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Bounds on the Hausdorff dimension of random attractors

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Zusammenfassung

In der vorliegenden Dissertation werden zufällige dynamische Systeme in Hilberträumen und deren Langzeitverhalten diskutiert. Der Schwerpunkt der Arbeit liegt auf der Abschätzung der Hausdorff-Dimension von zufälligen Attraktoren, welche ein wichtiges Merkmal für das Langzeitverhalten darstellen. Eine Besonderheit des ersten Teils der Arbeit ist, dass die Grundmenge des zugrunde liegenden Maßraums eine fraktale Menge ist. Eine solche Menge ist typischerweise eine Teilmenge eines euklidischen Raumes, hat ein leeres Inneres und keinen glatten Rand. Aufgrund dieser Eigenschaften ist eine klassische Differentiation von Funktionen auf diesen Mengen nicht möglich. Nach einer Einführung in die Analysis auf Fraktalen und dem zugehörigen Laplace-Operator wird ein zufälliges dynamisches System aus der Lösung einer stochastischen partiellen Differentialgleichung erzeugt und die Existenz eines eindeutigen zufälligen Attraktors diskutiert. Für die Hausdorff-Dimension dieses Attraktors wird im Anschluss eine obere Schranke hergeleitet, die von dem spektralen Exponent des Laplace-Operators abhängt. Insbesondere geben wir im Rahmen eines Beispiels einen numerischen Wert für die obere Schranke an. Der zweite Teil der Arbeit befasst sich mit einer stochastischen partiellen Differentialgleichung, welche von einem multiplikativen Rauschen getrieben wird. Wir beweisen die Existenz des zufälligen Attraktors der zugehörigen Dynamik und die Existenz einer invarianten instabilen Mannigfaltigkeit. Um eine untere Abschätzung für die Hausdorff-Dimension des Attraktors zu erhalten, projizieren wir eine Teilmenge der Mannigfaltigkeit, welche auch Teilmenge des Attraktors ist, auf den instabilen Teilraum des Hilbertraums.

Abstract

In this thesis we deal with random dynamical systems in Hilbert spaces and their long-time behavior. We focus on the derivation of bounds on the Hausdorff dimension of random attractors, which are characteristic for the long-time behavior of the dynamics. In the first part of the work the basic set of the underlying measure space is a fractal set. Such a subset of an Euclidean space has typically an empty interior and no smooth boundary. Therefore the classical differentiation of functions on these sets fails. After an introduction to the analysis on fractals and the related Laplacian we generate a random dynamical system from the solution of a stochastic partial differential equation and show the existence of an associated unique random attractor. In the sequel we derive an upper bound on the Hausdorff dimension of this random attractor, which depends on the spectral exponent of the considered Laplacian. In an example we compute a numerical value of the upper bound. Another result of the thesis deals with a different stochastic partial differential equation driven by a multiplicative noise. We prove the existence of the random attractor of the related dynamics as well as the existence of an unstable invariant manifold. Subsequently we project a subset of the manifold, which is also a subset of the random attractor, onto the unstable subspace of the Hilbert space. This approach allows us to obtain a lower bound on the Hausdorff dimension of the attractor.

Contents

Chapter 1

Introduction

In the present thesis the main object is to discuss the Hausdorff dimension for the random attractor of an associated random dynamical system.
A random dynamical system is a generalization of a (deterministic) dynamical system, which mathematically describes for example the motion of a particle over time, i.e. the change of its initial state in time. For a random dynamical system (RDS) we allow in addition a random influence which is modeled by a so called metric dynamical system. The main feature of a random dynamical system is the *cocycle* property, which replaces the semigroup property in the deterministic case.

As an illustrative example think of the pollution inside the oceans and imagine we could track the position of a plastic bottle that swims within some current. Since an ocean current behaves not like a steady stream and is disturbed for instance by the weather or the sea level, a good model for the trajectory of a plastic bottle within the current should allow random influence. If we are able to describe the position of the bottle at a certain time by the solution of a stochastic differential equation (e.g. stochastic Navier-Stokes equation), then a random dynamical system can give us a priori information on the possible trajectories of this bottle.

For finite-dimensional problems the generation of random dynamical systems from stochastic differential equations is described in detail in the book of Arnold [Arn10]. A main obstacle arises since the appearing stochastic integral is defined only almost surely whereas the cocycle property needs to hold for all ω of our probability space. In an infinite dimensional setup it is rather difficult to overcome this problem, but for equations with additive or multiplicative noise it is possible, see e.g. Caraballo, Langa and Robinson [CLR00], Crauel and Flandoli [CF94] or Schmalfuß [Sch92]. The standard method is to transform the SDE into a random (partial) differential equation (RDE) via the Ornstein-Uhlenbeck process. This RDE can be solved pathwise and the solution mapping generates a random dynamical system.
Having the classical theory in mind (see for instance Temam [Tem88] or Robinson [Rob01]), we can ask for qualitative properties that depict the long-time behaviour of the dynamical system such as attractors or invariant manifolds.
As a natural phenomena which describes a *random attractor* one can think of an ocean vortex, see e.g. [Ghi17]. Following the trajectory of the plastic bottle, discussed in the above example, we arrive after a sufficient long time in a vortex which changes its shape due to the random influences and the time. If we wait long enough the vortex attracts all the objects that are carried from the currents that started in an appropriate region. Mathematically these objects are characterized as random sets that are obtained by random dynamical systems. Note that an RDS is a non-autonoumous system, which makes it more difficult to predict time-dependent random sets, like the random attractor on large time scales. To work around this problem one defines the attractors in a pullback sense, cf. [Sch92]. The pullback dynamics gives us the advantage to study asymptotic behaviour for $t \to \infty$, since then the image of the initial sets under the RDS

are in a time-independent ω-fiber.
Invariant manifolds of an RDS are positively invariant subsets of the corresponding space that are constructed around a random fixed-point of the RDS and the set has a graph-like structure. In particular we distinguish between stable and unstable manifolds for instance we refer to [OS13], [LS07]. An unstable manifold at a fixed point is roughly speaking the set of initial points that stay inside the manifold under the image of the cocycle and tends exponentially fast to the fixed-point when $t \to -\infty$. Conversely, the points from the stable manifold tends exponentially fast to the fixed-point for $t \to \infty$.

We aim to estimate the Hausdorff dimension of the previously discussed random attractors. The dimension of a set gives an answer to the question how much this set fills the space. It also tells us how much an element in this set can move, i.e. the number of degrees of freedom. This perception becomes more difficult when a dimension is non-integer, but it can help us to compare sets or to decide if they are comparable. Note that the dimension is a geometric property and there is a big variety of different definitions under consideration of the given assumptions or the chosen properties of the set. These other definitions can give us again more infomation on the considered set.
The *Hausdorff dimension* goes back to Felix Hausdorff in 1919 and gives us the possibility to assign a dimension to any set in a metric space. Its definition is based on the Hausdorff measure, which is a generalization of the Lebesgue measure and it is constructed by a countable family of infinitesimal covering sets. A wide overview of the concepts 'Hausdorff measure' and 'Hausdorff dimension' can be found in [Fal90]. In the last decades the Hausdorff dimension has become more important due to the arising theory of fractal geometry, where often sets with non-integer Hausdorff dimension appear. In this context the most common and well-studied objects are *self-similar fractals* like the Cantor set, the Sierpinski gasket, the Mandelbrot set or the Menger sponge. Although these examples are purely mathematical, they have become more and more interesting for natural scientists. Examples such as the percolation through porous structures or the diffusion across conductive layers have supports that are modeled with fractal sets, see e.g. the references in [Fre05]. In the 80s and 90s there have been various approaches to define a meaningful analysis on these sets. For a diffusion the Laplacian is of particular interest. An overview of the analytic approach for the definition of a Laplacian on a fractal set, using graph approximations and energy forms, can be found in [Kig01]. This well-elaborated theory offers the possibility to study partial differential equations on fractal sets.

The thesis is organized as follows: in the second chapter we start with a short repetition of known statements concerning semigroups and linear unbounded operators in Hilbert spaces. Subsequently, we discuss the Wiener process with values in a Hilbert space and introduce SPDEs and their mild solution by taking an example. Finally, we make a brief sketch of the analysis on a class of fractals. As an example we discuss the Sierpinski gasket. Chapter 3 begins with a summary of the necessary terminology for random dynamical systems and metric dynamical systems and we introduce the stationary Ornstein-Uhlenbeck process. Moreover we show related properties of this process which are in our interest. We introduce the SPDE which we consider for our main result in Chapter 4. The SPDE gets transformed into an RDE and we prove the existence of a random dynamical system and its random attractor. In the end we use a proper conjugacy to obtain a random attractor for the original SPDE. The upper bound on the Hausdorff dimension of the random attractor is tackled in Chapter 4. After a summary of the methods for the estimate in the deterministic and the random case we name conditions which are necessary to obtain the aimed estimate. We consider the RDE of Chapter 3 with the Laplacian introduced in Chapter 2. Then the upper estimate of the Hausdorff dimension depends especially on the spectral properties of the Laplacian. In a last part of this chapter we give an example of the considered nonlinearity and derive, in the case the underlying set is the Sierpinski gasket, a numerical value for the upper bound. In the last chapter of this dissertation we present a technique for a lower bound on the Hausdorff dimension of another random attractor. For this purpose we begin introducing an SPDE with a multiplicative noise and acquire the conjugated RDE. We prove similarly to Chapter 3 the existence of an RDS and an associated random attractor. Since the dimension estimate is based on invariant manifolds we introduce the related setting together with the unstable and stable subspaces of the corresponding Hilbert space. After using a typical cut-off function and the Lyapunov-Perron transform of the mild solution of the associated RDE we obtain an invariant manifold for the truncated version of the RDE. In the last section we show that this manifold is in fact a local unstable manifold for the original RDE. Moreover, the definition the local unstable manifold allows to identify a non-trivial subset of the random attractor. Using the projection onto the unstable subspace we derive a lower bound on the Hausdorff dimension.

The main results are the following: as far as we know the presented work is the first connection between the analysis on fractals and the infinite dimensional random dynamics. Although the results in Chapter 3 are classical statements we prove and check that they are applicable for the space of square integrable functions on a fractal set with an appropriate measure. In the fourth chapter we use the known theory of Temam and Debussche to show the dependence of the Hausdorff dimension of the random attractor on the spectral exponent. This exponent is a constant stemming from the asymptotic spectrum of the Laplacian constructed on the fractal. Moreover we remark that we discuss the upper estimate on the Hausdorff dimension for an equation with a Lipschitz continuous nonlinearity. This has not been discussed in the literature. In particular we see the dependence of the upper bound on the Lipschitz constant. To the best of our knowledge there are no well-known examples of a numerical value of the upper bound on the Hausdorff dimension, hence Section 4.3 is completely new. The last chapter combines two concepts concerning random dynamics, random attractors and random invariant manifolds. First, we comment the proof of the global unstable manifold in Section 5.2. Although the proof is structured like similar proofs in the related literature we need to treat carefully the additional factor involving the time integral over the Ornstein-Uhlenbeck process. Analogously we have to regard this factor in all the subsequent statements of the local invariant manifold. The associated proofs have not been seen in publications that are known to the author. Finally, note that we take assumptions to obtain a random attractor and a local unstable manifold at zero. Typically these two objects are not considered together. Here we show that a subset of the manifold is included in the random attractor. The idea to use an orthogonal projection to estimate a rough lower bound on the Hausdorff dimension is a classical approach in the theory of fractal geometry. However it has not been used in the context of random attractors.

Chapter 2

Preliminaries

In this chapter we introduce the basics for our work. In the first section we discuss the theory of strongly continuous and analytic semigroups. The second section presents an overview of the noise that will drive our stochastic partial differential equations. In Section 3 we briefly illustrate the theory of solutions for stochastic partial differential equation, which are in our interest. The last section is devoted to a rather new topic, fractal geometry and analysis for functions on these sets. This section will support in particular our main theorem in Chapter 4.

2.1 Semigroup theory in Hilbert spaces

C_0-semigroups

Before we present the differential equations we are working with we introduce all the components that are necessary. The most results of the following section can be defined for Banach spaces but since we consider later random partial differential equations an L^2-space, we state all results for Hilbert spaces.
A *Hilbert space*, usually denoted by H, is a complete normed vector space equipped with an inner product $(\cdot,\cdot)_H : H \times H \to \mathbb{R}_+ = [0,\infty)$. The norm $\|\cdot\|_H := \sqrt{(\cdot,\cdot)_H}$ is induced by the inner product. In the following we omit the index H. If we consider norms resp. products in other spaces we will denote it by using the notation of the corresponding space as an index to the norm resp. product.

Later we will be mainly interested in the space of all functions which are square - integrable w.r.t. a σ-finite measure. More precisely, let $(E,\mathcal{E},\mu)$ be a complete σ-finite measure space. Then we define the L^2-space,

$$L^2(E,\mu) := \left\{ f : E \to \mathbb{R} : f \text{ is } \mu\text{-measurable}, \ \int_E |f|^2 \, d\mu < \infty \right\} .$$

Recall the following important property, cf. [Alt16, Theorem 9.8, p.294]. A Hilbert space H is *separable* if and only if there exists an *orthonormal basis (ONB)* $\{e_i\}_{i\in\mathbb{N}} \subset H$. The space H is isometrically isomorph to the space of square summable sequences, denoted by $\ell_2(\mathbb{R})$ (see [Alt16, Theorem 9.8 (Note), p.294]).

The following foundations stem from [Paz83, Chapter 1] and [SY02, Chapter 3].

Definition 2.1.1. Let H be a Hilbert space. A family $\{S(t)\}_{t\geq 0}$ of bounded linear operators mapping from H to H is called a *semigroup* of bounded linear operators, if the following two conditions hold:

(i) $\quad S(0) = \mathrm{Id}_H\,,$

(ii) $\quad S(t+s) = S(t)S(s), \quad \forall\, s,t \in \mathbb{R}_+\,.$

The second item is called the *semigroup property* of $\{S(t)\}_{t\geq 0}$. If in addition

(iii) $\lim\limits_{t\downarrow 0} S(t)x = x, \quad$ for every $x \in H\,,$

then $\{S(t)\}_{t\geq 0}$ is called a *strongly continuous semigroup (in zero)*, often referred to as C_0-semigroup.
A linear operator $A : D(A) \to H$ is called the *infinitesimal generator* of the semigroup $\{S(t)\}_{t\geq 0}$ with domain

$$D(A) = \left\{ x \in H : \lim_{t\downarrow 0} \frac{S(t)x - x}{t} \text{ exists} \right\} \subseteq H\,.$$

For $x \in D(A)$ we have

$$Ax = \lim_{t\downarrow 0} \frac{S(t)x - x}{t} = \left.\frac{d^+ S(t)x}{dt}\right|_{t=0}\,.$$

Note that for the definition of the infinitesimal generator the strongly continuity of $\{S(t)\}_{t\geq 0}$ is not necessary.
In the following we state some important properties of C_0-semigroups. By $L(H)$ we denote the space of linear bounded operators mapping from H into itself.
All statements can be found proven in Pazy [Paz83, p.4f]

Theorem 2.1.2. *Let $\{S(t)\}_{t\geq 0}$ be a C_0-semigroup. Then there exist constants $\alpha \geq 0$ and $M \geq 1$ such that*

$$\|S(t)\|_{L(H)} \leq M e^{\alpha t}$$

for $t \in \mathbb{R}_+$.

We mention that $\{S(t)\}_{t\geq 0}$ is called *uniformly bounded* if $\alpha = 0$. $\{S(t)\}_{t\geq 0}$ is called *C_0-semigroup of contractions* if in addition $M = 1$, i.e. $\|S(t)\|_{L(H)} \leq 1$ for every $t \geq 0$.

Corollary 2.1.3. *Let $\{S(t)\}_{t\geq 0}$ be a C_0-semigroup. The mapping*

$$S(\cdot)x : \mathbb{R}_+ \to H, \quad t \mapsto S(t)x \in H$$

is continuous for every $x \in H$.

Theorem 2.1.4. *Let $\{S(t)\}_{t\geq 0}$ be a C_0-semigroup and A be its infinitesimal generator. Then*

a) for $x \in H$, $t \geq 0$

$$\int_0^t S(s)x\,ds \in D(A) \quad \textit{and} \quad A\left(\int_0^t S(s)x\,ds\right) = S(t)x - x,$$

b) for $x \in D(A)$, $0 \leq s < t$

$$\int_s^t AS(r)x\,dr = \int_s^t S(r)Ax\,dr = S(t)x - S(s)x\,.$$

Definition 2.1.5. An operator A on a Hilbert space H is *bounded from below*, if there exists an $a \in \mathbb{R}$ such that

$$a\|u\|^2 \leq (Au, u)$$

for all $u \in D(A)$. If $(Au,u) \geq 0$, then A is said to be *non-negative*.

We introduce the concept of the resolvent of a linear operator A.

Definition 2.1.6. Let H be a Hilbert space and $A \in L(H)$. Then

$$\varrho(A) = \{\lambda \in \mathbb{C} : \exists (\lambda \mathrm{Id}_H - A)^{-1} \in L(H)\}$$

is called the *resolvent set* of A and

$$\sigma(A) = \mathbb{C} \setminus \varrho(A)$$

is called the *spectrum* of A. In general the spectrum of a closed operator consists of the point, residual and continuous spectrum [Kow09, p.70f]. We emphasize the point spectrum, since it will play an important role in the whole thesis.

The *point spectrum* of a closed operator A is given by the set

$$\{\lambda \in \mathbb{C} : \exists u \in H, u \neq 0, (\lambda \mathrm{Id}_H - A)u = 0\},$$

its elements are called *eigenvalues* of A. The condition above is equivalent to $u \in \mathrm{Ker}(\lambda \mathrm{Id}_H - A)$. This implies as well that $\lambda \mathrm{Id}_H - A$ is *not* injective.
The family of bounded linear operators

$$R(\lambda, A) := (\lambda \mathrm{Id}_H - A)^{-1}, \quad \text{for } \lambda \in \varrho(A)$$

is called the *resolvent* of A. For more information concerning resolvents see e.g. [SY02, p.66] and [EN99, p.133].
We say that an operator A has a *compact resolvent*, if the resolvent $R(\lambda, A)$ is compact for one $\lambda \in \varrho(A)$ (hence for every $\lambda \in \varrho(A)$), see [EN99, Chapter 2, Definition 4.24, p.117].

The following spectral theorem is presented entirely in [DSBB63, p.1331, Theorem 2] and shows the impressive consequence of having a compact resolvent of a self adjoint operator.

Theorem 2.1.7. *Let A be a self-adjoint operator such that $R(\lambda, A)$ is compact for non-real λ. Then*

(i) the spectrum of A is a sequence of points on $\mathbb{R}$ with no finite limit point and

(ii) each $\lambda \in \sigma(A)$ belongs to the point spectrum of A and has a finite geometric multiplicity.

We want to point out the following consequence of the *spectral mapping theorem*, which is an important tool for many applications concerning the mappings of (self-adjoint) operators. We present the general theorem of N.Dunford and J.T.Schwartz [DSBB63, XII Section 2 and 3] applied in our interest.

Theorem 2.1.8. *Let H be a separable Hilbert space. Assume T is a self-adjoint, positive operator on H and $f : \sigma(T) \to \mathbb{R}$ is a bounded and continuous mapping. Further let $(\lambda_i)_{i=1}^{\infty} \subset \sigma(T)$ with corresponding eigenvectors $(e_i)_{i=1}^{\infty}$ which form an ONB of H. Then we define for elements from the domain*

$$D(f(T)) := \left\{ u \in H : \sum_{i=1}^{\infty} |f(\lambda_i)|^2 \cdot |(u, e_i)|^2 < \infty \right\}$$

the operator $f(T) : D(f(T)) \to H$ by

$$f(T)u = \sum_{i=1}^{\infty} f(\lambda_i)(u, e_i)e_i \, .$$

We close the subsection with the theorem of Hille-Yoshida, which states the conditions guaranteeing that A is the infinitesimal generator of a C_0-semigroup.

Theorem 2.1.9. *A linear operator A is the infinitesimal generator of C_0-semigroup if and only if*

(i) A is a closed operator and $D(A)$ is dense in H

and

(ii) there exists an $\alpha \in \mathbb{R}$ such that $\{\lambda \in \mathbb{R} : \lambda > \alpha\} \subset \varrho(A)$ and for the resolvent of A it holds

$$\|R(\lambda, A)^n\|_{L(H)} \leq \frac{M}{(\lambda - \alpha)^n}$$

for $\lambda > \alpha$ and $n \in \mathbb{N}$.

Analytic semigroups

For the work with evolution equations in infinite dimensional spaces the theory of analytic semigroups is significant. This theory allows us to introduce fractional powers for generators of analytic semigroups. We point out a compactness argument between two interpolation spaces which will be of interest for the random attractor in Chapter 3. For this subsection we refer to [SY02, p.76ff] and [Paz83, p.60ff].

Definition 2.1.10. Let D be an open set of $\mathbb{C}$ and H a Hilbert space. An H-valued function $u : D \to H$ is called *analytic* if for any $z_0 \in D$ the limit

$$\lim_{z \to 0} \frac{1}{z} \left(f(z + z_0) - f(z_0) \right)$$

does exist in H.

We introduce for $\delta \in (0, \pi)$ and $a \in \mathbb{R}$ the following open sectors in the complex plane

$$\begin{aligned} \Delta_\delta &:= \{z \in \mathbb{C} : |\arg z| < \delta,\ z \neq 0\}, \\ \Delta_\delta(a) &:= a + \Delta_\delta = \{z \in \mathbb{C} : |\arg(z - a)| < \delta,\ z \neq a\}, \end{aligned}$$

where $z \in \mathbb{C}$ can be represented by $z = |z|e^{i \arg z}$. For a better understanding we outline the sector Δ_δ in the following picture.

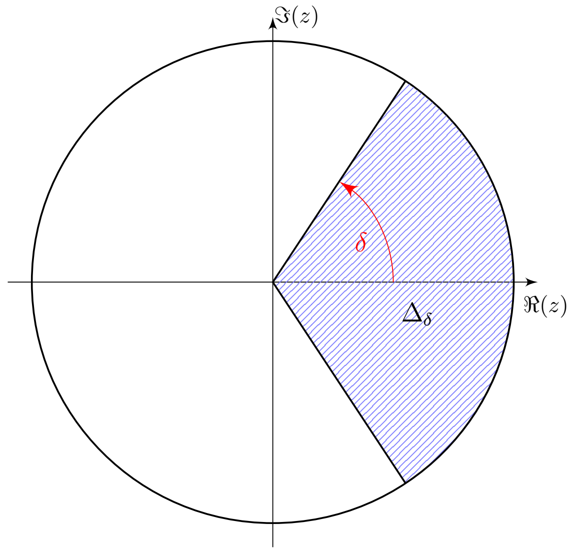


Figure 2.1: A sector in the complex plane for an angle $\delta > 0$.

Definition 2.1.11. Let $\{S(t)\}_{t\geq 0}$ be a C_0-semigroup on H. We call $\{S(t)\}_{t\geq 0}$ an *analytic semigroup*, if there exists an analytic extension of $\{S(t)\}_{t\geq 0}$ fulfilling the four conditions:

(i) for every $z \in \Delta_\delta \cup \{0\}$ the image $S(z)$ belongs to $L(H)$,

(ii) $S(z_1 + z_2) = S(z_1)S(z_2)$ for every $z_1, z_2 \in \Delta_\delta \cup \{0\}$,

(iii) for every $x \in H$ and $z \in \Delta_\delta \cup \{0\}$, $\lim_{z\to 0} S(z)x = x$,

(iv) the mapping $S(\cdot)x : \Delta_\delta \to H$, $z \mapsto S(z)x$ is an analytic mapping for each $x \in H$.

The next natural question that arises, is under which assumptions A generates an analytic semigroup. An answer to this question can for example be found in Pazy [Paz83, p.30, p.61] and in Sell and You [SY02, p.79]. We summarize the necessary assumptions on A in the next definition.

Definition 2.1.12. A linear operator $A : D(A) \to H$ is called a *sectorial operator* if it satisfies the following properties,

(i) $D(A)$ is dense in H and A is a closed operator,

(ii) there exist real numbers $a \in \mathbb{R}, \sigma \in (0, \frac{\pi}{2})$ and $M \geq 1$ such that

  (1) $\Sigma_\sigma(a) \subset \varrho(A)$ and
  (2) $\|R(\lambda, A)\|_{L(H)} \leq \frac{M}{|\lambda - a|}$

  for all $\lambda \in \Sigma_\sigma(a)$.

A sectorial operator is called *positive*, if inequality (2) holds for some $a > 0$.

According to Lemma 36.1 [SY02] if A is a sectorial operator then $-A$ generates an analytic semigroup. The following theorem is fundamental for our theory and is stated by Sell and You in [SY02, p.68, Theorem 32.1].

Theorem 2.1.13. *Let A be a self-adjoint operator on a Hilbert space H. Further let $(e_i)_{i=1}^\infty$ be the eigenfunctions of A with corresponding eigenvalues $(\lambda_i)_{i=1}^\infty$. Assume A is bounded from below and has a compact resolvent. Then we obtain the following.*

(1) The linear operator $-A$ generates a C_0-semigroup $\{S(t)\}_{t\geq 0}$, which can be represented by

$$S(t)u = \sum_{i=1}^{\infty} e^{-\lambda_i t}(u, e_i)e_i$$

where the sequence $(e_i)_{i=1}^\infty$ forms an orthonormal basis in H. Moreover for all $t \geq 0$

$$\|S(t)\|_{L(H)} \leq e^{-\lambda_1 t} \tag{2.1.1}$$

where λ_1 is the smallest eigenvalue of A.

(2) For each $t > 0$ the operator $S(t)$ is compact.

(3) The semigroup $\{S(t)\}_{t\geq 0}$ is analytic and the linear operator A is sectorial.

(4) For any fixed $t > 0$

$$\lim_{h\to 0} \|S(t+h) - S(t)\|_{L(H)} = 0\,,$$

which expresses the norm continuity of the semigroup $\{S(t)\}_{t\geq 0}$.

For the proof of these statements we refer to [SY02, p.67f, p.81f]. Since the assertions (1) and (3) are important for the following Theorem 2.1.16 we provide a proof.

Proof. We start with the statement (1).
Theorem 2.1.7 asserts that the spectrum consists of isolated points and has no finite accumulation point. For an orthonormal basis we choose the set of normalized eigenvectors e_i of the corresponding eigenvalues λ_i, $i \in \mathbb{N}$ of the operator A, such that

$$Ae_i = \lambda_i e_i$$

for every $i \in \mathbb{N}$ and every eigenvalue $\lambda_i \in [a, \infty)$, where $a \in \mathbb{R}$ is the lower bound presented in Definition 2.1.5. For any $u \in H$ the Fourier series expansion is given by

$$u = \sum_{i=1}^{\infty} (u, e_i) e_i .$$

Moreover the Parseval equality ([Alt16, Definition 9.7, p.293]) holds, that is

$$\|u\|^2 = \sum_{i=1}^{\infty} |(u, e_i)|^2 .$$

According to Theorem 2.1.8 the domain of A can be characterized by

$$D(A) = \left\{ u \in H : \sum_{i=1}^{\infty} |\lambda_i|^2 |(u, e_i)|^2 < \infty \right\}$$

and the operator A is then given by

$$Au = \sum_{i=1}^{\infty} \lambda_i (u, e_i) e_i \tag{2.1.2}$$

for $u \in D(A)$. Taking the inner product in (2.1.2) with u entails

$$(Au, u) = \sum_{i=1}^{\infty} \lambda_i |(u, e_i)|^2 \geq \lambda_1 \|u\|^2 .$$

Therefore we can assume w.l.o.g. that the constant a in Definition 2.1.5 is equal to λ_1. Now apply Theorem 2.1.8 with $f(\lambda) = e^{-\lambda t}$, $\lambda \in \mathbb{R}$ and we obtain

$$S(t)u := \sum_{i=1}^{\infty} e^{-\lambda_i t} (u, e_i) e_i . \tag{2.1.3}$$

We see easily that $S(t) \in L(H)$ and $\|S(t)\| \leq e^{-\lambda_1 t}$, $t \geq 0$,

$$\|S(t)u\|^2 = \sum_{i=1}^{\infty} e^{-2\lambda_i t} |(u, e_i)|^2 \leq \sum_{i=1}^{\infty} e^{-2\lambda_1 t} |(u, e_i)|^2 = e^{-2\lambda_1 t} \|u\|^2 .$$

The generated semigroup (2.1.3) fulfills trivially the identity property. The semigroup property follows since

$$S(t)S(s)u = \sum_{i=1}^{\infty} e^{-\lambda_i t} (S(s)u, e_i) e_i = \sum_{i=1}^{\infty} e^{-\lambda_i (t+s)} (u, e_i) e_i = S(t+s)u$$

for every $t, s \in \mathbb{R}_+$ and $u \in H$.
The strong continuity of $\{S(t)\}_{t\geq 0}$ follows by a typical $\varepsilon - \delta$ argument when considering the term $\|S(t)u-u\|^2$ for $t \downarrow 0$. In particular one can choose a number $N \geq 1$ such that $\lambda_i > 0$ for $i \geq N+1$ hence $(e^{-\lambda_i t} - 1)^2 \leq 1$ for every $t \geq 0$. For $1 \leq i \leq N$ we use the continuity of $e^{-\lambda_i t}$ in $t = 0$.
It remains to show $-A$ is the generator of the C_0-semigroup. For this purpose denote B as the infinitesimal generator of the semigroup $\{S(t)\}_{t\geq 0}$. For a fixed $N \in \mathbb{N}$ we define the orthogonal projection $P_N : H \to \operatorname{span}\{e_1, ..., e_N\}$ by $P_N u := \sum_{i=1}^N (u, e_i)e_i$. We only show $D(A) \subset D(B)$ and $Bu = -Au$ for every $u \in D(A)$. The inverse inclusion is very similar and can be reviewed in [SY02, Section 3.2, p.68]. First consider

$$-Au = \sum_{i=1}^{\infty} -\lambda_i (u, e_i)e_i = \lim_{N\to\infty} \sum_{i=1}^{N} -\lambda_i (u, e_i)e_i,$$

then since $\frac{1}{h}\left(e^{-\lambda_i h} - 1\right) \stackrel{h\downarrow 0}{\longrightarrow} -\lambda_i$ we obtain

$$\begin{aligned}
-Au &= \lim_{N\to\infty} \sum_{i=1}^{N} \lim_{h\to 0^+} \frac{1}{h}\left(e^{-\lambda_i h} - 1\right)(u, e_i)e_i = \lim_{N\to\infty} \lim_{h\to 0^+} P_N \left(\frac{1}{h}(S(h) - \mathrm{Id}_H)u\right) \\
&= \lim_{N\to\infty} P_N \left(\lim_{h\to 0^+} \frac{1}{h}(S(h) - \mathrm{Id}_H)u\right) = \lim_{N\to\infty} P_N(Bu) .
\end{aligned}$$

The final step is to conclude that $\lim_{N\to\infty} P_N(Bu) = Bu$. But this follows directly from the minimum property for Fourier coefficients $\alpha_i \in \mathbb{R}$, which is an application of the principle of uniform boundedness (see e.g. [KA78, Chapter 7, Theorem 1, p.202]),

$$\left\| Bu - \sum_{i=1}^{N} \alpha_i e_i \right\|^2 \text{ is minimal } \iff \alpha_i = (Bu, e_i),\ i = 1, ..., N .$$

Now we shortly discuss the assertion (3). We will not give a complete proof. We need to verify that for some $\delta \in (0, \pi)$ and arbitrary $z \in \Delta_\delta \cup \{0\}$, $S(z) \in L(H)$. First we can replace t by a complex variable z in (2.1.3), such that for every $u \in H$

$$S(z)u := \sum_{i=1}^{\infty} e^{-\lambda_i z}(u, e_i)e_i .$$

Similar to the proof of part (1)

$$\|S(z)u\|^2 = \sum_{i=1}^{\infty} \left|e^{-\lambda_i z}\right|^2 |(u, e_i)|^2 = \sum_{i=1}^{\infty} e^{-2\lambda_i \Re(z)} |(u, e_i)|^2 \leq e^{-2\lambda_1 \Re(z)} \|u\|^2 .$$

if $\Re(z) > 0$. Note that this condition is equal to choose an element $z \in \Delta_{\frac{\pi}{2}} \cup \{0\}$. The semigroup property follows exactly as before. Similar to the reell case the strong continuity of $S(\cdot)u$ on $\Delta_{\frac{\pi}{2}} \cup \{0\}$ for $u \in H$ can be obtained.
Since it is rather technical to show that the mapping $S(\cdot)u$ is analytic for all $u \in H$ on the sector $\Delta_{\frac{\pi}{2}}$ and that A is a sectorial operator, we refer to the general proofs in [SY02, Theorem 36.2] and [EN99, Proposition 4.3] which can be applied to the defined semigroup. $\square$

We introduce the notations of fractional powers for the operator A and the corresponding domains applying Theorem 2.1.8, see [SY02, p.92ff].

Definition 2.1.14. Let A be a self-adjoint, positive operator with a compact resolvent on a Hilbert space H. The eigenvalues of A are denoted by $(\lambda_i)_{i=1}^{\infty}$ and their corresponding eigenfunctions by $(e_i)_{i=1}^{\infty}$.
For any $\alpha \geq 0$ we define the domain of the fractional power A^α by

$$V^{2\alpha} := D(A^\alpha) = \left\{ u \in H : \sum_{i=1}^{\infty} \lambda_i^{2\alpha} |(u, e_i)|^2 < \infty \right\} \tag{2.1.4}$$

and the operator A^α by

$$A^\alpha u = \sum_{i=1}^{\infty} \lambda_i^\alpha (u, e_i) e_i$$

for all $u \in D(A^\alpha)$.

For each $\alpha \geq 0$ the set V^α is a linear subspace in H and $V^0 = H$. The space V^α becomes a Hilbert space, if it is equipped with the V^α-inner product

$$(u, v)_\alpha := \sum_{i=1}^{\infty} \lambda_i^\alpha (u, e_i)(v, e_i) \tag{2.1.5}$$

and the V^α-norm given by

$$\|u\|_\alpha^2 := \|A^\alpha u\|^2 = \sum_{i=1}^{\infty} \lambda_i^\alpha |(u, e_i)|^2 \tag{2.1.6}$$

for every $u, v \in V^\alpha$, $\alpha \in \mathbb{R}_+$. Let us summarize the following properties. If $\beta \leq \alpha$, then $V^\alpha \subseteq V^\beta$ and

$$\|u\|_\beta^2 \leq \lambda_1^{\beta-\alpha} \|u\|_\alpha^2$$

for every $u \in V^\alpha$. Hence V^α is continuously embedded in V^β, i.e. $V^\alpha \hookrightarrow V^\beta$. Moreover V^α is dense in V^β. For $\alpha \geq \beta$ let us identify each real number between them by $\gamma = \theta\alpha + (1-\theta)\beta$ for $\theta \in [0,1]$. One can easily check that $V^\alpha \hookrightarrow V^\gamma \hookrightarrow V^\beta$ and the following interpolation inequality holds

$$\|u\|_\gamma \leq \|u\|_\alpha^\theta \|u\|_\beta^{1-\theta} \tag{2.1.7}$$

for all $u \in V^\alpha$ and $\theta \in [0,1]$. A family of Banach spaces V^α with norms $\|\cdot\|_\alpha$ defined for $\alpha \in I \subset \mathbb{R}$ with the above properties is called a *family of interpolation spaces on I*. We point out a statement of Theorem 37.2 in Sell and You [SY02, p.94] concerning the embedding between two interpolation spaces.

Theorem 2.1.15. *Let A be a positive, self-adjoint linear operator with a compact resolvent on a Hilbert space H. Let the Hilbert space V^α be defined by* (2.1.4)*,* (2.1.5) *and* (2.1.6)*.*
If $\alpha > \beta$ then $V_\alpha \hookrightarrow V_\beta$ is a compact embedding.

The fundamental theorem on sectorial operators will be of use for the next chapter and is proven in Sell and You [SY02, p.97, Theorem 37.5]. We focus only on one property, which will help us with a compactness argument.

Theorem 2.1.16. *Let A be a positive sectorial operator on a Hilbert space H and $S(t)$ be an analytic semigroup generated by $-A$.*
Then for any $r \geq 0$, there exists a constant $M_r > 0$ such that for all $t > 0$

$$\|S(t)\|_{L(H, D(A^r))} = \|A^r S(t)\|_{L(H)} \leq \frac{M_r}{t^r} e^{-\alpha t}, \tag{2.1.8}$$

where $\alpha > 0$. According to Definition 2.1.5 and Theorem 2.1.13 (1) we can choose $\alpha \leq a = \lambda_1$ the smallest positive eigenvalue of A.

Remark 2.1.17. Later the operator A will be responsible for the diffusion of a (partial) differential equation. The usage of the algebraic sign of the operator A on right-hand side of a differential equation is not consistent in the literature, for instance compare A. Pazy [Paz83, Section 4.1, Section 6.1] or G. R. Sell and Y. You [SY02, Section 3.8.1].
For example let A be the negative Laplacian $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ on some smooth domain of $\mathbb{R}^n$, then the operator has to appear on the right-hand side of a differential equation without a minus sign. In particular the statements of the last theorem hold if we change everywhere A by $-A$. Then $-A = -\Delta$ is positive and (2.1.8) becomes

$$\|S(t)\|_{L(H,D((-A)^r))} = \|(-A)^r S(t)\|_{L(H)} \leq \frac{M_r}{t^r} e^{-\alpha t}\,.$$

Note that this is always a question of definition.

2.2 Wiener processes in a Hilbert space

The theory of canonical processes is of great relevance for random dynamical systems. Therefore we make a short introduction to these processes and in particular we consider the Q-Wiener process. The first part of this section is based on the general construction of stochastic processes in [Bau02, p.304f.].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with σ-algebra $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ and probability measure $\mathbb{P} : \mathcal{F} \to [0,1]$. Let $\mathbb{T} \subseteq \mathbb{R}$ be our non-empty time space endowed with a proper topology such that $\mathcal{B}(\mathbb{T})$ is well-defined. Further we consider the measurable space $(H, \mathcal{B}(H))$ with the Hilbert space H and the Borel-σ-algebra $\mathcal{B}(H)$.

Definition 2.2.1. An (H-valued) *stochastic process* is a family of H-valued random variables $X = (X_t)_{t\in\mathbb{T}}$, i.e. $X : \mathbb{T} \times \Omega \to H$. The stochastic process is called *jointly measurable* if $(t,\omega) \mapsto X_t(\omega) = X(t,\omega) \in H$ is $\mathcal{B}(\mathbb{T}) \otimes \mathcal{F} - \mathcal{B}(H)$-measurable.
The mapping $X(\cdot,\omega) : \mathbb{T} \to H$, $t \mapsto X_t(\omega)$ for fixed $\omega \in \Omega$ is called *path* of the process X and corresponds to an ω-wise evaluation of the process. A path is also called trajectory, sample or realization of X.

In the following let H be a separable Hilbert space. We define

$$X_{\mathbb{I}} = (X_{i_1}, X_{i_2}, ..., X_{i_n}) \in H^n := \{f : \{i_1, ..., i_n\} \to H\}$$

for a set $\mathbb{I} = \{i_1, ..., i_n\} \subset \mathbb{T}$. The probability measure $\mathbb{P}_{\mathbb{I}}$ on $\mathcal{B}(H^n)$ is defined as the distribution of $X_{\mathbb{I}}$ w.r.t. $\mathbb{P}$, i.e. $\mathbb{P}_{\mathbb{I}} := \mathbb{P}_{X_{\mathbb{I}}}$. Let $\mathbb{J}$ and $\mathbb{I}$ be two non-empty subsets of $\mathbb{T}$ and $\mathbb{J} = \{j_k\}_{k=1}^{l} \subset \mathbb{I}$, s.t. for every $k \in \{1,..,l\}$, $j_k \in \{i_1, ..., i_n\}$, $l < n$. We define the measurable projection

$$\begin{aligned} p_{\mathbb{J}}^{\mathbb{I}} :& H^n \to H^l, \\ H^n \ni B = B_{i_1} \times B_{i_2} \times ... \times B_{i_n} &\mapsto p_{\mathbb{J}}^{\mathbb{I}}(B) = B_{j_1} \times ... \times B_{j_l} \end{aligned}$$

Hence the projection maps each element in H^n to its restriction in H^l. We denote $\mathcal{T} = \mathcal{T}(\mathbb{T})$ as the collection of all non-empty finite subsets of $\mathbb{T}$. Then the family of measures $(\mathbb{P}_{\mathbb{I}})_{\mathbb{I}\in\mathcal{T}}$ is called *finite dimensional distributions* of the process X. The family $(\mathbb{P}_{\mathbb{I}})_{\mathbb{I}\in\mathcal{T}}$ is called *projective*, if for each two sets $\mathbb{I}, \mathbb{J} \in \mathcal{T}$

$$p_{\mathbb{J}}^{\mathbb{I}}(\mathbb{P}_{\mathbb{I}})(\cdot) := \mathbb{P}_{\mathbb{I}}\left((p_{\mathbb{J}}^{\mathbb{I}})^{-1}(\cdot)\right) = \mathbb{P}_{\mathbb{J}}(\cdot)\,.$$

Since H is independent of the time we define $H^{\mathbb{T}}$ as the set of all mappings from $\mathbb{T}$ to H.

Theorem 2.2.2 (Kolmogorov Existence- [Bau02], Satz 35.3)**.** *Let $(H, \mathcal{B}(H))$ be a separable Hilbert space and let $\mathbb{T} \subset \mathbb{R}$ be a non-empty time set. For each projective family of probability measures $(\mathbb{P}_{\mathbb{I}})_{\mathbb{I}\in\mathcal{T}}$ on $(H^{\mathbb{I}}, \mathcal{B}(H^{\mathbb{I}}))$ there exists a unique probability measure $\mathbb{P}_{\mathbb{T}}$ on $(H^{\mathbb{T}}, \mathcal{B}(H^{\mathbb{T}}))$ with*

$$p_{\mathbb{I}}^{\mathbb{T}}(\mathbb{P}_{\mathbb{T}}) = \mathbb{P}_{\mathbb{I}}\,.$$

The σ-algebra $\mathcal{B}(H^{\mathbb{T}})$ can be generated in the following way

$$\mathcal{B}(H^{\mathbb{T}}) = \sigma\left(\left\{\left(p_{\mathbb{I}}^{\mathbb{T}}\right)^{-1}(B) \subset H^{\mathbb{T}} \,|\, \mathbb{I} \in \mathcal{T},\, B \in H^{\mathbb{I}}\right\}\right)\,. \tag{2.2.1}$$

Endowed with the product topology, $H^{\mathbb{T}}$ forms a topological space. Recall that every sequence $(\omega_n)_{n\in\mathbb{N}} \subset H^{\mathbb{T}}$ converges to an element $\omega \in H^{\mathbb{T}}$ in this topology if and only if for each $t \in \mathbb{T}$, $(\omega_n(t))_{n\in\mathbb{N}} \subset H$ converges to $\omega(t) \in H$.

If we choose the probability space $\Omega := H^{\mathbb{T}}$, $\mathcal{F} := \mathcal{B}(H^{\mathbb{T}})$ and $\mathbb{P} := \mathbb{P}_{\mathbb{T}}$ as formulated in the proof of Korollar 35.4, [Bau02, p.309] we obtain a *canonical* probability space. First note that each mapping $\Omega \ni \omega : \mathbb{T} \to H$, $t \mapsto \omega(t)$ is $\mathcal{B}(\mathbb{T}) - \mathcal{B}(H)$ measurable. For a fixed $t \in \mathbb{T}$ the $\mathcal{B}(H^{\mathbb{T}}) - \mathcal{B}(H)$ measurable projection is given by

$$p_{\{t\}}^{\mathbb{T}}(\omega) = \omega(t) \in H$$

and we define $X_t(\omega) := \omega(t)$ as the corresponding random variable. The process X is called the *canonical* process and its set of paths is equal to Ω. Note that the canonical process consists of coordinate functions.

Definition 2.2.3 (Appendix A.3, p.545, [Arn10])**.** Let $\{t, t_1, t_2, ..., t_k\}$ be a subset $\mathbb{T}$ for $k \in \mathbb{N}$. A stochastic process $X : \mathbb{T} \times \Omega \to H$ is called *stationary* if for every $t \in \mathbb{T}$,

$$\mathbb{P}_{t_1+t,...,t_k+t} = \mathbb{P}_{t_1,...,t_k}, \qquad k \geq 1\,.$$

In the context of the Definition 2.2.1 and succeeding, the notation of the distribution $\mathbb{P}_{t_1,...,t_k}$ should be understood in the following way

$$\mathbb{P}_{t_1,...,t_k}(A) = \mathbb{P}_{X_{t_1},...,X_{t_k}}(A) = \mathbb{P}(X_{t_1} \in A_1, ..., X_{t_k} \in A_k)$$

for a set $A := (A_1, ..., A_k) \in \mathcal{B}(H^k)$.

Definition 2.2.4. Let $X = (X_t)_{t\in\mathbb{T}}$ and $Y = (Y_t)_{t\in\mathbb{T}}$ be two stochastic processes on one probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the same state space $(H, \mathcal{B}(H))$. If for each $t \in \mathbb{T}$,

$$\mathbb{P}(\omega \in \Omega : X_t(\omega) = Y_t(\omega)) = 1$$

we call X a *modification* or *version* of Y and vice versa.

If two processes are modifications of each other, then they are *equivalent* processes as well, i.e. they have the same finite dimensional distributions.
Let $\mathbb{T} = \mathbb{R}$ and H be a separable Hilbert space. If $(Y_t)_{t\in\mathbb{R}}$ is a modification of another process $(X_t)_{t\in\mathbb{R}}$ and possesses only continuous paths then the process $(Y_t)_{t\in\mathbb{R}}$ is called *continuous modification* of $(X_t)_{t\in\mathbb{R}}$.
The following fundamental theorem by Kolmogorov, Loève and Chentsov states that under a certain condition on the moments of the considered process there exists a continuous modification.

Theorem 2.2.5 (Kolmogorov Continuity- [Kal97], Theorem 2.23)**.** *Let X be a process with time set $\mathbb{R}$ and values in a Hilbert space H and assume there exist constants $\alpha, \beta, c > 0$ such that*

$$\mathbb{E}(\|X_t - X_s\|^{\alpha}) \leq c|t-s|^{1+\beta}$$

for all $s, t \in \mathbb{R}$. Then X has a continuous modification. If in particular $c \in (0, \frac{\beta}{\alpha})$ then X is Hölder continuous with exponent c.

We denote by

$$C = C(\mathbb{R}, H)$$

the set all continuous functions mapping from $\mathbb{R}$ to H and by

$$C_0 = C_0(\mathbb{R}, H)$$

the set of all continuous functions of the corresponding space and with value zero in zero i.e. for $f \in C_0$ we have $f(0) = 0$. We want to collect some results concerning the σ-algebra of the set C, see [Bau02, p.336f].

The set C is an *essential* subset of $H^{\mathbb{R}}$ for a given projective family of probability measures $(\mathbb{P}_{\mathbb{J}})_{\mathbb{J} \in \mathcal{T}}$, i.e. there exists a stochastic process $(X_t)_{t\in\mathbb{R}}$ whose finite dimensional distributions are $\mathbb{P}_{\mathbb{J}}$ and C is its set of paths. But in general $C \notin \mathcal{B}(H^{\mathbb{R}})$ cf. [Bau02, Korollar 38.5]. Considering the Borel σ-algebra we can still find a connection to $\mathcal{B}(H^{\mathbb{R}})$. By [Bau92, Satz 31.6] and [Arn10, Appendix A2, p.544] we know that C endowed with the compact open topology on $\mathbb{R}$, given by the complete metric

$$d(\omega, \overline{\omega}) = \sum_{n=1}^{\infty} \frac{\|\omega - \overline{\omega}\|_n}{2^n(1 + \|\omega - \overline{\omega}\|_n)}, \quad \|\omega - \overline{\omega}\|_n := \sup_{t\in[-n,n]} \|\omega(t) - \overline{\omega}(t)\|, \quad \omega, \overline{\omega} \in C \tag{2.2.2}$$

is a *polish* space, in particular it is a separable topological space. We consider $\mathcal{B}(C)$ the Borel σ-algebra on C w.r.t. this topology.

