First Integrals in Stationary and Axially Symmetric Space-Times and Sub-Riemannian Structures

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Inhaltsverzeichnis

Zusammenfassung

Erste Integrale haben sowohl in der Physik als auch in der Mathematik große Bedeutung. Sie sind konstant entlang Lösungen der geodätischen oder der Hamiltonschen Gleichungen und werden daher auch *Bewegungskonstanten* oder *Konstanten der Bewegung* genannt (man spricht auch verkürzt einfach von *Integralen*). Killing-Tensoren entsprechen Integralen, welche homogene Polynome in den Impulsen sind.

Killing-Tensoren und erste Integrale im Allgemeinen sind aus mehreren Gründen interessant. Zunächst einmal sind sie ein Hilfsmittel, um in der Physik Lösungen für die Bewegung von Teilchen zu finden und besser zu verstehen. Ein klassisches Beispiel ist das Kepler-Problem, das ein zusätzliches Integral erlaubt (*Runge-Lenz-Vektor*), welches mit dem Coulomb-Potential verknüpft ist. Killing-Vektorfelder sind die einfachsten Killing-Tensorfelder und infinitesimale Erzeuger von Isometrien. Daher werden Killing-Tensoren auch als *versteckte Symmetrien* bezeichnet. Zweitens schränkt die Existenz einer ausreichenden Anzahl an Integralen die geodätischen Orbits im Phasenraum auf Tori oder Zylinder ein (unter gewissen Zusatzbedingungen). Umgekehrt zeigen nicht-integrable Hamiltonsche Systeme typischerweise chaotisches Verhalten, d.h. die Lösungskurven hängen stark von den Anfangsbedingungen ab. Drittens gibt es den Begriff der (maximalen) Superintegrabilität, der die Existenz einer maximalen Anzahl von Integralen voraussetzt. Viele bekannte dynamische Systeme der Physik sind superintegrabel; außerdem gibt es Querverbindungen zur Multiseparabilität (Koexistenz verschiedener Koordinaten, für welche die Hamilton-Jacobi-Gleichung durch Variablenseparation gelöst werden kann) und zu *speziellen Funktionen.* Ein weiterer Aspekt ist die Verbindung zwischen Killing-Tensoren und geodätischer Äquivalenz. Zwei Metriken heißen geodätisch äquivalent, wenn sie bis auf Umparametrisierung dieselben Geodäten haben. Die Bedingung, wann zwei Metriken geodätisch äquivalent sind, lässt sich über Killing-Tensoren formulieren.

Prinzipiell existiert nur eine begrenzte Anzahl an Methoden, mit welchen sich systematisch die Existenz bzw. Nicht-Existenz von Integralen nachprüfen lässt, und typischerweise beschränken sich diese Methoden auf bestimmte Arten der Integrabilität. Wir verwenden in dieser Arbeit einen Ansatz aus der klassischen Prolongations-Projektionstheorie (vgl. Abschnitt 1.2). Dieser Ansatz erfuhr zuletzt mehr Aufmerksamkeit [MS10; KM12], da er rechnergestützt verwendet werden kann und keine Approximation beinhaltet. Die Kernergebnisse der Arbeit sind:

(1) Zwei neue Algorithmen werden vorgestellt, mit denen die Existenz von Killing-Tensoren für stationär-axialymmetrische Vakuum-Metriken überprüft werden kann. Beide Algorithmen erreichen eine hohe Effizienz (vgl. Abschnitt 2.4). In Abschnitt 3.2 zeigen wir an Hand eines Beispiels, wie die Algorithmen auch für Metriken mit einem reellen Parameter verwendet werden können.

Stationär-axialsymmetrische Metriken dienen in der Astrophysik als Modelle kompakter Objekte wie Neutronensternen und Schwarzen Löchern. Typischerweise werden solche Ob-

jekte durch Kerr-Metriken beschrieben, die ein zusätzliches quadratisches Integral besitzen und daher Liouville-integrabel sind [Car68b; WP70]. Ob andere stationär-axialsymmetrische Metriken z.B. für Schwarze Löcher in der Natur realisiert sind (Existenz sogenannter *bumpy black holes*) ist ein offenes Problem [Bri08a; BL14]. Mit Hilfe der Algorithmen untersuchen wir Zipoy-Voorhees-Metriken und eine Tomimatsu-Sato-Metrik. Zipoy-Voorhees-Metriken stellen eine Verallgemeinerung der Schwarzschild-Metrik dar (statischer Grenzfall der Kerr-Metriken). Tomimatsu-Sato-Metriken sind eine nicht-statische Verallgemeinerung von Zipoy-Voorhees-Metriken. Als ersten Schritt geben wir in Abschnitt 3.2 einen neuen, einfachen Beweis für die Tatsache, dass der flache Raum und die Schwarzschild-Raumzeit die einzigen integrablen Zipoy-Voorhees-Metriken mit zusätzlichem quadratischem Integral sind, vgl. [Car68a; WP70].

In Abschnitt 3.3 zeigen wir dann für eine spezielle Zipoy-Voorhees-Metrik, die Darmois-Lösung, die Nicht-Existenz zusätzlicher Killing-Tensoren bis zum Grad 11. Dies verallgemeinert ein entsprechendes Resultat in [KM12] und ergänzt die Ergebnisse in [MPS13; LG12].

In Abschnitt 3.4 untersuchen wir eine Tomimatsu-Sato-Metrik und zeigen die Nicht-Existenz eines zusätzlichen Killing-Tensors bis zum Grad 7.

(2) In Kapitel 4 zeigen wir Reduzibilität für involutive kubische Integrale in beliebigen Weyl-Metriken, d.h. wir zeigen, dass Killing-Tensoren vom Rang 3 für solche Metriken als Linearkombination symmetrischer Produkte von Killing-Tensoren niedrigeren Rangs geschrieben werden können (Erstpublikation in [Vol15b]). Weyl-Metriken sind statische Vakuum-Metriken aus der Klasse stationär-axialsymmetrischer Metriken und im Wesentlichen durch eine Parameter-Funktion bestimmt.

Für den Beweis stellen wir eine notwendige Bedingung für die Existenz nichttrivialer kubischer Killing-Tensoren auf und lösen das relevante System partieller Differentialgleichungen anschließend explizit.

(3) In Kapitel 5 untersuchen wir Liouville-Integrabilität für einige sub-Riemannsche Strukturen auf Rang-2-Distributionen in Carnot-Gruppen der Dimension 6, 7 und 8. Carnot-Gruppen approximieren allgemeine sub-Riemannsche Strukturen in typischen Punkten. Dieser Teil der Dissertation beruht auf einer gemeinsamen Arbeit mit Boris Kruglikov und Georgios Lukes-Gerakopoulos [KVL15].

Wir zeigen, dass es solche sub-Riemannschen Strukturen gibt, die zwar ein hohes Maß an Symmetrie, aber nicht genügend Integrale für Liouville-Integrabilität haben. Diesen unerwarteten Effekt beobachten wir für die Symmetriealgebren der sub-Riemannscher Strukturen wie auch ihrer zu Grunde liegenden Distributionen. Wir verwenden dazu einen ähnlichen Algorithmus wie für stationär-axialsymmetrische Metriken.

Offene Probleme und Perspektiven für weitere Forschung besprechen wir in Kapitel 6.

Abstract

First integrals play a crucial role in physics and mathematics. They remain constant along geodesics or solutions of Hamilton's equations, and are also known as *constants of motion*, *orbital invariants*, or simply *integrals*. Killing tensors correspond to (first) integrals that are homogeneous polynomials in the momenta.

Killing tensors, and first integrals in general, are interesting for a number of reasons. Firstly, they help in finding and understanding solutions to the equations of particle motion in physics. A classic example is the Kepler problem, which admits the *Runge-Lenz vector*, an integral connected with a Coulomb-type potential. Since the simplest Killing tensors (i.e. Killing vectors) are the infinitesimal generators of isometries, Killing tensor fields are often called *hidden symmetries*. Secondly, the existence of a sufficient number of integrals (complete integrability) restricts geodesic orbits in phase space to tori or cylinders (under certain additional conditions). On the other hand, non-integrable Hamiltonian systems typically show chaotic behavior, i.e. the solution curves depend heavily on the initial conditions. Next, the existence of a maximal number of integrals is called maximal superintegrability. Many famous dynamical systems in physics are superintegrable. Superintegrability is also related to multiseparability (coexistence of several coordinate systems for which the Hamilton-Jacobi equation can be solved by separation of variables) and to special functions. Another aspect is the link between Killing tensors and geodesic equivalence. Two metrics are called geodesically equivalent, if they have the same (unparametrized) geodesics. The requirement for two metrics to be geodesically equivalent can be formulated in terms of Killing tensors.

There is only a limited number of methods to check systematically the existence or nonexistence of integrals, and these methods typically are confined to studying certain types of integrability. We are going to work with an approach based on classical Cartan-Kähler prolongation-projection theory (cf. Section 1.2). This approach has received more attention lately [MS10; KM12], since it has the advantage that it can be implemented on a computer to rigorously check the number of independent integrals polynomial in momenta with smooth coefficient functions. The key results of this thesis are:

(1) Two new algorithms are presented for checking the existence of Killing tensors in metrics of the stationary and axially symmetric vacuum class. Both algorithms achieve a high computational efficiency (cf. Section 2.4). In Section 3.2 we also demonstrate with an example how the algorithms can be used with metrics that depend on a real parameter.

Stationary and axially symmetric metrics are interesting as a model for compact astrophysical objects such as neutron stars and black holes. A standard model for such objects is the Kerr metric, which is Liouville integrable with an additional quadratic Killing tensor [Car68b; WP70]. Whether other stationary and axially symmetric metrics are realized in nature for instance for black holes (existence of so-called *bumpy black holes*) is an open problem in astrophysics [Bri08a; BL14]. Using the algorithms, we examine Zipoy-Voorhees

metrics and a Tomimatsu-Sato metric. Zipoy-Voorhees metrics generalize the Schwarzschild metric (the static limit of Kerr). The Tomimatsu-Sato family is a non-static generalization of Zipoy-Voorhees metrics. In Section 3.2, we give a novel proof for the fact that flat space and the Schwarzschild metric are the only integrable cases with a quadratic Killing tensor in the family of Zipoy-Voorhees metrics, cf. [Car68a; WP70], as a first step towards higher-order integrability.

In Section 3.3, we then prove the non-existence of additional Killing tensors up to valence 11, for a particular Zipoy-Voorhees metric (the Darmois solution). This extends the result from [KM12] and complements the findings of [MPS13; LG12].

In Section 3.4, a Tomimatsu-Sato metric is examined and nonexistence of an additional Killing tensor is proven up to valence 7.

(2) In Chapter 4, we prove reducibility of all involutive cubic integrals in arbitrary Weyl metrics, i.e. we show that all involutive valence-3 Killing tensors for such metrics can be written as a linear combination of symmetrized products of lower-valence Killing tensors (the result was first published in [Vol15b]). Weyl metrics form the static vacuum subclass of stationary and axially symmetric metrics and involve a smooth parametrizing function that essentially characterizes the metric.

For the proof, we establish a necessary criterion for the existence of non-trivial valence-3 Killing tensors and subsequently solve the relevant system of partial differential equations.

(3) In Chapter 5, we explore Liouville integrability for certain sub-Riemannian structures on rank-2 distributions in Carnot groups of dimension 6, 7, and 8. Carnot groups are the *nilpotent approximations* of general sub-Riemannian structures in typical points. This part of the thesis is based on collaborative research with Boris Kruglikov and Georgios Lukes-Gerakopoulos [KVL15].

We show that sub-Riemannian structures exist that have a high degree of symmetry, but do not possess enough integrals for Liouville integrability. This surprising effect is observed for the symmetry algebras of sub-Riemannian structures as well as of their underlying distributions. For the proof we use an algorithm similar to that used for the stationary and axially symmetric metrics.

Open problems and perspectives for further research are discussed in Chapter 6.

Chapter 1

Introduction

In the following, we consider a D-dimensional differentiable manifold M with (pseudo-) Riemannian[1] metric g. Its cotangent bundle T^*M is endowed with a natural symplectic form σ [Arn89]. Coordinates on M are usually denoted by q, and coordinates on T^*M by (q, p). The q_i are called position coordinates, while we speak of the p_i as momenta or momentum coordinates. Sometimes we also take $p \in T^*M$, since there is no risk of confusion.

We begin with a definition of Killing tensors and integrals, followed by a brief discussion of their importance and applications. In Section 1.1, we are then turning to the concept of integrability, especially Liouville integrability. Other major definitions are given in the same section. Cartan-Kähler prolongation-projection is introduced in Section 1.2, and in Section 1.3 we discuss the decomposition of the Poisson equation for 2-dimensional manifolds. We conclude Chapter 1 with an overview of the assumptions that we make, see Section 1.4.

We discuss integrals that the geodesic flow of a given Hamiltonian system admits. Let us begin with a generalization of the concept of Killing vectors.

Definition 1 (Killing tensor field). *A Killing tensor field K of valence d on M is a symmetric $(0, d)$-tensor such that*

$$\nabla_{(a}K_{b_1\dots b_d)} = 0, \tag{1.1}$$

where ∇ denotes the Levi-Civita connection for the metric g. The round brackets denote symmetrization over the indices $a, b_1, \dots, b_d$.

From the definition, it is immediately clear that the metric g is a Killing tensor, because $\nabla g = 0$. Moreover, given two Killing tensors, their symmetrized product is also a Killing tensor.

The metric g provides an isomorphism between T^*M and TM, and therefore we are going to identify co- and contravariant tensor fields as well as the corresponding homomorphisms with mixed co- and contravariant indices.

Killing tensor fields are in 1-to-1 correspondence to (first) integrals that are homogeneous polynomials in the momenta[2]. The isomorphism that maps Killing tensors to homogeneous

[1] We are going to consider Lorentzian metrics in Chapter 3 and 4. Later, in Chapter 5, we broaden our view towards sub-Riemannian metrics. For ease of exposition, however, we focus on (pseudo-)Riemannian metrics first.

[2] Equivalently, via the isomorphism between T^*M and TM, we can consider polynomials in the velocities $\dot{\gamma}$ (where γ denotes a solution trajectory).

polynomial integrals is given by evaluation at the momenta,

$$\begin{aligned}&\textsf{Killing tensors} \to \textsf{Integrals},\\ K \mapsto I_K = K(&p,\ldots,p) = K^{i_1,\ldots,i_d}\, p_{i_1}\cdots p_{i_d}.\end{aligned} \tag{1.2}$$

Given a homogeneous polynomial integral, the corresponding Killing tensor can thus be reconstructed by identifying the coefficients of the polynomial.

In the language of integrals, the Killing equation (1.1) takes the form[3]

$$\{I_K, H\} = X_H(I_K) \equiv 0 \qquad \text{with} \quad H = g^{ij}p_ip_j \tag{1.3}$$

where $\{\cdot,\cdot\}$ denotes the usual *Poisson bracket* on T^*M, and where $X_H(I_K)$ is the derivative of the function in the direction of the Hamiltonian vector field X_H. Note that we made use of the usual summation convention implying summation over upper and lower indices. We are going to often resort to this convention if there is no risk of confusion.

In general, a function $I : T^*M \to \mathbb{R}$ is called an *integral* if it Poisson commutes with the Hamiltonian $H : T^*M \to \mathbb{R}$,

$$\{H, I\} = \frac{\partial H}{\partial q_i}\frac{\partial I}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial I}{\partial q_i} = 0. \tag{1.4}$$

We refer to Equation (1.4) or, accordingly, (1.3) as the *Poisson equation*. It is always satisfied for $I = H$. The product of two integrals is also an integral.

Dynamical systems with a Hamiltonian function H that are governed by the Hamiltonian equations,

$$\begin{aligned}\dot q &= \frac{dH}{dp},\\ \dot p &= -\frac{dH}{dq},\end{aligned} \tag{1.5}$$

are called *Hamiltonian systems*. Integrals satisfying Equation (1.4) remain constant along solution trajectories of (1.5). To distinguish between integrals of the Hamiltonian flow and those of the geodesic flow, we refer to integrals that obey Equation (1.4) with $H = g^{ij}p_ip_j$ as *geodesic invariants*. On the other hand, if the Hamiltonian is not of such form, the integral is called a *Hamiltonian invariant* and is constant along the flow of (1.5). Often, one considers Hamiltonians with a *potential* $V(q)$, i.e. $H = g^{ij}(q)\,p_ip_j + V(q)$. In this case, $T = g^{ij}\,p_ip_j$ is called the *kinetic term*. In the case of sub-Riemannian geometry, there are subtleties in the definition of the Hamiltonian; these are discussed in Chapter 5.

If the Hamiltonian is polynomial in the momenta, then the leading-degree component of a polynomial Hamiltonian invariant is a geodesic invariant. Moreover, for Hamiltonians that are homogeneous polynomials in momenta, the existence of a real-analytic integral implies the existence of an integral that is polynomial in momenta (tome III of [Dar87]).

It is well known that Killing vector fields and, via the isomorphism $\xi \mapsto \langle \xi, p\rangle = \xi^i p_i$, homogeneous linear integrals correspond to symmetries of the metric, more precisely to local isometries (Noether's theorem, [Arn89; FK88]).

Since our attention is exclusively on integrals that are polynomial in the momenta (and in fact we restrict to homogeneous integrals), let us agree on the following conventions from now on

[3] Equivalence of (1.3) and (1.1) is checked by straightforward computation of the corresponding system of PDEs.

— any (first) integral is assumed to be polynomial in the momenta, unless explicitly stated otherwise

— by the term *degree of the integral*, we refer to its degree w.r.t. momenta

— we usually use the language of integrals, rather than that of Killing tensors. This is because the formulation in terms of integrals is more instructive when we perform symplectic reduction and split the equations from (1.1) accordingly.

First integrals appear in several contexts and play an important role in fields like integrability, geometry or mechanics, for instance.

Firstly, the existence of integrals can help in answering natural questions about the behavior of trajectories of the geodesic flow. In Hamiltonian systems, integrals help in finding and understanding solutions to Hamilton's equations, i.e. explicit trajectories for the motion of particles in physics. A classic example is the Kepler problem of celestial mechanics, where the Coulomb-type $1/r$-potential leads to the *Runge-Lenz vector*, which allows us to solve Hamilton's equations (1.5) explicitly. Kepler's laws of planetary motion make implicit use of first integrals. Since Killing vector fields are the infinitesimal generators of isometries, Killing tensor fields for physical systems are also often described as *hidden symmetries* (compare again the the Runge-Lenz vector [Iro02]).

In the phase space T^*M, trajectories remain on level sets of the integrals. Existence of a *complete family* of integrals implies (under certain additional conditions) that solutions of a Hamiltonian system live on tori or cylinders in phase space [Arn89; FGS03], see Section 1.1 for details.

On the other hand, complete integrability implies, again under certain conditions, solvability by quadrature [Arn89; FGS03]. Integrability is a desired property in physics, since it allows one to write down the behavior of mechanical systems in an 'explicit' manner. There are many important systems in physics that are (Liouville) integrable, e.g. the harmonic oscillator, the Euler top or the Kepler problem [Arn89; Iro02]. However, integrability may fail even for physically relevant systems, e.g. the three-body problem is in general non-integrable. Such systems show chaotic behavior, i.e. the solution curves depend heavily on the initial conditions.

There are also systems with more integrals than those needed for integrability. Such systems are called superintegrable. In particular, *maximally superintegrable* systems are D-dimensional systems with $2D - 1$ integrals. Many such systems appear in physics, and superintegrable systems are related to multiseparability (existence of several coordinate systems in which the Hamilton-Jacobi equation can be solved by separation of variables) as well as to the theory of special functions [KMP07; KMP13].

Finally, Killing tensors also play a role in the theory of geodesic equivalence. Two metrics on the same manifold are called geodesically equivalent, if they have the same geodesics (disregarding reparametrizations). The requirement for two metrics to be geodesically equivalent can be formulated in terms of Killing tensors, e.g. [TM03].

In some way it therefore is a natural geometric problem whether a given metric has a sufficient number of integrals. Metrics that have this property may lead to interesting examples, as for instance in the case of the Kerr metric possessing the Carter constant [Car68a; Car68b], see also Chapter 3. According to [Mar14], Kerr-de Sitter metrics are at present the only known examples of integrable space-times[4] with an additional integral of

[4]With regard to the astrophysical applications, we use the term *space-time* for 4-dimensional manifolds equipped with a Lorentzian metric.

higher-than-linear degree, among the class of stationary and axially symmetric space-times (with or without cosmological constant).

The study of integrals in 2-dimensional manifolds is a classical problem in differential geometry and goes at least back to Darboux [Dar87]. A general characterization of 2-dimensional metrics admitting quadratic integrals exists in terms of a separation property, see the contribution by Kœnigs in Darboux's multi-volume work [Dar87]. For a 2-dimensional metric, and a Hamiltonian of the form

$$H = \frac{p_x^2 + p_y^2}{\Omega} + V(x,y), \tag{1.6}$$

an additional quadratic integral exists if there are coordinates (x, y) such that

$$\Omega = X(x) - Y(y) \quad \text{and} \quad V = \frac{\tilde{X}(x) - \tilde{Y}(y)}{X(x) - Y(y)}$$

where X, $\tilde{X}$ are functions of x and Y, $\tilde{Y}$ functions of y only. The quadratic integral is [Dar87; BMP09]

$$I = \frac{X\,p_y^2 + Y\,p_x^2}{X - Y} - \frac{X\,\tilde{Y} - Y\,\tilde{X}}{X - Y}. \tag{1.7}$$

For stationary and axially symmetric metrics, which are one major application in the following, integrability has been shown for the Kerr metric by Brandon Carter [Car68b; WP70].

A classical result on cubic integrals in dimension 2 that was given by Jules Drach can be found in [Dra35]. In particular, Drach obtained a list of 10 potentials that admit a cubic integral (of odd parity) on a Euclidean space with Hamiltonian $H = p_x^2 + p_y^2 + V$.

More recent results include proofs of nonexistence on the 2-torus for cubic and quartic integrals in [Bya87; DK00; BM11] (for higher degrees almost nothing is known). On the 2-sphere, in contrast, some integrable examples are known. For instance, a new integrable system has been presented, with the additional integral being of cubic degree, by Dullin and Matveev in [DM04].

A review of existing literature on stationary and axially symmetric metrics can be found in Chapter 3. Literature on sub-Riemannian structures is given in Chapter 5.

1.1 Integrability and reducibility

A family $(I_i)_{i=1,\dots,k}$ of integrals is *in involution*, if any two of them commute w.r.t. the Poisson bracket, i.e. if $\{I_i, I_j\} = 0$, for any $i, j \in \{1, 2, \dots, k\}$. We use this terminology on the level of Killing tensors as well and call a family of Killing tensor fields involutive if their corresponding integrals are in involution[5]. A family of functions $(f_1, \dots, f_k)$ is called *functionally independent* if the 1-forms df_i are linearly independent at each point [Arn89]. The family $(f_1, \dots, f_k)$ is said to be *algebraically dependent* if there is a vanishing non-trivial polynomial of the f_i.

If a Hamiltonian system possesses a sufficient number of functionally independent integrals in involution, it is called *Liouville integrable* or *completely integrable*. Complete integrability is an important concept in the theory of Hamiltonian systems.

[5]Involutivity for Killing tensors can equivalently be formulated using the Schouten-Nijenhuis bracket. Two Killing tensors are involutive if they commute w.r.t. the Schouten-Nijenhuis bracket.

Definition 2 (Liouville integrability)**.** *A Hamiltonian system with D degrees of freedom is called* Liouville integrable *if it admits D functionally independent integrals I_i in involution, i.e.*

$$\{H, I_i\} = 0,\ \forall\, i = 1, \dots, D \quad ; \qquad \{I_i, I_j\} = 0,\ \forall\, i, j = 1, \dots, D.$$

We first consider the case when the mechanical system is coming from a pseudo-Riemannian metric g on a manifold of dimension $D = 4$, i.e. $H(x,p) = g^{ij}p_ip_j$, and where the metric g possesses two commutative independent Killing vector fields (some additional assumptions are specified later). Therefore, because H itself is an integral, one additional integral functionally independent of the linear integrals coming from the Killing vector fields, and in involution with them, is sufficient for (Liouville-Arnold) integrability. We assume that this additional integral is a (w.l.o.g. homogeneous) polynomial in the momenta p, where the coefficients may depend on the position q. Similarly, we discuss examples in sub-Riemannian geometry for dimensions 6, 7 and 8 in Chapter 5.

We briefly note that Liouville integrability is not the only notion of integrability. Actually, one can distinguish various kinds of integrability for systems of PDEs. For instance, there is the notion of Frobenius integrability (existence of integral manifolds). Another kind of integrability can be found in the context of the Painlevé test, where integrability is defined via the nonexistence of movable critical singularities [Con97; CM03]. On the other hand, Liouville integrability is often only considered for a special class of integrals. For example, Stäckel integrability is a form of integrability involving only integrals that are quadratic polynomials in the momenta.

Another concept of integrability is *integrability by quadrature*. A system is called integrable by quadrature if its solutions can be obtained through basic operations like algebraic manipulations and the inversion or integration of known functions [Arn89]. Liouville integrability implies integrability by quadrature. Furthermore, Liouville integrability restricts the position of orbits in phase space T^*M. The exact statement is:

Fact 1 (Liouville-Arnold Theorem [Arn89])**.** *Let (M^D, σ) define a Hamiltonian system which admits D functionally independent integrals $(F_1, \dots, F_D)$ in involution. Then the level set $M_c = \{x : F_i(x) = c_i,\ i = 1, \dots, D\}$ is a smooth manifold and invariant under the flow of X_H with $H = F_1$. If M_c is compact and connected, it is diffeomorphic to the D-torus $\mathbb{T}^D$. The Hamiltonian equations $\dot{x} = X_H(x)$ are integrable by quadrature.*

So, given compactness, geodesic orbits lie on tori in phase space. One can relax the compactness requirement, and this leads to a weaker formulation of the theorem where orbits instead lie on cylinders [Arn89; FGS03].

For the cases that we consider in this thesis, there already is a family $\mathcal{I}$ of known (simple) integrals (functionally independent and in involution; homogeneous polynomial in momenta). We refer to these integrals as *trivial* or *standard integrals*. In such situations, we study whether or not $\mathcal{I}$ can be extended to an involutive family that makes the system integrable in the Liouville sense. For later reference, let us define

Definition 3. *Let $\mathcal{I} = (I_2, \dots, I_D)$ be a family of polynomial integrals in involution of degree at most d. We say that $\mathcal{I}$* can be extended to a Liouville-integrable family of integrals of degree at most d *if there is an additional integral I, polynomial in momenta, that makes $(\mathcal{I}, I) = (I, I_2, \dots, I_D)$ a family that ensures Liouville integrability, i.e. such that I is in involution with $\mathcal{I}$ and such that the integrals are functionally independent.*

Some of our results can be understood as reducibility results. By reducibility we mean:

Definition 4 (Reducible integral)**.** *Let I be an integral of degree d for a Hamiltonian system, and let $\mathcal{I} = (I_1, \ldots, I_k)$ be a family of integrals of lower degree than I. Then the integral I is called* reducible by $\mathcal{I}$ *(or, briefly,* $\mathcal{I}$-reducible*) if it can be written as a linear combination of products of integrals in $\mathcal{I}$, i.e. if I lies in the subalgebra generated by $\mathcal{I}$.*

The integral I is called reducible *if there is a family $\mathcal{I}$ of integrals such that I is $\mathcal{I}$-reducible. In addition, we allow that $\mathcal{I}$ contains the Hamiltonian.*

An integral is called *irreducible* if it is not reducible, i.e. if it cannot be written as a linear combination of products of integrals of lower degree in momenta and the Hamiltonian. If an integral I is reducible and if there is a family $\mathcal{I}$ containing solely linear integrals and the Hamiltonian, then I is called *totally reducible*.

In the context of Killing tensors, reducibility implies that a Killing tensor field can be written as a linear combination of symmetrized products of lower-valence Killing tensors and the metric. In the case of total reducibility, a reducible Killing tensor can be written as a linear combination of symmetrized products of Killing vector fields and the metric tensor field.

Often we make use of the term *additional integral* for the integrals we investigate. By this term we mean integrals that are irreducible w.r.t. a given family $\mathcal{I}$ of integrals, in involution with and linearly independent of the integrals of $\mathcal{I}$. Another term that is going to be used frequently is *final integral*. By this we mean an additional integral I_f to a family $\mathcal{I}$ of integrals, such that $(\mathcal{I}, I_f)$ makes the Hamiltonian system under consideration Liouville integrable (cf. Definition 3). For an instructive example, see Section 2.3.4 on pages 37ff. For instance, the Schwarzschild metric has an additional *quadratic* integral (in addition to those resulting from obvious symmetries), in spite of the fact that it has four linear integrals (which, however, are not in involution). Therefore, it has a final integral. This final (or additional) integral provides a complete family of four integrals. However, the *final* integral is reducible, because it can be written as a sum of squares of linear integrals.

1.2 Cartan-Kähler prolongation-projection

Let $\mathcal{E}$ be a differential equation and $\mathcal{S}$ a system of differential equations, i.e. a set of one or more differential equations. We call $\mathcal{E}$ an *algebraic consequence* of $\mathcal{S}$, if $\mathcal{E}$ can be obtained from $\mathcal{S}$ through algebraic manipulations. Similarly, we call $\mathcal{E}$ a *differential consequence* of $\mathcal{S}$ if it can be obtained from $\mathcal{S}$ by algebraic manipulations and partial differentiation, cf. [MS10]. Our exposition in this section follows [MS10], see also [KLV86].

The fundamental idea of prolongation-projection is as follows. Given a system $\mathcal{S}$ of PDEs, add differential consequences to it, i.e.

$$\begin{aligned}
\mathcal{S}^{(0)} &= \mathcal{S} \\
\mathcal{S}^{(1)} &= \mathcal{S}^{(0)} \cup \frac{\partial \mathcal{S}}{\partial x^1} \cup \cdots \cup \frac{\partial \mathcal{S}}{\partial x^D} \\
\mathcal{S}^{(2)} &= \mathcal{S}^{(1)} \cup \frac{\partial^2 \mathcal{S}}{\partial x^1 \partial x^1} \cup \cdots \cup \frac{\partial^2 \mathcal{S}}{\partial x^D \partial x^D} \\
&\ldots
\end{aligned} \tag{1.8}$$

Here, we assume to have D base variables. We denote by $\frac{\partial \mathcal{S}}{\partial x^i}$ the set of equations obtained by differentiating every equation in $\mathcal{S}$ w.r.t. the base variable x^i etc. Prolonging the system in this way does not add or remove solutions (we assume $\mathcal{C}^\infty$-differentiability). Since $\mathcal{S} =$

$\mathcal{S}^{(0)} \subset \mathcal{S}^{(k)}$, any solution of $\mathcal{S}^{(k)}$ solves $\mathcal{S}$ as well. Conversely, a smooth solution of $\mathcal{S}$ obviously is also a smooth solution of the k-th prolongation $\mathcal{S}^{(k)}$.

Now, in each $\mathcal{S}^{(k)}$, let us consider the derivatives of the unknown functions as new, independent variables (this converts the differential system $\mathcal{S}^{(k)}$ into an algebraic system of equations). This conversion does not lose solutions since a solution of the initial differential problem provides a solution of the algebraic problem derived from it. We are going to refer to the algebraic system obtained in this way as the *associated system*.

Geometric interpretation. *Jet spaces* provide a geometric understanding of differential equations, and we intend to mention this geometric formulation because it elucidates the terminology *prolongation-projection*. A precise treatment of this formalism can be found, for instance, in [KMS93; KLV86]. We follow the outline in [KMS93] .

In the general theory, jet spaces and jet bundles are a generalization of the concept of tangent spaces and tangent bundles. Basically, jet spaces are spaces of equivalence classes of mappings in $\mathcal{C}^\infty(M, N)$ between manifolds M and N. For our purposes, it suffices to consider the special situation when $N = \mathbb{R}$. For $\mathcal{C}^\infty$-functions $f : M \to \mathbb{R}$, one defines the equivalence relation of *r-th contact* as follows: Two smooth functions have r-th contact in a point $q \in M$, if all their derivatives in q up to order r are equal. This means that the r-th order Taylor polynomials of two functions agree if and only if they have r-th contact.

The jet of the function f is then defined as the equivalence class $j_q^r f := [f]_q$ of smooth functions that have r-th contact in $q \in M$. The respective jet space of the manifold M in the point q is the space of equivalence classes $j_q^r f$, i.e.

$$J_q^r M := \{j_q^r f : f \in \mathcal{C}^\infty(M, \mathbb{R})\}.$$

The corresponding jet bundle is $J^r M = \bigcup_{q \in M} J_q^r M$.

Given this construction, a partial differential equation (PDE) $\mathcal{E}$ on a function on M can be understood as an algebraic restriction on the corresponding jet in $J^r M$ (with r the order of the PDE). Now, if one differentiates $\mathcal{E}$, this *prolongs* the restriction to a higher-order jet space. On the other hand, eliminating highest derivatives is a *projection* to a lower-order jet space.

Overdetermined PDE systems of finite type. Systems of PDEs are said to be *of finite type* if after a finite number of differentiations, the equations allow us to express highest derivatives of the unknown functions by lower-order derivatives.

