Calculation of beam structures with the symbolic computer language Maple

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Summary
Beside the standard calculation programs for civil engineering buildings mathematical programs have been lately established for the solution of differential equations for the analysis of mechanical and static systems. Programs like Maple, Matlab, MathCAD and Mathematica are popular in this field. To the knowledge of the authors, the widest application functionality offers the program Maple. Its advantages are, e.g. the alternatively symbolic or numerical solution of differential equation systems, the easy handling of parameter studies, the immediate visualization of results, the definition of macros for selected calculation steps and their export in other computer languages and, not least, the automatically generated, very clear documentation of the mathematical calculus.

In this article a systematic approach with use of Maple is presented for the development of stiffness matrices for beam structures as well as the check of the solutions. Besides pure beam elements also beams with complicated loads and complex boundary conditions are considered.

A single beam is characterized, e.g. by a differential equation of the fourth order $EI \cdot w'''' = q$.

The solution of the differential equation becomes even more difficult if biaxial bending, elastic foundation or the second order analysis have to be taken into consideration. It is worth mentioning that the complexity of the solution with Maple does not increase as the complexity of differential equations increases.

The contribution also shows, that the presented method can easily be integrated in netbased teaching and learning systems.

1   Introduction
In the last time the development of new programs and the computer technology have led to radical changes in the methods of the calculation of engineering constructions. In the first place, the matrix methods are to be stated for necessary calculations of the tasks which permit a compact and general form of the presentation of equations and are suitable for the treatment on computers. The personal computers were enabled with it not only to solve systems of linear-algebraic equations with hundreds or thousands of unknowns, to take over the treatment from differential equations, as well as the formation of these equations also. With it the computers take over the majority of the process of the calculation and the project engineering of a construction. The questions about the use of these qualities have indubitable actuality for the tasks of the statics, mechanics, structural informatics and structural mechanics.

2 Systematic approach for the development of differential equations of beams with and without consideration of the transverse strain and the formulas for normal stress

In the practise of the project engineering a ready use of beam structures takes place (Fig.1). Beam structures exist of beam which are tied together in nodes. In the practise of civil engineering you often find suitable constructions, e.g. bridge span, buildings, ceilings or in the mechanical engineering. Development of an element library for the calculation of similar constructions with the symbolic computer language Maple permits effective project engineering.

Beam structures

Differential equation for single beam

$$\frac{d^4 w}{dx^4} = \frac{q}{EI}$$

with w- deflection, EI- bending stiffness (E- modulus of elasticity, I- moment of inertia), x- the coordinates of the cross section, q- line load.

Bems with comlicated loads and boundary conditions

1. Beam on elastic foundation

$$\frac{d^4 w}{dx^4} + 4 \cdot n^4 \cdot w = \frac{q}{EI} \text{ , mit } n = \sqrt[4]{\frac{kb}{4EI}}$$

2. Biaxial bending

$$\frac{d^4 w}{dx^4} \pm N \cdot \frac{d^2 w}{dx^2} = \frac{q}{EI}$$

3. Theory of second order

$$\left. \begin{array}{l} \dfrac{d^3 \varphi}{dx^3} = \dfrac{q}{EI} \\[2ex] \dfrac{dw}{dx} = \varphi - \dfrac{EI}{GF} \cdot \dfrac{d^2 \varphi}{dx^2} \end{array} \right\}$$