Remark 2.2.6. Remember the general representation of a Borel σ-algebra $\mathcal{B}(H^{\mathbb{T}})$ for the space of mappings from $\mathbb{T}$ to H in (2.2.1). Following the discussion in [Bil68, Chapter 1, Section 3, p.19ff.] we obtain in the case of C the Borel σ-algebra

$$\mathcal{B}(C) = \sigma\left\{p^{-1}_{t_1,\dots,t_k}(A) \subset C : t_1, \dots, t_k \in \mathbb{R} \text{ for } k \geq 1, A \in \mathcal{B}(H^k)\right\} \tag{2.2.3}$$

for the projections $p_{t_1,\dots,t_k}(\omega) := (\omega(t_1), \dots, \omega(t_k)) \in H^k$. Keeping this in mind we mention the next important theorem.

Theorem 2.2.7 ([Bau02] Satz 38.6)**.** *Let H be a separable Hilbert space. Then*

$$\mathcal{B}(C) = C \cap \mathcal{B}(H^{\mathbb{R}})$$

in respect of both topologies, the compact open topology and the product topology.

Theorem 2.2.8 ([Bau02] Lemma 39.2)**.** *Suppose a stochastic process $X = (X_t)_{t\in\mathbb{R}}$ with values in a separable Hilbert space H possesses a continuous modification. Then the set C is essential w.r.t. the family of finite dimensional distributions of the given process and the process X is equivalent to the related C-canonical process, i.e. a process on the space $(C, C \cap \mathcal{B}(H^{\mathbb{R}}), \widetilde{\mathbb{P}})$ given by $\widetilde{X}_t(\omega) = \omega(t)$ for all $(t, \omega) \in \mathbb{R} \times C$. The measure $\widetilde{\mathbb{P}}$ is the restriction of the outer measure $\mathbb{P}^*_{\mathbb{R}}$ on $C \cap \mathcal{B}(H^{\mathbb{R}})$ with $\mathbb{P}^*_{\mathbb{R}}(C) = 1$.*

Remark 2.2.9.

- The transition from $\mathbb{P}_{\mathbb{R}}$ to the outer measure is necessary since the essential set C is not in $\mathcal{B}(H^{\mathbb{R}})$.
- The above theorem allows us for every (H-valued) process with a continuous modification to switch to an equivalent (H-valued) canonical process whose path space is the set of (H-valued) continuous functions on $\mathbb{R}$.

We will now consider the most important example of stochastic processes: the Wiener process. In our case we consider a Q-Wiener process, which is an H-valued analogy to the Wiener process with state space $\mathbb{R}$ (see for example [Bau02, Chapter 8, § 40]). For the following concepts we refer to [DPZ92, Chapter 4] and [PR07, Appendix B].

Definition 2.2.10. Let $(U, \|\cdot\|_U)$ and $(H, \|\cdot\|)$ be two separable Hilbert spaces. An operator $T \in L(U, H)$ is said to be a *nuclear* operator, if there exists a sequence $(a_i)_{i\in\mathbb{N}} \subset H$ and a sequence $(b_i)_{i\in\mathbb{N}} \subset U$ such that

$$Tx = \sum_{i=1}^{\infty} a_i (b_i, x)_U, \qquad \text{for all } x \in U$$

and

$$\sum_{i=1}^{\infty} \|a_i\| \cdot \|b_i\|_U < \infty \,.$$

By $L_1(U, H)$ we denote the space of all nuclear operator mapping from U to H.

Definition 2.2.11 ([DPZ92], Appendix C)**.** Let us consider a separable Hilbert space H and denote by $(e_i)_{i=1}^{\infty}$ an orthonormal basis in H. If $T \in L_1(H, H)$ we define the *trace* of T,

$$\text{Tr } T := \sum_{i=1}^{\infty} (Te_i, e_i).$$

For every nuclear operator mapping from H to H the trace of an operator Tr T is well defined (see [DPZ92, Proposition C.1]). The definition of $\text{Tr}\,(\cdot)$ does not dependent of the choice of the ONB. In the finite dimensional case this definition coincide with the trace of a matrix.

Moreover according to [DPZ92, Proposition C.2], a non-negative operater $T \in L(H)$ is nuclear if and only if for an ONB $(e_i)_{i\in\mathbb{N}} \subset H$

$$\text{Tr } T = \sum_{i=1}^{\infty} (Te_i, e_i) < +\infty \,.$$

We call a non-negative, symmetric operator $T \in L_1(H, H)$ *trace class* operator.
Pursuant to [PR07, Proposition 2.1.5] if $T \in L(H)$ is given by a symmetric (i.e. as well self-adjoint), non-negative operator, there exists an ONB $(v_i)_{i\in\mathbb{N}}$ of H, such that

$$Tv_i = \lambda_i v_i$$

for every $i \in \mathbb{N}$ where zero is the only accumulation point. The sequence of eigenvalues $(\lambda_i)_{i\in\mathbb{N}} \subset \mathbb{R}_+$ is bounded, since the operator T is bounded. If in addition T is trace class, then

$$\text{Tr } T = \sum_{i=1}^{\infty} (Te_i, e_i) = \sum_{i=1}^{\infty} \lambda_i < \infty \,.$$

The following definition presents the link to stochastic processes in Hilbert spaces.

Definition 2.2.12. Let H be a separable Hilbert space, $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $Q \in L(H)$ a non-negative, symmetric and trace class operator. An H-valued stochastic process $W(t)$, $t \geq 0$ is called *Q-Wiener process* if

(i) $W(0) = 0$

(ii) W has continuous paths $\mathbb{P}$-a.s.

(iii) W has independent increments, i.e.

$$W(t_0), W(t_1) - W(t_2), ..., W(t_n) - W(t_{n-1})$$

are pairwise independent for all $0 \leq t_1 < t_2 < ... < t_n < \infty$, $n \in \mathbb{N}$ and

(iv) W has normally distributed increments, i.e.

$$\mathbb{P} \circ (W(t) - W(s))^{-1} = \mathcal{N}(0, (t-s)Q)$$

for all $0 \leq s \leq t$.

The next proposition gives us a representation of a Q-Wiener process by a sequence of real-valued Brownian motions.

Proposition 2.2.13 ([DPZ92], Proposition 4.1)**.** *Assume that W is a Q-Wiener process. Then the following statements hold.*

(i) W is a Gaussian process on H with mean 0 and covariance operator tQ, $t \geq 0$.

(ii) For any $t \geq 0$ the process W has the following representation

$$W(t) = \sum_{i=1}^{\infty} \sqrt{\lambda_i} \beta_i(t) e_i \tag{2.2.4}$$

where $\beta_i(t) := \frac{1}{\sqrt{\lambda_i}}(W(t), e_i)$, $i \in \mathbb{N}$ are real valued Brownian motions mutually independent on $(\Omega, \mathcal{F}, \mathbb{P})$ and the series in (2.2.4) converges in $L^2(\Omega, \mathcal{F}, \mathbb{P}; H)$. As previously introduced $(e_i)_{i=1}^{\infty}$ is the sequence of eigenvalues of the operator Q with the related eigenvalues $(\lambda_i)_{i=1}^{\infty}$.

Remark 2.2.14. The Q-Wiener process W defined above has stationary increments, that is for every $r \in \mathbb{R}_+$ and $0 \leq s \leq t$

$$\mathbb{P}_{W(t+r)-W(s+r)} = \mathbb{P}_{W(t)-W(s)} = \mathcal{N}(0, (t-s)Q)\,.$$

The variance resp. the second moment of a Q-Wiener process is determined as follows

$$\mathbb{E}\left(\|W(t)\|^2\right) = \mathbb{E}\left(\sum_{i=1}^{\infty} \lambda_i \beta_i^2(t)(e_i, e_i)\right) = \sum_{i=1}^{\infty} \lambda_i \mathbb{E}\beta_i^2(t) = t \operatorname{Tr} Q < +\infty\,.$$

Hence we obtain another reason for the operator Q to have a finite trace.

Proposition 2.2.15 ([DPZ92], Proposition 4.2)**.** *For any symmetric, non-negative trace class operator Q on a separable Hilbert space H there exists a Q-Wiener process $W = (W(t))_{t \in \mathbb{R}_+}$.*

As we will see later (in Chapter 3) we will need a corresponding Q-Wiener process on the time set $\mathbb{R}$ not only $\mathbb{R}_+$. The standard procedure to obtain a Wiener process with two-sided time is the following (cf. [Box88, Section 3.1, p.33]). Let $(W'(t))_{t \geq 0}$ and $(W''(t))_{t \geq 0}$ be two independent Q-Wiener processes, then we call $W = W(t)_{t \in \mathbb{R}}$ defined as

$$W(t) := \begin{cases} W'(t), & t \geq 0, \\ W''(-t), & t < 0 \end{cases}$$

a *two-sided* Q-Wiener process or a Q-Wiener process on time $\mathbb{R}$. We want to fix the following conclusion from Theorem 2.2.8.

Corollary 2.2.16. *A (two-sided) Q-Wiener process $W = (W_t)_{t\in\mathbb{R}}$ is equivalent to its C-canonical Wiener process $\omega(\cdot) \in C$.*

In addition to the definition of nuclear operators 2.2.10 we want to introduce another class of operators, the class of Hilbert-Schmidt operators.

Definition 2.2.17 ([PR07], Definition B.0.5)**.** Let U, H be two separable Hilbert spaces. A bounded linear operator $T : U \to H$ is called *Hilbert-Schmidt* if

$$\sum_{k=1}^{\infty} \|Te_k\|^2 < \infty$$

for an ONB $(e_k)_{k\in\mathbb{N}} \subset U$. The space of Hilbert-Schmidt operators is denoted by $L_2(U, H)$.

If $H = U$ we write $L_2(U)$ instead of $L_2(U, U)$. The definition of a Hilbert-Schmidt operator and the associated norm

$$\|T\|^2_{L_2(U,H)} := \sum_{k=1}^{\infty} \|Te_k\|^2$$

do not depend on the choice of the ONB and together they form a separable Hilbert space with inner product

$$(T, S)_{L_2(U,H)} := \sum_{k=1}^{\infty} (Se_k, Te_k)$$

for $S, T \in L_2(U, H)$. Moreover $\|T\|_{L(U,H)} \leq \|T\|_{L_2(U,H)}$ (see [PR07, Proposition B.0.6 & B.0.7]).

Let U be a separable Hilbert space with inner product $(\cdot, \cdot)_U$. We assume there exists a Q-Wiener process with values in U and suppose the eigenvalues of Q, $(\lambda_k)_{k\in\mathbb{N}}$, are positive. We know already that there exists a sequence $(e_i)_{i=1}^{\infty}$ of eigenvectors of Q that forms an ONB of U, then for any $u \in U$

$$Qu = Q\left(\sum_{i=1}^{\infty} (u, e_i)_U e_i\right) = \sum_{i=1}^{\infty} \lambda_i (u, e_i)_U e_i \,.$$

Similiar to the Definition 2.1.14 the operator $Q^{\frac{1}{2}} \in L(U)$,

$$Q^{\frac{1}{2}} u = \sum_{i=1}^{\infty} \lambda_i^{\frac{1}{2}} (u, e_i)_U e_i$$

can be defined.

Proposition 2.2.18 ([PR07], Proposition 2.3.4)**.** *If $Q \in L(U)$ is non-negative and symmetric, then there exists exactly one element $Q^{\frac{1}{2}} \in L(U)$, which is non-negative, symmetric and $Q^{\frac{1}{2}} \circ Q^{\frac{1}{2}} = Q$. If, in addition,* Tr $Q < \infty$ *we have that $Q^{\frac{1}{2}} \in L_2(U)$, where $\|Q^{\frac{1}{2}}\|^2_{L_2(U)} = \operatorname{Tr} Q$ and $L \circ Q^{\frac{1}{2}}$ is an element of $L_2(U, H)$ for every $L \in L(U, H)$.*

By $Q^{-\frac{1}{2}}$ we denote the pseudo inverse of $Q^{\frac{1}{2}}$. For the definition of pseudo inverses and their properties check [PR07, Appendix C]. Further we define the space $U_0 := Q^{\frac{1}{2}}(U)$, which is a subspace of U and we endow it with the inner product

$$(u, v)_0 := \sum_{i=1}^{\infty} \frac{1}{\lambda_i} (u, e_i)_U (v, e_i)_U = (Q^{-\frac{1}{2}} u, Q^{-\frac{1}{2}} v)_U,$$

such that $(U_0, (\cdot,\cdot)_0)$ forms a Hilbert space. This Hilbert space is separable since it possesses the ONB $(Q^{\frac{1}{2}}e_i)_{i=1}^{\infty} \subset U_0$. The space of Hilbert-Schmidt operators mapping from U_0 to H is very important for the stochastic integration in H. The norm of an operator $B \in L_2(U_0, H)$ can be rearranged and described by the norm of the space $L_2(U, H)$,

$$\|B\|^2_{L_2(U_0,H)} = \sum_{k=1}^{\infty} \|Bg_k\|^2 = \sum_{k=1}^{\infty} \|B \circ Q^{\frac{1}{2}} e_k\|^2 = \|B \circ Q^{\frac{1}{2}}\|^2_{L_2(U,H)} ,$$

where $(g_k)_{k=1}^{\infty} \subset U_0$ is an ONB with $g_k = Q^{\frac{1}{2}}e_k = \sqrt{\lambda_k}e_k$ for every $k \in \mathbb{N}$.

Note that by this short discussion we can interpret the characterization of a Q-Wiener process in (2.2.4) by

$$W(t) = \sum_{i=1}^{\infty} \beta_i(t) g_i$$

for an ONB $(g_i)_{i=1}^{\infty}$ in U_0. Necessary for the convergence of the above series in $L^2(\Omega, \mathcal{F}, \mathbb{P}; U)$ is the finite trace of Q (cf.[DPZ92, p.88]). If we consider more generally symmetric, non-negative operators $Q \in L(H)$ which have not a finite trace, the Definition 2.2.12 is not valid anymore. We can still define a Wiener process in a larger Hilbert space.

Let $(U_1, (\cdot,\cdot)_1)$ be a Hilbert space such that the embedding $J : U_0 \to U_1$ is Hilbert-Schmidt. Note by remark [PR07, Remark 2.5.1] there always exist such a space U_1 and an embedding J. Then the following proposition allows us to define the so called *cylindrical Wiener processes.*

Proposition 2.2.19 ([PR07, DPZ92], Proposition 2.5.2 resp. 4.11)**.** *Let $(e_k)_{k\in\mathbb{N}}$ be an ONB of $U_0 = Q^{\frac{1}{2}}(U)$ and $(\beta_k)_{k\in\mathbb{N}}$ be a family of mutually independent real-valued Brownian motions. Define $Q_1 := J \circ J^*$. Here J^* denotes the adjoint of J.*
Then the operator $Q_1 \in L(U_1)$ is non-negative, symmetric and has a finite trace. The series

$$W(t) = \sum_{k=1}^{\infty} \beta_k(t) J e_k, \qquad t \geq 0$$

is convergent in $L^2(\Omega, \mathcal{F}, \mathbb{P}; U_1)$ and defines a Q_1-Wiener process on U_1. Moreover $Q_1^{\frac{1}{2}}(U_1) = J(U_0)$ and $J : U_0 \to Q_1^{\frac{1}{2}}(U_1)$ is an isometry, i.e. for every $u \in U_0$

$$\|u\|_0 = \|Q_1^{-\frac{1}{2}} \circ Ju\|_1 = \|Ju\|_{Q_1^{1/2}(U_1)} .$$

Remark 2.2.20.

- If the non-negative, symmetric operator Q has not a finite trace, i.e. $\mathrm{Tr}\, Q = +\infty$ we call the Wiener process above a *cylindrical* Wiener process.
- If the operator Q is trace class in the above proposition, then by [PR07, Proposition 2.3.4] $Q^{\frac{1}{2}}$ is a Hilbert-Schmidt operator. In this case we can set $U = U_1$ to obtain again a Q-Wiener process.
- Also a Q-Wiener process with a trace class operator $Q \in L(U)$ can be considered as a cylindrical Wiener process, if we set $J = \mathrm{Id} : U_0 \to U$, see [PR07, Remark 2.5.3, p.42].

2.3 Stochastic partial differential equations in Hilbert spaces

We are interested in stochastic partial differential equations that are driven by an additive Q-Wiener process (see Chapter 3 & 4) and in SPDE with a multiplicative noise, of course driven by real standard Brownian motion (see Chapter 5). In particular we consider SPDEs that can be transformed by a proper conjugation into a random differential equation (RDE) which is a partial differential equation without a stochastic integral. Nevertheless we briefly discuss the theory of stochastic differential equations in Hilbert spaces using the example of an equation whose solution is the Ornstein-Uhlenbeck process. The process itself is discussed in detail in Chapter 3.

We start with the classical real-valued Ornstein-Uhlenbeck process.
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $W = (W(t))_{t\in\mathbb{R}_+}$ be a real-valued one-dimensional Wiener process and $\xi : \Omega \to \mathbb{R}$ be a random variable which is independent to W. Moreover we define $\mathcal{G}_t$ as the filtration generated by the standard Brownian filtration and $\sigma(\xi)$. By $\mathcal{F}_t$ we denote the filtration generated by $\mathcal{G}_t$ and $\sigma(\mathcal{N})$ where $\mathcal{N}$ are the $\mathbb{P}$-null sets in $\bigcup_{t\geq 0} \mathcal{G}_t$.
Under the previous assumptions we consider for $\alpha, \sigma > 0$ the following stochastic differential equation (SDE),

$$\begin{aligned} dX(t) &= -\alpha X(t)dt + \sigma dW(t), \\ X(0) &= \xi, \end{aligned} \tag{2.3.1}$$

according to Example 6.8 in [KS88, Section 5.6]. The SDE has the following integral representation

$$X(t) = \xi + \int_0^t (-\alpha X(s))\, ds + \int_0^t \sigma\, dW(s), \qquad t \geq 0, \tag{2.3.2}$$

where the last summand in (2.3.2) is a stochastic integral with respect to W. Now we define what we understand under a (strong) solution of the given SDE.

Definition 2.3.1 ([KS88], Definition 2.1, Section 5.2)**.** A continuous process $(X(t))_{t\geq 0}$ is called solution of the SDE (2.3.1), if

1) $(X(t))$ is $(\mathcal{F}_t)$-adapted,
2) $X(0) = \xi$ a.s.,
3) $\forall t \geq 0$: $\int_0^t \left(\alpha |X(s)| + \sigma^2\right)\, ds < \infty$,
4) $X(t)$ fulfills (2.3.2).

Since in this simple case the functions $\mu : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $\mu(t,x) := \alpha x$ and $\sigma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $\sigma(t,x) := \sigma$ are independent of the time, we obtain that they fulfill the typical Lipschitz continuity and linear growth conditions. Together with the assumption that $\xi \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ we obtain that there exists a unique solution of (2.3.1), according to the Theorem 2.9 in [KS88].

The solution of equation (2.3.1) has the following form:

$$X(t) = e^{-\alpha t}\xi + \sigma \int_0^t e^{-\alpha(t-s)}\, dW(s), \qquad t \geq 0\,. \tag{2.3.3}$$

In the next part we want to describe the Ornstein-Uhlenbeck process as an H-valued stochastic process in a separable Hilbert space H. We will see that the solution of a similar stochastic partial differential equation in H has basically the same structure like (2.3.3).

Like in the finite dimensional case we begin to describe the setup which is necessary.
All details of the following definitions and theorems can be found in [FK01, Chapter 2], [PR07,

Section 2.3] or [DPZ92, Section 3,4,5].

Succeeding we always fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and define for the Lebesgue measure dx ([Bau92, Section 5,6]),

$$\Omega_\infty := [0,\infty) \times \Omega \qquad \text{and} \qquad \mathbb{P}_\infty := dx \otimes \mathbb{P}\,.$$

We call a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ to an associated probability space *normal* if all null sets of $\mathcal{F}$ are contained in $\mathcal{F}_0$ and the filtration is right-continuous. Fix again a separable Hilbert space $(H, \|\cdot\|)$.

Definition 2.3.2 ([DPZ92], Section 3.3)**.** An H-valued stochastic process X on $\mathbb{R}_+ \times \Omega$ is called

a) *progressively measurable*, if for every $t \geq 0$ the mapping
$$[0,t] \times \Omega \to H, \qquad (s,\omega) \mapsto X(s,\omega)$$
is $\mathcal{B}([0,t]) \otimes \mathcal{F}_t - \mathcal{B}(H)$ measurable.

b) *predictable* if X is $\mathcal{P}_\infty - \mathcal{B}(H)$ measurable, where $\mathcal{P}_\infty$ is the so-called predictable σ-algebra
$$\begin{aligned}\mathcal{P}_\infty :&= \sigma\left(\{(s,t] \times F_s : 0 \leq s < t < \infty,\ F_s \in \mathcal{F}_s\} \cup \{\{0\} \times F_0 : F_0 \in \mathcal{F}_0\}\right)\\ &= \sigma\left(Y : \Omega_\infty \to H : Y \text{ is left-continuous and adapted to } \mathcal{F}_t, t \geq 0\right)\,.\end{aligned}$$

Remark 2.3.3. Note by this definition if the process is continuous and adapted to $\mathcal{F}_t$, $t \geq 0$, then it is predictable.

Let $(U, \|\cdot\|_U)$ be another separable Hilbert space. As proven in proposition 2.1.13 [PR07] a U-valued Q-Wiener process $W(t),\ t \geq 0$, defined like in Definition 2.2.12, is always a Q-Wiener process with respect to a normal filtration $\mathbb{F}$, i.e. $W(t)$ is adapted to $\mathcal{F}_t$ for every $t \geq 0$ and the increments are independent of $\mathcal{F}_\cdot$. Indeed the filtration can be described in the following way. First define $\mathcal{N} := \{A \in \mathcal{F} : \mathbb{P}(A) = 0\}$, $\mathcal{G}_t := \sigma(W(s) : s \leq t)$ and $\mathcal{G}_t^0 := \sigma(\mathcal{G}_t \cup \mathcal{N})$. Then for every $t \geq 0$

$$\mathcal{F}_t := \bigcap_{r>t} \mathcal{G}_r^0$$

is the normal filtration to a Q-Wiener process.
Then we consider the following SPDE in H

$$\begin{aligned} dX(t) &= AX(t)dt + BdW(t), \qquad t \in (0,\infty)\\ X(0) &= \xi\,. \end{aligned} \tag{2.3.4}$$

Such an equation is called a linear SPDE with additive noise. The operator $A : D(A) \to H$ is the infinitesimal generator of a C_0-semigroup $\{S(t)\}_{t\geq 0} \subset L(H)$ like we introduced in Section 2.1. The operator B is an element of $L(U,H)$ and ξ is an H-valued $\mathcal{F}_0$-measurable random variable. Note that by these definitions BW is an H-valued (BQB^*)-Wiener process with $\mathrm{Tr}\ BQB^* < \infty$ (cf. [DPZ92, Remark 5.1]).

Definition 2.3.4 ([PR07], Appendix F, Definition F.0.1)**.** An H-valued predictable process $X(t)$, $t \in [0,\infty)$, is called a *mild solution* of the SPDE (2.3.4) if

$$X(t) = S(t)\xi + \int_0^t S(t-s)BdW(s), \qquad \mathbb{P}\text{-a.s.} \tag{2.3.5}$$

for each $t \in [0,\infty)$.

Note that the integral on the right-hand side of equation (2.3.5) is a stochastic integral in H and has to be well-defined. The solution of this specific SPDE is called *stochastic convolution* (for $\xi \equiv 0$ cf. [DPZ92, Section 5.1.2]) or *Ornstein-Uhlenbeck process*. We will come back at the Ornstein-Uhlenbeck process in the following chapter, since this process plays a key role in the theory of random dynamical systems.
For more details concerning stochastic integrals in Hilbert space please consider [PR07, Section 2.3] and [DPZ92, Section 4.2].
For the sake of completeness we introduce the concept of weak and strong solutions.

Definition 2.3.5 ([PR07], Appendix F, Definition F.0.3)**.**

a) An H-valued predictable process $X(t)$, $t \in [0,\infty)$ is called *weak solution* of the SPDE (2.3.4) if

$$(X(t),\zeta) = (\xi,\zeta) + \int_0^t (X(s), A^*\zeta)\,ds + \int_0^t (\zeta, B\,dW(s)), \qquad \mathbb{P}\text{-a.s.} \tag{2.3.6}$$

for each $t \in [0,\infty]$ and $\zeta \in D(A^*)$. Here $(A^*, D(A^*))$ is the adjoint of $(A, D(A))$ on H.

b) An $D(A)$-valued predictable process $X(t)$, $t \in [0,\infty)$ is called *strong solution* of the SPDE (2.3.4) if

$$X(t) = \xi + \int_0^t AX(s)\,ds + \int_0^t B\,dW(s), \qquad \mathbb{P}\text{-a.s.} \tag{2.3.7}$$

for each $t \in [0,\infty]$.

Note again, that the integrals on the right-hand side of (2.3.6) and (2.3.7) have to be well-defined.

Remark 2.3.6.

- As formulated in proposition F.0.4 every strong solution of (2.3.4) is a weak solution. Under certain integrability assumptions on X and B which have to hold almost surely, we see that a weak solution of problem (2.3.4) is aswell a mild solution. The implications in the other direction follow from general integrability assumptions.
- In comparison to the literature the extension of the above definitions and SPDEs with an additional Lipschitz nonlinearity $F : H \to H$ is natural.
  For random differential equations the concepts of mild, weak and strong solutions are similar, in particular there appear only Bochner integrals. Their definitions can be found in [Paz83].

2.4 Analysis on fractals

Before we step on to the next chapter we like to give an insight on classes of sets that have interesting properties. We briefly introduce the theory of fractal sets. In our case these embedded subsets of $\mathbb{R}^n$, $n \in \mathbb{N}$, shall be used as the underlying space for our Hilbert space. We present an overview how to construct an associated Laplacian in the Hilbert space on a proper fractal set. One can find useful introductions to this topic for example in the books of J. Kigami [Kig01] or R. S. Strichartz [Str06].
The purpose is to consider later SPDEs depending on this Laplacian as well as the dynamics stemming from the solution of the SPDE. The Hausdorff dimension of a random attractor will be estimated in Chapter 4 and as we will see, it depends heavily on the spectrum of the constructed Laplacian, more precisely on the so called *spectral exponent*.

We introduce the concept of self-similar sets. They are a special class of fractal sets.

Theorem 2.4.1 (Theorem 1.1.4, [Kig01]). *Let $\{F_i\}_{i=1}^N$ be a family of contractions on a complete metric space (X,d). Then there exists a unique non-empty compact subset K of X that satisfies the* self-similiar property*, i.e.*

$$K = \bigcup_{i=1}^{N} F_i(K)\,.$$

In this case we call K a self-similiar *set w.r.t. the contractions $\{F_i\}_{i=1}^N$.*

Sometimes K is called the geometric attractor of the iterated function system $\{F_i\}_{i=1}^N$, cf. [Fal97, Section 2.2, p.29]. Here and in the sequel we consider all sets w.r.t. the Euclidean metric d.

Example 2.4.2. As one of *the standard examples* in the last decades we like to introduce the *Sierpinski gasket* or short *SG*, see for example [Kig01, p.3] and for a first impression see the Figure 2.2 below.

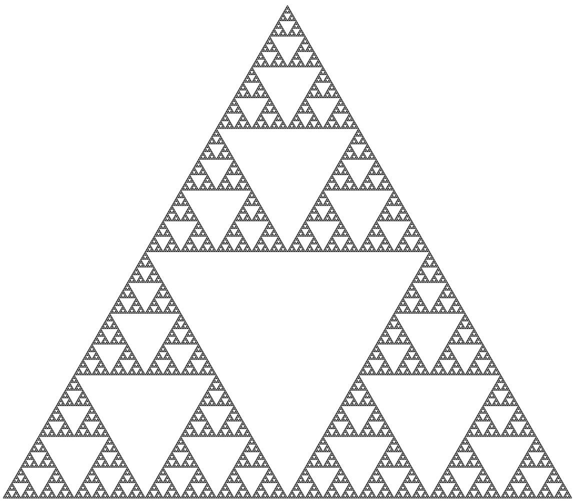


Figure 2.2: The Sierpinski gasket, a subset in the plane

A standard method to construct the Sierpinski gasket is based on a graph approximation. Choose three points p_1, p_2, p_3 in the Euclidean space $\mathbb{R}^2$ such that they form an equilateral triangle. Additionally define the contractions

$$\begin{aligned} F_i &: \mathbb{R}^2 \to \mathbb{R}^2, \\ F_i(x) &:= \frac{1}{2}(x - p_i) + p_i \end{aligned}$$

for $i = 1, 2, 3$.
Applying these three mappings to the set $V_0 := \{p_1, p_2, p_3\}$ we obtain three new points and the old points. We collect all points in the set V_1. The points in $V_1 \setminus V_0$ get connected to each other by edges. In total we obtain four copies within the first triangle scaled by 0.5.
To be more illustrative, we consider the first equilateral triangle as an undirected graph with vertices in V_0. Then applying the contractions multiple times we obtain for instance the vertex sets $V_1,\ V_2$ and V_3 as it can be read of the following picture.

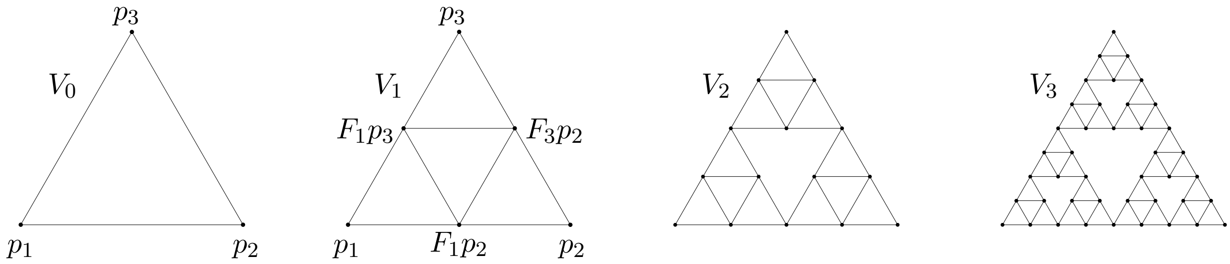


Figure 2.3: The first four 'pre-levels' of the SG as graphs with the vertex sets $V_0, ..., V_3$.

It is natural to call V_0 the set of boundary points. Then since $\{V_i\}_{i=1}^{\infty}$ is an increasing sequence of vertex sets we define inductively

$$V_m := \bigcup_{i=1}^{3} F_i(V_{m-1})$$

the set of vertices at the mth-level of the SG, such that $V_m \subset V_{m+1}$ for every $m \geq 0$. Furthermore we set

$$V_* = \bigcup_{m=0}^{\infty} V_m \,. \tag{2.4.1}$$

Now taking the closure of V_* w.r.t. the Euclidean metric in $\mathbb{R}^2$ we obtain the Sierpinski gasket (SG). Hence we have in particular that V_* is dense in SG as it is shown in a general setup in Lemma 1.3.11 [Kig01, p.22].

Other well-known examples are the Peano curve, the Koch curve, the Cantor set, the Sierpinski carpet and in $\mathbb{R}^3$ for example the Menger sponge. Note that together with the Sierpinski gasket the first two examples belong to a certain class - the class of (connected) p.c.f. fractals. P.c.f. fractals stands for post critically finite self-similiar structures, which are a special class of finitely ramified self-similar sets. For a precise definition see [Kig01, Definition 1.3.4 and Definition 1.3.13, p.19 resp. p.23].
In this work we restrict ourselfs to the case of connected p.c.f. fractals with a regular harmonic structure (see [Kig01, Definition 3.1.2., p.69]) and we take the SG as our standard example.

A well-known property of fractal sets is their often non-integer Hausdorff dimension. We like to recall the basic ideas to determine the dimension and start by introducing the Hausdorff measure. We refer to Section 1.5 in [Kig01, p.28] and Chapter 2 in [Fal90, p.25].

Definition 2.4.3. Let (X, d) be a metric space and $|U| = \sup_{x,y \in U} d(x, y)$ be the diameter of a set $U \subset X$. For any set $A \subseteq X$ and $\delta > 0,\ s \geq 0$ a family of sets $\{U_i\}_{i=1}^{\infty} \subset \mathcal{P}(X)$, which satiesfy

(i) $A \subset \bigcup_{i=1}^{\infty} U_i$ and

(ii) $|U_i| < \delta, \forall i \in \mathbb{N}$

is called a δ-cover of A. Then we call

$$\mathcal{H}_\delta^s(A) := \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta \text{ -cover of } A \right\}$$

the *δ-approximated s-dimensional Hausdorff measure* of A.
The mapping $\mathcal{H}^s : \mathcal{P}(X) \to [0, \infty]$ given by

$$\mathcal{H}^s(A) := \lim_{\delta \to 0} \mathcal{H}_\delta^s(A) \,,$$

is called the *s-dimensional Hausdorff measure* of a set $A \subseteq X$.

Definition 2.4.4. Let (X, d) be a metric space. For any $A \subseteq X$ we call

$$\dim_{\mathrm{H}}(A) := \sup\{s \geq 0 : \ \mathcal{H}^s(A) = \infty\} = \inf\{s \geq 0 : \ \mathcal{H}^s(A) = 0\}$$

the *Hausdorff dimension* of A.

Clearly the Hausdorff measure as well as the Hausdorff dimension depend on the chosen metric d. For the ease of notation we omit the dependence in our notation.
We remark as described in [Fal90, p.26] there exists a relation between the Hausdorff measure and the Lebesgue measure. More precisely, for a set $A \in \mathcal{B}(\mathbb{R}^n)$, it holds

$$\mathcal{H}^n(A) = \frac{2^n \cdot \Gamma(n/2+1)}{\pi^{n/2}} \cdot \lambda^n(A), \tag{2.4.2}$$

where λ^n presents the n-dimensional Lebesgue measure and the constant $\frac{\Gamma(n/2+1)}{\pi^{n/2}}$ is the reciprokal of the n-dimensional ball of radius 1 with gamma function Γ.

Next we give some examples for the above definitions.

Example 2.4.5.

1. If we consider the Cantor set $C \subset [0, 1]$ it is well-known that C is a λ^1-null set, although it is a non-empty and uncountable set. On the other side one can show, that the Hausdorff dimension of C is $\frac{\log 2}{\log 3}$ and $\mathcal{H}^{\log 2/\log 3}(C) = 1$.

2. Further examples are:
   - the Sierpinski gasket, $\dim_{\mathrm{H}}(SG) = \frac{\log 3}{\log 2}$,
   - the Koch curve $\mathcal{K}$, $\dim_{\mathrm{H}}(\mathcal{K}) = \frac{\log 4}{\log 3}$ and
   - the boundary of the Mandelbrot set $\partial M \subset \mathbb{C}$, $\dim_{\mathrm{H}}(\partial M) = 2$.

More examples and properties of the Hausdorff measure can be found e.g. in the book of K. Falconer [Fal90, p.25ff] or J. Kigami in [Kig01, Section 1.5].
In general it can be difficult to calculate the Hausdorff dimension of an arbitrary set. However for self-similiar sets under the *open set condition* we can determine the Hausdorff dimension. This result is based on Moran's theorem in [Mor45, Theorem II].
Let $N \geq 2$ and $\{F_i\}_{i=1}^N$ be a family of contractions on $\mathbb{R}^n$ w.r.t. the Euclidean metric d. The *open set condition* holds for $\{F_i\}_{i=1}^N$, if there exists an non-empty bounded open set $U \subset \mathbb{R}^n$ such that

1. $\bigcup\limits_{i=1}^{N} F_i(U) \subset U$ and
2. $F_i(U) \cap F_j(U) = \emptyset \ \forall i, j \in \{1, ..., N\}, i \neq j$.

Theorem 2.4.6. *Let $(K, \{1, 2, ..., N\}, \{F_i\}_{i=1}^N)$ be a self-similiar structure satisfying the open set condition, then $\dim_{\mathrm{H}}(K) = s$ where $s \geq 0$ has to fulfill*

$$\sum_{i=1}^{N} c_i^s = 1\,,$$

where $c_i \in (0, 1)$ are the contraction ratios of F_i.

For an even weaker condition than the open set condition we refer to [Kig01, Theorem 1.5.7., p.30ff].

Before we give the idea of the Laplacian for functions on these sets, we need to know which measure is useful concerning the analysis on these settings. We are always working with Borel sets in $\mathbb{R}^n$ therefore we consider a non-empty closed set $X \subseteq \mathbb{R}^n$ and the Borel σ-algebra $\mathcal{B}(X)$. Due to [Fal97, Theorem 2.8, p.36f.] we have the following.

Theorem 2.4.7. *Let $\{F_1, ..., F_N\}$ be a family of contractions ($N \geq 2$) on X. Further let $p_1, ..., p_N \in (0,1)$ be probabilities with the property $\sum_{i=1}^N p_i = 1$. Then there exists a unique Borel regular (probability) measure μ such that*

$$\mu(A) = \sum_{i=1}^{N} p_i \mu(F_i^{-1}(A)) \tag{2.4.3}$$

for every $A \in \mathcal{B}(X)$. Moreover in view of Theorem 2.4.1 the mass is concentrated on the compact set $K \subset X$, i.e. $\operatorname{supp} \mu = K$ *and consequently $\mu(K) = 1$.*

The measure defined in (2.4.3) is called *self-similar* measure w.r.t. the contractions $\{F_i\}_{i=1}^N$. When the probability weights $(p_i)_{i=1}^N$ are uniformly distributed we call μ the *standard* measure. For any function $f \in C(K)$, where $C(K)$ is the space of continuous functions mapping from K to $\mathbb{R}$, the integration w.r.t. a self-similar measure is given by

$$\int_X f(x)\, d\mu(x) = \sum_{i=1}^{N} p_i \int_X f(F_i(x))\, d\mu(x)\,,$$

see for instance [Fal97, p.37]. We consider all the measures in this section as complete measures, cf. [Kig01, Section 1.4, p.25]. With the constructed setting we are able to define an appropriate Hilbert space on $(K, \mathcal{B}(K), \mu)$,

$$L^2(K,\mu) := \left\{ f : K \to \mathbb{R} :\ f \text{ is measurable, } \int_K |f|^2\, d\mu < \infty \right\} \tag{2.4.4}$$

with inner product for $f, g \in L^2(K, \mu)$

$$(f,g)_\mu := \int_K f g\, d\mu\,,$$

which induces a corresponding norm $\|\cdot\|^2 = (\cdot,\cdot)$.

If we want to consider differential equations like the heat or wave equation on some domain in $\mathbb{R}^n$ it is natural that partial derivatives come across. If we consider a 'fractal' domain in $\mathbb{R}^n$ we can *not* define a differential operator like the classical Laplacian

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

since in general the appearing partial derivatives are not defined in K. We like to present the (functional) analytic approach presented in [Kig01] and [Str06] to overcome this problem. First we give an introduction what we mean by a Laplacian on the SG and later we summarize the general idea for self-similar structures that generate a Dirichlet form and for which we derive a proper Laplacian.

We want to use a sequence of energy functionals or short *energies* defined on the prestages of the SG, see [Str06, Section 1.3].

Definition 2.4.8. Let V be the vertex set of a simple, finite and connected graph in $\mathbb{R}^n$ and define the set of functions on V by

$$\ell(V) := \{f : V \to \mathbb{R}\}\,.$$

The *energy (form)* of a function $f \in \ell(V)$ on the graph with vertex set V is given by

$$\begin{aligned} E &: \ell(V) \to \mathbb{R}_+, \\ E(f) &:= \frac{1}{2} \sum_{p \in V} \sum_{\substack{q \in V \\ q \sim p}} [f(q) - f(p)]^2, \end{aligned}$$

where $\{q \in V : q \sim p\}$ symbolize all neighbor points of p, i.e. for which a joint edge exists. Applying the polarization identity we obtain a bilinear energy form

$$E(f, g) := \frac{1}{2} \sum_{p \in V} \sum_{\substack{q \in V \\ q \sim p}} [f(q) - f(p)][g(q) - g(p)] \,,$$

for two functions $f, g \in \ell(V)$.

We state some properties of the functional E. Apparently we have $E(f) \geq 0$ for every $f \in \ell(V)$ and $E(f) = 0$ if and only if f is constant on V. A direct consequence is that the bilinear form $E(f, g)$ forms a inner product on the space V modulo constants.

Definition 2.4.9 (Section 1.4, p.26, [Gri09])**.** Let V be the vertex set of a simple, finite and connected graph in $\mathbb{R}^n$ and $f : V \to \mathbb{R}$. Then the *discrete Laplacian* of a function f evaluated in $p \in V$ is defined by

$$\Delta f(p) = \frac{1}{\deg(p)} \sum_{\substack{q \in V \\ q \sim p}} [f(q) - f(p)]$$

where $\deg(p)$ counts the number of neighbor points of p. Of course we assume that $\deg(p) \in (0, \infty)$. For a sequence of vertex sets $(V_m)_{m=1}^{\infty}$ we define analogously the discrete m-th Laplacian Δ_m for a function $f : V_m \to \mathbb{R}$, where the appearing sum is taken over all neighbors of a point on the set V_m.

Example 2.4.10. Let us return to the case of the *SG*. In the previous example we constructed a sequence of vertex sets $(V_m)_{m=0}^{\infty}$ with corresponding graphs. Consider a function $f \in \ell(V_0)$ and denote $E_1(\cdot)$ resp. $E_0(\cdot)$ the energy for functions on V_1 resp. V_0.
The unique solution of the following discrete Dirichlet problem,

$$\begin{aligned} \Delta f^{[1]}(x) &= 0, \qquad & x &\in V_1 \setminus V_0 \\ f^{[1]}(x) &= f(x), \qquad & x &\in V_0 \end{aligned} \tag{2.4.5}$$

is the unique extension $f^{[1]} : V_1 \to \mathbb{R}$ of f, that minimizes the energy $E_1(f^{[1]})$ among all extensions of f with $f^{[1]}|_{V_0} = f$. This extension is called *harmonic extension*. For any other extension $\tilde{f}^{[1]} \in \ell(V_1)$ we have $E_1(f^{[1]}) \leq E_1(\tilde{f}^{[1]})$. As one can easily verify in the case of the *SG* we have the following connection between the energy forms of the first set V_1 and V_0,

$$E_1(f^{[1]}) = \frac{3}{5} E_0(f) \,.$$

for every function $f \in \ell(V_0)$. The appearing factor $r = \frac{3}{5}$ is called *renormalization* factor. We rescale the energies by this factor to conclude

$$\mathcal{E}_1(\tilde{f}^{[1]}) \geq \mathcal{E}_1(f^{[1]}) := \frac{5}{3} E_1(f^{[1]}) = E_0(f) \,. \tag{2.4.6}$$

Iterating the previous procedure we obtain for a function $f \in \ell(V_m)$ ($m \geq 0$) and its n-th harmonic extension $f^{[n]} \in \ell(V_{m+n})$,

$$E_{m+n}(f^{[n]}) = \left(\frac{3}{5}\right)^n E_m(f)$$

and similar to (2.4.6) we have (e.g. for $m = 0$)

$$\mathcal{E}_n(\tilde{f}^{[n]}) \geq \mathcal{E}_n(f^{[n]}) := \left(\frac{5}{3}\right)^n E_n(f^{[n]}) = E_0(f) \quad \text{for every } n \geq 0\,.$$

Consider now a real-valued function f on V_*, see (2.4.1). Then we have a non-decreasing sequence of energies $\left(\mathcal{E}_m(\cdot|_{V_m})\right)_{m\geq 0}$ considering the corresponding restrictions on each prestage V_m, $m \geq 0$,

$$\mathcal{E}_0(f|_{V_0}) \leq \mathcal{E}_1(f|_{V_1}) \leq \cdots \leq \mathcal{E}_m(f|_{V_m}) \leq \cdots$$

with limit in $[0, \infty]$, cf. [Str06, Section 1.4, p.18]. Hence we define for every $f \in \ell(V_*)$

$$\mathcal{E}(f) := \lim_{m\to\infty} \mathcal{E}_m(f|_{V_m})$$

the energy form on $\ell(V_*)$. The domain of the defined energy form is given by the functions with finite energy,

$$\mathcal{F}_{\mathcal{E}} := \{f \in \ell(V_*) : \mathcal{E}(f) < \infty\}\ . \tag{2.4.7}$$

Similar to Definition 2.4.8 we observe for two functions $f, g \in \mathcal{F}_{\mathcal{E}}$,

$$\mathcal{E}(f, g) := \lim_{m\to\infty} \mathcal{E}_m(f|_{V_m}, g|_{V_m})\ . \tag{2.4.8}$$

We denote by $\mathcal{F}_{\mathcal{E}^0}$ the set of functions with finite energy that vanish on the boundary V_0, i.e.

$$\mathcal{F}_{\mathcal{E}^0} := \{f \in \mathcal{F}_{\mathcal{E}} : f|_{V_0} = 0\}\,. \tag{2.4.9}$$

If no confusion occurs, we omit the restriction signs on the set V_m, $m \geq 0$.
Note that every function $f \in \mathcal{F}_{\mathcal{E}}$ is continuous on SG, [Str06, p.19]. To see this, first recall that $\mathcal{E}_m(f) \leq \mathcal{E}(f)$ and

$$\left(\frac{3}{5}\right)^{-m} |f(x) - f(y)|^2 \leq \left(\frac{3}{5}\right)^{-m} \frac{1}{2} \sum_{p\in V_m} \sum_{\substack{q\in V_m \\ q\sim p}} [f(q) - f(p)]^2 = \mathcal{E}_m(f) \leq \mathcal{E}(f)$$

for any $x, y \in V_m$ that are neighbors. Hence we have

$$|f(x) - f(y)| \leq \left(\frac{3}{5}\right)^{\frac{m}{2}} \mathcal{E}(f)^{\frac{1}{2}}$$

which gets arbitrary small for points x, y that are sufficient close to each other, when m rises. Therefore any function $f \in \mathcal{F}_{\mathcal{E}}$ is uniformly continuous on V_* and according to [AE06, Theorem 2.1, p.10] there exists a unique continuous extension on the SG. In the case of the Sierpinski gasket one obtain Hölder continuity with exponent $\log\left(\frac{5}{3}\right) / \log 2$ w.r.t. the Euclidean metric, see [Str06, p.19]. Moreover we have the following interesting connections.

Proposition 2.4.11 (p.19-21f., [Str06])**.** *The domain of the energy $\mathcal{F}_{\mathcal{E}}$ is dense in $C(SG)$ and subsequently $\mathcal{F}_{\mathcal{E}}$ is dense in $L^2(SG, \mu)$. Moreover the space $\mathcal{F}_{\mathcal{E}}$ modulo constants forms a Hilbert space with inner product* (2.4.8)*.*

A consequence of the above results is the following. Let $f \in C(SG)$ such that $f|_{V_*} \in \mathcal{F}_{\mathcal{E}}$, then we define the energy on $C(SG)$ by

$$\mathcal{E}(f) := \mathcal{E}(f|_{V_*})$$

and similar for two functions $f, g \in C(SG)$ with $f|_{V_*}, g|_{V_*} \in \mathcal{F}_{\mathcal{E}}$ by polarization

$$\mathcal{E}(f,g) := \mathcal{E}(f|_{V_*}, g|_{V_*})\,.$$

According to [Str06, Section 2.1] we define a proper Laplacian on the SG in a weak sense.

