THE ISOPERIMETRIC PROBLEM FOR A CLASS OF NON-RADIAL WEIGHTS AND APPLICATIONS

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Abstract. We study a class of isoperimetric problems on $\mathbb{R}^N_+$ where the densities of the weighted volume and weighted perimeter are given by two different non-radial functions of the type $|x|^k x_N^\alpha$. Our results imply some sharp functional inequalities, like for instance, Caffarelli-Kohn-Nirenberg type inequalities.

Key words: isoperimetric inequality, weighted rearrangement, functional inequalities

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1. Introduction

The last decades have seen an increasing interest in the study of "Manifolds with Density", which is a manifold where both perimeter and volume carry the same weight. To have an idea of the possible applications of that subject one can consult, for instance [36], [37] and the references therein. In particular, much attention has been devoted to find, for a given manifold with density, its isoperimetric set (see, e.g., [3], [5–11], [14], [15], [17], [21], [32], [34], [37], [38]). On the other hand, many authors have studied isoperimetric problems when volume and perimeter carry two different weights. A remarkable example is obtained when the manifold is $\mathbb{R}^N$ and the two weights are two different powers of the distance from the origin. More precisely, given two real numbers k and l, the problem is to find the set G in $\mathbb{R}^N$ which minimizes the weighted perimeter $\int_{\partial G} |x|^k \, \mathcal{H}_{N-1}(dx)$ once the weighted volume $\int_G |x|^l \, dx$ is prescribed. Such a problem is far from being artificial since its solution allows to compute, for instance, the best constants in the well-known Caffarelli-Kohn-Nirenberg inequalities as well as to establish the radiality of the corresponding minimizers. Several partial results have been obtained on such an issue (see, e.g., [1], [2], [4], [13], [16], [19], [20], [22], [23], [24], [30], [35], [36]).

Let $\mathbb{R}^N_+ := \{x \in \mathbb{R}^N : x_N > 0\}$. The problem that we address here is the following:

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Given $k, l \in \mathbb{R}$, $\alpha > 0$,

Minimize $\displaystyle\int_{\partial\Omega} |x|^k x_N^\alpha \, \mathcal{H}_{N-1}(dx)$ *among all smooth sets* $\Omega \subset \mathbb{R}^N_+$ *satisfying* $\displaystyle\int_\Omega |x|^l x_N^\alpha \, dx = 1.$

Let B_R denote the ball of $\mathbb{R}^N$ of radius R centered at the origin and let B and Γ denote the Beta and the Gamma function, respectively. Our main result, contained in Section 6, is the following.

Theorem 1.1. *Let $N \in \mathbb{N}$, $N \geq 2$, $k, l \in \mathbb{R}$, $\alpha > 0$ and $l + N + \alpha > 0$. Further, assume that one of the following conditions holds:*
(i) *$l + 1 \leq k$;*
(ii) *$k \leq l + 1$ and $l\frac{N+\alpha-1}{N+\alpha} \leq k \leq 0$;*
(iii) *$N \geq 2$, $0 \leq k \leq l + 1$ and*

$$l \leq l_1(k, N, \alpha) := \frac{(k+N+\alpha-1)^3}{(k+N+\alpha-1)^2 - \frac{(N+\alpha-1)^2}{N+\alpha}} - N - \alpha \,. \tag{1.1}$$

Then

$$\int_{\partial\Omega} |x|^k x_N^\alpha \, \mathcal{H}_{N-1}(dx) \geq C^{rad}_{k,l,N,\alpha} \left(\int_\Omega |x|^l x_N^\alpha \, dx \right)^{(k+N+\alpha-1)/(l+N+\alpha)} , \tag{1.2}$$

for all smooth sets Ω in $\mathbb{R}^N_+$, where

$$\begin{aligned} C^{rad}_{k,l,N,\alpha} \quad &:= \quad \frac{\displaystyle\int_{\partial B_1 \cap \mathbb{R}^N_+} |x|^k x_N^\alpha \, \mathcal{H}_{N-1}(dx)}{\left(\displaystyle\int_{B_1 \cap \mathbb{R}^N_+} |x|^l x_N^\alpha \, dx \right)^{(k+N+\alpha-1)/(l+N+\alpha)}} \\ &= \quad (l + \alpha + N)^{\frac{k+N+\alpha-1}{l+N+\alpha}} \left(B\left(\frac{N-1}{2}, \frac{\alpha+1}{2} \right) \frac{\pi^{\frac{N-1}{2}}}{\Gamma\left(\frac{N-1}{2}\right)} \right)^{\frac{l-k+1}{l+N+\alpha}} . \end{aligned} \tag{1.3}$$

Equality in (1.2) holds if $\Omega = B_R \cap \mathbb{R}^N_+$.

Note that if $N + \alpha \geq 3$, then **(iii)** covers the important range

$$l = 0 \leq k \leq 1.$$

However, we emphasize that this is not true when $2 \leq N + \alpha < 3$.

Note also that the weights we consider are not radial and it seems not trivial to use spherical symmetrization. So that we did not try to adapt the techniques contained in [17], and, depending on the regions where the three parameters lie, we use different methods. The proof in the case **(i)** is given in [2]. It is based on Gauss's Divergence Theorem. In the case **(ii)** (see Theorem 6.1) the proof uses an appropriate change of variables, which has been introduced in [28] and [29], together with the isoperimetric inequality with respect to the weight x_N^α. The case **(iii)** (see Theorem 6.2) is the most delicate and it requires several different arguments: again a suitable change of variables, then an interpolation argument, introduced for the first time in our previous paper [1] and, finally, the so-called starshaped rearrangement.

In Section 4 we provide some necessary conditions on k, l and α such that the half-ball centered at the origin is an isoperimetric set. In the proof we firstly evaluate the second variation of the perimeter functional. The claim is achieved using the fact that such a variation at a minimizing set must be nonnegative, together with a nontrivial weighted Poincaré inequality on the sphere derived in [8].
Part of these results have been announced in [2].

2. Notation and preliminary results

Throughout this article N will denote a natural number with $N \geq 2$, k and l are real numbers, while α is a nonnegative number and

$$l + N + \alpha > 0. \tag{2.1}$$

Let us introduce some notation.

$$\begin{aligned}
\mathbb{R}^N_+ &:= \left\{ x \in \mathbb{R}^N : x_N > 0 \right\}, \\
\mathbb{S}^{N-1}_+ &:= \left\{ x \in \mathbb{S}^{N-1} : x_N > 0 \right\}, \\
B_R(x_0) &:= \left\{ x \in \mathbb{R}^N : |x - x_0| < R \right\}, \quad (x_0 \in \mathbb{R}^N), \\
B_R &:= B_R(0), \quad (R > 0), \\
B_R^+ &:= B_R \cap \mathbb{R}^N_+.
\end{aligned}$$

Furthermore, $\mathcal{L}^m$ will denote the m-dimensional Lebesgue measure, $(1 \leq m \leq N)$, and

$$\begin{aligned}
\omega_N &:= \mathcal{L}^N(B_1), \\
\kappa(N, \alpha) &:= \int_{\mathbb{S}^{N-1}_+} x_N^\alpha \, \mathcal{H}_{N-1}(dx).
\end{aligned}$$

Note that

$$\kappa(N, \alpha) = B\left(\frac{N-1}{2}, \frac{\alpha+1}{2} \right) \frac{\pi^{\frac{N-1}{2}}}{\Gamma\left(\frac{N-1}{2}\right)}, \tag{2.2}$$

where B and Γ are the Beta function and the Gamma function, respectively, (see [9]).
We will use frequently N-dimensional spherical coordinates (r, θ) in $\mathbb{R}^N$:

$$\mathbb{R}^N \ni x = r\theta, \quad \text{where } r = |x|, \text{ and } \theta = x|x|^{-1} \in \mathbb{S}^{N-1}.$$

If M is any set in $\mathbb{R}^N_+$, then χ_M will denote its characteristic function.

Next, let k and l be real numbers satisfying (2.1). We define a measure $\mu_{l,\alpha}$ by

$$d\mu_{l,\alpha}(x) = |x|^l x_N^\alpha \, dx. \tag{2.3}$$

If $M \subset \mathbb{R}^N_+$ is a measurable set with finite $\mu_{l,\alpha}$-measure, then we define $M^\star$, the $\mu_{l,\alpha}$-symmetrization of M, as follows:

$$M^\star := B_R^+ \text{ with } R : \mu_{l,\alpha}\left(B_R^+\right) = \mu_{l,\alpha}(M) = \int_M d\mu_{l,\alpha}(x). \tag{2.4}$$

If $u : \mathbb{R}^N_+ \to \mathbb{R}$ is a measurable function such that

$$\mu_{l,\alpha}(\{|u(x)| > t\}) < \infty \qquad \forall t > 0,$$

then let $u^\star$ denote the weighted Schwarz symmetrization of u, or, in short, the $\mu_{l,\alpha}$–symmetrization of u, which is given by

$$u^\star(x) = \sup\left\{t \geq 0 : \mu_{l,\alpha}(\{|u(x)| > t\}) > \mu_{l,\alpha}\left(B^+_{|x|}\right)\right\}. \tag{2.5}$$

Note that $u^\star$ is radial and radially non-increasing, and if M is a measurable set with finite μ_l-measure, then

$$(\chi_M)^\star = \chi_{M^\star}.$$

The $\mu_{k,\alpha}$–*perimeter* of a measurable set M is given by

$$P_{\mu_{k,\alpha}}(M) := \sup\left\{\int_M \text{div}\left(x_N^\alpha |x|^k \mathbf{v}\right) dx : \mathbf{v} \in C_0^1(\mathbb{R}^N_+, \mathbb{R}^N),\ |\mathbf{v}| \leq 1 \text{ in } M\right\}. \tag{2.6}$$

It is well-known that if M is a smooth set, then

$$P_{\mu_{k,\alpha}}(M) = \int_{\partial M} x_N^\alpha |x|^k \, \mathcal{H}_{N-1}(dx) \tag{2.7}$$

where, here and throughout, $\mathcal{H}_{N-1}$ will denote the $(N-1)$-dimensional Hausdorff-measure.

We will call a set $\Omega \subset \mathbb{R}^N_+$ *smooth*, if for every $x_0 \in \partial\Omega \cap \mathbb{R}^N_+$, there is a number $r > 0$ such that $B_r(x_0) \subset \mathbb{R}^N_+$, $B_r(x_0) \cap \Omega$ has exactly one connected component and $B_r(x_0) \cap \partial\Omega$ is the graph of a C^1–function on an open set in $\mathbb{R}^{N-1}$.

Let $\Omega \subset \mathbb{R}^N_+$ and $p \in [1, +\infty)$. We will denote by $L^p(\Omega, d\mu_{l,\alpha})$ the space of all Lebesgue measurable real valued functions u such that

$$\|u\|_{L^p(\Omega, d\mu_{l,\alpha})} := \left(\int_\Omega |u|^p \, d\mu_{l,\alpha}(x)\right)^{1/p} < +\infty. \tag{2.8}$$

By $W^{1,p}(\Omega, d\mu_{l,\alpha})$ we denote the weighted Sobolev space consisting of all functions which together with their weak derivatives u_{x_i}, $(i = 1, ..., N)$, belong to $L^p(\Omega, d\mu_{l,\alpha})$. This space will be equipped with the norm

$$\|u\|_{W^{1,p}(\Omega, d\mu_{l,\alpha})} := \|u\|_{L^p(\Omega, d\mu_{l,\alpha})} + \|\nabla u\|_{L^p(\Omega, d\mu_{l,\alpha})}. \tag{2.9}$$

Finally, $\mathcal{D}^{1,p}(\Omega, d\mu_{l,\alpha})$ will stand for the closure of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\left(\int_\Omega |\nabla u|^p \, d\mu_{l,\alpha}(x)\right)^{1/p}.$$

We will often use the following well-known *Hardy-Littlewood inequality*

$$\int_{\mathbb{R}^N_+} uv \, d\mu_{l,\alpha}(x) \le \int_{\mathbb{R}^N_+} u^\star v^\star \, d\mu_{l,\alpha}(x), \tag{2.10}$$

which holds for any couple of functions $u, v \in L^2(\mathbb{R}^N_+, d\mu_{l,\alpha})$.

Now let us recall the so-called starshaped rearrangement (see [31]) which we will use in Section 5. For later convenience, we will write y for points in $\mathbb{R}^N_+$ and (z, θ) for corresponding N-dimensional spherical coordinates ($z = |y|$, $\theta = y|y|^{-1}$).

We call a measurable set $M \subset \mathbb{R}^N_+$ *starshaped* if the set

$$M \cap \{z\theta : \, z \ge 0\}$$

is either empty or a segment $\{z\theta : \, 0 \le z < m(\theta)\}$ for some number $m(\theta) > 0$, for almost every $\theta \in \mathbb{S}^{N-1}$.

If M is a bounded measurable set in $\mathbb{R}^N_+$, and $\theta \in \mathbb{S}^{N-1}_+$, then let

$$M(\theta) := M \cap \{z\theta : \, z \ge 0\}.$$

There is a unique number $m(\theta) \in [0, +\infty)$ such that

$$\int_0^{m(\theta)} z^{N-1} \, dz = \int_{M(\theta)} z^{N-1} \, dz.$$

We define

$$\widetilde{M}(\theta) := \{z\theta : \, 0 \le z \le m(\theta)\}, \quad (\theta \in \mathbb{S}^{N-1}_+),$$

and

$$\widetilde{M} := \{z\theta : \, z \in \widetilde{M}(\theta), \, \theta \in \mathbb{S}^{N-1}_+\}.$$

We call the set $\widetilde{M}$ the *starshaped rearrangement of* M.