Definition 5. *A system $\mathcal{S}$ of PDEs of order n in the unknown functions $u_1(x), \ldots, u_l(x)$ is said to be* of finite type *if, for a certain $k \in \mathbb{N}$, its k-th prolongation $\mathcal{S}^{(k)}$ allows for expressing all partial derivatives $\frac{d^{n+k} u_i}{dx^J}$, where J is a multiindex with $|J| = n + k$, in terms of lower-order derivatives, for all $i = 1, \ldots, l$.*

For systems of PDEs that are overdetermined and of finite type the number of equations grows faster than the number of unknowns, because after a finite number of prolongations no new unknowns are added to the system. Thus, after a finite number of prolongation steps, an overdetermined algebraic system is obtained.

For the structural equations that govern Killing tensors, finiteness has been proven in [Wol98]. For a Killing tensor K of valence d, this paper establishes the system $\mathcal{S}_d^{(k)}$ of structural equations of the first d derivatives ($k \leq d$) of the components of K, along with integrability and algebraic conditions for them. Structural equations for (valence-2) Killing tensors are given in [HM75a; HM75b]. In [Tho86], structural equations were employed to study Killing tensors in constant curvature spaces.

1.3 Decomposition of the Poisson equation for integrals

In [Hie87] methods have been collected by Jarmo Hietarinta for searching for polynomial integrals. For quadratic Hamiltonians with known geodesic invariants, especially for Hamiltonian systems on a flat manifold, the (polynomial) Poisson equation (1.3) is treated in a degree-wise manner. By this approach it is possible to construct Hamiltonian invariants from the already known geodesic invariants, and in addition to find restrictions to the Hamiltonians that admit them.

Consider the 2-dimensional case with a Hamiltonian in the form $H = H^{(2)} + H^{(0)}$, where $H^{(2)}$ is homogeneous quadratic and $H^{(0)}$ scalar w.r.t. momenta (p_x, p_y). In Section 2.2, we argue that for such systems it is sufficient to consider separately the integrals that have odd and even parity in the momenta p_x and p_y. Therefore consider an integral of the form

$$I = I^{(d)} + I^{(d-2)} + \cdots + I^{(e)}, \quad e \in \{0, 1\},$$

where each $I^{(k)}$ is a homogeneous component of I that has degree $\deg(I^{(k)}) = k$ in momenta. In [Hie87], the cases $d = 2, 3, 4$ are considered. The requirement (1.3) for I being an integral is a polynomial in the momenta. We decompose it into its homogeneous parts:

$$\begin{aligned} \{H^{(2)}, I^{(d)}\} &= 0 \\ \{H^{(2)}, I^{(d-2)}\} + \{H^{(0)}, I^{(d)}\} &= 0 \\ \{H^{(2)}, I^{(d-4)}\} + \{H^{(0)}, I^{(d-2)}\} &= 0 \\ &\cdots \end{aligned} \tag{1.9}$$

Note that the first of these equations is itself a Poisson equation (1.3) for the polynomial $I^{(d)}$. In [Hie87] the focus is on the 2-dimensional Euclidean case and Hamiltonians with a potential term, i.e. the Hamiltonian is taken in the form $H = H^{(2)} + H^{(H)}$ with $H^{(2)} = \frac{1}{2}(p_x^2 + p_y^2)$. The potential typically is of a given form $H^{(0)} = V(x, y)$. Since the geodesic invariants are known for Euclidean space [Tho86; Wol98], the first equation of (1.9) can be solved explicitly in terms of some parameters. Substituting this solution into the remaining equations, one obtains the other components of I as well as restrictions on the parameters and the potential V. In [Hie87] several examples are worked out, in which the equations can be solved step by step along this prescription (i.e. the system (1.9) is solved "from top to bottom"). During this procedure, the integrability conditions for the equations in (1.9) typically turn into restrictions on the potentials $V = H^{(0)}$.

A recent application of this method can be found in [Mar14], where a systematic search has been undertaken for integrals in degree 2 and 4 of the Newtonian analogue of stationary and axially symmetric gravitational fields.

Whereas the described method works very well for the situations discussed in [Hie87; Mar14], it is clear that this approach cannot always work since the problem of finding geodesic invariants in general is very difficult.

However, a similar procedure is used in Chapter 4, when we consider cubic integrals in Weyl metrics. To circumvent the problem of finding geodesic invariants, we use the prescription in a reverse manner, starting from the lowest-degree equation in a list similar to System (1.9). The decomposition (1.9) also plays an important role in the construction of Algorithms I and II in Chapter 2.

Bertrand-Darboux equation. We consider the integrability condition obtained, in the 2-dimensional case, from the equation

$$\{H^{(2)}, I^{(0)}\} + \{H^{(0)}, I^{(2)}\} = \{T, W\} + \{V, K\} = 0 \tag{1.10}$$

where $T = g^{ij}p_ip_j$, $K = K^{ij}p_ip_j$ and $V(x,y)$, $W(x,y)$ are functions ($i, j \in \{1,2\}$). Writing down the coefficients of the polynomial (1.10), which have to vanish independently, one obtains equations for the derivatives $W_i = \partial_i W$. Computing $W_{ij} - W_{ji} = 0$, the integrability condition for W is obtained.

If we identify the Killing tensor K^{ij} with the homomorphism K^i_j, the integrability condition can be written in the concise form

$$d(K(dV)) = 0 \tag{1.11}$$

and is called the *Bertrand-Darboux equation*. This integrability condition is used in Section 4.4 in the determination of integrals for the Zipoy-Voorhees metric.

1.4 Assumptions

In what follows, we are going to explore Hamiltonian systems on D-dimensional manifolds that already possess $D-2$ functionally independent linear integrals $I_i : T^*M^D \to \mathbb{R}$, where $i \in \{1, \ldots, D-2\}$, in addition to the Hamiltonian H. The question to answer is whether or not there exists a D-th (final) low-degree integral in addition to the family of $D-1$ known involutive integrals.

Our approach requires Hamiltonian systems with certain properties. The following paragraphs serve to summarize the general setting for which the method, in particular that developed in Chapter 2, is suitable. Moreover, our list contains some of the properties that can be exploited for a simplification of the problem.

The integrals. We consider integrals that are *homogeneous polynomials in the momentum coordinates* p_i, i.e.

$$I(x,p) = \sum_{i_1+\cdots+i_D=d} a_{i_1,\ldots,i_D}(x)\; p_1^{i_1}\cdots p_D^{i_D}.$$

Note that the coefficients depend (smoothly) on the position. The degree in momenta is referred to as the degree of the integral.

— We are looking for families of integrals $T^*M^D \to \mathbb{R}$ that are in involution, i.e. that commute pairwise w.r.t. the Poisson bracket $\{\cdot,\cdot\}$.

— The coefficients in these polynomials depend on position. They are smooth functions (smoothness is here understood as the differentiability class $\mathcal{C}^\infty$).

Recall that we adopted the shorthand convention to call integrals *polynomial integrals*, if they are polynomial in the momenta, and that we refer to the degree of the integral w.r.t. momenta as the *degree of the integral*. Thus, a *quadratic integral* is understood as an integral that is homogeneous polynomial in momenta of degree 2 with smooth coefficient functions, and so on for the other degrees.

Symmetry requirements. Since we consider Hamiltonian systems with only one integral missing for Liouville integrability, the obvious question is whether or not there is an additional integral I_D such that the family $(I_i)_{i=1,\dots,D}$ is involutive. The requirement for a homogeneous polynomial I in momenta to be an integral is the Poisson equation (1.3). It defines a system $\mathcal{S}$ of PDEs on the coefficient functions of I. The prolongation of $\mathcal{S}$ has to be of finite type in order that our methods can be applied. For our examples, this is ensured by [Wol98] or Lemma 1 in Section 3.1.1.

Vacuum Requirement. In Chapters 3 and 4, we consider stationary and axially symmetric metrics that satisfy the vacuum condition, i.e. the Einstein tensor or, equivalently, the Ricci tensor are required to vanish identically. This requirement is also called *Ricci-flatness*. In Chapter 4, we investigate Weyl metrics. For these metrics the vacuum condition is not satisfied automatically, but has to be imposed separately by requiring a certain set of equations to hold. For stationary and axially symmetric vacuum (SAV) space-times, this set of equations can be rewritten in terms of a differential equation for a complex function, the *Ernst potential*, plus two secondary equations.

Sub-Riemannian structures. In Chapter 5, we discuss the connection between symmetries and the existence of integrals for sub-Riemannian structures. We consider Carnot groups which are nilpotent approximations to general sub-Riemannian structures [Bel97; MSS97]. More specifically, we consider left-invariant rank-2 distributions on Carnot groups. Given a sub-Riemannian metric on the distribution, it is discussed on page 74f how to obtain the corresponding Hamiltonian. The Hamiltonians that we consider are left-invariant. Therefore, right-invariant vector fields automatically lead to integrals since multiplication from the left and multiplication from the right commute.

Chapter 2

Algorithmic Method

In this chapter we are going to establish the general method and develop two algorithms, which are applied to stationary and axially symmetric metrics in Chapter 3. Another algorithm is only briefly mentioned in the present chapter and is presented in more detail in Chapter 5 (Algorithm III on page 79), where it is applied to left-invariant sub-Riemannian structures on Carnot groups. The concept behind the algorithms (especially Algorithms I and II) is also the foundation for Chapter 4.

Only in very few cases can the existence or nonexistence of integrals be checked by direct integration. For Hamiltonian systems, the method described in Section 1.3 allows us to find Hamiltonian invariants in systems for which the geodesic invariants are already known. For his construction of the quadratic integral in the Kerr metric, Carter starts from the separability of the Hamilton-Jacobi and the Schrödinger equation [Car68a]. Other approaches for checking integrability have been used, for instance, in [WP70] and [GHKW11].

As noted on page 12, there are several notions of (non-)integrability. The Painlevé test, for instance, is a method to check the existence of movable critical singularities, see [Con97; CM03]. Another approach (e.g. Ziglin theory, Morales-Ramis theory) invokes Differential Galois theory [MR99; Gor01; MPS13]. There are also approaches using properties of the topological entropy [Taĭ94]. Numerical evidence of (non-)integrability can be obtained using Poincaré surfaces of section, e.g. [Bri08b; LG12]. However, some caution is required with numerical observations; see [Bri08b] for an example where numerical observation misleadingly suggested integrability [KM12; MPS13; LG12].

We use a computer implementation of the Cartan-Kähler prolongation-projection method to make statements about the integrability of Hamiltonian systems on a D-dimensional manifold M with a (Lorentzian) metric g. The method itself is not restricted to Hamiltonian systems. By integrability we refer to the existence of a final D-th integral $I : T^*M \to \mathbb{R}$ of a certain homogeneous polynomial degree d in momenta, in addition to a family $\mathcal{I}$ of known (simple) integrals, cf. Definition 3.

The Poisson equation (1.3) defines a system of partial differential equations on a set of independent variables (*base variables*, *coordinates*) and dependent variables (*unknown functions*). We consider integrals that are homogeneous polynomials in momenta with smooth coefficient functions, see also Section 1.4. The method that we use can rigorously establish nonexistence of smooth integrals that are polynomial in momenta of a given degree and in involution with a family of given standard integrals. Moreover, the method can detect additional (involutive) integrals if they exist. We study homogeneous polynomial integrals (w.r.t. momenta) because we are interested in Killing tensors. However, studying the homogeneous case is sufficient if one is interested in polynomial integrals for Hamiltonians that are homogeneous polynomial in the momenta (geodesic invariants). It is a classical result that nonexistence of homogeneous polynomial integrals implies the nonexistence of real-analytic integrals, if the Hamiltonian is a homogeneous polynomial in the momenta (tome III of [Dar87]).

The possibility of a computer implementation has clear benefits that make the method a promising new technique. First, it is computationally efficient and allows us to address complicated problems that cannot easily be accessed by other methods. Since the number of equations and unknowns involved in the problems we study grows significantly with increasing dimension and with increasing degree of the integral, manual computation is hopeless for many interesting cases.

Second, computer implementation allows us to give a completely rigorous computer-assisted proof in a conceptually concise way (the method does not involve any approximations). Since an application to parametrized metrics is also possible (see Section 3.3), this opens up for a large range of applications. In recent years, computer-assisted proofs have already helped in solving interesting and hard problems in mathematical physics and pure mathematics, e.g. the four color theorem in the Euclidean plane, and it is likely that the role of computer-based mathematics will increase further in the years to come.

2.1 General idea

The problem defined by the Poisson equation (1.3) formulates the requirement that a function $I : T^*M^D \to \mathbb{R}$ that is homogeneous polynomial in momenta is a geodesic invariant for a Hamiltonian dynamical system with Hamilton function $H = g^{ij}\, p_i p_j$. Assume I is polynomial of degree d. The problem can then be rewritten as a system $\mathcal{S} = \mathcal{S}_d$ of partial differential equations on the coefficient functions of I. This system of PDEs is overdetermined[1], having $\binom{D+d}{d+1}$ differential equations on $\binom{D+d-1}{D-1}$ unknown functions, depending on $D = \dim(T^*M)/2$ coordinates. Moreover, it is of finite type, cf. [Wol98] or the proof to Lemma 1 on page 26. We adopt an approach via prolongation-projection and study the associated algebraic system of the prolongations of $\mathcal{S}_d$. We make use of the structure of $\mathcal{S}$ in order to reduce the number of unknowns and equations (projection step).

In Section 1.2 we have outlined the idea of the prolongation-projection method along general lines. Let us now study this in more detail. Provided the Hamiltonian is given explicitly, the associated system becomes a linear system of equations, because in Equation (1.3) the coefficients of the unknowns are determined by components of the metric and their first derivatives. Studying the number of solutions of this linear system provides a bound to the number of independent integrals.

When we develop the algorithm, we make use of the structural properties of the considered system of PDEs. In particular, we take into account the form of the structural equations for Killing tensors as discussed in [Wol98]. Moreover, we group the equations in a component-wise manner w.r.t. momenta.

Prerequisites. We consider integrals that are homogeneous polynomials of degree d in the momenta. A number of $D-2$ linear integrals $p_3, \ldots, p_D$ is assumed to exist in addition

[1] Note that the initial system of PDEs is overdetermined as a differential system, i.e. the number of equations is larger than the number of *unknown functions*. However, if derivatives of unknown functions are counted as independent algebraic objects, the number of *unknowns* in the initial system is larger (at least in general) than the number of equations. Only after taking enough prolongations we obtain a system with more equations than unknowns.

to the Hamiltonian. Imposing involutivity with the linear integrals, any final integral has to be independent of the coordinates $x^3, \dots, x^D$ corresponding to the linear integrals, because in suitable coordinates $I_i = p_i$ (with ignorable coordinates x^i, $i = 3, \dots, D$, see on page 23), and thus

$$0 = \{I_i, I_{\text{final}}\} = \{p_i, I_{\text{final}}\} = \partial_i I_{\text{final}}.$$

Moreover, the parity properties of the Hamiltonian can be used in order to decompose the system of PDEs into smaller subsystems (cf. Section 2.2.2). Since computational efficiency heavily depends on the number of equations and unknowns of the problem, this can significantly improve the performance of the algorithm.

The associated linear system. Let $\mathcal{S}_d^{(k)}$ denote the k-th prolongation of $\mathcal{S}_d$, i.e. the system of PDEs obtained by adding derivatives of order $\le k$ of the differential equations of $\mathcal{S}_d$. The total number of equations in $\mathcal{S}_d^{(k)}$ equals

$$m_{d,k} = \binom{D+d}{D-1} \cdot \binom{k+2}{2}, \tag{2.1}$$

but usually not all of them are independent (D denotes the dimension of the underlying manifold).

The unknowns of the associated linear system are the derivatives of the unknown functions evaluated at a point P, i.e. $a_{\tau,\sigma}(P) = \partial_\sigma a_\tau(P)$ where τ and σ are multiindices[2]. Since we study systems with enough symmetry, only derivatives w.r.t. the two non-ignorable coordinates x^1 and x^2 are involved. We denote the set of unknowns by $\mathcal{V}_d^{(k)}$, and represent it as a column vector. The number of entries is equal to

$$n_{d,k} = \binom{d+D-1}{D-1} \cdot \binom{k+3}{2}. \tag{2.2}$$

The associated (linear) system can be represented by a $m_{d,k} \times n_{d,k}$ matrix $M = M_d^{(k)}$, and this matrix system associated with $\mathcal{S}_d^{(k)}$ has the form

$$M\, v = \begin{pmatrix} \cdots \\ \cdots \\ \cdots \\ \cdots \end{pmatrix} \begin{pmatrix} \vdots \end{pmatrix} = 0 \tag{2.3}$$

with v being the vector of the unknowns $\mathcal{V}_d^{(k)}$. Note that the matrix in general has many more rows than columns since we deal with an overdetermined system.

The rank computation. Computation of the rank of the matrix describing the associated linear system provides the dimension of the space of solutions via the usual dimension formula. These are the independent solutions for the unknown functions and their derivatives in the chosen point of reference, see Observation 2 on page 28. The number of solutions is an upper bound $\overline{\Lambda}_d$ to the number of linearly independent polynomial integrals of degree d, and in general much greater than the number of integrals that the system admits.

[2] The multiindex τ is a tuple $\tau = (\tau_1, \dots, \tau_D)$ labeling the coefficients of the integral (by denoting the exponents of the corresponding moment monomials), while the tuple $\sigma = (\sigma_1, \sigma_2)$ denotes the order of differentiation w.r.t. the base variables x^1 and x^2, respectively. Therefore, $|\tau| = \sum \tau_i = d$ is the degree of the integral and $|\sigma| = \sigma_1 + \sigma_2$ the order of prolongation.

We can easily count the number of trivial integrals arising from the Hamiltonian and the $D-2$ linear integrals. It is given by the formula (for degree d of the integral)

$$\Lambda_d^0 = \sum_{l=0}^{\lfloor d/2 \rfloor} \binom{D+d-2l-3}{d-2l}. \tag{2.4}$$

Now, let Λ_d be the number of linearly independent first integrals of degree d (including the trivial integrals Λ_d^0). We have $\Lambda_d \leq \overline{\Lambda}_d^{(k)}$. If we achieve, after a certain number k_0 of prolongations, that the upper bound coincides with the number of known (trivial) integrals,

$$\Lambda_d \leq \overline{\Lambda}_d^{(k_0)} = \Lambda_d^0,$$

then this proves $\Lambda_d = \Lambda_d^0$, because $\Lambda_d^0 \leq \Lambda_d$. Thus, there cannot be additional integrals. In the examples considered in Chapter 3, the upper bound coincides with the number of known integrals after $k_0 = d$ steps of prolongation. In Chapter 5, the method concludes after $k_0 = d+1$ steps.

2.1.1 The Kruglikov-Matveev algorithm

A first implementation of the algorithm along the above lines has been given in [KM12]. It determines, for a static and axially symmetric metric, more specifically the Zipoy-Voorhees metric with $\delta = 2$, the number of independent integrals that are homogeneous polynomials in momenta and in involution with the integrals that follow from staticity and axial symmetry of the system.

Nonexistence of an additional integral is shown up to degree 6 in [KM12]. For the computation, coordinates are used that are adapted to the symmetries. In particular, [KM12] employs prolate spheroidal coordinates. We are going to use the same coordinates in our computations for the Zipoy-Voorhees metric, see Equation (3.17). Part of the techniques that we describe in the following section have already been applied in [KM12].

2.2 Techniques

With increasing degree of the integral, the matrix system (2.3) grows and becomes harder and harder to handle directly. It is therefore crucial to use tools of some kind to reduce the number of equations and unknowns. Thereby, efficiency of the general algorithm is increased, but in some cases the computations are only possible if suitable techniques are applied to reduce the problem.

On the following pages, we give a short synopsis of the techniques that we employ. In part, these tools are already used in [KM12], namely symplectic reduction, parity decomposition w.r.t. non-ignorable coordinates and the choice of a generic, rational point. New techniques[3] include the parity decomposition w.r.t. ignorable coordinates and the use of the inner structure (block structure for equations and unknowns). The reduction of the number of unknowns by addition of trivial integrals is also new. Efficiency of these new techniques is discussed in Section 2.4.

[3] Algorithms I, II in the form given in this thesis have not been published before. Results obtained with a first version of Algorithm II have been presented by the author at the FDIS conference 2015 at Będlewo, Poland [Vol15a]. Algorithm III, in slightly different form, has been presented and applied as part of the collaborative work [KVL15].

2.2.1 Symplectic reduction

Ignorable coordinates and symplectic reduction. The presence of symmetries allows one to reduce a Hamiltonian dynamical problem to a lower-dimensional one. This fact has been known classically, see, e.g., the book by Whittaker [Whi04].

In our context, symplectic reduction is a tool that we use to reduce the problem to 2 dimensions and it is also instructive for finding a suitable viewpoint on the equations. When we use coordinates, some of them may not appear in the Hamiltonian, i.e. they might be *ignorable* (sometimes also called *cyclic*). More precisely, this means that certain coordinates do not appear in the Hamiltonian of the system, though their corresponding momenta do. This allows us to define a new *reduced* Hamiltonian by fixing the ignorable momenta. The ignorable coordinates can be recovered by an integral formula [Whi04]. If there are k ignorable coordinates in a dynamical system with D degrees of freedom ($2D$-dimensional phase space), the reduced dynamical system has $(D-k)$ degrees of freedom left. The corresponding coordinates are called *non-ignorable*.

In more modern terms, this can be understood in a coordinate-free manner via actions of Lie groups on symplectic spaces. This is known as *symplectic reduction*, see e.g. [Mar92]. More precisely, assume that there is a global symmetry group G acting on a Hamiltonian symplectic manifold (N, σ, H) and preserving the symplectic form σ (N is a manifold and by H we denote the Hamiltonian). Any element of the Lie algebra $\mathfrak{g}$ of G defines a natural vector field on N via the exponential map. A moment map for G is a mapping $\mu : N \to \mathfrak{g}^*$ such that $d(\mu, \xi) = \imath_v \sigma$, where we define a function $N \to \mathbb{R}$ by $(\mu, \xi)(x) = \langle \mu(x), \xi \rangle$ and denote the vector field corresponding to $\xi \in \mathfrak{g}$ by the symbol v). The statement of the Marsden-Weinstein reduction theorem is:

Fact 2 (Reduction theorem [Mar92])**.** *Consider a level set for μ and identify any two points that can be transferred into one another by a group transformation. This quotient space inherits a symplectic structure from N, and thus it can be used as a new phase space. Also, dynamical trajectories of the Hamiltonian H on N determine corresponding trajectories on the reduced space.*

Under an additional compactness assumption, this construction is also referred to as the Marsden-Weinstein quotient, see e.g. [MS95].

In our context, symplectic reduction is advantageous since we study systems with enough symmetry to be able to view the problems as effectively 2-dimensional ones, involving only two base variables. The other coordinates are adjusted to the symmetries. This limits the number of derivatives that we need to take into account for the prolongation. Since the number of m-th order partial derivatives w.r.t. D independent coordinates is $\binom{D+m-1}{m}$, the number of equations grows heavily with D, and reducing the number D therefore can clearly decrease the necessary computational effort.

The Hamiltonian is always an integral and we can thus consider level sets of the Hamiltonian in addition to using symplectic reduction. This further reduces the dimension of phase space by one (classical, cf. [Whi04]). We take this into account in Section 5.4, where we deduce an explicit low-dimensional representation for the Poisson equation (1.3).

2.2.2 Parity decomposition

The reduction to a 2-dimensional space comes at a cost. In the reduced picture, the Hamiltonian is no longer a homogeneous polynomial in the momenta. In the initial problem, we are concerned with finding geodesic invariants, i.e. integrals that are homogeneous polynomials in momenta for Hamiltonians that are defined solely by the metric. When performing symplectic reduction, we go over to working on level sets, where some of the momenta turn into constants. Therefore, on the reduced manifold we have a Hamiltonian that is non-homogeneous,

$$
\begin{aligned}
H|_{p_i=c_i,\ i\in\{3,\ldots,D\}} &= \left(\sum_{i,j\in\{1,\ldots,D\}} g^{ij}\, p_i p_j\right)\Bigg|_{p_i=c_i,\ i\in\{3,\ldots,D\}} \\
&= \underbrace{\sum_{i,j\in\{1,2\}} g^{ij}\, p_i p_j}_{\text{kinetic}} + \underbrace{2\sum_{\substack{i\in\{1,2\}\\ j\in\{3,\ldots,D\}}} g^{ij}\, p_i c_j}_{\text{linear}} + \underbrace{\sum_{i,j\in\{3,\ldots,D\}} g^{ij}\, c_i c_j}_{\text{potential term}} \quad .
\end{aligned} \tag{2.5}
$$

In this way, the reduction leads to a Hamiltonian that has a potential term, and in general also a linear term, in addition to the kinetic term defined by the (reduced) metric. As a consequence, we have to consider Hamiltonian invariants instead of geodesic invariants. Thus, in a sense, we trade off dimension against homogeneity.

We now present how this change of setting allows us to simplify the computational problem.

Parity in the non-ignorable momenta. In the examples of stationary and axially symmetric vacuum metrics, the reduced Hamiltonian has even parity in the non-ignorable coordinates, i.e. the reduced Hamiltonian consists of a quadratic and a scalar component only. This allows one to consider integrals of odd and even parity separately, since the system of PDEs decomposes accordingly. The restriction to pure-parity integrals is possible in all of our cases except for the sub-Riemannian examples in Chapter 5.

Observation 1 (Parity decomposition)**.** *If the Hamiltonian is of pure even parity, then for all homogeneous terms in the Poisson equation* (1.3)*:*

- *Terms of* (1.3) *that are of* even parity *in the non-ignorable momenta contain only coefficients of the integral from terms with* odd parity *in these momenta.*
- *Terms of* (1.3) *that are of* odd parity *in the non-ignorable momenta contain only coefficients of the integral from terms with* even parity *in these momenta.*

Proof. The polynomial $\{H, I\}$ is of degree $d+1$ in momenta provided that I is of degree d. Without loss of generality, let us assume that d is even, then $d+1$ is odd and the leading term w.r.t. non-ignorable momenta of $\{H, I\}$ is $\{H^{(2)}, I^{(d)}\}$, where superscripts indicate the homogeneous degree in non-ignorable coordinates. Therefore, this first term contains only coefficients in I from terms with even parity in the non-ignorable momenta. The next-to-leading term similarly contains only coefficients from $I^{(d-1)}$, which come from terms with odd parity, and so on. $\square$

Parity (of the potential) w.r.t. ignorable momenta. For SAV metrics we can split the Poisson equation (1.3) according to Observation 1. The Hamiltonian $H(x^1, x^2, p_1, \ldots, p_D)$ is of even parity in (x^1, x^2) and decomposes, through symplectic reduction, into a kinetic part $T(x^1, x^2, p_1, p_2)$ and a potential part $V(x^1, x^2, p_3, \ldots, p_D) = V(x^1, x^2, c_3, \ldots, c_D)$.

Now, in addition, we assume that the potential $V = V^{33}p_3^2 + V^{34}p_3p_4 + V^{44}p_4^2$ is invariant under inversion $p_a \mapsto \pm p_a$ for $a \in \{3, 4\}$. We do not require simultaneous inversion of both momenta. In our context, this additional symmetry requirement simply means that we assume a coordinate system such that the metric is diagonal (hypersurface-orthogonality[4]). Provided this additional requirement holds, the system of PDEs from (1.3) splits into four separate systems of PDEs, according to the parity in the two non-ignorable coordinates and the parity in p_3 (or p_4). This is worked out in detail in the context of Lemma 2 on page 36.

2.2.3 Inner structure of the system of PDEs

The system of PDEs that we consider actually has a lot of inner structure, which can be taken advantage of. Our system of PDEs emerges from the Poisson equation (1.3), which is a polynomial in the momenta.

This motivates considering the structure of and between equations obtained from degree-wise decomposition of this polynomial. Similar considerations are made in [Hie87] for several examples, see Section 1.3. By symplectic reduction, and after using Observation 1, we obtain a non-homogeneous polynomial equation

$$\{H, I\} = \{T + V, I^{(d)} + I^{(d-2)} + \cdots + I^{(e)}\} = 0 \tag{2.6}$$

where $e \in \{0, 1\}$ has to be chosen according to the parity of the integral I. As in [Hie87], we can decompose this polynomial requirement w.r.t. degree in the non-ignorable momenta. In this way, we obtain a list of equations

$$
\begin{array}{lll}
\mathcal{E}_0 & \{T, I^{(d)}\} = 0 & \text{(2.7a)} \\
\mathcal{E}_1 & \{T, I^{(d-2)}\} + \{V, I^{(d)}\} = 0 & \text{(2.7b)} \\
\vdots & \dots & \\
\mathcal{E}_{\text{fin}} & \begin{cases} \{T, I^{(0)}\} + \{V, I^{(2)}\} & = 0 \quad \text{(even parity branch)} \\ \{V, I^{(1)}\} & = 0 \quad \text{(odd parity branch)} \end{cases} & \begin{array}{l}\text{(2.7c)} \\ \text{(2.7d)}\end{array}
\end{array}
$$

These polynomials can be further decomposed by looking at components w.r.t. ignorable momenta. In Chapter 3, we have two ignorable momenta, and thus we can decompose the k-th equation $\mathcal{E}_{k-1}$ in (2.7) into $2k - 1$ new polynomial equations. The degree-wise decomposition leads to some helpful observations:

Structuring the unknowns. We organize the unknowns (i.e. the coefficients a_τ of I) into blocks in a natural way. We define blocks by first arranging the $I^{(i)}$ according to the value of l, where $2l = i - d + \tilde{e}$ with $\tilde{e} \in \{0, 1\}$ denoting the parity of $d + e$. Then, for constant i, we arrange them according to the order m of differentiation. The resulting block structure is sketched in Figure 2.1.

This block arrangement for the unknowns suggests to find a similar structure for the equations and then try to split the problem into smaller units. We can do this and use the smaller units of the linear-algebraic system to (partially) solve for some of the unknowns. This idea is invoked in the elimination scheme for SAV metrics in Algorithms I and II.

Structuring the equations. For the equations, it is possible to make an arrangement similar to that for the unknowns. Consider the equations $\mathcal{E}_l$ obtained from the decomposition (2.7). Consider derivatives $\frac{\partial^{|J|}\mathcal{E}_l}{\partial x^J}$, where J is a multiindex with $|J| = m$. When we arrange the sets of equations in blocks according to l and m, we obtain a structure as shown

[4]In the SAV examples, this hypersurface orthogonality requirement corresponds to the static limit.

Block structure of the unknowns

$l \backslash m$	0	1		$\cdots$	$M{-}2$	$M{-}1$	M
0	$I^{(d)}$	$\frac{\partial}{\partial x^J} I^{(d)}$			$\frac{\partial^{M-2}}{\partial x^J} I^{(d)}$	$\frac{\partial^{M-1}}{\partial x^J} I^{(d)}$	$\frac{\partial^{M}}{\partial x^J} I^{(d)}$
1	$I^{(d-2)}$						
$\vdots$				$\cdots \vdots \cdots$			$\vdots$
$L{-}1$	$I^{(d-L+1)}$						
L	$I^{(d-L)}$			$\cdots$			

Figure 2.1: Sketch of how we arrange the unknowns into the block structure. The multiindex J, in each block, runs over all combinations with $|J| = m$ according to the column number (level of prolongation).

in Figure 2.2. We use the same block structuring scheme for the partial differential equations that we obtained from the subpolynomials $\mathcal{E}_l$. Note, however, that the maximal number of rows in the figures, L and $\hat{L}$, need not be equal. They can differ by one.

Obviously, the blocks in Figures 2.1 and 2.2 are related, for $0 \le l \le L$, and the (l, m)-block of equations contains unknowns from the $(l, m+1)$-block of unknowns (e denoting the parity of the integral I). The crucial observation now is that for suitably large $m \ge 1$ all m-th derivatives of the unknown functions connected with the l-th row in Figure 2.1 can be solved for in the equations of the $(l, m-1)$-block in Figure 2.2 (analogously for higher derivatives in the blocks following to the right). It turns out that this is possible for $m \ge d - 2l - \tilde{e} + 1$ where the right-hand side of the inequality is the (p_1, p_2)-degree of the respective subpolynomial ($\tilde{e} = 0, 1$ denotes the parity of $d + e$). This fact can be seen from the structural equations as treated in [Wol98]. However, in our context (only 2 base variables), it can be understood much easier:

Lemma 1 (Partial solution of the blocks)**.** *For Hamiltonian invariants in 2-dimensional spaces with potential, all m-th derivatives of the unknowns ($m \ge d - \tilde{e} - 2l + 1$) in the l-th row of Figure 2.1 can be solved for (i.e. expressed by lower derivatives or unknowns with smaller l), given the equations in the $(l, m-1)$-block of Figure 2.2 with $0 \le l \le L$.*

Proof. The assertion is easily checked by inspecting the equations of the l-th row. The relevant term in the corresponding polynomial equations is of the form (by $\simeq 0$ we symbolize that the other terms are not of interest. They only involve lower-order derivatives of the unknowns or unknowns from blocks with lower value of l)

$$\{T, I^{(k)}\} = \{\Omega\,(p_1^2 + p_2^2), I^{(k)}\} \simeq 0, \tag{2.8}$$

where $I^{(k)}$ is a homogeneous polynomial of degree $k = d - 2l - \tilde{e}$ in momenta, and where $\Omega = \Omega(x^1, x^2)$ is a function (we assume isothermal coordinates). Such a term does not exist in the equations with $l = \hat{L}$ in the case of odd-parity integrals, and this is where the restriction on l appears.