Definition 2.4.12. Let $u \in C(SG)$ and $f \in \mathcal{F}_{\mathcal{E}}$. If for every $g \in \mathcal{F}_{\mathcal{E}^0}$

$$\mathcal{E}(f,g) = -\int_{SG} u \cdot g \, d\mu,$$

then we say $f \in D(\Delta_\mu)$ with $\Delta_\mu f = u$. Hence for functions $f \in D(\Delta_\mu)$ and $g \in \mathcal{F}_{\mathcal{E}^0}$ we write

$$\mathcal{E}(f,g) = -\int_{SG} \Delta_\mu f \cdot g \, d\mu\,.$$

Note that the definition of a Laplacian on SG is motivated by the integration by parts formula. Assume $f \in C^2([0,1])$ and $g \in C^1([0,1])$ with $g|_{\{0,1\}} = 0$. Then the integration by parts gives

$$\int_0^1 f'(x)g'(x)\,dx = -\int_0^1 f''(x)g(x)\,dx\,.$$

On the other hand, if we assume there exists a function $h \in C([0,1])$ and $f \in C^1([0,1])$ such that

$$\int_0^1 f'(x)g'(x)\,dx = -\int_0^1 h(x)g(x)\,dx$$

then $f \in C^2([0,1])$ and $f'' = h$. Here the left-hand side is the bilinear form of the related Dirichlet principle of a Laplace equation presented for instance in [Eva08, Section 8.1.2, p.434] and this integral is called *energy* functional, see [Eva08, Section 2.2.5, p.42].

For the class of p.c.f. fractals it is possible to define a Laplacian similar to the case of the SG. The analysis on these fractals has been intensively investigated, cf. [Str06] and [Kig01].
We still assume μ to be a self-similar measure on the measurable space $(K, \mathcal{B}(K))$ with weights $(p_i)_{i=1}^N \in (0,1)^N$. Further let $(V_m)_{m=0}^\infty$ be a non-decreasing sequence of vertex sets of graphs approximating K and $(\Delta_m)_{m=0}^\infty$ a sequence of discrete Laplacians on V_m such that according to Theorem 2.1 in [Gri09, p.33f.] and e.g. [Kig01, p.43],

$$\mathcal{E}_m(f,g) = -(\Delta_m f, g) \quad m \geq 0$$

for every $f, g \in \ell(V_m)$. Then we have similar to the example $\mathcal{E}(f,g) := \lim_{m\to\infty} \mathcal{E}_m(f,g)$ for functions $f, g \in \mathcal{F}_{\mathcal{E}} := \{u \in \ell(V_*) : \mathcal{E}(f) < \infty\}$. $\mathcal{F}_{\mathcal{E}^0}$ is analogously defined like before. The space $L^2(K,\mu) \cap \mathcal{F}_{\mathcal{E}}$ together with the inner product

$$\mathcal{E}_*(f,g) = \mathcal{E}(f,g) + \int_K f(x)g(x)\,d\mu(x)$$

forms a Hilbert space, cf. [Kig01, Theorem 2.4.1, p.63].
The energies satisfies the following rule [Kig01, Definition 3.1.1, p.69] for $f, g \in \ell(V_m)$,

$$\mathcal{E}_{m+1}(f,g) = \sum_{i=1}^N \frac{1}{r_i} \mathcal{E}_m(f \circ F_i, g \circ F_i)$$

and note that in general the rescaling factors $(r_i)_{i=1}^N$ of the energy differ depending on F_i (this is possible by weakening the symmetry of a self-similar set, see [Kig01, Example 3.1.6, p.71]). We will always assume that

$$0 < r_i < 1$$

for every $i \in \{1, ..., N\}$. Before we state one of the most important theorems in this context we recall the definition of a local regular Dirichlet form, see e.g. [Kig01, Appendix B.3, p.202].

Definition 2.4.13 (Section 1.1, pp.3-5, [FOT10])**.** Let H be a real Hilbert space and $\mathcal{E} : D(\mathcal{E}) \times D(\mathcal{E}) \to \mathbb{R}_+$ be a symmetric bilinear form with *domain* $D(\mathcal{E})$, a linear dense subset of H. Define another symmetric bilinear form $\mathcal{E}_*$ by

$$\mathcal{E}_*(f, g) = \mathcal{E}(f, g) + (f, g)$$

for any $f, g \in D(\mathcal{E})$ and $(\mathcal{E}_*, D(\mathcal{E}))$ forms a pre-Hilbert space. A symmetric bilinear form $\mathcal{E}$ is called *closed*, if the domain $D(\mathcal{E})$ is complete w.r.t. $\mathcal{E}_*$.
Consider now a σ-finite measure space $(X, \mathcal{B}(X), \mu)$ and the associated Hilbert space $L^2(X, \mu)$. Let $\mathcal{E}$ be a closed form on $L^2(X, \mu)$ with domain $\mathcal{D}(\mathcal{E})$, then $(\mathcal{E}, D(\mathcal{E}))$ is called a *Dirichlet form* if the so called *Markov* property holds, that is for any function $f \in D(\mathcal{E})$ the function defined by

$$\overline{f}(x) = \begin{cases} 1, & \text{if } f(x) \geq 1, \\ f(x), & \text{if } 0 < f(x) < 1, \\ 0, & \text{if } f(x) \leq 0 \end{cases}$$

is an element of $D(\mathcal{E})$ and it holds $\mathcal{E}(\overline{f}, \overline{f}) \leq \mathcal{E}(f, f)$.
Let us denote $\text{supp}\,(f) = \overline{\{x \in X : f(x) \neq 0\}}$ and

$$\mathcal{C}_0(X) = \{f : X \to \mathbb{R}, f \text{ is continuous and } \text{supp}\,(f) \text{ is compact}\}.$$

A Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is called *regular*, if it possesses a core, that is a set $\mathcal{C} \subset (D(\mathcal{E}) \cap \mathcal{C}_0(X))$ which is dense in $D(\mathcal{E})$ w.r.t. $\mathcal{E}_*$ and in $\mathcal{C}_0(X)$ w.r.t. $\|\cdot\|_\infty$. A Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is *local* if $\mathcal{E}(f, g) = 0$ for functions $f, g \in D(\mathcal{E})$ with disjoint compact supports.

Suppose we are in the setup described above.

Theorem 2.4.14 (Corollary 3.4.7 and Theorem 3.4.6, pp.92-94, [Kig01])**.** *Let $r_i p_i < 1$ for every $i \in \{1, ..., N\}$. Then the quadratic form $(\mathcal{E}, \mathcal{F}_{\mathcal{E}^0})$ is a local regular Dirichlet form on $L^2(K, \mu)$ and there exists a non-negative self-adjoint operator H_{D} on $L^2(K, \mu)$ with $D(H_{\mathrm{D}}^{1/2}) = \mathcal{F}_{\mathcal{E}^0}$ and $\mathcal{E}(f, g) = (H_{\mathrm{D}}^{1/2} f, H_{\mathrm{D}}^{1/2} g)$ for every $f, g \in \mathcal{F}_{\mathcal{E}^0}$. Moreover the operator H_{D} has a compact resolvent and $-H_{\mathrm{D}} =: \Delta_\mu$ is called the* Dirichlet *Laplacian on $L^2(K, \mu)$.*

A general version of this theorem can be found in [Kat95, Chapter 6, Section 2].

Analog to the above theorem we can define a Neumann Laplacian with Neumann boundary conditions, where of course one needs to explain what we understand under a Neumann derivative in our setup, see [Kig01, Definition 3.7.6, p.110].
In the classical case $(K, \mu) = (\mathbb{R}, \lambda)$ with λ the Lebesgue measure on $\mathbb{R}$, the domain of the energy coincides with the sobolev space $H^1(\mathbb{R})$. The energy is given by

$$\mathcal{E}(f, g) = \int_{\mathbb{R}} f'(x) g'(x)\, d\lambda(x)$$

for functions $f, g \in H^1(\mathbb{R})$. Then according to the last theorem the operator H_{D} coincides with the standard Laplacian $-\Delta = -\frac{d}{dx^2}$ on $\mathbb{R}$, see [Kig01, p.65f.] for more details.
We want to collect some more properties of the introduced (Dirichlet) Laplacian on a proper self-similar set. We refer to [Kig01] for details and proofs.

Proposition 2.4.15.

- *There exists a point-wise definition of the Laplacian Δ_μ and it holds the Gauß-Green Formula similar to the classical one.*
- *Due to Theorem 2.1.7 the eigenvalues $(\lambda_i)_{i=1}^\infty$ of the operator H_D are real positive numbers with no finite limit point and the corresponding eigenfunctions $\{\varphi_i\}_{i=1}^\infty$ build an ONB for $L^2(K,\mu)$. Hence the space $L^2(K,\mu)$ is separable, cf. [Alt16, Theorem 9.8, p.294].*
  In particular we want to mention that the operator Δ_μ fulfills all necessary assumptions on Theorem 2.1.13. Therefore Δ_μ is the generator of an analytic semigroup.
- *A unique property of the Laplacian on self-similar sets is the existence of* (pre-)localized *eigenfunctions, that is an eigenfunction that fulfill Dirichlet* and *Neumann boundary conditions for an appropriate eigenvalue.*

We like introduce another property which is of great relevance for our results. We summarize the asymptotic behaviour of the constructed Laplacian.

Definition 2.4.16 (Definition 4.1.3, p.133, [Kig01])**.** Let $(\lambda_i)_{i\geq 1}\subset(0,\infty)$ be the sequence of Dirichlet eigenvalues to the Laplacian $-\Delta_\mu$, depending on the self-similar measure μ. Then we define the *eigenvalue counting function* by

$$\rho(x,\mu) := \max\{k\in\mathbb{N} : \lambda_k \leq x\} = \#\{k : \lambda_k \leq x\} \tag{2.4.10}$$

for any $x\in\mathbb{R}$. If no confusion occurs we write $\rho(x)$ instead of $\rho(x,\mu)$. Let $E(\lambda)$ denote the eigenspace of the eigenvalue λ then (2.4.10) can be rewritten,

$$\rho(x,\mu) = \sum_{\lambda\leq x} \dim E(\lambda),$$

hence it is clear that we count the eigenvalues w.r.t. their geometric multiplicity.

For classical Laplacians (with Dirichlet boundary conditions) on bounded domains of $\mathbb{R}^n$ there exists the well-known result from Weyl about the asymptotic behaviour of the eigenvalue counting function, i.e. for $x\to\infty$

$$\rho(x)\sim x^{n/2}\,. \tag{2.4.11}$$

For a proof we refer to [Lap91, Theorem 2.3]. We like to present the 'fractal' analogue to Weyl's theorem, based on [Kig01, Theorem 4.1.5, p.134].

Theorem 2.4.17. *Let μ be a self-similar measure with weights $(p_i)_{i=1}^N$ and suppose $r_i p_i < 1$, $i\in\{1,...,N\}$, for the corresponding renormalization factors r_i of the energy $\mathcal{E}$. Then for x sufficiently large there exist constants $C_1, C_2 > 0$ such that*

$$C_1 x^{\frac{d_S}{2}} \leq \rho(x) \leq C_2 x^{\frac{d_S}{2}}\,. \tag{2.4.12}$$

Here d_S is the unique real number d that satisfies,

$$\sum_{i=1}^N (r_i p_i)^{d/2} = 1\,. \tag{2.4.13}$$

We call d_S the spectral exponent *of $(\mathcal{E},\mathcal{F}_\mathcal{E},\mu)$.*

We are able to deduce the asymptotic behaviour of the eigenvalues of $-\Delta_\mu$ according to Lemma 5.1.3 in [Kig01, p. 159] together with the previous theorem.

Lemma 2.4.18. *There exist constants $C_3, C_4 > 0$, such that for any $m \geq 1$,*

$$C_3 m^{\frac{2}{d_S}} \leq \lambda_m \leq C_4 m^{\frac{2}{d_S}} . \tag{2.4.14}$$

In particular, for λ_m large enough the constants C_3 and C_4 can be determined by the constants C_1 and C_2 of the previous theorem.

Example 2.4.19. First let us consider again the Sierpinski gasket introduced in Example 2.4.2. While constructing the energy we set the renormalization for each mapping F_1, F_2, F_3 equally, that is $r = r_1 = r_2 = r_3 = \frac{3}{5}$ as presented in (2.4.6). Moreover let μ be the standard (self-similar) measure acting on SG, therefore $p_1 = p_2 = p_3 = \frac{1}{3}$. Then according to (2.4.13) $d_S = d_S(SG) = \frac{\log 9}{\log 5}$ and hence $\lambda_m \sim m^{\log 25/\log 9}$.

The result of Theorem 2.4.17 is in view of (2.4.11) surprisingly. One could have expected that as a natural analogue the Hausdorff dimension appears in the corresponding exponent. We want to comment the spectral exponent a bit more since it plays an important role for one of our main results.

Remark 2.4.20.

- The original result of Kigami states a function rule describing the eigenvalue counting function $\rho(x)$ for large values of x.
- In the definition of the spectral exponent both the geometry of our setting (p_i) as well as the analysis constructed on the set (r_i) have influence on the actual value of d_S.
- For the SG the spectral exponent is often called spectral dimension and it holds the famous *Einstein relation*,

$$d_S = \frac{\dim_{\mathrm{H}}(SG)}{d_W}$$

  where d_W is the so called *walk dimension* which is connected to a Brownian motion on the SG (see [BP88]). In particular d_W can be characterized via the mean crossing time through the graphs building the SG, that is the expected time which a random walker needs to pass through the points of the boundary, cf. [FT13, Section 2.1].

Chapter 3

Random Dynamics

This chapter is devoted to random dynamical systems (RDS) that are generated by the solution operators of the associated SPDEs. We start with a short overview of all necessary concepts concerning this subject, e.g. metric dynamical system, invariant measure and properties of random dynamical systems. The second section explains all necessary properties of the stationary Ornstein-Uhlenbeck process for a certain metric dynamical system, which we constructed in advance. This section will help us to make the link between random dynamical systems for SPDEs and (partial) random differential equations, which we describe in detail in Section 3. In the last section we discuss the long-time behaviour of the considered RDS based on the theory of random attractors.

3.1 Fundamental concepts of random dynamical systems

In this section we want to introduce the basics of random dynamics in infinite-dimensional spaces.

We start with the classical theory of (deterministic) dynamical systems introduced by Birkhoff in [Bir27, Chapter VII]. A more modern definition of a dynamical system can be found for example in [Arn10, Appendix A.1] or [PW10, Section 4.4].

Definition 3.1.1. A *dynamical system* can be described by a tripel $(\mathbb{T}, E, \phi)$. The set $\mathbb{T}$ denotes a time set (typically $\mathbb{Z}_+, \mathbb{Z}, \mathbb{R}_+$ or $\mathbb{R}$) and E denotes a state space (for example $\mathbb{R}^n$, $n \in \mathbb{N}$ or H a Hilbert space). The mapping $\phi : \mathbb{T} \times E \to E$ has to fulfill the identity property and the semigroup property, i.e.

$$\phi(0, x) = x, \quad \text{and} \quad \phi(t+s, x) = \phi(t, \phi(s, x)) \qquad \text{for every } x \in E \text{ and } t, s \in \mathbb{T}\,.$$

The most common appearance of dynamical systems is in combination with differential equations. For example consider the following initial value problem

$$\begin{aligned} \frac{dx(t)}{dt} &= f(x(t)), \qquad t > 0 \\ x(0) &= x_0 \in E, \end{aligned}$$

where x_0 is a fixed value in E. Under sufficient regularity assumptions on f one can obtain a unique solution of this ordinary differential equation and the solution generates a dynamical system, see [PW10, Satz 4.4.2].

One could expect that we could simply add an ω from a reasonable probability space in the above definition to derive a random dynamical system. As we will see this would not be correct. In fact to arrive at the definition of a *random* dynamical system one has to respect that the randomness changes in time as well. Remember again the example of the introduction: the trajectory of a plastic bottle under the influence of the ocean current and the wind.

Motivated by this example we define now the tool that helps us to describe the evolution of the noise w.r.t. the time. Since we are only interested in the time-continuous case we fix for the following definition $\mathbb{T} = \mathbb{R}$.

Definition 3.1.2 (Appendix A.1, p.536ff., [Arn10])**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\theta = (\theta_t)_{t\in\mathbb{R}}$ be a family of measure-preserving mappings from Ω to itself, i.e. for all $A \in \mathcal{F}$ and $t \in \mathbb{R}$,

$$\mathbb{P}(\theta_t^{-1}A) = \mathbb{P}(A)$$

or shortly, $\theta_t\mathbb{P} = \mathbb{P}$.
The quadrupel $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ or sometimes just θ is called a *metric dynamical system* or shorter *MDS*, if the following properties are fulfilled.

(1) The mapping $(t, \omega) \mapsto \theta_t\omega$ is $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{F})$-measurable,

(2) $\theta_0 = \mathrm{Id}_\Omega$ and

(3) $\theta_{t+s} = \theta_t \circ \theta_s$ for all $t, s \in \mathbb{R}$.

Notice that for a (two-sided) time set like $\mathbb{R}$ we obtain directly from the properties (2) and (3) above that the inverse is given by $\theta_t^{-1} = \theta_{-t}$ for every $t \in \mathbb{R}$.

Definition 3.1.3 (Appendix A.1, p.536-539, [Arn10])**.** Assume we are in the same setting as in the definition above. A set $A \in \mathcal{F}$ is called *invariant* w.r.t. θ if $\theta_t^{-1}A = A$ for every $t \in \mathbb{R}$. The collection of all invariant sets in $\mathcal{F}$ is denoted by $\mathcal{I}$. Indeed $\mathcal{I}$ forms a sub σ-algebra of $\mathcal{F}$. Note that for $\omega \in A$ the whole orbits $((\theta_t\omega)_{t\in\mathbb{R}})$ belong to A, if A is θ-invariant.
A given MDS θ is *ergodic* if for all sets $A \in \mathcal{I}$ we have $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$.
A measurable function $f : \Omega \to \mathbb{R}$ is called *invariant* w.r.t. to the given MDS θ if $f(\theta_t\omega) = f(\omega)$ for every $t \in \mathbb{R}$ and any $\omega \in \Omega$.

The *Birkhoff-Chintchin Ergodic Theorem* is of great importance for our result in Chapter 4. We refer to [Arn10, Appendix A.1, p.538] and to [Ogr11, Theorem 2.30, p.13].

Theorem 3.1.4. *Let $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ be an MDS and suppose $f : \Omega \to \mathbb{R}$ is integrable on Ω. Then there exists a θ-invariant set Ω_f of full $\mathbb{P}$-measure, where the following limits exist and*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(\theta_r\omega)\, dr = \lim_{t\to\infty} \frac{1}{t} \int_{-t}^0 f(\theta_r\omega)\, dr =: \overline{f}(\omega)$$

for every $\omega \in \Omega_f$. Moreover $\overline{f}$ is invariant on Ω_f, $\overline{f} \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\mathbb{E}\overline{f} = \mathbb{E}f$$

where we set $\overline{f} \equiv 0$ outside of Ω_f.
If additionally θ is ergodic, then $\overline{f}$ is constant on Ω_f and it follows $\overline{f} = \mathbb{E}f$, which presents the well-known equality of time and space average.

Remark 3.1.5. Assume we are in the ergodic case of the above theorem. Let $\overline{f}$ be defined on Ω_f by the previous limits. Then by setting $\overline{f} := \mathbb{E}f$ on the exceptional set of measure zero (i.e. the complement of Ω_f) it holds $\overline{f} = \mathbb{E}f$ on the whole sample space Ω.

Example 3.1.6. For an example of a metric dynamical system recall the statement of Theorem 2.2.8 and consider a Q-Wiener process W on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in a separable Hilbert space H. By Corollary 2.2.16 we can change from $W(t, \omega)$ to the equivalent (two-sided) C_0-canonical Wiener process $\omega(t)$ for $t \in \mathbb{R}$ and $\omega(\cdot) \in C_0(\mathbb{R}; H)$. This process exists on the canonical space $(C_0(\mathbb{R}; H), \mathcal{B}(C_0(\mathbb{R}; H)), \mathbb{P}_0)$ with the associated *Wiener measure* $\mathbb{P}_0$ possessing the same finite dimensional distributions as our Wiener process. This probability space extends to an ergodic metric dynamical system, if we introduce the so called *Wiener shift* given by

$$\theta_t : C_0(\mathbb{R}; H) \to C_0(\mathbb{R}; H), \qquad \omega(\cdot) \mapsto \theta_t\omega(\cdot) := \omega(t+\cdot) - \omega(t)$$

for $t \in \mathbb{R}$.

In the following we will write for simplicity C_0 and $\mathcal{B}(C_0)$ instead of $C_0(\mathbb{R}, H)$ resp. $\mathcal{B}(C_0(\mathbb{R}, H))$.

Theorem 3.1.7. *The measure $\mathbb{P}_0$ is invariant and $(C_0, \mathcal{B}(C_0), \mathbb{P}_0, \theta)$ forms an ergodic metric dynamical system.*

Proof. The invariance will be shown according to Lemma 3.3 [Box88]. For $k \in \mathbb{N}$ define the finite set $S_k := \{s_1, ..., s_k\}$ whose (fixed) elements are in $\mathbb{R}$. For a set $A_{S_k} \in \mathcal{B}(H^k)$ we define the generalized cylindric sets

$$C(t, A_{S_k}) := \big\{\omega \in C_0 \colon \big(\omega(t+s_1) - \omega(t), ..., \omega(t+s_k) - \omega(t)\big) \in A_{S_k}\big\}$$

for every $t \in \mathbb{R}$. We want to show that $C(t, A_{S_k})$ is a generator of $\mathcal{B}(C_0)$. First recall in view of (2.2.3)

$$\mathcal{B}(C_0) = \sigma\left\{p^{-1}_{r_1,...,r_k}(A) \subset C_0 : r_1, ..., r_k \in \mathbb{R} \text{ for } k \geq 1, A \in \mathcal{B}(H^k)\right\}.$$

Next we clearly have a relation to the following (classical) cylindric sets

$$C(0, A_{S_k}) = Z(A_{S_k}) := \{\omega \in C_0 : \omega(s_1), ..., \omega(s_k) \in A_{S_k}\} \tag{3.1.1}$$

and we know by Proposition 4.1 from [IW81, p.16] that $\sigma(\mathcal{Z}) = \mathcal{B}(C_0)$, where $\mathcal{Z}$ is the collection over all cylindric sets of the form (3.1.1). Notice that by choosing $t = 0$,

$$\mathcal{Z} \subset \left\{C(t, A_{S_k}) : t \in \mathbb{R}, A_{S_k} \in \mathcal{B}(H^k), S_k \in \mathbb{R}^k, k \geq 1\right\} =: \mathcal{E}_C\,.$$

Since $C(t, A_{S_k}) \in \mathcal{B}(C_0)$ for arbitrary $t \in \mathbb{R}$ and $A_{S_k} \in \mathcal{B}(H^k)$ we finally obtain $\sigma(\mathcal{E}_C) = \mathcal{B}(C_0)$. Hence the sets $C(t, A_{S_k})$ can be measured by $\mathbb{P}_0$. Applying the shift to $C(t, A_{S_k})$ we conclude for $r \in \mathbb{R}$

$$\begin{aligned}
\theta_{-r}C(t, A_{s_k}) &= \{\omega \in C_0 : \theta_r\omega \in C(t, A_{S_k})\} \\
&= \{\omega \in C_0 : (\theta_r\omega(t+s_1) - \theta_r\omega(t), ..., \theta_r\omega(t+s_k) - \theta_r\omega(t)) \in A_{S_k}\} \\
&= \{\omega \in C_0 : (\omega(t+s_1+r) - \omega(t+r), ..., \omega(t+s_k+r) - \omega(t+r)) \in A_{S_k}\} \\
&= C(t+r, A_{S_k})\,.
\end{aligned}$$

To show the requested invariance we write

$$\mathbb{P}_0(\theta_{-r}C(t, A_{S_k})) = \mathbb{P}_0(C(t+r, A_{S_k})) = \mathbb{P}_0(C(t, A_{S_k}))$$

where the last equality follows since the increments of $\omega(\cdot)$ are stationary, see the Remark 2.2.14. The next part to show is that θ builds indeed an MDS.

1. The measurability of the mapping $(t, \omega) \mapsto \theta_t\omega$ for $t \in \mathbb{R}$ and $\omega \in C_0$ follows from the continuity of the mapping shown in Lemma 3.2 in [Box88]. In our case we would use the metric defined in (2.2.2), which induces the compact open topology and which is equivalent to the one given in [Box88] for C_0.

2. The identity property holds obviously, since $\omega \in C_0$.

3. For the last property it follows for $t, s \in \mathbb{R}$ and $r \in \mathbb{R}$

$$\begin{aligned}\theta_{t+s}\omega(r) &= \omega(t+s+r) - \omega(t+s) = \omega(t+s+r) - \omega(t) - (\omega(t+s) - \omega(t)) \\ &= \theta_t(\omega(s+r) - \omega(s)) = \theta_t \circ \theta_s\omega(r)\,.\end{aligned}$$

To close the proof we need to verify the ergodicity. We follow [Box88, Lemma 3.3, p.36ff.].
We need the following result for the proof. Let $t_1, t_2 \in \mathbb{R}$, $t_1 \leq t_2$ and consider the time sets $S_1 = \{s_1\}$ and $S_2 = \{s_2\}$ for arbitrary $s_1, s_2 \in \mathbb{R}$. We show the independence of an appropriated shifted generalized cylindric set and another generalized cylindric set w.r.t. $\mathbb{P}_0$. There exists a $r_0 \in \mathbb{R}$ such that for all $r \leq r_0$,

$$\mathbb{P}_0\left[(\theta_{-r}C(t_1, A_{S_1})) \cap C(t_2, A_{S_2})\right] = \mathbb{P}_0(C(t_1, A_{S_1}))\mathbb{P}_0(C(t_2, A_{S_2}))\,.$$

If we choose $r_0 := \min\{0, t_2 - t_1 - |s_1| - |s_2|\}$, then we notice at first $r \leq r_0 \leq 0$ and for these r we conclude

$$t_1 + r + |s_1| \leq t_1 + r_0 + |s_1| \leq t_2 - |s_2|\,.$$

Therefore we conclude that we can sort the times $(t_1 + r + s_1)$, $(t_1 + r)$, t_2 and $(t_2 + s_2)$ in such a way that we can apply the independence of the increments of the (canonical) Q-Wiener process,

$$\begin{aligned}\mathbb{P}_0\left[(\theta_{-r}C(t_1, A_{S_1})) \cap C(t_2, A_{S_2})\right] &= \mathbb{P}_0(\theta_{-r}C(t_1, A_{S_1}))\mathbb{P}_0(C(t_2, A_{S_2})) \\ &= \mathbb{P}_0(C(t_1, A_{S_1}))\mathbb{P}_0(C(t_2, A_{S_2}))\,.\end{aligned}$$

We can extend this result on multidimensional generalized cylindrical sets $C(t, A_S)$ and to finite unions of these cylindrical sets.
Choose a Borel set $A \in \mathcal{I} \subset \mathcal{B}(C_0)$ (w.r.t. our θ), i.e. A is θ_t-invariant for every $t \in \mathbb{R}$. As mentioned above $\sigma(\mathcal{E}_C) = \mathcal{B}(C_0)$, hence we conclude that we can approximate the set A by a finite union of disjoint (generalized) cylindric sets $B = B(\varepsilon) \in \mathcal{E}_C$ for any $\varepsilon > 0$ with $\mathbb{P}_0(A\Delta B) \leq \varepsilon$. For a proper r we obtain

$$\mathbb{P}_0((\theta_{-r}B^c) \cap B) = \mathbb{P}_0(B^c)\mathbb{P}_0(B) \tag{3.1.2}$$

and similar when the position of B and B^c are changed. Since $\mathbb{P}_0(\cdot\Delta\cdot)$ forms a pseudometric on $\mathcal{B}(C_0) \otimes \mathcal{B}(C_0)$, we have

$$\mathbb{P}_0((\theta_{-r}B)\Delta B) \leq \mathbb{P}_0((\theta_{-r}B)\Delta(\theta_{-r}A)) + \mathbb{P}_0((\theta_{-r}A)\Delta A) + \mathbb{P}_0(A\Delta B)$$

where second summand vanishs since $\theta_t A = A$ for every $t \in \mathbb{R}$. The first summand is due to the shift invariance identical to the last summand, hence

$$\mathbb{P}_0((\theta_{-r}B)\Delta B) = 2\mathbb{P}_0(A\Delta B) \leq 2\varepsilon\,. \tag{3.1.3}$$

Moreover using the definition of the symmetric difference and (3.1.2)

$$\begin{aligned}\mathbb{P}_0((\theta_{-r}B)\Delta B) &= \mathbb{P}_0((\theta_{-r}B) \cap B^c) + \mathbb{P}_0((\theta_{-r}B^c) \cap B) \\ &= 2\mathbb{P}_0(B)\mathbb{P}_0(B^c) = 2\mathbb{P}_0(B)(1 - \mathbb{P}_0(B))\,.\end{aligned} \tag{3.1.4}$$

Combining (3.1.3) and (3.1.4) we obtain

$$2\mathbb{P}_0(B)(1 - \mathbb{P}_0(B)) \leq 2\varepsilon$$

and since ε can be arbitrary small we conclude after some basic set-rearrangements

$$\mathbb{P}_0(A)(1 - \mathbb{P}_0(A)) = 0\,.$$

□

Remark 3.1.8. To motivate why we are always interested in θ-invariant sets, we refer to Remark 2 and Lemma 1 in [CGASV10]. Assume a certain property holds on a θ-invariant set $\Omega' \subset \Omega$ of full $\mathbb{P}$-measure (a.s.). Then we redefine the (ergodic) MDS in such a way, that the property holds for all $\omega \in \Omega'$. The 'new' (ergodic) MDS is given by $(\Omega', \Omega' \cap \mathcal{F}, \mathbb{P}', \theta')$, where for every $A' \in \Omega' \cap \mathcal{F}$, $A \in \mathcal{F}$ with $A' = A \cap \Omega'$ we have $\mathbb{P}'(A) = \mathbb{P}(A)$. Often the notation of the new MDS is overwritten by the old one $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$.

Let us return to the case of Example 3.1.6. We want to mention the next theorem which gives us an answer to the question, if there exists a θ-invariant subset of C_0 of full $\mathbb{P}_0$-measure, such that the trajectories of the noise have a certain growth condition, which is necessary for many estimates in the sequel. We recall the definition of Hölder continuous functions.

Definition 3.1.9 (Section 0.1, 0.2, pp.1-3, [Lun95])**.** Let X be a real Banach space with norm $\|\cdot\|_X$ and $I \subset \mathbb{R}$ be a finite interval. Then for any $\gamma \in (0,1)$ the (Banach) space of Hölder continuous functions on I is defined by

$$C^\gamma(I;X) = \left\{ f \in C(I;X) : \|f\|_{C^\gamma(I;H)} = \sup_{\substack{a,b \in I \\ a \neq b}} \frac{\|f(a) - f(b)\|}{|a-b|^\gamma} < \infty \right\}$$

with norm $[f]_{C^\gamma(I;H)} := \|f\|_{C^\gamma(I;H)} + \|f\|_\infty$. By adding the supremum norm to the seminorm $[\cdot]_{C^\gamma(I;H)}$ turns into a norm.
In the case $C^\gamma(I;H) \equiv C_0^\gamma(I;H)$ from Example 3.1.6, $\|\cdot\|_{C^\gamma(I;H)}$ is already a norm since the only possible constant function is identical zero.

Theorem 3.1.10 (Lemma 3.3, [GLR11])**.** *Let $(\omega(t))_{t\in\mathbb{R}}$ be the canoncial H-valued Wiener process on the metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ introduced in Example 3.1.6.*
Then there exists a θ-invariant set $\Omega' \subseteq \Omega$ with $\mathbb{P}(\Omega') = 1$ such that

1. *for any $\varepsilon > 0$, $\omega \in \Omega'$ there exists a constant $C_1 = C_1(\varepsilon, \omega) > 0$ such that for every $t \in \mathbb{R}$*

$$\|\omega(t)\| \leq \varepsilon |t|^2 + C_1 \,, \tag{3.1.5}$$

2. *on any interval $[r,s] \subseteq \mathbb{R}$ the paths are Hölder continuous with Hölder constant $0 < \gamma < \frac{1}{2}$. In particular there exists a constant $C_2 = C_2(\omega, \gamma, r, s) > 0$, such that*

$$\|\omega\|_{C^\gamma([r,s];H)} \leq C_2$$

with $C_2 \in L^1(\Omega)$.

A consequence of the inequality (3.1.5) is that the 'tails' of the trajectories are subexponentially growing on a θ-invariant set of full measure. For any $\varepsilon > 0$ and $\omega \in \Omega$ there exists a time $t_0 = t_0(\varepsilon, \omega)$ and an ω-wise constant $C_\varepsilon = C(\varepsilon, \omega) > 0$ such that for $|t| \geq t_0$

$$\|\omega(t)\| \leq C_\varepsilon e^{\varepsilon |t|} \,, \tag{3.1.6}$$

cf. [CGASV10, Lemma 11 ff.].
All in all we redefine the sample space

$$\Omega := \left\{ \omega \in C_0 : \lim_{t \to \pm\infty} \frac{\log^+ \|\omega(t)\|}{t} = 0 \right\} \tag{3.1.7}$$

The associated 'new' metric dynamical system is also denoted by $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$. Here $\mathcal{F}$ represents $\Omega \cap \mathcal{B}(C_0)$ and $\mathbb{P}$ the restriction of the Wiener measure to this new trace σ-algebra.

Definition 3.1.11 (Definition 1.1.1, [Arn10])**.** Let $(H, \mathcal{B}(H))$ be a measurable space and as in Definition 3.1.2 $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ an MDS. Then a *random dynamical system* is a mapping

$$\varphi : \mathbb{R}_+ \times \Omega \times H \to H, \quad (t, \omega, x) \mapsto \varphi(t, \omega, x),$$

such that

(1) φ is $(\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F} \otimes \mathcal{B}(H), \mathcal{B}(H))$-measurable,

(2) $\varphi(0, \omega, \cdot) = \mathrm{Id}_H$ for *all* $\omega \in \Omega$ and

(3) the *cocycle property* has to hold, i.e. we have
$\varphi(t + s, \omega, x) = \varphi(t, \theta_s\omega, \varphi(s, \omega, x))$ for *all* $\omega \in \Omega$, $x \in H$ and $s, t \in \mathbb{R}_+$.

If in addition the mapping $\varphi(t, \omega, \cdot) : H \to H$ is continuous for every $t \in \mathbb{R}_+$ and $\omega \in \Omega$, then φ is called a *continuous* random dynamical system.

The cocycle property is illustrated in Figure 3.1.

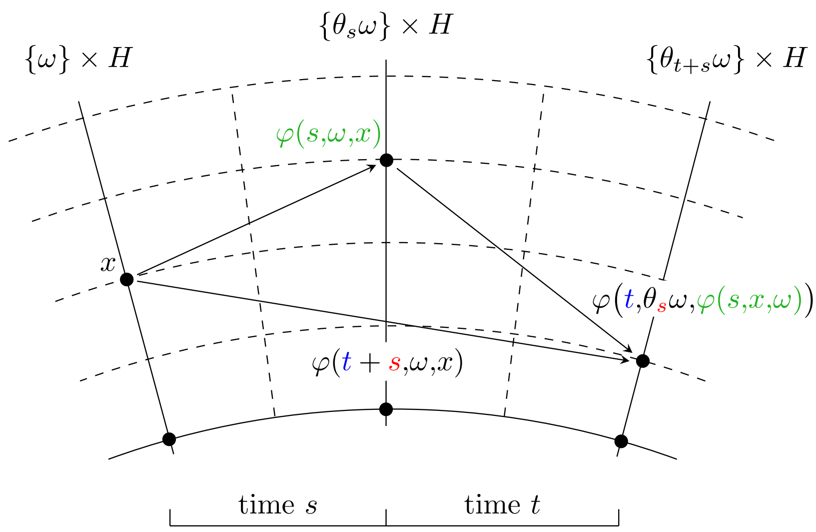


Figure 3.1: The cocycle property for two times $t, s > 0$. The space $\Omega \times H$ is called bundle and consists of the fibers $\theta.\omega \times H$ for $\omega \in \Omega$.

Remark 3.1.12.

- Note that the cocycle property in the last definition has to hold for all $s, t \in \mathbb{R}_+$, that is why this property is also called *perfect* cocycle. However there exist also *crude* and *very crude* cocycles, where (3) holds only for fixed $s \in \mathbb{R}$ and for all $t \in \mathbb{R}$, $\mathbb{P}$-a.s. or resp. only for fixed $s, t \in \mathbb{R}$, $\mathbb{P}$-a.s.

- We emphasize that the RDS is defined only for non-negative times, whereas the MDS θ is defined for the whole real line. This discrepancy arises because of the fact, that the solution operator of a proper SPDE will generated an RDS and this solution is only defined for non-negative times. If one pursues other aims for instance in finite-dimensional spaces, then according to Definition 1.1.1 in [Arn10] one can define the RDS for arbitrary time sets which form a group or a semigroup.

- We want to comment that as formulated in [Arn10, Remark 1.1.5(V)] we arrive at a deterministic dynamical system if we omit the ω-dependence in the above definition. In particular the cocycle property corresponds to the semigroup property.

- As we will see in the sequel the solution of a given random differential equation is a possible generator of an RDS. Notice that the solution of an SPDE seems to not be the right choice to generate an RDS since the appearing stochastic integrals and as a consequence the whole solution are only defined $\mathbb{P}$-a.s. (see Definitions 2.3.4 - 2.3.5), whereby the properties (2) and (3) have to hold for *every* $\omega \in \Omega$. The next part will help us to overcome this conflict.

3.2 The Ornstein-Uhlenbeck process and its application in the field of RDS

In this section we introduce the Ornstein-Uhlenbeck process with values in a Hilbert space, the properties of this process and we will see how it can be applied for the SPDE which is of our interest. In particular we discuss how to transform an SPDE into an RPDE (or short RDE) via a certain conjugacy. We show that the solution of the considered SPDE generates a continuous RDS.

Assume $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is the MDS constructed in the last section with the sample space in (3.1.7). Recall Section 2.3 and the SPDE constructed in (2.3.4). According to [CS01, Section 2, p.359], [DPZ92, Section 5.1.2, p.119] and [CGASV10, Section 3, Lemma 12] consider the equation

$$\begin{aligned} dZ(t) &= AZ(t)dt + dW(t), \quad t > s \\ Z(s) &= 0 \end{aligned} \tag{3.2.1}$$

for $t \in [s, \infty) \subset \mathbb{R}$. The noise $W = (W(t))_{t \in \mathbb{R}}$ is given by a two-sided Q-Wiener process with values in H and $\operatorname{Tr} Q < \infty$. The linear operator $-A$ is chosen like in Theorem 2.1.13. In particular the operator $-A$ is the infinitesimal generator of an analytic semigroup $S(t)$, $t \geq 0$. The unique mild solution of (3.2.1) is then given by the process $\widetilde{Z} : \mathbb{R}^2 \times \Omega \to H$ for $t \in [s, \infty)$ by

$$\widetilde{Z}(t, s, \omega) := \int_s^t S(t-r)\, dW(r) \quad \mathbb{P} - a.s.,$$

cf. [DPZ92, Theorem 5.4, p.121 and Theorem 6.5, p.156]. The process is defined by an Itô-integral so only a.s. To overcome the problem that ω lies possibly in a null set we redefine the integral by an ω-wise approach. We apply an H-valued *integration by parts for stochastic integrals*, see for instance Lemma 5.13, p.131 in [DPZ92] ($t \geq 0$) and Lemma 2.4, p.261 in [CM87] (arbitrary t),

$$\int_s^t S(t-r)\, dW(r) = W(t) - S(t-s)W(s) + A \int_s^t S(t-r)W(r)\, dr \qquad \mathbb{P} - \text{a.s.}$$

As a result from our chosen probability space we can rewrite this equality

$$\int_s^t S(t-r)\, d\omega(r) = \omega(t) - S(t-s)\omega(s) + A \int_s^t S(t-r)\omega(r)\, dr\,, \tag{3.2.2}$$

where the right-hand side holds for all $\omega \in \Omega$.
With the following lemma we introduce the so called *stationary Ornstein-Uhlenbeck* process. The eponymous stationarity of this process will be shown in the after next Lemma 3.2.3.

Lemma 3.2.1. *Let $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ be the MDS from before. The* stationary Ornstein-Uhlenbeck *process $Z : \mathbb{R} \times \Omega \to H$ given by*

$$(t, \omega) \mapsto Z(t, \omega) := Z(\theta_t \omega) := \int_{-\infty}^0 S(-r)\, d\theta_t \omega(r) \tag{3.2.3}$$

is well-defined for all $\omega \in \Omega$. Moreover for every $t \in \mathbb{R}$ we observe $Z(\theta_t\omega) = \int_{-\infty}^t S(t-r)\,d\omega(r)$ and for $t \geq 0$

$$Z(\theta_t\omega) = S(t)Z(\omega) + \int_0^t S(t-r)\,d\omega(r) \tag{3.2.4}$$

for every $\omega \in \Omega$.

Remark 3.2.2. Let $t \in \mathbb{R}_+$. The stationary Ornstein-Uhlenbeck process is the unique mild solution of the following equation

$$\begin{aligned} dZ(\theta_t\omega) &= AZ(\theta_t\omega)dt + dW(t), \quad t > 0, \\ Z(\theta_0\omega) &= Z(\omega) = \int_{-\infty}^0 S(-r)\,d\omega(r) \end{aligned} \tag{3.2.5}$$

for every $\omega \in \Omega$.

Proof of Lemma 3.2.1. We begin with the claimed equalities, provided $Z(\theta_t\omega)$ is well-defined. First notice that $d\theta_t\omega(r) = d(\omega(t+r) - \omega(t)) = d(\omega(t+r))$ for every fixed $t \in \mathbb{R}$. Then we obtain

$$Z(\theta_t\omega) = \int_{-\infty}^0 S(-r)d\omega(t+r) = \int_{-\infty}^t S(t-r)d\omega(r) \tag{3.2.6}$$

for $t \in \mathbb{R}$. In the case $t \geq 0$ we conclude using the semigroup property

$$\begin{aligned} Z(\theta_t\omega) &= \int_{-\infty}^t S(t-r)d\omega(r) = S(t)\int_{-\infty}^0 S(-r)d\omega(r) + \int_0^t S(t-r)d\omega(r) \\ &= S(t)Z(\omega) + \int_0^t S(t-r)d\omega(r)\,. \end{aligned} \tag{3.2.7}$$

With view on (3.2.6) it suffices to consider $Z(\omega)$ since we can shift $\omega(\cdot)$ appropriately. According to [Nea17, Lemma 2.2.7 and Lemma 2.2.10] we show that the random variable $Z(\cdot)$ is well-defined. First apply the H-valued integration by parts for stochastic integrals (3.2.2) which we used already above,

$$Z(\omega) = \lim_{t\to\infty}\left(\underbrace{\omega(0)}_{=0} - S(t)\omega(-t) + A\int_{-t}^0 S(-r)\omega(r)\,dr\right)\,.$$

According to Theorem 3.1.10 we can choose Ω to be a θ-invariant subset of C_0 such that the trajectories have a subexponential growth. In the last summand we have two possible singularities in 0 and in infinity. Therefore we write

$$Z(\omega) = \lim_{t\to\infty}\left(-S(t)\omega(-t) + A\int_{-t}^{-1} S(-r)\omega(r)\,dr\right) + A\int_{-1}^0 S(-r)\omega(r)\,dr\,. \tag{3.2.8}$$

Under consideration of the exponential decay of the semigroup (2.1.1) and the inequality (3.1.6) we estimate for every $\alpha > \varepsilon > 0$ and $t \geq t_0(\varepsilon,\omega)$,

$$\|S(t)\omega(-t)\| + \left\|A\int_{-t}^{-1} S(-r)\omega(r)\,dr\right\| \leq C_\varepsilon e^{-(\alpha-\varepsilon)t} + \int_{-t}^{-1} \|AS(-r)\omega(r)\|\,dr\,.$$

where we used the closedness of the operator A to take the operator inside the integral, see [Vra03, Theorem 1.2.2, p.8]. For the latest integral we obtain in view of the Theorem 2.1.16 for analytic semigroups,

$$\int_{-t}^{-1} \|AS(-r)\omega(r)\|\,dr \leq \int_{-t_0}^{-1} \|AS(-r)\omega(r)\|\,dr + M_1C_\varepsilon\int_{-t}^{-t_0} \frac{e^{(\alpha-\varepsilon)r}}{-r}\,dr\,.$$

Therefore the limit in (3.2.8) is finite. Now for the last summand in (3.2.8) we estimate using the Hölder norm,

$$\begin{aligned}\left\| A\int_{-1}^{0} S(-r)\omega(r)\,dr\right\| \le M_1\int_{-1}^{0}\frac{e^{\alpha r}}{-r}\cdot\|\omega(r)\|\,dr &\le M_1\|\omega\|_{C^\gamma([-1,0];H)}\int_{-1}^{0} e^{\alpha r}|r|^{\gamma-1}\,dr\\ &\le M_1C_2\int_{-1}^{0} e^{\alpha r}|r|^{\gamma-1}\,dr\end{aligned}$$

for $C_2 = C_2(\omega,\gamma,-1,0) > 0$. The last integral is finite, since $\gamma\in\left(0,\frac{1}{2}\right)$.

□

Now we want to collect some more properties concerning the stationary Ornstein-Uhlenbeck process.

Lemma 3.2.3. *The process defined in* (3.2.3) *is stationary, i.e. for a time set* $\{t,t_1,...,t_k\}$, $k\ge 1$ *and a set* $A\in\mathcal{F}$ *it holds that* $\mathbb{P}_{Z_{t_1+t},...,Z_{t_k+t}}(A) = \mathbb{P}_{Z_{t_1},...,Z_{t_k}}(A)$.

Proof. For the Ornstein-Uhlenbeck process it follows similar as before for arbitrary $s,t\in\mathbb{R}$,

$$Z(s+t,\omega) = \int_{-\infty}^{s+t} S(s+t-r)\,d\omega(r) = \int_{-\infty}^{s} S(s-r)\,d\theta_t\omega(r) = Z(s,\theta_t\omega)$$

for each $\omega\in\Omega$. From this equality we conclude for a proper time set and $A\in\mathcal{F}$,

$$\begin{aligned}\mathbb{P}_{Z_{t_1+t},...,Z_{t_k+t}}(A) &= \mathbb{P}\big[(Z(t_1+t,\omega),...,Z(t_k+t,\omega))\in A\big]\\ &= \mathbb{P}\big[(Z(t_1,\theta_t\omega),...,Z(t_k,\theta_t\omega))\in A\big]\\ &= \theta_t\mathbb{P}\big[(Z(t_1,\omega),...,Z(t_k,\omega))\in A\big]\\ &= \mathbb{P}_{Z_{t_1},...,Z_{t_k}}(A)\end{aligned}$$

where in the last equality we use the invariance of the measure $\mathbb{P}$. □

Lemma 3.2.4. *For the given MDS* $(\Omega,\mathcal{F},\mathbb{P},\theta)$ *the stationary Ornstein-Uhlenbeck process possesses trajectories in* $C(\mathbb{R}_+;H)$. *Moreover* Z *is a random fixed-point of the equation*

$$\varphi(t,\omega,Z(\omega)) = Z(\theta_t\omega) \tag{3.2.9}$$

for every $t\ge 0$ *and* $\omega\in\Omega$, *where* φ *is the random dynamical system generated by the equation* (3.2.5).

Proof. According to [DPZ92, Theorem 5.14, p.132] we know that $t\mapsto\int_0^t S(t-r)\,d\omega(r)$ is Hölder-continuous with Hölder exponent $\gamma\in(0,\frac{1}{2})$. The continuity of $Z(\theta_\cdot\omega)$ follows from the continuity of the semigroup, i.e. $0\le t\mapsto S(t)x$ is continuous for every $x\in H$ (cf. Corollary 2.1.3) and the decomposition in (3.2.4).