Note that $\widetilde{M}$ is Lebesgue measurable and starshaped, and we have

$$\mathcal{L}^N(M) = \mathcal{L}^N(\widetilde{M}). \tag{2.11}$$

If $v : \mathbb{R}^N_+ \to \mathbb{R}$ is a measurable function with compact support, and $t \ge 0$, then let E_t be the super-level set $\{y : \, |v(y)| \ge t\}$. We define

$$\widetilde{v}(y) := \sup\{t \ge 0 : \, y \in \widetilde{E_t}\}.$$

We call $\widetilde{v}$ the *starshaped rearrangement of* v . It is easy to verify that $\widetilde{v}$ is equimeasurable with v, that is, the following properties hold:

$$\widetilde{E_t} = \{y : \, \widetilde{v}(y) \ge t\}, \tag{2.12}$$

$$\mathcal{L}^N(E_t) = \mathcal{L}^N(\widetilde{E_t}) \quad \forall t \ge 0. \tag{2.13}$$

This also implies Cavalieri's principle: If $F \in C([0,+\infty))$ with $F(0)=0$ and if $F(v) \in L^1(\mathbb{R}^N)$, then

$$\int_{\mathbb{R}^N} F(v)\,dy = \int_{\mathbb{R}^N} F(\widetilde{v})\,dy \tag{2.14}$$

and if F is non-decreasing, then

$$\widetilde{F(v)} = F(\widetilde{v}). \tag{2.15}$$

Note that the mapping

$$z \longmapsto \widetilde{v}(z\theta), \quad (z \geq 0),$$

is non-increasing for all $\theta \in \mathbb{S}^{N-1}$.
If $v, w \in L^2(\mathbb{R}^N_+)$ are functions with compact support, then there holds Hardy-Littlewood's inequality:

$$\int_{\mathbb{R}^N_+} vw\,dy \leq \int_{\mathbb{R}^N_+} \widetilde{v}\widetilde{w}\,dy. \tag{2.16}$$

If $f : (0,+\infty) \to \mathbb{R}$ is a measurable function with compact support, then its (equimeasurable) *non-increasing rearrangement* , $\widehat{f} : (0,+\infty) \to [0,+\infty)$, is the monotone non-increasing function such that

$$\mathcal{L}^1\{t \in [0,+\infty) : |f(t)| > c\} = \mathcal{L}^1\{t \in [0,+\infty) : \widehat{f}(t) > c\} \quad \forall c \geq 0,$$

see [31], Chapter 2. A general Pólya-Szegö principle for non-increasing rearrangement has been given in [33], Theorem 2.1. For later reference we will only need a special case:

Lemma 2.1. *Let $\delta \geq 0$, and let $f : (0,+\infty) \to \mathbb{R}$ be a bounded, locally Lipschitz continuous function with bounded support, such that*

$$\int_0^{+\infty} t^\delta |f'(t)|\,dt < +\infty.$$

Then $\widehat{f}$ is locally Lipschitz continuous and

$$\int_0^{+\infty} t^\delta |\widehat{f}'(t)|\,dt \leq \int_0^{+\infty} t^\delta |f'(t)|\,dt. \tag{2.17}$$

3. The functionals $\mathcal{R}_{k,l,N,\alpha}$ and $\mathcal{Q}_{k,l,N,\alpha}$

Throughout this section we assume (2.1), i.e.

$$k+N+\alpha-1 > 0 \quad \text{and} \quad l+N+\alpha > 0.$$

If M is any measurable subset of $\mathbb{R}^N_+$, with $0 < \mu_{l,\alpha}(M) < +\infty$, we set

$$\mathcal{R}_{k,l,N,\alpha}(M) := \frac{P_{\mu_{k,\alpha}}(M)}{(\mu_{l,\alpha}(M))^{(k+N+\alpha-1)/(l+N+\alpha)}}. \tag{3.1}$$

Note that

$$\mathcal{R}_{k,l,N,\alpha}(M) = \frac{\displaystyle\int_{\partial M} x_N^\alpha |x|^k \, \mathcal{H}_{N-1}(dx)}{\left(\displaystyle\int_M x_N^\alpha |x|^l \, dx\right)^{(k+N+\alpha-1)/(l+N+\alpha)}} \tag{3.2}$$

if the set M is smooth.

If $u \in C_0^1(\mathbb{R}_+^N) \setminus \{0\}$, we set

$$\mathcal{Q}_{k,l,N,\alpha}(u) := \frac{\displaystyle\int_{\mathbb{R}_+^N} x_N^\alpha |x|^k |\nabla u| \, dx}{\left(\displaystyle\int_{\mathbb{R}_+^N} x_N^\alpha |x|^l |u|^{(l+N+\alpha)/(k+N+\alpha-1)} \, dx\right)^{(k+N+\alpha-1)/(l+N+\alpha)}}. \tag{3.3}$$

Note that the integrals in (3.3) converge due to assumption (2.1).

Finally, we define

$$C_{k,l,N,\alpha}^{rad} := \mathcal{R}_{k,l,N,\alpha}(B_1 \cap \mathbb{R}_+^N). \tag{3.4}$$

We study the following isoperimetric problem:

Find the constant $C_{k,l,N,\alpha} \in [0, +\infty)$, such that

$$C_{k,l,N,\alpha} := \inf\{\mathcal{R}_{k,l,N,\alpha}(M) : \ M \textit{ is measurable with } 0 < \mu_{l,\alpha}(M) < +\infty.\} \tag{3.5}$$

Moreover, we are interested in conditions on k, l and α such that

$$\mathcal{R}_{k,l,N,\alpha}(M) \geq \mathcal{R}_{k,l,N,\alpha}(M^\star) \tag{3.6}$$

holds for all measurable sets $M \subset \mathbb{R}_+^N$ with $0 < \mu_{l,\alpha}(M) < +\infty$.

Let us begin with some immediate observations.

If M is a measurable subset of $\mathbb{R}_+^N$ with finite $\mu_{l,\alpha}$-measure and $\mu_{k,\alpha}$-perimeter, then there exists a sequence of smooth sets $\{M_n\}$ such that

$$\lim_{n\to\infty} \mu_{l,\alpha}(M_n \Delta M) = 0 \ \text{ and } \ \lim_{n\to\infty} P_{\mu_{k,\alpha}}(M_n) = P_{\mu_{k,\alpha}}(M).$$

This property is well-known for Lebesgue measure (see for instance [27], Theorem 1.24) and its proof carries over to the weighted case. This implies that we also have

$$C_{k,l,N,\alpha} = \inf\{\mathcal{R}_{k,l,N,\alpha}(\Omega) : \ \Omega \subset \mathbb{R}_+^N, \ \Omega \text{ smooth}\}. \tag{3.7}$$

The functionals $\mathcal{R}_{k,l,N,\alpha}$ and $\mathcal{Q}_{k,l,N,\alpha}$ have the following homogeneity properties,

$$\mathcal{R}_{k,l,N,\alpha}(M) = \mathcal{R}_{k,l,N,\alpha}(tM), \tag{3.8}$$

$$\mathcal{Q}_{k,l,N,\alpha}(u) = \mathcal{Q}_{k,l,N,\alpha}(u^t), \tag{3.9}$$

where $t>0$, M is a measurable set with $0<\mu_{l,\alpha}(M)<+\infty$, $u\in C_0^1(\mathbb{R}^N_+)\setminus\{0\}$, $tM:=\{tx:\, x\in M\}$ and $u^t(x):=u(tx)$, $(x\in\mathbb{R}^N_+)$, and there holds

$$C^{rad}_{k,l,N,\alpha}=\mathcal{R}_{k,l,N,\alpha}(B_1^+). \tag{3.10}$$

Hence we have that

$$C_{k,l,N,\alpha}\le C^{rad}_{k,l,N,\alpha}, \tag{3.11}$$

and (3.6) holds if and only if

$$C_{k,l,N,\alpha}=C^{rad}_{k,l,N,\alpha}.$$

Finally, we recall the following weighted isoperimetric inequality proved, for example, in [8] (see also [11] and [34]).

Proposition 3.1. *For all measurable sets $M\subset\mathbb{R}^N_+$, with $0<\mu_{0,\alpha}(M)<+\infty$, the following inequality holds true*

$$\mathcal{R}_{0,0,N,\alpha}(M):=\frac{P_{\mu_{0,\alpha}}(M)}{\left(\mu_{0,\alpha}(M)\right)^{(N+\alpha-1)/(N+\alpha)}}\ge C^{rad}_{0,0,N,\alpha}:=\frac{P_{\mu_{0,\alpha}}(M^\star)}{\left(\mu_{0,\alpha}(M^\star)\right)^{(N+\alpha-1)/(N+\alpha)}}\,, \tag{3.12}$$

where $M^\star=B_R^+$ with R such that $\mu_{0,\alpha}(M)=\mu_{0,\alpha}(M^\star)$

We recall that the isoperimetric constant $C^{rad}_{0,0,N,\alpha}$ is explicitly computed in [8], see also [34] for the case $N=2$.

Lemma 3.1. *Let $l>l'>-N-\alpha$. Then*

$$\frac{\left(\mu_{l,\alpha}(M)\right)^{1/(l+N+\alpha)}}{\left(\mu_{l',\alpha}(M)\right)^{1/(l'+N+\alpha)}}\ge\frac{\left(\mu_{l,\alpha}(M^\star)\right)^{1/(l+N+\alpha)}}{\left(\mu_{l',\alpha}(M^\star)\right)^{1/(l'+N+\alpha)}} \tag{3.13}$$

for all measurable sets $M\subset\mathbb{R}^N_+$ with $0<\mu_{l,\alpha}(M)<+\infty$. Equality holds only for half-balls B_R^+, $(R>0)$.

Proof: Let $M^\star$ be the $\mu_{l,\alpha}$-symmetrization of M. Then we obtain, using the Hardy-Littlewood inequality,

$$\begin{aligned}\mu_{l',\alpha}(M)=\int_M x_N^\alpha|x|^{l'}\,dx &= \int_{\mathbb{R}^N_+}|x|^{l'-l}\chi_M(x)\,d\mu_{l,\alpha}(x)\\ &\le \int_{\mathbb{R}^N_+}\left(|x|^{l'-l}\right)^\star(\chi_M)^\star(x)\,d\mu_{l,\alpha}(x)\\ &= \int_{\mathbb{R}^N_+}|x|^{l'-l}\chi_{M^\star}(x)\,d\mu_{l,\alpha}(x)\\ &= \int_{M^\star}x_N^\alpha|x|^{l'}\,dx=\mu_{l',\alpha}(M^\star).\end{aligned}$$

This implies (3.13).

Next assume that equality holds in (3.13). Then we must have

$$\int_M |x|^{l'-l}\, d\mu_{l,\alpha}(x) = \int_{M^\star} |x|^{l'-l} d\mu_{l,\alpha}(x),$$

that is,

$$\int_{M\setminus M^\star} |x|^{l'-l}\, d\mu_{l,\alpha}(x) = \int_{M^\star\setminus M} |x|^{l'-l} d\mu_{l,\alpha}(x).$$

Since $l'-l<0$, this means that $\mu_l(M\Delta M^\star)=0$. The Lemma is proved. □

Lemma 3.2. *Let k,l,α satisfy (2.1). Assume that $l>l'>-N-\alpha$ and $C_{k,l,N,\alpha}=C^{rad}_{k,l,N,\alpha}$. Then we also have $C_{k,l',N,\alpha}=C^{rad}_{k,l',N,\alpha}$. Moreover, if $\mathcal{R}_{k,l',N,\alpha}(M)=C^{rad}_{k,l',N,\alpha}$ for some measurable set $M\subset\mathbb{R}^N_+$, with $0<\mu_{l',\alpha}(M)<+\infty$, then $M=B^+_R$ for some $R>0$.*

Proof: By our assumptions and Lemma 3.1 we have for every measurable set M with $0<\mu_{l,\alpha}(M)<+\infty$,

$$\begin{aligned}\mathcal{R}_{k,l',N,\alpha}(M) &= \mathcal{R}_{k,l,N,\alpha}(M)\cdot\left[\frac{(\mu_{l,\alpha}(M))^{1/(l+N+\alpha)}}{(\mu_{l',\alpha}(M))^{1/(l'+N+\alpha)}}\right]^{k+N+\alpha-1}\\ &\geq C^{rad}_{k,l',N,\alpha},\end{aligned}$$

with equality only if $M=B^+_R$ for some $R>0$. □

Lemma 3.3. *Assume that $k\leq l+1$. Then*

$$C_{k,l,N,\alpha}=\inf\left\{\mathcal{Q}_{k,l,N,\alpha}(u):\ u\in C^1_0(\mathbb{R}^N_+)\setminus\{0\}\right\}. \tag{3.14}$$

Proof: The proof uses classical arguments (see, e.g. [25]). We may restrict ourselves to nonnegative functions u. By (3.5) and the coarea formula we obtain,

$$\begin{aligned}\int_{\mathbb{R}^N_+} x_N^\alpha|x|^k|\nabla u|\,dx &= \int_0^\infty\int_{u=t} x_N^\alpha|x|^k\,\mathcal{H}_{N-1}(dx)\,dt\\ &\geq C_{k,l,N,\alpha}\int_0^\infty\left(\int_{u>t}x_N^\alpha|x|^l\,dx\right)^{(k+N+\alpha-1)/(l+N+\alpha)} dt.\end{aligned} \tag{3.15}$$