The equations obtained from the polynomials (2.8) (at prolongation level $m = 0$) yield a system of equations that has the qualitative structure (we denote the coordinates by $x = x^1$

Block structure of the equations

$l \backslash m$	0	1		$\cdots$	$M{-}2$	$M{-}1$	M
0	$\mathcal{E}_0$	$\frac{\partial \mathcal{E}_0}{\partial x^J}$			$\frac{\partial^{M-2}\mathcal{E}_0}{\partial x^J}$	$\frac{\partial^{M-1}\mathcal{E}_0}{\partial x^J}$	$\frac{\partial^{M}\mathcal{E}_0}{\partial x^J}$
1	$\mathcal{E}_1$						
$\vdots$				$\cdots \vdots \cdots$			$\vdots$
$\hat{L}{-}1$	$\mathcal{E}_{\hat{L}-1}$						
$\hat{L}$	$\mathcal{E}_{\hat{L}}$			$\cdots$			

Figure 2.2: Sketch of how we arrange the equations into the block structure. The multi-index J, in each block, runs over all combinations with $|J|$ according to the column number (level of prolongation).

and $y = x^2$)

$$\begin{aligned}(a_0)_x &\simeq 0\\ (a_1)_x + (a_0)_y &\simeq 0\\ &\vdots\\ (a_k)_x + (a_{k-1})_y &\simeq 0\\ (a_k)_y &\simeq 0,\end{aligned}$$

where a_i, $i = 0, \ldots, k$, are the coefficients of $I^{(k)}$. Differentiating once w.r.t. the coordinates x and y, obtain

$$(a_0)_{xx} \simeq 0, \qquad (a_0)_{xy} \simeq 0, \qquad (a_1)_{xx} = (a_1)_{xx} + (a_0)_{xy} \simeq 0 \qquad \text{etc.}$$

Continuing in a similar way, one can solve more and more of the equations obtained after differentiation, beginning from 'above' and 'below'. Obviously, one is done after k differentiations, when all highest derivatives of the a_i are explicitly obtained. □

The proof for finiteness of the system of PDEs (1.3) can be obtained by an analogous reasoning. It provides a result corresponding to that of [Wol98] for the 2-dimensional case.

For our present discussion, Lemma 1 states that we can solve the equations in the (l, m)-block of Figure 2.2 for the unknowns of the $(l, m+1)$-block in Figure 2.1 ($m = d - 2l - \tilde{e}$). This implies that we can immediately replace a number of unknowns, namely those in these blocks and in blocks to the right of them. We are going to see on pages 39ff that this simple observation already helps to speed up the computations considerably.

2.2.4 Choice of a specific point

We choose a generic point on the reduced space (level sets of the linear integrals), i.e. we consider the evaluation of (2.3) at a point, i.e.

$$M_0\, v_0 = M(x^1 = x_0^1, x^2 = x_0^2)\, v(x^1 = x_0^1, x^2 = x_0^2) = 0, \tag{2.9}$$

where M and v are the matrix and the vector of unknowns describing the associated linear system (2.3). Of course, there is the risk that the rank of M_0 drops if a 'wrong' point is chosen. This would misleadingly suggest the existence of an additional integral. Therefore we have to be cautious to pick a point such that the matrix M_0 has the generic rank of the matrix M describing the associated linear system (we call such points *generic*). This simplifies the problem from computing the rank of a matrix in $\mathsf{Mat}(\mathcal{C}^\infty)$ to one in $\mathsf{Mat}(\mathbb{R})$.

Observation 2 (choice of a generic point)**.** *Given a specific metric, we choose a point such that the rank of the matrix describing the (linear) associated system (cf. page 21) takes its generic value. Then the upper bound $\overline{\Lambda}$ obtained from the matrix evaluated at this point is the same as the upper bound generically obtained from the original matrix (before fixing the point).*

Rational metric: *If the metric has entries that are rational functions in the base variables, the resulting matrix $M_0 = M(x^1 = x^1_0, x^2 = x^2_0)$ of the associated system can w.l.o.g. be considered a matrix in $\mathsf{Mat}(\mathbb{Z})$ if a point with rational coordinates is chosen.*

Known integrals: *Assume we already know a family $\mathcal{I}$ of integrals. We have the freedom of adding linear combinations of these integrals to our supposed additional integral I. Choosing this linear combination accordingly, we can eliminate some of the coefficients of I at the point (x^1_0, x^2_0), i.e. we can eliminate some of the unknowns in Equation (2.9). If all known integrals are taken into account, we do not need to compute the exact rank of the matrix $M(x^1_0, x^2_0)$. In this case, it suffices to check whether this matrix has maximal rank or not.*

Sparse Matrices. In case of the sub-Riemannian structures considered in Chapter 5, the considerations of Section 2.2.3 still have an analogy. However, the described technique does not work very efficiently due to the additional linear term in the reduced Hamiltonian. On the other hand, this can be remedied if one can choose a point that makes the equations particularly simple. Through such a choice, we obtain a matrix M_0 with very few non-zero entries. This allows us to iteratively solve monomial and bi-monomial equations first, before computing the rank of the remaining matrix. In this way, the number of rows and columns for the matrix of the rank computation is reduced considerably. In Chapter 5, we can investigate systems with several 10,000 equations and unknowns using this tool.

2.3 Algorithm for an explicitly given metric

In this section, we develop an algorithm to check nonexistence of additional integrals (geodesic invariants) for metrics with certain symmetry properties. Our attention is towards stationary and axially symmetric vacuum metrics, which in so-called *Lewis-Papapetrou coordinates* x, y, ϕ, t can be written (see Chapter 3)

$$g_{\mathrm{SAV}} = e^{2U}\left(e^{-2\gamma}\left(dx^2 + dy^2\right) + x^2\,d\phi^2\right) + e^{-2U}\,(dt + A\,d\phi)^2, \tag{2.10}$$

with three parametrizing functions $U(x, y)$, $\gamma(x, y)$ and $A(x, y)$. We sometimes choose other coordinates, for instance prolate-spheroidal coordinates. Therefore, we consider metrics of the structurally more general form[5]

$$g = g_{11}\,dx^1 \otimes dx^1 + g_{22}\,dx^2 \otimes dx^2 + \sum_{i,j=3,4} g_{ij}\,dx^i \otimes dx^j. \tag{2.11}$$

[5]The tensor products are written explicitly to avoid double superscripts.

where, of course, $g_{34} = g_{43}$, and where all $g_{ij} = g_{ij}(x^1, x^2)$ depend on the first two coordinates only ($i, j \in \{1, \dots, 4\}$). Thus:

— We can, via symplectic reduction, ignore the coordinates x^3 and x^4. Then, the reduced Hamiltonian has no linear terms in the momenta.

— We use suitable coordinates on the reduced space. Often, though not always, we choose isothermal coordinates, i.e. $g_{11} = g_{22}$ [Gau73; Kor14; Che55].

The central question that we pose can be written as follows:

Question 1. *How many integrals of maximal degree d (and in involution with the standard integrals) does the geodesic flow of a metric of the form* (2.11) *admit?*

2.3.1 Outline of the algorithm

We consider metrics of the form (2.11) and explore how to address Question 1 for such metrics. The approach follows the prolongation-projection method outlined in Section 1.2, and we demonstrate how to use the techniques that we outlined on pages 22ff. Amongst all tools described in Section 2.2, we stress two basic observations that are exploited for the algorithmic computations:

- **Use of symmetries of the system.** On one hand, we use the action of symmetry groups to reduce dimensionality of the computational problem to effectively two dimensions by disregarding ignorable coordinates. We require involutivity of the integrals. On the other hand, additional symmetry properties concerning parity of the integrals allow us to split the system of PDEs into smaller subsystems, another important simplification.

- **Multi-step elimination scheme.** The general algorithm described in Section 2.1 completes the *projection* step of prolongation-projection solely by computing the rank of the matrix of the associated linear system. This works well for smaller matrices but usually becomes time-consuming with larger matrix dimensions. Therefore we choose to extend the mere rank computation by a preceding reduction step, which scales down the number of equations (and unknowns) before computing the rank. In this way, we can make use of the particular inner structure of the system of equations. In turn, this also proves useful as a preparation for the (non-algorithmic) computations in Chapter 4.

An application of the algorithm that is described in this section is given in Section 3.4 on page 53f, where nonexistence of a final integral up to degree 7 is proven for a Tomimatsu-Sato metric.

Use of symmetries

Metric (2.11) obviously admits two linear integrals, namely p_3 and p_4. Via ignoration of coordinates, we can treat these momenta as constants and obtain the Hamiltonian in the reduced picture on level sets as

$$H = g^{ij} p_i p_j = \underbrace{\frac{p_1^2}{g_{11}} + \frac{p_2^2}{g_{22}}}_{\substack{=T \\ \text{kinetic term}}} + \underbrace{\frac{g_{44}}{g_{33}g_{44} - g_{34}^2}\, p_3^2 + \frac{g_{33}}{g_{33}g_{44} - g_{34}^2}\, p_4^2 - \frac{g_{34}}{g_{33}g_{44} - g_{34}^2}\, p_3\, p_4}_{\substack{=V \\ \text{potential term}}} \,. \quad (2.12)$$

Hence, H has two terms of even parity in the momenta p_1, p_2 of the reduced problem. By Observation 1 this allows us w.l.o.g. to restrict to integrals of pure odd or even parity. Obviously, this yields two subproblems, namely to separately solve the problems

$$\{H, I_{\text{even}}\} = 0 \qquad \text{and} \qquad \{H, I_{\text{odd}}\} = 0$$

where the subscripts indicate the corresponding parity w.r.t. non-ignorable momenta.

In general, both subsystems have to be investigated to obtain the full picture. However, simplifications in case of Weyl (diagonal) metrics are exploited for Algorithm II. For such metrics, additional symmetry exists and admits a further decomposition of the system of PDEs. This is worked out in detail on pages 34ff.

Elimination scheme

Let us briefly review how we obtain the equations for the relevant system of PDEs. The Poisson bracket is a polynomial

$$\{H, I\} = \sum_{i=0}^{d+1} \sum_{j=0}^{i} \sum_{k=0}^{d+1-i} P_k^{(i,\,j)}\; p_1^{i-j}\, p_2^{j}\, p_3^{k}\, p_4^{d+1-i-k} = 0, \tag{2.13}$$

where each $P_k^{(i,\,j)}$ represents an equation in the system of PDEs. The prolongated system is obtained by taking derivatives of the $P_k^{(i,\,j)}$. Denote the resulting equation obtained after m differentiations (with μ derivatives w.r.t. x^1 and $m-\mu$ w.r.t. x^2) by

$$P_k^{(i,\,j,\,m,\,\mu)},$$

with $i \in [\![0, d+1]\!]$, $j \in [\![0, i]\!]$, $k \in [\![0, d+1-i]\!]$, and $\mu \in [\![0, m]\!]$, where

$$[\![a, b]\!] = \{n \in \mathbb{Z} : a \leq n \leq b\} \tag{2.14}$$

denotes the set of integers between (and including) a and b. In our cases, not all $P_k^{(i,\,j)}$ are non-zero if we consider only integrals of pure parity in (p_1, p_2). If we consider integrals of odd (even) (p_1, p_2)-parity, then only $P_k^{(i,\,j)}$ with even (odd) value of i can be non-zero. Now, the unknown functions are the coefficients in the polynomial that represents I,

$$I = \sum_{\substack{i=0\\ \text{par}(i)=e}}^{d} \sum_{j=0}^{i} \sum_{k=0}^{d-i} I_k^{(i,\,j)}\; p_1^{i-j}\, p_2^{j}\, p_3^{k}\, p_4^{d-i-k} \qquad \text{with } e = 0 \text{ or } e = 1. \tag{2.15}$$

For the derivatives of the unknown functions, use a notation analogous to that for the $P_k^{(i,\,j)}$, namely

$$I_k^{(i,\,j,\,m,\,\mu)}, \tag{2.16}$$

with $i \in [\![0, d]\!]$, $j \in [\![0, i]\!]$, $k \in [\![0, d-i]\!]$, $\mu \in [\![0, m]\!]$. Here, m denotes the order of partial differentiation, and μ is the order of differentiation w.r.t. the coordinate x^1. If we write out the polynomial (1.3) using subpolynomials obtained as coefficients w.r.t. p_1 and p_2, cf. Section 2.2.3, we obtain the following picture.

$$\{T, I^{(d)}\} = 0, \tag{2.17}$$

$$\{T, I_0^{(d-1)}\} = 0, \qquad \{T, I_1^{(d-1)}\} = 0,$$

$$\{T, I_0^{(d-2)}\} + \{V^{44}, I_0^{(d)}\} = 0, \quad \{T, I_1^{(d-2)}\} + \{V^{34}, I^{(d)}\} = 0, \quad \{T, I_2^{(d-2)}\} + \{V^{33}, I^{(d)}\} = 0,$$

$$\{T, I_0^{(d-3)}\} + \{V^{44}, I_0^{(d)}\} = 0, \quad \{T, I_1^{(d-3)}\} + \{V^{44}, I_1^{(d)}\} + \{V^{34}, I_0^{(d)}\} = 0, \quad \{T, I_2^{(d-3)}\} + \{V^{34}, I_1^{(d)}\} + \{V^{33}, I_0^{(d)}\} = 0, \quad \{T, I_3^{(d-3)}\} + \{V^{33}, I_1^{(d)}\} = 0,$$

etc.

where we use the shorthand notation $V = \sum V^{ij} p_i p_j = H^{(0)}$ and $T = H^{(2)}$ for the homogeneous components of the Hamiltonian to avoid double superscripts. As we argue above, the components of I of odd and even parity w.r.t. (p_1, p_2) can be treated separately.

We explain the elimination scheme for one of the two branches only, namely for the branch whose parity equals that of d (i.e. the branch of highest degree in (p_1, p_2)). This choice is only for simplicity and brevity. For the other branch, one continues analogously. From the typical form of the polynomials arising as coefficients w.r.t. (p_1, p_2) in (2.13),

$$\begin{aligned} \text{first subpolynomial:} \quad & \{H^{(2)}, I^{(d)}\} \\ & \vdots \\ (l+1)\text{-th subpolynomial:} \quad & \{H^{(2)}, I^{(d-2l)}\} + \{H^{(0)}, I^{(d-2l+2)}\} \\ & \vdots \end{aligned} \tag{2.18}$$

It follows that the equations are coupled in such a way that the $(l+1)$-th subpolynomial contains derivatives of $I^{(d-2l)}$, but contains $I^{(d-2l+2)}$ without differentiation. When we take prolongations, an analogous pattern remains with higher derivatives.

After some (say, m) prolongations, the highest derivatives of coefficients of I in the $(l+1)$-th subpolynomial can be expressed through lower-order derivatives, cf. the proof of Lemma 1 on page 26. Then, the same order of differentiation for coefficients in a certain component $I^{(d-2l)}$ appears in the $(l+1)$-th subpolynomial at prolongation step m and in the $(l+2)$-th subpolynomial at prolongation step $m+1$ (i.e. unknowns from the $(l, m+1)$-block appear in the blocks (l, m) and $(l+1, m+1)$ of the equations). These derivatives may also appear in these subpolynomials for higher steps of prolongation. The situation is illustrated in Figure 2.3, which also takes into account cases where not all respective unknowns can be expressed by lower-order expressions. By lower order, we refer to expressions involving unknowns that belong to blocks with a lower value of l or m in the tableau of Figure 2.1.

This interaction structure suggests the following on the level of the differential equations obtained from the subpolynomials (we reinterpret Figure 2.3 as a tabular organizing the equations, cf. Section 2.2.3). The elimination scheme is build on this procedure as follows:

- Organize the equations $P_k^{(i,j,m,\mu)}$ in a tableau as in Figure 2.2, with the numbers $l = \frac{d-i}{2}$ labeling the subpolynomials in vertical direction and the prolongation level m on the horizontal axis. (we refer to l as the *block number*). Any set $P^{[i,m]}$ of all equations[6] $P_k^{(i,j,m,\mu)}$ with the same m and i contains unknowns $I_k^{(i-2,j,m+1,\mu)}$ $(i \geq 2)$ with $m+1$ derivatives w.r.t. x^1, x^2.

- The subpolynomials usually[7] have a term $\{H^{(2)}, F\}$ where

$$F = I^{(i)} = \sum_j F_j \, p_x^j p_y^{i-j}$$

is some homogeneous component of I. Since we assume isothermal coordinates on the reduced space, $H^{(2)} = \Omega(p_x^2 + p_y^2)$, and the subpolynomials with term $\{H^{(2)}, F\}$ entail

[6] This is what we called the (l, m)-block in Section 2.2.3.

[7] We ignore equations of the form $\{H^{(0)}, I^{(1)}\} = 0$, which appear in the odd-parity branch.

Common unknowns in the subpolynomials, differentiated by (x^1, x^2)

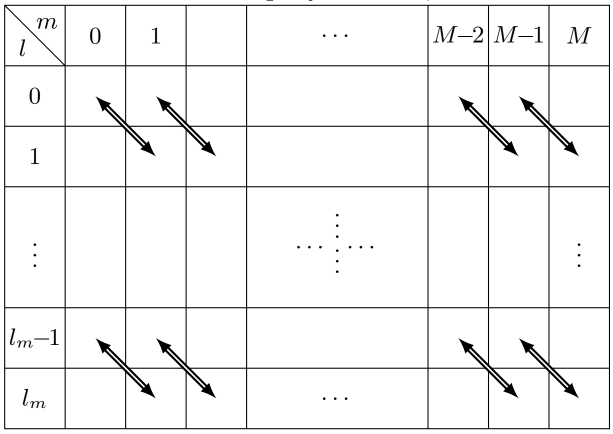

Figure 2.3: Sketch of the 'interaction' of the prolongated (w.r.t. (x^1, x^2)) subpolynomials of Equation (2.13). Horizontally, the order of differentiation by (x^1, x^2) is shown, vertically the number of the subpolynomial. The arrows indicate which subpolynomials contain the same coefficients in the same order of differentiation.

expressions of the form

$$\begin{aligned} (F_0)_x &\simeq 0 \\ (F_1)_x + (F_0)_y &\simeq 0 \\ &\vdots \\ (F_i)_x + (F_{i-1})_y &\simeq 0 \\ (F_i)_y &\simeq 0 \end{aligned} \tag{2.19}$$

where we use $\simeq$ as in Lemma 1 to denote terms that are not of interest. Performing prolongations, more and more of the expressions can be used to eliminate terms in the other equations. After $i+1$ steps, this allows us to solve for all highest derivatives of F_j, $j \in \{0, \dots, i\}$, as in Section 2.2.3.

- Having completed the previous steps, diagonal and horizontal replacements can be performed as follows.
  (i) For each pair of values (i_0, m_0), solve equations in $P^{[i_0, m_0]}$ for the $I_k^{(i_0,\, j,\, m_0\,+\,1,\, \mu)}$ if only one term with such unknown appears in the equation (j, k and μ run over all permissible values).
  (ii) Substitute the solutions in all equations $P_k^{(i_0\,-\,1,\, j,\, m,\, \mu)}$ with $m \le m_0$. Then substitute the solutions in all equations $P_k^{(i_0,\, j,\, m,\, \mu)}$ with $m \le m_0 - 1$.

To get a clearer impression of how this step works, consider the equations in the block $(l, m) = (l, d-2l)$ of Figure 2.3. These can be completely solved for the $(d-2l+1)$-th derivatives of $I_k^{(d\,-\,2l,\, j)}$.

Now:

- the obtained expressions for the $I_k^{(i,j,i,\mu)}$ can be substituted into the equations of the (l,m)-blocks with $m > d-2l$, and of the blocks with $(l+1,m)$, where $m > d-2l$.
- the previous substitution can be done for each value $1 < l \leq \lfloor \frac{d-e}{2} \rfloor$, where e is the (p_1,p_2)-parity of the integral under consideration.
- in each block (l,m) there are $(2l+1)(d-2l+2)(m+1)$ equations. For these equations, consider the $(2l+1)(d-2l+1)(m+2)$ unknowns from the block $(l,m+1)$. In the cases with $m < d-2l$, only some of the equations can be solved for the respective unknowns explicitly, but for $m \geq d-2l$ we can solve for all the unknowns of the respective block and obtain (at most) a number of $(2l+1)(m-d+2l)$ integrability equations (the reader may recall that this is for the case when $\mathrm{par}(e) = \mathrm{par}(d)$. In general, the expressions also contain $\tilde{e} \in \{0,1\}$, which denotes the parity $\mathrm{par}(d+e)$. Thus, $\tilde{e} = 0$ in the case displayed here).

Having performed this step, we can choose a generic point as described in Observation 2 on page 28. Thereby, we find a matrix with (in general, real) entries for the associated system. Naturally, we want to choose points that ensure self-contained computability on the computer. In our examples the metric usually has rational entries and we therefore choose points with integer or rational components. The trivial integrals are still solutions of this matrix equation (cf. page 28). Since we are only interested in additional integrals, we can make use of the freedom to add arbitrary constant multiples of trivial integrals to our function I, which we require via (2.13) to be an integral. This freedom can be interpreted as the freedom to choose the value of the coefficient in $I_k^{(i,0,0,0)}$ in our point of reference for each k and i according to our wishes. Naturally, we want to choose $I_k^{(i,0,0,0)} = 0$, because this eliminates these unknowns while at the same time keeping the system homogeneous (no constant terms).

Summary

Let us briefly summarize the algorithm that we constructed:

Algorithm I. *Nonexistence results for Question 1 on page 29 can be obtained for metrics of the form (2.11) through the following algorithm:*

(1) Consider pure-parity integrals. Run the algorithm for both alternatives separately.

(2) Compute the matrix of the linear system associated to the prolongation of the system of PDEs obtained from the Poisson equation (1.3).

(3) Choose a generic point P and evaluate the matrix of the associated linear system at this point.

(4) Add multiples of the known integrals to set as many of the unknowns as possible to zero (in P).

(5) Perform the vertical and horizontal reductions as described on the previous page.

(6) Perform other tricks, e.g. rewrite the matrix system such that the coefficients are integers.

(7) *Compute the rank of the resulting (reduced) matrix and determine the dimensionality of its kernel (in particular, determine whether the matrix has full rank).*

The algorithm confirms nonexistence of an additional integral if the obtained kernel dimension equals the minimal expected value (that is 0 if all trivial integrals are taken into account in (4)). Otherwise, it has to be checked that a truly generic point had been chosen. If the matrix kernel dimension stays larger than the minimal expected value, existence and independence of an extra integral have to be investigated by solving the associated linear system in all generality.

In what follows we often use the terms *initial matrix* and *reduced matrix* of the associated linear system. The initial matrix is the matrix describing the associated linear system obtained in Step (2). By the reduced matrix we mean the matrix obtained after completion of Step (6).

2.3.2 Algorithm for diagonal metrics

Algorithm I is applicable to metrics of type (2.11) with two ignorable coordinates. The downside of this generality is that the algorithm might miss particular properties of a considered system that simplify the problem. As a result, efficiency can be improved further if additional symmetries are present. We now consider diagonal metrics of type (2.11), i.e. we assume to have coordinates such that the metric tensor locally is represented by a diagonal matrix.

Equivalently, we can view this as an additional symmetry of the potential $V = H^{(0)}$ obtained from (2.11) as outlined on page 23. This is explored in more detail in Section 3.1.2, when we investigate Weyl metrics. The effect is that the potential component V^{34} vanishes, see also (3.15) (actually, we just need that V^{34} is constant).

The advantage of this additional symmetry, or structure of the metric tensor, is that (for integrals of degree d) the Poisson equation (1.3) splits into four separate subsystems which can be considered independently. Moreover, it turns out that only one of them is distinct to degree d, and this observation allows us to iteratively answer Question 1 on page 29 in a succinct way by regarding, for each degree, only the distinct subsystems $\mathcal{S}_r^{\mathsf{lead}}$, $r \leq d$. We make use of this observation to improve Algorithm I for diagonal metrics of type (2.11), especially Weyl metrics. The efficiency of this modification is discussed in Chapter 3.

Use of symmetries

We use the same techniques as those described in Section 2.3.1. However, we show that diagonal metrics admit a refined decomposition scheme, which allows for faster computations. The basic idea is as follows: Decompose the integral into its parts of odd and even parity in (p_1, p_2). These parts satisfy separate PDE systems (see Observation 1 the explanations for Algorithm I). The integral can thus be written in the following form, identifying the coefficients w.r.t. p_3 and p_4 for each of these parts,

$$I = \sum_{\substack{r \text{ even} \\ k\in\{0,\dots,d-r\}}} I_k^{(r)} p_3^k p_4^{d-r-k} + \sum_{\substack{r \text{ odd} \\ k\in\{0,\dots,d-r\}}} I_k^{(r)} p_3^k p_4^{d-r-k} \tag{2.20}$$

where $I_k^{(r)}$ is a polynomial in (p_1, p_2) of degree r (note that each of the sums satisfies a Poisson equation (1.3) of its own).

(a) Graph of the decomposition (parity of r equals parity of d). The thick-line part is the lead component, the lower component is drawn with thin lines.

(b) Graph of the decomposition (parity of r opposite to parity of d). The left component is the one with thick lines, the other is the right component.

Figure 2.4: The graphs show the structure of the equations. Vertices of the graph correspond to equations in (2.21). So, the top vertices correspond to a maximal value of r (according to the respective parity branch). Going down to child vertices reduces r by 2, compare $I_k^{(r)}$ in the lines of (2.21). For $r = \text{const}$, k increases from left to right in (2.21). This corresponds to the vertices on horizontal levels in the graphs. Edges indicate that equations have unknowns in common.

Using (1.3), consider coefficients w.r.t. the momenta p_3 and p_4. This produces a set of differential equations written in terms of Poisson brackets. We arrange these equations in an advantageous way similar to Equations (2.17)[8]:

$$\begin{gathered}
\{T, I^{(d)}\} = 0, \\
\{T, I_0^{(d-1)}\} = 0, \qquad \{T, I_1^{(d-1)}\} = 0, \\
\{T, I_0^{(d-2)}\} + \{V^{44}, I_0^{(d)}\} = 0, \qquad \{T, I_1^{(d-2)}\} = 0, \qquad \{T, I_2^{(d-2)}\} + \{V^{33}, I^{(d)}\} = 0, \\
\{T, I_0^{(d-3)}\} + \{V^{44}, I_0^{(d)}\} = 0, \quad \{T, I_1^{(d-3)}\} + \{V^{44}, I_1^{(d)}\} = 0, \quad \{T, I_2^{(d-3)}\} + \{V^{33}, I_0^{(d)}\} = 0, \quad \{T, I_3^{(d-3)}\} + \{V^{33}, I_1^{(d)}\} = 0, \\
\text{etc.}
\end{gathered} \tag{2.21}$$

These equations form a system of PDEs that we previously dubbed $\mathcal{S}_d$. As in Section 2.3.1, equations of odd and even parity w.r.t. (p_1, p_2) form separate systems of equations, named $\mathcal{S}_d^{\mathsf{odd}}$ and $\mathcal{S}_d^{\mathsf{even}}$, respectively. Moreover, we now find another decomposition that is due to the vanishing of V^{34}.

Since $V^{34} = 0$, the terms of the form $\{V^{34}, \cdot\}$ disappeared from the equations, compare (2.21) with (2.17). This entails that in any equation only those polynomials $I_k^{(r)}$ appear together that have their values r both of the same parity, and also k of the same parity. This latter property is new compared to (2.17).

We therefore have four subsystems that have to be satisfied separately. We classify them according to Table 2.1. A graphical display of the situation and the four subsystems is given in Figure 2.4. The names that we use for the subsystems become clear from Figure 2.4 in combination with Equations (2.21).

[8] We include terms both of odd and even parity in (p_1, p_2) here, because we need the exact structure later. Note the differences between (2.21) and (2.17).

Subsystems in the diagonal case

	k even	k odd
par(r)=par(d)	$\mathcal{S}_d^{\mathsf{lead}}$	$\mathcal{S}_d^{\mathsf{lower}}$
par(r)=par($d+1$)	$\mathcal{S}_d^{\mathsf{left}}$	$\mathcal{S}_d^{\mathsf{right}}$

Table 2.1: The subsystems obtained by decomposition of the system of PDEs by parity in ignorable and non-ignorable momenta. The entries in the table denote the names of the respective subsystems. The table indicates the restrictions to r and k that have to be met in order that the $I_k^{(r)}$ appear in the respective subsystems. Here, par(r) denotes the parity of the index r as a number.

For the subsystems left and right, one recognizes quickly that both systems are formally equivalent — they are simply two copies of the same system of PDEs. Moreover, it is fairly obvious that all the four systems are *"similar"* in the following sense. Consider the corresponding decomposition into subsystems for integrals of degree $d-1$ and $d-2$. Then, for instance, the lead subsystem for degree $d-1$ is equivalent to the left subsystem for degree d. Similarly, the lead subsystem for degree $d-2$ is equivalent to the lower subsystem for integrals of degree d.

Thus, out of the four subsystems of Table 2.1, only one is *distinct*, i.e. truly new for this degree. This distinct subsystem is $\mathcal{S}_d^{\mathsf{lead}}$. Knowing the lead subsystem for degrees $d-2$, $d-1$ and d, we can therefore reconstruct the complete system $\mathcal{S}_d$. For later reference, let us put this result into a lemma.

Lemma 2. *The solutions to $\mathcal{S}_d$ can be obtained by knowing the solutions to $\mathcal{S}_d^{\mathsf{lead}}$, $\mathcal{S}_{d-1}^{\mathsf{lead}}$, and $\mathcal{S}_{d-2}^{\mathsf{lead}}$. The solutions to $\mathcal{S}_d$ are given by the formula*

$$\mathsf{Sol}(\mathcal{S}_d) = \mathsf{Sol}(\mathcal{S}_d^{\mathsf{lead}}) \times \left(\mathsf{Sol}(\mathcal{S}_{d-1}^{\mathsf{lead}})\right)^2 \times \mathsf{Sol}(\mathcal{S}_{d-2}^{\mathsf{lead}}), \tag{2.22}$$

where $\mathsf{Sol}(\cdot)$ denotes the space of solutions of the respective subsystem.

Elimination scheme

The elimination scheme is basically the same as before, in Algorithm I. We represent the Poisson equation (1.3) by

$$\{H, I\} = \sum_{\substack{i=0\\ \mathrm{par}(i)=\mathrm{par}(d+1)}}^{d+1} \sum_{j=0}^{i} \sum_{\substack{k=0\\ \mathrm{par}(k)=\mathrm{even}}}^{d+1-i} P_k^{(i,j)}\; p_1^{i-j} p_2^j p_3^k p_4^{d-i-k}, \tag{2.23}$$

so each $P_k^{(i,j)}$ represents an equation of the system of PDEs that encodes the requirement for I to be an integral. The prolongated system is obtained by differentiation of the $P_k^{(i,j)}$. The equation resulting from $P_k^{(i,j)}$ after m differentiations, with μ derivatives w.r.t. x^1 and $m-\mu$ w.r.t. x^2, is again denoted by the symbol $P_k^{(i,j,m,\mu)}$. Analogously, for the derivatives of the unknown functions, write again $I_k^{(i,j,m,\mu)}$. Then we follow the same prescription as in the non-diagonal case.

Summary

Summarizing, we find the following algorithm, which is a refinement of Algorithm I for diagonal metrics.

Algorithm II. *Nonexistence results to Question 1 on page 29 can be obtained for diagonal metrics of type (2.11) and integrals of degree d through the following algorithm:*

(1) For $d_0 \in \{d-2, d-1, d\}$ complete the following computation:

- *(i) Consider only the* lead *subsystem of $\mathcal{S}_{d_0}$. It solely involves integrals of degree d_0 that have pure parity* $\operatorname{par}(d_0)$ *w.r.t. (p_1, p_2), and pure even parity in p_3.*
- *(ii) Compute the matrix of the linear system associated to the prolongation of $\mathcal{S}_{d_0}^{\mathsf{lead}}$.*
- *(iii) Choose a generic point P and evaluate the matrix of the associated linear system at this point.*
- *(iv) Add multiples of the known integrals (linear integrals, Hamiltonian) to set as many of the unknowns as possible to zero (in P).*
- *(v) Perform the vertical and horizontal reductions as described on page 32.*
- *(vi) Perform other tricks, e.g. rewrite the matrix system such that the coefficients are integers.*
- *(vii) Compute the rank of the reduced matrix and determine the dimensionality of the matrix kernel (and in particular decide whether the rank is maximal).*

(2) Use Formula (2.22) to compute the total number of independent solutions to $\mathcal{S}_{d_0}$.

The algorithm confirms nonexistence of an additional integral if the obtained number of solutions equals the minimal expected value, that is 0 if all trivial integrals are taken into account in (iv). Otherwise, it has to be checked whether a truly generic point has been chosen or whether additional integrals exist (and whether they are independent).

Again, we use the terms *initial* and *reduced matrix* for the matrices obtained after Steps (ii) and (vi), respectively.

2.3.3 Algorithm for sparse systems

The examples that we are going to discuss in Chapter 5 do not have symmetry properties as nice as with metric (2.11). In particular, the Hamiltonian in these situations is of the form $H = \sum_{i,j\in\{1,\dots,4\}} g^{ij} p_i p_j$ and has terms linear in the non-ignorable momenta (for the concrete situation, see Chapter 5). Thus, we are not able to decompose the system of PDEs into separate subsystems. On the other hand, although the elimination scheme of Algorithms I and II can still be used in principle, it is not very efficient anymore. We are therefore forced to deal with a larger system of equations that involve many more unknowns.