For the second part of the lemma we use again (3.2.4) and combine it with the integration by parts formula. Hence we have for all $\omega\in\Omega$

$$Z(\theta_t\omega) = S(t)Z(\omega) + \int_0^t S(t-r)\,d\omega(r) = S(t)Z(\omega) + \omega(t) + A\int_0^t S(t-r)\omega(r)\,dr\,.$$

Setting $\varphi:\mathbb{R}_+\times\Omega\times H\to H$ with

$$\varphi(t,\omega,x) = S(t)x + \int_0^t S(t-r)\,d\omega(r) = S(t)x + \omega(t) + A\int_0^t S(t-r)\omega(r)\,dr$$

we obtain directly the fixed-point property (3.2.9). It remains to prove that φ is indeed a random dynamical system. The measurability follows e.g. from [AB06, Lemma 4.51, p.153] and [CV06,

Lemma III.14, p.70]. Fix $x \in H$. The mapping $\varphi(\cdot,\cdot,x) : \mathbb{R}_+ \times \Omega \to H$ is *Carathéodory*, i.e. $\varphi(\cdot,\omega,x) : \mathbb{R}_+ \to H$ is continuous for each $\omega \in \Omega$ and $\varphi(t,\cdot,x) : \Omega \to H$ is measurable for each $t \in \mathbb{R}_+$. Hence the mapping $\varphi : \mathbb{R}_+ \times \Omega \to H$ is jointly $(\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}) - \mathcal{B}(H)$-measurable. Again because $\varphi(t,\omega,\cdot) : H \to H$ is continuous, $\varphi(\cdot,\cdot,\cdot)$ is Carathéodory, hence we obtain that $\varphi : \mathbb{R}_+ \times \Omega \times H \to H$ is jointly $(\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F} \otimes \mathcal{B}(H) - \mathcal{B}(H))$-measurable. The identity property is obviously.
The cocycle property follows by a direct computation

$$\begin{aligned}\varphi(t,\theta_s\omega,\varphi(s,\omega,x)) &= S(t)\left[S(s)x + \int_0^s S(s-r)\,d\omega(r)\right] + \int_0^t S(t-r)\,d\theta_s\omega(r)\\ &= S(t+s)x + \int_0^s S(t+s-r)d\omega(r) + \int_s^{t+s} S(t+s-r)\,d\omega(r)\\ &= \varphi(t+s,\omega,x)\end{aligned}$$

for every $t,s \in \mathbb{R}_+$, $\omega \in \Omega$ and $x \in H$. $\square$

Remark 3.2.5. Note that we can obtain more regularity results concerning the Ornstein-Uhlenbeck process, cf. [Nea17, Remark 2.2.14, p.16]. One can show that the process $Z(\theta_t\omega)$ is well-defined in $D((-A)^\beta)$ for every $0 \le \beta < \frac{1}{2}$ and $Z(\theta.\omega) \in C^\gamma(\mathbb{R}; D(-A)^\beta)$ for every $0 \le \gamma + \beta < \frac{1}{2}$, cf. [DPZ92, Theorem 5.16, p.134].

Now we come to a concept which plays an important role especially when we want to define random attractors in the pullback sense. We have already fixed our sample space Ω in (3.1.7). There we controlled the growth of our process such that it does not expand to infinity 'to fast', when the time tends to $\pm\infty$. Now we want extend this property to a formal definition for random variables. We refer to [Arn10, Definition 4.1.1, p.164] and [BGAS14, Section 3, p.3957].

Definition 3.2.6. Let $(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}})$ be some arbitrary metric dynamical system. A random variable $X : \Omega \to [0,\infty)$ is called *tempered* or more precisely *tempered from above*, if there exists a θ-invariant set of full $\mathbb{P}$-measure such that

$$\lim_{t\to\pm\infty} \frac{\log^+ X(\theta_t\omega)}{|t|} = 0 \tag{3.2.10}$$

for all $\omega \in \Omega$. Note that $\log^+(\cdot) := \max\{0, \log(\cdot)\}$.
A random variable $X : \Omega \to (0,\infty)$ is called *tempered from below* if $1/X$ is tempered from above or equivalently if

$$\lim_{t\to\pm\infty} \frac{\log^- X(\theta_t\omega)}{|t|} = 0 \tag{3.2.11}$$

for every $\omega \in \Omega$ with $\log^-(\cdot) := \max\{0, -\log(\cdot)\}$ on some θ-invariant set of full $\mathbb{P}$-measure.

Remark 3.2.7.

- One has to be careful with definition of the word 'tempered'. Sometimes a tempered random variable is referred to be tempered from above and from below, see [Arn10, Definition 4.1.1, p.164]. In our setup we understand under a tempered random variable always a random variable tempered from above as defined in the previous definition.

- According to Proposition 4.1.3 in [Arn10, p.165] the long-time behaviour of a random variable $X : \Omega \to \mathbb{R}$ on a metric dynamical system θ can be described by

$$\limsup_{t\to\pm\infty} \frac{X(\theta_t\omega)}{|t|} = \{0,\infty\}$$
$$\liminf_{t\to\pm\infty} \frac{X(\theta_t\omega)}{|t|} = \{-\infty,0\}$$

and in the ergodic case the lim sup's and lim inf's are constants for every $\omega \in \Omega$, i.e. 0 or $\pm\infty$ on a invariant subset of Ω. The only other possiblity for a *tempered* random variable on such a subset is

$$\limsup_{t\to\pm\infty} \frac{\log^+ X(\theta_t\omega)}{|t|} = \infty$$

and similar for the lim inf. Hence the assumption of temperedness for a random variable is not very strict.

- We can rearrange (3.2.10) such that the subexponential growth of a tempered random variable becomes more visible. Equivalent to the above definition is the following. For every $\varepsilon > 0$ and $\omega \in \Omega$ there exists a $t_0 = t_0(\varepsilon, \omega) > 0$ such that for $|t| \geq t_0$

$$X(\theta_t\omega) \leq e^{\varepsilon|t|}\,.$$

Other possible equivalent formulations can be found in [IS01, p.220] or [CDLS10, Lemma 2.1, (2.3)]. Note that if $t \mapsto X(\theta_t\omega)$ is continuous for fixed $\omega \in \Omega$, then for every $\varepsilon > 0$ there exists a constant $C(\varepsilon, \omega) > 0$ such that for all $t \in \mathbb{R}$

$$\frac{1}{C(\varepsilon,\omega)} e^{-\varepsilon|t|} \leq X(\theta_t\omega) \leq C(\varepsilon,\omega)e^{\varepsilon|t|} \tag{3.2.12}$$

see Proposition 4.3.3 [Arn10, p.188].

- Another more practical formulation of temperedness is given by the dichotomy for linear growth of stationary processes, see Proposition 4.1.3 [Arn10, p.165]. Let X be the non-negative random variable defined on a θ-invariant set Ω with $\mathbb{P}(\Omega) = 1$ like in Definition 3.2.6. If

$$\sup_{t\in[0,1]} X(\theta_t\omega) \in L^1(\Omega)$$

then X is a tempered random variable.

Example 3.2.8. Let $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ be the MDS which we already used in Lemma 3.2.1 together with the stationary Ornstein-Uhlenbeck process $Z : \mathbb{R} \times \Omega \to H$. Then $\|Z(\cdot)\| : \Omega \to [0,\infty)$ is a tempered random variable.

Proof. According to the last remark, it is sufficient to show that

$$\mathbb{E}\left(\sup_{t\in[0,1]} \|Z(\theta_t\omega)\|\right) < \infty\,.$$

Then it follows for all ω from a θ-invariant subset Ω of full $\mathbb{P}$-measure

$$\lim_{t\to\pm\infty} \frac{\log^+ \|Z(\theta_t\omega)\|}{|t|} = 0\,.$$

Under consideration of (3.2.7) and (3.2.2) we have for $t \in [0,1]$

$$\begin{aligned}
Z(\theta_t\omega) &= S(t)Z(\omega) + \omega(t) + A\int_0^t S(t-r)\omega(r)\,dr \\
&= S(t)Z(\omega) + \omega(t) + A\int_0^t S(t-r)(\omega(r) - \omega(t))\,dr + A\int_0^t S(t-r)\omega(t)\,dr \\
&= S(t)Z(\omega) + S(t)\omega(t) + A\int_0^t S(t-r)(\omega(r) - \omega(t))\,dr
\end{aligned}$$

where we applied Theorem 2.1.4(b). Hence we need to show that

$$\mathbb{E} \sup_{t\in[0,1]} \|Z(\theta_t\omega)\| \leq \underbrace{\mathbb{E}\|Z(\omega)\|}_{\text{(I)}} + \underbrace{\mathbb{E} \sup_{t\in[0,1]} \|\omega(t)\|}_{\text{(II)}} + \underbrace{\mathbb{E} \sup_{t\in[0,1]} \left\| A \int_0^t S(t-r)(\omega(r)-\omega(t))\, dr \right\|}_{\text{(III)}} .$$

For the first expectation (I) we use the Itô-isometry for the H-valued stochastic integration, see [PR07, Proposition 2.3.5, p.25]. We observe

$$\begin{aligned}\mathbb{E}\left(\|Z(\omega)\|^2\right) &= \mathbb{E}\left(\left\|\int_{-\infty}^0 S(-r)\, d\omega(r)\right\|^2\right) = \mathbb{E}\left(\int_{-\infty}^0 \|S(-r)\circ Q^{\frac{1}{2}}\|^2_{L_2(H)}\, dr\right)\\ &\leq \int_{-\infty}^0 e^{2\alpha r}\|Q^{\frac{1}{2}}\|^2_{L_2(H)}\, dr = \frac{1}{2\alpha}\operatorname{Tr} Q < \infty ,\end{aligned}$$

where in the last line the Hilbert-Schmidt norm $\|Q^{\frac{1}{2}}\|^2_{L_2(H)} = \operatorname{Tr} Q$ according to Proposition 2.2.18 . Hence $\mathbb{E}\|Z(\omega)\| < \infty$.

Now we consider (II). It follows readily from the Doob maximal inequality [KS88, Theorem 1.3.8 (iv), p.14] and Remark 2.2.14, that

$$\mathbb{E} \sup_{t\in[0,1]} \|\omega(t)\| \leq \sum_{i=1}^{\infty} \lambda_i \cdot \mathbb{E}\left(\sup_{t\in[0,1]} |\beta_i(t)|\right)^2 \leq 4\operatorname{Tr} Q < \infty ,$$

where β_i are the real-valued Brownian motions introduced in Proposition 2.2.13.

The term (III) is estimated by using the methods of the proof of Lemma 3.2.1,

$$\left\| A \int_0^t S(t-r)(\omega(r)-\omega(t))\, dr \right\| \leq M_1 \int_0^t \frac{e^{-\alpha(t-r)}}{t-r} \|\omega(r)-\omega(t)\|\, dr .$$

For the noise we have for $t, r \in [0,1]$, $r \neq t$

$$\|\omega(r)-\omega(t)\| = \frac{\|\omega(r)-\omega(t)\|}{|t-r|^\gamma} \cdot |t-r|^\gamma \leq \|\omega\|_{C^\gamma([0,1];H)} \cdot |t-r|^\gamma .$$

Therefore we conclude for every $t \in [0,1]$,

$$\left\| A \int_0^t S(t-r)\omega(r)\, dr \right\| \leq M_1 \|\omega\|_{C^\gamma([0,1];H)} \int_0^t e^{-\alpha(t-r)} \cdot |t-r|^{\gamma-1}\, dr < \infty$$

and $\|\omega\|_{C^\gamma([0,1];H)} \leq C_2(\omega,\gamma,0,1) \in L^1(\Omega)$ completes the proof. □

3.3 Random dynamical systems for SPDE's and conjugacy

Finally we introduce the SPDE we are interested in and for which we show that the solution operator generates an RDS. Let $H := L^2(K,\mu)$ where $(K,\mathcal{B}(K),\mu)$ is a σ-finite measure space of a bounded subset K of $\mathbb{R}^n$, $n \geq 1$ representing the domain of our problem. Then we consider

$$\begin{aligned} dv(t) &= Av(t)dt + F(v(t))dt + dW(t),\ \forall t > 0,\\ v(0) &= v_0 \in H . \end{aligned} \tag{3.3.1}$$

The linear operator A is the generator of an analytic semigroup in H, denoted by $\{S(t)\}_{t\geq 0}$, see Section 2.1. The appearing noise is again an H-valued Q-Wiener process with finite trace, cf. 2.2.12. On the mapping $F : H \to H$ we assume a global Lipschitz continuity with the associated Lipschitz constant $L > 0$, i.e. $\|F(v_1(t)) - F(v_2(t))\| \leq L\|v_1(t) - v_2(t)\|$ for every $t \in [0,\infty)$. Let

$(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}_+})$ be the metric dynamical system based on Ω the canonical sample space on $\mathbb{R}_+$ which is given by (3.1.7). Let us endow the trace σ-algebra $\mathcal{F}$ with an associated filtration $(\mathcal{F}_t)_{t\geq 0}$ and assume v_0 to be an H-valued $\mathcal{F}_0$-measurable random variable. Then the mild solution of the equation (3.3.1) is given according to (2.3.5)

$$v(t) = S(t)v_0 + \int_0^t S(t-r)F(v(r))\,dr + \int_0^t S(t-r)\,dW(r)$$

for all $t \in \mathbb{R}_+$, $\mathbb{P}$-a.s. Now we define for every $(t, \omega) \in \mathbb{R}_+ \times \Omega$

$$u(t) := v(t) - Z(\theta_t\omega)\,. \tag{3.3.2}$$

Together with Remark 3.2.2 we deduce

$$\begin{aligned} du(t) &= d(v(t) - Z(\theta_t\omega)) = [Av(t) + F(v(t))]dt + dW(t) - AZ(\theta_t\omega)dt - dW(t) \\ &= A(v(t) - Z(\theta_t\omega))dt + F(v(t))dt\,. \end{aligned}$$

Similar for the initial condition $u_0 := u(0) = v(0) - Z(\omega) = v_0 - Z(\omega)$.
Hence we arrive at an ω-wise differential equation which we call *random* (partial) differential equation, for short *RDE* or *RPDE*,

$$\begin{aligned} du(t) &= Au(t)dt + F(u(t) + Z(\theta_t\omega))dt, \quad t > 0, \\ u(0) &= u_0\,. \end{aligned} \tag{3.3.3}$$

We recall the well-known Gronwall Lemma, which will be used repeatedly in our context.

Lemma 3.3.1 (Lemma 29.2, p.436,[Wlo87])**.** *Let $g, v \in C([0,T])$, h be a non-negative integrable function on $(0,T)$ and suppose for $t \in [0,T]$ we have*

$$v(t) \leq g(t) + \int_0^t h(\tau)v(\tau)\,d\tau\,.$$

Then we have for $t \in [0,T]$

$$v(t) \leq g(t) + \int_0^t g(\tau)h(\tau)e^{H(t)-H(\tau)}\,d\tau = e^{H(t)}\left[g(0) + \int_0^t g'(\tau)e^{-H(\tau)}\,d\tau\right],$$

with $H(t) = \int_0^t h(\tau)\,d\tau$. The last equality in the statement holds for differentiable functions g.

The next theorem is crucial for our theory.

Theorem 3.3.2. *Under the previous assumptions there exists a unique solution of the equation (3.3.3) on every interval $[0,T]$, $T > 0$ given by the variation of constants formula for every $\omega \in \Omega$ and $u_0 \in H$*

$$u(t) = S(t)u_0 + \int_0^t S(t-r)F(u(r) + Z(\theta_r\omega))\,dr\,. \tag{3.3.4}$$

Moreover $u \in C([0,T];H)$. The solution operator (3.3.4) is the generator of a continuous random dynamical system $\varphi : \mathbb{R}_+ \times \Omega \times H \to H$ given by

$$\varphi(t,\omega,x) = u(t,\omega,x) = S(t)x + \int_0^t S(t-r)F(u(r) + Z(\theta_r\omega))\,dr \tag{3.3.5}$$

for $(t,\omega,x) \in \mathbb{R}_+ \times \Omega \times H$.

Proof. The existence and uniqueness of the solution of the RDE is a special case of Theorem 6.1.2 in [Paz83, p.184f.]. In particular the Banach fixed-point Theorem [AB06, Theorem 3.48, p.95] and the Gronwall Lemma mentioned above play key roles.

The measurability follows the same ideas we have seen in the proof of Lemma 3.2.4. For a fixed $x \in H$, $T > 0$ define the mapping

$$
\begin{aligned}
&T_{\cdot,x}(\cdot) : \Omega \times C([0,T];H) \to C([0,T];H) \\
&T_{\omega,x}(u)[t] = S(t)x + \int_0^t S(t-r)F(u(r) + Z(\theta_r\omega))\,dr
\end{aligned}
$$

for every $t \in [0,T]$. For fixed $\omega \in \Omega$ this mapping is Lipschitz continuous (see e.g. the proof of [Paz83, Theorem 6.1.2, p.184]) and for fixed $u \in C([0,T];H)$ the mapping $T_{\cdot,x}(u)$ is measurable since every involved mapping is continuous resp. measurable. Moreover according to [Bau92, Satz 31.6] the space $C([0,T];H)$ is separable. Then [AB06, Lemma 4.51, p.153] says that $T_{\cdot,x}(\cdot)$ is $[\mathcal{F} \otimes \mathcal{B}(C([0,T];H))] - \mathcal{B}(C([0,T];H))$ measurable. Taking into account that $x \in C([0,T];H)$ and the fixed-point argument in [AB06, Theorem 3.48, p.95], we obtain that the sequence $T^n_{\omega,x}(x) = T_{\omega,x} \circ \cdots \circ T_{\omega,x}(x)$ converges to a unique fixed-point $u(\cdot,\omega,x) \in C([0,T];H)$ which itself is $[\mathcal{F} \otimes \mathcal{B}(C([0,T];H))] - \mathcal{B}(C([0,T];H))$ measurable. Now evaluating this function at a time $t \in [0,T]$ we obtain that the mapping $\omega \mapsto u(t,\omega,x) \in H$ is $\mathcal{F} - \mathcal{B}(H)$ measurable.
Therefore $\varphi(\cdot,\cdot,x) : \mathbb{R}_+ \times \Omega \to H$ is jointly measurable, from the Carathéodory argument discussed in Lemma 3.2.4. As we will see below $\varphi(t,\omega,\cdot)$ is continuous. Finally the desired measurability follows again from the Carathéodory argument.

The identity property is evident. Concerning the cocycle property notice for $t,s \geq 0$ the following

$$
\begin{aligned}
\varphi(t+s,\omega,x) &= S(t+s)x + \int_0^{t+s} S(t+s-r)F(u(r,\omega,x) + Z(\theta_r\omega))\,dr \\
&= S(t)\left[S(s)x + \int_0^s S(s-r)F(u(r,\omega,x) + Z(\theta_r\omega))\,dr\right] \\
&\quad + \int_s^{t+s} S(t+s-r)F(u(r,\omega,x) + Z(\theta_r\omega))\,dr \\
&= S(t)u(s) + \int_0^t S(t-r)F(u(r+s,\omega,x) + Z(\theta_{r+s}\omega))\,dr \\
&= S(t)w(0) + \int_0^t S(t-r)F(w(r,\omega,x) + Z(\theta_{r+s}\omega))\,dr\,,
\end{aligned}
$$

where $w(t) := u(t+s)$ solves the RDE

$$
\begin{aligned}
dw(t) &= Aw(t)dt + F(w(t) + Z(\theta_t\theta_s\omega))dt \\
w(0) &= u(s)\,.
\end{aligned}
$$

By the uniqueness of the solution we obtain $\varphi(t+s,\omega,x) = \varphi(t,\theta_s\omega,u(s,\omega,x))$. The next goal is to show the continuity of the RDS. Moreover we even obtain Lipschitz continuity.
Let us define $\mathscr{D}\varphi(t) := \varphi(t,\omega,y) - \varphi(t,\omega,x)$. Then first notice that

$$
\begin{aligned}
\|\mathscr{D}\varphi(t)\| &\leq \|S(t)\|_{L(H)}\|(y-x)\| \\
&\quad + \int_0^t \|S(t-r)\|_{L(H)}\|F(u(r,\omega,y) + Z(\theta_r\omega)) - F(u(r,\omega,x) + Z(\theta_r\omega))\|\,dr \\
&\leq \|y-x\|e^{-\alpha t} + L\int_0^t e^{-\alpha(t-r)}\|\underbrace{u(r,\omega,y) - u(r,\omega,x)}_{=\mathscr{D}\varphi(r)}\|\,dr
\end{aligned}
$$

where we used (2.1.8) and the Lipschitz continuity of F. To apply the Gronwall Lemma we muliply the last inequality by $e^{\alpha t}$, such that

$$e^{\alpha t}\|\mathscr{D}\varphi(t)\| \le \|y-x\| + L\int_0^t e^{\alpha r}\|\mathscr{D}\varphi(r)\|\,dr\,.$$

We deduce from Lemma 3.3.1 (second statement),

$$e^{\alpha t}\|\mathscr{D}\varphi(t)\| \le \|y-x\|e^{Lt}$$

and hence

$$\|\varphi(t,\omega,y)-\varphi(t,\omega,x)\| = \|\mathscr{D}\varphi(t)\| \le \|y-x\|e^{(L-\alpha)t} \tag{3.3.6}$$

for fixed $t\in\mathbb{R}_+$ and $\omega\in\Omega$. The last expression gets arbitrary small if $x,y\in H$ are close enough to each other. □

Note that in this proof we do not have to assume anything on the positive constants L and α. Until now we have a random dynamical system φ for the corresponding RDE. Originally we were interested in an RDS generated by the solution operators of SPDEs of the type (3.3.1). The following lemma closes this gap and is often called *conjugacy* or *transformation of an RDS.* We refer, for instance, to [CDLS10, Lemma 2.1] or [DLS03, Lemma 2.2, p.2116] for the idea of the proof.

Lemma 3.3.3. *Let $T:\Omega\times H\to H$ be a mapping, such that $T(\omega,\cdot):H\to H$ is a homeomorphism for any $\omega\in\Omega$ and $T(\cdot,x):\Omega\to H$, $T^{-1}(\cdot,x):\Omega\to H$ are measurable for every $x\in H$. If φ is a random dynamical system, then the mapping*

$$\begin{aligned}&\psi:\mathbb{R}_+\times\Omega\times H\to H\\ &\psi(t,\omega,x) := T\big(\theta_t\omega,\varphi(t,\omega,T^{-1}(\omega,x))\big)\end{aligned}$$

is as well a random dynamical system.

Combining this lemma with the choice $T(\omega,x)=x+Z(\omega)$ and its inverse (w.r.t. the second component) $T^{-1}(\omega,x)=x-Z(\omega)$ we obtain in view of (3.3.2),

$$v(t) = T(\theta_t\omega,u(t)) = u(t)+Z(\theta_t\omega)\,. \tag{3.3.7}$$

We summarize our observations in the following corollary.

Corollary 3.3.4. *Let φ be the RDS generated by the unique solution of the RDE* (3.3.3). *Defining the conjugacy according to* (3.3.7) *gives us a random dynamical system $\psi:\mathbb{R}_+\times\Omega\times H\to H$ for the SPDE* (3.3.1)

$$\psi(t,\omega,x) = \varphi(t,\omega,x-Z(\omega)) + Z(\theta_t\omega)$$

for every $t\in\mathbb{R}_+$, $\omega\in\Omega$ and $x\in H$. Moreover ψ is continuous.

The idea of conjugacy can be visualized by the following figure.

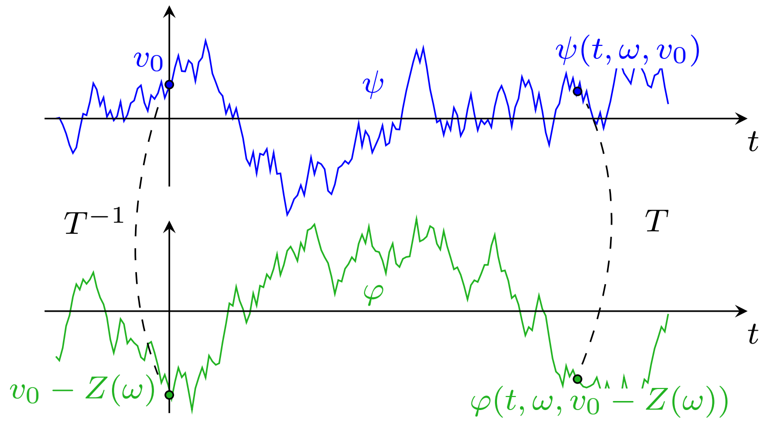


Figure 3.2: The initial data v_0 gets transformed via T^{-1} to be the initial data $u_0 = v_0 - Z(\omega)$ of an according RDS φ. Then we evole this system until a certain time t and conjugate the actual state via T to obtain $\psi(t,\omega,v_0)$. If we apply this transformation along the time $t \geq 0$ we arrive at the trajectories in the picture for a fixed $\omega \in \Omega$.

3.4 Random attractors for random dynamical systems

In this part, we discuss the concept of random attraction. Keeping in mind the example given in the introduction, we now describe mathematical objects that are similar to the vortex in the ocean.
For the basic definitions and ideas we refer to [Sch97] and [Sch92]. A more recent overview of the concepts can be found for example in [BGAS14].
We begin with the definition of a random closed set. For this definition we apply the theory of multifunctions also known as set-valued functions. A general introduction is given in e.g. [AB06, Chapter 18]. For a collection of results focusing our case we refer to [Ogr11, Chapter 2].

Definition 3.4.1. Let H be a separable Hilbert space. A multifunction $M : \Omega \to \mathcal{P}(H)$ which maps in particular to nonempty closed sets of $\mathcal{P}(H)$ is called a *closed random set* if for every $x \in H$

$$\omega \mapsto \inf_{y \in M(\omega)} \|x - y\|,$$

is $\mathcal{F} - \mathcal{B}(\mathbb{R}_+)$ measurable. A multifunction M is also denoted by $\{M(\omega)\}_{\omega\in\Omega}$ or simply $M(\omega)$ for $\omega \in \Omega$.

Remark 3.4.2. For abbreviation we say *random set* instead of closed random set. According to [CV06, Proposition III.4, p.63] for a sequence of compact-valued multifunctions M_n, $n \in \mathbb{N}$ the mapping $\omega \mapsto \bigcap_{n\in\mathbb{N}} M_n(\omega)$ is measurable and if in addition the sets M_n ,$n \geq 1$ are monotonically decreasing, then $\bigcap_{n\in\mathbb{N}} M_n$ has compact values. Moreover if $\overline{\bigcup_{n\in\mathbb{N}} M_n}^H$ is compact, then $\omega \mapsto \overline{\bigcup_{n\in\mathbb{N}} M_n(\omega)}^H$ is measurable.

In the following we identify the subset of $\mathcal{P}(H)$ which gives us later the sets that get attracted by the random attractor.

Definition 3.4.3. A set $\mathcal{D}$ consisting only of closed random sets of $\mathcal{P}(H)$ is called *universe* if it is *inclusion closed*, i.e. whenever an arbitrary random set $D_1 \in \mathcal{P}(H)$ is a subset of another random set $D_2 \in \mathcal{D}$ for every $\omega \in \Omega$, then $D_1 \in \mathcal{D}$.

The following definition of tempered sets is extracted from [Sch97, p.956], see also [BLW09, Definition 2.3, p.847].

Definition 3.4.4. Let $\theta = (\theta_t)_{t\in\mathbb{R}}$ be an MDS. A random set $D \in \mathcal{P}(H)$ is called a *tempered set* (w.r.t. θ) if for any $c > 0$

$$\lim_{t\to\infty} e^{-ct} \sup_{x\in D(\theta_{-t}\omega)} \|x\| = 0\,.$$

An equivalent definition is given accordingly to [BGAS14, p.3958] by

$$\lim_{t\to\pm\infty} \frac{1}{|t|} \log^+ \left(\sup_{x\in D(\theta_t\omega)} \|x\| \right) = 0\,. \tag{3.4.1}$$

Let $\mathcal{D}$ denote the universe of tempered sets in H. Now we define the random attractor in the pullback sense.

Definition 3.4.5 ([Sch97])**.** Let $\varphi : \mathbb{R}_+ \times \Omega \times H \to H$ be a random dynamical system and θ an associated MDS. We call the random compact set $\mathcal{A} \in \mathcal{D}$ a *random attractor* (of the RDS φ), if the following two conditions are fulfilled,

(i) $\mathcal{A}(\omega)$ is invariant for every $\omega \in \Omega$, i.e.

$$\varphi(t,\omega,\mathcal{A}(\omega)) = \mathcal{A}(\theta_t\omega) \quad \text{for all } t \in \mathbb{R}_+$$

(ii) $\mathcal{A}(\omega)$ attracts all sets in $\mathcal{D}$, that is for all $\omega \in \Omega$,

$$\lim_{t\to\infty} \operatorname{dist}\big(\varphi(t,\theta_{-t}\omega, D(\theta_{-t}\omega)), \mathcal{A}(\omega)\big) = 0 \qquad \text{for all } D \in \mathcal{D}\,, \tag{3.4.2}$$

where 'dist' is the *Hausdorff semi-distance* defined for two nonempty sets $A, B \subset H$ by

$$\operatorname{dist}(A,B) := \sup_{x\in A} \inf_{y\in B} \|x-y\|\,. \tag{3.4.3}$$

Note that the Hausdorff semi-distance is not a metric, since for every subset $A \subseteq B$ we have $\operatorname{dist}(A,B) = 0$.

Remark 3.4.6.

- The above attraction property (ii) happens in the pullback sense. The advantage of the convergence in this sense, is that the set $(\mathcal{A}(\omega))_{\omega\in\Omega}$ is not changing in time. Hence the name attractor is meaningful. Roughly speaking, if we go far enough backwards in time with our initial set, the property (3.4.2) tells us that we will arrive in the attractor when we wait long enough.

- It is possible to define a random attractor as the set that attracts every bounded deterministic set, see [CF94, Definition 3.9, p.370]. However this definition requires a $\mathbb{P}$-a.s. approach of the property (3.4.2). In particular we lose the uniqueness of the random attractor [CF94, p.372]. If we consider random sets in $\mathcal{D}$ we obtain the uniqueness of the random attractor, see [Sch97, Lemma 2.3, p.955].

- As stated in [Sch97, Remark 2.2] the random attractor is indeed a generalisation of the deterministic semigroup attractor defined e.g. in [Tem88, Chapter I, Definition 1.2, p.20]. The universe of the random attractor corresponds to the *basin of attraction* of the attractor in the deterministic case. Additionally the mentioned Remark 2.2 tells us that the forward convergence to the attractor holds in probability. More precisely, for every $\varepsilon > 0$ we have for any $D \in \mathcal{D}$ and $\omega \in \Omega$

$$\lim_{t\to\infty} \mathbb{P}\big[\operatorname{dist}\big(\overline{\varphi(t,\omega,D(\omega))}, \mathcal{A}(\theta_t\omega)\big) > \varepsilon\big] = 0\,.$$

The above limit does not hold $\mathbb{P}$-a.s. in the general case, see [Arn10, Remark 9.3.7, p.488] for a counterexample. An interesting overview of the different kinds of random attractors can be found in [Sch02].

- For comparison: in the case of a deterministic autonomous dynamical system the concepts of pullback and forward convergence coincide, whereas in nonautonomous systems the convergence differs in general, see [GK01, Section 4].

Before we state the important existence theorem for the random attractor of an RDS, we need one more definition giving us the concept of absorption.

Definition 3.4.7 ([Sch97], p.956)**.** Let $\varphi : \mathbb{R}_+ \times \Omega \times H \to H$ be a random dynamical system and θ an associated MDS.
A random set $B \in \mathcal{D}$ is called *random absorbing* for φ if for every $D \in \mathcal{D}$ and $\omega \in \Omega$ there exists a time $t_D(\omega) > 0$ such that

$$\varphi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)) \subset B(\omega) \text{ for all } t \geq t_D(\omega)\,.$$

An often more comfortable condition in finding an absorbing set is the following. If for every $x \in D(\theta_{-t}\omega)$, $D \in \mathcal{D}$ and $\omega \in \Omega$

$$\limsup_{t\to\infty} \|\varphi(t, \theta_{-t}\omega, x)\| \leq \frac{1}{2}\rho(\omega) \tag{3.4.4}$$

for an ω-wise constant $\rho(\omega) > 0$, then $B(\omega) = \mathscr{B}(0, \rho(\omega))$ is a random absorbing set.

The following existence theorem originate from [Sch92, Theorem 2.1, p.187].

Theorem 3.4.8. *Let φ be a continuous random dynamical system, θ an associated MDS and $\mathcal{D}$ the collection of tempered sets. Suppose φ has a compact random absorbing set $B \in \mathcal{D}$. Then the random dynamcial system φ has a unique random attractor in $\mathcal{D}$ (also called random $\mathcal{D}$-attractor) which is given by*

$$\mathcal{A}(\omega) := \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} \varphi(t, \theta_{-t}\omega, B(\theta_{-t}\omega))}^H\,. \tag{3.4.5}$$

Remark 3.4.9. We want to point out that, if we have in addition that the set B in the above definition is *positively invariant*, i.e. for every $t \geq 0$ and $\omega \in \Omega$

$$\varphi(t, \omega, B(\omega)) \subseteq B(\theta_t\omega)$$

then the random attractor in (3.4.5) has according to [Sch97, Theorem 2.4, p.956] the following representation

$$\mathcal{A}(\omega) = \bigcap_{t\geq 0} \varphi(t, \theta_{-t}\omega, B(\theta_{-t}\omega))\,.$$

An extension of Lemma 3.3.3 gives us the attractor of a conjugated dynamical system, cf. [IS01, Theorem 2.1].

Lemma 3.4.10 ([CS15]). *Let $T : \Omega \times H \to H$ be the mapping of Lemma 3.3.3 giving us the conjugacy between the RDS φ and ψ. If the random dynamical system φ possesses the random attractor $\mathcal{A}$ and if we have in addition,*

$$\omega \mapsto T(\omega, D(\omega)) \in \mathcal{D}, \qquad \omega \mapsto T^{-1}(\omega, D(\omega)) \in \mathcal{D}$$

for every $D \in \mathcal{D}$, then the random dynamical system ψ possesses the random attractor $\widetilde{\mathcal{A}}(\omega) = T(\omega, \mathcal{A}(\omega)) \in \mathcal{D}$ for every $\omega \in \Omega$.

Let us return to the equation (3.3.3)

$$\begin{aligned} du(t) &= Au(t)dt + F(u(t) + Z(\theta_t\omega))dt, \quad t > 0, \\ u(0) &= u_0 \,. \end{aligned}$$

which was transformed from the stochastic partial differential equation (3.3.1). To begin with we show the existence of a random absorbing set for the RDS φ generated by the mild solution, cf. Theorem 3.3.2.
The Lipschitz continuity of the nonlinearity implies a linear growth. We assume in particular that there exist constants $l, d > 0$ such that,

$$\|F(u(t))\| \le l\|u(t)\| + d \tag{3.4.6}$$

for $t \in \mathbb{R}_+$ and additionally $l < \alpha \le L$. Note that the following lemmas would also hold for $L < \alpha$ but then the random attractor in (3.4.5) reduces to a singleton $\mathcal{A}(\omega) = \{a^*(\omega)\}$ due to the continuity estimate in (3.3.6). In particular the singleton is a random fixed point, i.e.

$$\varphi(t, \omega, a^*(\omega)) = a^*(\theta_t\omega)$$

for every $t \in \mathbb{R}_+$ and $\omega \in \Omega$, cf. [Ogr11, Corollary 3.21, p.53].

Lemma 3.4.11. *The RDS φ given by* (3.3.5) *has a random absorbing set $B \in \mathcal{D}$.*

Proof. Having in mind the estimate (2.1.1) with $0 < \alpha = \lambda_1$ and the linear growth condition (3.4.6), we obtain together with the notation $u(t) = u(t, \omega, u_0)$

$$\begin{aligned} \|u(t)\| &\le e^{-\alpha t}\|u_0\| + \int_0^t e^{-\alpha(t-r)}\|F(u(r) + Z(\theta_r\omega))\| \, dr \\ &\le e^{-\alpha t}\|u_0\| + \int_0^t e^{-\alpha(t-r)}\big(l\|u(r) + Z(\theta_r\omega)\| + d\big) \, dr \,. \end{aligned}$$

In Example 3.2.8 we have seen that $\|Z(\theta_r\omega)\|$, $r \in \mathbb{R}$ is tempered. We introduce the tempered random variable

$$G(\theta_r\omega) := l\|Z(\theta_r\omega)\| + d, \qquad \text{for } r \in \mathbb{R} \,.$$

Using this random variable we obtain

$$\|u(t)\| \le e^{-\alpha t}\left[\|u_0\| + \int_0^t e^{\alpha r} l\|u(r)\| \, dr + \int_0^t e^{\alpha r} G(\theta_r\omega) \, dr\right] .$$

We are now in a similar situation to the proof of Theorem 3.3.2. We multiply the above inequality by $e^{\alpha t}$ and apply the Gronwall Lemma 3.3.1 to see that,

$$e^{\alpha t}\|u(t)\| \le e^{lt}\left[\|u_0\| + \int_0^t e^{\alpha r} G(\theta_r\omega) e^{-lr} \, dr\right] .$$

Finally we observe

$$\|u(t)\| \le e^{(l-\alpha)t}\|u_0\| + \int_0^t e^{(l-\alpha)(t-r)} G(\theta_r\omega)\, dr\,.$$

We replace ω by $\theta_{-t}\omega$ and let $u_0 \in D(\theta_{-t}\omega) \in \mathcal{D}$ for fixed $t \in \mathbb{R}_+$,

$$\begin{aligned}\|u(t,\theta_{-t}\omega,u_0)\| &\le e^{(l-\alpha)t}\|u_0\| + \int_0^t e^{(l-\alpha)(t-r)} G(\theta_{r-t}\omega)\, dr \\ &= e^{(l-\alpha)t}\|u_0\| + \int_{-t}^0 e^{-(l-\alpha)r} G(\theta_r\omega)\, dr\,.\end{aligned}$$

Following the criterion in (3.4.4) we consider the limit

$$\limsup_{t\to\infty} \|u(t,\theta_{-t}\omega,u_0)\| \le \int_{-\infty}^0 e^{(\alpha-l)r} G(\theta_r\omega)\, dr\,,$$

where the last integral exists since G is a tempered random variable. Note that the estimate is uniformly over the set $D(\theta_{-t}\omega)$. There exists a random absorbing set $B(\omega) = \mathscr{B}(0,\rho(\omega))$ with radius

$$\rho(\omega) := 2\int_{-\infty}^0 e^{(\alpha-l)r} G(\theta_r\omega)\, dr\,. \tag{3.4.7}$$

We want to remark that B is indeed a tempered set. According to Definition 3.4.4 we need to show that for every $c > 0$, $\omega \in \Omega$,

$$\lim_{t\to\infty} e^{-ct} \sup_{x\in B(\theta_{-t}\omega)} \|x\| \le \lim_{t\to\infty} e^{-ct}\rho(\theta_{-t}\omega) = 0\,.$$

Since $t \mapsto G(\theta_t\omega)$ is continuous we know by (3.2.12), that for every $\varepsilon > 0$, that there exists a $C(\varepsilon,\omega)$ such that $G(\theta_t\omega) \le C(\varepsilon,\omega)e^{\varepsilon|t|}$ for every $t \in \mathbb{R}$. For every $c > 0$ there exists a $\varepsilon = \varepsilon(c,\alpha,l) > 0$ such that $\varepsilon < c$ and $\varepsilon < (\alpha - l)$. Replacing ω by $\theta_{-t}\omega$ in (3.4.7) we observe

$$\rho(\theta_{-t}\omega) = 2\int_{-\infty}^{-t} e^{(\alpha-l)(r+t)} G(\theta_r\omega)\, dr \le 2e^{(\alpha-l)t}C(\varepsilon,\omega)\int_{-\infty}^{-t} e^{(\alpha-l-\varepsilon)r}\, dr\,.$$

Next we consider the product with e^{-ct} and determine the following integral

$$e^{-ct}\rho(\theta_{-t}\omega) = 2e^{(-c+\alpha-l)t}C(\varepsilon,\omega)\int_{-\infty}^{-t} e^{(\alpha-l-\varepsilon)r}\, dr = \frac{2C(\varepsilon,\omega)}{\alpha-l-\varepsilon}e^{(-c+\varepsilon)t}\,,$$

which tends to zero if $t \to \infty$ by our assumptions. This completes the proof. □

Before we state the main theorem of this section, we prove that there exists a compact absorbing set.
To show the compactness of our absorbing set, recall the interpolations spaces we introduced in Section 2.1. According to Theorem 2.1.15 the space $D(A^\beta)$ is compactly embedded in the space H for $\beta > 0$, i.e. every bounded set in $D(A^\beta)$ is relatively compact in H. We have the following lemma.

Lemma 3.4.12. *If $B(\omega)$ is a random absorbing set for our RDS φ, then the set $K(\omega) := \overline{\varphi(1,\theta_{-1}\omega,B(\theta_{-1}\omega))}^H \subset H$ is compact and random absorbing for $\omega \in \Omega$.*

Proof. That K is indeed absorbing follows from the cocycle property and that B is already absorbing. Moreover K is a tempered set which ensues from a similar argument as in the last proof. Now concerning the compactness we observe the following.
We know by Section 4.7.1 [SY02, p.233] that $u \in D(A^\beta)$ for $0 < \beta < 1$, hence

$$\|\varphi(1,\omega,u_0)\|_\beta = \|A^\beta \varphi(1,\omega,u_0)\| \le \|A^\beta S(1)u_0\| + \int_0^1 \|A^\beta S(1-r)F(\varphi(r,\omega,u_0) + Z(\theta_r\omega))\|\, dr\,.$$

Using the estimates for an analytic semigroup in (2.1.8), we have

$$\|\varphi(1,\omega,u_0)\|_\beta \le M_\beta e^{-\alpha}\|u_0\| + \int_0^1 M_\beta \frac{e^{-\alpha(1-r)}}{(1-r)^\beta}\|F(\varphi(r,\omega,u_0) + Z(\theta_r\omega))\|\, dr\ .$$

Similar to the previous lemma we estimate the latter integral,

$$\begin{aligned}\|\varphi(1,\omega,u_0)\|_\beta &\le M_\beta e^{-\alpha}\|u_0\| + \int_0^1 M_\beta \frac{e^{-\alpha(1-r)}}{(1-r)^\beta}(l\|\varphi(r,\omega,u_0) + Z(\theta_r\omega))\| + d)\, dr\\ &\le M_\beta e^{-\alpha}\|u_0\| + \int_0^1 M_\beta l \frac{e^{-\alpha(1-r)}}{(1-r)^\beta}\|\varphi(r,\omega,u_0)\|\, dr + \int_0^1 M_\beta \frac{e^{-\alpha(1-r)}}{(1-r)^\beta} G(\theta_r\omega)\, dr\,.\end{aligned}$$

The last integral is a constant (ω-wise), since $t \mapsto G(\theta_t\omega)$ is continuous by Lemma 3.2.4 and therefore it can be estimated by its maximum $C_{\max}(\omega)$ on $[0,1]$. We define

$$C_3(\beta,\omega) := \int_0^1 M_\beta \frac{e^{-\alpha(1-r)}}{(1-r)^\beta} G(\theta_r\omega)\, dr > 0\,.$$

Moreover since $e^{-\alpha(1-r)} \le 1$ for every $r \in [0,1]$,

$$\|\varphi(1,\omega,u_0)\|_\beta \le M_\beta\|u_0\| + M_\beta l \int_0^1 \frac{1}{(1-r)^\beta}\|\varphi(r,\omega,u_0)\|\, dr + C_3(\beta,\omega)\,.$$

Now replace ω by $\theta_{-1}\omega$ and choose $u_0 \in B(\theta_{-1}\omega)$ from an absorbing set. Then according to the previous lemma for $r \in [0,1]$,

$$\begin{aligned}\|\varphi(r,\theta_{-1}\omega,u_0)\| &\le e^{(l-\alpha)r}\|u_0\| + e^{(l-\alpha)r}\int_{-1}^{r-1} e^{-(l-\alpha)(s+1)} G(\theta_s\omega)\, ds\\ &\le \|u_0\| + e^{\alpha-l}\int_{-1}^{0} e^{(\alpha-l)s} G(\theta_s\omega)\, ds \le \rho(\theta_{-1}\omega) + \frac{e^{\alpha-l}}{\alpha-l} C_{\max}(\omega)\end{aligned}$$

and we obtain $\varphi(r,\theta_{-1}\omega,u_0) \in \mathscr{B}(0,R(\omega))$ with $R(\omega) := \rho(\theta_{-1}\omega) + \frac{e^{\alpha-l}}{\alpha-l}C_{\max}(\omega)$ for $r \in [0,1]$ and

$$\|\varphi(1,\theta_{-1}\omega,u_0)\|_\beta \le M_\beta \rho(\theta_{-1}\omega) + M_\beta l \int_0^1 \frac{1}{(1-r)^\beta} R(\omega)\, dr + C_3(\beta,\omega)\ < \infty\,.$$

We conclude by the compact embedding $D(A^\beta) \hookrightarrow H$, that the set $\varphi(1,\theta_{-1}\omega, B(\theta_{-1}\omega))$ is relatively compact and hence

$$K(\omega) = \overline{\varphi(1,\theta_{-1}\omega,B(\theta_{-1}\omega))}^H$$

is compact. □

We formulate the main result concerning the existence of the random attractor for our RDS.

Theorem 3.4.13. *The random dynamical system φ generated by the solution of the RDE* (3.3.3) *possesses a unique random $\mathcal{D}$-attractor $\mathcal{A}(\omega)$, $\omega \in \Omega$. Moreover, the conjugated RDS ψ of the original SPDE possesses a unique random attractor $\widetilde{\mathcal{A}}(\omega)$, $\omega \in \Omega$.*

Proof. The first statement follows by the Lemmas 3.4.11 and 3.4.12, such that the assumptions of Theorem 3.4.8 are fulfilled. The random attractor for the RDS ψ given by the SPDE (3.3.1) follows from Lemma 3.4.10 with the conjugacy $T(\omega, x) := x + Z(\omega)$ for $\omega \in \Omega$,

$$\widetilde{\mathcal{A}}(\omega) = T(\omega, \mathcal{A}(\omega)) = \mathcal{A}(\omega) + Z(\omega) = \{x + Z(\omega) : x \in \mathcal{A}(\omega)\}.$$

□

We close this chapter with a remark concerning the application of the introduced theory of random attractors to differential equations on fractal sets which are introduced in Section 2.4. Throughout the Sections 3.3 and 3.4 we did not impose any strong assumptions on the set $K \subset \mathbb{R}^n$ of the corresponding space $L^2(K, \mu)$. Since we constructed in Section 2.4 a proper Laplacian Δ_μ on a fractal set K with an associated (self-similar) measure μ, we obtain in particular that Δ_μ is the generator of an analytic semigroup, see Theorem 2.1.13. Hence we are able to consider a similar equation as (3.3.1) with $A = \Delta_\mu$ and a sufficiently smooth nonlinearity. The mild solution of this SPDE gives us according to the last two sections a random attractor for the dynamics that emerge in the Hilbert space.

Chapter 4

An upper bound on the Hausdorff dimension of the random attractor

In this chapter we present one of our main results. We give an estimate for the Hausdorff dimension of the random attractor $\mathcal{A}(\omega), \omega \in \Omega$ we constructed in the previous chapter. According to the further developed theory of B. Schmalfuß in [Sch97] and A. Debussche in [Deb98] for *random* attractors we have the general result concerning the Hausdorff dimension

$$\dim_{\mathrm{H}}(\mathcal{A}(\omega)) \leq m, \qquad \omega \in \Omega$$

where m is the smallest natural number, that fulfills $m^{2/n} > \frac{L}{C_\Delta}$. Here $n \geq 1$ is the dimension of the underlying set $K \subset \mathbb{R}^n$ of the corresponding Hilbert space $L^2(K, \lambda)$, where λ denotes the Lebesgue measure on K. Moreover L is the Lipschitz constant of the appearing nonlinearity and the constant C_Δ is related to the spectral asymptotics of the Laplacian on K.
In turns out in the classical results that the dimension n appears also in the asymptotics of the Laplacian defined on $L^2(K, \lambda)$. In the interesting case of a fractal domain $K \subset \mathbb{R}^n$, an associated self-similar measure μ and the corresponding Laplacian Δ_μ, recall in particular Section 2.4, we obtain a dependence on the spectral exponent d_S. The Hausdorff dimension of the random attractor is then less than or equal to the smallest $m \in \mathbb{N}$ satisfying

$$m^{2/d_S} > \frac{L}{C_{\Delta_\mu}}$$

where C_{Δ_μ} is a constant depending on the asymptotics of the Laplacian Δ_μ. Recall that the spectral exponent does not coincide with the Hausdorff dimension $\dim_{\mathrm{H}}(K)$ in general.
The described statement presents a generalization of the established results, since there are so far no results concerning more arbitrary (e.g. fractal) domains of the Hilbert space $L^2(K, \mu)$.