Further, Cavalieri's principle gives

$$u(x)=\int_0^\infty \chi_{\{u>t\}}(x)\,dt,\quad (x\in\mathbb{R}^N). \tag{3.16}$$

Hence (3.16) and Minkowski's inequality for integrals (see [40]) lead to

$$\begin{aligned}
&\int_{\mathbb{R}^N_+} x_N^\alpha |x|^l |u|^{(l+N+\alpha)/(k+N+\alpha-1)}\, dx \\
= \;& \int_{\mathbb{R}^N_+} x_N^\alpha |x|^l \left| \int_0^\infty \chi_{\{u>t\}}(x)\, dt \right|^{(l+N+\alpha)/(k+N+\alpha-1)} dx \\
\leq \;& \left(\int_0^\infty \left(\int_{\mathbb{R}^N_+} x_N^\alpha |x|^l \chi_{\{u>t\}}(x)\, dx \right)^{(k+N+\alpha-1)/(l+N+\alpha)} dt \right)^{(l+N+\alpha)/(k+N+\alpha-1)} \\
= \;& \left(\int_0^\infty \left(\int_{u>t} x_N^\alpha |x|^l \, dx \right)^{(k+N+\alpha-1)/(l+N+\alpha)} dt \right)^{(l+N+\alpha)/(k+N+\alpha-1)} .
\end{aligned} \tag{3.17}$$

Now (3.15) and (3.17) yield

$$\mathcal{Q}_{k,l,N,\alpha}(u) \geq C_{k,l,N,\alpha} \quad \forall u \in C_0^1 \setminus \{0\}(\mathbb{R}^N_+). \tag{3.18}$$

To show (3.14), let $\varepsilon > 0$, and choose a smooth set Ω such that

$$\mathcal{R}_{k,l,N,\alpha}(\Omega) \leq C_{k,l,N,\alpha} + \varepsilon. \tag{3.19}$$

It is well-known that there exists a sequence $\{u_n\} \subset C_0^\infty(\mathbb{R}^N) \setminus \{0\}$ such that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N_+} x_N^\alpha |x|^k |\nabla u_n|\, dx = \int_{\partial\Omega} x_N^\alpha |x|^k \, \mathcal{H}_{N-1}(dx), \tag{3.20}$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^N_+} x_N^\alpha |x|^l |u_n|^{(l+N+\alpha)/(k+N+\alpha-1)}\, dx = \int_\Omega x_N^\alpha |x|^l \, dx. \tag{3.21}$$

To do this, one may choose mollifiers of χ_Ω as u_n (see e.g. [41]). Hence, for large enough n we have

$$\mathcal{Q}_{k,l,N,\alpha}(u_n) \leq C_{k,l,N,\alpha} + 2\varepsilon. \tag{3.22}$$

Since ε was arbitrary, (3.14) now follows from (3.18) and (3.22). □

4. Necessary conditions

In this section we assume that

$$k + N + \alpha - 1 > 0 \;\text{ and }\; l + N + \alpha > 0.$$

The main result is Theorem 4.1 which highlights the phenomenon of symmetry breaking. The following result holds true.

Lemma 4.1. *A necessary condition for*

$$C_{k,l,N,\alpha} > 0 \tag{4.1}$$

is

$$l\frac{N+\alpha-1}{N+\alpha} \le k. \tag{4.2}$$

Proof: Assume that $k < l(N+\alpha-1)/(N+\alpha)$, and let $te_1 = (t, 0, \dots, 0)$, $(t > 2)$. Since for any $x \in B_1(te_1)$, it results $t-1 \le |x| \le t+1$, we have

$$\mathcal{R}_{k,l,N,\alpha}(B_1(te_1)) \le D\frac{(t+1)^k}{(t-1)^{l(k+N+\alpha-1)/(l+N+\alpha)}}.$$

where the positive constant $D = D(k,l,N,\alpha)$ is given by

$$D = \frac{\displaystyle\int_{\partial B_1(te_1)\cap\mathbb{R}^N_+} x_N^\alpha\, \mathcal{H}_{N-1}(dx)}{\left(\displaystyle\int_{B_1(te_1)\cap\mathbb{R}^N_+} x_N^\alpha\, dx\right)^{(k+N+\alpha-1)/(l+N+\alpha)}}$$

Since $k - l(k+N+\alpha-1)/(l+N+\alpha) < 0$, it follows that

$$\lim_{t\to\infty} \mathcal{R}_{k,l,N,\alpha}(B_1(te_1)) = 0.$$

□

Theorem 4.1. *A necessary condition for*

$$C_{k,l,N,\alpha} = C^{rad}_{k,l,N,\alpha} \tag{4.3}$$

is

$$l+1 \le k + \frac{N+\alpha-1}{k+N+\alpha-1}. \tag{4.4}$$

Remark 4.1. Theorem 4.1 means that if $l+1 \le k + \frac{N+\alpha-1}{k+N+\alpha-1}$, then symmetry breaking occurs, that is $C_{k,l,N,\alpha} < C^{rad}_{k,l,N,\alpha}$. Our proof relies on the fact that the second variation of the perimeter for smooth volume-preserving perturbations from the ball B_1^+ is non-negative if and only if (4.4) holds. Note that this also follows from a general second variation formula with volume and perimeter densities, see [38].

Proof: First we assume $N \ge 2$. Let (r,θ) denote N–dimensional spherical coordinates, such that

$$\theta_1 = \arccos\frac{x_N}{|x|}, \quad \theta_1 \in [0, \pi/2],$$

and $u \in C^2(\mathbb{S}^{N-1}_+)$, $s \in C^2(\mathbb{R})$ with $s(0) = 0$, and define

$$U(t) := \{x = r\theta \in \mathbb{R}^N_+ : \, 0 \le r < 1 + tu(\theta) + s(t)\}, \quad (t \in \mathbb{R}).$$

Note that $U(0)=B_1^+$. By the Implicit Function Theorem, we may choose s in such a way that

$$\int_{U(t)} x_N^\alpha |x|^l\,dx = \int_{B_1^+} x_N^\alpha |x|^l\,dx \quad \text{for } |t|<t_0, \tag{4.5}$$

for some number $t_0>0$. We set $s_1:=s'(0)$ and $s_2:=s''(0)$. Let $d\Theta$ be the surface element on the sphere and

$$h:=h(\theta_1)=\cos^\alpha\theta_1=\left(\frac{x_N}{|x|}\right)^\alpha. \tag{4.6}$$

Since

$$\int_{U(t)} x_N^\alpha |x|^l\,dx = \int_{\mathbb{S}_+^{N-1}} h\int_0^{1+tu(\theta)+s(t)} \rho^{l+N+\alpha-1}\,d\rho\,d\Theta,$$

a differentiation at $t=0$ of (4.5) leads to

$$\begin{aligned} 0 &= \int_{\mathbb{S}_+^{N-1}} (u+s_1)\,hd\Theta \quad \text{and} && (4.7)\\ 0 &= (l+N+\alpha-1)\int_{\mathbb{S}_+^{N-1}} (u+s_1)^2 h\,d\Theta + s_2\int_{\mathbb{S}_+^{N-1}} h\,d\Theta. && (4.8)\end{aligned}$$

Next we consider the perimeter functional

$$\begin{aligned} J(t) &:= \int_{\partial U(t)} x_N^\alpha |x|^k\,\mathcal{H}_{N-1}(dx) && (4.9)\\ &= \int_{\mathbb{S}_+^{N-1}} (1+tu+s(t))^{k+N+\alpha-2}\sqrt{(1+tu+s(t))^2+t^2|\nabla_\theta u|^2}\,h\,d\Theta, \end{aligned}$$

where ∇_θ denotes the gradient on the sphere. Differentiation at $t=0$ of (4.9) leads to

$$\begin{aligned} J'(0) &= (k+N+\alpha-1)\int_{\mathbb{S}_+^{N-1}} (u+s_1)\,h\,d\Theta, \quad \text{and}\\ J''(0) &= (k+N+\alpha-2)(k+N+\alpha-1)\int_{\mathbb{S}_+^{N-1}} (u+s_1)^2\,h\,d\Theta +\\ &\quad +(k+N+\alpha-1)s_2\int_{\mathbb{S}_+^{N-1}} h\,d\Theta + \int_{\mathbb{S}_+^{N-1}} |\nabla_\theta u|^2\,h\,d\Theta. \end{aligned}$$

By (4.7) and (4.8) this implies

$$J'(0)=0, \tag{4.10}$$

and

$$J''(0)=(k+N+\alpha-1)(k-l-1)\int_{\mathbb{S}_+^{N-1}} (u+s_1)^2\,h\,d\Theta + \int_{\mathbb{S}_+^{N-1}} |\nabla_\theta u|^2\,h\,d\Theta. \tag{4.11}$$

Now assume that (4.3) holds. Then we have $\mathcal{R}_{k,l,N,\alpha}(U(t)) \geq \mathcal{R}_{k,l,N,\alpha}(B_1^+)$ for all t with $|t| < t_0$. In view of (4.5) this means that $J(t) \geq J(0)$ for $|t| < t_0$, that is,

$$J''(0) \geq 0 = J'(0). \tag{4.12}$$

The second condition is (4.10), and the first condition implies, in view of (4.7) and (4.11), that

$$\begin{aligned} 0 \; &\leq \; (k+N+\alpha-1)(k-l-1)\int_{\mathbb{S}_+^{N-1}} v^2\, h\, d\Theta + \int_{\mathbb{S}_+^{N-1}} |\nabla_\theta v|^2\, h\, d\Theta \\ &\forall v \in C^2(\mathbb{S}_+^{N-1}) \;\text{ with }\; \int_{\mathbb{S}_+^{N-1}} v\, h\, d\Theta = 0. \end{aligned} \tag{4.13}$$

Applying Proposition 2.1 in [8], we get

$$\int_{\mathbb{S}_+^{N-1}} |\nabla_\theta v|^2\, h\, d\Theta \geq (N+\alpha-1)\int_{\mathbb{S}_+^{N-1}} v^2\, h\, d\Theta$$

for any $v \in C^2(\mathbb{S}_+^{N-1})$ with $\displaystyle\int_{\mathbb{S}_+^{N-1}} hv\, d\Theta = 0$. The conclusion follows. □

5. The case of negative α

In this section we firstly show that the relative isoperimetric problem in $\mathbb{R}_+^2$ for $\alpha \in (-1,0)$ and $k = l = 0$ has no solution. Nevertheless, in Theorem 5.2, we prove that, the second variation of the perimeter w.r.t. volume-preserving smooth perturbations at the half circle is nonnegative for such values of the parameters.
Throughout this section the points in $\mathbb{R}_+^2$ will be simply denoted by (x,y).

Theorem 5.1. *Let*

$$N = 2,\; \alpha \in (-1,0) \;\text{ and }\; k = l = 0. \tag{5.1}$$

Then there is no constant $C \in (0,+\infty)$ such that

$$\int_{\partial\Omega\setminus\{y=0\}} y^\alpha dl \geq C \left(\int_\Omega y^\alpha dxdy \right)^{\frac{\alpha+1}{\alpha+2}}, \text{ for any set } \Omega \subset \mathbb{R}_+^2.$$

Proof: Let $0 < a < b$ and

$$\Omega_{a,b} := \left\{ (x,y) \in \mathbb{R}_+^2 : 0 < x < 1,\; a < y < b \right\}.$$

We have

$$A_\alpha(\Omega_{a,b}) := \int_{\Omega_{a,b}} y^\alpha dxdy = \int_a^b t^\alpha dt = \frac{b^{\alpha+1} - a^{\alpha+1}}{\alpha+1}.$$

while

$$P_\alpha(\Omega_{a,b}) := \int_{\partial\Omega_{a,b}} y^\alpha dl = 2\int_a^b t^\alpha dt + a^\alpha + b^\alpha = \frac{2}{\alpha+1}\left(b^{\alpha+1} - a^{\alpha+1}\right) + a^\alpha + b^\alpha.$$

Setting

$$U := a^{\alpha+1}, \quad V := b^{\alpha+1} - a^{\alpha+1} \qquad (U, V > 0)$$

we have

$$A_\alpha(\Omega_{a,b}) = \frac{V}{\alpha+1} \quad \text{and} \quad P_\alpha(\Omega_{a,b}) = \frac{2}{\alpha+1}V + U^{\frac{\alpha}{\alpha+1}} + (U+V)^{\frac{\alpha}{\alpha+1}}.$$

In order to conclude to proof we claim that $\forall \epsilon > 0 \ \exists \ 0 < a < b$ such that

$$R_\alpha(\Omega_{a,b}) \equiv \frac{P_\alpha(\Omega_{a,b})}{\left[A_\alpha(\Omega_{a,b})\right]^{\frac{\alpha+1}{\alpha+2}}} < \epsilon.$$

First choose V small enough to have

$$2(\alpha+1)^{-\frac{1}{\alpha+1}} \; V^{\frac{1}{\alpha+2}} < \frac{\epsilon}{2}$$

and then U large enough to have

$$\frac{U^{\frac{\alpha}{\alpha+1}} + (U+V)^{\frac{\alpha}{\alpha+1}}}{\left(\frac{1}{\alpha+1}\right)^{\frac{\alpha+1}{\alpha+2}} V^{\frac{\alpha+1}{\alpha+2}}} < \frac{\epsilon}{2}.$$

Then

$$R_\alpha(\Omega_{a,b}) = 2(\alpha+1)^{-\frac{1}{\alpha+1}} \; V^{\frac{1}{\alpha+2}} + \frac{U^{\frac{\alpha}{\alpha+1}} + (U+V)^{\frac{\alpha}{\alpha+1}}}{\left(\frac{1}{\alpha+1}\right)^{\frac{\alpha+1}{\alpha+2}} V^{\frac{\alpha+1}{\alpha+2}}} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

Now let $\alpha \in (-1, 0)$ and consider the measure $d\nu = \cos^\alpha t\, dt$. We introduce the weighted Sobolev space $H^1\left(\left(-\frac{\pi}{2}, \frac{\pi}{2}\right); d\nu\right)$ which is made of functions $\phi : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ such that

$$\begin{aligned} \|\phi\|^2_{H^1\left(\left(-\frac{\pi}{2},\frac{\pi}{2}\right); d\nu\right)} &= \|\phi\|^2_{L^2\left(\left(-\frac{\pi}{2},\frac{\pi}{2}\right); d\nu\right)} + \|\phi'\|^2_{L^2\left(\left(-\frac{\pi}{2},\frac{\pi}{2}\right); d\nu\right)} \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \phi(t)^2\, d\nu + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \phi'(t)^2\, d\nu < \infty. \end{aligned}$$

Finally let

$$V := \left\{ \phi \in H^1\left(\left(-\frac{\pi}{2}, \frac{\pi}{2}\right); d\nu\right) : \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \phi\, d\nu = 0 \right\}.$$

In the following Lemma we prove that V is compactly embedded in $L^2\left(\left(-\frac{\pi}{2}, \frac{\pi}{2}\right); d\nu\right)$.