Yet, there is a cure for this problem for the case of the sub-Riemannian structures that we discuss in Chapter 5. A suitable choice of the point of reference for the computation yields a linear system that is described by a sparse matrix. Some equations in this system turn out to have only one or two entries, and this allows us to solve some of the equations immediately. Also, the partial solution can eliminate some of the redundant equations of the overdetermined system.

This is exploited for Algorithm III on pages 77ff. Below, in Section 2.4, we see that this simple trick actually proves very efficient in reducing the matrices of the considered linear systems.

2.3.4 Examples

In order to give an impression of the nature of the result that the algorithm yields, let us consider the Kerr family of metrics.

(Extreme) Kerr metric. The Kerr family is a particular example of a family of metrics of the type (2.11). It is discussed in more detail in Section 3.1. We use Boyer-Lindquist coordinates (r, θ, ϕ, t) and consider the extreme Kerr metric (rotation parameter $a = 1$) [BL67; Ste03]:

$$g = \begin{pmatrix} \frac{r^2+\cos^2(\theta)}{r^2-2r+1} & & & \\ & r^2+\cos^2(\theta) & & \\ & & \frac{P_a(r,\theta)\sin^2(\theta)}{r^2+\cos^2(\theta)} & \frac{-2\sin^2(\theta)r}{r^2+\cos^2(\theta)} \\ & & \frac{-2\sin^2(\theta)r}{r^2+\cos^2(\theta)} & -\frac{r^2-2r+\cos^2(\theta)}{r^2+\cos^2(\theta)} \end{pmatrix} \tag{2.24}$$

with $P_a(r,\theta) = \cos^2(\theta)r^2 + r^4 - 2\cos^2(\theta)r + \cos^2(\theta) + r^2 + 2r$. Applying Algorithm I, we obtain the following results (we choose the point $(r, \theta) = (2, \pi/4)$).

Degree	1	2	3	4
$e = 0$ integrals	2	**5**	8	14
$e = 1$ integrals	0	**0**	0	0
All integrals	2	**5**	8	14

The labels $e = 0$ and $e = 1$ mark results applying to the subsystems $\mathcal{S}^{\mathsf{even}}$ and $\mathcal{S}^{\mathsf{odd}}$, respectively. In the first column (degree 1), we have two integrals, which are the independent linear integrals p_ϕ and p_t that both are of even parity in (p_r, p_θ). In second degree, we find 5 integrals (H, p_ϕ^2, $p_t p_\phi$, p_t^2, and the Carter constant). Again, all are of even parity in (p_r, p_θ).

Schwarzschild metric. The Schwarzschild metric is a particular example of a Kerr metric (namely, its static limit). In anticipation of the applications in Chapter 3, we note that it is also a special case of the Zipoy-Voorhees family (with parameter value $\delta = 1$). Using prolate spheroidal coordinates (x_1, x_2, ϕ, t), the metric reads

$$g = \begin{pmatrix} \frac{x_1+1}{x_1-1} & & & \\ & \frac{(x_1+1)^2}{1-x_2^2} & & \\ & & (x_1+1)^2(1-x_2^2) & \\ & & & \frac{1-x_1}{x_1+1} \end{pmatrix} \tag{2.25}$$

Algorithm II yields the following results (with $(x_1, x_2) = (1/2, 2)$ as the point of reference).

Degree	0	1	2	3	4
lead integrals	1	0	**4**	0	10
All integrals	1	2	**5**	8	14

The appearance of the quadratic integral can be understood as follows. Besides the Hamiltonian, the Schwarzschild metric admits four linear integrals, but only two of them commute. However, a linear combination of their squares commutes with the two commuting linear integrals. Essentially this is the fact that the angular momenta l_1, l_2, l_3 on $\mathbb{S}^2$ (or the generators of $\mathfrak{so}(3)$) do not commute w.r.t. the Poisson bracket, but each of them commutes with the total angular momentum $L = \sum_i l_i^2$, which is a Casimir function for the system.

This demonstrates that the algorithm may find *reducible* integrals, if some of the reduction integrals are not in involution with one another. On the other hand, if we prove nonexistence of additional integrals, this proves the nonexistence of corresponding irreducible integrals.

2.4 Computational efficiency

As we indicate previously, the computations as outlined above are to be performed with computer assistance. All computations are rigorous and do not involve any kind of approximation[9]. Let us give a quick overview on the computational efficiency of the algorithms, including a qualitative assessment of how it is affected by the tricks and tools of Section 2.2. As we remark earlier in this chapter, the rank computation is crucial for computation speed. We use usual Gauß elimination, but of course one could also work with alternative algorithms, such as the Bareiss algorithm, but it appears that this does not significantly change the computation time.

In the previous sections, we describe in detail how we perform the computations. Recall that we use two major techniques to facilitate our computations: We decompose the system of PDEs using its symmetries, and we eliminate unknowns by partial solution of the system via reasonably simple equations. Roughly speaking, the linear system obtained after taking into account the symmetry has been called the *initial matrix system*, the one obtained after applying the elimination scheme (and other tools) was dubbed *reduced matrix system*, cf. the notes beneath Algorithms I and II.

Let us briefly look at how these tricks affect computation time, i.e. the efficiency of the algorithms. Time indications in this section, and throughout the text, refer to a (relatively) standard desktop computer with a 3.4GHz processor and 32GB RAM. The computations were performed using Maple 18.

Effect of the degree on the computation time. Of course, computation time quickly increases with growing degree d of the integral. This has two reasons. On one hand, the number of unknowns and equations increases because, respectively, the degree of the integral and of the Poisson bracket increases. For instance, for the number of unknown functions, we find the following formula in dimension $D = 4$:

$$\binom{D+d-1}{D-1} \overset{D=4}{=} \frac{1}{6}\,(d+3)(d+2)(d+1) = \mathcal{O}(d^3).$$

On the other hand, if we increase the degree d, then we also have to perform more prolongation steps, and this further increases the size of the associated matrix system.

Effect of symmetry decomposition schemes. Algorithms I and II differ mainly concerning the implementation of the additional decomposition w.r.t. ignorable momenta that Weyl (diagonal) metrics offer. We run both algorithms on the Zipoy-Voorhees metric with $\delta = 2$ to get an impression of how much this additional symmetry improves performance. We compare the computation speed on our computer for several degrees of the integral.

Zipoy-Voorhees metric with $\delta = 2$

d	Algorithm I (e=1)	Algorithm II
4	4s	1s
6	2.7m	54s
8	1.5h	25.5m

One might object that in case of Algorithm II we have to run the algorithm three times to obtain the overall result for degree d, while in case of Algorithm I we only have to run

[9] We work with integers or at least rational numbers in the cases of a specific metric. In the case of metrics with a parameter, Section 3.2, the computations are also rigorous.

it twice. However, adding the computation times for all necessary computations for given degree d, we obtain the following computation times.

Zipoy-Voorhees metric with $\delta = 2$
all branches

d	Algorithm I	Algorithm II
6	5.4m	67s
8	2.9h	32m

Hence we see that the additional decomposition clearly speeds up the computation.

Effect of the elimination scheme. Algorithm I differs from the algorithm used in [KM12], see page 22, mainly in the elimination scheme used to reduce the matrix of the associated system. Algorithm II and [KM12] also differ in this respect, although, of course, the differences are larger in this case.

For instance, the algorithm from [KM12] needs around 3 minutes for the case $d = 4$ on our computer, while we achieve the same result in 8 and less than 3 seconds with Algorithms I and II, respectively.[10]

For Algorithm I, the elimination scheme also works quite well. For instance, take the case $d = 7$ with $e = 0$ in Table 3.2 on page 55 of Section 3.4.

Numbers of equations and unknowns before and after reduction

initial		reduced	
equations	unknowns	equations	unknowns
2880	2700	556	356

This implies a reduction of the number of equations by about $4/5$ (to 19%), while the number of unknowns is reduced by about $7/8$ (to 13% of the initial number).

Effect of dimension (sub-Riemannian examples). Let us compare the number of equations and unknowns before and after applying our tools (refer to Chapter 5 for details on Algorithm III). For instance, we obtain (at prolongation $M = d + 1$) the following table for the sub-Riemannian structures.

	Original		Reduced		Percentage	
example	equations	unknowns	equations	unknowns	equations	unknowns
$6D_p$ (d=6)	28512	20790	11816	9155	41%	44%
$6D_p$ (d=5)	12936	9072	2840	2262	22%	25%
$6D_h$ (d=5)	12936	9072	5360	4013	41%	44%
7D (d=5)	25872	16632	9397	6993	36%	42%
$8D_1$ (d=5)	48048	28512	4439	3514	9%	12%

The percentage figures give the ratio of the reduced and the initial numbers of, respectively, equations or unknowns. We see that for the same degree $d = 5$, our techniques may even work *better* for higher dimensions. Although they become less efficient when going from dimension 6 ($6D_p$) to 7, they reduce the number of equations in the 8D case *to* around 10 percent of the original number (and as an effect of this, computation time in both 7D

[10] The mentioned figures are the sums of all relevant branches of the computation. The detailed computation times are (i) algorithm from [KM12]: 1.8m ($e = 0$), 1.2m ($e = 1$), (ii) Algorithm I: 4s ($e = 0$), 4s ($e = 1$), (iii) Algorithm II: each computation less or around 1s.

and 8D for degree 5 lies at around 10 hours, in spite of the fact that initially we had almost twice the number of equations and unknowns in 8D.

In addition, we mention that it is only the tools that enabled us to perform the computations at all. Without the reduction, available memory on the computer is quickly exhausted.

Conclusions. The comparisons made above suggest that additional structure and the tools from Section 2.2 contribute significantly to the performance of Algorithms I, II and III. In particular, we observed the following effects:

— Additional symmetry can significantly speed up the computation when implemented into the algorithm, e.g. the structure of the system of equations (pages 25ff) clearly improves performance for Weyl metrics.

— The elimination scheme, extending mere rank computation by a preceding partial solution procedure, accelerates the projection step considerably.

— At least in some cases it is only due to the techniques and tools outlined in this chapter that computation is possible with the available memory on a standard computer.

In the next chapter, we are going to apply Algorithms I and II to stationary and axially symmetric metrics. Also, we demonstrate how to use Algorithm II in almost unchanged form to study metrics with a real-valued parameter.

Chapter 3

Stationary and Axially Symmetric Space-Times

Algorithms I and II outlined in Chapter 2 are fairly general for 4-dimensional metrics and have a wide range of applicability (and some of the tools are not restricted to the symmetry assumptions made). However, they were developed with applications to stationary and axially symmetric metrics in mind. Such metrics are relevant to the theory of compact astrophysical objects. In particular, our focus is on vacuum metrics of this class, i.e. we make the additional assumption of a vanishing Ricci tensor.

Stationary and axially symmetric metrics are characterized by the existence of two Killing vector fields.[1] *Stationarity* is characterized by the presence of a timelike Killing vector field ξ [Ste03]. If this Killing vector field is hypersurface-orthogonal (see below), then it is called a *static* metric [CAM90; Ste03]. *Axial symmetry*, on the other hand, is defined via the action of the 1-torus S^1 via isometries such that the set of fixed points (called the *axis of rotation*) is a (regular) 2-dimensional surface [Ste03; MS93]. Let the corresponding Killing vector field be called η. It can be shown that for any point on the axis there is a neighborhood such that $|\eta|$ is non-negative and zero only at points on the axis, see [MS93] (spacelike character of the symmetry). We assume (as done in [Ste03]) that η is spacelike everywhere outside the axis and that η and ξ are involutive.

Stationary and axially symmetric metrics can (under some additional assumption) be written in a Lewis-Papapetrou form, see Equation (3.3) and [Lew32; Pap66; Ste03]. We examine, using Algorithms I and II, some metrics of this class to determine whether they are Liouville integrable with integrals of low degree. In addition, we demonstrate how to use Algorithm II (with modifications) for checking that flat space and the Schwarzschild space are the only Zipoy-Voorhees metrics admitting an additional quadratic integral. Applying the algorithmic approach to parameter-dependent metrics is also an important preparation for the general nonexistence proof for Theorem 4 for additional cubic integrals in Weyl metrics, see Chapter 4. We prove nonexistence of additional involutive integrals up to a certain degree d for the Zipoy-Voorhees metric. The following list is a summary of the results proven in the current chapter.

[1] In astrophysics, it is common to refer to the integrals connected with stationarity and axial symmetry as respectively the energy and the angular momentum integrals. The third trivial conservation law is then linked to the conservation of rest mass, cf. [Bri08b].

Result. *(1) We give a novel, algorithmic proof of the fact that flat space and the Schwarzschild space-time are the only Zipoy-Voorhees space-times that are Liouville integrable with H, p_t, p_ϕ and an additional quadratic integral (this issue is also addressed in [Car68a; Car68b; WP70]). Moreover, we see in Chapter 4 that any integral at most of cubic degree and in involution with the standard integrals is reducible by linear integrals and the Hamiltonian*[2].

(2) The Zipoy-Voorhees metric with parameter $\delta = 2$ (Darmois solution [Dar27; Ste03]) has no additional[3] *irreducible integral polynomial in momenta with smooth coefficient functions, up to degree 11 of the integral, i.e. the family (H, p_ϕ, p_t) of integrals cannot be extended to a Liouville-integrable family of integrals of degree ≤ 11.*

(3) The Tomimatsu-Sato metric from [Man12], with $\delta = 2$, $\kappa = 2$ and $p = 3/5$, $q = 4/5$, has no additional irreducible integral polynomial in momenta with smooth coefficients up to degree 7 of the integral, i.e. the family (H, p_ϕ, p_t) cannot be extended to a Liouville-integrable family of integrals at most of degree 7.

In Chapter 4, these results are complemented by a general nonexistence proof for additional cubic integrals.

3.1 Stationary and axially symmetric vacuum metrics

The class of stationary and axially symmetric vacuum (SAV) metrics is well known in astrophysics for its significance in the description of compact astrophysical objects. The probably most prominent example, or rather family of examples, is the Kerr family of metrics, which provides a model for (the exterior of) rotating neutron stars and black holes. The metric depends on a parameter a that describes rotation. The Kerr family is also special because it admits an additional quadratic integral, the Carter constant, that makes it Liouville integrable [Car68a; Car68b; WP70].

Fact 3. *The Kerr family of metrics in Boyer-Lindquist coordinates reads [BL67; Ste03]*

$$g_a(r,\theta) = \begin{pmatrix} \frac{r^2+a^2\cos^2(\theta)}{r^2-2r+a^2} & & & \\ & r^2+a^2\cos^2(\theta) & & \\ & & \frac{P_a(r,\theta)\sin^2(\theta)}{r^2+a^2\cos^2(\theta)} & \frac{-2a\sin^2(\theta)r}{r^2+a^2\cos^2(\theta)} \\ & & \frac{-2a\sin^2(\theta)r}{r^2+a^2\cos^2(\theta)} & -\frac{r^2-2r+a^2\cos^2(\theta)}{r^2+a^2\cos^2(\theta)} \end{pmatrix}, \tag{3.1}$$

with $P_a(r,\theta) = a^2\cos^2(\theta)r^2 + r^4 - 2a^2\cos^2(\theta)r + a^4\cos^2(\theta) + a^2r^2 + 2a^2r$, and is integrable with the additional integral being the quadratic Carter constant

$$C = p_\theta^2 + \cos^2(\theta)\left(\left(\frac{p_\phi}{\sin(\theta)}\right)^2 - a^2\left(H + p_\theta^2\right)\right) \tag{3.2}$$

[2] Note that for $\delta = 0$, we have flat space and therefore any integral is totally reducible [Tho86]. For the parameter values ± 1, it is the Schwarzschild metric. The Schwarzschild metric is integrable and possesses the Carter integral as its final integral, which is in involution with the three standard integrals (and functionally independent of them). However, the Carter integral is not irreducible in the Schwarzschild case. The Schwarzschild metric admits four (non-involutive) linear integrals and the Carter integral is reducible by them, cf. Section 2.3.4.

[3] Recall that this refers to integrals in addition to the standard integrals H, p_t and p_ϕ, and in involution with them, see Chapter 1.

where p_ϕ is the integral connected with axial symmetry, and p_θ the one connected with stationarity.

The original proof can be found in [Car68a; Car68b]. Another proof is given in [WP70].

A generalization of the Kerr family of metrics is the class of Tomimatsu-Sato metrics [TS72; TS73]. Zipoy-Voorhees metrics [Zip66; Voo70] are special cases of Tomimatsu-Sato metrics [TS73]. Other examples of SAV metrics include the exterior of the Meinel-Neugebauer disk of dust [NM94; NM95] or the Manko-Novikov metrics [MN92]. Static and axially symmetric vacuum metrics form *Weyl's class* [Ste03]. Examples of Weyl metrics are, for instance, the Manko-Novikov metrics and the Zipoy-Voorhees metric.

The Zipoy-Voorhees family has recently gained some particular attention, because numerical studies suggested integrability for metrics of this family [Bri08a; Bri08b; Bri10a; Bri10b]. However, later studies with different techniques provide contradicting evidence [KM12; LG12; MPS13]. We continue the approach followed in [KM12]. We note that this approach considers polynomial integrals with smooth coefficient functions and that these integrals are not completely covered by [MPS13]. In addition, we present a way to check nonexistence of an additional integral for arbitrary values of the parameter δ of the Zipoy-Voorhees metric. We suggest that this approach can be extended to higher degrees.

Concerning stationary and axially symmetric metrics, we restrict our attention to integrals that respect stationarity and axial symmetry (involutivity assumption).

3.1.1 General properties

SAV metrics always admit two involutive Killing vectors, a timelike one (representing stationarity) and a spacelike one (representing axial symmetry, cf. page 43). Under certain conditions, there exists a standard form for these metrics, cf. Equation (3.3). We already saw that the finiteness requirement we mentioned in Section 1.4 is met, cf. [Wol98] and the considerations on page 26. We can therefore apply Algorithms I and II from Chapter 2.

Standard form of SAV metrics

Stationary and axially symmetric vacuum metrics can be brought into the following standard form by means of suitable coordinate transformations [Lew32; Pap66], see also [Ste03]. The adapted coordinates are called Lewis-Papapetrou coordinates [Ste03].

$$g = e^{2U}\left(e^{-2\gamma}\left(dx^2+dy^2\right)+R^2\,d\phi^2\right)-e^{-2U}\left(dt-A\,d\phi\right)^2 \tag{3.3}$$

where U, γ, R and A depend on the non-ignorable coordinates x and y only. This coordinate choice is, under certain additional assumptions, also possible if we do not require the vacuum conditions [Ste03]. Let us also note that the static limit of metrics (3.3) is obtained when the rotation parameter function $A\equiv 0$. This turns the metric (3.3) diagonal, due to hypersurface-orthogonality (i.e. the block-diagonal form with vanishing components $g_{ti}=0,\ \forall i$).

We solely consider vacuum metrics. The vacuum restriction is the requirement that the Ricci tensor of the metric g is identically zero (*Ricci-flatness*). Requiring the vacuum property is a fair assumption for the movement of test particles around astrophysical objects as long as external fields (like electro-magnetic fields) are ignored. Since not all metrics of the form (3.3) satisfy this requirement, we have to impose Ricci-flatness separately. We refer to these additional requirements on the parameter functions U, γ, R, A as the *vacuum conditions*.

Vacuum conditions and the Ernst equation

For SAV space-times, Ricci-flatness implies a set of equations for the parameter functions U, γ, R and A. From the Ricci-flatness requirement, five differential equations are obtained, see also [Ste03; GP09],

$$\triangle R = R_{xx} + R_{yy} = 0, \tag{3.4a}$$

$$\frac{A_x^2 + A_y^2}{R^2}\, e^{-4U} - 2\blacktriangle U = 0, \tag{3.4b}$$

$$\frac{d}{dx}\left(\frac{A_x}{R} e^{-4U}\right) + \frac{d}{dy}\left(\frac{A_y}{R} e^{-4U}\right) = 0, \tag{3.4c}$$

$$4e^{4U}\left(\frac{R_{yy}}{R} - \frac{R_x}{R}\gamma_x + \frac{R_y}{R}\gamma_y - U_x^2 + U_y^2\right) - \frac{A_y^2 - A_x^2}{R^2} = 0, \tag{3.4d}$$

$$2e^{4U}\left(\frac{R_{xy}}{R} + \frac{R_x}{R}\gamma_y + \frac{R_y}{R}\gamma_x + 2\,U_x U_y\right) - \frac{A_y A_x}{R^2} = 0. \tag{3.4e}$$

We denote by $\triangle$ the usual (flat) Laplace operator $\triangle f = f_{xx} + f_{yy}$ in Lewis-Papapetrou coordinates, while we use $\blacktriangle$ for the differential operator defined by

$$\blacktriangle f = f_{xx} + f_{yy} + \frac{R_x}{R} f_x + \frac{R_y}{R} f_y.$$

The Equations (3.4) break up into two sets of equations, (3.4a)–(3.4c) and (3.4d)–(3.4e), which we refer to as *primary* and *secondary equations*, respectively. The secondary equations contain (derivatives of) γ, while the primary equations do not. Let us first consider the primary equations. Equation (3.4c) is the integrability condition for a function Φ that solves the equations

$$\Phi_x = -\frac{A_y}{R} e^{-4U}, \qquad \Phi_y = \frac{A_x}{R} e^{-4U}. \tag{3.5}$$

The integrability condition for A obtained from this definition in terms of Φ leads to the equation

$$e^{2U}\left(\blacktriangle\Phi - 2\,U_x\,\Phi_x - 2\,U_y\,\Phi_y\right) = 0. \tag{3.6}$$

Equations (3.6) and (3.4b) can be combined to form one equation

$$\Re(\mathcal{E})\,\blacktriangle\mathcal{E} - (\nabla\mathcal{E})^2 = 0, \tag{3.7}$$

where $\Re(\mathcal{E})$ denotes the real part of $\mathcal{E}$ and where $\mathcal{E}$ is the complex *Ernst potential*,

$$\mathcal{E} = e^{-2U} + i\,\Phi. \tag{3.8}$$

Equation (3.7) is usually referred to as the *Ernst equation*. In addition to the Ernst equation, the two secondary equations (3.4d) and (3.4e) are needed explicitly in Chapter 4, and we are going to recast them into a simpler form.

The primary equations pose restrictions on U and R. Provided that R is non-constant, the equation $\triangle R = 0$ allows for setting $R = x > 0$ by a change of coordinates [Ste03]. These coordinates are called *Weyl's canonical coordinates* [Ste03]. If R is constant, this change of coordinates is impossible, but one can show that $\triangle\gamma = \gamma_{xx} + \gamma_{yy} = 0$ holds and hence that the metric is flat. In case of non-constant R, we perform the mentioned coordinate transformation, so $R = x$, and we obtain from the two remaining equations, (3.4d) and (3.4e), the following concise expressions for γ_x and γ_y:

$$\gamma_x = x\,(U_y^2 - U_x^2) - \frac{e^{-4U}}{4x}\,(A_x^2 - A_y^2) \tag{3.9a}$$

$$\gamma_y = -2x\,U_x\,U_y + \frac{e^{-4U}}{2x}\,A_x\,A_y \tag{3.9b}$$

In the static case, the vacuum conditions simplify significantly. We have $\mathcal{E} = e^{-2U}$ and require $A = 0$ for staticity. Then, the vacuum conditions read as follows

$$R_y\,U_y + R_x\,U_x + R\,U_{yy} + R\,U_{xx} = 0 \tag{3.10a}$$

$$\triangle R = R_{xx} + R_{yy} = 0 \tag{3.10b}$$

$$2R\,U_x^2 - 2R_y\,\gamma_y + 2R_x\,\gamma_x + R_{xx} - R_{yy} - 2R\,U_y^2 = 0 \tag{3.10c}$$

$$2R\,U_x\,U_y + R_y\,\gamma_x + R_x\,\gamma_y + R_{xy} = 0 \tag{3.10d}$$

In case of non-constant R, the secondary equations enable us to express derivatives of γ in terms of derivatives of U, allowing us to eliminate them, and finally γ, from the equations. We obtain the equations:

$$U_{yy} = -U_{xx} - \frac{1}{x}\,U_x \tag{3.11a}$$

$$R = x \tag{3.11b}$$

$$\gamma_x = -x\,U_x^2 + x\,U_y^2 \tag{3.11c}$$

$$\gamma_y = -2x\,U_x\,U_y \tag{3.11d}$$

Note that for $R = x$, $\blacktriangle f = 0$ becomes the *Euler-Darboux equation* $f_{xx} + \frac{f_x}{x} + f_{yy} = 0$ [KR05; Zwi98].

Symmetry properties

As discussed above on page 43, stationary and axially symmetric metrics are characterized by the existence of two Killing vector fields. The corresponding (global) symmetry group is $h = \mathbb{R} \times \mathbb{S}^1$. We have 2 ignorable coordinates, ϕ and t in Lewis-Papapetrou coordinates, that are adjusted to these symmetries (see also Section 2.2.1). We therefore restrict to level surfaces $\{p_\phi = c_\phi, p_t = c_t\}$ with constant c_ϕ, c_t (these need to be regular values). In Lewis-Papapetrou coordinates, h acts along coordinate directions and we are able to identify the reduced coordinates easily. The same applies for prolate-spheroidal coordinates, which we use for the Zipoy-Voorhees metric, and for the Tomimatsu-Sato metric.

The 4-dimensional problem thus can be reformulated as a 2-dimensional problem with metric g_{red}, and the Hamiltonian H splits into a kinetic term $T := H_{\mathrm{red}} = \sum_{i,j\in\{x,y\}} g_{\mathrm{red}}^{ij} p_i p_j$ along with a potential V, which is polynomial in p_ϕ and p_t. Note that the highest-degree component w.r.t. (p_x, p_y) of a Hamiltonian integral is a geodesic invariant (i.e. it commutes with T).

The commutative Killing vector fields are ∂_t and ∂_ϕ, connected with, respectively, staticity and axial symmetry of the metric. Via symplectic reduction, this allows us to reduce the problem to 2 dimensions, see Section 2.2.1. Assume we work on the level hypersurface with $p_\phi = c_\phi$ and $p_t = c_t$. This implies a splitting of the Hamiltonian into a kinetic part (quadratic in p_x, p_y), and a potential $V = V(x, y, c_\phi, c_t)$ as established in more generality in Equation (2.5),

$$H = \underbrace{\sum_{i,j=x,y} g^{ij} p_i p_j}_{\substack{=:T \\ \text{kinetic part}}} + \underbrace{\sum_{i,j=\phi,t} g^{ij} p_i p_j}_{\substack{=:V \\ \text{potential}}} . \tag{3.12}$$

The two parts are again denoted by $H^{(2)} = T$ and $H^{(0)} = V$, as in Chapter 2. The metric and the potential on the reduced space read

$$g_{\text{red}} = e^{2U-2\gamma} \left(dx^2 + dy^2\right) \tag{3.13a}$$

$$V = \frac{1}{e^{2U}x^2}\, p_\phi^2 + \frac{2\,A}{e^{2U}x^2}\, p_\phi\, p_t + \left(\frac{A^2}{e^{2U}x^2} - e^{2U}\right) p_t^2 . \tag{3.13b}$$

Our interest in this chapter is in integrals of the form

$$I(x,y) = \sum_{i=0}^{d} \sum_{j=0}^{i} \sum_{k=0}^{d-i} a_{ijk}(x,y)\, p_x^{i-j}\, p_y^j\, p_\phi^k\, p_t^{d-i-k} \tag{3.14}$$

for metrics of type (3.3).

Finiteness

In Section 1.4 we stated that finiteness of the overdetermined system of PDEs is required for the considered method to work. We saw in Section 2.2.3 that the system of PDEs we consider here is of finite type, see also [Wol98]. The computations in [Wol98] and in the proof of Lemma 1 suggest that for integrals of degree d we typically need $d+1$ steps of prolongation to achieve a conclusion if no additional integral exists in this degree. However, for the stationary and axially symmetric vacuum metrics under consideration, a conclusion is reached after d steps of prolongation, compare also [KM12].

3.1.2 Weyl metrics

In the larger part of this and the following chapter, we are concerned with static SAV metrics, which have a vanishing rotation parameter $A(x,y) = 0$ (so-called Weyl metrics). In terms of the potential, $V = V^{\phi\phi}p_\phi^2 + V^{t\phi}p_t p_\phi + V^{tt}p_t^2$, the vanishing of A implies the vanishing of $V^{t\phi}$. On the other hand, if $V^{t\phi} = \frac{2\,A}{e^{2U}R^2}$ vanishes for a SAV metric, clearly A must vanish. We have

$$V = V^{\phi\phi}p_\phi^2 + \underbrace{V^{t\phi}}_{=0} p_t p_\phi + V^{tt}p_t^2 . \tag{3.15}$$

From Equations (3.13), we obtain the metric and the potential on the reduced space for Weyl metrics:

$$g_{\text{red}} = e^{2U-2\gamma} \left(dx^2 + dy^2\right) \tag{3.16a}$$

$$V = R^{-2} e^{-2U}\, p_\phi^2 - e^{2U}\, p_t^2 \tag{3.16b}$$

3.2 Zipoy-Voorhees family

Let us first consider Zipoy-Voorhees metrics with arbitrary parameter value δ. In prolate spheroidal coordinates they have the parametrized form [Zip66; Voo70]

$$g = \left(\frac{x+1}{x-1}\right)^{\delta} \left((x^2-y^2)\left(\frac{x^2-1}{x^2-y^2}\right)^{\delta^2}\left(\frac{dx^2}{x^2-1}+\frac{dy^2}{1-y^2}\right)+(x^2-1)(1-y^2)d\phi^2\right) - \left(\frac{x-1}{x+1}\right)^{\delta} dt^2. \quad (3.17)$$

We do not use a Lewis-Papapetrou form here, but the metric still has the form (2.11) assumed on page 28.

Zipoy-Voorhees metrics are a special case of Weyl metrics. The parameter δ can be viewed as a variational parameter (often $\delta \geq 0$ is assumed; the metric is transformed into itself if the sign of δ and x are changed simultaneously).

For $\delta = 0$, one obtains the flat metric. In case of $\delta = 1$ (or $\delta = -1$), the Schwarzschild metric is obtained [Ste03; KH03]. Both flat space and the Schwarzschild metric are integrable. In the flat case, there is an involutive set of 4 functionally independent linear integrals [Tho86]. For the Schwarzschild metric the situation is a bit more subtle: there are four linear integrals in addition to the Hamiltonian, but only two of them form an involutive set. A final involutive integral is provided by a sum of squares of linear integrals, see also Section 2.3.4.

Theorem 1. *A Zipoy-Voorhees metric (3.17) can admit an additional quadratic integral $I = K^{ij}(x,y)\,p_i\,p_j$ only for parameter values $\delta = 0, \pm 1$.*

Note that we prove nonexistence of an additional integral of degree 3 in Chapter 4. For degree 2, we present three possible proofs using two different approaches. One approach is presented in anticipation of Chapter 4. The other two proofs are based on Algorithm II.

We begin with a sketch of a proof that follows the procedure in Chapter 4. Afterwards, we are going to see two other proofs that are both based on Algorithm II. We consider only the even-parity case here, i.e.

$$I = \sum_{i,j\in\{1,2\}} K^{ij}(x_1,x_2)\,p_i\,p_j + \sum_{i,j\in\{3,4\}} K^{ij}(x_1,x_2)\,p_i\,p_j\,.$$

The odd-parity part is straightforward. We look into it in Example 1 in Chapter 4 (see page 66) where we also go into the case of cubic degree.

Proof 1: Degree-wise approach

This version of the proof follows the lines along which we prove nonexistence of an additional cubic integral. This is discussed in detail in Chapter 4 on pages 70f. The idea is as follows:

— The integral is of the form

$$I(x,y) = a_0 p_x^2 + a_1 p_x p_y + a_2 p_y^2 + b_0 p_\phi^2 + b_2 p_t^2.$$

Here, we already made use of the additional decomposition properties of Weyl metrics, see pages 34ff.

— Use the degree-1 components of the Poisson equation (1.3) and compute the integrability conditions for b_0 and b_2 (the *Bertrand-Darboux equations*). We take derivatives of these integrability conditions and add the integrability requirements to the remaining system of PDEs (which involves only a_0, a_1, a_2 and the metric).

— We solve for derivatives of the a_i. We prove that $a_1 = 0$ and express derivatives of a_0 and a_2 in terms of the functions themselves.

— We integrate the relations to find a_0 and a_2, and then b_0 and b_2

— We compare the solution with the Hamiltonian. The result is that the integral is a linear combination of the Hamiltonian and the linear integrals

In the course of the computations, we have to require $\delta \neq 0, \pm 1$, as should be expected.

Proof 2: Algorithm-based approach

The second proof is closer to Algorithm II. For parametrized metrics, we obtain a matrix system with entries that depend on the parameter (so we have a parameter-dependent matrix instead of $\mathsf{Mat}(\mathbb{R})$). The rank computation is the main obstacle to directly apply Algorithm II. The routines that are implemented in typical computer algebra systems usually appear to compute a generic matrix rank and do not recognize if the rank drops for certain values of the parameter. We give two possible ways to circumvent this issue. Proof 2 follows the block structure given in Section 2.2.3. Proof 3 uses usual Gauß elimination as far as possible.