In the first section we summarize the classical concepts and ideas of the deterministic theory to arrive at the upper bound of the Hausdorff dimension. In the next section we state the main ideas of a similar approach in the probabilistic case. Moreover we apply these ideas to the RDE we considered before and show the main result of this chapter. In a last section we present a mathematical example for the nonlinearity of the related random differential equation (3.3.3). We show that for this class of operators the arising dynamics possess a random attractor and we discuss the value of the upper bound of the associated Hausdorff dimension.

4.1 Technical preparations for the estimate

In this section we follow the introduction in [Tem88, Chapter V, VI]. Since we will use several times Fréchet derivatives we recall its definition.

Definition 4.1.1 (Definition 12.1.2, p.267, [Wou79])**.** Let X, Y be two Banach spaces and $T : X \to Y$. If there exists an operator $L(x) \in L(X, Y)$ such that for some $x \in D(T)$

$$\|T(x+h) - T(x) - L(x)h\| = \mathrm{o}(\|h\|) \quad \forall h \in X$$

then $L(x)h$ is called *Fréchet differential* of $T(x)$, denoted often by $DT(x)[h]$ or $T'(x)[h]$. The operator $L(x) = T'(x)$ is called the *Fréchet derivative* of T at $x \in D(T)$. The expression $\mathrm{o}(\|h\|)$ means clearly that $\|T(x+h) - T(x) - L(x)h\|$ goes faster to zero than $\|h\|$ when h approaches zero.

The approach to estimate the Hausdorff dimension requires the concept of an exterior product of Hilbert spaces.

Definition 4.1.2 (Chapter V, Section 1,[Tem88])**.** Let H be a Hilbert space. The *m-exterior product* of H is given by

$$\bigwedge^m H := \underbrace{H \wedge \cdots \wedge H}_{m \text{ times}} = \operatorname{span}\left\{ \sum_{\sigma \in \{1,\ldots,m\}^m} (-1)^\sigma x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(m)} \colon x_1, \ldots, x_m \in H \right\},$$

where $\sigma = (\sigma(1), \ldots, \sigma(m))$ is a permutation of $\{1, \ldots, m\}$, $(-1)^\sigma = \pm 1$ denotes the sign of the permutation and the tensor product for m elements $x_i \in H$, $i \in \{1, \ldots, m\}$ is given by

$$(x_1 \otimes \cdots \otimes x_m)(y_1, \ldots, y_m) := \prod_{i=1}^m (x_i, y_i) \quad \forall y_i \in H\,.$$

For $x_i \in H$, $i \in \{1, \ldots, m\}$ we call

$$x_1 \wedge \cdots \wedge x_m := \sum_{\sigma \in \{1,\ldots,m\}^m} (-1)^\sigma x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(m)}$$

the *exterior product* of $x_1, \ldots, x_m$. Moreover we obtain a positive definite inner product by defining

$$(x_1 \wedge \cdots \wedge x_m, y_1 \wedge \cdots \wedge y_m)_{\bigwedge^m H} := \det[(x_i, y_i)]_{1 \le i,j \le m}$$

for all $x_i, y_i \in H$, $i \in \{1, \ldots, m\}$ and define the associated norm $\|\cdot\|_{\bigwedge^m H} := \big((\cdot,\cdot)_{\bigwedge^m H}\big)^{1/2}$.

We like to stress the idea for the estimate of the Hausdorff dimension for the attractor. A detailed proof is presented for the deterministic case in [Tem88, Chapter V, Section 3.3, p.368] and for the probabilistic case in [Deb98, Theorem 2.4, p.972] and [Sch97, Theorem 3.2, p.959].

The main question that arises is: How can we estimate the (d-dimensional) Hausdorff measure $\mathcal{H}^d(\mathcal{A})$ for an appropriate number d of the attractor $\mathcal{A}$?

Since the attractor is a compact set we can cover it with a finite amount of balls in H. In particular for every $\varepsilon > 0$ each ball has a radius smaller than ε. The next step is to use the invariance property of the attractor, which infers that we need to know how a ball in a Hilbert space behaves when it gets distorted by the (random) dynamical system. In fact we need also results on the Hausdorff measure for these distorted balls. In the following we will provide some ideas to these problems.

We denote by $T^* : H \to H$ the adjoint of an operator T on H. In preparation for the next proposition we want to recall some of the results in Section 1.3.2 [Tem88, Chapter V]. The linear self-adjoint operator L^*L of a linear bounded (not necessary compact) operator L in H possesses a sequence of non-increasing numbers

$$\nu_n(L^*L) = \inf_{\substack{G\subset H \\ \dim G\leq n-1}} \sup_{\substack{x\in G^\perp \\ \|x\|=1}} ((L^*L)x,x), \quad n\geq 1\,.$$

If L^*L is compact, then ν_n would be the eigenvalues of L^*L. Suppose L^*L is not compact and set $\nu_\infty(L^*L) = \inf_{n\in\mathbb{N}} \nu_n(L^*L)$, then two possiblities arise. First suppose there exists a $n_0 \in \mathbb{N}$ such that

$$\nu_1(L^*L) \geq ... \geq \nu_{n_0}(L^*L) > \nu_{n_0+1}(L^*L) = \nu_m(L^*L) = \nu_\infty(L^*L), \quad \forall m \geq n_0+1\,.$$

The other possiblity is $\nu_m(L^*L) > \nu_\infty(L^*L)$ for every $m \in \mathbb{N}$. In the first case $\nu_1,...,\nu_{n_0}$ are eigenvalues of L^*L and in the second case each ν_m, $m\in\mathbb{N}$ is an eigenvalue of L^*L, see proposition 1.1 [Tem88, p.348 ff.].
Now it is useful to introduce the index set I which is either equal to $\{1,...,n_0\}$ in the first case or equal to $\mathbb{N}$ for the other case. In both cases we can decompose H into H_ν and $H_\nu^\perp$, where $H_\nu = \operatorname{span}\{e_i, i\in I : L^*Le_i = \nu_i e_i\}$. By proposition 1.2 [Tem88, p.350] we have for every $n\in\mathbb{N}$

$$\nu_n(L^*L) = \sup_{\substack{G\subset H \\ \dim G\leq n}} \inf_{\substack{x\in G \\ \|x\|=1}} ((L^*L)x,x)\,.$$

After these short explanations and notations we have the following result concerning the image of a ball in H.

Proposition 4.1.3 (Chapter V, Proposition 1.3, [Tem88])**.** *Let H be a Hilbert space and $\mathscr{B}$ the unit ball therein. Let L be a linear bounded operator in H and suppose we have the decomposition, $H = H_\nu \oplus H_\nu^\perp$ described above. The image of the ball $\mathscr{B}$ under the mapping L, i.e. $L(\mathscr{B})$ is a subset of an ellipsoid $\mathscr{E}$ in H. Depending on the properties of L we distinguish between two cases:*

(i) Either L is compact or L being not compact but $H_\nu = H$, then the axes of $\mathscr{E}$ are directed along Le_i, $i\in\mathbb{N}$ and their length is given by the numbers

$$\alpha_i(L) = \nu_i\left((L^*L)^{\frac{1}{2}}\right) = \sup_{\substack{G\subset H \\ \dim G\leq i}} \inf_{\substack{x\in G \\ \|x\|=1}} \|Lx\|, \qquad i\in\mathbb{N}$$

where e_i, $i\in\mathbb{N}$ are the eigenvectors of L^*L.

(ii) If L is not compact and $H_\nu \neq H$, then $\mathscr{E}$ is the product of a ball around zero with radius

$$\alpha_\infty(L) = \nu_\infty\left((L^*L)^{\frac{1}{2}}\right) = \inf_{i\in\mathbb{N}} \alpha_i(L)$$

in the space $H_\nu^\perp$ and the ellipsoid in the space H_ν with axes along to Le_i and lengths $\alpha_i(L)$, $i\in I$. Similar as in (i) the eigenvectors of L^*L are denoted by $e_i, i\in I$. Note that in this case I can be finite or infinite.

We remark that $\alpha_1(L) = \|L\|_{L(H)}$ and we define additionally the product of the first m axes by

$$P_m(L) = \alpha_1(L)\ldots\alpha_m(L)\,.$$

According to proposition 1.4 in [Tem88, p.353] we have the following connection between the axes of an ellipsoid, which is generated by a linear mapping L and the m-exterior product.

Lemma 4.1.4. *If $L \in L(H)$, then for every $m \in \mathbb{N}$*

$$P_m(L) = \sup_{\substack{x_1,\dots,x_m \in H \\ \|x_i\| \le 1, \forall i}} \|Lx_1 \wedge \cdots \wedge Lx_m\|_{\bigwedge^m H} .$$

Until now we considered general techniques and results how a volume behaves under a linear bounded mapping. But the solution operator of an associated evolution equation is general not linear. To close this gap we use the Frechet derivative of the solution. We have the following discussion.

Suppose we are given a similar situation as in the beginning of Section 3.3. In particular recall the assumptions on the linear operator A and the nonlinearity F. We consider now a similar equation to (3.3.1) but without the noise. We have the following (deterministic) initial value problem

$$\begin{aligned} \frac{du(t)}{dt} &= G(u(t)), \qquad t > 0, \\ u(0) &= u_0 , \end{aligned} \tag{4.1.1}$$

for every $u_0 \in H$ and $G := A + F : H \to H$. By Theorem 3.3.2 it is possible to state the unique mild solution of this equation. Moreover we know that the solution operator generates a dynamical system $\varphi : \mathbb{R}_+ \times H \to H$ with $u(t) = \varphi(t, u_0)$. We want to remark that in [Tem88, Chapter V, Section 2.3, p.362] the author considers the concept of weak solutions for equations of the type (4.1.1). In our case the two concepts of solutions coincide according to the result in [Bal77].
We assume that G is Fréchet differentiable with derivative DG and we arrive at the following linearized equation,

$$\begin{aligned} \frac{dU(t)}{dt} &= DG(u(t)) \cdot U(t), \qquad t > 0, \\ U(0) &= \xi , \end{aligned} \tag{4.1.2}$$

which is well-posed for every $u_0, \xi \in H$. In addition we suppose that the dynamical system $\varphi(t, u_0)$ is Fréchet differentiable with derivative $L(t, u_0)$ given by

$$U(t) := D\varphi(t, u_0)\xi = L(t, u_0)[\xi]$$

for every $\xi \in H$, where U is the solution of (4.1.2).
If we consider m solutions $U_1, ..., U_m$ of the equation (4.1.2) for m initial elements $\xi_1, ..., \xi_m \in H$ we obtain by further calculations in the norm $\bigwedge^m H$ ([Tem88, p.362f.]) for fixed $t \in \mathbb{R}_+$ and $u_0 \in H$,

$$\|U_1(t) \wedge \cdots \wedge U_m(t)\|_{\bigwedge^m H} = \|\xi_1 \wedge \cdots \wedge \xi_m\|_{\bigwedge^m H} \cdot \exp\left\{ \int_0^t \mathrm{Tr}\left(DG(\varphi(r, u_0)) \circ Q_m(r)\right) dr \right\} , \tag{4.1.3}$$

where $\mathrm{Tr}\left(DG(\varphi(t, u_0)) \circ Q_m(t)\right)$ describes the trace of the linear operator obtained by applying the derivative $DG(\varphi(t, u_0))$ on the orthogonal projector $Q_m(t) = Q_m(t, u_0; \xi_1, ..., \xi_m)$. This projector maps H onto the space spanned by $(U_1(t), ..., U_m(t))$, cf. [Tem88, Chapter V, Lemma 1.2, p.344]. Combining Lemma 4.1.4 (applied on the Fréchet derivatives $U_i(t)$) with the equation (4.1.3) we have

$$\begin{aligned} P_m(L(t, u_0)) &= \sup_{\substack{\xi_1,\dots,\xi_m \in H \\ \|\xi_i\| \le 1, \forall i}} \|U_1(t) \wedge \cdots \wedge U_m(t)\|_{\bigwedge^m H} \\ &= \sup_{\substack{\xi_1,\dots,\xi_m \in H \\ \|\xi_i\| \le 1, \forall i}} \exp\left\{ \int_0^t \mathrm{Tr}\left(DG(\varphi(r, u_0)) \circ Q_m(r)\right) dr \right\} , \end{aligned}$$

where we also applied proposition 1.4 [Tem88, Chapter V, p.353].
For the following let X be an invariant set in H w.r.t. to the dynamical system φ in the sense of definition 1.1 in [Tem88, Chapter 1, p.19], i.e.

$$\varphi(t, X) = X \qquad \forall t \geq 0.$$

Further define the numbers

$$\begin{aligned}\overline{P_m}(t) &:= \sup_{u_0 \in X} P_m(L(t, u_0)) \\ q_m(t) &:= \sup_{u_0 \in X} \sup_{\substack{\xi_1, \ldots, \xi_m \in H \\ \|\xi_i\| \leq 1, \forall i}} \left(\frac{1}{t} \int_0^t \operatorname{Tr}\big(DG(\varphi(r, u_0)) \circ Q_m(r)\big)\, dr \right)\end{aligned}$$

for every $t \in \mathbb{R}_+$ and $m \in \mathbb{N}$. In fact it follows,

$$\frac{1}{t} \log \overline{P_m}(t) \leq q_m(t)\,.$$

Moreover we introduce for every $m \geq 2$ the *global Lyapunov exponents* on X, cf. [Tem88, p.361],

$$\nu_m = \log \left\{ \lim_{t \to \infty} \left(\frac{\overline{P_m}(t)}{\overline{P_{m-1}}(t)} \right)^{1/t} \right\}$$

and for $m = 1$ the Lyapunov exponent becomes $\nu_1 = \lim_{t\to\infty}(\overline{P_1}(t))^{1/t}$.
We can now formulate the conditions that are necessary to obtain the estimate of the Hausdorff dimension for the (deterministic) attractor, see [Tem88, Chapter V, Section 3.4, p.377].

DC (I) For any $t \in \mathbb{R}_+$ there exists a constant $C_t > 0$, such that the following *uniformly differentiability* is fulfilled. For any $\varepsilon > 0$ we have

$$\sup_{\substack{u,v \in X \\ \|u-v\| \leq \varepsilon}} \frac{\|\varphi(t,u) - \varphi(t,v) - L(t,v)[u-v]\|}{\|u - v\|} \leq C_t\, \varepsilon\,.$$

DC (II) For some $t_0 > 0$ it holds

$$\sup_{u \in X} \|L(t_0, u)\|_{L(H)} < \infty\,.$$

DC (III) For some $t_0 > 0$ and $m \geq 1$

$$\overline{P_m}(t_0) = \sup_{u \in X} P_m(L(t_0, u)) < 1\,.$$

Remark 4.1.5.

- It suffices to show that the number $q_m(t_0) < 0$ for some $t_0 > 0$ since then condition DC (III), $\overline{P_m}(t_0) < 1$, is fulfilled.
- Proposition 2.1 [Tem88, Chapter V, p.364] tells us that if for some $m \in \mathbb{N}$ and t sufficiently large $q_m(t) < 0$, then the m-volume of the parallelepiped spanned by $(U_1(t), ..., U_m(t))$, that is

  $$\|U_1'(t) \wedge \cdots \wedge U_m'(t)\|_{\bigwedge^m H},$$

  decays exponentially. That means that the m-dimensional volumes decrease, which gives us information on the Hausdorff dimension, since it is defined by m-dimensional sets. Moreover we have $\log \lim_{t\to\infty}(\overline{P_m}(t))^{1/t} = \nu_1 + ... + \nu_m < 0$.

- The numbers

$$\lim_{t\to\infty}\left(\frac{\overline{P_m}(t)}{\overline{P_{m-1}}(t)}\right)^{1/t}$$

are called *global Lyapunov numbers* which can be defined iteratively starting with $\lim\limits_{t\to\infty}\left(\overline{P_1}(t)\right)^{1/t}$ and they exist on an invariant set X according to [Tem88, p.359].

- It follows readily from the equality

$$\varphi(t)=\varphi(t-\lfloor t\rfloor)\circ\underbrace{\varphi(1)\circ\cdots\circ\varphi(1)}_{\lfloor t\rfloor\text{ - times}},\quad t\geq 0$$

that we can replace the condition DC (I) by

$$\sup_{t\in[0,1]}\sup_{\substack{u,v\in X\\ \|u-v\|\leq\varepsilon}}\frac{\|\varphi(t,u)-\varphi(t,v)-L(t,v)[u-v]\|}{\|u-v\|}\leq C\varepsilon$$

for some constant $C>0$ and $\varepsilon>0$. Instead of the condition DC (II) we can assume the slightly stronger condition, that

$$\sup_{t\in[0,1]}\sup_{u\in X}\|L(t,u)\|_{L(H)}<\infty\,.$$

We state now the theorem of the Hausdorff dimension for an invariant set $X\subset H$ of a dynamical system φ.

Theorem 4.1.6. *Under the deterministic conditions DC (I)–DC (III) we infer*

$$\dim_{\mathrm{H}}(X)\leq m$$

for the smallest natural number m that fulfills DC (III).

Remark 4.1.7.

- There are several slightly more general results. For instance in Theorem 3.1 in [Tem88, p.369] the Hausdorff dimension is bounded by some real number $d>0$, where this number has to fulfill DC (III) instead of $m\in\mathbb{N}$. However in practical applications d often equals a natural number, cf. [Tem88, p.378].

- We have also a little more precise result using the Lyapunov exponents. If the conditions DC (I), DC (II) hold and for some $N\in\mathbb{N}$

$$\nu_1+\ldots+\nu_{N+1}<0\,, \tag{4.1.4}$$

then the upper bound of the Hausdorff dimension is given by

$$N+\frac{(\nu_1+\ldots+\nu_N)\wedge 0}{|\nu_{N+1}|}$$

see [Tem88, Chapter V, Theorem 3.3, p.374]. Of course we are especially interested in the smallest natural number that fulfills (4.1.4).

- The idea of the proof of Theorem 4.1.6 is based on iterated coverings including two covering results. The first result gives an estimate how many balls are necessary to cover an ellipsoid, introduced in Proposition 4.1.3. The second result states that the sum of an ellipsoid and a ball (like it was appearing in Proposition 4.1.3 (ii)) is covered by a larger ellipsoid with specified properties on the axes of this ellipsoid.

4.2 The upper estimate for the random attractor

Due to the ω-dependence we have to adjust the deterministic assumptions and a general theorem on the dimension of a random attractor has also to treat the evolution of the noise in time by the MDS $\theta = (\theta_t)_{t\in\mathbb{R}}$.

Suppose we are given an RDS φ in H with an appropriate MDS $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ and assume φ possesses a random attractor $\mathcal{A}(\omega)$, $\omega \in \Omega$ like introduced in Chapter 3.

First note that the results of Proposition 4.1.3 hold identically for the random case, i.e. in an ω-wise analogue. Similar to the Proposition 4.1.3 we denote by $\alpha_i(L(t,\omega))$, $i \in I$ ($I = \mathbb{N}$ or a finite set) the length of the axes of a corresponding ellipsoid and it holds $\alpha_1(L(t,\omega)) = \|L(t,\omega)\|_{L(H)}$ for every $(t,\omega) \in \mathbb{R}_+ \times \Omega$. In particular we introduce

$$P_m(L(t,\omega)) := \alpha_1(L(t,\omega)) \cdots \alpha_m(L(t,\omega))$$

for every $(t,\omega) \in \mathbb{R}_+ \times \Omega$.

The following conditions are presented for discrete time in [Deb98] and they are the *random* analogue to the deterministic conditions DC (I)–DC (III).

RC (I) There exists a random variable $C = (C(\omega))_{\omega\in\Omega} \geq 1$, such that φ is *uniformly differentiable* on $\mathcal{A}(\omega)$ with Fréchet derivative $L(t,\omega,\cdot)$ for $t \in \mathbb{R}_+$ and $\omega \in \Omega$, i.e.

$$\sup_{t\in[0,1]} \sup_{\substack{u,v\in\mathcal{A}(\omega)\\ \|u-v\|\leq\varepsilon}} \frac{\|\varphi(t,\omega,u) - \varphi(t,\omega,v) - L(t,\omega,v)[u-v]\|}{\|u-v\|} \leq C(\omega)\varepsilon\,.$$

and additionally it holds

$$\mathbb{E}(\ln C) < \infty\,.$$

RC (II) There exists a random variable $\widetilde{\alpha}_1 = (\widetilde{\alpha}_1(\omega))_{\omega\in\Omega} \geq 1$ such that,

$$\sup_{t\in[0,1]} \sup_{u\in\mathcal{A}(\omega)} \alpha_1(L(t,\omega,u)) \leq \widetilde{\alpha}_1(\omega) \quad \text{and} \quad \mathbb{E}(\ln \widetilde{\alpha}_1) < \infty\,.$$

RC (III) For every $t \in \mathbb{R}_+$ there exists a random variable $\widetilde{P}_m(t,\cdot)$ and for some $m \in \mathbb{N}$ there exists a time $t_0(\omega) > 0$ such that,

$$\sup_{u\in\mathcal{A}(\omega)} P_m(L(t,\omega,u)) \leq \widetilde{P}_m(t,\omega) \quad \text{and} \quad \mathbb{E}(\ln \widetilde{P}_m(t,\cdot)) < 0\,,$$

for $t \geq t_0(\omega)$ and every $\omega \in \Omega$.

Now we are able to formulate the generalization of Theorem 4.1.6 to the random case.

Theorem 4.2.1. *Let $\varphi : \mathbb{R}_+ \times \Omega \times H \to H$ be a random dynamical system on a Hilbert space H and let $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ be the associated MDS. Moreover assume that the RDS possesses a unique random attractor $(\mathcal{A}(\omega))_{\omega\in\Omega}$ and the random conditions RC (I)–RC (III) are satisfied. Then we conclude for all $\omega \in \Omega$,*

$$\dim_{\mathrm{H}}(\mathcal{A}(\omega)) \leq m,$$

where $m \in \mathbb{N}$ is the smallest number satisfying RC (III).

For the proof of this theorem we refer to [Deb98, Theorem 2.4, p.972] and for a continuous version see [CF98, Theorem 4.3, p.457].

The next step is to apply the above theorem to the RDS generated by the mild solution of the RDE (3.3.3),

$$\begin{aligned} du(t) &= Au(t)dt + F(u(t) + Z(\theta_t\omega))dt, \quad t > 0 \\ u(0) &= u_0 \in H \,, \end{aligned}$$

and its random attractor $\mathcal{A}(\omega)$, $\omega \in \Omega$ discussed in Theorem 3.4.13. From here on we assume, unless stated otherwise, that we are on the Hilbert space $H = L^2(K, \mu)$ for a fractal set K and a self-similar measure μ introduced in (2.4.4). In particular we consider the (Dirichlet) Laplacian presented in Theorem 2.4.14,

$$A = \Delta_\mu : D(\Delta_\mu) \to H \,.$$

In remembering the Proposition 2.4.15, we know that A is the generator of an analytic semigroup denoted by $\{S(t)\}_{t\geq 0}$. Recall the mild solution of (3.3.3) which is given by (3.3.4),

$$u(t) = u_{\omega,u_0}(t) = S(t)u_0 + \int_0^t S(t-r)F(u(r) + Z(\theta_r\omega))\, dr \,,$$

for $t \in \mathbb{R}_+$, $\omega \in \Omega$ and $u_0 \in H$. Before we verify the conditions RC (I)–RC (III), we show the Fréchet differentiability of the RDS φ generated by u.

Theorem 4.2.2. *Let $u : \mathbb{R}_+ \times \Omega \times H \to H$ be the mild solution of the RDE* (3.3.4) *with a Lipschitz-continuous nonlinearity $F : H \to H$ with constant $L > 0$. Let $\mathcal{A}(\omega)$, $\omega \in \Omega$ be the random attractor of the corresponding RDS φ that is generated by u. We assume that F is twice Fréchet differentiable with a uniform bound $C_F > 0$, such that*

$$\sup_{u\in H} \|D^2F(u)\|_{BL(H\times H,H)} := \sup_{u\in H} \sup_{\substack{h_1,h_2\in H \\ \|h_1\|\leq 1, \|h_2\|\leq 1}} \|D^2F(u)[h_1,h_2]\| \leq C_F,$$

where $D^2F(u)$ is the second Fréchet derivative at u and $BL(H \times H, H)$ is the space of bounded bilinear forms, see [Wou79, Section 12.2].
Then the RDS $\varphi(t, \omega, u_0)$ is Fréchet differentiable at $u_0 \in \mathcal{A}(\omega)$ with derivative

$$\begin{aligned} &D\varphi(t,\omega,u_0) : \mathcal{A}(\omega) \to H, \\ &\xi \mapsto D\varphi(t,\omega,u_0)\xi =: U(t) \end{aligned}$$

given by

$$U(t) = S(t)\xi + \int_0^t S(t-r)DF(\varphi(r,\omega,u_0) + Z(\theta_r\omega))U(r)\, dr$$

for every $t \in [0, 1]$, $\omega \in \Omega$ and $\xi \in \mathcal{A}(\omega)$.

Proof. According to Definition 4.1.1 we need to show for $t \in [0, 1]$, $\omega \in \Omega$ and ξ close to zero,

$$\|\varphi(t,\omega,u_0+\xi) - \varphi(t,\omega,u_0) - U(t)\| = \mathrm{o}(\|\xi\|)\,.$$

For a better readability we introduce the following abbreviations for this proof,

$$\begin{aligned} \varphi(r, u_0+\xi) + Z &:= \varphi(r,\omega,u_0+\xi) + Z(\theta_r\omega) \\ \varphi(r,u_0) + Z &:= \varphi(r,\omega,u_0) + Z(\theta_r\omega)\,. \end{aligned}$$

Following Theorem 2.1.13 we have the constant $\alpha > 0$ giving us the exponential stability of the semigroup $\{S(t)\}_{t\geq 0}$. Then we obtain for the latter difference

$$\begin{aligned}\|\varphi(t,\omega,u_0+\xi)-\varphi(t,\omega,u_0)-U(t)\| \leq & \int_0^t \|S(t-r)\left\{F(\varphi(r,u_0+\xi)+Z)-F(\varphi(r,u_0)+Z)\right.\\ & \left.-DF(\varphi(r,u_0)+Z)U(r)\right\}\| \, dr\\ \leq & \int_0^t e^{-\alpha(t-r)} \|F(\varphi(r,u_0+\xi)+Z)-F(\varphi(r,u_0)+Z)\\ & -DF(\varphi(r,u_0)+Z)U(r)\| \, dr\,.\end{aligned} \tag{4.2.1}$$

Inside the last norm we insert an artificial 'zero' by

$$\pm DF(\varphi(r,u_0)+Z)[\varphi(r,u_0+\xi)-\varphi(r,u_0)]\,. \tag{4.2.2}$$

We obtain two expressions in norm that we want to estimate. Using the notation $\mathscr{D}\varphi(u_0,\xi) := \varphi(\cdot,u_0+\xi)-\varphi(\cdot,u_0)$ the first expression becomes

$$\|F(\varphi(\cdot,u_0+\xi)+Z)-F(\varphi(\cdot,u_0)+Z)-DF(\varphi(\cdot,u_0)+Z)[\mathscr{D}\varphi(u_0,\xi)]\|\,,$$

which can be estimated using Taylor's formula for Frechet derivatives, see [Wou79, p.276]. Since F is twice Fréchet differentiable, we observe

$$\begin{aligned}&\|F(\varphi(\cdot,u_0+\xi)+Z)-F(\varphi(\cdot,u_0)+Z)-DF(\varphi(\cdot,u_0)+Z)[\mathscr{D}\varphi(u_0,\xi)]\|\\ =&\left\|D^2F(\varphi(\cdot,u_0+Z))[\mathscr{D}\varphi(u_0,\xi),\mathscr{D}\varphi(u_0,\xi)]+\mathrm{o}(\|\mathscr{D}\varphi(u_0,\xi)\|^2)\right\|\\ \leq&\|D^2F(\varphi(\cdot,u_0)+Z)\|_{BL(H\times H,H)}\|\mathscr{D}\varphi(u_0,\xi)\|^2+\mathrm{o}(\|\mathscr{D}\varphi(u_0,\xi)\|^2)\,.\end{aligned}$$

Using now our assumption on D^2F we obtain,

$$\begin{aligned}&\|F(\varphi(\cdot,u_0+\xi)+Z)-F(\varphi(\cdot,u_0)+Z)-DF(\varphi(\cdot,u_0)+Z)[\mathscr{D}\varphi(u_0,\xi)]\|\\ \leq& C_F\|\mathscr{D}\varphi(u_0,\xi)\|^2+\mathrm{o}(\|\mathscr{D}\varphi(u_0,\xi)\|^2)\,.\end{aligned}$$

For simplicity we assume now that ξ is close to zero such that

$$\mathrm{o}(\|\mathscr{D}\varphi(u_0,\xi)\|^2)=\mathrm{o}\big(\|\varphi(\cdot,u_0+\xi)-\varphi(\cdot,u_0)\|^2\big)\leq\|\varphi(\cdot,u_0+\xi)-\varphi(\cdot,u_0)\|^2=\|\mathscr{D}\varphi(u_0,\xi)\|^2$$

and if we set $C_F' := C_F+1$ we have

$$\begin{aligned}&\|F(\varphi(\cdot,u_0+\xi)+Z)-F(\varphi(\cdot,u_0)+Z)-DF(\varphi(\cdot,u_0)+Z)[\mathscr{D}\varphi(u_0,\xi)]\|\\ \leq& C_F'\|\varphi(\cdot,u_0+\xi)-\varphi(\cdot,u_0)\|^2\,.\end{aligned}$$

Similar to the proof of the continuity of the RDS φ in Theorem 3.3.2 we obtain

$$\|\varphi(r,\omega,u_0+\xi)-\varphi(r,\omega,u_0)\|\leq\|\xi\|e^{(L-\alpha)r}\,,$$

hence

$$\begin{aligned}&\|F(\varphi(r,u_0+\xi)+Z)-F(\varphi(r,u_0)+Z)-DF(\varphi(r,u_0)+Z)[\varphi(r,u_0+\xi)-\varphi(r,u_0)]\|\\ \leq& C_F'\|\xi\|^2e^{2(L-\alpha)r}\,.\end{aligned} \tag{4.2.3}$$

If we combine the second expression of the norm in (4.2.1) combined with (4.2.2) we see

$$\begin{aligned}&\|DF(\varphi(r,u_0)+Z)[\varphi(r,u_0+\xi)-\varphi(r,u_0)]-DF(\varphi(r,u_0)+Z)U(r)\|\\ \leq&\|DF(\varphi(r,u_0)+Z)\|_{L(H)}\|\varphi(r,u_0+\xi)-\varphi(r,u_0)-U(r)\|\,.\end{aligned}$$

Since F is Lipschitz continuous we infer that $\|DF(\varphi(r,u_0)+Z)\|_{L(H)} \leq L$ and therefore

$$\begin{aligned}&\|DF(\varphi(r,u_0)+Z)[\varphi(r,u_0+\xi)-\varphi(r,u_0)] - DF(\varphi(r,u_0)+Z)U(r)\| \\ &\leq L\|\varphi(r,u_0+\xi)-\varphi(r,u_0)-U(r)\| .\end{aligned} \tag{4.2.4}$$

Finally we conclude by (4.2.3) and (4.2.4)

$$\begin{aligned}&\|\varphi(t,\omega,u_0+\xi)-\varphi(t,\omega,u_0)-U(t)\| \\ &\leq C'_F\|\xi\|^2\int_0^t e^{-\alpha(t-r)+2(L-\alpha)r}\,dr \\ &\quad+\int_0^t Le^{-\alpha(t-r)}\|\varphi(r,\omega,u_0+\xi)-\varphi(r,\omega,u_0)-U(r)\|\,dr .\end{aligned} \tag{4.2.5}$$

For the moment suppose $L \neq \alpha$. Now we multiply the latter by $e^{\alpha t}$ to arrive at the following inequality,

$$\begin{aligned}&e^{\alpha t}\|\varphi(t,\omega,u_0+\xi)-\varphi(t,\omega,u_0)-U(t)\| \\ &\leq \frac{C'_F}{2L-\alpha}\|\xi\|^2\left(e^{(2L-\alpha)t}-1\right)+\int_0^t Le^{\alpha r}\|\varphi(r,\omega,u_0+\xi)-\varphi(r,\omega,u_0)-U(r)\|\,dr .\end{aligned}$$

Applying the Gronwall Lemma 3.3.1 we obtain

$$\|\varphi(t,\omega,u_0+h)-\varphi(t,\omega,u_0)-U(t)\| \leq \frac{C'_F\|\xi\|^2}{L-\alpha}e^{(L-\alpha)t}\left[e^{(L-\alpha)t}-1\right] = \mathrm{o}(\|\xi\|)$$

for every $t\in[0,1]$, $\omega\in\Omega$ and $u_0\in\mathcal{A}(\omega)$. In the case $\alpha = L$ we have starting in (4.2.5),

$$\|\varphi(t,\omega,u_0+h)-\varphi(t,\omega,u_0)-U(t)\| \leq C'_F\|\xi\|^2 t = \mathrm{o}(\|\xi\|)$$

Note that our estimates are independent of $u_0 \in \mathcal{A}(\omega)$. □

In general it can be challenging to show condition RC (I). Due to the chosen assumptions on the nonlinearity F the condition follows readily from the last proof. As discussed in Section 3.4 we choose $L \geq \alpha$, otherwise the random attractor reduces to a singleton. Hence we distinguish between two cases. First let $L > \alpha$, then

$$\sup_{t\in[0,1]}\frac{C'_F\|\xi\|^2}{L-\alpha}e^{(L-\alpha)t}\left[e^{(L-\alpha)t}-1\right] \leq \frac{C'_F}{L-\alpha}e^{2(L-\alpha)}\|\xi\|^2 .$$

and if $L=\alpha$, then $\sup_{t\in[0,1]} C'_F\|\xi\|^2 t \leq C'_F\|\xi\|^2$.

Finally we identify the random variable C and we see that it is actually a constant, for $\varepsilon > 0$

$$\sup_{t\in[0,1]}\sup_{\substack{u_0\in\mathcal{A}(\omega)\\ \|\xi\|\leq\varepsilon}}\frac{1}{\|\xi\|}\|\varphi(t,\omega,u_0+\xi)-\varphi(t,\omega,u_0)-U(t)\| \leq C(\omega)\varepsilon$$

where

$$C(\omega) := \begin{cases}\frac{C'_F}{L-\alpha}e^{2(L-\alpha)}, & L>\alpha,\\ C'_F, & L=\alpha .\end{cases}$$

We conclude that $C(\omega) \geq 1$ for every $C_F > 0$, since $C'_F = C_F + 1$ and $e^{2(L-\alpha)} > L-\alpha$. Clearly we have as well $\mathbb{E}(\ln C) < \infty$.

For the second condition RC (II) we have the following discussion. First, bear in mind the notation $U(t) = D\varphi(t,\omega,u_0)\xi$ for fixed $(t,\omega,u_0,\xi) \in \mathbb{R}_+ \times \Omega \times H \times H$ and consider

$$\begin{aligned} \frac{dU(t)}{dt} &= AU(t) + DF(\varphi(t,\omega,u_0) + Z(\theta_t\omega))U(t), \quad t > 0 \\ U(0) &= \xi \in H\,. \end{aligned} \tag{4.2.6}$$

We know from [Hen81, Theorem 7.1.4, p.193 and Exercise 5, p.195] that the above linear differential equation possesses a unique mild solution. As decribed in the previous section this mild solution U coincides with a weak solution by a result in [Bal77]. The unique weak solution U is an element of $L^2(0,T;\mathcal{F}_{\mathcal{E}^0})$ for every $T > 0$ in the sense of [Tem88, Chapter II, Section 3.4, p.74] and [Sho97, Chapter III, Proposition 4.1 and Corollary 4.1, p.122]. Here $\mathcal{F}_{\mathcal{E}^0} = D\left((\Delta_\mu)^{\frac{1}{2}}\right)$ is the set of functions with finite energy and value zero on the boundary defined in (2.4.9). Note that this space corresponds for the classical Laplacian on a smooth domain to the Sobolev space H_0^1.

If we multiply (4.2.6) with its solution $U(t)$, we observe since $(\Delta_\mu U, U) \leq 0$, that

$$\begin{aligned} \frac{d}{dt}\|U(t)\|^2 = 2\left(\frac{dU(t)}{dt}, U(t)\right) &= 2(AU(t),U(t)) + 2\big(DF(\varphi(t,\omega,u_0) + Z(\theta_t\omega))U(t),U(t)\big) \\ &\leq 2\big(DF(\varphi(t,\omega,u_0) + Z(\theta_t\omega))U(t),U(t)\big) \\ &\leq 2\|DF(\varphi(t,\omega,u_0) + Z(\theta_t\omega))\|_{L(H)} \cdot \|U(t)\|^2\,. \end{aligned}$$

From which we deduce the bound of the Fréchet derivative of the solution for every $t \geq 0$, $\omega \in \Omega$,

$$\|U(t)\| \leq \|U(0)\| \cdot \exp\left\{\int_0^t \|DF(\varphi(r,\omega,u_0) + Z(\theta_r\omega))\|_{L(H)}\,dr\right\} \leq \|\xi\|e^{Lt}\,.$$

Hence the condition RC (II) holds by choosing $\widetilde{\alpha}_1 = e^L$, since

$$\begin{aligned} \sup_{t\in[0,1]}\sup_{u_0\in\mathcal{A}(\omega)} \alpha_1(D\varphi(t,\omega,u_0)) &= \sup_{t\in[0,1]}\sup_{u_0\in\mathcal{A}(\omega)} \|D\varphi(t,\omega,u_0)\|_{L(H)} \\ &= \sup_{t\in[0,1]}\sup_{u_0\in\mathcal{A}(\omega)}\sup_{\substack{\xi\in H\\ \|\xi\|=1}} \|U(t)\| \leq e^L\,. \end{aligned}$$

Clearly we have

$$\mathbb{E}(\ln\widetilde{\alpha}_1) < \infty\,.$$

The last condition RC (III) is most important for our result. The following lemma is of great use for the proof. A more general version can be found in [Tem88, Chapter VI, Lemma 2.1, p.390].

Lemma 4.2.3. *Suppose $A = \Delta_\mu$ is the Laplacian on $L^2(K,\mu)$ with the properties introduced in Section 2.4. Let $\psi_1, ..., \psi_m$ be a family of $\mathcal{F}_{\mathcal{E}^0}$ which is orthonormal in H, then*

$$\sum_{i=1}^m (A\psi_i,\psi_i) \leq -(\lambda_1 + ... + \lambda_m),$$

where $(\lambda_i)_{i\in\mathbb{N}}$ is the sequence of eigenvalues of $-A$.
Taking into account the asymptotics of the Laplacian $-A$ given in Lemma 2.4.18, we estimate

$$\sum_{i=1}^m \lambda_i \geq C_3\sum_{i=1}^m i^{\frac{2}{d_S}} \geq \frac{C_3}{1+2/d_S} m^{1+\frac{2}{d_S}}\,.$$

for a constant $C_3 > 0$.

According to the previous section we consider first $P_m(D\varphi(t,\omega,u_0))$ for fixed $t \in \mathbb{R}_+$, $\omega \in \Omega$ and $u_0 \in \mathcal{A}(\omega)$ for some $m \in \mathbb{N}$. For this purpose consider m solutions of the equation (4.2.6) for m initial elements $\xi_i \in H$, $i \in \{1,...,m\}$. Let $Q_m(r) = Q_m(r,\omega,u_0;\xi_1,...,\xi_m)$ be the orthogonal projector that maps H onto $\mathrm{span}(U_1(r),...,U_m(r))$ for $r \in \mathbb{R}_+$. Choose an orthonormal basis of H denoted by $(\psi_i(r))_{i=1}^{\infty}$ such that $\psi_1(r),...,\psi_m(r)$ is an orthonormal basis of $Q_m(r)H$. Note that since $U_1(r),...,U_m(r) \in \mathcal{F}_{\mathcal{E}^0}$ we have that $\psi_1(r),...,\psi_m(r) \in \mathcal{F}_{\mathcal{E}^0}$. Hence we conclude accordingly to the previous lemma,

$$\begin{aligned}
&\mathrm{Tr}\ ([A + DF(\varphi(r,\omega,u_0) + Z(\theta_r\omega))] \circ Q_m(r)) \\
&= \sum_{i=1}^{\infty} \big([A + DF(\varphi(r,\omega,u_0) + Z(\theta_r\omega))] \circ Q_m(r)\psi_i(r), \psi_i(r)\big) \\
&= \sum_{i=1}^{m} \big[(A\psi_i(r), \psi_i(r)) + \big(DF(\varphi(r,\omega,u_0) + Z(\theta_r\omega))\psi_i(r), \psi_i(r)\big)\big] \\
&\leq -\sum_{i=1}^{m} \lambda_i + mL \leq -\frac{C_3}{1+2/d_S} m^{1+\frac{2}{d_S}} + mL
\end{aligned} \tag{4.2.7}$$

with L being the Lipschitz constant of the nonlinearity F. We obtain for $m \in \mathbb{N}$,

$$\begin{aligned}
P_m(D\varphi(t,\omega,u_0)) &= \sup_{\substack{\xi_1,\dots,\xi_m \in H \\ \|\xi_i\| \leq 1, \forall i}} \|U_1(t) \wedge \cdots \wedge U_m(t)\|_{\bigwedge^m H} \\
&= \sup_{\substack{\xi_1,\dots,\xi_m \in H \\ \|\xi_i\| \leq 1, \forall i}} \exp\left\{ \int_0^t \mathrm{Tr}\ ([A + DF(\varphi(r,\omega,u_0) + Z(\theta_r\omega))] \circ Q_m(r))\ dr \right\}.
\end{aligned}$$

Together with the estimate (4.2.7) we infer

$$\begin{aligned}
P_m(D\varphi(t,\omega,u_0)) &\leq \sup_{\substack{\xi_1,\dots,\xi_m \in H \\ \|\xi_i\| \leq 1, \forall i}} \exp\left\{ \int_0^t -\frac{C_3}{1+2/d_S} m^{1+\frac{2}{d_S}} + mL\, dr \right\} \\
&= \exp\left\{ t\left(-\frac{C_3}{1+2/d_S} m^{1+\frac{2}{d_S}} + mL \right) \right\} =: \widetilde{P}_m(t,\omega),
\end{aligned} \tag{4.2.8}$$

where in our case $\widetilde{P}_m(t,\omega)$ is independent of ω.
Finally we observe for any $t > 0$,

$$\mathbb{E}(\ln \widetilde{P}_m(t,\cdot)) = \ln \widetilde{P}_m(t,\cdot) < 0$$

if and only if

$$m^{\frac{2}{d_S}} > \frac{L}{C_s}, \tag{4.2.9}$$

for $C_s := \frac{C_3}{1+2/d_S}$ and $m \in \mathbb{N}$ sufficiently large. Hence the condition RC (III) is fulfilled.

Theorem 4.2.4. *Let φ be the RDS presented in Theorem 3.3.2, where the domain K of the Hilbert space $L^2(K,\mu)$ is a connected p.c.f. fractal introduced in Section 2.4. Suppose φ possesses a random attractor $\mathcal{A}(\omega)$, $\omega \in \Omega$ as stated in Theorem 3.4.13. Moreover assume that the conditions RC (I)–RC (III) hold.*
Then the Hausdorff dimension of the random attractor is bounded by the smallest number $m \in \mathbb{N}$ which fulfills

$$m > \left(\frac{L}{C_s}\right)^{\frac{d_S}{2}}.$$

The proof follows directly from the previous discussion and Theorem 4.2.1.

Remark 4.2.5. In the arguments succeeding Lemma 4.2.3 we can choose $t_0(\omega) = t_0$ to be an arbitrary positive constant, such that for every $t \geq t_0$ the condition RC (III) is fulfilled.
However in the case of an ω-depending Lipschitz constant $L(\theta_t\omega)$ for $t \in \mathbb{R}_+$ and $\omega \in \Omega$ it is possible to obtain a similar result as in (4.2.9). Assume in addition that $\mathbb{E}L < \infty$. Then (4.2.8) becomes

$$P_m(D\varphi(t,\omega,u_0)) \leq \sup_{\substack{\xi_1,\ldots,\xi_m \in H \\ \|\xi_i\| \leq 1, \forall i}} \exp\left\{ -\frac{C_3 t}{1+2/d_S} m^{1+\frac{2}{d_S}} + mt \cdot \frac{1}{t}\int_0^t L(\theta_r\omega)\,dr \right\} .$$

According to the Birkhoff-Chintchin Ergodic Theorem 3.1.4 there exists a time $t_0(\omega) > 0$ such that for every $t \geq t_0(\omega)$,

$$P_m(D\varphi(t,\omega,u_0)) \leq \exp\left\{ t\left(-\frac{C_3}{1+2/d_S} m^{1+\frac{2}{d_S}} + m\mathbb{E}L \right) \right\} =: \widetilde{P}_m(t,\omega) .$$

Hence we obtain a similar condition as above,

$$m^{\frac{2}{d_S}} > \frac{\mathbb{E}L}{C_s} .$$

Clearly we have to adjust as well the other conditions, that are necessary for the result in Theorem 4.2.4.

For the random attractor of the SPDE (2.3.4) discussed in Theorem 3.4.13 it holds

$$\widetilde{\mathcal{A}}(\omega) = T(\omega, \mathcal{A}(\omega))$$

where $T(\omega,\cdot) : H \to H$ is the conjugation between the RDE and the SPDE. By Corollary 2.4 [Fal90, (b), p.30] we know that for every $\omega \in \Omega$

$$\dim_{\mathrm{H}}(\widetilde{\mathcal{A}}(\omega)) = \dim_{\mathrm{H}}(\mathcal{A}(\omega)) ,$$

since $\|T(\omega,x) - T(\omega,y)\| = \|x-y\|$ for every $x, y \in \mathcal{A}(\omega)$.

We shortly discuss that the Hausdorff dimension of $\mathcal{A}(\omega)$ is constant a.s. This statement is presented e.g. in [CF98, Lemma 4.2, p.456] and [Sch97, Remark 3.3, p.962].
According to Theorem 3.3.2 the mapping $\varphi(t,\omega,\cdot) : H \to H$ is Lipschitz continuous for every $t \in \mathbb{R}_+$ and $\omega \in \Omega$. Therefore Corollary 2.4 in [Fal90, (a), p.30] together with the invariance of the random attractor in Definition 3.4.5 tell us that

$$\dim_{\mathrm{H}}(\mathcal{A}(\theta_t\omega)) = \dim_{\mathrm{H}}(\varphi(t,\omega,\mathcal{A}(\omega))) \leq \dim_{\mathrm{H}}(\mathcal{A}(\omega))$$

for fixed $t \in \mathbb{R}_+$ and $\omega \in \Omega$. Since the measure $\mathbb{P}$ is invariant w.r.t. θ we have $\mathbb{E}[\dim_{\mathrm{H}}(\mathcal{A}(\theta_t\cdot))] = \mathbb{E}[\dim_{\mathrm{H}}(\mathcal{A}(\cdot))]$. Hence we see directly that

$$\mathbb{P}(\dim_{\mathrm{H}}(\mathcal{A}(\theta_t\omega)) = \dim_{\mathrm{H}}(\mathcal{A}(\omega)), \forall t \geq 0) = 1 .$$

Then applying the Ergodic Theorem 3.1.4 and that θ is ergodic we obtain that $\dim_{\mathrm{H}}(\mathcal{A}(\cdot))$ is a.s. constant.