Lemma 5.1. *If* $\{w_n\}_{n\in N} \subset V$ *is such that*

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} w_n'(t)^2 \, d\nu \leq C \quad \forall n \in \mathbb{N}$$

then there exists $w \in V$ *such that there holds*

$$\lim_{n\to\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |w_n(t) - w(t)|^2 \, d\nu = 0.$$

Proof: Note that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} w_n'(t)^2 dt \leq \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} w_n'(t)^2 \cos^\alpha t dt \leq C \quad \forall n \in \mathbb{N}.$$

By the definition of V we can infer that for each $n \in \mathbb{N}$, there exists $t_n \in (-\frac{\pi}{2}, \frac{\pi}{2})$ such that, up to a subsequence, $w_n(t_n) = 0$. So we have

$$w_n(t) = \int_{t_n}^{t} w_n'(\sigma) d\sigma$$

and therefore

$$|w_n(t)|^2 \leq \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |w_n'(\sigma)| \, d\sigma \right)^2 \leq \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |w_n'(\sigma)|^2 \, d\sigma \leq C \quad \forall n \in \mathbb{N}.$$

So w_n is bounded in $H^1\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and, therefore, there exists $w \in C^0\left(\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right) \cap H^1\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that, up to a subsequence,

$$w_n(t) \to w(t) \text{ uniformly in } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

The assertion easily follows, since

$$\cos^\alpha t \in L^1\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \quad \forall \alpha \in (-1, 0).$$

□

Now define the Rayleigh quotient

$$Q(v) := \frac{\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} v'(t)^2 \cos^\alpha t dt}{\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} v(t)^2 \cos^\alpha t dt}, \quad \text{with} \quad v \in V.$$

Lemma 5.2. *There holds*

$$\mu := \min_{\phi \in V} Q(v) = 1 + \alpha.$$

Proof: Note that $\sin t \in V$. An integration by parts gives

$$Q(\sin t) = \frac{\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{\alpha+2} t dt}{\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 t \cos^\alpha t dt} = \frac{(\alpha+1)\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 t \cos^\alpha t dt}{\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 t \cos^\alpha t dt} = \alpha + 1, \tag{5.2}$$

and, therefore

$$\mu \le \alpha + 1.$$

Now, by contradiction, assume that

$$\mu < 1 + \alpha.$$

By Lemma 5.1 there exists a function $u \in V$ such that $Q(u) = \mu$ which satisfies the Euler equation

$$-\left(u' \cos^\alpha(t)\right)' = \mu u \cos^\alpha(t) \quad \text{on} \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \tag{5.3}$$

We set

$$R(v) := \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} v'(t)^2 \, d\nu - \mu \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} v(t)^2 \, d\nu, \quad v \in V,$$

and

$$u_1(t) = \frac{u(t) - u(-t)}{2}, \quad u_2(t) = \frac{u(t) + u(-t)}{2}.$$

We have

$$R(u) = R(u_1) + R(u_2) = 0.$$

Hence at least one of the following statements must be true

(i) $$R(u_1) \le 0,$$

or

(ii) $$R(u_2) \le 0.$$

Our aim is to reach a contradiction by showing that (i) and (ii) are both false.

Case (i): Assume $R(u_1) \le 0$.
Since u_1 is odd we have

$$v_1 := \frac{u_1(t)}{\sin t} \in C^1\left(\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right)$$

and

$$R(u_1) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(v_1' \sin t + v_1 \cos t\right)^2 \cos^\alpha t dt - \mu \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} v_1^2 \sin^2 t \cos^\alpha t dt =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2v_1' v_1 \sin t \cos^{\alpha+1} t dt + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (v_1')^2 \sin^2 t \cos^\alpha t dt +$$

$$+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} v_1^2\cos^{\alpha+2} tdt + -\mu\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} v_1^2\sin^2 t\cos^{\alpha} tdt$$

$$= (\alpha+1)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} v_1^2\sin^2 t\cos^{\alpha} tdt - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} v_1^2\cos^{\alpha+2} tdt +$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (v_1')^2\sin^2 t\cos^{\alpha} tdt + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} v_1^2\cos^{\alpha+2} tdt - \mu\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} v_1^2\sin^2 t\cos^{\alpha} tdt$$

Recalling the assumption $\alpha+1-\mu>0$, we have

$$\begin{aligned} R(u_1) &= (\alpha+1)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} v_1^2\sin^2 t\cos^{\alpha} tdt + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (v_1')^2\sin^2 t\cos^{\alpha} tdt - \mu\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} v_1^2\sin^2 t\cos^{\alpha} tdt \\ &= (\alpha+1-\mu)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} v_1^2\sin^2 t\cos^{\alpha} tdt + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (v_1')^2\sin^2 t\cos^{\alpha} tdt \geq 0, \end{aligned}$$

where equality holds if and only if $\mu=\alpha+1$ and v_1 is a constant. This contradicts our assumption.

Case (ii): Assume $R(u_2)\leq 0$.
Since u_2 is even function belonging to V, we have

$$0=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u_2\cos^{\alpha} tdt = 2\int_{0}^{\frac{\pi}{2}} u_2\cos^{\alpha} tdt.$$

Then there exists $c\in\left(0,\frac{\pi}{2}\right)$ such that

$$u_2(c)=u_2(-c)=0.$$

From (5.3) we deduce that

$$\int_{-c}^{c}(u_2')^2\cos^{\alpha} tdt = -\int_{-c}^{c} u_2\left(u_2'\cos^{\alpha} t\right)' dt = \mu\int_{-c}^{c} u_2^2\cos^{\alpha} tdt. \tag{5.4}$$

On the other hand, setting

$$v_2 := u_2\cos^{\frac{\alpha}{2}} t,$$

we obtain from (5.4)

$$\begin{aligned} \int_{-c}^{c}(u_2')^2\cos^{\alpha} tdt &= \int_{-c}^{c}\left(v_2'\cos^{-\frac{\alpha}{2}} t + \frac{\alpha}{2}v_2\cos^{-\frac{\alpha}{2}-1} t\sin t\right)^2\cos^{\alpha} tdt \\ &= \int_{-c}^{c}(v_2')^2 dt + \alpha\int_{-c}^{c} v_2v_2'\tan tdt + \frac{\alpha^2}{4}\int_{-c}^{c} v_2^2\tan^2 tdt. \end{aligned} \tag{5.5}$$

Since $v_2(\pm c) = 0$ and $v_2 \in C^1[-c, c]$, the classical one-dimensional Wirtinger inequality implies that

$$\int_{-c}^{c} (v_2')^2\, dt \geq \left(\frac{\pi}{2c}\right)^2 \int_{-c}^{c} v_2^2 dt, \tag{5.6}$$

where equality holds if and only if v_2 is proportional to $\sin\left(\frac{\pi t}{2c}\right)$

Inequalities (5.4) and (5.6) ensure

$$\begin{aligned}\int_{-c}^{c} (u_2')^2 \cos^\alpha t dt &\geq \left(\frac{\pi}{2c}\right)^2 \int_{-c}^{c} v_2^2 dt \\ &\quad -\frac{\alpha}{2}\int_{-c}^{c} v_2^2 \left(1+\tan^2 t\right) dt + \frac{\alpha^2}{4}\int_{-c}^{c} v_2^2 \tan^2 t dt \\ &= \left(\frac{\pi^2}{4c^2} - \frac{\alpha}{2}\right)\int_{-c}^{c} v_2^2 dt + \left(\frac{\alpha^2}{4} - \frac{\alpha}{2}\right)\int_{-c}^{c} v_2^2 \tan^2 t dt \\ &> \left(\frac{\pi^2}{4c^2} - \frac{\alpha}{2}\right)\int_{-c}^{c} v_2^2 dt \\ &= \left(\frac{\pi^2}{4c^2} - \frac{\alpha}{2}\right)\int_{-c}^{c} u_2^2 \cos^\alpha t dt.\end{aligned} \tag{5.7}$$

Finally equation (5.3) implies

$$1 + \alpha > \mu > \frac{\pi^2}{4c^2} - \frac{\alpha}{2} \geq 1 - \frac{\alpha}{2}$$

and therefore $\frac{3}{2}\alpha > 0$, a contradiction. □

Theorem 5.2. *Let $N = 2$, $\alpha \in (-1, 0)$ and $k = l = 0$. Then the functional J defined in (4.9), satisfies $J''(0) \geq 0$.*

Proof: The assertion follows from Lemma 5.2 and taking into account (4.11). □

6. Main results

This section is devoted to the proof of Theorem 1.1, that is, we obtain sufficient conditions on k, l and N such that $C_{k,l,N,\alpha} = C^{rad}_{k,l,N,,\alpha}$ holds, or equivalently,

$$\mathcal{R}_{k,l,N,\alpha}(M) \geq C^{rad}_{k,l,N,\alpha} \quad \text{for all measurable sets } M \subset \mathbb{R}^N_+ \text{ with } 0 < \mu_{l,\alpha}(M) < +\infty. \tag{6.1}$$

Proofs of Theorem 1.1, cases **(ii)**, **(iii)**, are given in the following two subsections 6.1 and 6.2.

First let us recall that the proof of case (i) of Theorem 1.1 has been given in [2].

Remark 6.1. *Condition (4.2), i.e.* $l\frac{N+\alpha-1}{N+\alpha} \le k$ *is a necessary and sufficient condition for* $C_{k,l,N,\alpha} > 0$*.*

Proof: The necessity follows from Lemma 4.1, and the sufficiency in the case $l+1 \le k$ follows from case (i) in Theorem 1.1. Finally, assume that $k < l+1$. Then (3.5) is equivalent to (3.14), by Lemma 3.3. Now the main Theorem of [12] tells us that condition (4.2) is also sufficient for $C_{k,l,N,\alpha} > 0$. □

6.1. **Proof of Theorem 1.1, case (ii).** The case $k \le 0$ and $\alpha = 0$ has been addressed in [18], Theorem 1.3. We significantly extend such a result by considering all nonnegative values of α and treating, at least for some values of the parameters, the equality case in (4.3).