Let us consider the fairly general Hamiltonian with

$$H = \frac{1}{2}\left(\frac{p_x^2}{\Omega_1} + \frac{p_y^2}{\Omega_2}\right) + V^{\phi\phi}p_\phi^2 + V^{tt}p_t^2 \tag{3.18}$$

The integral is supposed to be polynomial in the momenta with smooth coefficients, i.e.

$$F = a_0 p_x^2 + a_1 p_x p_y + a_2 p_y^2 + b_0 p_\phi^2 + b_1 p_\phi p_t + b_2 p_t^2. \tag{3.19}$$

Without loss of generality, we assume $b_1 = 0$, because b_1 belongs to the subsystem $\mathcal{S}^{\mathsf{lower}}$ and therefore must be constant, see pages 34ff. The Poisson equation is a polynomial and we extract the system of PDEs as before (one set of equations comes from the component of the polynomial with degree 3, one from degree 1).

$$\begin{aligned} \{T, I^{(2)}\} &= 0 && \text{degree 3} \\ \{T, I^{(0)}\} + \{V, I^{(2)}\} &= 0 && \text{degree 1} \end{aligned}$$

Differentiating the system twice already allows us to solve the equations for all third derivatives $\frac{\partial^3 a_i}{\partial x^i \partial x^j \partial x^k}$, $i, j, k = 1, 2$, $(x^1, x^2) = (x, y)$. Thus, we only need to differentiate once the equations coming from the degree 3 part of the Poisson equation. We solve the equations for all second order derivatives of the a_i except the mixed derivative $\frac{\partial^2 a_1}{\partial x \partial y}$. Thus, the pure derivatives $\frac{\partial^2 a_0}{\partial x^2}$ and $\frac{\partial^2 a_2}{\partial y^2}$ are expressed through the mixed derivative of a_1.

From the degree-1 part we need the first and the second derivatives. However, since the initial equations constitute differential equations for b_0 and b_2, we can restrict to the integrability conditions for those and first derivatives of these integrability conditions. Eliminating second derivatives of the a_i (those obtained in the first step), we obtain four relations for $\frac{\partial^2 a_1}{\partial x \partial y}$:

$$\Omega_1 V_y^{\phi\phi} \frac{\partial^2 a_1}{\partial x \partial y} = \text{lower-order terms} \tag{3.20}$$

$$\Omega_2 V_x^{\phi\phi} \frac{\partial^2 a_1}{\partial x \partial y} = \text{lower-order terms} \tag{3.21}$$

$$\Omega_1 V_y^{tt} \frac{\partial^2 a_1}{\partial x \partial y} = \text{lower-order terms} \tag{3.22}$$

$$\Omega_2 V_x^{tt} \frac{\partial^2 a_1}{\partial x \partial y} = \text{lower-order terms} \tag{3.23}$$

For Weyl metrics and in particular for the Zipoy-Voorhees family, we have $V_y^{\phi\phi} \neq 0$. Thus, we can solve for $\frac{\partial^2 a_1}{\partial x \partial y}$.

The next step is to solve for the first derivatives of the a_i (out of which $\frac{\partial a_0}{\partial x}$ and $\frac{\partial a_2}{\partial y}$ have already been found). We can do this using the remaining two equations from the degree 3 polynomial and the integrability conditions (we use that $\langle \nabla V^{\phi\phi}, \nabla^\perp V^{tt} \rangle \neq 0$ when there is no additional linear integral, see Lemma 4 in Chapter 4).

After this step, we are left with a system of linear equations (regarding the potential as fixed) in the unknowns a_0, a_1, a_2. Three of the equations originate in the derivatives of the integrability conditions. We can consider these equations as a matrix equation

$$M \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = 0. \tag{3.24}$$

The determinant of the 3×3 matrix M is zero, so its rank is either 2 (this is the trivial case, since we know that there is at least one quadratic integral, the Hamiltonian), or the rank is at most 1 (this is necessary for the existence of an additional irreducible integral). The requirement for the rank being at most one is that the determinants of all 2×2 submatrices of M vanish simultaneously. This leads to $\binom{3}{2}^2 = 9$ differential equations for the potential.

— For Weyl metrics, we can use the vacuum conditions (3.11) and we obtain, from the nine determinant equations, three (non-linear) differential equations in the unknowns U_{xxx}, U_{xxy}, U_{xx}, U_{xy}, U_x, U_y.

— Let us restrict ourselves to Zipoy-Voorhees metrics. We use Observation 2 and choose the point $(1/2, 2)$. We are left with only three equations. They admit only the solutions that we already know for δ:
$$\delta = 0 \quad \text{and} \quad \delta = \pm 1.$$

Proof 3: Algorithm II

The third possible proof is a direct implementation of Algorithm II. Proof 2 kept full control over the elimination of unknowns during the projection step. On the other hand, in many cases computer-implemented solving routines can be used to check whether a given parameter-dependent expression is always non-zero. In this case, one has to make sure that they work reliably, e.g. by checking the pivots manually afterwards. Thus, instead of doing the above computation, one can use usual Gauß elimination for the matrix rank computation. The Gauß elimination algorithm can be used as long as there are coefficients that are non-zero for arbitrary values of the parameter. In this way, one can obtain a reduced matrix equation with fewer equations and fewer unknowns. We choose the point $(1/2, 2)$ and subtract suitable multiples of trivial integrals to reduce the matrix. For this reduced matrix, it is not necessary to compute the exact rank. Instead, it suffices to determine whether and for which values of the parameter, the rank of the reduced matrix is maximal. This can be checked by looking at determinants of the submatrices that are obtained by deleting a sufficient number of rows from the reduced matrix.

In the degree-2 case, we obtain through Gauß elimination a 4×3 matrix (we remove lines and columns that are no longer of interest). Then, we compute the determinants of all 4 submatrices. Additional Killing tensors can only exist when all determinants are zero. This leads to the immediate solutions $\delta = 0$ and $\delta = \pm 1$, and a remaining system of algebraic equations on δ:

$$\begin{aligned}
96\delta^6 - 144\delta^5 + 160\delta^4 - 105\delta^3 - 23\delta^2 + 24\delta + 28 &= 0 \\
96\delta^4 - 324\delta^3 + 420\delta^2 - 459\delta + 159 &= 0 \\
192\delta^6 - 720\delta^5 + 1288\delta^4 - 1266\delta^3 + 835\delta^2 + 114\delta - 56 &= 0 \\
3328\delta^8 - 10560\delta^7 + 21664\delta^6 - 28104\delta^5 + 28788\delta^4 - 16665\delta^3 + 11306\delta^2 - 2451\delta + 1064 &= 0
\end{aligned}$$

This system of equations has no solutions and this concludes the proof. □

3.3 A specific Zipoy-Voorhees metric

In Chapter 2 we developed an algorithm to answer Question 1 (how many integrals exist in addition to a given family of standard integrals?) for metrics in Weyl's class, see Algorithm II on page 37.

We now apply Algorithm II to a certain example of a Weyl metric, namely the Zipoy-Voorhees metric with parameter $\delta = 2$. This is the so-called Darmois solution [Ste03; Dar27]. The existence of integrals for this metric has been studied in the literature [Bri08b; KM12; LG12; MPS13], cf. also Chapter 1. Based on numerical analysis, [Bri08b] suggests Liouville integrability for this metric. In [KM12], this suggestion is challenged with a proof of nonexistence of an additional integral polynomial in momenta of degree at most 6. Non-integrability is as well suggested by numerical methods [LG12]. Non-integrability for meromorphic integrals is established in [MPS13]. We prove nonexistence of an additional integral of degree at most 11 for integrals that are polynomials in momenta with smooth coefficient functions (hence our result does not follow from [MPS13]). The computation also illustrates the efficiency of Algorithm II.

The Darmois metric has the following form in prolate spheroidal coordinates:

$$g = \left(\tfrac{x+1}{x-1}\right)^2 \left((x^2 - y^2) \left(\tfrac{x^2-1}{x^2-y^2}\right)^4 \left(\tfrac{dx^2}{x^2-1} + \tfrac{dy^2}{1-y^2}\right) + (x^2-1)(1-y^2)d\phi^2 \right) - \left(\tfrac{x-1}{x+1}\right)^2 dt^2. \tag{3.25}$$

Now, let $\mathcal{S}_d$ be the system of equations obtained from the Poisson equation (1.3) by considering coefficients with respect to momenta. For Weyl metrics, the system of equations $\mathcal{S}_d$ splits into four separate subsystems, which have been introduced as $\mathcal{S}_d^{\mathsf{lead}}$, $\mathcal{S}_d^{\mathsf{lower}}$, $\mathcal{S}_d^{\mathsf{right}}$, $\mathcal{S}_d^{\mathsf{left}}$ in Chapter 2. Each of the subsystems can be solved independently (unknowns of one of the subsystems do not appear in other subsystems). In Lemma 2 on page 36, we establish that two of these subsystems are equivalent (namely, $\mathcal{S}_d^{\mathsf{left}} \sim \mathcal{S}_d^{\mathsf{right}}$). Therefore, the Poisson problem (1.3) transforms into three separate subproblems defined by the subsystems

Results for the Zipoy-Voorhees metric with $\delta = 2$

degree d	0	1	2	3	4	5	6	7	8	9	10	11
all equations	2	15	48	140	300	630	1120	1980	3150	5005	7392	10920
all unknowns	3	12	50	120	294	560	1080	1800	3025	4620	7098	10192
matrix rows	0	0	10	38	64	157	228	452	608	1058	1358	2162
matrix columns	0	0	9	26	52	106	178	306	468	726	1043	1510
matrix rank	0	0	9	26	52	106	178	306	468	726	1043	1510
computation time	<1s	<1s	<1s	<1s	1s	12s	54s	5.1m	26m	1.8h	13h	7days
integrals in lead	1	0	3	0	6	0	10	0	15	0	21	0
all integrals of degree d	1	2	4	6	9	12	16	20	25	30	36	42
known integrals Λ_d^0	1	2	4	6	9	12	16	20	25	30	36	42

Table 3.1: Results computed with Algorithm II. The numbers are for the associated linear system obtained after $M = d$ prolongations. For the computation, we chose the point $(x, y) = (1/2, 2)$

$\mathcal{S}_d^{\mathsf{lead}}$, $\mathcal{S}_d^{\mathsf{lower}}$ and $\mathcal{S}_d^{\mathsf{right}}$ (or $\mathcal{S}_d^{\mathsf{left}}$). It is also already mentioned in Section 2.3.2 in the context of Lemma 2 that the three subsystems structurally are very similar, and that in fact only the subsystem $\mathcal{S}_r^{\mathsf{lead}}$ is *distinct* to a certain degree r. Recall that the subsystem $\mathcal{S}_r^{\mathsf{lead}}$ for degree r is identified as the subsystem

— of parity w.r.t. (p_x, p_y) equal to the parity of d.

— of even parity w.r.t. p_ϕ (or, equivalently, p_t).

Iteratively running Algorithm II, we prove:

Theorem 2. *The Zipoy-Voorhees space-time with parameter $\delta = 2$ has no additional integral polynomial in momenta of degree $d \leq 11$ that is functionally independent of and in involution with the linear integrals p_ϕ, p_t, and the Hamiltonian.*

Algorithmic results. The considerations above suggest the following procedure:

— Consider the integral as a polynomial in momenta that includes only components of leading parity in (p_x, p_y) and even degree in p_ϕ (or p_t)

— Iteratively run Algorithm II beginning at degree 0 using this form of the integral

— At each step, use Formula (2.22) to compute the total number of integrals for this degree

We need d steps of prolongation to reach a conclusion (where d is the degree of the integral), cf. also [KM12]. Table 3.1 on this page gives an overview on the results of the algorithm for the Zipoy-Voorhees metric with $\delta = 2$. Obviously, the results agree with the values obtained by Formula (2.4),

$$\Lambda_d^0 \overset{(2.4)}{=} d + 1 + d\left\lfloor \frac{d}{2} \right\rfloor - \left\lfloor \frac{d}{2} \right\rfloor^2. \tag{3.26}$$

This concludes the proof of Theorem 2.

3.4 A Tomimatsu-Sato metric

We finish this chapter with an application of Algorithm I from Section 2.3.1 to a Tomimatsu-Sato metric. As we are going to see, this algorithm is still very efficient, although the metric has less symmetry than Weyl metrics.

We consider a metric from the Tomimatsu-Sato family, which is a natural thing to do as the Tomimatsu-Sato family generalizes the Zipoy-Voorhees family to the case with rotation. In particular, Tomimatsu-Sato metrics with $\delta = 1$ return Kerr metrics.

We are going to investigate a Tomimatsu-Sato metric with $\delta = 2$. In the *Ernst-Perjés representation*, it has the general form [Ern76; Per89; Man12]

$$g_{\mathrm{TS}} = \kappa^2 f^{-1} \left(e^{2\gamma} (x^2 - y^2) \left(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2} \right) + (x^2-1)(1-y^2)\, d\phi^2 \right) - f\, (dt - \omega\, d\phi)^2. \quad (3.27)$$

The functions f, γ and ω are defined by

$$f = \frac{\mu^2 - (x^2-1)(1-y^2)\sigma^2}{\mu^2 + \mu\nu - (1-y^2)((x^2-1)\sigma^2 - \sigma\tau)}, \quad (3.28a)$$

$$e^{2\gamma} = \frac{\mu^2 - (x^2-1)(1-y^2)\sigma^2}{p^4 (x^2-y^2)^4}, \quad (3.28b)$$

$$\omega = -\frac{\kappa (1-y^2)((x^2-1)\sigma\nu + \mu\tau)}{\mu^2 - (x^2-1)(1-y^2)\sigma^2}, \quad (3.28c)$$

where μ, ν, σ and τ are the polynomials

$$\mu = p^2 (x^2-1)^2 + q^2 (1-y^2)^2, \quad (3.28d)$$

$$\nu = 4x (px^2 + 2x + p), \quad (3.28e)$$

$$\sigma = 2pq (x^2 - y^2), \quad (3.28f)$$

$$\tau = -4qp^{-1} (1-y^2)(px+1). \quad (3.28g)$$

In addition, p and q have to obey the restriction, $p^2 + q^2 = 1$. The other free parameter, κ, can in principle be removed through redefinition of some quantities, but we keep it in view of [Man12]. The particular example that we are studying, is the one with parameter values $\delta = 2$, $\kappa = 2$, and $p = 3/5$ ($q = 4/5$). These parameters have also been chosen in [Man12], where some physical properties of the Tomimatsu-Sato metric for $\delta = 2$ are being discussed.

Theorem 3. *For the Tomimatsu-Sato metric (3.27) with parameter values $\delta = 2$, $\kappa = 2$, and $p = 3/5$ ($q = 4/5$), the family (H, p_t, p_ϕ) of integrals cannot be extended to a Liouville-integrable family of integrals of degree at most 7 in momenta.*

Running Algorithm I for the metric (3.27) with these specified parameter values, we obtain Table 3.2. Inspection of the column for $\overline{\Lambda}_d^{(d)}$ quickly shows that we only detect trivial integrals, and this implies nonexistence of additional integrals up to degree 7.

In Table 3.2, note that the number of equations is the same for both branches, $e = 0$ and $e = 1$, if d is even (recall that e is the parity of the integral in (p_x, p_y), which we have to specify for the computation). On the other hand, the numbers of unknowns agree for both branches if d is odd. This effect is a result of the following formulas governing the numbers of equations and unknowns. For the number of equations, we find

$$\begin{aligned} m_{d,M} &= \sum_{l=0}^{\frac{d+e-\tilde{e}}{2}} (2l+1+\tilde{e})(d+2-\tilde{e}-2l) \binom{M+2}{2} \quad \text{(for degree } d \text{ after } M \text{ prolongations)} \\ &= \frac{1}{24} (d+2+\Delta)(d^2 - d\Delta - 2\Delta^2 + 6e\Delta + 7d - 5\Delta + 12)\,(M+1)(M+2) \end{aligned} \quad (3.29)$$

where $\tilde{e}$ is the parity of $d+e$, i.e. $\tilde{e}=0$ (resp. 1) if $(d+e) \mod 2 = [0]$ (resp. [1]), and where we define $\Delta = e - \tilde{e}$. The number of unknowns follows the formula

$$\begin{aligned} n_{d,M} &= \sum_{l=0}^{\frac{d-e-\tilde{e}}{2}} (2l+1+\tilde{e})(d+1-\tilde{e}-2l)\binom{M+3}{2} \quad \text{(for degree } d \text{ after } M \text{ prolongations)} \\ &= \frac{1}{24}\,(d+2-\Sigma)(d^2+d\Sigma-2\Sigma^2+4d+6e(\Sigma-1)+2\Sigma+6)\,(M+2)(M+3), \end{aligned} \tag{3.30}$$

where we define $\Sigma = e+\tilde{e}$. Hence, e cancels from (3.29) if d is even, because $\tilde{e}=e$ and thus $\Delta = 0$. Likewise, if d is odd, then e cancels from (3.30), because $\tilde{e}+e=1$ and thus $\Sigma = 1$.

Results Tomimatsu-Sato metric

d	e	$\overline{\Lambda}_d^{(d)}$	$m_{d,d}$	$n_{d,d}$	rows of M	columns of M	rk(M)	time
1	0	2	12	12	0	0	0	0.1s
	1	0	18	12	10	4	4	0.4s
2	0	4	60	60	13	9	9	2s
	1	0	60	40	28	8	8	1s
3	0	6	160	150	34	18	18	3s
	1	0	190	150	72	32	32	5s
4	0	9	420	399	91	61	61	18s
	1	0	420	336	140	56	56	18s
5	0	12	840	784	172	104	104	1.5m
	1	0	924	784	276	136	136	1.9m
6	0	16	1680	1584	342	230	230	32m
	1	0	1680	1440	456	216	216	28m
7	0	20	2880	2700	556	356	356	21h
	1	0	3060	2700	776	416	416	24h

Table 3.2: Complete table of results for the Tomimatsu-Sato metric with the parameter values $\delta = 2$, $\kappa = 2$, and $p = 3/5$ ($q = 4/5$). The degree of the integral is denoted by d, its parity in (p_x, p_y) by e. The results are obtained after d steps of prolongation. The symbol M denotes the reduced matrix. The symbols $m_{d,d}$ and $n_{d,d}$ denote, respectively, the number of equations and unknowns of the initial matrix system. The point of reference for the computations is $(x,y) = (1/2, 2)$. The last column provides the (approximate) computation times, cf. Section 2.4.

Chapter 4

Cubic Integrals in Arbitrary Weyl Metrics

In this chapter, we discuss cubic integrals in Weyl metrics. Theorem 4 is the main result of this chapter, proving reducibility for involutive integrals of degree 3 for arbitrary Weyl metrics. Weyl metrics and their properties have already been discussed in detail in Chapter 3, especially on pages 44ff. The contents of the present chapter are an adaptation of the author's paper [Vol15b]. The chapter is organized as follows: first, we consider the existence of an additional linear integral of odd parity in (p_x, p_y) and study its implications for the system of equations describing the existence of an additional cubic integral. For cubic integrals we then characterize the case of an additional linear integral. In case such an additional integral does not exist, we construct a necessary criterion for existence of an additional (irreducible and involutive) cubic integral. Finally, we combine this necessary criterion with the remaining equations to prove general nonexistence of an additional cubic integral.

Recall from Section 1.1 on page 14f that when using the term *additional integrals*, we count only integrals irreducible with respect to and in involution with what we consider to be standard integrals. Since we prove nonexistence of such additional integrals in degree 3, this implies $\mathcal{I}$-reducibility according to Definition 4. For Weyl metrics, we prove reducibility of degree-3 integrals by one degree. In addition, we prove total reducibility of degree-3 integrals for the family of Zipoy-Voorhees metrics, continuing the considerations in Chapter 3 on pages 49ff.

Theorem 4. *Let (M, g) be a 4-dimensional manifold M endowed with a Weyl metric g. Then, any involutive integral of third degree for the geodesic flow of the Weyl metric g is reducible.*

4.1 Method

The proof exploits prolongation-projection techniques as well as the particular structure of the system $\mathcal{S}$ of PDEs obtained from the Poisson equation (1.3). For Weyl metrics, we establish useful properties in Section 2.3.2 and these properties become helpful again in this chapter. For instance, by Lemma 2 on page 36, we can restrict ourselves to a smaller system of PDEs. In the following, we adopt the terminology introduced in Section 2.2.3 and refer to homogeneous components of the Poisson equation (1.3) as subpolynomials.

We assume a manifold M with Weyl metric g, and we search for integrals in addition to the standard integrals $\mathcal{I} = (H, p_\phi, p_t)$. As before, we restrict to integrals coming from a Killing tensor (i.e. integrals that are homogeneous polynomial w.r.t. momenta) that are irreducible w.r.t. $\mathcal{I}$ and in involution with $\mathcal{I}$. We proceed according to the following directions:

(i) Use symplectic reduction to transform the 4-dimensional homogeneous problem without potential to a 2-dimensional (non-homogeneous) problem with potential (page 23f). The assumption that the additional integral we look for is in involution with the standard integrals means that it can only depend on the non-ignorable coordinates x and y, rather than ϕ, t. Moreover, since we work with Weyl metrics, we need only consider the distinct subsystem $\mathcal{S}^{\mathsf{lead}}$, cf. Section 2.3.2 on pages 34ff.

(ii) The (overdetermined) system of PDEs describing the requirement that $I : T^*M \to \mathbb{R}$ is an integral is obtained from the Poisson equation (1.3), i.e. from $\{H, I_K\} = 0$. In Section 1.3, we explain how this system of PDEs splits according to the degree in the momenta (p_x, p_y), and how this yields three (sub-)polynomials. We have also seen how to decompose these polynomials further w.r.t. the momenta (p_ϕ, p_t) to obtain a coupled system of polynomials as in Figure 2.4 on page 35.

(iii) The equations obtained from zeroth degree in momenta (p_x, p_y) can be understood as simple scalar product relations. We assume that there is no linear integral in addition to (p_ϕ, p_t) (the case with an additional linear integral is simple and is discussed separately). Two of the relations are orthogonality relations and can be solved for components of the integral I. Together with the third equation they leave one unknown function α that parametrizes the degree-1 component of the integral I.

(iv) The equations obtained from second degree in (p_x, p_y) lead to an integrability condition for α.

(v) The remaining system of PDEs involves components of the metric and their derivatives. We can eliminate the parameter function γ using the vacuum conditions (3.11) from page 47, leaving one unknown function $U(x, y)$ determining the metric. We treat derivatives of U as being new, independent unknowns. The system of PDEs on the coefficients of U is overdetermined and of finite type. By computing its differential consequences (prolongation) and subsequently eliminating the highest derivatives of U (projection), we end up with a simple differential equation for U.

(vi) It remains to solve an ordinary differential equation to obtain U, which can be done explicitly and leads to a solution that describes flat space. This excludes additional (non-trivial) cubic integrals.

The Poisson equation (1.3) is a polynomial in momenta of degree $d + 1$, and I is a (homogeneous) polynomial of degree d. We obtain the system of PDEs from the coefficients of the Poisson equation (1.3) in momenta. In addition, we add the vacuum conditions (3.11) to the system, because the Ricci tensor does not vanish automatically when using Equations (3.16).

The computations made here are completely explicit, though cumbersome. The first part can reasonably be done manually, while the second part involves tedious and lengthy expressions. These can easily be treated using computer algebra.

Step (i). The coordinates ϕ and t are ignorable, cf. page 23f. Symplectic reduction w.r.t. the corresponding symmetry group (defined by stationarity and axial symmetry) suggests to regard level surfaces with constant $p_\phi = c_\phi$ and $p_t = c_t$. We distinguish the equations of the system of PDEs according to the momenta monomials from which they emerged as a coefficient (compare the subpolynomials in Section 2.3.1 on pages 29ff). Since we work with Weyl metrics, the equations arrange in a structure as shown in Equation (2.21) on page 35. We write down the Hamiltonian in component-wise form:

$$H = T + \underbrace{V^{\phi\phi}p_\phi^2 + V^{tt}p_t^2}_{=V} \tag{4.1}$$

where $T \equiv H^{(2)}$ is the reduced Hamiltonian, i.e. a homogeneous polynomial in p_x, p_y, and where V^{ab} are the smooth coefficient functions of the monomials $p_a\,p_b$ ($a, b \in \{\phi, t\}$). In the 2-dimensional picture, we deal with Hamiltonian invariants instead of geodesic invariants, and decomposition of the integral I w.r.t. (p_x, p_y) leads to

$$I = I^{(d)} + \underbrace{I_\phi^{(d-1)}p_\phi + I_t^{(d-1)}p_t}_{=I^{(d-1)}} + \underbrace{I_{\phi\phi}^{(d-2)}p_\phi^2 + I_{t\phi}^{(d-2)}p_tp_\phi + I_{tt}^{(d-2)}p_t^2}_{=I^{(d-2)}} + \cdots + I_{tt\ldots t}^{(0)}p_t^d, \tag{4.2}$$

where each $I^{(k)}$ is of degree k in the momenta p_x, p_y. We require the metric to be non-flat such that we can choose coordinates with $R = x$ (R has to be non-constant, see Section 3.1.1).

Step (ii). As discussed in Section 2.3 on pages 28ff, the Poisson equation (1.3) can be split into subpolynomials w.r.t. degree in the non-ignorable momenta. On the level of the system of PDEs obtained from (1.3), this corresponds to splitting the system into smaller (but coupled) systems of equations. Let us first consider only the split of (1.3) into three subpolynomials according to the degree w.r.t. (p_x, p_y):

$$\{T, I^{(3)}\} = 0 \qquad \textit{degree 4} \tag{4.3a}$$

$$\{T, I^{(1)}\} + \{V, I^{(3)}\} = 0 \qquad \textit{degree 2} \tag{4.3b}$$

$$\{V, I^{(1)}\} = 0 \qquad \textit{degree 0} \tag{4.3c}$$

The equations of odd parity in (p_x, p_y) split off from this system and form a separate system decoupled from the other equations (cf. the split into $\mathcal{S}^{\mathsf{even}}$ and $\mathcal{S}^{\mathsf{odd}}$ in Section 2.3.1). Equation (4.3a) is the condition that must hold for a geodesic invariant $I^{(3)}$ on the reduced manifold. However, only some of these integrals 'ascend' to integrals on the initial manifold M, due to the restrictions (4.3b) and (4.3c). We take the following view on the equations: Equations (4.3b) and (4.3c) constitute relations that define $I^{(3)}$ and $I^{(1)}$, while (4.3a) is a restriction on $I^{(3)}$, and hence on $I^{(1)}$ (note that this is in a sense reverse to the view taken in [Hie87], where the leading-degree equation is solved first).

We split Equations (4.3) further considering coefficients w.r.t. (p_t, p_ϕ). The polynomial (4.3a) does not involve these momenta, and hence does not decompose further. Equation (4.3b) is of degree 2 in (p_t, p_ϕ), and so it can be split into three parts:

$$\{T, I_{\phi\phi}^{(1)}\} + \{V^{\phi\phi}, I^{(3)}\} = 0$$

$$0 = \{T, I_{t\phi}^{(1)}\}$$

$$\{T, I_{tt}^{(1)}\} + \{V^{tt}, I^{(3)}\} = 0$$

We write the equations in this form to hint at the fact that for Weyl metrics the subsystem $\mathcal{S}^{\mathsf{odd}}$ of our problem decomposes into separate subsystems $\mathcal{S}^{\mathsf{lead}}$ and $\mathcal{S}^{\mathsf{lower}}$, as discussed in Section 2.3.2. The second of the equations implies that $I^{(1)}_{t\phi}$ is a geodesic invariant on the reduced space. From (4.3c) we now see that it ascends to a Hamiltonian invariant on the level of the initial manifold M (of course, we already know this from Lemma 2). The polynomial equation (4.3c) splits into five parts:

$$
\begin{aligned}
\{V^{\phi\phi}, I^{(1)}_{\phi\phi}\} &= 0 \\
0 &= \{V^{\phi\phi}, I^{(1)}_{t\phi}\} \\
\{V^{tt}, I^{(1)}_{\phi\phi}\} + \{V^{\phi\phi}, I^{(1)}_{tt}\} &= 0 \\
0 &= \{V^{tt}, I^{(1)}_{t\phi}\} \\
\{V^{tt}, I^{(1)}_{tt}\} &= 0
\end{aligned}
$$

The second and fourth of these relations again belong to $\mathcal{S}^{\mathsf{lower}}$, and we remove this subsystem, since it is not interesting for our problem (see, however, Section 4.2).

Step (iii). The remaining three equations from (4.3c) can be interpreted nicely as scalar product relations for the components of $I^{(1)}$. For instance,

$$
\{V^{\phi\phi}, I^{(1)}_{\phi\phi}\} = V^{\phi\phi}_x\, b^{\phi\phi}_1 + V^{\phi\phi}_y b^{\phi\phi}_2 = e^{-2U+2\gamma}\, \langle \nabla V^{\phi\phi}, b^{\phi\phi}\rangle = \langle dV^{\phi\phi}, b^{\phi\phi}\rangle
$$

where $I^{(1)}_{\phi\phi} = b^{\phi\phi}_1 p_x + b^{\phi\phi}_2 p_y$ and where $\nabla V^{\phi\phi}$ denotes the gradient vector corresponding to the differential $dV^{\phi\phi}$. In this way, (4.3c) turns into the scalar product relations

$$
\begin{aligned}
\langle \nabla V^{\phi\phi}, b^{\phi\phi}\rangle &= 0 \\
\langle \nabla V^{tt}, b^{\phi\phi}\rangle + \langle \nabla V^{\phi\phi}, b^{tt}\rangle &= 0 \\
\langle \nabla V^{tt}, b^{tt}\rangle &= 0
\end{aligned}
$$

that can be solved for $b^{\phi\phi}$ and b^{tt} in terms of a parameter function, yielding

$$
b^{\phi\phi} = \alpha_1\, \nabla^{\perp} V^{\phi\phi} \qquad \text{and} \qquad b^{tt} = \alpha_2\, \nabla^{\perp} V^{tt}, \tag{4.4}
$$

where we introduced the shorthand notation $\nabla^{\perp} f = e^{2U-2\gamma}(-f_y, f_x)$, for a function f on the reduced space, i.e. $\nabla^{\perp} f$ is the vector field obtained from ∇f via rotation by $\pi/2$. Defining the angle Ψ between ∇V^{tt} and $\nabla V^{\phi\phi}$,

$$
\cos\Psi = \frac{\langle \nabla V^{tt}, \nabla V^{\phi\phi}\rangle}{\|\nabla V^{\phi\phi}\|\, \|\nabla V^{tt}\|}, \tag{4.5}
$$

the second of the three scalar product relations can be brought into the form

$$
(\alpha_2 - \alpha_1)\ \sin\Psi = 0. \tag{4.6}
$$

Let us summarize this:

Lemma 3. *The potentials $\nabla V^{\phi\phi}$ and ∇V^{tt} are proportional, or the parameter functions α_1 and α_2 are equal.*

The case $\sin\Psi = 0$ ($\nabla V^{\phi\phi}$ and ∇V^{tt} proportional) is discussed shortly, in Sections 4.2 and 4.3.1. But first, let us briefly address Step (iv) of our procedure list.

Step (iv). Return to the equations obtained from (4.3b). Consider the term $\{V^{\phi\phi}, I^{(3)}\}$ and denote $I^{(3)} = \sum_{i,j,k\in\{x,y\}} I^{ijk}\, p_i\, p_j\, p_k$. We obtain

$$\begin{aligned}\{V^{\phi\phi}, I^{(3)}\} &= 3\left(V^{\phi\phi}_x I^{xij} p_i p_j + V^{\phi\phi}_y I^{yij} p_i p_j\right)\\ &= 3\,(V^{\phi\phi}_k I^{kij} p_i p_j),\end{aligned}$$

and an analogous computation for $\{V^{tt}, I^{(3)}\}$. With this in mind, we interpret (4.3b) as defining equations for the tensor field $K^{(3)}$ corresponding to $I^{(3)}$, namely as equations for $K^{(3)}(\nabla V^{\phi\phi}, \cdot, \cdot)$ and $K^{(3)}(\nabla V^{tt}, \cdot, \cdot)$. There are two more equations than components of $K^{(3)}$ and in fact we are able to find expressions for all components of $K^{(3)}$ if the differentials dV^{tt} and $dV^{\phi\phi}$ are linearly independent (in this way, we express $K^{(3)}$ in terms of derivatives of the function $\alpha := \alpha_1 = \alpha_2$). We have the following obvious identities:

$$K^{(3)}(\nabla V^{\phi\phi}; \nabla V^{\phi\phi}, \nabla V^{tt}) = K^{(3)}(\nabla V^{tt}; \nabla V^{\phi\phi}, \nabla V^{\phi\phi}) \tag{4.7a}$$

$$K^{(3)}(\nabla V^{\phi\phi}; \nabla V^{tt}, \nabla V^{tt}) = K^{(3)}(\nabla V^{tt}; \nabla V^{\phi\phi}, \nabla V^{tt}) \tag{4.7b}$$

These can be used to obtain expressions for the derivatives of the function α in terms of the parameter function U of the Weyl metric. Before we investigate this further, let us take a step back and first consider linear integrals.

We proceed as follows:

1. Consider the case $\sin\Psi = 0$. This can be linked to the existence of an additional Killing vector.

2. In case $\sin\Psi \neq 0$, determine $K^{(3)}$ in terms of $\alpha = \alpha_1 = \alpha_2$ and its derivatives. Express the derivatives of α via the parameter function U of the Weyl metric, using the symmetry in the arguments of $K^{(3)}$. Then derive an integrability condition for α.

3. Combine the integrability condition with (4.3a) and the vacuum conditions (3.11). Show that the system does not have any solution other than flat space, using algebraic manipulations as well as prolongation-projection arguments.

We begin with a look at Killing vectors, continuing Step (iii) of our procedure from page 58.