4.3 A numerical value for the bound in the case of the SG

Until now we have not chosen a particular nonlinearity F of the RDE (3.3.3),

$$\begin{aligned} du(t) &= Au(t)dt + F(u(t) + Z(\theta_t\omega))dt, \quad t > 0 \\ u(0) &= u_0 \in H\,. \end{aligned}$$

In the previous section we made several assumptions on the nonlinearity and we justify them is this section using integral operators.
We have to check that F is Lipschitz continuous with constant $L > 0$. According to the discussion prior to Lemma 3.4.11 there has to exist a small linear growth constant such that $l < \alpha \leq L$, where $\alpha = \lambda_1$ is the smallest positive eigenvalue of $-\Delta_\mu = -A$. The nonlinearity needs to be twice Fréchet differentiable and its second derivative has to be bounded by a constant $C_F > 0$, cf. Theorem 4.2.2.
As a traditional fractal set K to start with, we choose the Sierpinski gasket described in Chapter 2. In Remark 2.4.20 we stated that it is possible to construct the Laplacian $-\Delta_\mu$ pointwise by a sequence of iterative graph Laplacians similar to Definition 2.4.9. Using the spectral decimation method described e.g. in [FS92] or [Str06, Section 3.3] we have the following results.
Solving an elementary discrete Dirichlet problem on the first graph, that is the pre-level of the SG, with vertex set V_1 gives us that the corresponding Laplacian Δ_1 has the eigenvalues $\lambda_1^{\Delta_1} = 2$ and $\lambda_2^{\Delta_1} = 5$ with multiplicity 1 resp. 2. Suppose we are given an eigenvalue $\lambda^{\Delta_{m-1}}$ on the graph with vertex set V_{m-1}, then we calculate the so called *continued* eigenvalues λ^{Δ_m} of the Laplacian Δ_m on the graph of V_m by

$$\lambda^{\Delta_{m-1}} = \lambda^{\Delta_m}(5 - \lambda^{\Delta_m}),$$

or equivalent

$$\lambda^{\Delta_m} = \frac{5}{2}\left[1 \pm \sqrt{1 - \frac{4}{25}\lambda^{\Delta_{m-1}}}\right]\,. \tag{4.3.1}$$

Using this recursive formula we obtain the eigenvalues λ of the Laplacian $-\Delta_\mu$ on the SG by

$$\lambda := \lim_{m\to\infty} 5^m \lambda^{\Delta_m},$$

whenever the above limit exists. Of particular interest are those limits which we obtain with all but a finite sequence of minus signs at the bifurcation in (4.3.1). Following the discussion in [Str06, p.78], the lowest positive eigenvalue λ_1 occurs, when we start with $\lambda_1^{\Delta_1} = 2$ and take at every occuring bifurcation in (4.3.1) the minus sign. Using for example a classical calculation program like *Mathematica* the limit converges to $\alpha = \lambda_1 \approx 2.242$. For details on the used code we refer to [BE17].

We introduce now integral operators according to [KA78, Chapter XVII, Section 3, p.548].

Definition 4.3.1. Let $H = L^2(K, \mu)$ be the Hilbert space introduced in the previous chapters. The operator $F : H \to H$ given by

$$\begin{aligned} &F(u) : K \to \mathbb{R}, \\ &s \mapsto F(u)(s) = \int_K k(s, x, u(x))\, d\mu(x) \end{aligned}$$

is called *integral operator* at $u \in H$ and $k : K \times K \times \mathbb{R} \to \mathbb{R}$ is called *integral kernel.*

For the Fréchet derivatives of this operator we have the following theorem.

Theorem 4.3.2 ([KA78], Satz 3.XVII, p.548)**.** *Let the function $k(s,x,\cdot):\mathbb{R}\to\mathbb{R}$ be twice differentiable with continuous second derivative $k''(s,x,\cdot)$ for fixed $s,x\in K$. In particular for every $(s,x,y)\in K\times K\times\mathbb{R}$ we have*

$$|k''(s,x,y)|\le C''_k$$

for some constant $C''_k>0$. Then the integral operator F is twice Fréchet differentiable at $u\in H$ with derivatives

$$[DF(u)\xi](s)=\int_K k'(s,x,u(x))\xi(x)\,d\mu(x)$$

and

$$[D^2F(u)(\xi,\bar{\xi})](s)=\int_K k''(s,x,u(x))\xi(x)\bar{\xi}(x)\,d\mu(x)$$

for every $s\in K$, $\xi,\bar{\xi}\in H$.

In the following example we will choose specific integral kernels to check the assumptions we took for the nonlinearity.

Example 4.3.3. For simplicity we assume that we have a partial separation of the variables, i.e. we suppose

$$(1)\quad k(s,x,y)=f(s,x)+g(y)\qquad\text{or}\qquad(2)\quad k(s,x,y)=f(s,x)g(y)$$

where $f\in C(K\times K;\mathbb{R})$ and $g\in C^2(\mathbb{R};\mathbb{R})$, such that there exist constants $C'_g,C''_g>0$ with

$$|g'(y)|\le C'_g\qquad\text{and}\qquad|g''(y)|\le C''_g$$

for every $y\in\mathbb{R}$. Moreover interpreting g as a mapping from H to H, it has a linear growth, i.e there exist constants $l_1,c>0$ such that $\|g(u)\|\le l_1\|u\|+c$ for $u\in H$.
We estimate the (uniform) bound of the first derivative in the case (1)

$$\begin{aligned}\|DF(u)\xi\|^2&=\int_K\left|\int_K k'(s,x,u(x))\xi(x)\,d\mu(x)\right|^2 d\mu(s)\\&=\underbrace{\mu(K)}_{=1}\left|\int_K g'(u(x))\xi(x)\,d\mu(x)\right|^2\\&\le(C'_g)^2\cdot\|\xi\|^2\end{aligned}$$

and therefore $\|DF(u)\|_{L(H)}\le C'_g$ for every $u\in H$. Together with the mean value theorem for Fréchet derivatives (see [Wou79, Section 12.1, Corollary 3, p.266]) we obtain that there exists a $\tau\in(0,1)$ such that for $u,v\in H$

$$\|F(u)-F(v)\|\le\|DF(v+\tau(u-v))\|_{L(H)}\|u-v\|\le C'_g\|u-v\|$$

and we take $L=C'_g$ as the Lipschitz constant. Similar we obtain for the second derivative,

$$\begin{aligned}\|D^2F(u)(\xi,\bar{\xi})\|^2&=\left|\int_K g''(u(x))\xi(x)\bar{\xi}(x)\,d\mu(x)\right|^2\\&\le(C''_g)^2\cdot\|\xi\|^2\|\bar{\xi}\|^2\end{aligned}$$

and so $\|D^2F(u)\|_{BL(H\times H,H)} \leq C''_g =: C_F < \infty$ for every $u \in H$. The linear growth of F follows for $u \in H$ and $s \in K$ from

$$\begin{aligned}|F(u)(s)|^2 &\leq \left(\left|\int_K f(s,x)\,d\mu(x)\right| + \left|\int_K g(u(x))\,d\mu(x)\right|\right)^2\\ &\leq 2f_{\max}^2 + 2\|g(u)\|^2\end{aligned}$$

where $f_{\max}$ is the maximum of the continuous function $h : K \to \mathbb{R}$, $h(s) := \max_{x\in K}|f(s,x)|$. We conclude with the linear growth of the function g, that

$$\|F(u)\| \leq l\|u\| + d, \quad d^2 := 2f_{\max}^2 + 4c^2 \text{ and } l := 2l_1 .$$

For an integral kernel as in (2) we obtain in a similar way the Lipschitz constant

$$\|DF(u)\|_{L(H)} \leq f_{\max}C'_g =: L$$

and the bound of the second derivative,

$$\|D^2F(u)\|_{BL(H\times H,H)} \leq f_{\max}C''_g =: C_F .$$

For the linear growth we observe

$$\|F(u)\| \leq l\|u\| + d, \quad d := f_{\max}c \text{ and } l := f_{\max}l_1 .$$

If we are interested in particular numbers, then choose for example

$$k(s,x,u(x)) = f(s,x) + g(u(x)) = e^{-|s|^2} + \sin(3u(x))$$

with $s, x \in K$ and $u \in L^2(K,\mu)$. Then we observe $|g'(y)| \leq 3 =: L$ and for the second derivative $|g''(y)| \leq 9 =: C_F$ for every $y \in \mathbb{R}$. Concerning the linear growth we can choose for instance $\|g(u)\| \leq \|u\| + 1$ such that $l = 2l_1 = 2$. Therefore we have found an example of a nonlinearity F with $2 < \alpha \approx 2.24 < 3$ and a bounded second derivative. In view of Lemma 3.4.11 and Theorem 3.4.13 we obtain a random attractor which is not a singleton.

Now we are ready to discuss the value of the upper bound on the Hausdorff dimension we gave in Theorem 4.2.4.

Theorem 4.3.4. *Let φ be the random dynamical system in* (3.3.5) *with a nonlinearity given by $F(u)(s) = \int_K e^{-|s|^2} + \sin(3u(x))\,d\mu(x)$ for $s \in K$, where K is the Sierpinski gasket. Then the associated random attractor $\mathcal{A}(\omega)$, $\omega \in \Omega$ has a bounded Hausdorff dimension and in particular,* $\dim_{\mathrm{H}}(\mathcal{A}(\omega)) \leq 11$.

Proof. We need the approximate size of the constant $C_s = \frac{C_3}{1+2/d_S}$. We recall from Example 2.4.19 that $d_S = \frac{\log 9}{\log 5} \approx 1.365$ for the Sierpinski gasket. Let us rewrite (4.2.9),

$$m^{\frac{2}{d_S}} > \frac{L}{C_s} = \frac{L}{C_3}\left(1 + \frac{2}{d_S}\right)$$

where the constant $C_3 > 0$ is given by Lemma 2.4.18. This number $C_3 > 0$ is often not stated explicitly in the literature. For the SG there is however an interesting result ([FS92, p.27]), which originates from the eigenvalue counting function $\rho : (0,\infty) \to [0,\infty)$, that counts the eigenvalues w.r.t. their multiplicities, i.e. $\rho(x) := \#\{j \in \mathbb{N} : \lambda_j \leq x\}$. We start with presenting this result,

$$\frac{\rho(x)}{x^{d_S/2}} \leq 9\cdot\zeta(5)^{-d_S/2}, \qquad x > \zeta(5) \tag{4.3.2}$$

where the function $\zeta : \left(-\infty, \frac{25}{4}\right] \to \mathbb{R}_+$ is given by

$$\zeta(z) := \lim_{m\to\infty} 5^{m+1}\phi_-^{(m)}(z)\,.$$

The term $\phi_-^{(m)}$ denotes the m-th composition of the function

$$\phi_-(x) = \frac{5}{2}\left(1 - \sqrt{1 - \frac{4}{25}x}\right)$$

for $x \in \left(0, \frac{25}{4}\right]$. Using again a reasonable program we calculate $\zeta(5) \approx 37.257$ and therefore

$$\frac{\rho(x)}{x^{d_S/2}} \leq 9 \cdot \zeta(5)^{-d_S/2} \approx 0.762, \qquad x > \zeta(5)\,. \tag{4.3.3}$$

From the inequality (4.3.2) we also obtain a bound for $x \leq \zeta(5)$ since

$$\rho(x) \leq \rho(\zeta(5)) \leq \rho(y) \leq 9\zeta(5)^{-d_S/2}y^{d_S/2}$$

for every $y > \zeta(5)$. Since $\rho(y)$ is right-continuous, we conclude

$$\frac{\rho(x)}{x^{d_S/2}} \leq \frac{9}{x^{d_S/2}} \leq \frac{9}{\lambda_1^{d_S/2}} \approx 5.187, \qquad x \leq \zeta(5) \tag{4.3.4}$$

and note that for $x < \lambda_1$ we have clearly $\rho(x) = 0$. In comparison of (4.3.3) and (4.3.4) we have the general result for every $x > 0$,

$$\frac{\rho(x)}{x^{d_S/2}} \leq \frac{9}{\lambda_1^{d_S/2}} =: C_2\,. \tag{4.3.5}$$

Although this estimate is not very sharp, we obtain a general result with this constant. Analogously to the proof of Lemma 5.1.3 in [Kig01, p.159] we observe for every $i \geq 1$,

$$i \leq \rho(\lambda_i) \leq C_2\lambda_i^{\frac{d_S}{2}}\,.$$

Hence choosing $C_3 := 1/(C_2)^{\frac{2}{d_S}}$ we have $C_3 i^{\frac{2}{d_S}} \leq \lambda_i$ for every $i \in \mathbb{N}$. Finally we are looking for the smallest $m \in \mathbb{N}$ such that

$$m > \left(\frac{(1+\frac{2}{d_S})}{C_3}\right)^{d_S/2} \cdot L^{d_S/2} = C_2\left(1 + \frac{2}{d_S}\right)^{d_S/2} \cdot L^{d_S/2} \approx 9.602 \cdot L^{d_S/2}$$

and since $L = 3$ we have $\dim_{\mathrm{H}}(\mathcal{A}(\omega)) \leq 21 = m$. We also observe by the last discussion that this upper bound on the Hausdorff dimension depends strongly on the Lipschitz constant L resp. the bound for the first derivative of F, more precisely $m \sim L^{d_S/2}$.

To obtain the estimate stated in the theorem, we construct a more precise estimate than (4.3.5). Although it is quite difficult to state the complete (ordered) spectrum of Δ_μ on the SG we know by (4.3.4) that $\rho(x) \leq 9$ for $x < \zeta(5)$. That means in particular that up to $\zeta(5)$ there are at most 9 eigenvalues. For each of these eigenvalues we have as a consequence of (4.3.5),

$$\lambda_i \geq \frac{\lambda_1}{9^{2/d_S}} i^{2/d_S} =: C_3' i^{2/d_S}, \qquad i \in \{1, ..., 9\}$$

and for every eigenvalue larger than $\zeta(5)$ we have, following (4.3.3),

$$\lambda_i \geq \frac{\zeta(5)}{9^{2/d_S}} i^{2/d_S} =: C_3'' i^{2/d_S}\,.$$

Considering now the sum of the first m ($m > 9$) eigenvalues, we observe

$$\sum_{i=1}^{m} \lambda_i \geq C_3' \sum_{i=1}^{9} i^{2/d_S} + C_3'' \sum_{i=10}^{m} i^{2/d_S} \geq \frac{9C_3' + (m-9)C_3''}{m} \sum_{i=1}^{m} i^{2/d_S}$$
$$\geq \frac{(m-9)C_3''}{m} \sum_{i=1}^{m} i^{2/d_S} \geq \frac{C_3''}{1+2/d_S} \frac{m-9}{m} \cdot m^{1+\frac{2}{d_S}}$$

since $C_3'' > C_3'$. Then if we follow the reasoning in (4.2.7) and (4.2.8) we conclude that m has to fulfill

$$-\frac{C_3''}{1+2/d_S} \frac{m-9}{m} \cdot m^{1+\frac{2}{d_S}} + mL < 0\,,$$

which is equivalent to

$$m^{\frac{2}{d_S}} > \frac{(1+2/d_S)}{C_3''} L \cdot \frac{m}{m-9}\,. \tag{4.3.6}$$

Hence we are looking for the smallest $m \in \mathbb{N}$ that fulfills (4.3.6) and $m > 9$. Together with the chosen nonlinearity and the related Lipschitz constant $L = 3$, the inequality (4.3.6) becomes

$$m^{\frac{2}{d_S}} > 1.654 \cdot L \cdot \frac{m}{m-9} = 4.962 \cdot \frac{m}{m-9}\,,$$

which holds for $\mathbb{N} \ni m > 10.652$, hence $\dim_{\mathrm{H}}(\mathcal{A}(\omega)) \leq 11$. Note again if $L \to \infty$ then $m \to \infty$. $\square$

Chapter 5

A lower bound on the Hausdorff dimension of the random attractor

In this chapter we discuss a lower bound for the random attractor of a given SPDE with a multiplicative noise. In the first section we state the SPDE and we show similar to Chapter 3 the existence of an RDS and its associated random attractor. In doing so we use a conjugation to obtain a related RDE.
We follow the idea to use invariant manifolds as a subset of the random attractor, as presented in the deterministic theory, e.g. in [Tem88, Chapter VII, Section 3, p.482 ff.]. Similar to this deterministic theory we show that a subset of the random unstable manifold is included in the random attractor.
Our approach rely on the theory of (local) random unstable manifolds described e.g. in [GALS10, Section 2]. Therefore we start the second section with a short introduction of the important concepts concerning these manifolds. Later in this section we truncate the given RDE to show the existence of a global unstable Lipschitz manifold for this truncated differential equation. This construction helps us in the third and last section to show that this manifold is in fact a local unstable manifold for the original RDE. Finally we are able to establish our result. We show that the intersection of the local manifold with a certain neighborhood of zero in H is included in the attractor. Using this and a projection theorem we obtain a lower bound for Hausdorff dimension of the random attractor.

5.1 The random attractor for an SPDE with multiplicative noise

Let H be the space $L^2(K,\mu)$ where $(K,\mathcal{B}(K),\mu)$ is a σ-finite measure space of an (open) bounded subset of $\mathbb{R}^n$, $n \geq 1$. Then let us consider the following SPDE with a multiplicative noise,

$$\begin{aligned} dv(t) &= (Av(t) + \mathcal{F}(v(t)))dt + v(t) \circ dW(t),\ \forall t > 0, \\ v(0) &= v_0 \in H\,. \end{aligned} \tag{5.1.1}$$

Here the linear operator A is the generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$ with assumptions like in Theorem 2.1.13 such that in particular $-A$ has discrete positive spectrum. For the nonlinearity term we assume a global Lipschitz continuity with constant $L > 0$. W is a two-sided Wiener process on the canonical space $\Omega := C_0(\mathbb{R},\mathbb{R})$. We equipp this set with the compact open topology and consider the related Wiener shift $\theta = (\theta_t)_{t\in\mathbb{R}}$ with $\theta_t\omega(\cdot) = \omega(t+\cdot) - \omega(t)$ similar to Example 3.1.6. Together with the Wiener measure $\mathbb{P}_0 : \mathcal{B}(C_0(\mathbb{R};\mathbb{R})) \to [0,1]$ given by the finite dimensional distributions of W, we obtain the probability space $(\Omega,\mathscr{F},\mathbb{P}) := (C_0(\mathbb{R};\mathbb{R}),\mathcal{B}(C_0(\mathbb{R};\mathbb{R})),\mathbb{P}_0)$. Similar to Theorem 3.1.7 we can show that $\mathbb{P}$ is invariant and that $(\Omega,\mathscr{F},\mathbb{P},\theta)$ forms an ergodic metric dynamical system.
For more informations on the appearing Stratonovich differential $v\circ dW$ we refer to [Pro04, Chapter II, Section 7]. It is also well-known (e.g. in [Pro04, Chapter V, Section 5]) that the equation

(5.1.1) has the corresponding Itô stochastic differential equation

$$\begin{aligned} dv(t) &= (Av(t) + \mathcal{F}(v(t)))dt + \frac{v(t)}{2}dt + v(t)dW(t), \ \forall t > 0, \\ v(0) &= v_0 \in H\,. \end{aligned}$$

The theory of solutions for the above stochastic evolution equation is formulated in [DPZ92, Chapter 7]. As in Chapter 3 we have the problem that the appearing stochastic integrals are only defined almost surely and that the associated probability space has to be complete. Since we are interested in random dynamical systems that are generated by the solution operators of equations like (5.1.1) we have to overcome this problem. The following lemma is proven in [DLS04, Lemma 2.1, p.2114] and serves as a corresponding result to the statements in Section 3.2 and Theorem 3.1.10.

Lemma 5.1.1.

(i) There exists a θ-invariant subset $\Omega' \subseteq \Omega$ of full $\mathbb{P}_0$-measure with the property that each element in Ω' grows sublinear, i.e. for every $\omega \in \Omega'$

$$\lim_{t\to\pm\infty} \frac{|\omega(t)|}{|t|} = 0\,.$$

(ii) Suppose we are given the θ-invariant set Ω' of (i). Then for every $\omega \in \Omega'$,

$$z(\omega) := \int_{-\infty}^{0} e^r\, d\omega(r) = -\int_{-\infty}^{0} e^r \omega(r)\, dr$$

is well-defined and the unique stationary solution of the equation

$$\begin{aligned} dz(t) &= -z(t)dt + d\omega(t), \ t > 0, \\ z(0) &= z(\omega) \end{aligned}$$

is given by the Ornstein-Uhlenbeck *process $z : \mathbb{R} \times \Omega' \to \mathbb{R}$,*

$$\begin{aligned} z(t,\omega) := z(\theta_t\omega) &= \int_{-\infty}^{0} e^r\, d\theta_t\omega(r) = -\int_{-\infty}^{0} e^r \theta_t\omega(r)\, dr \\ &= -\int_{-\infty}^{0} e^r \omega(t+r)\, dr + \omega(t)\,. \end{aligned}$$

Moreover for every $\omega \in \Omega'$ the paths of $z(\theta.\omega)$ are continuous and $\lim_{t\to\pm\infty} \frac{|z(\theta_t\omega)|}{|t|} = 0$.

(iii) There exists a θ-invariant set $\Omega'' \subseteq \Omega$ of full $\mathbb{P}_0$-measure such that for every $\omega \in \Omega''$

$$\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t z(\theta_r\omega)\, dr = 0\,.$$

For the following we replace Ω by $\widetilde{\Omega} := \Omega' \cap \Omega''$, the σ-algebra $\mathcal{B}(C_0(\mathbb{R};\mathbb{R}))$ by the trace with $\widetilde{\Omega}$, that is $\widetilde{\Omega} \cap \mathcal{B}(C_0(\mathbb{R};\mathbb{R}))$ and the Wiener measure $\mathbb{P}_0$ by its restriction on the new trace σ-algebra. For simplicity we rename the associated 'new' metric dynamical system by the old notation, i.e.

$$(\Omega, \mathscr{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})\,.$$

We consider now an RDE which can be transformed by the Ornstein-Uhlenbeck process into the SPDE (5.1.1),

$$\begin{aligned} \frac{du(t)}{dt} &= Au(t) + z(\theta_t\omega)u(t) + e^{-z(\theta_t\omega)}\mathcal{F}\left(u(t)e^{z(\theta_t\omega)}\right), \\ u(0) &= u_0\,, \end{aligned} \tag{5.1.2}$$

for every $t \in \mathbb{R}_+$, $\omega \in \Omega$ and $u_0 \in H$. The substitution is identified by,

$$u(t) = v(t)e^{-z(\theta_t\omega)}$$

for every $t \in \mathbb{R}_+$, $\omega \in \Omega$. This substitution gives us in fact the conjugacy $T : \Omega \times H \to H$ with $T(\omega, x) := xe^{z(\omega)}$ and its inverse $T^{-1}(\omega, x) = xe^{-z(\omega)}$.

For simplicity we introduce for all $\omega \in \Omega$ and $u \in H$ the notation

$$F(\omega, u) := e^{-z(\omega)}\mathcal{F}\left(ue^{z(\omega)}\right) . \tag{5.1.3}$$

Clearly this nonlinearity F is Lipschitz continuous with the same Lipschitz constant L from the originial nonlinearity $\mathcal{F}$. In recall of Theorem 3.3.2 we have the following similar statement, see also [DLS04, p.2115].

Theorem 5.1.2. *Under the previous assumptions the equation* (5.1.2) *has a unique mild solution on every interval* $[0, T]$*,* $T > 0$ *given by*

$$u(t) = u(t, \omega, u_0) = S(t)e^{\int_0^t z(\theta_s\omega)\,ds}u_0 + \int_0^t S(t-r)e^{\int_r^t z(\theta_s\omega)\,ds}F(\theta_r\omega, u(r))\,dr \tag{5.1.4}$$

for all $t \in \mathbb{R}_+$*,* $u_0 \in H$ *and* $\omega \in \Omega$*. The solution operator given in* (5.1.4) *generates a continuous random dynamical system* $\varphi : \mathbb{R}_+ \times \Omega \times H \to H$*.*

To prove the previous theorem we refer to the proof of Theorem 3.3.2 and the references therein.

In view of Theorem 2.1.13 we choose the constant $\alpha = \lambda_1$ the smallest positive eigenvalue of $-A$. For the following we assume that $\mathcal{F}$ has at most a linear growth with a small growth constant. More precisely, we assume the existence of the constants $l, c_1 > 0$ such that,

(A1) $$\|\mathcal{F}(u)\| \le l\|u\| + c_1, \qquad \text{and} \qquad l < \alpha\,.$$

In case we use F instead of $\mathcal{F}$ the last condition (A1) becomes

$$\|F(\omega, u)\| = e^{-z(\omega)}\|\mathcal{F}(ue^{z(\omega)})\| \le l\|u\| + c_1e^{-z(\omega)} \tag{5.1.5}$$

for every $\omega \in \Omega$ and $u \in H$.

The following lemma is a preparation for the existence theorem of the random attractor for the questioned evolution equation.

Lemma 5.1.3. *The RDS* φ *generated by the solution* (5.1.4) *has a random absorbing set* B*.*

Proof. We estimate in norm

$$\|u(t)\| \le e^{-\alpha t + \int_0^t z(\theta_s\omega)\,ds}\|u_0\| + \int_0^t e^{-\alpha(t-r) + \int_r^t z(\theta_s\omega)\,ds}\|F(\theta_r\omega, u(r))\|\,dr$$

and define $f(t) := e^{\alpha t - \int_0^t z(\theta_s\omega)\,ds}\|u(t)\|$ and $Y(\omega) := c_1e^{-z(\omega)}$ for $\omega \in \Omega$. Note that Y is a tempered random variable. In view of the linear growth (5.1.5) we obtain

$$f(t) \le \|u_0\| + \int_0^t lf(r)\,dr + \int_0^t Y(\theta_r\omega)e^{\alpha r - \int_0^r z(\theta_s\omega)\,ds}\,dr\,.$$

Applying the Gronwall Lemma 3.3.1, we have

$$f(t) \le e^{lt}\left[\|u_0\| + \int_0^t e^{-\int_0^r z(\theta_s\omega)\,ds + (\alpha - l)r}Y(\theta_r\omega)\,dr\right] .$$

Hence we see that

$$\|u(t,\omega,u_0)\| \le e^{(l-\alpha)t+\int_0^t z(\theta_s\omega)\,ds}\|u_0\| + \int_0^t e^{(l-\alpha)(t-r)+\int_r^t z(\theta_s\omega)ds} Y(\theta_r\omega)\,dr\,.$$

We substitute ω by $\theta_{-t}\omega$,

$$\begin{aligned}\|u(t,\theta_{-t}\omega,u_0)\| &\le e^{(l-\alpha)t+\int_{-t}^0 z(\theta_s\omega)\,ds}\|u_0\| + \int_{-t}^0 e^{-(l-\alpha)r+\int_{r+t}^t z(\theta_{s-t}\omega)\,ds} Y(\theta_r\omega)\,dr \\ &\le e^{(l-\alpha)t+\int_{-t}^0 z(\theta_s\omega)\,ds}\|u_0\| + \int_{-\infty}^0 e^{-(l-\alpha)r-\int_0^r z(\theta_s\omega)\,ds} Y(\theta_r\omega)\,dr\,.\end{aligned}$$

The first summand converges for $t\to\infty$ to zero, since we have together with property (iii) of Lemma 5.1.1 that for every $\varepsilon>0$ there exist $t_0(\omega,\varepsilon)>0$ such that for every $|t|\ge t_0(\omega,\varepsilon)$,

$$\int_{-t}^0 z(\theta_s\omega)\,ds = t\left[\frac{1}{-t}\left(-\int_{-t}^0 z(\theta_s\omega)\,ds\right)\right] = t\left[\frac{1}{-t}\int_0^{-t} z(\theta_s\omega)\,ds\right] \le t\varepsilon. \tag{5.1.6}$$

We choose now $\varepsilon>0$ such that $l-\alpha+\varepsilon<0$. Taking into account that $u_0\in D(\theta_{-t}\omega)$, where D is a tempered set, we see that the limit $t\to\infty$ of the first summand is indeed zero.
Finally we obtain

$$\limsup_{t\to\infty}\|u(t,\theta_{-t}\omega,u_0)\| \le \int_{-\infty}^0 e^{(\alpha-l)r-\int_0^r z(\theta_s\omega)\,ds} Y(\theta_r\omega)\,dr. \tag{5.1.7}$$

The following discussion ensures the existence of the latest integral. We identify for every $\varepsilon>0$ with $\frac{\alpha-l}{2}>\varepsilon$ the (joint) time $t_0(\omega,\varepsilon)>0$ such that for every $|t|\ge t_0(\omega,\varepsilon)$

$$Y(\theta_t\omega)\le e^{-\varepsilon t},$$

since Y is a tempered random variable (see in particular Remark 3.2.7) and it holds as well

$$-\int_0^t z(\theta_r\omega)\,dr \le -\varepsilon t\,.$$

The integral in (5.1.7) becomes

$$\begin{aligned}\int_{-\infty}^0 e^{(\alpha-l)r-\int_0^r z(\theta_s\omega)\,ds} Y(\theta_r\omega)\,dr = {} & c_1\int_{-t_0(\omega,\varepsilon)}^0 e^{(\alpha-l)r-\int_0^r z(\theta_s\omega)\,ds} e^{-z(\theta_r\omega)}\,dr \\ & + \int_{-\infty}^{-t_0(\omega,\varepsilon)} e^{(\alpha-l-2\varepsilon)r}\,dr < \infty\,.\end{aligned}$$

The function inside the first integral in the latter inequality is a composition of continuous functions on $[-t_0(\omega,\varepsilon),0]$ and has a maximum. Hence we choose the absorbing set $B(\omega):=\mathscr{B}(0,\rho(\omega))$ with radius

$$\rho(\omega):=2\int_{-\infty}^0 e^{(\alpha-l)r-\int_0^r z(\theta_s\omega)\,ds} Y(\theta_r\omega)\,dr\,.$$

The temperedness of this set is analogue to the proof of Lemma 3.4.11. □

Like we have seen in Lemma 3.4.12 we show the compactness of an absorbing set.

Lemma 5.1.4. *Let B be a random absorbing set as in the previous lemma. Then the set $K(\omega):=\overline{\varphi(1,\theta_{-1}\omega,B(\theta_{-1}\omega))}^H$ is compact and random absorbing.*

Proof. We will only discuss the compactness argument. It is sufficient to show the boundedness of the set $\varphi(1, \theta_{-1}\omega, B(\theta_{-1}\omega))$ in $D(A^\beta)$, $\beta \in (0, 1)$. The solution at time one is

$$\varphi(1, \omega, u_0) = e^{\int_0^1 z(\theta_s\omega)\,ds} S(1)u_0 + \int_0^1 e^{\int_r^1 z(\theta_s\omega)\,ds} S(1-r)F(\theta_r\omega, \varphi(r, \omega, u_0))\,dr\,.$$

Since the process z has continuous paths (see (ii) in Lemma 5.1.1) we have, by defining $z_{\max}(\omega) := \max_{t\in[0,1]} z(\theta_t\omega) \in \mathbb{R}$, that $\exp\left(\int_r^1 z(\theta_s\omega)\,ds\right) \leq \exp(|z_{\max}(\omega)|)$ for every $r \in [0, 1]$ and $\omega \in \Omega$. Using additionally that the semigroup is analytic we obtain

$$\begin{aligned} \|\varphi(1, \omega, u_0)\|_\beta \leq M_\beta e^{-\alpha+|z_{\max}(\omega)|}\|u_0\| + \int_0^1 M_\beta e^{|z_{\max}(\omega)|} l \frac{e^{-\alpha(1-r)}}{(1-r)^\beta}\|\varphi(r, \omega, u_0)\|\,dr \\ + \int_0^1 M_\beta e^{|z_{\max}(\omega)|}\frac{e^{-\alpha(1-r)}}{(1-r)^\beta}Y(\theta_r\omega)\,dr \end{aligned} \tag{5.1.8}$$

where $M_\beta > 0$ is a constant as formulated in Theorem 2.1.16 and $Y(\omega) = c_1 e^{-z(\omega)}$ like in Lemma 5.1.3. Now replacing ω by $\theta_{-1}\omega$ and assuming $u_0 \in B(\theta_{-1}\omega)$ we can argue exactly like in Lemma 3.4.12. We have that $\varphi(r, \theta_{-1}\omega, u_0) \in \mathscr{B}(0, R(\omega))$ for a proper radius $R(\omega) > 0$ and $r \in [0, 1]$. Using in the end again continuity for the last integral in (5.1.8), we conlude that $\|\varphi(1, \theta_{-1}\omega, u_0)\|_\beta < \infty$. The compact embedding of $D(A^\beta)$ into H (cf. Theorem 2.1.15) gives the statement. □

We complete this section with the existence theorem for the random attractor. This result follows from the previous two lemmas and the general Theorem 3.4.8. Moreover we use the Lemmas 3.3.3 and 3.4.10.

Theorem 5.1.5. *The random dynamical system φ generated by the solution of the RDE* (5.1.2) *possesses a unique random $\mathcal{D}$-attractor $\mathcal{A}(\omega), \omega \in \Omega$. Given the conjugation $T : \Omega \times H \to H$*

$$(\omega, x) \mapsto T(\omega, x) = xe^{z(\omega)},$$

SPDE (5.1.1) *possesses the conjugated RDS*

$$\begin{aligned} &\psi : \mathbb{R}_+ \times \Omega \times H \to H \\ &\psi(t, \omega, x) = \varphi\left(t, \omega, xe^{-z(\omega)}\right) e^{z(\theta_t\omega)}\,. \end{aligned}$$

Moreover, the conjugated RDS ψ possesses a unique random attractor given by

$$\widetilde{\mathcal{A}}(\omega) = \mathcal{A}(\omega)e^{z(\omega)} := \{xe^{z(\omega)} : x \in \mathcal{A}(\omega)\}, \quad \omega \in \Omega\,.$$

5.2 Existence of a global unstable manifold

This section starts with a short introduction to invariant manifolds. In particular we define the concept of a global unstable (Lipschitz) manifold. Then we consider the RDE (5.1.2) and truncate it to obtain a global unstable manifold inside the truncated ball. The final result is achieved by a fixed-point argument using the Lyapunov-Perron approach.
We begin with describing a necessary splitting of the Hilbert space to discuss stability resp. instability of manifolds.

Let $A = \Delta$ be a Laplacian on $H = L_2(K, \mu)$ where $(K, \mathcal{B}(K), \mu)$ is a σ-finite measure space of an (open) bounded subset of $\mathbb{R}^n$, $n \geq 1$. This operator is the generator of an analytic semigroup $\{S(t)\}_{t\in\mathbb{R}_+}$ and its spectrum shall be given by

$$\cdots < \lambda_i < \cdots < \lambda_2 < \lambda_1 < 0$$

for all $i \in \mathbb{N}$ and it has no finite accumulation point, $\lim_{i\to\infty} \lambda_i = -\infty$.
For any given positive number $k > |\lambda_1|$ there exists an index $N \in \mathbb{N}$ with

$$\lambda^{\mathbf{u}} := \lambda_N + k > 0, \qquad \lambda^{\mathbf{s}} := \lambda_{N+1} + k < 0\,. \tag{5.2.1}$$

These are the corresponding smallest positive and largest negative eigenvalues of the shifted spectrum. In the case an operator A has not a complete negative spectrum, we assume that $A+k$ is hyperbolic, i.e. 0 is not an eigenvalue of $A+k$.
The associated eigenfunctions of $\Delta + k$ are denoted by $v_i \in H$, $i \in \mathbb{N}$ and form an orthonormal basis of H. Since we avoid the eigenvalue zero the Hilbert space H is splitted into an unstable subspace $H^{\mathbf{u}}$ and a stable subspace $H^{\mathbf{s}}$ and in particular we have $H = H^u \oplus H^{\mathbf{s}}$. These subspaces are spanned by their corresponding eigenfunctions,

$$H^u = \operatorname{span}(\{v_1, v_2, ..., v_N\}) \qquad \text{and} \qquad H^{\mathbf{s}} = \operatorname{span}(\{v_{N+1}, ...\})\,. \tag{5.2.2}$$

Note that H^u is a finite dimensional and $H^{\mathbf{s}}$ is an infinite dimensional subspace of H. These spaces are invariant under the mapping $\{S(t)\}_{t\in\mathbb{R}}$ for $H^{\mathbf{u}}$ resp. $\{S(t)\}_{t\in\mathbb{R}_+}$ for H^s, see the following lemma. We introduce the corresponding orthogonal projections $\pi^{\mathbf{u}} : H \to H^{\mathbf{u}}$ and $\pi^{\mathbf{s}} : H \to H^{\mathbf{s}}$ by

$$x \mapsto \pi^{\mathbf{u}} x = \sum_{i=1}^{N} (x, v_i) v_i\,,$$
$$x \mapsto \pi^{\mathbf{s}} x = \sum_{i=N+1}^{\infty} (x, v_i) v_i\,.$$

Clearly we have $\pi^{\mathbf{s}} = \mathrm{Id}_H - \pi^{\mathbf{u}}$. We want to point out the following general lemma, see for instance [DLS04, p. 2112f.] or [OS13, p. 1667].

Lemma 5.2.1. *Let $A : D(A) \to H$ be a linear unbounded operator which is the generator of a C_0-semigroup S on H. The semigroup S satisfies the* exponential dichotomy *with exponents $\lambda^{\mathbf{u}} > \lambda^{\mathbf{s}}$ if the following statements hold*

(i) $\pi^{\mathbf{u}} S(t) = S(t)\pi^{\mathbf{u}}$ for $t \in \mathbb{R}$ and $\pi^{\mathbf{s}} S(t) = S(t)\pi^{\mathbf{s}}$ for $t \geq 0$,

and there exists a constant $M \geq 1$ such that,

(ii) for $t \leq 0$

$$\|\pi^{\mathbf{u}} S(t)\|_{L(H)} \leq M e^{\lambda^{\mathbf{u}} t}$$

and for $t \geq 0$

$$\|\pi^{\mathbf{s}} S(t)\|_{L(H)} \leq M e^{\lambda^{\mathbf{s}} t}\,.$$

Remark 5.2.2. The proof follows from some functional analytic considerations. Note in the case of $H^{\mathbf{u}}$ that the semigroup is still well-defined for $t \leq 0$, since the space is finite dimensional. In the following we set $M = 1$.

For the next part recall the Definition 3.4.1 of a random set and the Remark 3.4.9 on a positively invariant random set.

Definition 5.2.3. Let H be a divided Hilbert space like in (5.2.2). Assume that the cocycle $\varphi(t, \omega, \cdot) \in H$ is zero in zero for all $t \geq 0$ and $\omega \in \Omega$.
Then the set $\mathcal{M}(\omega)$ is called a *global unstable (Lipschitz) manifold* at zero, if the following properties are satisfied:

(i) $\mathcal{M}$ is positively invariant,

(ii) $\mathcal{M}$ is exponentially attracting, i.e for all $t \geq 0$, $\omega \in \Omega$ and $x \in \mathcal{M}(\omega)$, there exists $x_{-t} \in \mathcal{M}(\theta_{-t}\omega)$ such that,

$$\varphi(t, \theta_{-t}\omega, x_{-t}) = x$$

and x_{-t} tends exponentially fast to zero for $t \to \infty$.

(iii) $\mathcal{M}$ has a Lipschitz graph structure, i.e. there exists a function $h^{\mathbf{u}} : \Omega \times H^{\mathbf{u}} \to H^{\mathbf{s}}$ with $h^{\mathbf{u}}(\omega, 0) = 0$ for every $\omega \in \Omega$, such that

$$\mathcal{M}(\omega) = \{\xi + h^{\mathbf{u}}(\omega, \xi) : \xi \in H^{\mathbf{u}}\} .$$

The function $h^{\mathbf{u}}$ has to be measurable in its first component for fixed $\xi \in H^{\mathbf{u}}$ and Lipschitz continuous in the second component for fixed $\omega \in \Omega$.

Remark 5.2.4. Analogously we could define a global *stable* Lipschitz manifold. Then we would need to replace the mapping $h^{\mathbf{u}}$ by a function $h^{\mathbf{s}} : \Omega \times H^{\mathbf{s}} \to H^{\mathbf{u}}$ enjoying the same properties as in (iii) of the previous definition.

We begin studying the following differential equation for $t \in \mathbb{R}_+$ and $\omega \in \Omega$,

$$\begin{aligned} \frac{du}{dt} &= (\Delta + k)u + z(\theta_t\omega)u + F(\theta_t\omega, u) - ku, \quad t > 0, \\ u(0) &= u_0 \in H \end{aligned} \tag{5.2.3}$$

which is identical to (5.1.2) under consideration of (5.1.3). Let us define

$$G(\omega, u) := F(\omega, u) - ku \tag{5.2.4}$$

for all $\omega \in \Omega$ and $u \in H$. By the Lipschitz continuity of F the operator G is Lipschitz continuous with Lipschitz constant smaller or equal to $L + k > 0$ for every $\omega \in \Omega$.
In the previous section we assumed a linear growth of the nonlinearity F as stated in (A1). As introduced in the above definition we want to consider unstable manifolds at zero. Therefore we give a condition on the nonlinearity $\mathcal{F}$ which ensures the fixed-point at zero,

(A2) $$\mathcal{F}(0) = 0 \,,$$

and then of course $F(\omega, 0) = 0$ for every $\omega \in \Omega$. An application of the Banach fixed-point theorem yields that zero is a fixed-point, i.e. $\varphi(t, \omega, 0) = 0$ for every $t \in \mathbb{R}_+$ and $\omega \in \Omega$.

We introduce a modified equation from (5.2.3) by using a cut-off function similar to the ideas in [GALS10, Section 4, p.1650].
We define the following *cut-off* function $\sigma : \mathbb{R}_+ \to [0, 1]$ given by

$$\sigma(s) = \begin{cases} 1, & s \leq 1, \\ 2 - s, & 1 < s < 2, \\ 0, & s \geq 2 \,. \end{cases}$$

Let $\rho : \Omega \to (0, \infty)$ be a random variable tempered from below, which we use as the radius of the cut-off area. We modify the nonlinearity defined in (5.2.4) in the following way,

$$G_\rho(\omega, u) := G_{\rho(\omega)}(\omega, u) := G\left(\omega, u \cdot \sigma\left(\frac{\|u\|}{\rho(\omega)}\right)\right) \tag{5.2.5}$$

for $u \in H$ and $\omega \in \Omega$, where we often write for simplicity G_ρ instead of $G_{\rho(\omega)}$.
Let $f_{\rho(\omega)}(\omega, x) := x \cdot \sigma\left(\frac{\|x\|}{\rho(\omega)}\right)$ for $x \in H$. Then $\|f_\rho(\omega)(\omega, x)\| \leq \rho(\omega)$ and therefore we are in fact inside of a ball with radius $\rho(\omega)$, $\omega \in \Omega$.

We consider the following truncated equation for $\omega \in \Omega$

$$\begin{aligned} \frac{du}{dt} &= (\Delta + k)u + z(\theta_t\omega)u + G_\rho(\theta_t\omega, u), \quad t > 0, \\ u(0) &= u_0 \in H\,. \end{aligned} \tag{5.2.6}$$

We add two more standard assumptions to the original nonlinearity $\mathcal{F}$. Assume that

(A3) the Frechét derivative of $\mathcal{F}$ exists and

$$D\mathcal{F}(0) = k\mathrm{Id} \text{ for } k > |\lambda_1|\,,$$

(A4) the Frechét derivative $D\mathcal{F}(\cdot)$ is Lipschitz continuous with Lipschitz constant $L' > 0$.

The appearing constant k in the assumption (A3) is chosen sensible, since we are now able to divide H into the stable resp. unstable subspaces which are generated by the shifted spectrum of the Laplacian $\Delta + k$, see (5.2.1) and (5.2.2).

The following lemma is important for the next statements. A similar version can be found in [Ogr11, p.57] and [CDLS10, Lemma 4.1].

Lemma 5.2.5. *The nonlinearity $G_\rho(\omega, \cdot)$ is Lipschitz continuous with an ω-independent Lipschitz constant smaller or equal to $3(L + k)$. Moreover we have also a ω-dependent constant, i.e.*

$$\|G_\rho(\omega, u_1) - G_\rho(\omega, u_2)\| \leq 9L'e^{z(\omega)}\rho(\omega)\|u_1 - u_2\| \tag{5.2.7}$$

for all $\omega \in \Omega$ and $u_1, u_2 \in H$ and in particular we obtain

$$\|G_\rho(\omega, u)\| \leq 18L'e^{z(\omega)}\rho^2(\omega)$$

for all $\omega \in \Omega$ and $u \in H$.

Proof. We write ρ instead of $\rho(\omega)$ since we only make ω-wise estimates.
We remark apriori

$$\sigma\left(\frac{\|u\|}{\rho}\right) = \begin{cases} 1, & \|u\| \leq \rho \\ 2 - \frac{\|u\|}{\rho}, & \rho < \|u\| < 2\rho \\ 0, & \|u\| \geq 2\rho \end{cases}$$

and let us introduce the following abbreviation, $\widetilde{u} := u \cdot \sigma\left(\frac{\|u\|}{\rho}\right)$. Therefore for every $u_1, u_2 \in H$, $\omega \in \Omega$,

$$\|\widetilde{u}_1 - \widetilde{u}_2\| = \left\|u_1\sigma\left(\frac{\|u_1\|}{\rho}\right) - u_2\sigma\left(\frac{\|u_2\|}{\rho}\right)\right\| \leq L_\sigma\|u_1 - u_2\| \tag{5.2.8}$$

with $L_\sigma = 3$. To see this, we only verify the case $\|u_1\| \in (0, \rho)$ and $\|u_2\| \in (\rho, 2\rho)$. The other cases follow by similar arguments. We consider the following convex combination $u_3 := (1-\tau)u_1 + \tau u_2$, $\tau \in [0, 1]$ and choose τ such that $\|u_3\| = \rho$ then

$$\begin{aligned} \|\widetilde{u}_1 - \widetilde{u}_2\| &= \left\|u_1 - u_3 + u_3 - u_2\left(2 - \frac{\|u_2\|}{\rho}\right)\right\| \\ &\leq \|u_1 - u_3\| + \left\|u_3\left(2 - \frac{\|u_3\|}{\rho}\right) - u_2\left(2 - \frac{\|u_2\|}{\rho}\right)\right\|\,. \end{aligned} \tag{5.2.9}$$

Inside the last norm we insert the term $\pm u_3\left(2-\frac{\|u_2\|}{\rho}\right)$ and obtain

$$\begin{aligned}\left\|u_3\left(2-\frac{\|u_3\|}{\rho}\right)-u_2\left(2-\frac{\|u_2\|}{\rho}\right)\right\| &\leq \big|\|u_2\|-\|u_3\|\big|\cdot\frac{\|u_3\|}{\rho}+\left|2-\frac{\|u_2\|}{\rho}\right|\|u_3-u_2\| \\ &\leq \left(\frac{\|u_3\|}{\rho}+2-\frac{\|u_2\|}{\rho}\right)\|u_3-u_2\| \leq 2\|u_3-u_2\|\,.\end{aligned}$$

Combining this estimate with (5.2.9) we have with $\tau\in[0,1]$

$$\|\widetilde{u}_1-\widetilde{u}_2\|\leq\|u_1-u_3\|+2\|u_3-u_2\|\leq(|\tau|+2|1-\tau|)\,\|u_1-u_2\|\leq 2\|u_1-u_2\|\,.$$

The Lipschitz constant $L_\sigma=3$ arises from the case $\|u_1\|\in(\rho,2\rho)$ and $\|u_2\|\in(\rho,2\rho)$. Finally we conclude the Lipschitz continuity for G_ρ from the above discussion,

$$\|G_\rho(\omega,u_1)-G_\rho(\omega,u_2)\|=\|G(\omega,\widetilde{u}_1)-G(\omega,\widetilde{u}_2)\|\leq(L+k)L_\sigma\|u_1-u_2\|\,.$$

To obtain the estimate (5.2.7) we use the mean value theorem [Wou79, Section 12.1, Corollary 3, p.266],

$$\begin{aligned}\|G_\rho(\omega,u_1)-G_\rho(\omega,u_2)\| &= \|G(\omega,\widetilde{u}_1)-G(\omega,\widetilde{u}_2)\| \\ &\leq \sup_{\tau\in(0,1)}\|DG(\omega,\widetilde{u}_1+\tau[\widetilde{u}_1-\widetilde{u}_2])\|_{L(H)}\cdot\|\widetilde{u}_1-\widetilde{u}_2\|\,.\end{aligned}$$

Now we observe by our assumption (A3) and the definition of G,

$$DG(\omega,x)=DF(\omega,x)-k\mathrm{Id}=DF(\omega,x)-D\mathcal{F}(0)=D\mathcal{F}\left(xe^{z(\omega)}\right)-D\mathcal{F}(0)$$

for every $x\in H$ and $\omega\in\Omega$. Using this and the Lipschitz continuity assumed in (A4) we see that

$$\begin{aligned}\|G_\rho(\omega,u_1)-G_\rho(\omega,u_2)\| &\leq \sup_{\tau\in(0,1)}\|DG(\omega,\widetilde{u}_1+\tau[\widetilde{u}_1-\widetilde{u}_2])\|_{L(H)}\cdot\|\widetilde{u}_1-\widetilde{u}_2\| \\ &\leq L'e^{z(\omega)}\sup_{\tau\in(0,1)}\|\widetilde{u}_1+\tau[\widetilde{u}_1-\widetilde{u}_2]\|\cdot\|\widetilde{u}_1-\widetilde{u}_2\|\,.\end{aligned}$$

We use the Lipschitz continuity in (5.2.8) again to observe

$$\begin{aligned}\|G_\rho(\omega,u_1)-G_\rho(\omega,u_2)\| &\leq L'L_\sigma e^{z(\omega)}\sup_{\tau\in(0,1)}\|\widetilde{u}_1+\tau[\widetilde{u}_1-\widetilde{u}_2]\|\cdot\|u_1-u_2\| \\ &\leq 3L'L_\sigma e^{z(\omega)}\rho\|u_1-u_2\|=9L'e^{z(\omega)}\rho\|u_1-u_2\|\,,\end{aligned}$$

where the estimate for the supremum is obtained from $\|\widetilde{u}\|\leq\rho$, for every $u\in H$. As a consequence of $G_\rho(\omega,0)=0$ we have for $\|u\|\leq 2\rho$

$$\|G_\rho(\omega,u)\|\leq 9L'e^{z(\omega)}\rho\|u\|\leq 18L'e^{z(\omega)}\rho^2\,.$$

In the case $\|u\|>2\rho$ it is clear that $\|G_\rho(\omega,u)\|=\|G(\omega,0)\|=0$. □

Remark 5.2.6.