Theorem 6.1. *Let* k, l *satisfy*

$$l\frac{N+\alpha-1}{N+\alpha} \le k \le \min\{0, l+1\}. \tag{6.2}$$

Then (4.3) holds. Moreover if $l\frac{N+\alpha-1}{N+\alpha} < k$ *and*

$$\mathcal{R}_{k,l,N,\alpha}(M) = C^{rad}_{k,l,N,\alpha} \text{ for some measurable set } M \text{ with } 0 < \mu_l(M) < +\infty, \tag{6.3}$$

then $M = B_R^+$ *for some* $R > 0$*.*

Proof : Let $u \in C_0^\infty(\mathbb{R}^N_+) \setminus \{0\}$. We set

$$y := x|x|^{\frac{k}{N+\alpha-1}}, \quad v(y) := u(x), \quad s := r^{\frac{k+N+\alpha-1}{N+\alpha-1}}.$$

Using N-dimensional spherical coordinates, denoting with ∇_θ the tangential part of the gradient on $\mathbb{S}^{N-1}$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N_+} x_N^\alpha |x|^l |u|^{(l+N+\alpha)/(k+N+\alpha-1)}\, dx \\
&= \int_{\mathbb{S}^{N-1}_+} \int_0^\infty r^{l+N+\alpha-1} |u|^{(l+N+\alpha)/(k+N+\alpha-1)}\, h dr\, d\Theta \\
&= \frac{N+\alpha-1}{k+N+\alpha-1} \int_{\mathbb{S}^{N-1}_+} \int_0^\infty s^{\frac{l+N+\alpha}{k+N+\alpha-1}(N+\alpha-1)-1} |v|^{(l+N+\alpha)/(k+N+\alpha-1)}\, h ds\, d\Theta \\
&= \frac{N+\alpha-1}{k+N+\alpha-1} \int_{\mathbb{R}^N_+} y_n^\alpha |y|^{\frac{l+N+\alpha}{k+N+\alpha-1}(N+\alpha-1)-N} |v|^{(l+N+\alpha)/(k+N+\alpha-1)}\, dy \\
&= \frac{N+\alpha-1}{k+N+\alpha-1} \int_{\mathbb{R}^N_+} |y|^{(l(N+\alpha-1)-k(N+\alpha))/(k+N+\alpha-1)} |v|^{(l+N+\alpha)/(k+N+\alpha-1)}\, dy\,.
\end{aligned} \tag{6.4}$$

Further we calculate

$$
\begin{aligned}
(6.5)\quad & \int_{\mathbb{R}^N_+} x_N^\alpha |x|^k |\nabla_x u|\, dx \\
&= \int_{\mathbb{S}^{N-1}_+} \int_0^\infty r^{k+N+\alpha-1} \left(u_r^2 + \frac{|\nabla_\theta u|^2}{r^2} \right)^{1/2} h\, dr\, d\Theta \\
&= \int_{\mathbb{S}^{N-1}} \int_0^\infty s^{N+\alpha-1} \left(v_s^2 + \frac{|\nabla_\theta v|^2}{s^2} \left(\frac{N+\alpha-1}{k+N+\alpha-1} \right)^2 \right)^{1/2} h\, ds\, d\Theta \\
&\geq \int_{\mathbb{S}^{N-1}} \int_0^\infty s^{N+\alpha-1} \left(v_s^2 + \frac{|\nabla_\theta v|^2}{s^2} \right)^{1/2} h\, ds\, d\Theta \\
&= \int_{\mathbb{R}^N_+} y_N^\alpha |\nabla_y v|\, dy\,,
\end{aligned}
$$

where we have used (6.2). By (6.4) and (6.5) we deduce,

$$
\begin{aligned}
(6.6)\quad & \mathcal{Q}_{k,l,N,\alpha}(u) \\
\geq\ & \frac{\int_{\mathbb{R}^N_+} y_N^\alpha |\nabla_y v|\, dy}{\left(\int_{\mathbb{R}^N_+} y_N^\alpha |y|^{l'} |v|^{(l+N+\alpha)/(k+N+\alpha-1)}\, dy \right)^{(k+N+\alpha-1)/(l+N+\alpha)}} \left(\frac{k+N+\alpha-1}{N+\alpha-1} \right)^{(k+N+\alpha-1)/(l+N+\alpha)} \\
=\ & \left(\frac{k+N+\alpha-1}{N+\alpha-1} \right)^{(k+N+\alpha-1)/(l+N+\alpha)} \mathcal{Q}_{0,l',N,\alpha}(v)\,,
\end{aligned}
$$

where we have set $l' := \frac{l(N+\alpha-1)-k(N+\alpha)}{k+N+\alpha-1}$. Note that we have $-1 \leq l' \leq 0$ by the assumptions (6.2).

Hence we may apply Lemma 3.3 to both sides of (6.6). This yields

$$
(6.7)\qquad C_{k,l,N,\alpha} \geq \left(\frac{k+N+\alpha-1}{N+\alpha-1} \right)^{(k+N+\alpha-1)/(l+N+\alpha)} C_{0,l',N,\alpha}.
$$

Furthermore, Lemma 3.2 tells us that

$$
(6.8)\qquad C_{0,l',N,\alpha} = C^{rad}_{0,l',N,\alpha}.
$$

Since also

$$
\left(\frac{k+N+\alpha-1}{N+\alpha-1} \right)^{(k+N+\alpha-1)/(l+N+\alpha)} C^{rad}_{0,l',N,\alpha} = C^{rad}_{k,l,N,\alpha}\,.
$$

From this, (6.7) and (6.8), we deduce that $C_{k,l,N,\alpha} \geq C^{rad}_{k,l,N,\alpha}$. Since $C_{k,l,N,\alpha} \leq C^{rad}_{k,l,N,\alpha}$ by definition, (4.3) follows.

Next assume that $\mathcal{R}_{k,l,N,\alpha}(M) = C^{rad}_{k,l,N,\alpha}$ for some measurable set $M \subset \mathbb{R}^N_+$ with $0 < \mu_l(M) <$

$+\infty$. If $l(N+\alpha-1)/(N+\alpha)<k$, then Lemma 3.2 tells us that we must have $M=B_R^+$ for some $R>0$. □

Remark 6.2.
(a) A well-known special case of Theorem 6.1 is $k=0=l$, see [34], [7] and [11].
(b) The idea to use spherical coordinates, and in particular the inequality (6.5) in our last proof, appeared already in some work of T. Horiuchi, see [28] and [29].

6.2. **Proof of Theorem 1.1, case (iii).** Now we treat the case when k assumes non-negative values. Throughout this subsection we assume $k\le l+1$. The main result is Theorem 6.2. Its proof is long and requires some auxiliary results. But the crucial idea is an interpolation argument that occurs in the proof of the following Lemma 6.1, formula (6.11).

Lemma 6.1. *Assume $l(N+\alpha-1)/(N+\alpha)\le k$ and $k\ge 0$. Let $u\in C_0^1(\mathbb{R}_+^N)\setminus\{0\}$, $u\ge 0$, and define y,z and v by*

$$y:=x|x|^{\frac{k}{N+\alpha-1}},\ z:=|y| \quad \text{and} \quad v(y):=u(x), \qquad x\in\mathbb{R}_+^N. \tag{6.9}$$

Then for every $A\in\left[0,\frac{(N+\alpha-1)^2}{(k+N+\alpha-1)^2}\right]$,

$$\mathcal{Q}_{k,l,N,\alpha}(u)\ge\left(\frac{k+N+\alpha-1}{N+\alpha-1}\right)^{\frac{k+N+\alpha-1}{l+N+\alpha}}\cdot\frac{\left(\int_{\mathbb{R}_+^N}y_N^\alpha|\nabla_y v|\,dy\right)^A\cdot\left(\int_{\mathbb{R}_+^N}y_N^\alpha|v_z|\,dy\right)^{1-A}}{\left(\int_{\mathbb{R}_+^N}y_N^\alpha|y|^{\frac{l(N+\alpha-1)-k(N+\alpha)}{k+N+\alpha-1}}v^{\frac{l+N+\alpha}{k+N+\alpha-1}}\,dy\right)^{\frac{k+N+\alpha-1}{l+N+\alpha}}}. \tag{6.10}$$

Proof: We calculate as in the proof of Theorem 6.1 ,

$$\int_{\mathbb{R}_+^N}x_N^\alpha|x|^k|\nabla_x u|\,dx=\int_{\mathbb{S}_+^{N-1}}\int_0^\infty s^{N+\alpha-1}\left(v_s^2+\frac{|\nabla_\theta v|^2}{s^2}\left(\frac{N+\alpha-1}{k+N+\alpha-1}\right)^2\right)^{1/2}h\,ds\,d\Theta$$

Since the mapping

$$t\longmapsto\log\left(\int_{\mathbb{S}_+^{N-1}}\int_0^{+\infty}z^{N+\alpha-1}\sqrt{v_z^2+t\frac{|\nabla_\theta v|^2}{z^2}}\,h\,dz\,d\Theta\right)$$

is concave, we deduce that for every $A \in \left[0, \frac{(N+\alpha-1)^2}{(k+N+\alpha-1)^2}\right]$,

$$
\begin{aligned}
(6.11) & \int_{\mathbb{R}^N_+} x_N^\alpha |x|^k |\nabla_x u| \, dx \\
\geq & \left(\int_{\mathbb{S}^{N-1}_+} \int_0^{+\infty} z^{N+\alpha-1} \sqrt{v_z^2 + \frac{|\nabla_\theta v|^2}{z^2}} \, h \, dz \, d\Theta \right)^A \cdot \left(\int_{\mathbb{S}^{N-1}_+} \int_0^{+\infty} z^{N+\alpha-1} |v_z| \, h \, dz \, d\Theta \right)^{1-A} \\
= & \left(\int_{\mathbb{R}^N_+} y_N^\alpha |\nabla_y v| \, dy \right)^A \cdot \left(\int_{\mathbb{R}^N_+} y_N^\alpha |v_z| \, dy \right)^{1-A}.
\end{aligned}
$$

Finally, we have

$$
(6.12) \qquad \int_{\mathbb{R}^N_+} x_N^\alpha |x|^l u^{\frac{l+N+\alpha}{k+N+\alpha-1}} \, dx = \frac{N+\alpha-1}{k+N+\alpha-1} \int_{\mathbb{R}^N_+} y_N^\alpha |y|^{\frac{l(N+\alpha-1)-k(N+\alpha)}{k+N+\alpha-1}} v^{\frac{l+N+\alpha}{k+N+\alpha-1}} \, dy.
$$

Now (6.10) follows from (6.11) and (6.12). □

Next we want to estimate the right-hand-side of (6.10) from below. We will need a few more properties of the starshaped rearrangement.

Lemma 6.2. *Assume $l(N+\alpha-1)/(N+\alpha) \leq k$. Then we have for any function $v \in C_0^1(\mathbb{R}^N_+) \setminus \{0\}$ with $v \geq 0$,*

$$
(6.13) \qquad \int_{\mathbb{R}^N_+} y_N^\alpha |y|^{\frac{l(N+\alpha-1)-k(N+\alpha)}{k+N+\alpha-1}} v^{\frac{l+N+\alpha}{k+N+\alpha-1}} \, dy \leq \int_{\mathbb{R}^N_+} y_N^\alpha |y|^{\frac{l(N+\alpha-1)-k(N+\alpha)}{k+N+\alpha-1}} \widetilde{v}^{\frac{l+N+\alpha}{k+N+\alpha-1}} \, dy,
$$

$$
(6.14) \qquad \frac{y \cdot \nabla \widetilde{v}}{|y|} \equiv \frac{\partial \widetilde{v}}{\partial z} \in L^1(\mathbb{R}^N_+) \quad \text{and}
$$

$$
(6.15) \qquad \int_{\mathbb{R}^N_+} y_N^\alpha \left| \frac{\partial v}{\partial z} \right| dy \geq \int_{\mathbb{R}^N_+} y_N^\alpha \left| \frac{\partial \widetilde{v}}{\partial z} \right| dy.
$$

Proof: Let us prove (6.13). Set

$$
w(y) := |y|^{\frac{l(N+\alpha-1)-k(N+\alpha)}{l+N+\alpha}}.
$$

Since $l(N+\alpha-1) - k(N+\alpha) \leq 0$, we have $w = \widetilde{w}$. Hence (6.13) follows from (2.16) and (2.15).

Next let $\zeta := z^N$ and define V and $\hat{V}$ by $V(\zeta, \theta) := v(z\theta)$, and $\widehat{V}(\zeta, \theta) := \widetilde{v}(z\theta)$. Observe that for each $\theta \in \mathbb{S}^{N-1}_+$, $\widehat{V}(\cdot, \theta)$ is the equimeasurable non-increasing rearrangement of $V(\cdot, \theta)$. Further we have

$$
\frac{\partial v}{\partial z} = N \zeta^{\frac{N-1}{N}} \frac{\partial V}{\partial \zeta} \quad \text{and} \quad \frac{\partial \widetilde{v}}{\partial z} = N \zeta^{\frac{N-1}{N}} \frac{\partial \widehat{V}}{\partial \zeta}.
$$

Since $\frac{\partial v}{\partial z} \in L^\infty(\mathbb{R}^N)$, Lemma 2.1 tells us that for every $\theta \in \mathbb{S}^{N-1}$,

$$\begin{aligned}\int_0^{+\infty} z^{N+\alpha-1}\left|\frac{\partial v}{\partial z}(z\theta)\right| dz &= \int_0^{+\infty} \zeta^{\frac{N+\alpha-1}{N}}\left|\frac{\partial V}{\partial \zeta}(\zeta,\theta)\right| d\zeta \\ &\geq \int_0^{+\infty} \zeta^{\frac{N+\alpha-1}{N}}\left|\frac{\partial \widehat{V}}{\partial \zeta}(\zeta,\theta)\right| d\zeta \\ &= \int_0^{+\infty} z^{N+\alpha-1}\left|\frac{\partial \widetilde{v}}{\partial z}(z\theta)\right| dz.\end{aligned}$$

Integrating this over $\mathbb{S}_+^{N-1}$, we obtain (6.15). □

A final ingredient is

Lemma 6.3. *Assume that $l(N+\alpha-1)/(N+\alpha) \leq k$, and let $M \subset \mathbb{R}_+^N$ be a bounded starshaped set. Then*

$$\begin{aligned}(6.16)\qquad & \left(\int_M y_N^\alpha |y|^{\frac{l(N+\alpha-1)-k(N+\alpha)}{k+N+\alpha-1}}\, dy\right)^{\frac{k+N+\alpha-1}{l+N+\alpha}} \\ &\leq d_1 \left(\int_M y_N^\alpha\, dy\right)^{\frac{(N+\alpha-1)(l-k+1)}{l+N+\alpha}} \cdot \left(\int_M y_N^\alpha |y|^{-1}\, dy\right)^{\frac{k(N+\alpha)-l(N+\alpha-1)}{l+N+\alpha}}, \qquad \textit{where}\end{aligned}$$

$$(6.17)\qquad d_1 = \left(\frac{k+N+\alpha-1}{l+N+\alpha}\right)^{\frac{k+N+\alpha-1}{l+N+\alpha}} \cdot \left(\frac{N+\alpha}{N+\alpha-1}\right)^{\frac{(N+\alpha-1)(l-k+1)}{l+N+\alpha}}.$$

Moreover, if $k < l+1$ and $l(N+\alpha-1)/(N+\alpha) < k$, then equality in (6.16) holds only if $M = B_R^+$ for some $R > 0$.

