2\Theta), i = 1 \dots N) = (\exp(-|\mathbf{v} - \epsilon\overline{\mathbf{v}}|^2/2\epsilon^2\Theta), i = 1 \dots N)$ (leaving the macroscopic bulk velocity $\overline{\mathbf{v}}$ unchanged) which itself makes only sense if we rescale the temperature as $T = \epsilon^2\Theta$. From now on we define

$$\mathbf{f}_{\overline{v}}^{(\epsilon)} = \mathbf{f}_{\epsilon\overline{v}}^{(1)} = (\exp(-|\mathbf{v} - \epsilon\overline{\mathbf{v}}|^2/2T), i = 1 \dots N) \tag{4.2}$$

with $T > 0$ constant. For convenience we assume in the following $\lambda'\overline{v} > 0$.

(4.1) Remark: Associated to $\mathbf{f}_{\overline{v}}^{(\epsilon)}$ are the moments

$$\begin{array}{ll} \textit{density} & \rho_{\overline{v}}^{(\epsilon)} = \langle \mathbb{1}\, \mathbf{f}_0^{(1)} \rangle + \mathcal{O}(\epsilon^2) \\ \textit{flux} & \phi_{\overline{v}}^{(\epsilon)} = \langle w_x \mathbf{f}_{\overline{v}}^{(\epsilon)} \rangle = (\overline{v}/T) \cdot \langle v_x^2 \mathbf{f}_0^{(1)} \rangle + \mathcal{O}(\epsilon^2) \end{array}$$

$F_{\overline{v}}^{(\epsilon)}$ is the diagonal matrix with the coefficients of $\mathbf{f}_{\overline{v}}^{(\epsilon)}$ as entries,

$$F_{\overline{v}}^{(\epsilon)} = F_{\epsilon\overline{v}}^{(1)} = \operatorname{diag}(\mathbf{f}_{\overline{v}}^{(\epsilon)}) = F_0 \left(I + \frac{\epsilon\overline{v}}{T} V_x \right) + \mathcal{O}(\epsilon^2) \tag{4.3}$$

The steady version of (4.1) is

$$v_x \partial_x \mathbf{g} = \epsilon^{-1} J[\mathbf{f}]\mathbf{g} \tag{4.4}$$

Thus we have to study the rescaled transport operator

$$L_{\overline{v}}^{(\epsilon)} = \epsilon^{-1} V_x^{-1} C_{\overline{v}}^{(\epsilon)} (F_{\overline{v}}^{(\epsilon)})^{-1} = \epsilon^{-1} L_{\epsilon\overline{v}}^{(1)} \tag{4.5}$$

with

$$C_{\overline{v}}^{(\epsilon)} = C_{\epsilon\overline{v}}^{(1)} = \sum_{\alpha \in R} \pi_\alpha^{(\epsilon\overline{v})} P_\alpha \Gamma P_\alpha^T \tag{4.6}$$

and

$$\pi_\alpha^{(\epsilon\overline{v})} = \gamma_\alpha \mathbf{f}_{\epsilon\overline{v}}^{(1)}(\mathbf{v}_i)\mathbf{f}_{\epsilon\overline{v}}^{(1)}(\mathbf{v}_l) = \pi_\alpha^{(0)} \cdot \left(1 + \frac{\epsilon\overline{v}}{T} \cdot (v_x^{(i)} + v_x^{(l)}) \right) + \mathcal{O}(\epsilon^2) \tag{4.7}$$

$L^{(1)}_{\overline{v}}$ is identical to the operator $L_{\overline{v}}$ investigated in the previous section. Finally define

$$\begin{aligned}\mathbf{r}^{(\epsilon)}_{\overline{v}} &= \epsilon \mathbf{r}_{\epsilon\overline{v}} \\ \lambda^{(\epsilon)}_{\overline{v}} &= \epsilon^{-1}\lambda(\epsilon\overline{v})\end{aligned}$$

where $\mathbf{r}_{\epsilon\overline{v}} = \mathbf{r}_0 + \epsilon\overline{v}\Delta\mathbf{r}(\epsilon\overline{v})$ is given as in Lemma (3.9), and

$$\lambda^{(\epsilon)}_{\overline{v}} \to \lambda'\overline{v} > 0 \quad \text{for} \quad \epsilon \to 0$$

with λ' given by (3.4). The following results are easy to prove from the Main Theorem (3.10).

(4.2) Corollary: (a) $L^{(\epsilon)}_{\overline{v}}$ is similar to the diagonal matrix

$$\operatorname{diag}(\epsilon^{-1}\lambda^+_1(\epsilon\overline{v}),\dots,\epsilon^{-1}\lambda^+_{n-1}(\epsilon\overline{v}),-\epsilon^{-1}\lambda^-_1(\epsilon\overline{v}),\dots,-\epsilon^{-1}\lambda^-_{n-1}(\epsilon\overline{v}),0,\lambda^{(\epsilon)}_{\overline{v}})$$

The corresponding eigenvectors are

$$\mathbf{t}^+_i(\epsilon\overline{v}) \quad (i=1\dots n-1), \quad \mathbf{t}^-_i(\epsilon\overline{v}) \quad (i=1\dots n-1), \quad \mathbf{f}^{(\epsilon)}_{\overline{v}}, \quad \mathbf{r}^{(\epsilon)}_{\overline{v}} + (\lambda^{(\epsilon)}_{\overline{v}})^{-1}\mathbf{f}^{(\epsilon)}_{\overline{v}}$$