- We will use the ω-wise Lipschitz constant

$$\widetilde{L}(\omega):=9L'e^{z(\omega)}\rho(\omega)$$

  of the function $G_\rho(\omega,\cdot)$ in the next lemma. There we will choose a suitable radius $\rho(\omega)$ such that we can control the appearing gap condition between $\lambda^{\mathbf{u}}>0$ and $\lambda^{\mathbf{s}}<0$.

- Note that in the estimate (5.2.7) we have no dependence on the constant k responsable for the spectral shift nor on the original Lipschitz constant L.

Proposition 5.2.7. *Under the previous assumptions the truncated RDE* (5.2.6) *possesses, similar to Theorem 5.1.2, a unique mild solution given by*

$$u_\rho(t,\omega,u_0) = S(t)e^{\int_0^t z(\theta_s\omega)\,ds}u_0 + \int_0^t S(t-r)e^{\int_r^t z(\theta_s\omega)\,ds}G_\rho(\theta_r\omega, u_\rho(r,\omega,u_0))\,dr \tag{5.2.10}$$

for all $t \in \mathbb{R}_+$, $u_0 \in H$ and $\omega \in \Omega$. The solution operator (5.2.10) *generates a random dynamical system φ_ρ.*

We want to construct unstable manifolds based on the Lyapunov-Perron method applied on random evolution equations like in [OS13, Section 3]. This method gives us a fixed-point of an integral equation (in the subsequent function space). The projection of this fixed-point will later describe the graph structure of the unstable manifold, cf. (iii) in Definition 5.2.3. We introduce the Banach space

$$C_\gamma := \left\{ u \in C((-\infty,0],H) : \sup_{t\le 0} e^{-\gamma t - \int_0^t z(\theta_s\omega)\,ds}\|u(t)\| < \infty \right\}$$

for $\gamma > 0$ with the related norm

$$\|u\|_\gamma := \sup_{t\le 0} e^{-\gamma t - \int_0^t z(\theta_s\omega)\,ds}\|u(t)\|\,.$$

This space describes the continuous functions in H, such that the norms are decreasing with a rate controlled by $e^{\gamma t + \int_0^t z(\theta_s\omega)\,ds}$ when t goes to $-\infty$.

Lemma 5.2.8. *We define $\gamma := \frac{\lambda^{\mathbf{u}}+\lambda^{\mathbf{s}}}{2}$ and assume $\gamma > 0$. For any $c > 0$ that fulfills the gap condition $4c < \lambda^{\mathbf{u}} - \lambda^{\mathbf{s}}$, we can choose the radius of ball $\mathscr{B}(0,\rho(\omega))$ sufficiently small, i.e.*

$$\rho(\omega) \le \frac{e^{-z(\omega)}}{9L'}c \tag{5.2.11}$$

for each $\omega \in \Omega$. Then for $t \le 0$, $\omega \in \Omega$ and $\xi \in H^{\mathbf{u}}$ the function $J_{\omega,\xi}$ on C_γ given by the integral equation (Lyapunov-Perron transform)

$$\begin{aligned} J_{\omega,\xi}(g)[t] := {} & \pi^{\mathbf{u}}S(t)e^{\int_0^t z(\theta_s\omega)\,ds}\xi - \int_t^0 \pi^{\mathbf{u}}S(t-r)e^{\int_r^t z(\theta_s\omega)\,ds}G_\rho(\theta_r\omega, g(r))\,dr \\ & + \int_{-\infty}^t \pi^{\mathbf{s}}S(t-r)e^{\int_r^t z(\theta_s\omega)\,ds}G_\rho(\theta_r\omega, g(r))\,dr \end{aligned} \tag{5.2.12}$$

has a unique fixed-point denoted by $\Gamma(\omega,\xi) \in C_\gamma$, which is Lipschitz continuous in $\xi \in H^{\mathbf{u}}$ for each $\omega \in \Omega$.

Proof. Since we want to apply the Banach fixed-point Theorem ([AB06, Theorem 3.48, p.95]) we need to see that $J_{\omega,\xi}$ maps from C_γ into itself. This property follows from very similar arguments as the ones the subsequent proof of the contraction property uses. The main ideas are based on the equality $S(\cdot)\xi = \pi^{\mathbf{u}}S(\cdot)\xi$, $\xi \in H^{\mathbf{u}}$ for the first summand in equation (5.2.12), the exponential dichotomy of the semigroup S, see Lemma 5.2.1 and the Lipschitz continuity of G_ρ discussed in Lemma 5.2.5 in combination with the assumptions of this theorem.
For simplicity we only show the contraction property. For each $(\omega,\xi) \in \Omega \times H^{\mathbf{u}}$ and $g, \bar{g} \in C_\gamma$ we consider

$$\begin{aligned} \|J_{\omega,\xi}(g) - J_{\omega,\xi}(\bar{g})\|_\gamma = \Bigg\| & -\int_\cdot^0 \pi^{\mathbf{u}}S(\cdot - r)e^{\int_r^\cdot z(\theta_s\omega)\,ds}[G_\rho(\theta_r\omega, g(r)) - G_\rho(\theta_r\omega, \bar{g}(r))]\,dr \\ & + \int_{-\infty}^\cdot \pi^{\mathbf{s}}S(\cdot - r)e^{\int_r^\cdot z(\theta_s\omega)\,ds}[G_\rho(\theta_r\omega, g(r)) - G_\rho(\theta_r\omega, \bar{g}(r))]\,dr \Bigg\|_\gamma \,. \end{aligned}$$

We estimate the last norm by Lemma 5.2.1 and the triangle inequality, which yields to

$$\sup_{t\leq 0}\Bigg[\int_t^0 e^{\lambda^{\mathbf{u}}(t-r)}\cdot e^{-\gamma(t-r)}e^{-\gamma r-\int_0^r z(\theta_s\omega)\,ds}\|G_\rho(\theta_r\omega,g(r))-G_\rho(\theta_r\omega,\bar g(r))\|\,dr$$
$$+\int_{-\infty}^t e^{\lambda^{\mathbf{s}}(t-r)}\cdot e^{-\gamma(t-r)}e^{-\gamma r-\int_0^r z(\theta_s\omega)\,ds}\|G_\rho(\theta_r\omega,g(r))-G_\rho(\theta_r\omega,\bar g(r))\|\,dr\Bigg]\,.$$

Taking the Lipschitz estimate for G_ρ (5.2.7) into account we obtain

$$\sup_{t\leq 0}\left[\int_t^0 e^{(\lambda^{\mathbf{u}}-\gamma)(t-r)}9L'e^{z(\theta_r\omega)}\rho(\theta_r\omega)\,dr+\int_{-\infty}^t e^{(\lambda^{\mathbf{s}}-\gamma)(t-r)}9L'e^{z(\theta_r\omega)}\rho(\theta_r\omega)\,dr\right]\|g-\bar g\|_\gamma\,.$$

Now applying the condition on the radius ρ from (5.2.11) we have

$$\begin{aligned}\|J_{\omega,\xi}(g)-J_{\omega,\xi}(\bar g)\|_\gamma &\leq c\|g-\bar g\|_\gamma\sup_{t\leq 0}\left[\int_t^0 e^{(\lambda^{\mathbf{u}}-\gamma)(t-r)}\,dr+\int_{-\infty}^t e^{(\lambda^{\mathbf{s}}-\gamma)(t-r)}\,dr\right]\\ &\leq\frac{4c}{\lambda^{\mathbf{u}}-\lambda^{\mathbf{s}}}\|g-\bar g\|_\gamma\,.\end{aligned}$$

From the assumptions we have the consequence that the mapping $J_{\omega,\xi}$ is Lipschitz continuous with Lipschitz constant smaller than one. Hence $J_{\omega,\xi}$ is a uniform contraction w.r.t. the norm $\|\cdot\|_\gamma$. The Banach fixed-point Theorem states that there exists a *unique fixed-point* of $J_{\omega,\xi}$ denoted by $\Gamma(\omega,\xi)\in C_\gamma$, i.e. the sequence $\{(J_{\omega,\xi}(v))^n\}_{n\geq 0}$ converges to $\Gamma(\omega,\xi)$ as $n\to\infty$ for every $v\in C_\gamma$.

It remains to show that the fixed-point is Lipschitz continuous in the space $H^{\mathbf{u}}$. Therefore consider for all $\xi,\bar\xi\in H^{\mathbf{u}}$, $\omega\in\Omega$,

$$\begin{aligned}\|\Gamma(\omega,\xi)-\Gamma(\omega,\bar\xi)\|_\gamma &= \|J_{\omega,\xi}(\Gamma(\omega,\xi))-J_{\omega,\bar\xi}(\Gamma(\omega,\bar\xi))\|_\gamma\\ &\leq \|J_{\omega,\xi}(\Gamma(\omega,\xi))-J_{\omega,\xi}(\Gamma(\omega,\bar\xi))\|_\gamma+\|J_{\omega,\xi}(\Gamma(\omega,\bar\xi))-J_{\omega,\bar\xi}(\Gamma(\omega,\bar\xi))\|_\gamma\\ &\leq\frac{4c}{\lambda^{\mathbf{u}}-\lambda^{\mathbf{s}}}\|\Gamma(\omega,\xi)-\Gamma(\omega,\bar\xi)\|_\gamma+\|\pi^{\mathbf{u}}S(\cdot)e^{\int_0^\cdot z(\theta_s\omega)\,ds}(\xi-\bar\xi)\|_\gamma\,.\end{aligned}$$

Similar as before Lemma 5.2.1 gives us,

$$\|\Gamma(\omega,\xi)-\Gamma(\omega,\bar\xi)\|_\gamma\leq\frac{4c}{\lambda^{\mathbf{u}}-\lambda^{\mathbf{s}}}\|\Gamma(\omega,\xi)-\Gamma(\omega,\bar\xi)\|_\gamma+\|\xi-\bar\xi\|\,,$$

which implies

$$\|\Gamma(\omega,\xi)-\Gamma(\omega,\bar\xi)\|_\gamma\leq\frac{\lambda^{\mathbf{u}}-\lambda^{\mathbf{s}}}{\lambda^{\mathbf{u}}-\lambda^{\mathbf{s}}-4c}\|\xi-\bar\xi\|=L_\Gamma\|\xi-\bar\xi\|\,. \tag{5.2.13}$$

with $L_\Gamma:=\frac{\lambda^{\mathbf{u}}-\lambda^{\mathbf{s}}}{\lambda^{\mathbf{u}}-\lambda^{\mathbf{s}}-4c}\geq 1$. □

Next we want to prove the main result of this section. The solution of the truncated equation (5.2.6) generates a global unstable manifold at zero, like we introduced in Definition 5.2.3.

Theorem 5.2.9. *Let $\mathcal{F}:H\to H$ be a nonlinear function satisfying the assumptions (A2) – (A4) and further the gap condition in Lemma 5.2.8 holds.*
Then the truncated differential equation (5.2.6) *possesses a global unstable manifold $\mathcal{M}(\omega)$ in the sense of Definition 5.2.3. In particular*

$$\mathcal{M}(\omega)=\{\xi+h^{\mathbf{u}}(\omega,\xi),\ \xi\in H^{\mathbf{u}}\}$$

where the function $h^{\mathbf{u}}:\Omega\times H^{\mathbf{u}}\to H^{\mathbf{s}}$ has to be measurable for fixed $\xi\in H^{\mathbf{u}}$ and Lipschitz continuous for $\omega\in\Omega$.

Proof. The proof is adapted from [Ogr11, p.62f.]. We begin with the Lipschitz graph structure of the manifold, i.e. (iii) in Definition 5.2.3.
For all $\omega \in \Omega$ and $\xi \in H^{\mathbf{u}}$ we define the mapping

$$\begin{aligned} H^{\mathbf{s}} \ni h^{\mathbf{u}}(\omega,\xi) &:= \pi^{\mathbf{s}}\Gamma(\omega,\xi)[0] \\ &= \int_{-\infty}^{0} \pi^{\mathbf{s}} S(-r) e^{\int_r^0 z(\theta_s\omega)\,ds} G_\rho(\theta_r\omega, \Gamma(\omega,\xi)[r])\,dr, \end{aligned}$$

where $\Gamma(\omega,\xi)$ is the fixed-point of equation (5.2.12). Note that $h^{\mathbf{u}}(\omega,\xi) \in C_\gamma$.
The Lipschitz continuity of $h^{\mathbf{u}}(\omega,\cdot)$ follows from the Lipschitz continuity of the fixed-point $\Gamma(\omega,\cdot)$ and we prove the main ideas. Similiar to the previous proof we deduce

$$\begin{aligned} \|h^{\mathbf{u}}(\omega,\xi) - h^{\mathbf{u}}(\omega,\bar{\xi})\| &\le \int_{-\infty}^{0} \left\| \pi^{\mathbf{s}} S(-r) e^{\int_r^0 z(\theta_s\omega)\,ds} \left[G_\rho(\theta_r\omega, \Gamma(\omega,\xi)[0]) - G_\rho(\theta_r\omega, \Gamma(\omega,\bar{\xi})[0]) \right] \right\| dr \\ &\le \int_{-\infty}^{0} e^{\gamma r} e^{-\gamma r - \int_0^r z(\theta_s\omega)\,ds} e^{-\lambda^{\mathbf{s}} r} \times \\ &\qquad \times \left\| G_\rho(\theta_r\omega, \Gamma(\omega,\xi)[0]) - G_\rho(\theta_r\omega, \Gamma(\omega,\bar{\xi})[0]) \right\| \,dr\,. \end{aligned}$$

Now applying the Lipschitz continuity of G_ρ and using the continuity of $\Gamma(\omega,\xi)$ in C_γ, see (5.2.13), we have

$$\begin{aligned} \|h^{\mathbf{u}}(\omega,\xi) - h^{\mathbf{u}}(\omega,\bar{\xi})\| &\le \int_{-\infty}^{0} e^{-(\lambda^{\mathbf{s}}-\gamma)r} 9L' e^{z(\theta_r\omega)} \rho(\theta_r\omega) \cdot L_\Gamma \|\xi - \bar{\xi}\|\,dr \\ &\le L_\Gamma c \|\xi - \bar{\xi}\| \int_{-\infty}^{0} e^{-(\lambda^{\mathbf{s}}-\gamma)r}\,dr \le \frac{1}{2} L_\Gamma \|\xi - \bar{\xi}\| \end{aligned} \tag{5.2.14}$$

where we applied the condition on ρ and the gap condition from Lemma 5.2.8.
Since $\Gamma(\omega,0) = 0$ (in view of assumption (A2)) we conclude that $h^{\mathbf{u}}(\omega,0) = 0$.

We will briefly discuss the measurability of the function $h^{\mathbf{u}}(\cdot,\xi)$ for fixed $\xi \in H^{\mathbf{u}}$. By the definition of $h^{\mathbf{u}}$ it suffices to discuss the measurability of the mapping $\Omega \ni \omega \mapsto \Gamma(\omega,\xi)[0] \in H$ for fixed $\xi \in H^{\mathbf{u}}$. We obtain $\Gamma(\omega,\xi)$ as the limit of compositions of the functions $J_{\omega,\xi}(v)$ for arbitrary $v \in C_\gamma$. But the mapping $\omega \mapsto J_{\omega,\xi}(v)[t]$ given in (5.2.12) is $\mathscr{F}$-measurable, since the only ω-depending terms are $e^{z(\omega)}$ and $G_\rho(\omega, v)$. Due to their definitions these terms are random variables, if the other variables are fixed. Hence we conclude that $h^{\mathbf{u}}$ is measurable in its first component and we know already that it is continuous in its second component. Thus $h^{\mathbf{u}}$ is a Carathéodory function, see [AB06, Definition 4.50, p.153].
For the following we identify ξ with $\pi^{\mathbf{u}}u_0$, where $u_0 = u(0) \in H$ is the given initial data. We already know

$$\begin{aligned} \Gamma(\omega,\xi)[0] &= \pi^{\mathbf{u}}\Gamma(\omega,\xi)[0] + \pi^{\mathbf{s}}\Gamma(\omega,\xi)[0] \\ &= \xi + h^{\mathbf{u}}(\omega,\xi), \end{aligned} \tag{5.2.15}$$

so it's seems that the set of fixed-points $\Gamma(\omega,\xi)[0]$, $\xi \in H^{\mathbf{u}}$ describe the desired manifold. To show (i) and (ii) of the Definition 5.2.3 we need to understand the image of these fixed-points under the cut-off system φ_ρ given by (5.2.10).

<u>Positive invariance</u>
Next we prove the positive invariance of the manifold. Let φ_ρ be the RDS resp. the solution operator in (5.2.10) and we want to show,

$$\varphi_\rho(t,\omega,\Gamma(\omega,\xi)[0]) = \Gamma\big(\theta_t\omega, \pi^{\mathbf{u}}\varphi_\rho(t,\omega,\Gamma(\omega,\xi)[0])\big)[0] \tag{5.2.16}$$

for each $t \geq 0$ and $\omega \in \Omega$. We define the function

$$\Psi_{t,\omega}[s] := \begin{cases} \Gamma(\omega,\xi)[s+t], & s+t<0 \\ \varphi_\rho(s+t,\omega,\Gamma(\omega,\xi)[0]) & s+t \geq 0\,, \end{cases}$$

for all $t \geq 0$. The previous function allows us to prove the more general result,

$$\Psi_{t,\omega}[s] = \Gamma\big(\theta_t\omega, \pi^{\mathbf{u}}\varphi_\rho(t,\omega,\Gamma(\omega,\xi)[0])\big)[s] \tag{5.2.17}$$

for all $s \leq 0$. Setting $s = 0$ yields the desired positive invariance. Remember that Γ is the fixed point of the integral equation, hence for $s \leq 0$,

$$\begin{aligned} \Gamma(\omega,\xi)[s] = \pi^{\mathbf{u}} S(s) e^{\int_0^s z(\theta_q\omega)\,dq}\xi - \int_s^0 \pi^{\mathbf{u}} S(s-r) e^{\int_r^s z(\theta_q\omega)\,dq} G_\rho(\theta_r\omega, \Gamma(\omega,\xi)[r])\,dr \\ + \int_{-\infty}^s \pi^{\mathbf{s}} S(s-r) e^{\int_r^s z(\theta_q\omega)\,dq} G_\rho(\theta_r\omega,\Gamma(\omega,\xi)[r])\,dr\,. \end{aligned} \tag{5.2.18}$$

First case: $s < -t$.
We derive for the stable part

$$\begin{aligned} \pi^{\mathbf{s}}\Psi_{t,\omega}[s] &= \pi^{\mathbf{s}}\Gamma(\omega,\xi)[s+t] \\ &= \int_{-\infty}^{s+t} \pi^{\mathbf{s}} S(s+t-r) e^{\int_r^{s+t} z(\theta_q\omega)\,dq} G_\rho(\theta_r\omega,\Gamma(\omega,\xi)[r])\,dr \\ &= \int_{-\infty}^{s} \pi^{\mathbf{s}} S(s-r) e^{\int_r^{s} z(\theta_{q+t}\omega)\,dq} G_\rho(\theta_{r+t}\omega,\underbrace{\Gamma(\omega,\xi)[r+t]}_{=\Psi_{t,\omega}[r]})\,dr\,. \end{aligned}$$

For the unstable part we obtain

$$\begin{aligned} \pi^{\mathbf{u}}\Psi_{t,\omega}[s] &= \pi^{\mathbf{u}}\Gamma(\omega,\xi)[s+t] \\ &= \pi^{\mathbf{u}} S(s+t) e^{\int_0^{s+t} z(\theta_q\omega)\,dq}\xi - \int_{s+t}^0 \pi^{\mathbf{u}} S(s+t-r) e^{\int_r^{s+t} z(\theta_q\omega)\,dq} G_\rho(\theta_r\omega,\Gamma(\omega,\xi)[r])\,dr \\ &= \pi^{\mathbf{u}} S(s) e^{\int_0^s z(\theta_{q+t}\omega)\,dq} \\ &\qquad \cdot \left[S(t) e^{\int_0^t z(\theta_q\omega)\,dq}\xi + \int_0^t \pi^{\mathbf{u}} S(t-r) e^{\int_r^t z(\theta_q\omega)\,dq} G_\rho(\theta_r\omega,\varphi_\rho(r,\omega,\Gamma(\omega,\xi)[0]))\,dr \right] \\ &\quad - \int_0^t \pi^{\mathbf{u}} S(s+t-r) e^{\int_r^{s+t} z(\theta_q\omega)\,dq} G_\rho(\theta_r\omega,\varphi_\rho(r,\omega,\Gamma(\omega,\xi)[0]))\,dr \\ &\quad - \int_{s+t}^0 \pi^{\mathbf{u}} S(s+t-r) e^{\int_r^{s+t} z(\theta_q\omega)\,dq} G_\rho(\theta_r\omega,\Gamma(\omega,\xi)[r])\,dr\,, \end{aligned}$$

where we recognize in the square bracket the projected cocycle of our cut-off system. Thus

$$\begin{aligned} \pi^{\mathbf{u}}\Psi_{t,\omega}[s] &= \pi^{\mathbf{u}} S(s) e^{\int_0^s z(\theta_{q+t}\omega)\,dq} \pi^{\mathbf{u}}\varphi_\rho(t,\omega,\Gamma(\omega,\xi)[0]) \\ &\quad - \int_{-t}^0 \pi^{\mathbf{u}} S(s-r) e^{\int_r^s z(\theta_{q+t}\omega)\,dq} G_\rho(\theta_{r+t}\omega,\underbrace{\varphi_\rho(r+t,\omega,\Gamma(\omega,\xi)[0])}_{=\Psi_{t,\omega}[r]})\,dr \\ &\quad - \int_s^{-t} \pi^{\mathbf{u}} S(s-r) e^{\int_r^s z(\theta_{q+t}\omega)\,dq} G_\rho(\theta_{r+t}\omega,\underbrace{\Gamma(\omega,\xi)[r+t]}_{=\Psi_{t,\omega}[r]})\,dr \\ &= \pi^{\mathbf{u}} S(s) e^{\int_0^s z(\theta_{q+t}\omega)\,dq} \pi^{\mathbf{u}}\varphi_\rho(t,\omega,\Gamma(\omega,\xi)[0]) \\ &\quad - \int_s^0 \pi^{\mathbf{u}} S(s-r) e^{\int_r^s z(\theta_{q+t}\omega)\,dq} G_\rho(\theta_{r+t}\omega,\Psi_{t,\omega}[r])\,dr\,. \end{aligned}$$

Second case: $-t \le s \le 0$.
Similar to the first case we observe for the stable part,

$$\begin{aligned}
\pi^{\mathbf{s}}\Psi_{t,\omega}[s] &= \pi^{\mathbf{s}}\varphi_\rho(s+t,\omega,\Gamma(\omega,\xi)[0]) \\
&= S(s+t)e^{\int_0^{s+t} z(\theta_q\omega)\,dq}\pi^{\mathbf{s}}\Gamma(\omega,\xi)[0] \\
&\quad + \int_0^{s+t} \pi^{\mathbf{s}}S(s+t-r)e^{\int_r^{s+t} z(\theta_q\omega)\,dq}G_\rho(\theta_r\omega,\varphi_\rho(r,\omega,\Gamma(\omega,\xi)[0]))\,dr \\
&= S(s+t)e^{\int_0^{s+t} z(\theta_q\omega)\,dq}\int_{-\infty}^0 \pi^{\mathbf{s}}S(-r)e^{\int_r^0 z(\theta_q\omega)\,dq}G_\rho(\theta_r\omega,\Gamma(\omega,\xi)[r])\,dr \\
&\quad + \int_0^{s+t} \pi^{\mathbf{s}}S(s+t-r)e^{\int_r^{s+t} z(\theta_q\omega)\,dq}G_\rho(\theta_r\omega,\varphi_\rho(r,\omega,\Gamma(\omega,\xi)[0]))\,dr
\end{aligned}$$

and hence

$$\begin{aligned}
\pi^{\mathbf{s}}\Psi_{t,\omega}[s] &= \int_{-\infty}^{-t} \pi^{\mathbf{s}}S(s-r)e^{\int_r^s z(\theta_{q+t}\omega)\,dq}G_\rho(\theta_{r+t}\omega,\underbrace{\Gamma(\omega,\xi)[r+t]}_{=\Psi_{t,\omega}[r]})\,dr \\
&\quad + \int_{-t}^{s} \pi^{\mathbf{s}}S(s-r)e^{\int_r^s z(\theta_{q+t}\omega)\,dq}G_\rho(\theta_{r+t}\omega,\underbrace{\varphi_\rho(r+t,\omega,\Gamma(\omega,\xi)[0])}_{=\Psi_{t,\omega}[r]})\,dr \\
&= \int_{-\infty}^{s} \pi^{\mathbf{s}}S(s-r)e^{\int_r^s z(\theta_{q+t}\omega)\,dq}G_\rho(\theta_{r+t}\omega,\Psi_{t,\omega}[r])\,dr\,.
\end{aligned}$$

The unstable projection of the function $\Psi_{t,\omega}$ rearranges to

$$\begin{aligned}
\pi^{\mathbf{u}}\Psi_{t,\omega}[s] &= \pi^{\mathbf{u}}\varphi_\rho(s+t,\omega,\Gamma(\omega,\xi)[0]) \\
&= S(s+t)e^{\int_0^{s+t} z(\theta_q\omega)\,dq}\underbrace{\pi^{\mathbf{u}}\Gamma(\omega,\xi)[0]}_{=\xi} \\
&\quad + \int_0^{s+t} \pi^{\mathbf{u}}S(s+t-r)e^{\int_r^{s+t} z(\theta_q\omega)\,dq}G_\rho(\theta_r\omega,\varphi_\rho(r,\omega,\Gamma(\omega,\xi)[0]))\,dr\,.
\end{aligned}$$

We split the last summand into two integrals and factorize the semigroup and the exponential factor,

$$\begin{aligned}
\pi^{\mathbf{u}}\Psi_{t,\omega}[s] &= \pi^{\mathbf{u}}S(s)e^{\int_0^s z(\theta_{q+t}\omega)\,dq} \\
&\qquad \cdot\left[S(t)e^{\int_0^t z(\theta_q\omega)\,dq}\xi + \int_0^t \pi^{\mathbf{u}}S(t-r)e^{\int_r^t z(\theta_q\omega)\,dq}G_\rho(\theta_r\omega,\varphi_\rho(r,\omega,\Gamma(\omega,\xi)[0]))\,dr\right] \\
&\quad - \int_{s+t}^{t} \pi^{\mathbf{u}}S(s+t-r)e^{\int_r^{s+t} z(\theta_q\omega)\,dq}G_\rho(\theta_r\omega,\varphi_\rho(r,\omega,\Gamma(\omega,\xi)[0]))\,dr \\
&= \pi^{\mathbf{u}}S(s)e^{\int_0^s z(\theta_{q+t}\omega)\,dq}\pi^{\mathbf{u}}\varphi_\rho(t,\omega,\Gamma(\omega,\xi)[0]) \\
&\quad - \int_s^0 \pi^{\mathbf{u}}S(s-r)e^{\int_r^s z(\theta_{q+t}\omega)\,dq}G_\rho(\theta_{r+t}\omega,\underbrace{\varphi_\rho(r+t,\omega,\Gamma(\omega,\xi)[0])}_{=\Psi_{t,\omega}[r]})\,dr\,,
\end{aligned}$$

which finishes the second case.

We conclude from both cases

$$\begin{aligned}
\Psi_{t,\omega}[s] &= \pi^{\mathbf{s}}\Psi_{t,\omega}[s] + \pi^{\mathbf{u}}\Psi_{t,\omega}[s] \\
&= \pi^{\mathbf{u}} S(s) e^{\int_0^s z(\theta_{q+t}\omega)\,dq} \pi^{\mathbf{u}} \varphi_\rho(t,\omega,\Gamma(\omega,\xi)[0]) \\
&\quad - \int_s^0 \pi^{\mathbf{u}} S(s-r) e^{\int_r^s z(\theta_{q+t}\omega)\,dq} G_\rho(\theta_{r+t}\omega, \Psi_{t,\omega}[r])\,dr \\
&\quad + \int_{-\infty}^s \pi^{\mathbf{s}} S(s-r) e^{\int_r^s z(\theta_{q+t}\omega)\,dq} G_\rho(\theta_{r+t}\omega, \Psi_{t,\omega}[r])\,dr\,.
\end{aligned}$$

Therefore $\Psi_{t,\omega}[s]$ is a solution to the equation (5.2.12) for all $s \le 0$ and each $t \ge 0,\ \omega \in \Omega$. The statement (5.2.17) follows from the uniqueness of the solution.

Exponentially attracting

To show the exponentially attracting property of (ii), we mention that we can not simply go backwards in time, since a priori the cocycle φ_ρ is only defined for non-negative times. That is the reason why we need to find an element (x_{-t}) in the negative shifted manifold $\mathcal{M}(\theta_{-t}\omega)$ such that if we evolve the system until time t, we are inside of $\mathcal{M}(\omega)$ again.
Let us define in view of (5.2.18) for any $\omega \in \Omega$ and $\xi \in H^{\mathbf{u}}$

$$x_s^{\mathbf{u}} := \pi^{\mathbf{u}}\Gamma(\omega,\xi)[s] = \pi^{\mathbf{u}} S(s) e^{\int_0^s z(\theta_q\omega)\,dq}\xi - \int_s^0 \pi^{\mathbf{u}} S(s-r) e^{\int_r^s z(\theta_q\omega)\,dq} G_\rho(\theta_r\omega, \Gamma(\omega,\xi)[r])\,dr$$

for $s \le 0$. Similar to the proof of the positive invariance we show that for $t \ge 0$ and $s \le 0$

$$\Gamma(\omega,\xi)[s-t] = \Gamma(\theta_s\omega, x_s^{\mathbf{u}})[-t]\,. \tag{5.2.19}$$

We start with the left hand side

$$\begin{aligned}
\Gamma(\omega,\xi)[s-t] &= \pi^{\mathbf{u}} S(s-t) e^{\int_0^{s-t} z(\theta_q\omega)\,dq}\xi \\
&\quad - \int_{s-t}^0 \pi^{\mathbf{u}} S(s-t-r) e^{\int_r^{s-t} z(\theta_q\omega)\,dq} G_\rho(\theta_r\omega, \Gamma(\omega,\xi)[r])\,dr \\
&\quad + \int_{-\infty}^{s-t} \pi^{\mathbf{s}} S(s-t-r) e^{\int_r^{s-t} z(\theta_q\omega)\,dq} G_\rho(\theta_r\omega, \Gamma(\omega,\xi)[r])\,dr\,.
\end{aligned}$$

We split the first integral in two integrals and factorize the semigroup $S(-t)$ and the corresponding exponential term,

$$\begin{aligned}
\Gamma(\omega,\xi)[s-t] &= \pi^{\mathbf{u}} S(-t) e^{\int_0^{-t} z(\theta_{q+s}\omega)\,dq} \\
&\qquad \cdot \left[\pi^{\mathbf{u}} S(s) e^{\int_0^s z(\theta_q\omega)\,dq}\xi - \int_s^0 \pi^{\mathbf{u}} S(s-r) e^{\int_r^s z(\theta_q\omega)\,dq} G_\rho(\theta_r\omega, \Gamma(\omega,\xi)[r])\,dr\right] \\
&\quad - \int_{s-t}^s \pi^{\mathbf{u}} S(s-t-r) e^{\int_r^{s-t} z(\theta_q\omega)\,dq} G_\rho(\theta_r\omega, \Gamma(\omega,\xi)[r])\,dr \\
&\quad + \int_{-\infty}^{s-t} \pi^{\mathbf{s}} S(s-t-r) e^{\int_r^{s-t} z(\theta_q\omega)\,dq} G_\rho(\theta_r\omega, \Gamma(\omega,\xi)[r])\,dr\,.
\end{aligned}$$

We recognize the bracket as our $x_s^{\mathbf{u}}$. Further we shift the last two integrals by a substitution to

$$\begin{aligned}
\Gamma(\omega,\xi)[s-t] &= \pi^{\mathbf{u}} S(-t) e^{\int_0^{-t} z(\theta_{q+s}\omega)\,dq} x_s^{\mathbf{u}} \\
&\quad - \int_{-t}^0 \pi^{\mathbf{u}} S(-t-r) e^{\int_r^{-t} z(\theta_{q+s}\omega)\,dq} G_\rho(\theta_{r+s}\omega, \Gamma(\omega,\xi)[r+s])\,dr \\
&\quad + \int_{-\infty}^{-t} \pi^{\mathbf{s}} S(-t-r) e^{\int_r^{-t} z(\theta_{q+s}\omega)\,dq} G_\rho(\theta_{r+s}\omega, \Gamma(\omega,\xi)[r+s])\,dr\,.
\end{aligned}$$

Now the right-hand side solves the integral equation (5.2.12) at time $-t$, fiber $\theta_s\omega$ and with initial value $x_s^{\mathbf{u}} \in H^{\mathbf{u}}$. Due to uniqueness of this solution we showed the statement.

We want to express $x_s^{\mathbf{u}}$ for $s \leq 0$ with the solution of a finite dimensional random differential equation, which we then express by the unstable projection of the truncated differential equation. The shifted fixed-point at $x_{-t}^{\mathbf{u}}$ evaluated in zero will then be our x_{-t} of (ii) in the Definition 5.2.3 of the unstable manifold.

Consider the following RDE on the finite dimensional space $H^{\mathbf{u}}$ for $t \in \mathbb{R}$ and $\omega \in \Omega$,

$$\begin{aligned} \frac{du_\rho^{\mathbf{u}}}{dt} &= (\Delta + k)u_\rho^{\mathbf{u}} + z(\theta_t\omega)u_\rho^{\mathbf{u}} + \pi^{\mathbf{u}} G_\rho(\theta_t\omega, \Gamma(\theta_t\omega, u_\rho^{\mathbf{u}})[0]), \quad t \neq 0 \\ u_\rho^{\mathbf{u}}(0) &= \xi \in H^{\mathbf{u}}\,, \end{aligned} \tag{5.2.20}$$

where in general $u_\rho^{\mathbf{u}} \neq \pi^{\mathbf{u}} u_\rho$. But we show below for certain initial values this characterization is possible. The Lipschitz continuity of G_ρ and $\Gamma(\omega, u)$ guarantees the uniqueness of the solution and defines a random dynamical system $\varphi_\rho^{\mathbf{u}}$ in $H^{\mathbf{u}}$. The solution is given by the variation of constants formula

$$\begin{aligned} u_\rho^{\mathbf{u}}(t,\omega,\xi) &= \varphi_\rho^{\mathbf{u}}(t,\omega,\xi) \\ &= \pi^{\mathbf{u}} S(t) e^{\int_0^t z(\theta_s\omega)\,ds}\xi + \int_0^t \pi^{\mathbf{u}} S(t-r) e^{\int_r^t z(\theta_q\omega)\,dq} G_\rho\big(\theta_r\omega, \Gamma(\theta_r\omega, \varphi_\rho^{\mathbf{u}}(r,\omega,\xi))[0]\big)\,dr \end{aligned} \tag{5.2.21}$$

for *all* $t \in \mathbb{R}$. Combining the definition of $x_s^{\mathbf{u}}$ with the shown equality (5.2.19) we have

$$x_s^{\mathbf{u}} = \pi^{\mathbf{u}} S(s) e^{\int_0^s z(\theta_q\omega)\,dq}\xi - \int_s^0 \pi^{\mathbf{u}} S(s-r) e^{\int_r^s z(\theta_q\omega)\,dq} G_\rho(\theta_r\omega, \Gamma(\theta_r\omega, x_r^{\mathbf{u}})[0])\,dr$$

for $s \leq 0$. Hence it is clear that $\mathbb{R}_- \ni s \mapsto x_s^{\mathbf{u}}$ is a solution to (5.2.20) and due to the uniqueness of the solution we obtain

$$\varphi_\rho^{\mathbf{u}}(-t,\omega,\xi) = x_{-t}^{\mathbf{u}}$$

for $t \geq 0$. The cocycle property of φ leads to

$$\varphi_\rho^{\mathbf{u}}(t, \theta_{-t}\omega, x_{-t}^{\mathbf{u}}) = \varphi_\rho^{\mathbf{u}}(0,\omega,\xi) = \xi \qquad \text{for each } t \geq 0\,. \tag{5.2.22}$$

Now we show $\varphi_\rho^{\mathbf{u}}(t,\omega,\xi) = \pi^{\mathbf{u}}\varphi_\rho(t,\omega,\Gamma(\omega,\xi)[0])$, $t \geq 0$. The unstable part of the truncated solution (5.2.10) for $t \geq 0$ becomes in view of the positive invariance (5.2.16),

$$\begin{aligned} &\pi^{\mathbf{u}}\varphi_\rho(t,\omega,\Gamma(\omega,\xi)[0]) \\ &= \pi^{\mathbf{u}} S(t) e^{\int_0^t z(\theta_s\omega)\,ds}\Gamma(\omega,\xi)[0] + \int_0^t \pi^{\mathbf{u}} S(t-r) e^{\int_r^t z(\theta_s\omega)\,ds} G_\rho(\theta_r\omega, \varphi_\rho(r,\omega,\Gamma(\omega,\xi)[0]))\,dr \\ &= \pi^{\mathbf{u}} S(t) e^{\int_0^t z(\theta_s\omega)\,ds}\xi + \\ &\quad + \int_0^t \pi^{\mathbf{u}} S(t-r) e^{\int_r^t z(\theta_s\omega)\,ds} G_\rho\Big(\theta_r\omega, \Gamma\big(\theta_r\omega, \pi^{\mathbf{u}}\varphi_\rho(r,\omega,\Gamma(\omega,\xi)[0])\big)[0]\Big)\,dr\,, \end{aligned}$$

and is therefore also a solution to (5.2.20). Again by uniqueness of the solution we conclude

$$\varphi_\rho^{\mathbf{u}}(t,\omega,\xi) = \pi^{\mathbf{u}}\varphi_\rho(t,\omega,\Gamma(\omega,\xi)[0]) \qquad t \geq 0\,. \tag{5.2.23}$$

Combining the last equation with equation (5.2.22) we observe for $t \geq 0$,

$$\xi = \varphi_\rho^{\mathbf{u}}(t,\theta_{-t}\omega, x_{-t}^{\mathbf{u}}) = \pi^{\mathbf{u}}\varphi_\rho(t,\theta_{-t}\omega, \Gamma(\theta_{-t}\omega, x_{-t}^{\mathbf{u}})[0])\,. \tag{5.2.24}$$

We summarize the result. For every $x \in \mathcal{M}(\omega)$, $\omega \in \Omega$ we find a $\xi \in H^{\mathbf{u}}$ with $x = \Gamma(\omega, \xi)[0]$ represented as in (5.2.15). By the previous discussion we notice, that for every $\xi \in H^{\mathbf{u}}$ there exists $x^{\mathbf{u}}_{-t} \in H^{\mathbf{u}}$ for $t \geq 0$, such that (5.2.24) holds. Now we define

$$x_{-t} := \Gamma(\theta_{-t}\omega, x^{\mathbf{u}}_{-t})[0]$$

for every $t \geq 0$, $\omega \in \Omega$ and therefore we achieve

$$x = \Gamma(\omega, \xi)[0] = \Gamma(\omega, \pi^{\mathbf{u}}\varphi_\rho(t, \theta_{-t}\omega, x_{-t}))[0] = \varphi_\rho(t, \theta_{-t}\omega, x_{-t}) \,,$$

where for the last equality we applied the positive invariance of $\mathcal{M}$. In particular we replace ω by $\theta_{-t}\omega$ and ξ by $x^{\mathbf{u}}_{-t}$ in (5.2.16), i.e.

$$\varphi_\rho(t, \theta_{-t}\omega, \Gamma(\theta_{-t}\omega, x^{\mathbf{u}}_{-t})[0]) = \Gamma\big(\omega, \pi^{\mathbf{u}}\varphi_\rho(t, \theta_{-t}\omega, \Gamma(\theta_{-t}\omega, x^{\mathbf{u}}_{-t})[0]\big)[0] \,.$$

What remains to show is that x_{-t} tends exponentially fast to zero when $t \to \infty$. Choosing independently of each other $-t \equiv 0$ and $s \equiv -t$ in (5.2.19) we obtain the following equality

$$x_{-t} = \Gamma(\theta_{-t}\omega, x^{\mathbf{u}}_{-t})[0] = \Gamma(\omega, \xi)[-t] \,.$$

As a preparation we remember (5.1.6). We have that for every $\varepsilon > 0$, $\gamma > \varepsilon$ there exists a $t_0(\omega, \varepsilon) > 0$ such that if $|t| \geq t_0(\omega, \varepsilon)$, then

$$\int_0^{-t} z(\theta_s\omega)\, ds = -t \cdot \left[\frac{1}{-t}\left(\int_0^{-t} z(\theta_s\omega)\, ds\right)\right] \leq t\varepsilon.$$

Therefore we estimate x_{-t} in norm, that is

$$\begin{aligned} \|\Gamma(\omega, \xi)[-t]\| &= \|\Gamma(\omega, \xi)[-t] - \Gamma(\omega, 0)[-t]\| \\ &\leq e^{-\gamma t + \int_0^{-t} z(\theta_s\omega)\, ds}\|\Gamma(\omega, \xi) - \Gamma(\omega, 0)\|_\gamma \\ &\leq e^{-(\gamma-\varepsilon)t} L_\Gamma \|\xi\|, \qquad |t| \geq t_0(\omega, \varepsilon) \end{aligned}$$

where $\Gamma(\omega, 0) = 0$ is the fixed-point of (5.2.12) for $\xi = 0$. Hence $\lim\limits_{t\to\infty} x_{-t} = 0$ exponentially fast. □

Remark 5.2.10. As a result of the last theorem we can represent the unstable manifold as a set of fixed-points in H. Following the equality (5.2.15) we observe for all $\omega \in \Omega$

$$\begin{aligned} \mathcal{M}(\omega) &= \{\xi + h^{\mathbf{u}}(\omega, \xi),\ \xi \in H^{\mathbf{u}}\} \\ &= \{\Gamma(\omega, \xi)[0],\ \xi \in H^{\mathbf{u}}\} \,. \end{aligned}$$

We note additionally that although we only consider global unstable manifolds at zero, it's not restrictive. In the most cases we can deduce manifolds at some point from the ones at zero, see [GALS10, p.1643] and [Arn10, Lemma 7.2.1, p.310].

Without proof we want to state a result giving us the unstable manifold at zero for a conjugated RDS. For more details we refer to [DLS04, Theorem 3.3, p.962]

Theorem 5.2.11. *Let $\mathcal{M}$ be a global unstable manifold at zero for an RDS φ generated by an RDE like in* (5.2.3)*. Suppose $T : \Omega \times H \to H$, $T(\omega, x) = xe^{z(\omega)}$ is the associated conjugation with inverse $T^{-1}(\omega, x) = xe^{-z(\omega)}$. Then the conjugated RDS ψ possesses the global unstable manifold,*

$$\widetilde{\mathcal{M}}(\omega) := T(\omega, \mathcal{M}(\omega))$$

at zero for $\omega \in \Omega$.

5.3 Local unstable manifolds and the lower estimate

In this section we show the existence of a local unstable (Lipschitz) manifold for our originial differential equation

$$\begin{aligned} \frac{du}{dt} &= (\Delta + k)u + z(\theta_t\omega)u + F(\theta_t\omega, u) - ku, \qquad t > 0, \\ u(0) &= u_0 \in H \end{aligned}$$

which is the transformed equation of the beginning in Chapter 5. The following definition can be found e.g. in [GALS10, Definition 2.6, p.1642].

Definition 5.3.1. Let $\varphi : \mathbb{R}_+ \times \Omega \times H \to H$ be a random dynamical system with a fixed-point in zero, i.e. $\varphi(t, \omega, 0) = 0$. The random set $\mathcal{M}^{\mathrm{loc}}(\omega)$ is called a *local unstable manifold* at zero, if the following properties are satisfied:

(i) $\mathcal{M}^{\mathrm{loc}}(\omega)$ has a graph-like structure on a closed ball $\mathscr{B}(0, r(\omega)) \subset H^{\mathbf{u}}$ with radius $r : \Omega \to \mathbb{R}_+$, i.e. there exists a function $h^{\mathbf{u}}(\omega, \cdot) : \mathscr{B}(0, r(\omega)) \to H^{\mathbf{s}}$ with $h^{\mathbf{u}}(\omega, 0) = 0$ such that,

$$\mathcal{M}^{\mathrm{loc}}(\omega) = \left\{\xi + h^{\mathbf{u}}(\omega, \xi) : \xi \in \mathscr{B}(0, r(\omega))\right\}.$$

(ii) For every $t \geq 0$, $\omega \in \Omega$ and $x \in \mathcal{M}^{\mathrm{loc}}(\omega) \cap V(\omega)$ there exists $x_{-t} \in \mathcal{M}^{\mathrm{loc}}(\theta_{-t}\omega)$ such that,

$$\varphi(t, \theta_{-t}\omega, x_{-t}) = x$$

and x_{-t} tends exponentially fast to zero for $t \to \infty$. $V(\omega) \subset H$ is a random set and a neighborhood of zero.

(iii) $\mathcal{M}^{\mathrm{loc}}(\omega)$ is locally positively invariant, i.e. for all $\omega \in \Omega$ and $x \in \mathcal{M}^{\mathrm{loc}}(\omega)$

$$\lim_{\|x\| \to 0} \tau(\omega, x) = \infty$$

where

$$\tau(\omega, x) := \inf\{t \geq 0 : \varphi(t, \omega, x) \notin \mathcal{M}^{\mathrm{loc}}(\theta_t\omega)\}.$$

Remark 5.3.2. The property (iii) is called locally positively invariant since if we are close enough to zero with our initial value x, then the image under φ stays inside the (shifted) manifold $\mathcal{M}(\theta_t\omega)$ for a finite time $t > 0$.

We collect some useful results in the following lemmas. All used notations are introduced in the previous section.

Lemma 5.3.3. *If $s \leq 0$, $\omega \in \Omega$ and $\xi \in H^{\mathbf{u}}$, then*

$$\|\Gamma(\omega, \xi)[s]\| \leq L_\Gamma e^{\gamma s + \int_0^s z(\theta_r\omega)\, dr} \|\xi\|.$$

In particular we have that $\|\Gamma(\omega, \xi)[0]\| \leq L_\Gamma \|\xi\|$.