Proof: Since M is starshaped, there is a bounded measurable function $m : \mathbb{S}_+^{N-1} \to [0,+\infty)$, such that

$$(6.18)\qquad M = \{z\theta :\, 0 \leq z < m(\theta),\ \theta \in \mathbb{S}_+^{N-1}\}.$$

Using Hölder's inequality we obtain

$$
\begin{aligned}
(6.19)\qquad & \int_M y_N^\alpha |y|^{\frac{l(N+\alpha-1)-k(N+\alpha)}{k+N+\alpha-1}}\,dy \\
= \; & \frac{k+N+\alpha-1}{(l+N+\alpha)(N+\alpha-1)} \int_{\mathbb{S}_+^{N-1}} m(\theta)^{\frac{(l+N+\alpha)(N+\alpha-1)}{k+N+\alpha-1}}\, h\, d\Theta \\
= \; & \frac{k+N+\alpha-1}{(l+N+\alpha)(N+\alpha-1)} \int_{\mathbb{S}_+^{N-1}} m(\theta)^{\frac{k(N+\alpha)-l(N+\alpha-1)}{k+N+\alpha-1}(N+\alpha-1)} m(\theta)^{\frac{(N+\alpha-1)(l-k+1)}{k+N+\alpha-1}(N+\alpha)}\, h\, d\Theta \\
\leq \; & \frac{k+N+\alpha-1}{(l+N+\alpha)(N+\alpha-1)} \left(\int_{\mathbb{S}_+^{N-1}} m(\theta)^{N+\alpha}\, h\, d\Theta \right)^{\frac{(N+\alpha-1)(l-k+1)}{k+N+\alpha-1}} \\
\times \; & \left(\int_{\mathbb{S}_+^{N-1}} m(\theta)^{N+\alpha-1}\, h\, d\Theta \right)^{\frac{k(N+\alpha)-l(N+\alpha-1)}{k+N+\alpha-1}} \\
= \; & \frac{k+N+\alpha-1}{(l+N+\alpha)(N+\alpha-1)} \left((N+\alpha) \int_M y_N^\alpha dy \right)^{\frac{(N+\alpha-1)(l-k+1)}{k+N+\alpha-1}} \times \\
\times \; & \left((N+\alpha-1) \int_M |y|^{-1} y_N^\alpha\, dy \right)^{\frac{k(N+\alpha)-l(N+\alpha-1)}{k+N+\alpha-1}},
\end{aligned}
$$

and (6.16) follows. If $k < l+1$ and $l(N+\alpha-1)/(N+\alpha) < k$, then (6.19) holds with equality only if $m(\theta) = \text{const}$. □

Now we are ready to prove our main result.

Theorem 6.2. *Assume* $0 \leq k \leq l+1$ *and*

$$
l \leq \frac{(k+N+\alpha-1)^3}{(k+N+\alpha-1)^2 - \frac{(N+\alpha-1)^2}{N+\alpha}} - N - \alpha. \tag{6.20}
$$

Then (4.3) holds. Furthermore, if inequality (6.20) is strict, then (6.3) holds only if $M = B_R^+$ *for some* $R > 0$.

Proof: First observe that the conditions $k \geq 0$ and (6.20) also imply $l(N+\alpha-1)/(N+\alpha) \leq k$. Let $u \in C_0^\infty(\mathbb{R}_+^N) \setminus \{0\}$, $u \geq 0$, and let v be given by (6.9). In view of (6.20), we may choose

$$
A = \frac{(N+\alpha)(l-k+1)}{l+N+\alpha}
$$

to obtain

$$(6.21)\quad \mathcal{Q}_{k,l,N,\alpha}(u) \geq \left(\frac{k+N+\alpha-1}{N+\alpha-1}\right)^{\frac{k+N+\alpha-1}{l+N+\alpha}} \times$$

$$\times \frac{\left(\int_{\mathbb{R}^N} y_N^\alpha |\nabla_y v|\, dy\right)^{\frac{(N+\alpha)(l-k+1)}{l+N+\alpha}} \cdot \left(\int_{\mathbb{R}^N_+} y_N^\alpha |v_z|\, dy\right)^{\frac{k(N+\alpha)-l(N+\alpha-1)}{l+N+\alpha}}}{\left(\int_{\mathbb{R}^N_+} y_N^\alpha |y|^{\frac{l(N+\alpha-1)-k(N+\alpha)}{k+N+\alpha-1}} v^{\frac{l+N+\alpha}{k+N+\alpha-1}}\, dy\right)^{\frac{k+N+\alpha-1}{l+N+\alpha}}}$$

.

Further, (6.15) and Hardy's inequality yield

$$(6.22)\qquad \int_{\mathbb{R}^N} y_N^\alpha |v_z|\, dy \geq \int_{\mathbb{R}^N_+} y_N^\alpha |\widetilde{v}_z|\, dy \geq (N+\alpha-1)\int_{\mathbb{R}^N_+} y_N^\alpha \frac{\widetilde{v}}{|y|}\, dy\,,$$

where $\widetilde{v}$ denotes the starshaped rearrangement of v. Together with (6.21) and (6.13) this leads to

$$(6.23)\quad \mathcal{Q}_{k,l,N,\alpha}(u) \geq (N+\alpha-1)^{\frac{k(N+\alpha)-l(N+\alpha-1)}{l+N+\alpha}} \left(\frac{k+N+\alpha-1}{N+\alpha-1}\right)^{\frac{k+N+\alpha-1}{l+N+\alpha}} \cdot$$

$$\cdot \frac{\left(\int_{\mathbb{R}^N_+} y_N^\alpha |\nabla_y v|\, dy\right)^{\frac{(N+\alpha)(l-k+1)}{l+N+\alpha}} \cdot \left(\int_{\mathbb{R}^N_+} y_N^\alpha \frac{\widetilde{v}}{|y|}\, dy\right)^{\frac{k(N+\alpha)-l(N+\alpha-1)}{l+N+\alpha}}}{\left(\int_{\mathbb{R}^N_+} y_N^\alpha |y|^{\frac{l(N+\alpha-1)-k(N+\alpha)}{k+N+\alpha-1}} \widetilde{v}^{\frac{l+N+\alpha}{k+N+\alpha-1}}\, dy\right)^{\frac{k+N+\alpha-1}{l+N+\alpha}}}.$$

Now let M be a bounded measurable subset of $\mathbb{R}^N_+$. Then combining (3.20), (3.21) and the argument leading to (3.7) we deduce that there exists a sequence of non-negative functions $\{u_n\} \subset C^1_0(\mathbb{R}^N_+)$ such that

$$(6.24)\qquad \lim_{n\to\infty} \int_{\mathbb{R}^N_+} x_N^\alpha |x|^k |\nabla u_n|\, dx = P_{\mu_k,\alpha}(M)$$

and

$$(6.25)\qquad u_n \longrightarrow \chi_M \quad \text{in } L^p(\mathbb{R}^N_+) \text{ for every } p \geq 1.$$

We define $M' := \{y = x|x|^{\frac{k}{N+\alpha-1}} : x \in M\}$ and $v_n(y) := u_n(x)$.

Let $\widetilde{v_n}$ and $\widetilde{M'}$ be the starshaped rearrangements of v_n and M' respectively. Then (6.24) and (6.25) also imply

$$\lim_{n\to\infty}\int_{\mathbb{R}^N_+} y_N^\alpha |\nabla_y v_n|\,dy = P_{\mu_0,\alpha}(M'), \quad \text{and} \tag{6.26}$$

$$\widetilde{v_n} \longrightarrow \chi_{\widetilde{M'}} \ \text{ in } L^p(\mathbb{R}^N_+) \text{ for every } p\geq 1. \tag{6.27}$$

Choosing $u = u_n$ in (6.23) and passing to the limit $n\to\infty$, we obtain, using (6.24), (6.25), (6.26), (6.27) and Proposition 3.1

$$\begin{aligned} &\mathcal{R}_{k,l,N,\alpha}(M) \\ \geq\ & (N+\alpha-1)^{\frac{k(N+\alpha)-l(N+\alpha-1)}{l+N+\alpha}}\left(\frac{k+N+\alpha-1}{N+\alpha-1}\right)^{\frac{k+N+\alpha-1}{l+N+\alpha}}\cdot \\ & \cdot\frac{\left(P_{\mu_0,\alpha}(\widetilde{M'})\right)^{\frac{(N+\alpha)(l-k+1)}{l+N+\alpha}}\cdot\left(\int_{\widetilde{M'}}\frac{y_N^\alpha dy}{|y|}\right)^{\frac{k(N+\alpha)-l(N+\alpha-1)}{l+N+\alpha}}}{\left(\int_{\widetilde{M'}} y_N^\alpha |y|^{\frac{l(N+\alpha-1)-k(N+\alpha)}{k+N+\alpha-1}}\,dy\right)^{\frac{k+N+\alpha-1}{l+N+\alpha}}} \\ \geq\ & (N+\alpha-1)^{\frac{k(N+\alpha)-l(N+\alpha-1)}{l+N+\alpha}}\left(\frac{k+N+\alpha-1}{N+\alpha-1}\right)^{\frac{k+N+\alpha-1}{l+N+\alpha}}\left(C^{rad}_{0,0,N,\alpha}\right)^{\frac{(N+\alpha)(l-k+1)}{l+N+\alpha}}\times \\ & \times\frac{\left(\mu_{0,\alpha}(\widetilde{M'})\right)^{\frac{(N+\alpha-1)(l-k+1)}{l+N+\alpha}}\cdot\left(\int_{\widetilde{M'}}\frac{y_N^\alpha dy}{|y|}\right)^{\frac{k(N+\alpha)-l(N+\alpha-1)}{l+N+\alpha}}}{\left(\int_{\widetilde{M'}} y_N^\alpha |y|^{\frac{l(N+\alpha-1)-k(N+\alpha)}{k+N+\alpha-1}}\,dy\right)^{\frac{k+N+\alpha-1}{l+N+\alpha}}}. \end{aligned} \tag{6.28}$$

In view of (6.16) and since $\mu_0(M') = \mu_0(\widetilde{M'})$ we finally get from this

$$\begin{aligned} &\mathcal{R}_{k,l,N,\alpha}(M) \\ \geq\ & (N+\alpha-1)^{\frac{k(N+\alpha)-l(N+\alpha-1)}{l+N+\alpha}}\left(\frac{k+N+\alpha-1}{N+\alpha-1}\right)^{\frac{k+N+\alpha-1}{l+N+\alpha}}\left(C^{rad}_{0,0,N,\alpha}\right)^{\frac{(N+\alpha)(l-k+1)}{l+N+\alpha}}\frac{1}{d_1} \\ =\ & \left(\int_{\mathbb{S}^{N-1}_+} h\,d\Theta\right)^{\frac{l-k+1}{l+N+\alpha}}\cdot(l+N+\alpha)^{\frac{k+N+\alpha-1}{l+N+\alpha}} = C^{rad}_{k,l,N,\alpha}, \end{aligned} \tag{6.29}$$

and (4.3) follows by (3.7).
Now assume that (6.3) holds. If inequality (6.20) is strict, then Lemma 3.2 tells us that we must have $M = B_R^+$ for some $R>0$. □

7. Applications

In this section we provide some applications of our results.

7.1. **Pólya-Szegö principle.** First we obtain a Pólya-Szegö principle related to our isoperimetric inequality (4.3) (cf. [42]) Assume that the numbers k, l and α satisfy (2.1) and one of the conditions **(i)**-**(iii)** of Theorem 1.1. Then (1.2) implies

$$\int_{\partial\Omega} |x|^k x_N^\alpha \mathcal{H}_{N-1}(dx) \geq \int_{\partial\Omega^\star} |x|^k x_N^\alpha \mathcal{H}_{N-1}(dx) \tag{7.1}$$

for every smooth set $\Omega \subset \mathbb{R}^N_+$, where $\Omega^\star$ is the $\mu_{l,\alpha}$-symmetrization of Ω. We will use (7.1) to prove the following

Theorem 7.1. *(Pólya-Szegö principle) Let the numbers k, l and α satisfy one of the conditions* **(i)**-**(iii)** *of Theorem 1.1. Further, let $p \in [1, +\infty)$ and $m := pk + (1-p)l$. Then there holds*

$$\int_{\mathbb{R}^N_+} |\nabla u|^p \, d\mu_{m,\alpha}(x) \geq \int_{\mathbb{R}^N_+} |\nabla u^\star|^p \, d\mu_{m,\alpha}(x) \quad \forall u \in \mathcal{D}^{1,p}(\mathbb{R}^N_+, d\mu_{m,\alpha}), \tag{7.2}$$

where $u^\star$ denotes the $\mu_{l,\alpha}$-symmetrization of u.