4.2 Killing vector fields

Let us first drop our assumption of a Weyl metric and allow non-staticity (in particular arbitrary non-constant $V^{t\phi} \neq 0$). Consider a 4-dimensional SAV metric and assume there is an additional linear integral. We characterize the existence of linear integrals in terms of the rank of the 2×3 matrix whose columns are given by gradients of the potential components $V^{\phi\phi}$, $V^{t\phi}$ and V^{tt}:

$$\mathcal{M} := (dV^{\phi\phi}, dV^{tt}, dV^{t\phi}). \tag{4.8}$$

Since the rank of $\mathcal{M}$ is the dimension of the linear space spanned by $dV^{\phi\phi}$, dV^{tt}, $dV^{t\phi}$, it is a geometric object and independent of the choice of coordinates on the reduced space. If the differential $dV^{t\phi} = 0$, the matrix $\mathcal{M}$ might be replaced by the 2×2 matrix $(dV^{\phi\phi}, dV^{tt})$, which is also denoted by $\mathcal{M}$ in the following. Then, instead of the rank of the matrix, the determinant can be used with the obvious correspondences. In case there is an additional linear integral present for a non-flat SAV metric (i.e. in addition to p_ϕ and p_t), the rank of $\mathcal{M}$ cannot be full. More precisely,

Lemma 4.

(a) Let (M, g) be in the SAV class.

- *If there is an additional linear integral (Killing vector field), then the rank of $\mathcal{M}$ is 1, or g is a flat metric.*
- *Let $\mathrm{rk}(\mathcal{M}) = 1$. Then p_y is a linear integral (Killing vector field) when using Weyl canonical coordinates[1].*

(b) Let (M, g) be in Weyl's class.

- *Let $\mathrm{rk}(\mathcal{M}) \leq 1$ be constant. Then there is an additional linear integral on M. In case $\mathrm{rk}(\mathcal{M}) = 1$ this vector field corresponds to p_y in Weyl canonical coordinates. If $\mathrm{rk}(\mathcal{M}) = 0$ the metric is flat.*
- *If there is an additional linear integral, it is given by p_y in Weyl canonical coordinates, if the metric is non-flat.*

Proof. Part (a). For linear integrals we only have two polynomials after taking coefficients w.r.t. (p_x, p_y), and they are similar to the polynomials of degree 0 and 2 in (4.3). Let us denote the components of the (p_x, p_y)-linear part of the integral as

$$I^{(1)} = b_1\, p_x + b_2\, p_y, \qquad b = (b_1, b_2). \tag{4.9}$$

The zeroth order equations read

$$\langle \nabla V^{\phi\phi}, b\rangle = 0, \qquad \langle \nabla V^{t\phi}, b\rangle = 0, \qquad \langle \nabla V^{tt}, b\rangle = 0. \tag{4.10}$$

Thus we conclude that the following relations must hold[2]:

$$\langle \nabla V^{\phi\phi}, \nabla^{\perp} V^{t\phi}\rangle = 0, \qquad \langle \nabla V^{\phi\phi}, \nabla^{\perp} V^{tt}\rangle = 0, \qquad \langle \nabla V^{t\phi}, \nabla^{\perp} V^{tt}\rangle = 0. \tag{4.11}$$

This means that the potential gradients are pairwise linearly dependent, i.e. they are pointing all in the same direction. Hence, the rank of the potential gradient matrix is 1, provided the metric is not flat. In the flat case we may have Lewis-Papapetrou coordinates with the parameter R constant[3], making it impossible to choose $R = x$. In this case we may choose another representation of the metric, or just exclude the flat case from our considerations. This concludes the proof of the first claim.

Now, let the rank of the matrix $\mathcal{M}$ be 1, so its rows, or all columns, have to be linearly dependent. This again gives us relations of the form (4.11), meaning that $\nabla V^{\phi\phi}$, $\nabla V^{t\phi}$ and ∇V^{tt} are pairwise linearly dependent. First, let us assume $\nabla V^{t\phi}$ and $\nabla V^{\phi\phi}$ to be non-zero[4]. We consider

$$\langle \nabla V^{\phi\phi}, \nabla^{\perp} V^{t\phi}\rangle = 0$$

This equation amounts to the requirement

$$x\, A_x\, U_y - (1 + x\, U_x)\, A_y = 0,$$

[1] Recall that Weyl canonical coordinates are Lewis-Papapetrou coordinates with $R = x$.

[2] In case that all potential gradients vanish, the metric is flat. Then, any geodesic invariant is a Hamiltonian invariant for the reduced space.

[3] The flat metric can also be described with Weyl canonical coordinates, cf. Remark 2.

[4] The first assumption basically is $A \neq 0$ (for constant A the metric is static). The second assumption is true except for the case when $U = -\ln(x)$, which implies constant A.

or the relation

$$\begin{pmatrix} A_x \\ A_y \end{pmatrix} = \kappa \begin{pmatrix} 1 + x\,U_x \\ x\,U_y \end{pmatrix},$$

with a scalar function κ to be determined. Inserting this into the requirement

$$\langle \nabla\, V^{\phi\phi}, \nabla^{\perp} V^{tt} \rangle = 0$$

yields the relation

$$U_y\, x^2\, e^{4U} = 0$$

and forces U to be a function of x only. Turning back to the relations for A, we see that $A_y = 0$, and thus A also is a function of x only.

Recalling the convention $R = x$, the metric depends on x only, if γ only depends on x, or if it is constant. We invoke the vacuum conditions (3.11c) and (3.11d),

$$\begin{aligned} 4x^2\, e^{4U}\, U_x^2 - A_x^2 - 4x\, e^{4U}\, \gamma_x &= 0, \\ \gamma_y\; x\, e^{4U} &= 0. \end{aligned}$$

Consider the latter equation. It means $\gamma_y = 0$, so we are done. Since the metric does not depend on y, p_y must be an integral, and therefore provides a Killing vector field.

Now, assume that $A = 0$. Then $\langle \nabla V^{\phi\phi}, \nabla^{\perp} V^{t\phi} \rangle = 0$ trivially and we have to go a slightly different way of reasoning. We consider

$$\langle \nabla V^{\phi\phi}, \nabla^{\perp} V^{tt} \rangle = 0.$$

It follows that

$$x^2\, e^{4U}\, U_y = 0,$$

which means $U_y = 0$ (on the entire neighborhood). Conclude $U = U(x)$, and then $\gamma = \gamma(x)$. Thus, the metric is a function of x only and p_y is a linear integral. This concludes the proof of part (a).

For part (b) let us first remark that if an additional linear integral exists in Weyl's class, it must be a multiple of p_y, or the metric is flat. Two cases need to be checked: Firstly, if there is exactly one additional linear integral, it is a multiple of p_y. Secondly, if there are two (independent) additional linear integrals, there are three (say $b^{(k)}, k = 1, 2, 3$). Looking at the equations $\langle \nabla\, V^{ij}, b^{(k)} \rangle = 0$, this forces all gradients ∇V^{ij} to be zero (or, equivalently, $dV^{ij} = 0$). Hence, V is constant. Thus U and A are constant and the metric is flat.

With this remark, the first claim of part (b) follows immediately from part (a), keeping in mind that rank 0 corresponds to flat space. The second claim of part (b) follows immediately from the second statement of part (a). □

4.3 Proof of Theorem 4

For the proof we assume, w.l.o.g., constant rank for the matrix $\mathcal{M}$. If we do not have constant $\operatorname{rk} \mathcal{M}$, we may still consider the subsets of points in the manifold M with constant rank 0, 1, or 2. Then, we consider the sets of their inner points ignoring the remaining points of M, which amount only to a null set w.r.t. the measure induced by the volume form on the manifold M. If we prove that a degree-3 polynomial integral is identical to a product of H, p_ϕ and p_t on an open subset, this is true everywhere.

4.3.1 Preliminaries and preparations

We first consider the case $\mathrm{rk}(\mathcal{M}) = 1$ and then continue with the case $\mathrm{rk}(\mathcal{M}) = 2$. As we have just seen, rank 1 is the case when there is one additional Killing vector field. Rank 2, on the other hand, is the case when no additional Killing vector field exists (assuming non-flatness).

Lemma 5. *If* $\mathrm{rk}\,\mathcal{M} = 1$*, then a third degree integral is reducible by at least one degree.*

Proof. By the hypothesis, there is the linear integral p_y in Weyl's canonical coordinates. Consider equations as in (4.3b), i.e. the equations

$$\begin{aligned}\{V^{\phi\phi}, F^{(3)}\} + \{T, F^{(1)}_{\phi\phi}\} &= 0,\\ \{V^{t\phi}, F^{(3)}\} + \{T, F^{(1)}_{t\phi}\} &= 0,\\ \{V^{tt}, F^{(3)}\} + \{T, F^{(1)}_{tt}\} &= 0.\end{aligned} \tag{4.12}$$

Each $F^{(1)}_{ab}$, $a, b \in \{\phi, t\}$, is a multiple of p_y, so we have

$$F^{(1)}_{\phi\phi} = h_1\, p_y, \qquad F^{(1)}_{t\phi} = h_2\, p_y, \qquad F^{(1)}_{tt} = h_3\, p_y,$$

implying for the Equations (4.12) that

$$\{V^{\phi\phi}, F^{(3)}\} + \{T, h_1\}\, p_y = 0 \tag{4.13a}$$

$$\{V^{t\phi}, F^{(3)}\} + \{T, h_2\}\, p_y = 0 \tag{4.13b}$$

$$\{V^{tt}, F^{(3)}\} + \{T, h_3\}\, p_y = 0 \tag{4.13c}$$

Hence, the leading order term $F^{(3)}$ takes the form

$$F^{(3)} = p_x\ ((\dots)\, p_y) + f\, p_y^3 =: F\, p_y \tag{4.14}$$

where the leading factor p_x is there because the potential gradients (or, equivalently, the differentials dV^{ab}) have only p_x components. The final contribution, $f\, p_y^3$, accounts for the fact that (4.12) only specifies components with at least one p_x. Now, considering (4.3a),

$$\{T, F^{(3)}\} = \{T, F\, p_y\} = \{T, F\}\, p_y = 0.$$

This means $\{T, F\} = 0$, so F is a quadratic integral on the reduced space. It follows that it can be extended to an integral on the initial space-time, because of the fact that

$$\{V^{\phi\phi}, F^{(3)}\} = \{V^{\phi\phi}, F\, p_y\} = \{V^{\phi\phi}, F\}\, p_y$$

and so on. Thus, we obtain from (4.12) the equations

$$\begin{aligned}\{V^{\phi\phi}, F\} + \{T, h_1\} &= 0\\ \{V^{t\phi}, F\} + \{T, h_2\} &= 0\\ \{V^{tt}, F\} + \{T, h_3\} &= 0,\end{aligned}$$

which determine h_1, h_2 and h_3 and therefore turn $\tilde{F} = F + h_1\, p_\phi^2 + h_2\, p_\phi\, p_t + h_3\, p_t^2$ into a quadratic integral on the initial space-time (see Remark 1 below). Note that $\tilde{F}$ might still be reducible or non-reducible. □

Remark 1. *An even-parity quadratic integral*

$$I = I^{(2)} + I^{(0)}_{\phi\phi}\, p_\phi^2 + I^{(0)}_{t\phi}\, p_t\, p_\phi + I^{(0)}_{tt}\, p_t^2$$

satisfies the polynomial equations

$$\{T, I^{(2)}\} = 0, \tag{4.15a}$$

$$\{V^{\phi\phi}, I^{(2)}\} + \{T, I^{(0)}_{\phi\phi}\} = 0, \tag{4.15b}$$

$$\{V^{t\phi}, I^{(2)}\} + \{T, I^{(0)}_{t\phi}\} = 0, \tag{4.15c}$$

$$\{V^{tt}, I^{(2)}\} + \{T, I^{(0)}_{tt}\} = 0. \tag{4.15d}$$

Proof. Decompose $\{H, I\} = 0$ w.r.t. (x, y, ϕ, t) and consider components homogeneous in the momenta (p_x, p_y). The first equation is the component of degree 3, the other three equations are components of degree 1 in (p_x, p_y). □

Steps (iii) and (iv) continued. Let us return to our considerations on Steps (iii) and (iv) in the list on page 58. The previous discussion already covers the case when there is an additional linear integral in involution with the standard integrals, so we can now focus on the case with no such additional integral. We assume a metric of Weyl's class. Keeping in mind the considerations of the previous sections, we see that this case requires $\mathrm{rk}\,\mathcal{M} = 2$. In case of rank 2, $\nabla V^{\phi\phi}$ and ∇V^{tt} must not be proportional. Then, recalling equation (4.6), the scaling functions α_1 and α_2 are equal for Weyl metrics. For simplicity we therefore introduce the new function $\alpha = \alpha_1 = \alpha_2$ (cf. Step (iii) on page 60), and obtain

$$b^{\phi\phi} = \alpha\, \nabla^\perp V^{\phi\phi} \qquad \text{and} \qquad b^{tt} = \alpha\, \nabla^\perp V^{tt}, \tag{4.16}$$

where $b^{\phi\phi}$ and b^{tt} are defined analogously to b in (4.9) on page 62.

Lemma 6. *Derivatives of α are determined by differential equations of the form*

$$\alpha_x = B\, \alpha$$
$$\alpha_y = \tilde{B}\, \alpha,$$

where B and $\tilde{B}$ are algebraic expressions containing no higher-than-second derivatives of components of U.

Proof. We use the relations (4.7), i.e. we basically use the six equations following from Equation (4.3b), and combine them in a straightforward way to find expressions for the coefficients a_0 through to a_3 of $I_T = \sum_i a_i\, p_x^{d-i} p_y^i$. In this way, we find two different expressions for a_1 and two for a_2, corresponding to the identities (4.7). The expressions contain no higher-than-second derivatives of U and γ (via T and derivatives of $V^{\phi\phi}$ and V^{tt}). The functions U and γ do not appear themselves, but only via derivatives, and therefore we can write B and $\tilde{B}$ in terms of first and second derivatives of U. The coefficients of the a_i are simply integer multiples of $\nu = \langle \nabla V^{tt}, \nabla^\perp V^{\phi\phi} \rangle = \frac{4}{x^3} U_y$, which is non-zero because we required ∇V^{tt} and $\nabla V^{\phi\phi}$ not to be proportional. We can then eliminate a_1 and a_2 and deduce two equations of the following form:

$$\langle \nabla V^{tt}, \nabla^\perp V^{\phi\phi} \rangle\ \alpha_x = B'\, \alpha$$
$$\langle \nabla V^{tt}, \nabla^\perp V^{\phi\phi} \rangle\ \alpha_y = \tilde{B}'\, \alpha$$

The expressions B' and $\tilde{B}'$ are polynomials in derivatives of U and γ (the derivatives are at most of second order). Components of the reduced metric appear at most with one differentiation. Dividing by the non-zero coefficient of the α-derivatives yields the desired result. $\square$

The integrability condition for α, i.e. $\alpha_{xy} - \alpha_{yx} = 0$, must hold and constitutes a *necessary criterion* for the existence of non-reducible Killing tensor fields.

Lemma 7. *Let* $\operatorname{rk}\mathcal{M} = 2$ *and* $A = 0$ *(*$\nabla V^{\phi\phi}, \nabla V^{tt} \neq 0$*). If there is an additional Killing tensor field of valence 3, then* $B_y - \tilde{B}_x = 0$.

Proof. Compute

$$\begin{aligned}(\alpha_x)_y - (\alpha_y)_x &= B_y\,\alpha + B\,\alpha_y - \tilde{B}_x\,\alpha - \tilde{B}\,\alpha_x \\ &= (B_y - \tilde{B}_x)\,\alpha + (B\tilde{B} - \tilde{B}B)\,\alpha \\ &= (B_y - \tilde{B}_x)\,\alpha\end{aligned}$$

Thus, we need $\alpha = 0$ or $B_y - \tilde{B}_x = 0$. In case that $\alpha = 0$, the integral $F_3 = F^{(3)} + F^{(1)} = 0$. Hence, the necessary criterion for the existence of non-trivial Killing tensor fields of valence 3 is $B_y - \tilde{B}_x = 0$. $\square$

We give an example where this idea already provides information on the reducibility of cubic integrals:

Example 1. *The Zipoy-Voorhees family of metrics is a family of Weyl metrics that is parametrized by a number* δ*, see also Chapter 3.*
We can use the method as described above, but we take H *in a modified form, namely*

$$H = \frac{p_x^2}{2\Omega_1} + \frac{p_y^2}{2\Omega_2} + V^{\phi\phi}\,p_\phi^2 + V^{tt}\,p_t^2.$$

The Zipoy-Voorhees metric satisfies, in prolate spheroidal coordinates:

$$\begin{aligned}\Omega_1 &= \frac{1}{2}\left(\frac{x^2-1}{x^2-y^2}\right)^{\delta^2}\left(\frac{x+1}{x-1}\right)^{\delta}\frac{x^2-y^2}{x^2-1} \\ \Omega_2 &= \frac{1}{2}\left(\frac{x^2-1}{x^2-y^2}\right)^{\delta^2}\left(\frac{x+1}{x-1}\right)^{\delta}\frac{x^2-y^2}{1-y^2} \\ V^{\phi\phi} &= \left(\left(\frac{x+1}{x-1}\right)^{\delta}(x^2-1)\,(1-y^2)\right)^{-1} \\ V^{tt} &= -\left(\frac{x+1}{x-1}\right)^{\delta}\end{aligned}$$

Proceeding as in Lemma 7, we first check that $\det\mathcal{M} \neq 0$*. Using computer algebra, we find the following:*

$$\det\mathcal{M} = 0 \quad\Leftrightarrow\quad -4y\delta = 0.$$

which obviously is not true for $\delta \neq 0$ *and generic* x, y*. Then we compute the necessary criterion as in Lemma 7. We find*

$$B_y - \tilde{B}_x = -\frac{2}{3}\frac{4x\delta^2 - 3(x^2-1)\delta + 2x}{y(x^2-1)^2},$$

which is nonzero for generic x, y*. We therefore must conclude* $\alpha = 0$*, which means that the integral must be zero (though it was supposed to be of degree 3).*

4.3.2 Completion of the proof

With all these preparations at hand, we can now complete the proof of Theorem 4. This comprises Steps (v) and (vi) of the list on page 58.

Lemma 8. *A polynomial equation of degree $N > 0$ for a function $f(x, y)$ with coefficients that depend on x only, is independent of y, i.e. $f = f(x)$.*

Proof. Denote the equation by $\sum_{n=0}^{N} a_n(x)\, f^n(x, y) = 0$, $a_N(x) \neq 0$. The assertion holds if $f \equiv 0$. Thus assume $f \not\equiv 0$. Differentiate once w.r.t. y and obtain $\sum_{n=1}^{N} a_n(x)\, n\, f^{n-1}\, f_y = 0$. Then either $f_y \equiv 0$ (and the assertion is proven) or we divide by f_y (in a neighborhood where it is non-zero) and proceed similarly. At some point we end up with $a_N = 0$, which contradicts the hypothesis that the polynomial equation is of degree N. Thus, we need $f_y = 0$ and f is a function of x only. □

To simplify notation, we use the following convention to refer to the components of Equation (4.3a), cf. the terminology in Chapter 2.

$$\{T, I^{(3)}\} = \sum_{i=0}^{4} P^{(i)}\, p_1^{4-i}\, p_2^{i}. \tag{4.17}$$

Thus, by $P^{(0)}$ we denote the p_1^4 component of (4.3a), by $P^{(1)}$ its $p_1^3 p_2$ component etc.

Lemma 9. *Let $U_x = U_x(x)$ be a function of x only. Assume that the Weyl metric admits an additional third-degree integral. Then $U_y = 0$.*

Proof. The proof has two parts: (1) Show that U_y has to be constant, (2) Show that the constant is zero.
For the first part, consider the $p_1^3 p_2$ component $P^{(1)}$ of (4.3a). Use the Ernst equation (3.7) to replace derivatives U_{yy}. In this way, obtain the equation

$$\begin{aligned} 5x^3\, U_y^4 + 78x^2\, U_x\, (1 + x\, U_x)\, U_y^2 + (18x^2\, U_x + 9x)\, U_{xx} \\ - 63x^3\, U_x^4 - 63x\, U_x^2 - 9\, U_x - 126x^2\, U_x^3 = 0. \end{aligned} \tag{4.18}$$

This is a polynomial equation of degree 4 for U_y, and all coefficients are functions of x only. By Lemma 8, this means $U_{yy} = 0$, so $U_y = \text{const} =: c$.

For the second part of the proof, we insert this result into the p_1^4 component $P^{(0)}$ of (4.3a). If we substitute U_{xx} with the help of the Ernst equation (3.7), we find

$$U_x\, (1 + x\, U_x)\, (1 + 2x\, U_x) = 0.$$

Hence, there are 3 cases: $U_x = 0$, $U_x = -\frac{1}{x}$ and $U_x = -\frac{1}{2x}$. We treat them separately:

- If $U_x = 0$, use again the $p_1^3 p_2$ component $P^{(1)}$, which reads

  $$5x^3\, c^4 = 0,$$

  so $c = 0$.

- For $U_x = -\frac{1}{x}$ we have the same equation, so again $c = 0$.

- In case $U_x = -\frac{1}{2x}$, the $p_1^3 p_2$ component $P^{(1)}$ reads

$$\frac{9 - 312x^2\,c^2 + 80x^4\,c^4}{x} = 0.$$

Therefore, $xc = \text{const}$ and hence $c = 0$.

$\square$

Remark 2. *There are several possible choices of the parametrizing functions U and γ that lead to flat metrics when using Weyl canonical coordinates ($R = x$). Obviously, $U \equiv 0$ and $\gamma \equiv 0$ is a choice that represents a flat metric (in cylindrical polar coordinates). Likewise, the choice $U = -\ln(x)$, $\gamma = \ln(x)$ leads to a flat metric [GP09]. However, a flat metric can even look more complicated. Gautreau and Hoffman [GH69] show that the choice*

$$U = -\frac{1}{2}\ln\left(\sqrt{x^2+y^2}+y\right) \qquad \text{and} \qquad \gamma = -\frac{1}{2}\ln\left(\frac{\sqrt{x^2+y^2}+y}{\sqrt{x^2+y^2}}\right), \tag{4.19}$$

also gives rise to a flat metric, cf. also [GP09].

Lemma 10. *Let* $\operatorname{rk}\mathcal{M} = 2$ *and* $A = 0$ *($\nabla V^{\phi\phi}, \nabla V^{tt} \neq 0$). Assume $\alpha \neq 0$. Then there is no non-trivial Killing tensor of valence 3.*

Proof. We assume there is such a Killing tensor. Then, by the necessary criterion (see Lemma 7), $B_y - \tilde{B}_x = 0$. In addition, consider (4.3a) and the vacuum conditions (3.11). Assuming no additional linear integral and Weyl canonical coordinates, we have $U_y \neq 0$.

Consider (4.3a) in combination with the necessary criterion from Lemma 7, plus the vacuum conditions. The vacuum conditions (3.11) are basically invoked in order to substitute U_{yy}, as well as γ_x and γ_y, which reduces the number of derivatives of U involved in the equations (and eliminates γ). We take derivatives w.r.t. x and y of (4.3a). Then, we have 18 equations (those obtained from (4.3a), plus the necessary criterion $B_y - \tilde{B}_x = 0$ from Lemma 7). Using the vacuum conditions (3.11), we have only the following unknowns left:

$$U_{xxxx},\ U_{xxxy},\ U_{xxx},\ U_{xxy},\ U_{xx},\ U_{xy},\ U_x,\ U_y,\ U.$$

Use the x-derivative of the $p_1^3 p_2$ component, $\frac{\partial}{\partial x}P^{(1)}$, to substitute U_{xxxy}, and the x-derivative of the $p_1^2 p_2^2$ component, $\frac{\partial}{\partial x}P^{(2)}$, to substitute U_{xxxx} in terms of lower order derivatives. The third-order derivative U_{xxy} can be substituted via the x-derivative of the integrability criterion, but only if

$$(1 + 2x\,U_x)\,(x\,U_x^2 - 3x\,U_y^2 + U_x) \neq 0. \tag{4.20}$$

In this case, we can proceed as follows: Substitute U_{xxx} by the help of the x-derivative of the p_1^4 component $P^{(0)}$, and use this component to substitute U_{xx}. Finally, substitute U_{xy} using the integrability condition.

With all these substitutions at hand, we are only left with equations in the unknowns U_x and U_y. For instance, the derivative w.r.t. y of the p_1^4 component of (4.3a), i.e. $\frac{\partial}{\partial y}P^{(0)}$, reads

$$x\,U_x^2\,(1 + 2x\,U_x)\,(1 + x\,U_x)^2\,(x\,U_x^2 + U_x + x\,U_y^2)^3 = 0.$$

Therefore, either $U_x = 0$ or $U_x = -\frac{1}{x}$ or $U_x = -\frac{1}{2x}$, or $x\,U_x^2 + U_x + x\,U_y^2 = 0$. The three cases that we mentioned first are covered by Lemma 9, and obviously they are incompatible with the hypothesis $U_y \neq 0$. Thus, we are left with the fourth case. We can use the equation to express U_y^2 in terms of U_x,

$$U_y^2 = -\frac{1}{x}\, U_x\, (1 + x\, U_x). \tag{4.21}$$

Substitute this into the integrability condition and obtain an expression for U_{xy} in terms of U_y,

$$\frac{\partial}{\partial x} U_y = -4x\, U_y^3, \tag{4.22}$$

which is a differential equation for U_y and can be solved in a straightforward way. The solution is given by

$$U_y = \pm \frac{1}{2\sqrt{x^2 - f_1(y)^2}}. \tag{4.23}$$

Use this to replace U_y^2 in (4.21),

$$f_1(y)^2 = -\frac{x(1 + 4x\, U_x + 4x^2\, U_x^2)}{4U_x\, (1 + x\, U_x)},$$

and solve this for U_x. There are two branches of possible solutions:

$$U_x = -\frac{1}{2x}\left(1 \pm \frac{f_1(y)}{\sqrt{x^2 + f_1^2}}\right).$$

We can use the integrability criterion to find an explicit form for f_1. First, obtain two differential equations:

$$(f_1)_y \pm 1 = 0$$

Without loss of generality, we therefore obtain

$$f_1 = \pm y.$$

Using Equation (4.21) for U_y^2, in combination with the Ernst equation (3.7) and with Equation (4.23), one finds, after integration,

$$U = \frac{1}{2}\, \ln(\sqrt{x^2 + y^2} \pm y) - \ln(x) + c_2 \tag{4.24}$$

with an additional integration constant $c_2 \in \mathbb{R}$. In addition, the solution (4.19) from Remark 2 is recovered. The corresponding formula for γ can be computed using the secondary vacuum conditions (3.11c) and (3.11d).

It is straightforward to check that the Riemann curvature of the metric obtained from Formula (4.24) is zero. Thus, the metric is flat, cf. the discussion in Remark 2. As a result, all Killing tensors are reducible.

To complete the proof, we still have to take into account the case where Equation (4.20) is not satisfied. Here, either $U_x = -\frac{1}{x}$ (this is covered by Lemma 9) or

$$U_x\, (1 + x\, U_x) - 3x\, U_y^2 = 0.$$

We solve for U_y^2 and obtain

$$U_y^2 = \frac{U_x\, (1 + x\, U_x)}{3x}.$$

From the p_1^4 component $P^{(0)}$ and the integrability criterion we can also obtain another expression for U_y^2:

$$U_y^2 = \frac{3\,U_x\,(1 + x\,U_x)}{x}.$$

The only way to allow both solutions to be true is if $U_x = 0$ or $U_x = -\frac{1}{x}$. Both cases are covered by Lemma 9. □

We have considered odd-parity third-degree integrals for Weyl metrics. If M is flat on a neighborhood, then it is totally reducible there [Tho86]. Thus, assume M to be non-flat. Briefly summarizing, we find the following lemma.

Lemma 11. *Let M be non-flat with $A = 0$. Let F be a third-degree involutive integral of odd parity on M. Then F is reducible by at least one degree.*

Proof. First, let us consider the case of an additional Killing vector field. As we have seen in Lemma 5, this implies that the odd-parity third-degree integral is reducible by the (linear) integral p_y. Hence, the assertion is proven in this case. Second, if there is no additional Killing vector field, Proposition 10 tells us (provided $\alpha \neq 0$) that there is no odd-parity third-degree integral. In the case $\alpha = 0$, we have $F = 0$. Thus, the assertion is proven. □

To conclude the proof of Theorem 4, we finally have to take into account the quadratic part of the integral (3.14). We already know from Section 2.3.2 that this relates to even-parity quadratic integrals. More specifically, the quadratic contributions $F_\phi^{(2)}$ and $F_t^{(2)}$ must obey equations

$$\{T, F_k^{(2)}\} = 0 \qquad \text{and} \qquad \{T, F_{abk}^{(0)}\} + \{V^{ab}, F_k^{(2)}\} = 0, \tag{4.25}$$

where $a, b, k \in \{\phi, t\}$. These equations are precisely the equations for quadratic integrals with leading terms $F_\phi^{(2)}$ and $F_t^{(2)}$, respectively. Previously, we have denoted the corresponding systems of PDEs by $\mathcal{S}^{\mathsf{left}}$ and $\mathcal{S}^{\mathsf{right}}$, see equations (4.15). Thus the polynomials

$$p_a\,(F_a^{(2)} + F_{a\phi\phi}^{(0)}\,p_\phi^2 + F_{at\phi}^{(0)}\,p_t\,p_\phi + F_{att}^{(0)}\,p_t^2), \tag{4.26}$$

with $a \in \{\phi, t\}$, are products of a linear and a quadratic integral, and therefore the even-parity contributions to the cubic integral F are reducible by p_ϕ and p_t, respectively. Hence, also the entire integral F combining the parts of odd and even parity is reducible by one degree.

4.4 Zipoy-Voorhees metrics

Theorem 4 proves reducibility (by one degree) for involutive cubic integrals for arbitrary Weyl metrics. Using the same methods, we can also prove complete reducibility for arbitrary Zipoy-Voorhees metrics.

Corollary 1. *Let M_{ZV} be a Zipoy-Voorhees metric and let I be an integral of the form (3.14) of third degree on M_{ZV}. Then I is totally reducible, i.e. is generated by linear integrals (i.e. Killing vector fields) and the Hamiltonian (i.e. the metric).*

For a general coordinate representation of the family of Zipoy-Voorhees metrics see (3.17) on page 49. The statement of Corollary 1 follows from Theorem 4 in combination with Theorem 1. We have proven Theorem 1 in Section 3.2 based on Algorithm II. In addition, we sketched another possible proof with degree-wise reasoning. We now develop this latter proof in more detail.

Proof of the corollary. Recall the form of Zipoy-Voorhees metrics from (3.17) on page 49:

$$g = \left(\tfrac{x+1}{x-1}\right)^{\delta} \left((x^2-y^2)\left(\tfrac{x^2-1}{x^2-y^2}\right)^{\delta^2} \left(\tfrac{dx^2}{x^2-1}+\tfrac{dy^2}{1-y^2}\right) + (x^2-1)(1-y^2)\, dz^2 \right) - \left(\tfrac{x-1}{x+1}\right)^{\delta} dt^2. \tag{4.27}$$

For the case $\delta = 1$, also refer to Section 2.3.4. Third-degree odd-parity integrals are discussed in Example 1 on page 66. Let us now consider even-parity components, assuming that $\delta \neq 0$ and $\delta \neq \pm 1$. We represent the Hamiltonian H in the form

$$H = \Omega_1\, p_x^2 + \Omega_2\, p_y^2 + V_\phi\, p_\phi^2 + V_t\, p_t^2, \tag{4.28}$$

and denote the integral by

$$F = a_0\, p_x^2 + a_1\, p_x\, p_y + a_2\, p_y^2 + b_0\, p_\phi^2 + b_1\, p_\phi\, p_t + b_2\, p_t^2.$$

From each polynomial of degree 1 we obtain integrability conditions for b_0 and b_2, after split w.r.t. (p_x, p_y) similar to Remark 1. The coefficient b_1 belongs to the subsystem $\mathcal{S}^{\mathsf{lower}}$ and is therefore constant and not of interest.

For quadratic integrals in 2-dimensional spaces, Equation (1.11) on page 17 is the integrability condition for the zeroth-order component of the integral. Similarly, we have two Bertrand-Darboux integrability conditions here, one for b_0 and one for b_2.

Combining these Bertrand-Darboux integrability conditions with the equations obtained from the degree-3 polynomial after splitting w.r.t. (p_x, p_y), we have a system of six (independent) linear equations for six unknowns (the derivatives of a_0, a_1 and a_2). The system can be solved for all these unknowns. Subsequently, by computing the expressions $(a_i)_{xy} - (a_i)_{yx}$ for $i = 0, 1, 2$, we obtain three linear equations for a_0, a_1 and a_2. These three equations are not independent, but are equivalent to a system of two equations. One of the resulting equations amounts to $a_1 = 0$, if $\delta^2 - 1 \neq 0$. The second equation relates a_0 and a_2 to one another (again assuming $\delta^2 - 1 \neq 0$),

$$(y^2-1)a_2 + (x^2-1)a_0 = 0.$$

From the Bertrand-Darboux equations for b_0, b_2, we can now deduce the differential $d(a_0)$ in terms of a_0 and solve the corresponding system of differential equations, obtaining

$$a_0 = c_1\, (y^2-x^2)^{1-\delta^2} (x+1)^{\delta^2+\delta-1} (x-1)^{\delta^2-\delta-1}.$$

Then, we can immediately compute a_2:

$$a_2 = -c_1 \left(\frac{x^2-1}{x^2-y^2}\right)^{\delta^2} \left(\frac{x+1}{x-1}\right)^{\delta} \frac{x^2-y^2}{y^2-1}.$$

Finally, from the equations obtained from the degree-1 polynomial after splitting w.r.t. the momenta p_x, p_y, we obtain the derivatives of b_0, b_2, and by integration

$$b_0 = -c_1\, (y^2-1)\, (x^2-1) \left(\frac{x+1}{x-1}\right)^{\delta} + c_2,$$

$$b_2 = -c_1 \left(\frac{x-1}{x+1}\right)^{\delta} + c_3.$$

Comparing this result to the Hamiltonian shows that

$$F = c_1\, H + c_2\, p_\phi^2 + c_3\, p_\phi\, p_t + c_4\, p_t^2.$$

This means that every involutive quadratic integral is (H, p_ϕ, p_t)-reducible provided $\delta \notin \{0, \pm 1\}$. For $\delta = \pm 1$ there is an additional involutive integral that is reducible by linear integrals, including non-involutive ones. Together with Theorem 4, this proves the assertion.