(b) The general solution of the rescaled system (4.4) is

$$\begin{aligned}&\sum_{i=1}^{n-1}\gamma^+_i \exp\left(-\epsilon^{-1}\lambda^+_i(\epsilon\overline{v})(1-x)\right)\cdot\mathbf{t}^+_i(\epsilon\overline{v}) + \sum_{i=1}^{n-1}\gamma^-_i \exp\left(-\epsilon^{-1}\lambda^-_i(\epsilon\overline{v})x\right)\cdot\mathbf{t}^-_i(\epsilon\overline{v}) \\ &+\gamma_n\mathbf{f}^{(\epsilon)}_{\overline{v}} + \gamma_r \exp\Big(-\lambda^{(\epsilon)}_{\overline{v}}(1-x)\Big)\cdot(\mathbf{r}^{(\epsilon)}_{\overline{v}} + (\lambda^{(\epsilon)}_{\overline{v}})^{-1}\mathbf{f}^{(\epsilon)}_{\overline{v}}) \qquad (4.8)\end{aligned}$$

Recall that the only term in (4.8) with nonzero flux is that related to the eigenvector $\mathbf{f}^{(\epsilon)}_{\overline{v}}$ (see Lemma (2.5)(c)), and that $\langle w_x\mathbf{f}^{(\epsilon)}_{\overline{v}}\rangle$ converges to a nonzero value for $\epsilon \searrow 0$ (Remark (4.1)). Suppose the pair $(\mathbf{g}^{(\epsilon)}_{\overline{v}}, \mathbf{h}^{(\epsilon)}_{\overline{v}}) = (\mathbf{f}^{(\epsilon)}_{\overline{v}} - \mathbf{h}^{(\epsilon)}_{\overline{v}}, \mathbf{h}^{(\epsilon)}_{\overline{v}})$ be a solution of the rescaled steady evaporation condensation problem (in the sense of Definition (2.1)). Then $\mathbf{h}^{(\epsilon)}_{\overline{v}}$ is of the form

$$\begin{aligned}\mathbf{h}^{(\epsilon)}_{\overline{v}} &= \sum_{i=1}^{n-1}\gamma^+_i \exp\left(\epsilon^{-1}\lambda^+_i(\epsilon\overline{v})x\right)\cdot\mathbf{t}^+_i(\epsilon\overline{v}) + \sum_{i=1}^{n-1}\gamma^-_i \exp\left(-\epsilon^{-1}\lambda^-_i(\epsilon\overline{v})(1-x)\right)\cdot\mathbf{t}^-_i(\epsilon\overline{v}) \\ &\quad+\gamma_r \exp\Big(-\lambda^{(\epsilon)}_{\overline{v}}(1-x)\Big)\cdot(\mathbf{r}^{(\epsilon)}_{\overline{v}} + (\lambda^{(\epsilon)}_{\overline{v}})^{-1}\mathbf{f}^{(\epsilon)}_{\overline{v}}) \qquad (4.9)\end{aligned}$$

with coefficients $\gamma^\pm_i$ and γ_r depending on ϵ. We call a family of pairs $(\mathbf{f}^{(\epsilon)}_{\overline{v}} - \mathbf{h}^{(\epsilon)}_{\overline{v}}, \mathbf{h}^{(\epsilon)}_{\overline{v}})_{\epsilon>0}$ of solutions *asymptotically bounded*, if $\gamma^\pm_i, \gamma_r$ are bounded for $\epsilon \searrow 0$.

(4.3) Corollary: Suppose $(\mathbf{f}^{(\epsilon)}_{\overline{v}} - \mathbf{h}^{(\epsilon)}_{\overline{v}}, \mathbf{h}^{(\epsilon)}_{\overline{v}})_{\epsilon>0}$ is an asymptotically bounded family of solutions to the evaporation condensation problem with a prescribed amount of noncondensable,

$$\int_0^1 \langle \mathbb{1}\, \mathbf{h}_{\overline{v}}^{(\epsilon)}(x)\rangle dx = H = \text{const}$$

Then it converges for $\epsilon \searrow 0$ pointwise in x to $(\mathbf{f}_0 - \mathbf{h}_0, \mathbf{h}_0)$ given by

$$\mathbf{h}_0 \;=\; \gamma_H \exp(-\lambda'\overline{v}(1-x)) \cdot \mathbf{f}_0^{(1)}, \quad \gamma_H = H(\lambda'\overline{v})^2(\exp(\lambda'\overline{v}) - 1)^{-1} = H \cdot \lambda'\overline{v} + \mathcal{O}(\overline{v}^2)$$

The limits of the associated moments are

density	$\langle \mathbb{1}\, \mathbf{f}_{\overline{v}}^{(\epsilon)}\rangle \to \langle \mathbb{1}\, \mathbf{f}_0^{(1)}\rangle$
flux	$\langle w_x \mathbf{f}_{\overline{v}}^{(\epsilon)}\rangle \to \overline{v} \cdot \langle v_x^2 \mathbf{f}_0^{(1)}\rangle / T$

The concentration profile of the noncondensable, i.e. the layer at the wall point $x = 1$ is given as

$$\textit{noncondensable concentration} \quad \langle \mathbb{1}\, \mathbf{h}_{\overline{v}}^{(\epsilon)}\rangle / \langle \mathbb{1}\, \mathbf{f}_{\overline{v}}^{(\epsilon)}\rangle \to \gamma_H \exp(-\lambda'\overline{v}(1-x))$$

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