Proof: A proof of this result would follow from the same arguments used in [42]. Here we give a different proof which holds true under the additional assumption that $u^\star$ is a Lipschitz continuous function. It is sufficient to consider the case that u is non-negative. Further, by an approximation argument we may assume that $u \in C_0^\infty(\mathbb{R}^N)$. Let

$$\begin{aligned} I &:= \int_{\mathbb{R}^N_+} |\nabla u|^p |x|^{pk+(1-p)l} x_N^\alpha \, dx \quad \text{and} \\ I^\star &:= \int_{\mathbb{R}^N_+} |\nabla u^\star|^p |x|^{pk+(1-p)l} x_N^\alpha \, dx. \end{aligned}$$

The coarea formula yields

$$I = \int_0^\infty \int_{u=t} |\nabla u|^{p-1} |x|^{pk+(1-p)l} x_N^\alpha \, \mathcal{H}_{N-1}(dx) \, dt \quad \text{and} \tag{7.3}$$

$$I^\star = \int_0^\infty \int_{u^\star=t} |\nabla u^\star|^{p-1} |x|^{pk+(1-p)l} x_N^\alpha \, \mathcal{H}_{N-1}(dx) \, dt. \tag{7.4}$$

Further, Hölder's inequality gives

$$\int_{u=t} |x|^k x_N^\alpha \, \mathcal{H}_{N-1}(dx) \leq \left(\int_{u=t} |x|^{kp+l(1-p)} |\nabla u|^{p-1} x_N^\alpha \, \mathcal{H}_{N-1}(dx) \right)^{\frac{1}{p}} \cdot \left(\int_{u=t} \frac{|x|^l x_N^\alpha}{|\nabla u|} \, \mathcal{H}_{N-1}(dx) \right)^{\frac{p-1}{p}}, \tag{7.5}$$

for a.e. $t \in [0, +\infty)$. Hence (7.3) together with (7.5) tells us that

$$I \geq \int_0^\infty \left(\int_{u=t} |x|^k x_N^\alpha \, \mathcal{H}_{N-1}(dx) \right)^p \cdot \left(\int_{u=t} \frac{|x|^l x_N^\alpha}{|\nabla u|} x_N^\alpha \, \mathcal{H}_{N-1}(dx) \right)^{1-p} dt. \tag{7.6}$$

Since $u^\star$ is a radial function, we obtain in an analogous manner,

$$I^\star = \int_0^\infty \left(\int_{u^\star=t} |x|^k x_N^\alpha \, \mathcal{H}_{N-1}(dx) \right)^p \cdot \left(\int_{u^\star=t} \frac{|x|^l x_N^\alpha}{|\nabla u^\star|} \, \mathcal{H}_{N-1}(dx) \right)^{1-p} dt. \tag{7.7}$$

Observing that

$$\int_{u>t} |x|^l x_N^\alpha \, dx = \int_{u^\star>t} |x|^l x_N^\alpha \, dx \quad \forall t \in [0, +\infty), \tag{7.8}$$

Fleming-Rishel's formula yields

$$\int_{u=t} \frac{|x|^l x_N^\alpha}{|\nabla u|} \, \mathcal{H}_{N-1}(dx) = \int_{u^\star=t} \frac{|x|^l x_N^\alpha}{|\nabla u^\star|} \, \mathcal{H}_{N-1}(dx) \tag{7.9}$$

for a.e. $t \in [0, +\infty)$. Hence (7.9) and (7.1) give

$$\begin{aligned} &\int_0^\infty \left(\int_{u=t} |x|^k x_N^\alpha \, \mathcal{H}_{N-1}(dx) \right)^p \cdot \left(\int_{u=t} \frac{|x|^l x_N^\alpha}{|\nabla u|} \, \mathcal{H}_{N-1}(dx) \right)^{1-p} dt \\ \geq \ &\int_0^\infty \left(\int_{u^\star=t} |x|^k x_N^\alpha \, \mathcal{H}_{N-1}(dx) \right)^p \cdot \left(\int_{u^\star=t} \frac{|x|^l x_N^\alpha}{|\nabla u^\star|} \, \mathcal{H}_{N-1}(dx) \right)^{1-p} dt. \end{aligned}$$

Now (7.2) follows from this, (7.6) and (7.7). □

An important particular case of Theorem 7.1 is

Corollary 7.1. *Let $p \in [1, +\infty)$, $N + \alpha \geq 3$, $a \geq 0$, $u \in \mathcal{D}^{1,p}(\mathbb{R}^N_+, d\mu_{ap,\alpha})$, and let $u^\star$ be the $\mu_{0,\alpha}$-symmetrization of u. Then*

$$\int_{\mathbb{R}^N_+} |\nabla u|^p \, d\mu_{ap,\alpha}(x) \geq \int_{\mathbb{R}^N_+} |\nabla u^\star|^p \, d\mu_{ap,\alpha}(x). \tag{7.10}$$

Proof: We choose $k := a$ and $l := 0$. If $a \in [0, 1]$ then k, l satisfy either one of the conditions **(ii)** or **(iii)**, see also Remark 5.2. If $a \geq 1$, then k, l satisfy condition **(i)** of Theorem 1.1. Hence (7.10) follows from Theorem 7.1. □

7.2. **Caffarelli-Kohn-Nirenberg-type inequalities.** Next we will use Theorem 7.1 to obtain best constants in some inequalities of Caffarelli-Kohn-Nirenberg-type.

Let p, q, a, b be real numbers such that

$$1 \le p \le q \begin{cases} \le \frac{(N+\alpha)p}{N+\alpha-p} & \text{if } \ p < N+\alpha \\ < +\infty & \text{if } \ p \ge N+\alpha \end{cases}, \tag{7.11}$$

$$a > 1 - \frac{N+\alpha}{p}, \quad \text{and}$$

$$b = b(a,p,q,N,\alpha) = (N+\alpha)\left(\frac{1}{p} - \frac{1}{q}\right) + a - 1.$$

We define

$$p^* \ := \ \begin{cases} \frac{(N+\alpha)p}{N+\alpha-p} & \text{if } p < N+\alpha \\ +\infty & \text{if } p \ge N+\alpha \end{cases}, \tag{7.12}$$

$$E_{a,p,q,N,\alpha}(v) \ := \ \frac{\displaystyle\int_{\mathbb{R}^N_+} |x|^{ap} |\nabla v|^p x_N^\alpha \, dx}{\left(\displaystyle\int_{\mathbb{R}^N_+} |x|^{bq} |v|^q x_N^\alpha \, dx\right)^{p/q}}, \quad v \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}, \tag{7.13}$$

$$S_{a,p,q,N,\alpha} \ := \ \inf\{E_{a,p,q,N,\alpha}(v) : \ v \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}\}, \quad \text{and} \tag{7.14}$$

$$S^{rad}_{a,p,q,N,\alpha} \ := \ \inf\{E_{a,p,q,N,\alpha}(v) : \ v \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}, \ v \text{ radial } \}. \tag{7.15}$$

Note that with this new notation we have

$$E_{k,1,\frac{l+N+\alpha}{k+N+\alpha-1},N,\alpha}(v) \ = \ \mathcal{Q}_{k,l,N,\alpha}(v) \quad \forall v \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}, \tag{7.16}$$

$$S_{k,1,\frac{l+N+\alpha}{k+N+\alpha-1},N,\alpha}(v) \ = \ C_{k,l,N,\alpha} \quad \text{and} \tag{7.17}$$

$$S^{rad}_{k,1,\frac{l+N+\alpha}{k+N+\alpha-1},N,\alpha} \ = \ C^{rad}_{k,l,N,\alpha}. \tag{7.18}$$

We are interested in the range of values a (depending on p, q, N and α) for which

$$S_{a,p,q,N,\alpha} = S^{rad}_{a,p,q,N,\alpha} \tag{7.19}$$

holds.

First observe that the case $1 < p = q$ (which is equivalent to $a - b = 1$) corresponds to a weighted Hardy-Sobolev-type inequality. Note that inequality (7.20) below was already known when $\alpha = 0$ (see, for example [29] and references therein). We have:

Theorem 7.2. *Let $p \ge 1$, $\alpha \ge 0$ and $k \in \mathbb{R}$ be such that $N - p + \alpha + k > 0$. Then we have*

$$\int_{\mathbb{R}^N_+} |\nabla u(x)|^p \, d\mu_{k,\alpha}(x) \ge \left(\frac{N-p+k+\alpha}{p}\right)^p \int_{\mathbb{R}^N_+} \frac{|u(x)|^p}{|x|^p} \, d\mu_{k,\alpha}(x) \tag{7.20}$$

for all $u \in \mathcal{D}^{1,p}(\mathbb{R}^N_+, d\mu_{k,\alpha})$ *and*

$$S^{rad}_{a,p,p,N,\alpha} = S_{a,p,p,N,\alpha} = \left(\frac{N-p+k+\alpha}{p}\right)^p . \tag{7.21}$$

Moreover there is no function $u \in \mathcal{D}^{1,p}(\mathbb{R}^N_+, d\mu_{k,\alpha})$ *satisfying equality in* (7.20) *and such that* $\int_{\mathbb{R}^N_+} |\nabla u|^p d\mu_{k,\alpha} \neq 0$.

Proof: The first two steps follow the line of proof of [26], Lemma 2.1.
Step 1. Assume first that $u \in C_0^\infty(\mathbb{R}^N)$. Then we have for every $x \in \mathbb{R}^N_+$,

$$|u(x)|^p = -\int_1^\infty \frac{d}{dt}|u(tx)|^p\, dt = -\int_1^\infty p|u(tx)|^{p-2}u(tx)\langle x, \nabla u(tx)\rangle\, dt.$$

Multiplying this with $x_N^\alpha |x|^{k-p}$ and integrating over $\mathbb{R}^N_+$ we find

$$\begin{aligned}
\int_{\mathbb{R}^N_+} |u(x)|^p x_N^\alpha |x|^{k-p}\, dx &= -p\int_1^\infty \left[\int_{\mathbb{R}^N_+} |u(tx)|^{p-2}u(tx)\langle x, \nabla u(tx)\rangle x_N^\alpha |x|^k\, dx\right] dt \\
&= -p\int_1^\infty \frac{1}{t^{N-p+\alpha+k}}\left[\int_{\mathbb{R}^N_+} \frac{|u(y)|^{p-2}u(y)}{|y|^p}\langle y, \nabla u(y)\rangle y_N^\alpha |y|^k\, dy\right] dt \\
&= -\frac{p}{N-p+\alpha+k}\int_{\mathbb{R}^N_+} \frac{|u(x)|^{p-2}u(x)}{|x|^p}\langle x, \nabla u(x)\rangle x_N^\alpha |x|^k\, dx. \qquad (7.22)
\end{aligned}$$

Note that by a density argument (7.22) still holds for functions $u \in \mathcal{D}^{1,p}(\mathbb{R}^N_+, d\mu_{k,\alpha})$. In view of the inequality

$$-u(x)\langle x, \nabla u(x)\rangle \leq |u(x)||x||\nabla u(x)| \tag{7.23}$$

this leads to

$$\int_{\mathbb{R}^N_+} |u(x)|^p x_N^\alpha |x|^{k-p}\, dx \leq \frac{p}{N-p+k+\alpha}\int_{\mathbb{R}^N_+} \frac{|u(x)|^{p-1}}{|x|^{p-1}}|\nabla u(x)| x_N^\alpha |x|^k\, dx. \tag{7.24}$$

Using Hölder's inequality, with p' being the conjugate exponent of p, we obtain that (this step is not necessary if $p = 1$)

$$\begin{aligned}
&\int_{\mathbb{R}^N_+} \frac{|u(x)|^{p-1}}{|x|^{p-1}}|\nabla u(x)| x_N^\alpha |x|^k\, dx \\
= &\int_{\mathbb{R}^N_+} \left\{\frac{|u(x)|^{p-1}}{|x|^{p-1}}\left[x_N^\alpha |x|^k\right]^{1/p'}\right\}\left\{|\nabla u(x)|\left[x_N^\alpha |x|^k\right]^{1/p}\right\} dx \\
\leq &\left(\int_{\mathbb{R}^N_+} |u(x)|^p x_N^\alpha |x|^{k-p}\, dx\right)^{1/p'} \cdot \left(\int_{\mathbb{R}^N_+} |\nabla u(x)|^p x_N^\alpha |x|^k\, dx\right)^{1/p}. \qquad (7.25)
\end{aligned}$$

Plugging this estimate into (7.24) concludes the first statement of the theorem.

Step 2. Next we show (7.21). Let $\varepsilon > 0$ and define

$$M_\epsilon = \frac{N-p+k+\alpha+\epsilon}{p}, \qquad u_\epsilon(x) = \begin{cases} 1 & \text{if } |x| \le 1 \\ |x|^{-M_\epsilon} & \text{if } |x| > 1. \end{cases}$$

Note that

$$\int_{\mathbb{R}^N_+} |\nabla u_\epsilon|^p x_N^\alpha |x|^k \, dx = {M_\epsilon}^p \int_{\mathbb{R}^N_+ \setminus B_1} x_N^\alpha |x|^{k-(M_\epsilon+1)p} \, dx.$$

Hence, by Lemma 7.1 (ii) below we obtain for any $\epsilon > 0$ that $u_\epsilon \in \mathcal{D}^{1,p}(\mathbb{R}^N_+, d\mu_{k,\alpha})$. On the other hand, we have that

$$\int_{\mathbb{R}^N_+} |u_\epsilon(x)|^p x_N^\alpha |x|^{k-p} \, dx = \int_{\mathbb{R}^N_+ \setminus B_1} x_N^\alpha |x|^{k-(M_\epsilon+1)p} \, dx + \beta,$$

where, by Lemma 7.1 (i),

$$\beta = \int_{B_1^+} x_N^\alpha |x|^{k-p} < \infty.$$

Now set

$$Q_\epsilon = \frac{\int_{\mathbb{R}^N_+} |\nabla u_\epsilon|^p x_N^\alpha |x|^k \, dx}{\int_{\mathbb{R}^N_+} |u_\epsilon|^p x_N^\alpha |x|^{k-p} \, dx} = \frac{\int_{\mathbb{R}^N_+ \setminus B_1} x_N^\alpha |x|^{k-(M_\epsilon+1)p} \, dx}{\beta + \int_{\mathbb{R}^N_+ \setminus B_1} |x|^{k-(M_\epsilon+1)p}} \, dx.$$

Note also that $(M_\epsilon + 1)p = N + k + \alpha + \epsilon$. Therefore we obtain from Lemma 7.1 (iii) that

$$\lim_{\epsilon \to 0} Q_\epsilon = (M_0)^p = \left(\frac{N-p+k+\alpha}{p} \right)^p.$$

This proves the second equality in (7.21). The first equality in (7.21) follows from the fact that the approximating functions u_ε are radial.