□

Chapter 5

Sub-Riemannian Structures

In the previous chapters, we discuss astrophysical examples from the class of stationary and axially symmetric vacuum metrics. We are now going to deal with sub-Riemannian structures, and we address these with the algorithm briefly outlined in Section 2.3.3.

We consider left-invariant sub-Riemannian structures on rank-2 distributions in *Carnot groups*, and study Liouville integrability for such structures. The results and the algorithm presented in this chapter are part of a joint paper with Boris Kruglikov and Georgios Lukes-Gerakopoulos [KVL15]. The exposition follows the deliberations in the paper.

Carnot groups have been described as the sub-Riemannian analog of what Euclidean spaces are within Riemannian geometry (the term *flat* is used, for instance, in [Sac04]). Carnot groups are, in typical points, an infinitesimal model (nilpotent approximation) for generic sub-Riemannian structures [Bel97], which is compared to the fact that Riemannian geometries are infinitesimally Euclidean in [MSS97]. Starting from this analogy, Montgomery, Shapiro & Stolin discuss Liouville integrability for sub-Riemannian structures on Carnot groups in [MSS97], asking whether Carnot groups in general are always integrable. They observe that this is not the case and illustrate it with a precise example [MSS97]. Other examples are discussed in [Kru02].

We observe a similarly unexpected effect. Intuitively, one expects that symmetry is related to the existence of integrals. For instance, the Noether theorem links symmetries of the Hamiltonian to linear integrals. In Chapter 1, we also mention that higher-degree integrals are often referred to as *hidden symmetries* (cf. also the Runge-Lenz vector in the Kepler problem). However, we are going to see examples of sub-Riemannian structures on Carnot groups where a high degree of symmetry does not guarantee the existence of additional integrals of low-degree, even if similar structures with a lower-dimensional symmetry algebra are integrable with integrals of low degree. This effect is observed both with regard to the symmetry algebra of the sub-Riemannian structure and that of its underlying distribution (see page 76 for the definition).

5.1 Introduction

A *Carnot group* G is a simply connected, finite-dimensional Lie group whose Lie algebra is nilpotent and decomposes into a direct sum $\bigoplus L_i$ of vector spaces L_i such that $[L_i, L_j] \subset L_{i+j}$, $[L_i, L_1] = L_{i+1}$, and $L_s = 0$ for $s > r \in \mathbb{N}$.

Furthermore, we require that $\dim L_1 = 2$ and that there is a sub-Riemannian metric $g \in \Gamma(S^2_+ L_1^*)$. A *sub-Riemannian structure* on a connected differentiable manifold M is a bracket-generating (also called completely non-holonomic) vector distribution $\Delta = L_1 \subset TM$, together with a Riemannian metric $g \in \Gamma(S^2_+\Delta^*)$ on the dual of the distribution. The *bracket-generating* condition is the requirement that in any local frame for Δ, TM is spanned by the frame and all of its iterated Lie brackets [Mon06].

There is a standard construction that endows G with a *sub-Riemannian distance*, i.e. a mapping that assigns a distance to two points $x, y \in G$. This standard construction is the *Carnot-Carathéodory metric*

$$d_g(x, y) = \inf_{\gamma \in \mathcal{H}(x,y)} \int_0^1 \|\dot{\gamma}\|_g \, dt.$$

Here, $\|\cdot\|_g$ denotes the norm induced by g. $\mathcal{H}(x, y)$ is the space of integral (horizontal) curves $\gamma : [0, 1] \to G$, $\dot{\gamma} \in \Delta$, that join x and y, i.e. $\gamma(0) = x$, $\gamma(1) = y$. Obviously, we require that $\mathcal{H}(x, y)$ is non-empty for any two points $x, y \in G$. This is assured by the Chow-Rashevskii theorem.

Fact 4 (Chow-Rashevskii Theorem [Cho40; Mon06])**.** *If Δ is a bracket-generating distribution on a connected manifold M, then any two points of M can be joined by a horizontal path.*

Hamiltonian. For a given sub-Riemannian metric, we can define a Hamiltonian. However, the construction is somewhat more subtle than in the (pseudo-)Riemannian case, because the metric is only defined for horizontal vector fields.

Consider the completely non-holonomic distribution $\Delta \subset TG$ with the sub-Riemannian metric $g \in \Gamma(S^2_+\Delta^*)$. Via this sub-Riemannian metric, define the isomorphism

$$\sharp^g : \Delta^* \to \Delta.$$

Now, consider the cotangent bundle T^*G equipped with the standard symplectic structure, and the inclusion $i : \Delta \hookrightarrow TG$. In the sub-Riemannian case, we cannot find an isomorphism between TG and T^*G. However, we can still define a vector bundle morphism Ψ_g by the composition

$$T^*G \xrightarrow{i^*} \Delta^* \xrightarrow{\sharp^g} \Delta \xrightarrow{i} TG,$$

where i^* is the dual mapping of i. With this construction at hand, we define the Hamiltonian $H : T^*G \to \mathbb{R}$ by

$$H : T^*G \xrightarrow{i^*} \Delta^* \xrightarrow{\sharp_g} \Delta \xrightarrow{\frac{1}{2}\|\cdot\|_g^2} \mathbb{R}. \tag{5.1}$$

Note that this definition of the Hamiltonian uses an additional factor $\frac{1}{2}$ compared to our previous convention. We make this choice to be consistent with the terminology in [KVL15]. We also adopt the following notation from [KVL15]: A basis of the Lie algebra $\mathfrak{g}$ is denoted by (e_i). The corresponding mapping $\mathfrak{g}^* \to \mathbb{R}$ is denoted by (ω_i), with $\omega_i(p) = \langle e_i, p\rangle$. The same symbols e_i and ω_i are used, respectively, for their corresponding left-invariant vector fields on G or left-invariant linear functions $T^*G \to \mathbb{R}$. The right-invariant counterparts of the ω_i are denoted by θ_i.

Locally, in an orthonormal basis of vector fields $(\xi_i)_{i=1,\dots,k}$ ($\xi_i \in \mathcal{H}$), the Hamiltonian can be written as $H = \frac{1}{2}\sum_1^k \xi_i^2$. A formula in local coordinates (q, p) is obtained by realizing the basis (e_i) as left-invariant vector fields on G.

This can be done as follows. Through the *Baker-Campbell-Hausdorff formula*,

$$\begin{aligned}
\log(e^X e^Y) = X + Y &+ \frac{1}{2}[X, Y] \\
&+ \frac{1}{12}([X, [X, Y]] + [Y, [Y, X]]) \\
&- \frac{1}{24}[Y, [X, [X, Y]]] \\
&- \frac{1}{720}([Y, [Y, [Y, [Y, X]]]] + [X, [X, [X, [X, Y]]]]) \\
&+ \frac{1}{360}([X, [Y, [Y, [Y, X]]]] + [Y, [X, [X, [X, Y]]]]) \\
&+ \frac{1}{120}([Y, [X, [Y, [X, Y]]]] + [X, [Y, [X, [Y, X]]]]) \\
&+ \dots,
\end{aligned} \tag{5.2}$$

one obtains the group law for G. The series expansion of the Baker-Campbell-Hausdorff formula terminates in our cases because of the nilpotency of the algebra of G. Then, consider a left-invariant vector field X with local representation $X_g = \sum_i X_g^i \partial_i$, where $\partial_i = \frac{\partial}{\partial x^i}$. A left-invariant basis can be constructed by computing components of X via

$$X_g^i = X_g(x^i) = (L_g)_* X_e(x^i) = X_e(x^i \circ L_g) \tag{5.3}$$

and collecting coefficients w.r.t. the $X_e^i = X_e(x^i)$, e.g. [CCG07].

Pontrjagin minimum principle. A (horizontal) curve $\gamma \in \mathcal{H}$ is called a *geodesic* if it minimizes the distance d_g for neighboring points that are suitably close. Such extremals of the sub-Riemannian distance functional are given by the Pontrjagin minimum (or maximum) principle, see [Pon87] or any book on optimal control theory.

Typically, geodesics are described by the Euler-Lagrange variational equation, which is a special case of the Pontrjagin principle. Such geodesics are called *normal* geodesics. In sub-Riemannian geometry not all geodesics are normal. There are also abnormal geodesics, and both cases do not exclude each other. Geodesics that are abnormal and not normal are called strictly abnormal, see e.g. [Mon14].

In our context, abnormal geodesics play no role. Thus, we are not considering them further and we study only geodesics that are described by the Euler-Lagrange equation, making the procedure analogous to the previous computations for pseudo-Riemannian metrics.

Integrability. As in the standard theory, the metric g is said to be (Liouville) *integrable* if the geodesic flow is Liouville integrable on T^*G, i.e. if there are functionally independent[1] integrals $I_1 = H, I_2, \dots, I_D$ that Poisson commute: $\{I_i, I_j\} = 0$, $\forall\, i, j \in \{1, \dots, D\}$.

Another, and maybe more common, kind of integrability on distributions is Frobenius integrability, i.e. the existence of integral manifolds whose tangent spaces are spanned by the distribution. This is not the kind of integrability addressed here and the bracket-generating distributions cannot be integrable in the Frobenius sense, since Frobenius integrability would require that $[L_1, L_1] \subset L_1$.

Usually, we are concerned with geodesic invariants. Now, for integrals analytic in the momenta and a Hamiltonian that is a homogeneous (quadratic) polynomial in the momenta,

[1] This means that the differentials are almost everywhere linearly independent.

the existence of such an analytic integral implies the existence of an integral that is a homogeneous polynomial in momenta (classical, e.g. tome III of [Dar87]). Since we are looking for only one additional involutive integral, we can thus always assume that this integral is a homogeneous polynomial in the momenta.

We are interested here in the integrability of left-invariant sub-Riemannian structures on Carnot groups, such that the sub-Riemannian structures are associated with rank-2 distributions. Since we consider bracket-generating systems, the smallest rank possible is 2. We start from dimension 6, because in our context lower-dimensional cases always have integrable geodesic flows. For a more thorough treatment of these low-dimensional cases, refer to [KVL15].

Symmetry dimensions. We consider both the symmetry dimensions for the underlying distribution and the sub-Riemannian structure defined on it. A vector field on G is a symmetry for the distribution $\Delta = L_1$ if its flow preserves Δ. It is a symmetry for the sub-Riemannian structure if its flow not only preserves Δ, but also the sub-Riemannian metric g. The dimension of the symmetry algebra for the underlying distribution $\Delta = L_1$ is denoted by $\dim \mathrm{Sym}(\Delta)$, while we denote the dimension of the symmetry algebra for the sub-Riemannian structure by $\dim \mathrm{Sym}(\Delta, \mathrm{g})$.

The dimension of the symmetry algebra $\mathrm{Sym}(\Delta)$ can be obtained as in [AK11], cf. Theorem 4 therein and also Theorem 8.4 in [Tan70]. Some of the results can also be found in [DZ09; AK11]. The symmetry dimensions are obtained by computing the *Tanaka prolongation*[2] of the graded nilpotent Lie algebras for the Carnot groups we consider. In Tanaka theory, it is common to use negative indices for the grading components, so the algebras decompose as

$$\mathfrak{g} = \mathfrak{g}_{-n} \oplus \cdots \oplus \mathfrak{g}_{-1}. \tag{5.4}$$

The Tanaka prolongation

$$\mathrm{pr}(\mathfrak{g}, \mathfrak{g}_{-1}) = \mathfrak{g}_{-n} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_M$$

is obtained by computing, iteratively for $0 \leq m \leq M$, the spaces of homomorphisms $\psi_m : \mathfrak{g}_{-k} \to \mathfrak{g}_{m-k}$ $(k = 1, \ldots, n)$ that act as derivations, i.e. that obey the Leibniz rule $\psi([e_1, e_2]) = [\psi(e_1), e_2] + [e_1, \psi(e_2)]$. Thus, the first prolongation $\mathfrak{g}_0$ is the algebra of grading-preserving derivations of $\mathfrak{g}$. The dimension of the symmetry algebra $\mathrm{Sym}(\Delta, \mathrm{g})$ is then obtained by identifying symmetries that also preserve g. In our cases, the dimension of the symmetry algebra $\mathrm{Sym}(\mathrm{L}_1, \mathrm{g})$ is bounded from above by $\dim \mathrm{Sym}(\mathrm{L}_1, \mathrm{g}) \leq \dim \mathfrak{g} + 1$. This statement is proven in [KVL15]. A more general theorem can be found in [Mor08].

In our cases, the Tanaka prolongation is trivial after at most four prolongations and thus $\mathfrak{g}_2$ is the last component that may be non-trivial. Therefore, the computations can still be done by hand. There also exist computer-algebra implementations to compute the Tanaka prolongation.

The examples. We start from dimension 6. This is the first interesting dimension, because in lower dimensions all left-invariant sub-Riemannian structures on Carnot groups are integrable with integrals of low degree, see [KVL15].

[2] The author wishes to thank Boris Kruglikov for discussions on symmetry algebras and Tanaka theory.

We consider the following sub-Riemannian structures in Carnot groups of dimension 6, 7 and 8:

$6\mathrm{D}_\mathrm{p}$ The parabolic sub-Riemannian structure with growth vector $(2,3,5,6)$. The underlying distribution has an 11-dimensional symmetry algebra, see [DZ09; AK11]. Moreover, $\dim \mathrm{Sym}(\mathrm{L}_1, \mathrm{g}) = 6$.

$6\mathrm{D}_\mathrm{h}$ A hyperbolic sub-Riemannian structure with growth vector $(2,3,5,6)$ and symmetry dimension $\dim \mathrm{Sym}(\mathrm{L}_1, \mathrm{g}) = 6$. Its underlying distribution has an 8-dimensional symmetry algebra [DZ09; AK11].

$6\mathrm{D}_\mathrm{e}$ The elliptic sub-Riemannian structure with growth vector $(2,3,5,6)$ and 7-dimensional symmetry algebra $\mathrm{Sym}(\mathrm{L}_1, \mathrm{g})$. Its underlying distribution also has 8-dimensional symmetry algebra.

7D The sub-Riemannian structure with growth vector $(2,3,5,7)$. It has $\dim \mathrm{Sym}(\mathrm{L}_1) = 9$. Its symmetry algebra $\mathrm{Sym}(\mathrm{L}_1, \mathrm{g})$ has maximal dimension, $\dim \mathrm{Sym}(\mathrm{L}_1, \mathrm{g}) = 8$.

$8\mathrm{D}_1$ An example of a sub-Riemannian structure on a truncated free graded nilpotent Lie algebra with 2 generators and growth vector $(2,3,5,8)$. The structure has a 9-dimensional symmetry algebra, $\dim \mathrm{Sym}(\mathrm{L}_1, \mathrm{g}) = 9$. The underlying distribution has symmetry dimension 12 [KVL15].

$8\mathrm{D}_2$ A sub-Riemannian structure with growth vector $(2,3,5,6,8)$ and $\dim \mathrm{Sym}(\mathrm{L}_1, \mathrm{g}) = 9$. It has a 10-dimensional symmetry algebra $\mathrm{Sym}(\mathrm{L}_1)$.

We also refer to the cases via their growth vector, if there is no risk of confusion.

5.2 Method

We prove nonexistence of a *final* integral of degree up to 5 or 6, for some of the examples given in the list on this page. In these examples, there is only one integral missing for Liouville integrability, but this final integral does not exist. Specifically, in the examples $6\mathrm{D}_\mathrm{p}$, $6\mathrm{D}_\mathrm{h}$, 7D and $8\mathrm{D}_1$, we have $D-1$ involutive integrals (D is the dimension of the Carnot group), but there is no final integral of low degree. On the other hand, we prove Liouville integrability for the cases $6\mathrm{D}_\mathrm{e}$ and $8\mathrm{D}_2$ with integrals of at most quadratic degree in the momenta.

As we already pointed out in the introduction, low-degree (Liouville) integrability is an interesting question of its own, see, for instance, the wide literature on second-order integrability (existence of second-degree integrals). On the other hand, virtually all known integrable Hamiltonian systems have polynomial integrals at most of degree 4. Although low-degree non-integrability does not exclude the possibility of higher-degree integrability, if we take low-degree non-integrability as a hint, we might ask ourselves whether these systems show other forms of non-integrability. Indeed, we conduct numerical analysis in [KVL15] and find that the above examples have Poincaré sections with chaotic behavior, cf. the figures in [KVL15]. This corroborates our finding that a high level of symmetry does not imply Liouville integrability of sub-Riemannian structures on Carnot groups. For the 8D example, numerical analysis has been performed in [Sac14] and our results concur with the observations made there.

Algorithm. We use the algorithm outlined in Section 2.3.3. This algorithm is different from Algorithms I and II regarding the elimination scheme that is used to reduce the associated linear system (2.3) of page 21. The reason why we should follow a different strategy for the sub-Riemannian structures is that the Hamiltonian, after symplectic reduction, is not of even parity w.r.t. non-ignorable momenta. It has a linear term in addition (cf. Chapters 2 and 3). Specifically, we obtain a Hamiltonian $H = T_c(q_1, q_2, p_1, p_2) + V_c(q_1, q_2)$ after symplectic reduction, with a kinetic term T and a potential V (the subscript c denotes dependence on $p_3 = c_3, \ldots, p_D = c_D$). Yet for the sub-Riemannian examples we consider here, the reduced Hamiltonian contains a term linear in (p_1, p_2):

$$H_c(q_1, q_2, p_1, p_2) = T^{ij}(q_1, q_2)\, p_i p_j + L^i(q_1, q_2)\, p_i + V(q_1, q_2), \tag{5.5}$$

where for the indices i, j we use the summation convention over $i, j \in \{1, 2\}$. The existence of the non-vanishing term L entails complications. We do not have the parity decomposition of Section 2.2.2 any longer. Moreover, since we work in higher dimension, the number of involved equations is (significantly) higher than those with the stationary and axially symmetric examples of Chapter 3.

However, we can increase the efficiency compared to the standard algorithm of Section 2.1 (page 20ff). On one hand, we choose a 'good' point of reference for the computations. Due to this choice, the equations become very simple and contain only very few terms. For this reason, the matrix of the associated system is rather sparse, i.e. contains many zero entries. This allows us to consider a special elimination scheme as follows.

First, the Hamiltonian H evaluated in a point and scaled by some integer factor, is a polynomial with integer coefficients. As discussed in Section 2.3, we therefore do not need to handle rational expressions, which already improves the speed of the rank computation. Then, because of the sparsity of the matrix, we can reduce the numbers of equations and unknowns through solving the simplest equations immediately.

We briefly outline the adapted algorithm. Let $D = \dim G$ and let d denote the degree of the integral, i.e.

$$F = \sum_{|\tau|=d} a_\tau(x_1, x_2)\, p^\tau, \tag{5.6}$$

where τ is a multiindex $\tau = (\tau_1, \ldots, \tau_D)$. Using the multiindex notation, we have $p^\tau = \prod_{i=1}^{D} p_i^{\tau_i}$, and $|\tau| = \sum_{i=1}^{D} \tau_i$, cf. Chapters 1 and 2. Recall that the Poisson bracket $\{H, F\}$ is a homogeneous polynomial in momenta of degree $d+1$, and that the coefficients of $\{H, F\}$ yield a system of PDEs. It is a system of $\binom{d+D}{D-1}$ first order partial differential equations on $\binom{d+D-1}{D-1}$ unknown functions (these are the coefficients a_τ). As discussed in Section 1.2, the system is of finite type (analogously to [Wol98]). This suggests that we should typically need $d+1$ prolongation steps to arrive at a conclusion if the considered structure is non-integrable, and this is the number of computations that we actually need. After performing k steps of prolongation, the total number of equations is

$$m_{d,k} = \binom{d+D}{D-1} \cdot \binom{k+2}{2},$$

and after all $d+1$ steps, we have

$$m_{d,d+1} = \binom{d+D}{D-1} \cdot \binom{d+3}{2}$$

equations in the associated linear system (2.3).

As in Chapter 2, Λ_d denotes the number of linearly independent first integrals of degree d. We have, after k prolongation steps,

$$\Lambda_d \leq \overline{\Lambda}_d^{(k)} := n_{d,k} - \mathrm{rk}(M_d^{(k)}) \tag{5.7}$$

The value $\overline{\Lambda}_d^{(k)}$ is the bound found from the associated matrix equation that corresponds to the system of PDEs obtained after k prolongations. Nonexistence of an additional integral of degree d requires, for a certain $k \in \mathbb{N}$, both equalities in the following relation,

$$\overline{\Lambda}_d^{(k)} \geq \Lambda_d \geq \Lambda_d^0 := \sum_{i=0}^{[d/2]} \binom{d-2i+D-3}{D-3}, \tag{5.8}$$

and we achieve this for our examples after $k = d+1$ steps of prolongation.

Algorithm III. *The algorithm that we use for sub-Riemannian structures on the Carnot groups is as follows:*

(1) Compute the associated linear system, and evaluate it at $(x_1, x_2) = (0, 0)$. Rewrite the equations such that all coefficients become integers. Then remove all redundant equations.

(2) Remove unknowns that can be set to zero in a point by adding suitable multiples of known integrals. Denote the set of respective unknowns by V_{triv}.

(3) Perform a partial solution of the system, iteratively using monomial and bi-monomial equations. Each iteration consists of two steps. First, solve all monomial equations. Second, analogously solve the bi-monomial equations. Repeat. During each iteration, remove duplicate equations. The iteration procedure stops when there are no more monomial or bi-monomial equations left.

(4) Denote the matrix obtained through this reduction by M_{red}, and define the sets V_{mon} and V_{bimon} of unknowns that were removed during the first or second step of the iteration, respectively. Furthermore, let $V_{red} = V \setminus (V_{triv} \cup V_{mon} \cup V_{bimon})$ denote the set of unknowns in the reduced matrix system. The upper bound to the number of integrals can be computed by the formula

$$\overline{\Lambda} = \#V - \mathrm{rk}(M) = \#V_{red} + \#V_{triv} - \mathrm{rk}(M_{red}). \tag{5.9}$$

If the matrix M_{red} has full rank, then no additional integrals exist.

For the examples of this chapter, we do not have standard integrals like in the SAV case. However, simple integrals can be computed as follows. First, we obtain linear integrals from right-invariant fields for the given (left-invariant) Hamiltonian. Next, integrals that are low-degree homogeneous polynomials in the left-invariant basis can be found directly. In addition, Casimir functions can be computed for the system. They can be obtained as solutions for the set of differential equations (see e.g. [CS04])

$$C_{ij}^k \, X_k \, \frac{\partial}{\partial X_j} F = 0 \qquad \forall \, i = 1, \ldots, n, \tag{5.10}$$

where C_{ij}^k are the structure constants of the Lie algebra in the basis $(X_1, \ldots, X_n)$ (the usual summation convention applies). The maximal number of Casimirs is given by the codimension of a generic orbit of the coadjoint action. Once the solutions are obtained, we still have to check whether the Casimir functions are functionally independent of other integrals of the respective structures.

5.3 Results

We compare the examples given on page 77 concerning the existence of low-degree integrals and the dimension of their symmetry algebras. As a result of this comparison, we find the following:

— a higher degree of symmetry does not imply the existence of additional integrals of low degree in momenta

— while a sub-Riemannian structure with a maximal degree of symmetry may lack a final integral, similar structures with a lower degree of symmetry can still be Liouville integrable with integrals of low degree

Specifically, we prove

6D There is no final integral of degree at most 6 for the left-invariant parabolic sub-Riemannian structure with Hamiltonian (5.12) and growth vector $(2, 3, 5, 6)$ on the maximally symmetric (w.r.t. dim $\mathrm{Sym}(\mathrm{L}_1)$), non-holonomic rank-2 distribution that has $\dim \mathrm{Sym}(\mathrm{L}_1) = 11$. Similarly, there is no final integral of degree ≤ 5 for the hyperbolic structure with Hamiltonian (5.16). However, the corresponding elliptic structure with Hamiltonian (5.14) is Liouville integrable with integrals of at most second degree, in spite of its lower symmetry dimension $\dim \mathrm{Sym}(\mathrm{L}_1) = 8$.

7D The maximally symmetric (w.r.t. dim $\mathrm{Sym}(\mathrm{L}_1, \mathrm{g})$), left-invariant elliptic sub-Riemannian structure with Hamiltonian (5.18) and growth vector $(2, 3, 5, 7)$ has no final integral of degree ≤ 5.

8D The left-invariant sub-Riemannian structure ($\dim \mathrm{Sym}(\mathrm{L}_1) = 12$) with the Hamiltonian (5.19) and growth vector $(2, 3, 5, 8)$ has no final integral of degree ≤ 5. Yet, the sub-Riemannian structure ($\dim \mathrm{Sym}(\mathrm{L}_1) = 10$) with growth vector $(2, 3, 5, 6, 8)$ is Liouville integrable with integrals of at most quadratic degree.

The exact statements can be found on the following pages, see especially Theorems 5 to 7. In [KVL15], non-integrability of examples $6\mathrm{D}_\mathrm{p}$, $6\mathrm{D}_\mathrm{h}$, 7D and $8\mathrm{D}_1$ is corroborated by numerical analysis. It is shown that the respective Poincaré sections exhibit chaotic behavior.

5.3.1 Dimension 6

In 6D, we compare three cases with growth vector $(2, 3, 5, 6)$, but different structure equations for the Lie algebras.

The parabolic structure with growth vector $(2, 3, 5, 6)$**.** First, the *parabolic* distribution with growth vector $(2, 3, 5, 6)$ has maximal symmetry dimension $\dim \mathrm{Sym}(\mathrm{L}_1) = 11$ [AK11]. Its structure equations are

$$[e_1, e_2] = e_3, \ [e_1, e_3] = e_4, \ [e_2, e_3] = e_5, \ [e_1, e_4] = e_6. \tag{5.11}$$

One obtains a set of standard integrals as follows:

- Trivially, the (left-invariant) Hamiltonian $H = \frac{1}{2}(\omega_1^2 + \omega_2^2)$ itself is an integral. Its coordinate form can be obtained using the Baker-Campbell-Hausdorff formula as described on page 75. The result is, cf. [KVL15],

$$2H = \left(p_1 - \tfrac{1}{2}x_2 p_3 - x_1 x_2 p_4 - \tfrac{1}{2}x_1^2 x_2 p_6\right)^2 + \left(p_2 + \tfrac{1}{2}x_1 p_3 + x_1 x_2 p_5\right)^2. \tag{5.12}$$

- From the right-invariant integrals, one immediately identifies the linear integrals $\theta_3 = p_3$, $\theta_4 = p_4$, $\theta_5 = p_5$, $\theta_6 = p_6$, which form an involutive family of four integrals.
- There are two Casimir functions, $\omega_5 = \theta_5$ and $\omega_6 = \theta_6$, which we already found together with the other right-invariant linear integrals.

The five integrals $\mathcal{I}^{\mathrm{par}}_{(2,3,5,6)} := (H, p_3, \dots, p_6)$ form an involutive family of left- and right-invariant integrals.We are going to prove that there is no other (irreducible and involutive) integral of low degree (up to 6) in addition to $\mathcal{I}^{\mathrm{par}}_{(2,3,5,6)}$ that would turn the family into a Liouville-integrable family of six involutive integrals.

Theorem 5. *The family* $(H, \theta_3, \dots, \theta_6)$ *cannot be extended to a Liouville integrable family of integrals of degree* ≤ 6 *for the left-invariant sub-Riemannian structure* (5.12) *on the parabolic* $(2,3,5,6)$*-distribution, i.e. there exists no irreducible integral of degree* ≤ 6*, in addition to the identified 5 involutive integrals.*

The assertion is proven using Algorithm III by performing the algorithmic computations searching for a final integral of degree $d = 1, \dots, 6$. We denote by $\#\mathcal{S} = m_{d,d+1}$ the number of equations in the initial system of PDEs.

Sextic integral $(d = 6)$; $\Lambda^0_6 = 130$
seventh step of prolongation

$\#\mathcal{S}$	$\#V$	$\overline{\Lambda}^{(7)}_6$	computation time
28512	20790	130	ca. 27 hours

This proves that there are no additional irreducible and involutive integrals of the studied type. Note that additional integrals of lower degree would appear in the number $\overline{\Lambda}^{(7)}_6$ shown in the table, because we can construct integrals from products of lower-degree integrals. The full results of the computation (all lower degrees) can be found in Appendix A.1.1.

The elliptic structure with growth vector $(2,3,5,6)$**.** The elliptic[3] $(2,3,5,6)$-distribution has symmetry dimension $\dim \mathrm{Sym}(\mathrm{L}_1) = 8$ [AK11]. Its structure equations are

$$[e_1, e_2] = e_3,\ [e_1, e_3] = e_4,\ [e_2, e_3] = e_5,\ [e_1, e_4] = e_6,\ [e_2, e_5] = e_6, \tag{5.13}$$

and the Hamiltonian (in local coordinates) reads

$$2H = \left(p_1 - \tfrac{1}{2}x_2p_3 - x_1x_2p_4 - \tfrac{1}{2}x_1^2x_2p_6\right)^2 + \left(p_2 + \tfrac{1}{2}x_1p_3 + x_1x_2p_5 + \tfrac{1}{2}x_1x_2^2p_6\right)^2. \tag{5.14}$$

The involutive family of standard integrals is obtained as follows. We find

- two Casimir functions $I_6 = \omega_6$ and $C = \frac{1}{2}(\omega_4^2 + \omega_5^2) - \omega_3\omega_6$,
- the Hamiltonian $I_1 = H = \frac{1}{2}(\omega_1^2 + \omega_2^2)$,
- the right-invariant linear functions $I_3 = \theta_3$, $I_4 = \theta_4$, $I_5 = \theta_5$ and $I_6 = \omega_6$, and
- the quadratic integrals $I_2 = \omega_1\omega_5 - \omega_2\omega_4 + \frac{1}{2}\omega_3^2$ and $I_2' = \theta_1\theta_5 - \theta_2\theta_4 + \frac{1}{2}\theta_3^2$.

[3]The considered elliptic case has a lower-dimensional symmetry algebra for L_1 than the parabolic case. However, its symmetry algebra $\mathrm{Sym}(\mathrm{L}_1, \mathrm{g})$ has maximal possible dimension 7. We come back to this on the next page.

The integral C is not independent, but obviously satisfies $C = \frac{1}{2}(I_4^2 + I_5^2) - I_3 I_6$. However, the quadratic integral I_2 turns the family $\mathcal{I} = (I_1, \dots, I_6)$ into a family of six functionally independent and involutive integrals. Therefore, the elliptic $(2, 3, 5, 6)$-structure is Liouville integrable. Actually, it is superintegrable, taking into account the integrals I_2' and the linear integral K obtained via $I_2 - I_2' = I_6 \cdot K$ (its coordinate representation is $K = x_1 p_2 - x_2 p_1 + x_4 p_5 - x_5 p_4$). Neither I_2' nor K commute with $I_1, \dots, I_6$.

The hyperbolic structure with growth vector $(2, 3, 5, 6)$**.** For the hyperbolic distribution, the underlying distribution has the same symmetry dimension $\dim \mathrm{Sym}(\mathrm{L}_1) = 8$ as the elliptic case (but it has $\dim \mathrm{Sym}(\mathrm{L}_1, \mathrm{g}) = 6$). The structure equations read

$$[e_1, e_2] = e_3,\ [e_1, e_3] = e_4,\ [e_2, e_3] = e_5,\ [e_1, e_5] = e_6,\ [e_2, e_4] = e_6, \tag{5.15}$$

and the Hamiltonian is, in local coordinates,

$$2H = \left(p_1 - \tfrac{1}{2}x_2 p_3 - x_1 x_2 p_4 - \tfrac{1}{4}x_1 x_2^2 p_6\right)^2 + \left(p_2 + \tfrac{1}{2}x_1 p_3 + x_1 x_2 p_5 + \tfrac{1}{4}x_1^2 x_2 p_6\right)^2. \tag{5.16}$$

Again, there are two Casimir functions $I_6 = \omega_6$ and $C = \omega_4\omega_5 - \omega_3\omega_6$, and together with $I_1 = H = \frac{1}{2}(\omega_1^2 + \omega_2^2)$, $I_3 = \theta_3$, $I_4 = \theta_4$, and $I_5 = \theta_5$ $(\theta_6 = \omega_6)$ one obtains six involutive integrals. However, not all of them are functionally independent, since we have the integral $C = I_4 I_5 - I_3 I_6$. We do not find another left-invariant integral that is corresponding to the integral I_2 of the elliptic structure.