Proof. The last result is a consequence of the first inequality setting $s \equiv 0$. We use the Lipschitz continuity of the fixed-point to show the first inequality,

$$e^{-\gamma s - \int_0^s z(\theta_r\omega)\, dr} \|\Gamma(\omega, \xi)[s]\| \leq \sup_{t \leq 0} e^{-\gamma t - \int_0^t z(\theta_r\omega)\, dr} \|\Gamma(\omega, \xi)[t]\| = \|\Gamma(\omega, \xi)\|_\gamma \leq L_\Gamma \|\xi\|$$

for $s \leq 0$. $\square$

Lemma 5.3.4. *Let $\omega \in \Omega$, $u_0 \in H$ and $\rho : \Omega \to (0,\infty)$ be a tempered from below random variable. If $\|\varphi_\rho(t,\omega,u_0)\| \le \rho(\theta_t\omega)$ then*

$$\varphi_\rho(t,\omega,u_0) \equiv \varphi(t,\omega,u_0)$$

for every $t \ge 0$.

Proof. The statement follows from the definition of the truncated nonlinearity in (5.2.5). We conclude from the assumption that

$$G_\rho(\theta_t\omega, u_\rho(t,\omega,u_0)) = G(\theta_t\omega, u_\rho(t,\omega,u_0))\,.$$

The definition of G implies that the mild solution of equation (5.2.6) is also a solution to the original equation (5.1.2). By the uniqueness of the solution we obtain the result. □

We will need also the following lemma, which is similar to [GALS10, Lemma 5.1, p.1659].

Lemma 5.3.5. *Let $\omega \in \Omega$, $C > 0$ some constant and $r : \Omega \to (0,\infty)$ some positive mapping which is tempered from below. Then the mapping $R : \Omega \to (0,\infty)$,*

$$R(\omega) := \frac{1}{C}\inf_{t\le 0} e^{-\gamma t - \int_0^t z(\theta_s\omega)\,ds} r(\theta_t\omega)$$

is tempered from below and satisfies $CR(\omega)e^{\gamma s + \int_0^s z(\theta_q\omega)\,dq} \le r(\theta_s\omega)$ for all $s \le 0$.

Proof. The definition of *temperedness from below* in (3.2.11) says us that for every $\varepsilon > 0$ there exists $s_0(\omega,\varepsilon) > 0$ such that for all $s \le -s_0(\omega,\varepsilon)$

$$\ln^+\left(\frac{1}{r(\theta_s\omega)}\right) \le -\varepsilon s\,. \tag{5.3.1}$$

Note that if $r(\theta_s\omega) \ge 1$ (5.3.1) reads $0 \le -\varepsilon s$ for all $s \le 0$. In the second case $r(\theta_s\omega) < 1$ we observe for $s \le -s_0(\omega,\varepsilon)$

$$r(\theta_s\omega) \ge e^{\varepsilon s}\,.$$

We need to show the same for R in the case of $s \le -s_0(\omega,\varepsilon)$. The definition of R yields to

$$\begin{aligned} R(\theta_s\omega) &= \frac{1}{C}\inf_{t'\le 0} e^{-\gamma t' - \int_0^{t'} z(\theta_{r+s}\omega)\,dr} r(\theta_{t'+s}\omega) \;= \frac{1}{C}\inf_{t\le s} e^{-\gamma(t-s) - \int_0^{t-s} z(\theta_{r+s}\omega)\,dr} r(\theta_t\omega) \\ &= \frac{1}{C} e^{\gamma s}\inf_{t\le s} e^{-\gamma t - \int_0^{t-s} z(\theta_{r+s}\omega)\,dr} r(\theta_t\omega)\,. \end{aligned}$$

The latter integral in the exponent can be rewritten,

$$-\int_0^{t-s} z(\theta_{r+s}\omega)\,dr = -\int_s^t z(\theta_r\omega)\,dr = -\int_0^t z(\theta_r\omega)\,dr + \int_0^s z(\theta_r\omega)\,dr\,.$$

For every $\varepsilon > 0$ we define $v_0(\omega) := v_0(\omega,\varepsilon) := \max(s_0(\omega,\varepsilon), t_0(\omega,\varepsilon))$, where $t_0(\omega,\varepsilon) > 0$ has been chosen such that for all $t \le s \le -t_0(\omega,\varepsilon)$,

$$\frac{1}{t}\int_0^t z(\theta_r\omega)\,dr \ge -\varepsilon \quad \text{and} \quad \frac{1}{s}\int_0^s z(\theta_r\omega)\,dr \le \varepsilon\,,$$

according to (iii) in Lemma 5.1.1.
We remember in the case that $r(\theta_t\omega) < 1$, we obtain $r(\theta_t\omega) \ge e^{\varepsilon t}$ for all $t \le -v_0(\omega)$. Note that in the other case we have $r(\theta_t\omega) \ge 1 \ge e^{\varepsilon t}$ for all $t \le 0$. Hence we deduce

$$R(\theta_s\omega) \ge \frac{1}{C} e^{\gamma s}\inf_{t\le s} e^{-\gamma t + 2\varepsilon t + \varepsilon s} = \frac{1}{C} e^{\varepsilon s} e^{\gamma s}\inf_{t\le s} e^{(-\gamma+2\varepsilon)t}\,.$$

If we choose $\varepsilon < \frac{\gamma}{2}$ then $\inf_{t \leq s} e^{(-\gamma+2\varepsilon)t} = e^{(-\gamma+2\varepsilon)s}$. Therefore it is clear that

$$R(\theta_s\omega) \geq \frac{1}{C} e^{3\varepsilon s} \text{ for all } s \leq -v_0(\omega)\,.$$

The temperedness follows then from

$$\ln^+\left(\frac{1}{R(\theta_s\omega)}\right) \leq \ln^+ C - 3\varepsilon s\,.$$

for $s \leq -v_0(\omega)$. To end the proof of the lemma we show the claimed inequality.
By the definition of R we conclude for any fixed $s \leq 0$

$$\begin{aligned} CR(\omega)e^{\gamma s + \int_0^s z(\theta_r\omega)\,dr} &= \inf_{t \leq 0} e^{-\gamma t - \int_0^t z(\theta_r\omega)\,dr} r(\theta_t\omega) e^{\gamma s + \int_0^s z(\theta_r\omega)\,dr} \\ &\leq e^{-\gamma s - \int_0^s z(\theta_r\omega)\,dr} r(\theta_s\omega) e^{\gamma s + \int_0^s z(\theta_r\omega)\,dr} = r(\theta_s\omega)\,. \end{aligned}$$

□

For convenience we suppose in the remaining part of this chapter that

$$\rho(\omega) = \frac{e^{-z(\omega)}}{9L'} c, \tag{5.3.2}$$

being the largest value w.r.t. the condition (5.2.11). Note in particular that ρ is a tempered random variable both from above and from below.

The next question is how to choose the set that could be our local unstable manifold and respects the cut-off we did?

To answer this question we have to choose a suitable radius of the ball in $H^{\mathbf{u}}$ such that the local manifold has a Lipschitz graph-like structure as stated in (i) of Definition 5.3.1. We aim to use the global unstable manifold for the truncated equation since every point therein has already a graph-like representation.
First we notice the following. If we choose an element $\xi \in \mathscr{B}_{H^{\mathbf{u}}}(0, r(\omega))$, where $\mathscr{B}_{H^{\mathbf{u}}}(0, r(\omega))$ denotes the closed ball in $H^{\mathbf{u}}$ for some radius $r > 0$, then according to the proof of Theorem 5.2.9 the fixed-point $\Gamma(\omega, \xi)[0] \in \mathcal{M}(\omega)$ and

$$\Gamma(\omega, \xi)[0] = \xi + h^{\mathbf{u}}(\omega, \xi) = \varphi_\rho(t, \theta_{-t}\omega, x_{-t})$$

for every $t \geq 0$ and $\omega \in \Omega$.
Provided $\|\Gamma(\omega, \xi)[0]\| = \|\varphi_\rho(t, \theta_{-t}\omega, x_{-t})\| \leq \rho(\omega)$ we obtain under consideration of Lemma 5.3.4 that

$$\varphi_\rho(t, \theta_{-t}\omega, x_{-t}) = \varphi(t, \theta_{-t}\omega, x_{-t})$$

for every $t \geq 0$. This implies that every fixed-point of the Lyapunov-Perron transform (5.2.12) (building the global unstable manifold), which is bounded in norm by $\rho(\omega)$, is a fixed-point for the original equation (5.1.2) and it can be expressed by the graph Lipschitz structure of the global manifold.

We use these ideas as a motivation to choose the radius $r(\omega)$ of the ball in $H^{\mathbf{u}}$ such that we stay inside the ball $\mathscr{B}(0, \rho(\omega)) \subset H$. Since the spaces $H^{\mathbf{u}}$ and $H^{\mathbf{s}}$ are orthogonal we conclude for $\xi \in \mathscr{B}_{H^{\mathbf{u}}}(0, r(\omega))$ that

$$\|\xi + h^{\mathbf{u}}(\omega, \xi)\|^2 = \|\xi\|^2 + \|h^{\mathbf{u}}(\omega, \xi)\|^2 \leq \|\xi\|^2 + \frac{L_\Gamma^2}{4}\|\xi\|^2 \leq r(\omega)^2 + \frac{L_\Gamma^2}{4} r(\omega)^2$$

where we used the Lipschitz continuity of $h^{\mathbf{u}}(\omega, \cdot)$ in (5.2.14).
If we want to stay inside the $\rho(\omega)$-ball we need to fulfill at least

$$\rho(\omega)^2 \overset{!}{=} r(\omega)^2 + \frac{L_\Gamma^2}{4} r(\omega)^2$$

which is equivalent to

$$r(\omega) = \frac{2}{\sqrt{4 + L_\Gamma^2}} \rho(\omega) =: \rho^{\mathbf{u}}(\omega)\,. \tag{5.3.3}$$

The maximal value of the radius $\rho^{\mathbf{u}}(\omega)$ can only be $\frac{2}{\sqrt{5}}\rho(\omega)$, since $L_\Gamma \geq 1$. A possible situation could be the following.

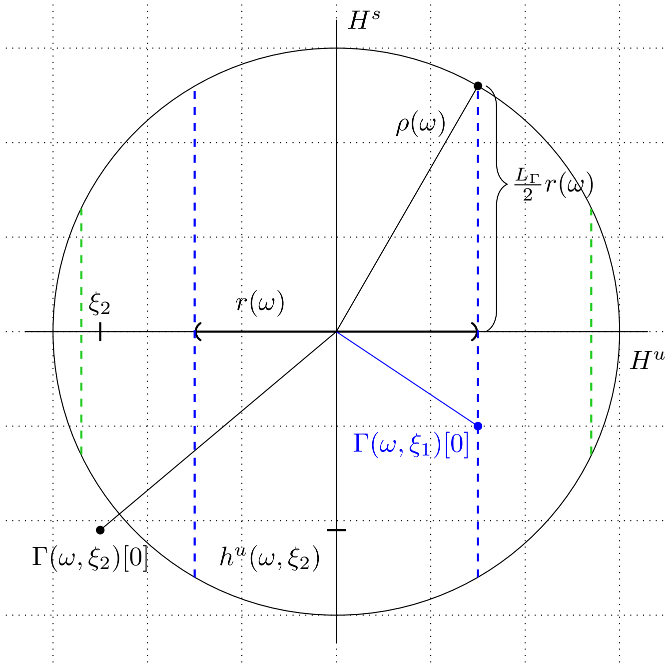


Figure 5.1: For a given radius $\rho(\omega)$ and a fixed Lipschitz constant $L_\Gamma \geq 1$ we obtain a zone (the part of the circle between the blue dashed lines) of possible fixed-points $\Gamma(\omega, \xi)[0]$ with $\|\xi\| \leq r(\omega)$. The fixed-point $\Gamma(\omega, \xi_1)[0]$ is a choice with a maximal norm, $\|\xi_1\| = r(\omega)$. If L_Γ gets smaller, but still larger or equal to one, then we observe e.g. the area between the green dashed lines. For fixed $\rho(\omega)$ the fixed-point $\Gamma(\omega, \xi_2)[0]$ is never a suitable choice.

Now we define for each $\omega \in \Omega$ the set

$$\mathcal{M}^{\mathrm{loc}}(\omega) := \{\Gamma(\omega, \xi)[0],\ \xi \in \mathscr{B}_{H^{\mathbf{u}}}(0, \rho^{\mathbf{u}}(\omega))\}\,.$$

Obviously $\mathcal{M}^{\mathrm{loc}}(\omega) \subset \mathcal{M}(\omega)$, which denotes the unstable global manifold of the truncated manifold. We will show that $\mathcal{M}^{\mathrm{loc}}(\omega)$ is our local unstable manifold.
Further let us define

$$R^{\mathbf{u}}(\omega) := \frac{1}{L_\Gamma} \inf_{t \leq 0} e^{-\gamma t - \int_0^t z(\theta_s \omega)\, dr} \rho^{\mathbf{u}}(\theta_t \omega) \tag{5.3.4}$$

such that by Lemma 5.3.5

$$L_\Gamma R^{\mathbf{u}}(\omega) e^{\gamma s + \int_0^s z(\theta_r \omega)\, dr} \leq \rho^{\mathbf{u}}(\theta_s \omega) \text{ for } s \leq 0.$$

In particular note that

$$R^{\mathbf{u}}(\omega) \leq \frac{1}{L_\Gamma}\rho^{\mathbf{u}}(\omega) \leq \frac{2}{\sqrt{5}L_\Gamma}\rho(\omega)\,. \tag{5.3.5}$$

The radius $R^{\mathbf{u}}(\omega)$ will be of particular interest for the next theorem. It will garantuee that in the end the solution stays inside the local unstable manifold.

Now we are prepared to formulate and show one of the main statements of the section. We follow the proof in [Ogr11, p.70 ff.].

Theorem 5.3.6. *The unstable global manifold for the truncated equation of Theorem 5.2.9 is a local unstable manifold for the original differential equation* (5.2.3) *in the sense of Definition 5.3.1.*

Proof. The set given by

$$\mathcal{M}^{\text{loc}}(\omega) := \{\Gamma(\omega,\xi)[0],\ \xi \in \mathscr{B}_{H^{\mathbf{u}}}(0,\rho^{\mathbf{u}}(\omega))\} \tag{5.3.6}$$

for $\omega \in \Omega$ has due to (5.2.15) already the requested local graph structure of (i), Definition 5.3.1. The next part to show is (ii). Choose a random neighborhood $V(\omega)$ of zero such that the unstable projection of $\mathcal{M}^{\text{loc}}(\omega) \cap V(\omega)$ onto $H^{\mathbf{u}}$ includes the ball with radius $R^{\mathbf{u}}(\omega)$ given by (5.3.4). As we will show below the radius $R^{\mathbf{u}}(\omega)$ is chosen such that for the complete trajectory $\|\varphi(t,\theta_{-t}\omega,x_{-t})\| \leq \rho(\omega)$ and $x_{-t} \in \mathcal{M}^{\text{loc}}(\theta_{-t}\omega)$ for $t \geq 0$. Moreover x_{-t} goes to zero for $t \to \infty$, which describes in a pullback sense the attracting property of the unstable manifold in zero.

Let $x = \Gamma(\omega,\xi)[0] \in \mathcal{M}^{\text{loc}}(\omega) \cap V(\omega)$ and $\xi \in \mathscr{B}_{H^{\mathbf{u}}}(0,R^{\mathbf{u}}(\omega))$, where $R^{\mathbf{u}}(\omega) \leq \rho^{\mathbf{u}}(\omega)$. From the discussion prior to this theorem we know that $x \in \mathcal{M}(\omega)$ and by Theorem 5.2.9 there exists $x_{-t} \in \mathcal{M}(\theta_{-t}\omega)$ such that $\varphi_\rho(t,\theta_{-t}\omega,x_{-t}) = x$ for every $t \geq 0$ and $\omega \in \Omega$. Moreover Lemma 5.3.3 tells us that $\|x\| = \|\Gamma(\omega,\xi)[0]\| \leq L_\Gamma\|\xi\| \leq \rho(\omega)$.
According to Lemma 5.3.4 we conclude for all $t \geq 0$ and $\omega \in \Omega$

$$\varphi_\rho(t,\theta_{-t}\omega,x_{-t}) = \varphi(t,\theta_{-t}\omega,x_{-t}) = x\,.$$

It remains to prove that $x_{-t} \in \mathcal{M}^{\text{loc}}(\theta_{-t}\omega)$. Recall the definition

$$x_{-t} := \Gamma(\theta_{-t}\omega, x^{\mathbf{u}}_{-t})[0]$$

where $x^{\mathbf{u}}_{-t} = \pi^{\mathbf{u}}\Gamma(\omega,\xi)[-t]$ and by (5.2.19) $\Gamma(\theta_{-t}\omega,x^{\mathbf{u}}_{-t})[0] = \Gamma(\omega,\xi)[-t]$. To show the statement we need to prove that $x^{\mathbf{u}}_{-t} \in \mathscr{B}_{H^{\mathbf{u}}}(0,\rho^{\mathbf{u}}(\theta_{-t}\omega))$ for fixed $\omega \in \Omega$.
Thanks to the choice that $\xi \in \mathscr{B}_{H^{\mathbf{u}}}(0,R^{\mathbf{u}}(\omega))$ and applying the Lemmas 5.3.3 and 5.3.5, we obtain

$$\|x^{\mathbf{u}}_{-t}\| = \|\pi^{\mathbf{u}}\Gamma(\omega,\xi)[-t]\| \leq \|\Gamma(\omega,\xi)[-t]\| \leq L_\Gamma e^{-\gamma t + \int_0^{-t} z(\theta_r\omega)\,dr}\|\xi\| \leq \rho^{\mathbf{u}}(\theta_{-t}\omega)\,.$$

We will show property (iii) of Definition 5.3.1 in the following. In [Ogr11] this property is only shown for discrete times. Following the approach therein we face an intersection taken over a discrete time set which we can not extend to the case of continuous time, since an uncountable intersection could be trivial.

Let $x = \Gamma(\omega,\xi)[0] \in \mathcal{M}^{\text{loc}}(\omega)$ und fix a time interval $[0,T]$ for $T > 0$. Then if $\xi \in H^{\mathbf{u}}$ is close to zero, we know according to Lemma 5.3.3 that $\|x\|$ is also close to zero. Before we specify how small we have to choose ξ, we discuss the necessary condition such that $\varphi(t,\omega,x) \in \mathcal{M}^{\text{loc}}(\theta_t\omega)$ for $t \in [0,T]$.
We begin with the truncated RDS in (5.2.10) and use the positive invariance given in (5.2.16),

$$\|\varphi_\rho(t,\omega,\Gamma(\omega,\xi)[0])\| = \|\Gamma(\theta_t\omega,\pi^{\mathbf{u}}\varphi_\rho(t,\omega,\Gamma(\omega,\xi)[0]))[0]\|\,.$$

Equation (5.2.23) tells us that we can replace the projected RDS $\pi^{\mathbf{u}}\varphi_\rho$ with the solution of the equation (5.2.20) $\varphi_\rho^{\mathbf{u}}$. Therefore we obtain

$$\|\Gamma(\theta_t\omega, \pi^{\mathbf{u}}\varphi_\rho(t,\omega,\Gamma(\omega,\xi)[0]))[0]\| = \|\Gamma(\theta_t\omega, \varphi_\rho^{\mathbf{u}}(t,\omega,\xi))[0]\| .$$

Together with Lemma 5.3.3 we infer

$$\|\Gamma(\theta_t\omega, \varphi_\rho^{\mathbf{u}}(t,\omega,\xi))[0]\| \le L_\Gamma \|\varphi_\rho^{\mathbf{u}}(t,\omega,\xi)\| .$$

Now if we choose $\xi \in H^{\mathbf{u}}$ so small that $\varphi_\rho^{\mathbf{u}}(t,\omega,\xi) \in \mathscr{B}_{H^{\mathbf{u}}}(0, R^{\mathbf{u}}(\theta_t\omega))$, then we observe

$$\|\varphi_\rho(t,\omega,\Gamma(\omega,\xi)[0])\| \le L_\Gamma \|\varphi_\rho^{\mathbf{u}}(t,\omega,\xi)\| \le \rho^{\mathbf{u}}(\theta_t\omega) , \tag{5.3.7}$$

in view of (5.3.5). Consequently, Lemma 5.3.4 implies

$$\varphi_\rho(t,\omega,\Gamma(\omega,\xi)[0]) = \varphi(t,\omega,\Gamma(\omega,\xi)[0]) .$$

for all $t \in [0,T]$. To determine how small we have to choose ξ notice the following discussion. Estimations of the mild solution in (5.2.21) lead for $t \in [0,T]$ to

$$\|\varphi_\rho^{\mathbf{u}}(t,\omega,\xi)\| \le e^{\lambda^{\mathbf{u}}t + \int_0^t z(\theta_s\omega)\,ds}\|\xi\| + L_\Gamma c \int_0^t e^{\lambda^{\mathbf{u}}(t-r) + \int_r^t z(\theta_s\omega)\,ds}\|\varphi_\rho^{\mathbf{u}}(r,\omega,\xi)\|\,dr ,$$

where we applied the usual estimates we used frequently in this chapter. We substitute $w(r) := \exp\left\{-\lambda^{\mathbf{u}}t - \int_0^t z(\theta_s\omega)\,ds\right\} \cdot \|\varphi_\rho^{\mathbf{u}}(r,\omega,x)\|$ for $r \in [0,t]$ and apply the Gronwall Lemma 3.3.1 to obtain

$$\|\varphi_\rho^{\mathbf{u}}(t,\omega,\xi)\| \le \|\xi\| e^{(L_\Gamma c + \lambda^{\mathbf{u}})t + \int_0^t z(\theta_s\omega)\,ds} .$$

Notice that the right-hand side of the latter estimate is not monotonic in t due to the Ornstein-Uhlenbeck process. Since the process has continuous paths on $[0,T]$, it attains its maxima resp. minima. Let us denote $M_T(\omega) = \max_{t\in[0,T]} |z(\theta_t\omega)|$. Moreover with $L_\Gamma, c, \lambda^{\mathbf{u}} > 0$ we conclude

$$\sup_{t\in[0,T]} \|\varphi_\rho^{\mathbf{u}}(t,\omega,\xi)\| \le \|\xi\| e^{(L_\Gamma c + \lambda^{\mathbf{u}} + M_T(\omega))T} .$$

We define $C_T(\omega) := L_\Gamma c + \lambda^{\mathbf{u}} + M_T(\omega)$.
Now consider on the other hand the radii $R^{\mathbf{u}}(\theta.\omega) : [0,T] \to \mathbb{R}_+ \setminus \{0\}$. They are continuous in time along the definition (5.3.4) and since $\rho(\theta.\omega) = \frac{c}{9L'} e^{-z(\theta.\omega)}$ is continuous. Hence there exists $t_0 := t_0(\omega) \in [0,T]$ such that

$$0 < R^{\mathbf{u}}(\theta_{t_0}\omega) = \min_{t\in[0,T]} R^{\mathbf{u}}(\theta_t\omega)$$

is well-defined. If we choose $\xi \in H^{\mathbf{u}}$ appropriately, that is

$$\|\xi\| \le R^{\mathbf{u}}(\theta_{t_0}\omega) e^{-C_T(\omega)T} =: \delta(\omega, T)$$

then

$$\varphi_\rho^{\mathbf{u}}(t,\omega,\xi) \in \mathscr{B}_{H^{\mathbf{u}}}(0, R^{\mathbf{u}}(\theta_t\omega))$$

for all $t \in [0,T]$. The following picture should emphasize the choice of ξ.

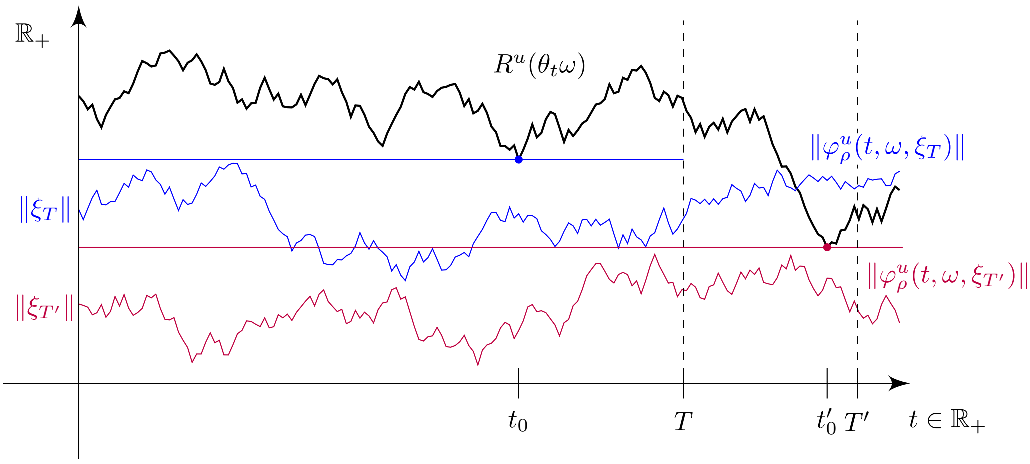


Figure 5.2: Expanding the time interval gives us a possible smaller minimal radius $R^{\mathbf{u}}(\theta_{t_0}\omega)$ and so we need to adjust the radius of initial value $\|\xi\|$. Please note that in this picture we only present the behavior of norms and radii.

Therefore we obtain that the inequality (5.3.7) holds for every $t \in [0, T]$ whenever $\xi \in \mathscr{B}_{\mathbb{R}_+}(0, \delta(\omega, T))$. Hence the definition of the conjectured local manifold in (5.3.6) tells us for $t \in [0, T]$ that

$$\varphi(t, \omega, \Gamma(\omega, \xi)[0]) \in \mathcal{M}^{\text{loc}}(\theta_t\omega)\,.$$

Recall from the Definition 5.3.1 with $x = \Gamma(\omega, \xi)[0]$ that

$$\tau(\omega, x) = \inf\{t \geq 0\,|\,\varphi(t, \omega, \Gamma(\omega, \xi)[0]) \notin \mathcal{M}^{\text{loc}}(\theta_t\omega)\}\,.$$

For every $T > 0$ there exists $\delta(\omega, T) > 0$ such that for $\xi \in H^{\mathbf{u}}$ with $\|\xi\| \leq \delta(\omega, T)$ we have

$$\tau(\omega, x) \geq \tau^{\mathbf{u}}(\omega, \xi) := \inf\{t \geq 0\,|\,\varphi^{\mathbf{u}}_\rho(t, \omega, \xi) \notin \mathscr{B}_{H^{\mathbf{u}}}(0, R^{\mathbf{u}}(\theta_t\omega))\} > T\,,$$

where the last inequality implies $\lim_{\|\xi\|\to 0} \tau^{\mathbf{u}}(\omega, \xi) = \infty$. Finally we conclude

$$\lim_{\|x\|\to 0} \tau(\omega, x) = \infty\,.$$

□

The upcoming theorem will show the connection between the local unstable manifold and the random attractor.

Theorem 5.3.7. *Assume the random dynamical system φ of the differential equation* (5.1.2) *possesses a local unstable manifold*

$$\mathcal{M}^{loc}(\omega) = \{\Gamma(\omega, \xi)[0],\ \xi \in \mathscr{B}_{H^{\mathbf{u}}}(0, \rho^{\mathbf{u}}(\omega))\}$$

in the sense of Definition 5.3.1 and a random attractor $\mathcal{A}(\omega)$, $\omega \in \Omega$ like we showed in Theorem 5.1.5.
In particular there exists a neighborhood of zero $V(\omega) \subset H$, such that

$$\left(\mathcal{M}^{loc}(\omega) \cap V(\omega)\right) \subseteq \mathcal{A}(\omega),$$

for all $\omega \in \Omega$.

Proof. Let $x \in \mathcal{M}^{\mathrm{loc}}(\omega) \cap V(\omega)$ for each $\omega \in \Omega$, then we know by (ii) of Definition 5.3.1 $\forall t \geq 0$ there exists a $x_{-t} \in \mathcal{M}^{\mathrm{loc}}(\theta_{-t}\omega)$ such that

$$\varphi(t, \theta_{-t}\omega, x_{-t}) = x\,.$$

According to Definition 3.4.5 the random attractor $\mathcal{A}$ given by Theorem 5.1.5 attracts every tempered random set $D \in \mathcal{D}$.
We prove that $\mathcal{M}^{\mathrm{loc}}(\omega)$ is a tempered set, i.e. for every $\omega \in \Omega$

$$\lim_{t\to\pm\infty} \frac{1}{|t|} \ln^+ \left(\sup_{x\in\mathcal{M}^{\mathrm{loc}}(\theta_t\omega)} \|x\| \right) = 0\,,$$

cf. (3.4.1). We use the definition of $\rho^{\mathbf{u}}(\theta_t\omega)$ in (5.3.3) such that for every $x \in \mathcal{M}^{\mathrm{loc}}(\theta_t\omega)$

$$\|x\| = \sqrt{\|\xi\|^2 + \|h^{\mathbf{u}}(\theta_t\omega, \xi)\|^2} \leq \left(1 + \frac{L_\Gamma^2}{4}\right)^{\frac{1}{2}} \rho^{\mathbf{u}}(\theta_t\omega) = \rho(\theta_t\omega)\,.$$

Therefore for every $t \in \mathbb{R}$

$$\sup_{x\in\mathcal{M}^{\mathrm{loc}}(\theta_t\omega)} \|x\| \leq \rho(\theta_t\omega)\,.$$

Since ρ, given by (5.3.2), is a tempered (from above) random variable, we conclude that

$$\lim_{t\to\infty} \mathrm{dist}(\varphi(t, \theta_{-t}\omega, \mathcal{M}^{\mathrm{loc}}(\theta_{-t}\omega)), \mathcal{A}(\omega)) = 0\,. \tag{5.3.8}$$

We know by the definition of the local manifold $\mathcal{M}^{\mathrm{loc}}(\omega) \cap V(\omega) \subseteq \varphi(t, \theta_{-t}\omega, \mathcal{M}^{\mathrm{loc}}(\theta_{-t}\omega))$ for all $t \geq 0$.
The limit in (5.3.8) can be rewritten. For every $\varepsilon > 0$ there exists $t_0 = t_0(\omega, \varepsilon) \geq 0$ such that $\forall t \geq t_0$

$$\mathrm{dist}(\varphi(t, \theta_{-t}\omega, \mathcal{M}^{\mathrm{loc}}(\theta_{-t}\omega)), \mathcal{A}(\omega)) < \varepsilon\,.$$

The property of the supremum in the Hausdorff semi-distance (defined in (3.4.3)) provides

$$\begin{aligned}
\mathrm{dist}(\mathcal{M}^{\mathrm{loc}}(\omega) \cap V(\omega), \mathcal{A}(\omega)) &\leq \mathrm{dist}(\varphi(t, \theta_{-t}\omega, \mathcal{M}^{\mathrm{loc}}(\theta_{-t}\omega)), \mathcal{A}(\omega)) < \varepsilon\,,\\
\mathrm{dist}(\mathcal{M}^{\mathrm{loc}}(\omega) \cap V(\omega), \mathcal{A}(\omega)) &= \sup_{x\in\mathcal{M}^{\mathrm{loc}}(\omega)\cap V(\omega)} \inf_{y\in\mathcal{A}(\omega)} \|y - x\| < \varepsilon\,.
\end{aligned}$$

Hence for every $x \in \mathcal{M}^{\mathrm{loc}}(\omega) \cap V(\omega)$ we have $\inf_{y\in\mathcal{A}(\omega)} \|x - y\| < \varepsilon$. Now choose $0 < \varepsilon_n = \frac{1}{2n}$, then for every $n \in \mathbb{N}$ there exists $y_n \in \mathcal{A}(\omega)$ with

$$\|x - y_n\| - \frac{1}{2n} < \inf_{y\in\mathcal{A}(\omega)} \|x - y\| < \frac{1}{2n}$$

and so $\lim_{n\to\infty} y_n = x$. Thus the claim $\mathcal{M}^{\mathrm{loc}}(\omega) \cap V(\omega) \subseteq \mathcal{A}(\omega)$ is proven. $\square$

Finally we prepare the desired result, i.e. the lower bound for the Hausdorff dimension of the random attractor. Therefore we will use the above lemma. We begin with some geometric discussions.
The monotonicity property, that every measure possesses (see e.g. [Rog98, p.2]) implies under consideration of the last theorem, that

$$\dim_H(\mathcal{M}^{\mathrm{loc}}(\omega) \cap V(\omega)) \leq \dim_H(\mathcal{A}(\omega))\,. \tag{5.3.9}$$

Now we want to find an *open* subset of $\mathcal{M}^{\mathrm{loc}}(\omega) \cap V(\omega)$. At first, we observe since $V(\omega)$ is a neighborhood of zero in the normed vector space H, therefore there exists an open ball

$$\mathscr{B}^{\mathrm{op}}(0, \mathcal{R}(\omega)) = \{x \in H : \|x\| < \mathcal{R}(\omega)\} \subset V(\omega)$$

for each $\omega \in \Omega$ with the positive radius $\mathcal{R} : \Omega \to (0, \infty)$. Hence we have $\mathcal{M}^{\mathrm{loc}}(\omega) \cap \mathscr{B}^{\mathrm{op}}(0, \mathcal{R}(\omega)) \subset \mathcal{M}^{\mathrm{loc}}(\omega) \cap V(\omega)$.
We define the set

$$\mathcal{O}^{\mathrm{loc}}(\omega) := \{\xi + h^{\mathbf{u}}(\omega, \xi) : \xi \in \mathscr{B}^{\mathrm{op}}_{H^{\mathbf{u}}}(0, \rho^{\mathbf{u}}(\omega))\},$$

which is an open subset of $\mathcal{M}^{\mathrm{loc}}(\omega)$. Indeed using the graph structure of the set $\mathcal{O}^{\mathrm{loc}}(\omega)$ we notice at first that the continuous mapping $p : \mathcal{O}^{\mathrm{loc}}(\omega) \to \mathscr{B}^{\mathrm{op}}_{H^{\mathbf{u}}}(0, \rho^{\mathbf{u}}(\omega))$, $p(x) = \pi^{\mathbf{u}}(x)$ is bijective. Then we obtain that the pre-image of the open set $\mathscr{B}^{\mathrm{op}}_{H^{\mathbf{u}}}(0, \rho^{\mathbf{u}}(\omega))$, i.e. $\mathcal{O}^{\mathrm{loc}}(\omega)$, is open (see [AB06, Definition 2.26, p.36]).
Therefore we have now

$$\mathcal{O}^{\mathrm{loc}}(\omega) \cap \mathscr{B}^{\mathrm{op}}(0, \mathcal{R}(\omega)) \subset \mathcal{M}^{\mathrm{loc}}(\omega) \cap V(\omega), \tag{5.3.10}$$

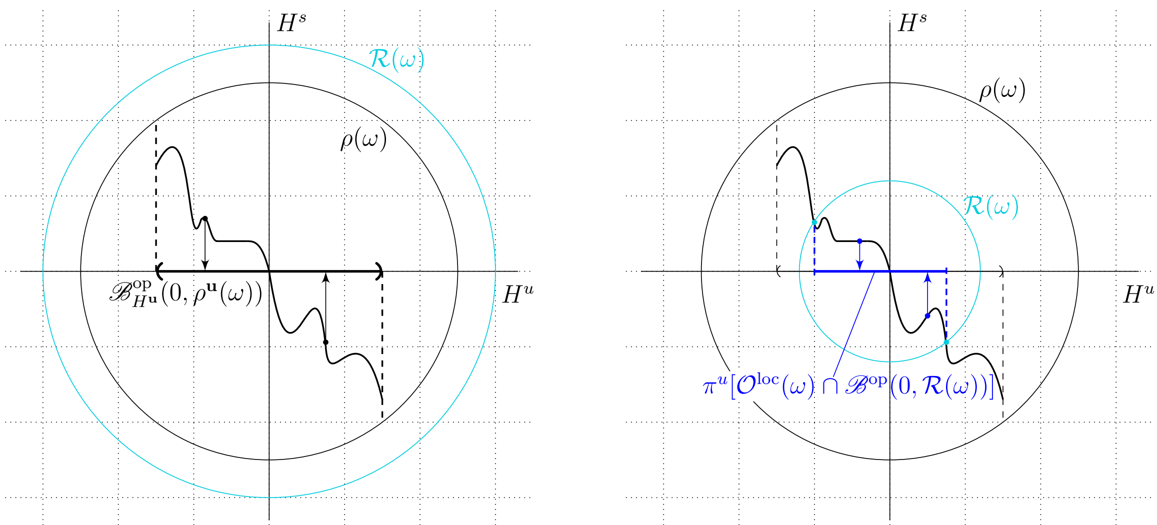
where the set $\mathcal{O}^{\mathrm{loc}}(\omega) \cap \mathscr{B}^{\mathrm{op}}(0, \mathcal{R}(\omega))$ is open as intersection of two open sets. To obtain a lower bound on the Hausdorff dimension we want to project this set onto the unstable subspace $H^{\mathbf{u}}$. In general two situations are of particular interest.
If $\mathcal{R}(\omega) \geq \rho(\omega)$ (cf. Figure 5.3a), then $\mathcal{O}^{\mathrm{loc}}(\omega) \subseteq \mathscr{B}^{\mathrm{op}}(0, \mathcal{R}(\omega))$ and therefore

$$\pi^{\mathbf{u}}\left[\mathcal{O}^{\mathrm{loc}}(\omega) \cap \mathscr{B}^{\mathrm{op}}(0, \mathcal{R}(\omega))\right] = \mathscr{B}^{\mathrm{op}}_{H^{\mathbf{u}}}(0, \rho^{\mathbf{u}}(\omega)) .$$

If on the other hand $\mathcal{R}(\omega) < \rho(\omega)$, cf. Figure 5.3b, then the projection of $\mathcal{O}^{\mathrm{loc}}(\omega) \cap \mathscr{B}^{\mathrm{op}}(0, \mathcal{R}(\omega))$ onto $H^{\mathbf{u}}$ does not necessary need to be a ball in $H^{\mathbf{u}}$, however the projection is still an open set in $H^{\mathbf{u}}$, which follows by a similar argument as before. Define the bijective continuous mapping

$$\begin{aligned} &q : \pi^{\mathbf{u}}\left[\mathcal{O}^{\mathrm{loc}}(\omega) \cap \mathscr{B}^{\mathrm{op}}(0, \mathcal{R}(\omega))\right] \to \mathcal{O}^{\mathrm{loc}}(\omega) \cap \mathscr{B}^{\mathrm{op}}(0, \mathcal{R}(\omega)), \\ &q(\xi) = \xi + h^{\mathbf{u}}(\omega, \xi), \end{aligned}$$

as the inverse mapping of the projection $\pi^{\mathbf{u}}$. Then, since $\mathcal{O}^{\mathrm{loc}}(\omega) \cap \mathscr{B}^{\mathrm{op}}(0, \mathcal{R}(\omega))$ is open, the pre-image $\pi^{\mathbf{u}}\left[\mathcal{O}^{\mathrm{loc}}(\omega) \cap \mathscr{B}^{\mathrm{op}}(0, \mathcal{R}(\omega))\right]$ is open as well.



(a) An example for the case $\mathcal{R}(\omega) \geq \rho(\omega)$ with the one-dimensional subspace $H^{\mathbf{u}}$.

(b) The case when $\mathcal{R}(\omega) < \rho(\omega)$ and the projection of $\mathcal{O}^{\mathrm{loc}}(\omega) \cap \mathscr{B}^{\mathrm{op}}(0, \mathcal{R}(\omega))$ on an open subset of $H^{\mathbf{u}}$.

Figure 5.3: An overview of the cases that can possible occur, when projecting the open subset of the local manifold.

Applying a mapping theorem for metric spaces shown by C.A.Rogers in [Rog98, p.53] to the spaces $(H, \|\cdot\|)$ and $(H^{\mathbf{u}}, \|\cdot\|)$ and (5.3.10) we have

$$\begin{aligned}\dim_H \left\{\pi^{\mathbf{u}} \left[\mathcal{O}^{\mathrm{loc}}(\omega) \cap \mathscr{B}^{\mathrm{op}}(0, \mathcal{R}(\omega))\right]\right\} &\leq \dim_H\{\mathcal{O}^{\mathrm{loc}}(\omega) \cap \mathscr{B}^{\mathrm{op}}(0, \mathcal{R}(\omega))\} \\ &\leq \dim_H(\mathcal{M}^{\mathrm{loc}}(\omega) \cap V(\omega))\,.\end{aligned}$$

We know that there exists an isometric isomorphism $L : H^{\mathbf{u}} \to \mathbb{R}^N$ between the N-dimensional vector space $H^{\mathbf{u}}$ and $\mathbb{R}^N$, see e.g. [Alt16, Theorem 9.8 (Note), p.294]. Hence the image of the open subset $\pi^{\mathbf{u}}\left[\mathcal{O}^{\mathrm{loc}}(\omega) \cap \mathscr{B}^{\mathrm{op}}(0, \mathcal{R}(\omega))\right] \subset H^{\mathbf{u}}$ under the mapping L is an open subset of $\mathbb{R}^N$. The following lemma presents the connection between the Hausdorff dimension and the dimension of the vector space $\mathbb{R}^N$. The statement can be found for example in [Fal90, p.29] and its proof makes use of the relation described in (2.4.2).

Lemma 5.3.8. *Let F be an open subset of $\mathbb{R}^N$, $N \in \mathbb{N}$. Then $\dim_H F = N$.*

Following the last lemma, we obtain in combination with (5.3.9) for each $\omega \in \Omega$,

$$N = \dim_H \left\{\pi^{\mathbf{u}} \left[\mathcal{O}^{\mathrm{loc}}(\omega) \cap \mathscr{B}^{\mathrm{op}}(0, \mathcal{R}(\omega))\right]\right\} \leq \dim_H(\mathcal{M}^{\mathrm{loc}}(\omega) \cap V(\omega)) \leq \dim_H(\mathcal{A}(\omega))\,.$$

Summarizing we have shown the following theorem.

Theorem 5.3.9. *Suppose the RDS φ is given by the solution operator in* (5.1.4). *Moreover assume $\mathcal{F}$ resp. $D\mathcal{F}$ fulfills the assumptions (A1) – (A4) and $\gamma = \frac{1}{2}(\lambda^{\mathbf{u}} + \lambda^{\mathbf{s}}) > 0$. We denote the related random attractor $\mathcal{A}(\omega)$, $\omega \in \Omega$. Then the Hausdorff dimension of the random attractor is bounded from below by $N \in \mathbb{N}$.*
The number N appears as the index of the smallest positive eigenvalue $\lambda^{\mathbf{u}}$ of the shifted spectrum of the Laplacian, recall (5.2.1).

Remark 5.3.10.

- Remember that the number $N \in \mathbb{N}$ depends in particular on the value of the Fréchet derivative of the nonlinearity at zero (see (A3)), i.e. $D\mathcal{F}(0) = k\mathrm{Id}$, where $k > |\lambda_1|$. Hence we observe for the considered equations, with this particular method, $N \geq 1$.
- An interesting consequence from the proof is that it seems not to be important which particular open subset of the random attractor we choose. The succeeding projection onto the unstable subspace reduces the information we gained before from the open subset in the Hilbert space. So even if we find another proper subset of the random attractor we would obtain the same estimate.

Conclusion and outlook

We summarize the results of this thesis and discuss possible extensions.

Chapter 2 introduces the concepts and definitions which allow us to derive the results that are obtained in the subsequent chapters.

In Chapter 3 we described the theory of random dynamics in Hilbert spaces. We checked the classical results such as the existence of a random dynamical system for a specific SPDE with additive noise and a Laplacian given by the analysis on fractals. Moreover, we proved that the RDS possesses a unique random attractor. All statements in Chapter 3 are valid for elements in an L^2- space on a (not necessary open) bounded subset of $\mathbb{R}^n$, $n \geq 1$, equipped with an associated measure.
Possible extensions of the obtained statements are the Banach space-valued results collected in [Nea17]. For instance one could also consider unbounded (fractal) domains, as they are presented in [Tep98], [CMT15], [Sab00] or random recursive fractals, where the set of possible contractive similarities is chosen according to a given probability distribution, see [Ham97] or [Ham00].
It is worth considering other types of equations. In particular, we think of equations that are driven by a more general noise such as the fractional Brownian motion, see [GALS10], or equations with other nonlinearities that are not necessary Lipschitz continuous. Provided new results in the theory of differential operators on fractal sets one can also deal with equations that use these operators. Interesting primary works in this direction are e.g. [HT13] and [HRT13].

We derived in the subsequent Chapter 4 an upper bound on the Hausdorff dimension of the random attractor obtained in Chapter 3. For this purpose we introduced a classical deterministic method presented e.g. in [Tem88]. This approach has been extended to the more general case of a random attractor, see [Sch97], [CF98] and [Deb98]. In our setting the upper bound on the Hausdorff dimension does not depend on the dimension of the underlying set but on the spectral exponent of the related Laplacian. This result is reasonable, since the considered estimates in the relevant proofs depend on the spectrum of the related operator. Giving an example, we have seen that the upper bound on the Hausdorff dimension grows, when the Lipschitz constant of the nonlinearity becomes larger.
For future research one can try to improve or replace the assumptions that we imposed on the nonlinearity. For example Theorem 4.2.2 holds even if the nonlinearity is only C^1. But then the essential *uniform* differentiability of the solution in condition RC (I) can not be established and we expect that it is necessary to have at least Hölder continuity for the first derivative of the nonlinearity. As we have seen in Section 4.3 it is technical to obtain a numerical value for the upper bound of the Hausdorff dimension. A better knowledge of the complete spectrum, in particular on the distribution of the eigenvalues of the considered Laplacians Δ_μ would lead directly to a more precise bound. The lower estimate of the sum of the first m ($m \geq 1$) eigenvalues has to be of the type m^{1+a} with exponent $a > 0$, if we want to apply the related theory of [Tem88]. Besides the SG that we considered, there are similar results for other fractal sets (see e.g. [ABC$^+$17]) which state an upper and lower bound on the eigenvalues. These results are not of the type $C \cdot i^\beta \leq \lambda_i$,

where $C > 0$ and $\beta > 0$ are suitable constants and even if we have estimates of the desired type, it is still a challenging question to compute the values of the constants explicitly.
We can also examine reaction-diffusion equations with a Laplacian defined on a fractal set and a polynomial nonlinearity. Obtaining meaningful results would enable us to compare them with the corresponding statements in [Deb98].

Finally in the last chapter we focused on an SPDE driven by a multiplicative noise and obtained an associated random attractor. Thanks to a suitable shift of the spectrum of the Laplacian, using the Lyapunov-Perron transform of the solution and a cut-off function, we were able to show the existence of local unstable invariant manifolds. We proved that the intersection of the local unstable manifold with a neighborhood of zero is contained in the random attractor. We managed to establish a lower bound on the Hausdorff dimension using the projection of a subset of the latter intersection onto the unstable subspace of the Hilbert space.
There exist other methods to obtain a lower bound of the Hausdorff dimension, thus a joint work with experts in the field of fractal geometry with a focus on Hausdorff dimensions would be beneficial. Of course these techniques has to be used with care since we are working with *dynamic* attractors not *geometric* attractors.
A next question for ongoing works is of course: Is it possible to combine the results of Chapter 4 and 5, i.e. can we find a bracketing result for the Hausdorff dimension of the random attractor. In the setup of the lower estimate, it seems possible to apply the method described in Chapter 4 to the attractor of Chapter 5, in order to obtain an upper bound on its Hausdorff dimension. However in the setting of Chapter 4 we assumed an additive noise which led to an additive Ornstein-Uhlenbeck process in the nonlinearity. Hence the theory of invariant manifolds fails to apply, since the solution does not generate a fixed-point at zero. Having a bracketing for the Hausdorff dimension brings us closer to the true value of the Hausdorff dimension. Note that having an upper bound of the Hausdorff dimension means firstly that the dimension is finite, but the true value of the Hausdorff dimension can still be very low.
Besides the Hausdorff dimension it could be interesting to consider other types of dimensions such as packing dimensions or box-counting dimensions. Although they are typical larger than the Hausdorff dimension they can afford new information due to their definition.

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