Step 3. Let us now show that there is no nontrivial function satisfying equality in (7.20). Assume that equality holds in (7.20). Then there holds equality in (7.24) and (7.25). Hence we must have

$$-u(x)\langle x, u(x) \rangle = |u(x)||x|\,|\nabla u(x)| \quad \text{and} \tag{7.26}$$

$$\frac{|u(x)|}{|x|} = \frac{p}{N-p+k+\alpha} |\nabla u(x)| \quad \text{for a.e. } x \in \mathbb{R}^N_+. \tag{7.27}$$

An integration of this leads to

$$u(x) = |x|^{-(N-p+k+\alpha)/p} h\left(x|x|^{-1}\right), \tag{7.28}$$

with a measurable function $h : \mathbb{S}^{N-1}_+ \to \mathbb{R}$. Since $|x|^{-1}u \in L^p(\mathbb{R}^N_+, d\mu_{k,\alpha})$, this implies that $h = 0$ a.e. on $\mathbb{S}^{N-1}_+$. The claim is proved. □

Lemma 7.1. *Let $\delta > 0$. Then*

$$\text{(i)} \qquad \int_{B_1^+} x_N^\alpha |x|^{-N-\alpha+\delta}\, dx < \infty, \qquad \text{and}$$

$$\text{(ii)} \qquad \int_{\mathbb{R}_+^N \setminus B_1} x_N^\alpha |x|^{-N-\alpha-\delta}\, dx < \infty.$$

Further, there holds

$$\lim_{\delta \to 0+0} \int_{\mathbb{R}_+^N \setminus B_1} x_N^\alpha |x|^{-N-\alpha-\delta}\, dx = \infty.$$

Proof: We use N-dimensional spherical coordinates to show that

$$\begin{aligned} \int_{B_1^+} x_N^\alpha |x|^{-N-\alpha+\delta} &= \int_{\mathbb{S}_+^{N-1}} \left(\int_0^1 \left(\frac{x}{|x|} \right)^\alpha r^{-1+\delta} dr \right) d\mathcal{H}^{N-1}(x) \\ &= \int_{\mathbb{S}_+^{N-1}} \left(\frac{x}{|x|} \right)^\alpha d\mathcal{H}^{N-1}(x) \left(\int_0^1 r^{-1+\delta} dr \right). \end{aligned}$$

From this (i) follows. (ii) and (iii) follow similarly. □

From now on let us assume that

$$1 < p < q \begin{cases} \le p^* & \text{if } \; p < N+\alpha \\ < +\infty & \text{if } \; p \ge N+\alpha \end{cases} . \tag{7.29}$$

We begin with the following

Lemma 7.2. *Assume that a, b, p, q, N and α satisfy the conditions (7.11) and (7.29). Further, assume that there exist real numbers k and l which satisfy $l + N + \alpha > 0$ and one of the conditions* **(i)**-**(iii)** *of Theorem 1.1, and such that*

$$ap = kp + l(1-p) \quad \text{and} \tag{7.30}$$

$$bq \le l. \tag{7.31}$$

Then (7.19) holds.

Proof: Let $u \in \mathcal{D}^{1,p}(\mathbb{R}_+^N, d\mu_{ap,\alpha}) \setminus \{0\}$, and let $u^\star$ be the $\mu_{l,\alpha}$-symmetrization of u. Then we have by Theorem 7.1 and (7.30),

$$\int_{\mathbb{R}_+^N} |x|^{ap} |\nabla u|^p x_N^\alpha\, dx \ge \int_{\mathbb{R}_+^N} |x|^{ap} |\nabla u^\star|^p x_N^\alpha\, dx. \tag{7.32}$$

Further, it follows from (2.10) and (7.31) that

$$\int_{\mathbb{R}_+^N} |x|^{bq} |u|^q x_N^\alpha\, dx \le \int_{\mathbb{R}^N} |x|^{bq} |u^\star|^q x_N^\alpha\, dx. \tag{7.33}$$

Finally, (7.32) together with (7.33) yield

$$E_{a,p,q,N,\alpha}(u) \ge E_{a,p,q,N,\alpha}(u^\star), \tag{7.34}$$

and the assertion follows. □

Now we define

$$a_1 := \frac{N+\alpha-1}{q-\frac{q}{p}+1}+1-\frac{N+\alpha}{p}, \quad \text{and} \tag{7.35}$$

$$a_2 := \frac{N+\alpha-1}{(q-\frac{q}{p}+1)\sqrt{(N+\alpha)(\frac{1}{p}-\frac{1}{q})}}+1-\frac{N+\alpha}{p}. \tag{7.36}$$

Observe that the conditions (7.29) imply that

$$a_2 \geq a_1 \geq 0, \tag{7.37}$$

and equality in the two inequalities holds iff $p < N+\alpha$ and $q = p^*$.
Moreover, an elementary calculation shows that

$$a_1 = \max\Big\{a : \ a = k + l\left(\frac{1}{p}-1\right), \ bq \leq l, \ -N-\alpha < l \leq k\frac{N+\alpha}{N+\alpha-1} \leq 0\Big\} \quad \text{and} \tag{7.38}$$

$$a_2 = \max\Big\{a : \ a = k + l\left(\frac{1}{p}-1\right), \ bq \leq l, \ k \geq 0, \ 0 < l+N+\alpha \leq \frac{(k+N+\alpha-1)^3}{(k+N+\alpha-1)^2 - \frac{(N+\alpha-1)^2}{N+\alpha}}\Big\}. \tag{7.39}$$

The main result of this section is the following

Theorem 7.3. *Assume that (7.29) holds. Then we have*

$$S_{a,p,q,N,\alpha} = S^{rad}_{a,p,q,N,\alpha} \qquad \forall a \in \Big(1-\frac{N+\alpha}{p}, a_2\Big]. \tag{7.40}$$

Proof: Let $a \in \Big(1-\frac{N+\alpha}{p}, a_2\Big]$. We define

$$l := q\left(a+\frac{N+\alpha}{p}-1\right)-N-\alpha, \quad \text{and} \tag{7.41}$$

$$k := \left(1+q-\frac{q}{p}\right)\left(a+\frac{N+\alpha}{p}-1\right)-N-\alpha+1. \tag{7.42}$$

This implies

$$\begin{aligned} a &= k + l\left(\frac{1}{p} - 1\right), \\ bq &= l \quad \text{and} \\ l + N + \alpha &= \frac{k + N + \alpha - 1}{\frac{1}{q} - \frac{1}{p} + 1} > 0. \end{aligned}$$

Now we split into two cases:
1. Let $a \leq a_1$.
Then

$$k \leq 0,$$

and since $q \leq p^*$ if $p < N + \alpha$ and $q < +\infty$ otherwise, we have

$$\begin{aligned} l\frac{N + \alpha - 1}{N + \alpha} - k &= (k + N + \alpha - 1)\frac{-\frac{1}{N+\alpha} - \frac{1}{q} + \frac{1}{p}}{\frac{1}{q} - \frac{1}{p} + 1} \\ &\leq 0. \end{aligned}$$

Hence we are in case **(ii)** of Theorem 1.1, so that the assertion follows by Lemma 7.2, for $a \leq a_1$.
2. Next let $a_1 \leq a \leq a_2$.
This implies

$$\begin{aligned} k &\geq 0 \quad \text{and} \\ k + N + \alpha - 1 &\leq \frac{N + \alpha - 1}{\sqrt{(N + \alpha)\left(\frac{1}{p} - \frac{1}{q}\right)}}. \end{aligned} \tag{7.43}$$

Now, from (7.43) we deduce

$$\begin{aligned} & l + N + \alpha - \frac{(k + N + \alpha - 1)^3}{(k + N + \alpha - 1)^2 - \frac{(N+\alpha-1)^2}{N+\alpha}} \\ = \ & \frac{(k + N + \alpha - 1)\left((k + N + \alpha - 1)^2\left(\frac{1}{p} - \frac{1}{q}\right) - \frac{(N+\alpha-1)^2}{N+\alpha}\right)}{\left(\frac{1}{q} - \frac{1}{p} + 1\right)\left((k + N + \alpha - 1)^2 - \frac{(N+\alpha-1)^2}{N+\alpha}\right)} \\ \leq \ & 0. \end{aligned}$$

Hence we are in case **(iii)** of Theorem 1.1, so that the assertion follows again by Lemma 7.2
. □

Remark 6.1: The characterizations (7.38) and (7.39) and the inequalities (7.37) show that the bound a_2 cannot be improved using our method.

Finally we evaluate the constants $S^{rad}_{a,p,q,N,\alpha}$ and the corresponding radial minimizers. For any radial function $v \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}$, it is easy to check the following equality

$$E_{a,p,q,N,\alpha}(v) = \left[B\left(\frac{N-1}{2}, \frac{\alpha+1}{2}\right)\right]^{1-\frac{p}{q}} \frac{\pi^{\frac{N-1}{2}\frac{q-p}{q}}}{\left(\Gamma\left[\frac{N-1}{2}\right)\right]^{\frac{q-p}{q}}} \frac{\displaystyle\int_{\mathbb{R}^N_+} |x|^{ap+\alpha} |\nabla v|^p \, dx}{\left(\displaystyle\int_{\mathbb{R}^N_+} |x|^{bq+\alpha} |v|^q \, dx\right)^{p/q}},$$

Therefore by Theorem 1.4 in [39], we deduce that the function

$$U(x) = \left(1 + |x|^{\frac{(N-p+ap+\alpha)(q-p)}{p(p-1)}}\right)^{\frac{p}{p-q}} .$$

achieves the infimum of $E_{a,p,q,N,\alpha}$, that is $S^{rad}_{a,p,q,N,\alpha} = E_{a,p,q,N,\alpha}(U)$.

7.3. **Problems in an orthant.** Among the possible extensions of our isoperimetric results we would like to address a problem in an orthant with monomial weights. Let O_+ denote the orthant

$$O_+ := \{x \in \mathbb{R}^N : x_i > 0, \, i = 1, \dots, N\},$$

and let $a_1, \dots, a_N$ be positive numbers. Using multi-index notation we have

$$\begin{aligned} \mathbf{a} &:= (a_1, \dots, a_N), \\ |\mathbf{a}| &:= a_1 + \dots + a_N, \\ x^{\mathbf{a}} &:= x_1^{a_1} \cdots x_N^{a_N}, \quad (x \in \mathbb{R}^N). \end{aligned}$$

Following the lines of proof of Theorem 1.1 we obtain the following isoperimetric result. We leave the details to the reader.

Theorem 7.4. *Let $N \in \mathbb{N}$, $N \geq 2$, $k, l \in \mathbb{R}$, $\mathbf{a} = (a_1, \dots, a_N)$ where $a_i > 0$, $(i = 1, \dots, N)$, and $l + N + |\mathbf{a}| > 0$. Further, assume that one of the following conditions holds:*
(i) *$l + 1 \leq k$;*
(ii) *$k \leq l + 1$ and $l\frac{N+|\mathbf{a}|-1}{N+|\mathbf{a}|} \leq k \leq 0$;*
(iii) *$N \geq 2$, $0 \leq k \leq l + 1$ and*

$$l \leq \frac{(k+N+|\mathbf{a}|-1)^3}{(k+N+|\mathbf{a}|-1)^2 - \frac{(N+|\mathbf{a}|-1)^2}{N+|\mathbf{a}|}} - N - |\mathbf{a}| \,. \tag{7.44}$$

Then

$$\int_{\partial\Omega} |x|^k x^{\mathbf{a}} \, \mathcal{H}_{N-1}(dx) \geq D \left(\int_\Omega |x|^l x^{\mathbf{a}} \, dx\right)^{(k+N+|\mathbf{a}|-1)/(l+N+|\mathbf{a}|)}, \tag{7.45}$$

for all smooth sets Ω in O_+, where

$$D = D(k,l,N,\mathbf{a}) \ := \ \frac{\displaystyle\int_{\partial B_1} |x|^k x^{\mathbf{a}}\, \mathcal{H}_{N-1}(dx)}{\left(\displaystyle\int_{B_1\cap O_+} |x|^l x^{\mathbf{a}}\, dx\right)^{(k+N+|\mathbf{a}|-1)/(l+N+|\mathbf{a}|)}}. \tag{7.46}$$

Equality in (7.45) holds if $\Omega = B_R \cap O_+$.

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