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Limit-point / limit-circle classification of second-order differential operators arising in $\mathcal{PT}$ quantum mechanics

Florian Büttner and Carsten Trunk

Abstract

We consider a second-order differential equation $-y'' + q(x)y(x) = \lambda y(x)$ with complex-valued potential q and eigenvalue parameter $\lambda \in \mathbb{C}$. In $\mathcal{PT}$ quantum mechanics the potential has the form $q(x) = -(ix)^{N+2}$ and is defined on a contour $\Gamma \subset \mathbb{C}$. Via a parametrization we obtain two differential equations on $[0, \infty)$ and $(-\infty, 0]$. With a WKB-analysis we classify this problem according to the limit-point/ limit-circle scheme.

Keywords: non-Hermitian Hamiltonian, Stokes wedges, limit point, limit circle, $\mathcal{PT}$ symmetric operator, spectrum, eigenvalues

1 Introduction

We consider a quantum system described by the Non-Hermitian Hamiltonian (see [3])

$$H = \frac{1}{2m}p^2 - (iz)^{N+2}, \tag{1.1}$$

with a natural number N. The associated Schrödinger eigenvalue problem

$$-y''(z) - (iz)^{N+2}y(z) = \lambda y(z),\ z \in \Gamma \tag{1.2}$$

is defined on a contour Γ in the complex plane and Γ is symmetric with respect to the imaginary axis. For simplicity we choose

$$\Gamma := \left\{ z = xe^{i\phi sgn(x)} : x \in \mathbb{R} \right\}, \quad \phi \in (-\pi/2, \pi/2), \tag{1.3}$$

cf. [2]. Via the parametrization

$$z(x) := xe^{i\phi sgn(x)}$$

we obtain two Sturm-Liouville differential equations on $[0, \infty)$ and on $(-\infty, 0]$, repectively. In 1957 A. R. Sims developed a limit-point/ limit-circle classification for complex potentials, see [7]. A further refinement was obtained in [4], see also [6]. For the eigenvalue problem (1.2) we give a full classification into limit-point/ limit-circle according to the angle ϕ in (1.3). In particular we show limit-point at Stokes line and limit-circle at Stokes wedges. With (1.1) we associate an operator in a $L^2(\mathbb{R})$ space. The associated operator is a $\mathcal{PT}$-symmetric operator, where $\mathcal{P}$ is the parity operator and $\mathcal{T}$ is time reversal, cf. [3] and [1].

2 Limit-point/ Limit-circle classification

We recall the limit-point/ limit-circle-classification from [4, Theorem 2.1]. We consider

$$-w(x)'' + q(x)w(x) \quad \text{ on } [0, \infty) \tag{2.1}$$

with q locally integrable and complex valued. We assume

$$Q := \text{clconv}\, \{q(x) + r : x \in [0, \infty),\ 0 < r < \infty\} \neq \mathbb{C}, \tag{2.2}$$

where clconv denotes the closed convex hull. For $\lambda_0 \notin \mathbb{C}\backslash Q$ is K the nearest point in Q and L a line touching Q in K. We translate K via $z \mapsto z - K$ in the origin and rotate via the angle $\eta \in (-\pi, \pi]$ so that L coincide with the imaginary axis and λ_0 and Q lie in the negative and non-negative half-planes. For such K and η define $\Lambda_{K,\eta} := \{\lambda \in \mathbb{C} : \text{Re}(\lambda - K)e^{i\eta} < 0\}$. The following theorem is taken from [4, Theorem 2.1].

Theorem 2.1. *For $\lambda \in \Lambda_{K,\eta}$, exactly one of the following holds.*

(I) There exists a, up to a constant, unique solution w of (2.1) satisfying

$$\int_0^\infty \mathrm{Re}\left[e^{i\eta}\left(|w'|^2+(q-K)|w|^2\right)\right]\,dx+\int_0^\infty |w|^2\,dx<\infty \tag{2.3}$$

and this is the only solution satisfying $w \in L^2(\mathbb{R}_+)$.

(II) There exists a, up to a constant, unique solution w of (2.1) satisfying (2.3) but all solutions satisfy $w \in L^2(\mathbb{R}_+)$.

(III) All solutions w of (2.1) satisfy (2.3) and $w \in L^2(\mathbb{R}_+)$.

Cases (I) and (II) are called limit-point cases and case (III) is called limit-circle case.

3 $\mathcal{PT}$-symmetric Problem

We can decompose the complex plane with the angle $\phi = -\frac{N+2}{2N+8}\pi + \frac{2k}{4+N}\pi$ in $N+4$ sectors, so-called Stokes wedges,

$$S_k := \left\{ z \in \mathbb{C} : -\frac{N+2}{2N+8}\pi + \frac{2k-2}{4+N}\pi < \arg(z) < -\frac{N+2}{2N+8}\pi + \frac{2k}{4+N}\pi \right\}, \quad k = 0,\dots,N+3$$

and the $N+4$ Stokes lines

$$L_k := \left\{ z \in \mathbb{C} : \arg(z) = -\frac{N+2}{2N+8}\pi + \frac{2k}{4+N}\pi \right\},\ k = 0,\dots,N+3.$$

Therefore Γ is either contained in two Stokes wedges or corresponds to two Stokes lines.