We thus apply Algorithm III for the hyperbolic $(2, 3, 5, 6)$-structure. Indeed, we find nonexistence of an additional integral up to degree 5.

Quintic integral $(d = 5)$; $\Lambda_d^0 = 80$
sixth step of prolongation

$\#\mathcal{S}$	$\#V$	$\overline{\Lambda}_5^{(6)}$	computation time
12936	9072	80	ca. 1.7 hours

5.3.2 Dimension 7

The sub-Riemannian structure that we study in dimension 7 has growth vector $(2, 3, 5, 7)$. Note that in 6D, we were considering the parabolic case $6\mathrm{D}_\mathrm{p}$ that had maximal symmetry dimension $\dim \mathrm{Sym}(\mathrm{L}_1) = 11$ for the underlying distribution. However, the $6\mathrm{D}_\mathrm{p}$ example has only $\dim \mathrm{Sym}(\mathrm{L}_1, \mathrm{g}) = 6$, while the maximally symmetric sub-Riemannian structure in 6D is the elliptic[4] $6\mathrm{D}_\mathrm{e}$ example (with $\dim \mathrm{Sym}(\mathrm{L}_1, \mathrm{g}) = 7$). While the parabolic case turned out to be non-integrable, we found the elliptic case to be integrable with integrals of a most degree 2. Thus, let us now explore a 7-dimensional case with maximal dimension of the symmetry algebra for (L_1, g).

Its structure equations are given in [AK11; KVL15] and read

$$\begin{gathered}[e_1, e_2] = e_3,\ [e_1, e_3] = e_4,\ [e_2, e_3] = e_5, \\ [e_1, e_4] = -[e_2, e_5] = e_6,\ [e_1, e_5] = [e_2, e_4] = e_7.\end{gathered} \tag{5.17}$$

[4]There are also elliptic cases with $\dim \mathrm{Sym}(\mathrm{L}_1, \mathrm{g}) = 6$ in 6 dimensions, depending on whether the rotation endomorphism yields a grading-preserving derivation of L_1.

In local coordinates, the left-invariant Hamiltonian given by $H = \frac{1}{2}(\omega_1^2 + \omega_2^2)$ can be written as follows [KVL15]

$$2H = \left(p_1 - \tfrac{1}{2}x_2p_3 - x_1x_2p_4 - \tfrac{1}{2}x_1^2x_2p_6 - \tfrac{1}{4}x_1x_2^2p_7\right)^2 + \left(p_2 + \tfrac{1}{2}x_1p_3 + x_1x_2p_5 - \tfrac{1}{2}x_1x_2^2p_6 + \tfrac{1}{4}x_1^2x_2p_7\right)^2. \quad (5.18)$$

We can find a family of six involutive integrals for this structure.

- The (right-invariant) linear integrals $\theta_3, \dots, \theta_7$ (resp. $p_3, \dots, p_7$ in local coordinates) are in involution

- The Hamiltonian extends this family to an involutive family of six functionally independent integrals, $\mathcal{I}_{(2,3,5,7)} = (H, \theta_3, \dots, \theta_7)$

- There are 3 Casimir functions: ω_6, ω_7 and

$$\omega_3(\omega_6^2 + \omega_7^2) - \frac{1}{2}(\omega_4^2 - \omega_5^2)\omega_6 - \omega_4\omega_5\omega_7.$$

  However, these Casimir functions are all generated by the integrals $\theta_3, \dots, \theta_7$, and therefore we still lack one additional integral for Liouville integrability.

We prove that there are no low-degree integrals in addition to $\mathcal{I}_{(2,3,5,7)}$ to ensure Liouville integrability:

Theorem 6. *The family $(H, p_3, \dots, p_7)$ of involutive integrals can not be extended to a Liouville-integrable family of integrals of degree ≤ 5, i.e. there is no irreducible involutive integral of degree ≤ 5, in addition to the identified 6 involutive integrals, for the considered left-invariant sub-Riemannian structure (5.18) on the $(2,3,5,7)$-distribution.*

This result is obtained using again Algorithm III.

Quintic integral $(d = 5)$; $\Lambda_5^0 = 166$
sixth step of prolongation

$\#\mathcal{S}$	$\#V$	$\overline{\Lambda}_5^{(6)}$	computation time
25872	16632	166	ca. 10.3 hours

5.3.3 Dimension 8

We compare two sub-Riemannian structures that have a 9-dimensional symmetry algebra $\mathrm{Sym}(\mathrm{L}_1, \mathrm{g})$. The examples have growth vectors $(2,3,5,6,8)$ and $(2,3,5,8)$, and their underlying distributions have symmetry dimension 10 and 12, respectively. In spite of its lower symmetry dimension, the $(2,3,5,6,8)$-structure turns out to be Liouville integrable with integrals of low degree, while the other does not.

The $(2,3,5,8)$-structure. The sub-Riemannian structure with growth vector $(2,3,5,8)$ that we investigate has also been considered in [Sac13; Sac14]. It is the free truncated graded nilpotent Lie algebra with the structure equations

$$[e_1,e_2]=e_3,\ [e_1,e_3]=e_4,\ [e_2,e_3]=e_5,$$
$$[e_1,e_4]=e_6,\ [e_1,e_5]=[e_2,e_4]=e_7,\ [e_2,e_5]=e_8.$$

The left-invariant Hamiltonian reads in local coordinates:

$$2H=\left(p_1-\tfrac{1}{2}x_2p_3-\tfrac{1}{2}(x_1^2+x_2^2)p_5-\tfrac{1}{4}x_1x_2^2p_7-\tfrac{1}{6}x_2^3p_8\right)^2 + \left(p_2+\tfrac{1}{2}x_1p_3+\tfrac{1}{2}(x_1^2+x_2^2)p_4+\tfrac{1}{6}x_1^3p_6+\tfrac{1}{4}x_1^2x_2p_7\right)^2. \quad (5.19)$$

For this structure we find a family of seven involutive integrals:

- We obtain an involutive family of six right-invariant linear integrals besides the Hamiltonian, namely $I_2=\theta_3$, $I_3=\theta_4$, $I_4=\theta_5$, $I_5=\theta_6$, $I_6=\theta_7$, $I_7=\theta_8$ (or $I_i=p_i$, $i=3,\dots,8$ in coordinates).
- There is one cubic Casimir function (along with θ_6, θ_7 and θ_8)

  $$C=(\omega_5^2-2\omega_3\omega_8)\,\omega_6+2\omega_3\omega_7^2+\omega_4^2\omega_8-2\omega_4\omega_5\omega_7,$$

  but it is not independent of the linear integrals. We find

  $$C=2((I_6^2-I_5I_7)I_2-I_6I_4I_3)+I_7I_3^2+I_5I_4^2).$$

We prove that there is no additional (final) integral in low degree:

Theorem 7. *The family $\mathcal{I}_{(2,3,5,8)}=(H,\theta_3,\dots,\theta_8)$ cannot be extended to a Liouville-integrable family of integrals of degree ≤ 5, i.e. there exists no irreducible integral of degree ≤ 5, in addition to the identified 7 involutive integrals, for the considered left-invariant sub-Riemannian structure on the $(2,3,5,8)$-distribution with Hamiltonian (5.19).*

Applying Algorithm III, we obtain the table

Quintic integral $(d=5)$; $\Lambda_5^0=314$
sixth step of prolongation

$\#\mathcal{S}$	$\#V$	$\overline{\Lambda}_5^{(6)}$	computation time
48048	28512	314	ca. 10.2 hours

The full list of results is provided in the appendix, see Section A.1.3.

The $(2,3,5,6,8)$-structure. Finally, we investigate a sub-Riemannian structure that has growth vector $(2,3,5,6,8)$. Its symmetry algebra for (L_1,g) has the same dimension as the previous example, while the underlying distribution has less symmetry than its $(2,3,5,8)$ counterpart. The structure with growth vector $(2,3,5,6,8)$ has the structure equations

$$[e_1,e_2]=e_3,\ [e_1,e_3]=e_4,\ [e_2,e_3]=e_5,$$
$$[e_1,e_4]=[e_2,e_5]=e_6,$$
$$[e_1,e_6]=[e_3,e_5]=e_7,$$
$$[e_2,e_6]=[e_4,e_3]=e_8.$$

The left-invariant Hamiltonian $H=\frac{1}{2}(\omega_1^2+\omega_2^2)$ admits the following integrals

- The five commuting right-invariant linear integrals $I_2 = \theta_4$, $I_3 = \theta_5$, $I_4 = \theta_6$, $I_5 = \theta_7$, $I_6 = \theta_8$, plus the Hamiltonian itself. These form an involutive family of six functionally independent integrals.

- There are 4 Casimir functions, namely ω_7, ω_8 and

$$I_7 = \omega_1\omega_8 - \omega_2\omega_7 + \omega_3\omega_6 - \frac{\omega_4^2+\omega_5^2}{2} = \theta_1\theta_8 - \theta_2\theta_7 + \theta_3\theta_6 - \frac{\theta_4^2+\theta_5^2}{2},$$
$$C = \omega_4\omega_7 + \omega_5\omega_8 - \tfrac{1}{2}\omega_6^2 = \theta_4\theta_7 + \theta_5\theta_8 - \tfrac{1}{2}\theta_6^2.$$

  The second one of the latter, C, is generated by $I_2, \ldots, I_6$ and thus reducible. However, we can add I_7 to the family of involutive integrals.

- We have the quadratic integral $I_8 = \omega_1\omega_5 - \omega_2\omega_4 + \frac{1}{2}\omega_3^2$. It can be checked that $I_1 = H, I_2, \ldots, I_8$ are involutive and functionally independent almost everywhere on T^*G, and therefore that the (2,3,5,8)-structure is Liouville integrable.

- There is another quadratic integral: $I_8' = \theta_1\theta_5 - \theta_2\theta_4 + \frac{1}{2}\theta_3^2$. This integral, together with C, renders the structure superintegrable.

The family $\mathcal{I}_{(2,3,5,6,8)} = (H, I_2, \ldots, I_8)$ ensures that the considered $(2,3,5,6,8)$-structure is Liouville integrable.

5.4 Reduction to 2 degrees of freedom

It is possible to reconsider the four non-integrable examples in a comprehensive manner. The same uniform writing is possible with the elliptic 6D structure that we considered. To see this, we express the Hamiltonians (5.12), (5.14), (5.16), (5.18) and (5.19) as

$$H = \frac{1}{2}\,\rho^2\left(\cos^2 z + \sin^2 z\right) = \frac{1}{2}\,\rho^2 \tag{5.20}$$

and use the conservation of H to reduce the dimension of the problem by one, setting $\rho = 1$. This makes the Hamiltonian constant. All p_i with $i \geq 3$ are linear integrals and can be considered as constants in the reduced picture [Whi04], i.e. we work on a level hypersurface with constant $p_i = c_i$ $(i = 1, \ldots, D)$. The Hamiltonian equation $\dot{\eta} = \{H, \eta\}$ reduces to the normal form[5]

$$\dot{x} = \cos(z), \quad \dot{y} = \sin(z), \quad \dot{z} = Q(x, y), \tag{5.21}$$

where Q is a quadratic polynomial taking the normal form

$$Q = \begin{cases} Q_1(x,y) = ax^2 + by & \text{for } D = 6 \text{ (parabolic)} \\ Q_2(x,y) = ax^2 + by^2 + c & \text{for } D = 7, 8 \end{cases}, \qquad \text{with constants } a, b, c. \tag{5.22}$$

In case of the elliptic 6D system, we obtain $Q = Q_2$ with $a = b \neq 0$, and a degenerate case with $Q = Q_1$ and $a = 0$. For the 6D hyperbolic case, we similarly have $Q = Q_2$ with $a = -b \neq 0$, and $Q = Q_1$ with $a = 0$ in a degenerate case.

[5] To obtain these normal forms, express $\dot{z}$ in terms of the original coordinates x_1, x_2 and their corresponding momenta. The normal forms are then obtained by a simple coordinate transformation.

For non-integrable examples, the constants a and b have to be nonzero, otherwise the 3D flow reduces to a 2D flow and cannot be chaotic. The respective integrals in such cases are

	$a = 0$	$b = 0$
Q_1	$I = \cos(z) + {}^{b}/_{2}\, y^2$	$I = \sin(z) - {}^{a}/_{3}\, x^3$
Q_2	$I = \cos(z) + c\, y + {}^{b}/_{3}\, y^3$	$I = \sin(z) - {}^{a}/_{3}\, x^3 - c\, x$

Moreover, non-integrability in the case $Q = Q_2$ requires $a \neq b$. This is because we can, for $a = b$ (cf. the 6D elliptic case), perform a polar change of coordinates [KVL15],

$$x = r\cos(\psi), \quad y = r\sin(\psi),$$

followed by the change to $s = z - \psi$. In this way, we obtain:

$$\dot{r} = \cos(s), \quad \dot{z} = ar^2 + c, \quad \dot{s} = ar^2 + c - \frac{\sin(s)}{r}.$$

Obviously, the system reduces to a 2D flow governed by the first and the third equation (we can isolate the coordinate z). It follows, in case that $Q = Q_2$ and $a = b$, that we have the additional integral

$$F = \frac{a}{4}\, r^4 + \frac{c}{2}\, r^2 - r\sin(s), \tag{5.23}$$

which corresponds to the integral I_2 of the 6D elliptic example, cf. page 81.

Chapter 6

Conclusions and Outlook

In the previous chapters, we demonstrate how the prolongation-projection method, in combination with the structural properties of the systems considered, is a very effective tool. We apply this tool to decide whether given Hamiltonian systems, defined via a (pseudo-) Riemannian or sub-Riemannian metric, admit additional low-degree integrals. We briefly summarize the results and point out some perspectives and open problems for further investigation.

Stationary and Axially Symmetric Metrics. With the help of the prolongation-projection method, we provide computer-assisted and rigorous proofs for difficult problems in differential geometry. Specifically, we

- extend the results from [KM12; LG12; MPS13] for the Zipoy-Voorhees metric with parameter $\delta = 2$,
- prove nonexistence of a final integral of degree 7 for a Tomimatsu-Sato metric with parameter $\delta = 2$,
- confirm algorithmically that flat space and the Schwarzschild metric are the only Zipoy-Voorhees metrics with an additional involutive quadratic integral (these integrals are reducible in both cases),
- give a proof of total reducibility for arbitrary Zipoy-Voorhees metrics in degree up to 3, and
- prove reducibility for involutive cubic integrals for *arbitrary* Weyl metrics.

These results confirm that our method, the algorithms and the techniques in Chapter 4, can efficiently answer questions about integrability of Hamiltonian systems in physics. Important perspectives for future research include the generalization of the mentioned results and the application of the methods to other contexts. For instance, we plan to relax the involutivity assumption. For non-involutive integrals, the methods of Chapter 2 and to some extend those of Chapter 4 can still be employed, but their application is more tedious since more equations and unknowns are involved.

Another interesting perspective is the generalization of Theorem 4 to the stationary case. Here, the methods from Chapter 4 work in principle, but the system is more complicated since it involves more equations and additional parametrizing functions.

Sub-Riemannian structures on Carnot groups. Concerning sub-Riemannian structures on Carnot groups, we explore the connection between Liouville integrability and the degree of symmetry for the sub-Riemannian structure and its underlying rank-2 distribution in Chapter 5.

- We observe the counter-intuitive effect that a higher degree of symmetry does not imply the existence of additional low-degree integrals, beginning from dimension 6. Moreover, structures of lower symmetry dimension can be integrable whereas their maximally symmetric counterparts are not.

- The reduced picture presented in Section 5.4 provides a uniform formulation for these cases and elucidates how the integrable cases relate to the non-integrable cases.

Since many of the tricks outlined in Chapter 2 do not work for the examples of Chapter 5, the computations are much more demanding than the examples in Chapter 3 as the degree of the integral increases. The associated linear systems (2.3) for the cases $6\mathrm{D}_{\mathrm{p}}$, $6\mathrm{D}_{\mathrm{h}}$, 7D and $8\mathrm{D}_1$ on page 77 involve a huge number of equations and unknowns — in fact several 10,000 equations and unknowns. Although we could already examine integrals up to degree 5 or 6 with Algorithm III, new tricks are needed when one aims to explore higher degrees.

The algorithmic method. The approach via prolongation-projection provides an efficient method to check the existence of low-degree integrals. Three implementations are established, namely Algorithms I, II and III on pages 33, 37 and 79, respectively. These algorithms are adapted to certain contexts and several computer-based proofs (e.g. Theorems 2, 3 and 5–7) can be obtained in an entirely self-contained, computer-based manner with these algorithms. The proof for Theorem 1 is also completed in this algorithmic way. To have such an algorithmic method is interesting for a number of reasons:

— A quick and easy-to-use way to prove nonexistence of low-degree integrals is provided (see the appendix for the Maple worksheets).

— For integrable systems, additional integrals can be detected by degree-wise application of the algorithmic check.

— Parameter-dependent families of metrics can systematically be examined for integrability with low-degree integrals (cf. the procedure used in Proof 3 of Theorem 1).

These properties are likely to make the method interesting for many applications in mathematics as well as in physics. The algorithm can be used as a single method (as we did e.g. with Theorems 1–3), but also in combination with other methods. For instance, we could use the algorithm with a parameter-dependent metric to find candidates for integrability, and subsequently explore the integrals with other, more tedious methods.

The degree of the integrals that can be studied in this way is conceptually unlimited, but restricted by computer strength. If one needs to explore higher-degree integrals, other methods (or additional tricks) might be needed. Comparing the examples of Chapter 3 with those of Chapter 5, we see that new tools could especially be helpful for systems that do not have the properties discussed on pages 23–27. It is possible that such tools can be found amongst the zoo of existing matrix factorization techniques.

The direct method of Chapter 4. In Chapter 4, we prove nonexistence of an additional involutive integral of third degree for arbitrary Weyl metrics. For Weyl metrics, the situation is particularly nice because we can restrict to one of the subsystems in the decomposition of Lemma 2 on page 36. Moreover, we are in the lucky situation that the metric is characterized by effectively one parametrizing function (see Step (v) on page 58), and that this system has a particularly simple solution.

The method itself, however, does in principle also work for more involved systems. For instance, we are not aware of major conceptual problems in the case of stationary and axially symmetric vacuum metrics that are non-static. The difficulties in this case are rather that there are more equations and that an additional parametrizing function is needed to account for rotations.

At the same time, there are promising perspectives for the method to be applied, besides the theory of integrable systems. For instance, we suggest that it can be used in the area of superintegrability. There is a lot of literature on second-order superintegrability in 2-dimensional spaces, i.e. superintegrable systems with two quadratic integrals in addition to the Hamiltonian, e.g. [CK14; KKM05; KKPM01; Mil14]. For superintegrable systems with integrals of higher degree, however, there are only some partial results known to date. Superintegrable systems involving a cubic integral have recently received more attention, e.g. [GW02; Mar09; MS11]. These results sometimes only extend to cases connected with certain coordinate choices [Gra04]. Problems of this kind are structurally very similar to the situations that we consider in this thesis, and we plan to examine such problems with our methods in the near future.

Appendix A

Appendix

A.1 Carnot groups: list of results

A.1.1 6D Carnot groups: list of results

In the following, the detailed results of the algorithmic computations for the non-integrable sub-Riemannian structures are provided, including the time intervals needed to complete the computations on our computer (the time that is needed naturally might vary if the algorithm is run again). Recall that Λ_d^0 denotes the number of integrals of degree d that arise from products of the known integrals. This number is bounded from above by $\overline{\Lambda}_d^{(k)}$, and the equality $\overline{\Lambda}_d^{(k_0)} = \Lambda_d^0$, for certain $k_0 \in \mathbb{N}$, implies that there is no additional integral for the Hamiltonian system.

Parabolic structure

d	$\#\mathcal{S}$	$\#V$	rows of M_{red}	$\#V_{\text{red}}$	$\text{rk}(M_{\text{red}})$	Λ_d^0	$\overline{\Lambda}_d^{(d+1)}$	time
1	126	60	0	0	0	4	4	0.1s
2	560	315	0	0	0	11	11	1s
3	1890	1176	116	86	86	24	24	13s
4	5292	3528	604	464	464	46	46	3.3m
5	12936	9072	2840	2262	2262	80	80	45m
6	28512	20790	11816	9155	9155	130	130	27.2h

Hyperbolic structure

d	$\#\mathcal{S}$	$\#V$	rows of M_{red}	$\#V_{\text{red}}$	$\text{rk}(M_{\text{red}})$	Λ_d^0	$\overline{\Lambda}_d^{(d+1)}$	time
1	126	60	0	0	0	4	4	0.1s
2	560	315	0	0	0	11	11	1s
3	1890	1176	196	143	143	24	24	15s
4	5292	3528	1203	891	891	46	46	6m
5	12936	9072	5360	4013	4013	80	80	1.7h

A.1.2 7D Carnot group: list of results

d	$\#\mathcal{S}$	$\#V$	rows of M_{red}	$\#V_{\text{red}}$	$\text{rk}(M_{\text{red}})$	Λ_d^0	$\overline{\Lambda}_d^{(d+1)}$	time
1	168	70	0	0	0	5	5	0.1s
2	840	420	0	0	0	16	16	1.5s
3	3150	1764	253	187	187	40	40	36s
4	9702	5880	1684	1272	1272	86	86	14m
5	25872	16632	9397	6993	6993	166	166	10.3h

A.1.3 8D Carnot group: list of results

d	$\#\mathcal{S}$	$\#V$	rows of M_{red}	$\#V_{\text{red}}$	$\text{rk}(M_{\text{red}})$	Λ_d^0	$\overline{\Lambda}_d^{(d+1)}$	time
1	216	80	0	0	0	6	6	0.1s
2	1200	540	0	0	0	22	22	2.3s
3	4950	2520	62	47	47	62	62	1.4m
4	16632	9240	97	82	82	148	148	33m
5	48048	28512	4439	3514	3514	314	314	10.2h

A.2 Bases in the sub-Riemannian structures

For completeness, we list the left-invariant bases in coordinate form for the left-invariant sub-Riemannian structures in Chapter 5.

A.2.1 6D sub-Riemannian structures

Left-invariant basis for the parabolic case.

$$\begin{aligned}
e_1 &= \partial_1 - \frac{1}{2}x_2\partial_3 - x_1x_2\partial_4 - \frac{1}{2}x_1^2x_2\partial_6\\
e_2 &= \partial_2 + \frac{1}{2}x_1\partial_3 + x_1x_2\partial_5\\
e_3 &= \partial_3 + x_1\partial_4 + x_2\partial_5 + \frac{1}{2}x_1^2\partial_6\\
e_4 &= \partial_4 + x_1\partial_6\\
e_5 &= \partial_5\\
e_6 &= \partial_6
\end{aligned}$$

Left-invariant basis for the elliptic case.

$$\begin{aligned}
e_1 &= \partial_1 - \frac{1}{2}x_2\partial_3 - x_1x_2\partial_4 - \frac{1}{2}x_1^2x_2\partial_6\\
e_2 &= \partial_2 + \frac{1}{2}x_1\partial_3 + x_1x_2\partial_5 + \frac{1}{2}x_1x_2^2\partial_6\\
e_3 &= \partial_3 + x_1\partial_4 + x_2\partial_5 + \frac{x_1^2 + x_2^2}{2}\partial_6\\
e_4 &= \partial_4 + x_1\partial_6\\
e_5 &= \partial_5 + x_2\partial_6\\
e_6 &= \partial_6
\end{aligned}$$

Left-invariant basis for the hyperbolic case.

$$\begin{aligned}
e_1 &= \partial_1 - \frac{1}{2}x_2\partial_3 - x_1x_2\partial_4 - \frac{1}{4}x_1x_2^2\partial_6\\
e_2 &= \partial_2 + \frac{1}{2}x_1\partial_3 + x_1x_2\partial_5 + \frac{1}{4}x_1^2x_2\partial_6\\
e_3 &= \partial_3 + x_1\partial_4 + x_2\partial_5 + x_1x_2\partial_6\\
e_4 &= \partial_4 + x_2\partial_6\\
e_5 &= \partial_5 + x_1\partial_6\\
e_6 &= \partial_6
\end{aligned}$$

A.2.2 7D sub-Riemannian structure

$$
\begin{aligned}
e_1 &= \partial_1 - \frac{1}{2}x_2\partial_3 - x_1x_2\partial_4 - \frac{1}{2}x_1^2x_2\partial_6 - \frac{1}{4}x_1x_2^2\partial_7 \\
e_2 &= \partial_2 + \frac{1}{2}x_1\partial_3 + x_1x_2\partial_5 - \frac{1}{2}x_1x_2^2\partial_6 + \frac{1}{4}x_1^2x_2\partial_7 \\
e_3 &= \partial_3 + x_1\partial_4 + x_2\partial_5 + \frac{x_1^2 - x_2^2}{2}\partial_6 + x_1x_2\partial_7 \\
e_4 &= \partial_4 + x_1\partial_6 + x_2\partial_7 \\
e_5 &= \partial_5 - x_2\partial_6 + x_1\partial_7 \\
e_6 &= \partial_6 \\
e_7 &= \partial_7
\end{aligned}
$$

A.2.3 8D sub-Riemannian structures

Left-invariant basis for the $(2,3,5,8)$-structure.

$$
\begin{aligned}
e_1 &= \partial_1 - \frac{1}{2}x_2\partial_3 - \frac{x_1^2 + x_2^2}{2}\partial_5 - \frac{1}{4}x_1x_2^2\partial_7 - \frac{1}{6}x_2^3\partial_8 \\
e_2 &= \partial_2 + \frac{1}{2}x_1\partial_3 + \frac{x_1^2 + x_2^2}{2}\partial_4 + \frac{1}{6}x_1^3\partial_6 + \frac{1}{4}x_1^2x_2\partial_7 \\
e_3 &= \partial_3 + x_1\partial_4 + x_2\partial_5 + \frac{1}{2}x_1^2\partial_6 + x_1x_2\partial_7 + \frac{1}{2}x_2^2\partial_8 \\
e_4 &= \partial_4 + x_1\partial_6 + x_2\partial_7 \\
e_5 &= \partial_5 + x_1\partial_7 + x_2\partial_8 \\
e_6 &= \partial_6 \\
e_7 &= \partial_7 \\
e_8 &= \partial_8
\end{aligned}
$$

Left-invariant basis for the $(2,3,5,6,8)$-structure.

$$
\begin{aligned}
e_1 = \partial_1 &- x_2\partial_3 + \left(-\frac{1}{3}x_1x_2 - \frac{1}{2}x_3\right)\partial_4 - \frac{1}{12}x_2^2\partial_5 + \left(-\frac{1}{24}x_2^3 - \frac{5}{24}x_1^2x_2 - \frac{1}{3}x_1x_3\right)\partial_6 \\
&+ \left(-\frac{1}{30}x_1x_2^3 - \frac{1}{24}x_2^2x_3 - \frac{3}{40}x_1^3x_2 - \frac{1}{8}x_1^2x_3\right)\partial_7 \\
&+ \left(\frac{7}{240}x_1^2x_2^2 + \frac{1}{4}x_1x_2x_3 + \frac{1}{3}x_3^2 - \frac{1}{80}x_2^4\right)\partial_8 \\
e_2 = \partial_2 &+ \frac{1}{12}x_1^2\partial_4 + \left(-\frac{1}{6}x_1x_2 - \frac{1}{2}x_3\right)\partial_5 + \left(\frac{1}{24}x_1^3 - \frac{1}{8}x_1x_2^2 - \frac{1}{3}x_2x_3\right)\partial_6 \\
&+ \left(\frac{1}{80}x_1^4 - \frac{9}{80}x_1^2x_2^2 - \frac{5}{12}x_1x_2x_3 - \frac{1}{3}x_3^2\right)\partial_7 \\
&+ \left(-\frac{1}{120}x_1^3x_2 - \frac{1}{24}x_1^2x_3 - \frac{1}{20}x_1x_2^3 - \frac{1}{8}x_2^2x_3\right)\partial_8
\end{aligned}
$$

$$
\begin{aligned}
e_3 &= \partial_3 + \frac{1}{2}x_1\partial_4 + \frac{1}{2}x_2\partial_5 + \frac{x_1^2+x_2^2}{3}\partial_6 \\
&\quad + \left(\frac{1}{8}x_1^3 + \frac{7}{24}x_1x_2^2 + \frac{1}{3}x_2x_3\right)\partial_7 + \left(-\frac{1}{24}x_1^2x_2 - \frac{1}{3}x_1x_3 + \frac{1}{8}x_2^3\right)\partial_8 \\
e_4 &= \partial_4 + x_1\partial_6 + \frac{1}{2}x_1^2\partial_7 - x_3\partial_8 \\
e_5 &= \partial_5 + x_2\partial_6 + (x_1x_2 + x_3)\partial_7 + \frac{1}{2}x_2^2\partial_8 \\
e_6 &= \partial_6 + x_1\partial_7 + x_2\partial_8 \\
e_7 &= \partial_7 \\
e_8 &= \partial_8
\end{aligned}
$$

A.3 Maple worksheets

Computer-algebra calculations for this thesis have been performed using Maple 18.

The Maple worksheets can be obtained from the author (andreas.vollmer@uni-jena.de) or on-line under the following address:

https://bitbucket.org/av122/dissertation/downloads

The archive with the files is password protected. The password is: AV15thesis

The md5 checksum is: ebca0cbf13924af1daae5dc12e6ca93c *worksheets.zip

Files in the archive:

Algorithm1.mw
Algorithm I

Algorithm2.mw
Algorithm II

Algorithm3.mw
Algorithm III

Theorem1_Proof1.mw
Proof 1 for Theorem 1; see also Section 4.4

Theorem1_Proof2.mw
Proof 2 for Theorem 1

Theorem1_Proof3.mw
Proof 3 for Theorem 1; cf. Algorithm III

Chapter4_rank1.mw
computations for Chapter 4 concerning the rank-1 case

Chapter4_rank2.mw
computations for Chapter 4 concerning the rank-2 case; see also the list on page 58

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Parts of this thesis, in particular Chapters 4, 5 and parts of Chapter 3, are based on the author's publication [Vol15b] and the collaborative paper [KVL15], and contain results published in these works.

Algorithms I, II and III are based on [KM12], doi: 10.1103/PhysRevD.85.124057

Symbols and Abbreviations

D	usually the dimension of a manifold.
$\mathcal{E}$	a differential equation
$F^{(i)}$	denotes the homogeneous polynomial component of degree i w.r.t. non-ignorable coordinates of a polynomial F, see Section 2.2 for details
$F^{(i,j)}$	The notation $F_k^{(i,j)} = F_k^{(i,j,0,0)}$ is used for coefficients in a polynomial w.r.t. momenta in the 4-dimensional situations, i.e. $F = \sum F_k^{(i,j)} p_1^{i-j} p_2^j p_3^k p_4^{d-i-k}$. See also $F^{(i)}$ and $F_k^{(i,j,M,m)}$.
$F_k^{(i,j,M,m)}$	differentiated version of $F^{(i,j)}$. The additional superscripts M and m denote the total level of prolongation and the number of x^1-derivatives, respectively.
$\mathcal{I}$	a family of integrals, typically assumed to be involutive
M	a differentiable manifold of dimension D, usually endowed with a (pseudo-) Riemannian or sub-Riemannian metric
N	a symplectic manifold, usually $N = T^*M$
$\mathcal{S}$	a system of differential equations; sometimes used synonymous to $\mathcal{S}_d$, if the degree d of the integral is clear
$\mathcal{S}_d$	the system of PDEs obtained from the Poisson equation (1.3), encoding the requirement for a function polynomial in momenta of degree d to be an integral
$\mathcal{S}_d^{(k)}$	k-th prolongation of a system $\mathcal{S}_d$ of PDEs (the subscript d denotes the degree of the integral)
T	quadratic part of the Hamiltonian, i.e. the kinetic part of a Hamiltonian in the reduced picture. The symbol is used in the stationary and axially symmetric examples after symplectic reduction; mostly synonymous to $H^{(2)}$
V	the potential in the reduced picture; mostly synonymous to $H^{(0)}$.
∂_i	partial derivative w.r.t. the i-th coordinate, shortcut for $\frac{\partial}{\partial q^i}$ or $\frac{\partial}{\partial x^i}$
f_x	partial derivative of f w.r.t. x, i.e. $f_x = \partial_x = \frac{\partial f}{\partial x}$
$m_{d,k}$	number of equations in the prolongated system of PDEs $\mathcal{S}_d^{(k)}$

$n_{d,k}$	number of unknowns in the prolongated system of PDEs $\mathcal{S}_d^{(k)}$
p	By p_i, we denote the momentum coordinates associated with the position coordinates q_i.
par	parity of a number or polynomial. If par is applied to a polynomial, we understand the parity to be w.r.t. nonignorable momenta only.
q	By q_i, we denote coordinates on the manifold, on which a Hamiltonian system is defined.
x	By x_i, we denote coordinates on the phase space T^*M, i.e. $x = (q, p)$.
Δ	sometimes denotes a (rank-2) distribution on a Lie group (Chapter 5)
$\triangle$	the Laplace operator, i.e. $\triangle f = f_{xx} + f_{yy}$ (see Chapter 3)
Λ_d	number of integrals of degree d that the system admits
Λ_d^0	number of trivial integrals of degree d that are known for the system
$\overline{\Lambda}_d^{(k)}$	upper bound to the number of integrals of degree d, at the k-th step of prolongation
ODE	ordinary differential equation
PDE	partial differential equation
SAV	stationary and axially symmetric (axisymmetric) vacuum space-times or metrics
smooth	Smoothness is understood as the differentiability class $\mathcal{C}^\infty$.

Index