We map the problem back to the real line via the parametrization

$$z : \mathbb{R} \to \mathbb{C}, \quad z(x) := xe^{i\phi sgn(x)}.$$

Thus y solves (1.2) for $z \neq 0$ if and only if w, $w(x) := y(z(x))$ solves

$$-e^{\mp 2i\phi} w''(x) - (ix)^{N+2} e^{\pm(N+2)i\phi} w(x) = \lambda w(x),\ x \in \mathbb{R}_\pm.$$

This differential equation can be written as

$$-w''(x) - (ix)^{N+2} e^{\pm(N+4)i\phi} w(x) = \tilde{\lambda} w(x),\ x \in \mathbb{R}_\pm \tag{3.1}$$

with $\tilde{\lambda} := \lambda e^{\pm 2i\phi}$.

Proposition 3.1. *(i) If $\phi \neq -\frac{N+2}{2N+8}\pi + \frac{2k}{4+N}\pi$, $k = 0, \dots, N+3$, then (3.1) is in the limit-point case, cf. case (I) in Theorem 2.1. In particular this implies that only one solution of (3.1) is in $L^2(\mathbb{R}_+)$ resp. $L^2(\mathbb{R}_-)$.*

(ii) If $\phi = -\frac{N+2}{2N+8}\pi + \frac{2k}{4+N}\pi$, $k = 0, \dots, N+3$, then (3.1) is in the limit-circle case, cf. case (III) in Theorem 2.1. In particular this implies that all solutions of (3.1) are in $L^2(\mathbb{R}_+)$ resp. $L^2(\mathbb{R}_-)$.

Proof. The two corresponding linear independent solutions w_1 and w_2 of the Schrödinger eigenvalue differential equation $-w''(x) - (ix)^{N+2}e^{(N+4)i\phi}w(x) = \tilde{\lambda}w(x)$, $x \in \mathbb{R}_+$ satisfy [5, Corollary 2.2.1]

$$w_{1,2}(x) \sim q(x)^{-1/4}\exp\left(\pm \int_1^x \mathrm{Re}(q(t)^{1/2})\,dt\right), \text{ for } x \to \infty$$

with $q(x) := -(ix)^{N+2}e^{(N+4)i\phi} - \lambda e^{2i\phi}$. The notation $f(x) \sim g(x)$ means that $f(x)/g(x) \to 1$ as $x \to \infty$. The same holds for the solutions as $x \to -\infty$ with $q(x) := -(ix)^{N+2}e^{-(N+4)i\phi} - \lambda e^{-2i\phi}$, which is easily seen by replacing x by $-x$.

If $\phi \neq -\frac{N+2}{2N+8}\pi + \frac{2k}{4+N}\pi$ and $\lambda = 0$ then $\mathrm{Re}(q(t)^{1/2}) \neq 0$ and there exists exactly one solution in $L^2(\mathbb{R}_+)$ resp. $L^2(\mathbb{R}_-)$. This implies, see [4, Remark 2.2], that we have case (I), limit point case, in Theorem 2.1.

For $\phi = -\frac{N+2}{2N+8}\pi + \frac{2k}{4+N}\pi$ we obtain $-w''(x) - x^{N+2}w(x) = \tilde{\lambda}w(x)$ and therefore we are in the limit-circle case with [8, Remark 7.4.2], if $N > 0$, i. e. case (III) in Theorem 2.1. In particular case (II) in Theorem 2.1 is not possible. □

Let ϕ be as in Proposition 3.1(i), limit-point case. Consider the following operators (cf. [4, Theorem 4.4])

$$\mathrm{dom}(A_\pm) := \left\{y \in L^2(\mathbb{R}_\pm) : A_\pm y \in L^2(\mathbb{R}_\pm), y, y' \text{ loc. abs. cont.}, y(0) = 0\right\}$$

$$A_\pm y(x) := -y''(x) - (ix)^{N+2}e^{\pm(N+4)i\phi}y(x).$$

Theorem 3.2. *The spectrum $\sigma(A_\pm)$ is contained in Q, cf. (2.2), and consists only of isolated eigenvalues of finite algebraic multiplicity.*

A similar conclusion holds for ϕ is as in Proposition 3.1(ii) (limit-circle case), cf. [4].

Remark 3.3. One can show that the operator $A_+ \oplus A_-$ with the coupling $y'(0+) = \alpha y'(0-)$ ($\alpha \in \mathbb{C}$) in zero is $\mathcal{PT}$-symmetric if and only if $|\alpha| = 1$. This gives a way to characterize all $\mathcal{PT}$-symmetric operators associated with (1.2).

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