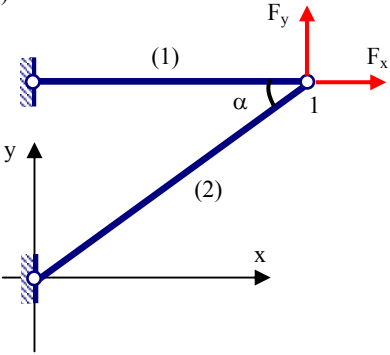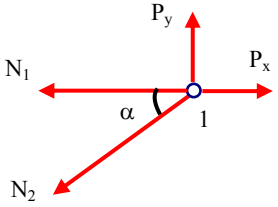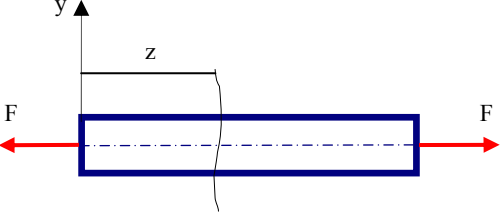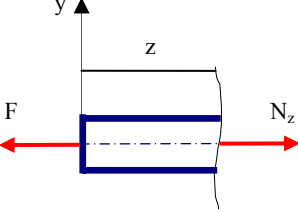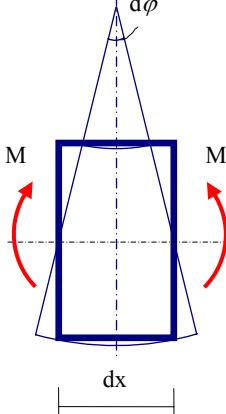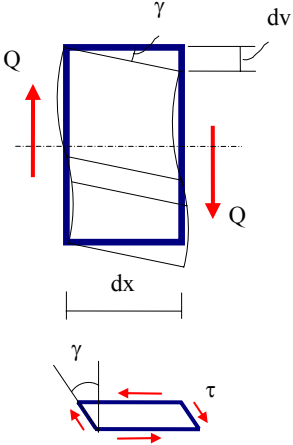Fig.1: Short overview for the calculations of beam structures

Table 1

Systematic approach for the development of the differential equations for beams or and without consideration of the transverse strain and the formulas for normal stress

Type of the development	Matrix equations Beam structure	Equations from the strength of materials Tension-compression	Bending without consideration of the transverse strain	Bending with consideration of the transverse strain	
Equation of equilibrium	$A \cdot S = F$	$N = \int_A \sigma dA$	$M = -\int_A \sigma \cdot y dA$	$M = -\int_A \sigma \cdot y dA$	1
Geometrical relations	$A^T \cdot U = \Delta$	$\varepsilon = \dfrac{du}{dx}$	$\varepsilon = \dfrac{du}{dx} = -\dfrac{d^2 w}{dx^2} y$	$\varepsilon = \dfrac{du}{dx} = -\dfrac{d\varphi}{dx} y$, mit $\varphi = \dfrac{dw}{dx} + \gamma$	2
Material law	$B^{-1} \cdot \Delta = S$	$\sigma = E \cdot \varepsilon$	$\sigma = E \cdot \varepsilon$	$\sigma = E \cdot \varepsilon$	3
(2)→(3)	$B^{-1} \cdot A^T \cdot U = S$	$\sigma = E \cdot \dfrac{du}{dx}$	$\sigma = -E \cdot \dfrac{d^2 w}{dx^2} y$	$\sigma = -E \cdot \dfrac{d\varphi}{dx} y$, mit $\varphi = \dfrac{dw}{dx} + \gamma$	4
(4)→(1)	$A \cdot B^{-1} \cdot A^T \cdot U = F$	$N = EA \cdot \dfrac{du}{dx}$	$M = EI \cdot \dfrac{d^2 w}{dx^2}$	$M = EI \cdot \dfrac{d\varphi}{dx}$, mit $\varphi = \dfrac{dw}{dx} + \gamma$	5
	$(A \cdot B^{-1} \cdot A^T) \cdot F = U$	$\dfrac{du}{dx} = \dfrac{N}{EA}$	$\dfrac{d^2 w}{dx^2} = \dfrac{M}{EI}$	$\dfrac{M}{EI} = \dfrac{d\varphi}{dx}$, mit $\varphi = \dfrac{dw}{dx} + \gamma$	6
(6)→(4)	$B^{-1} \cdot A^T \cdot (A \cdot B^{-1} \cdot A^T)^{-1} \cdot F = S$	$\sigma = \dfrac{N}{A}$	$\sigma = -\dfrac{M}{I} y$	$\sigma = -\dfrac{M}{I} \cdot y$	7

The matrix were calculated with the computer programs, e.g. MAPLE.

Table 1

Type of the deformation

Beam structure	Tension-compression	Bending without consideration of the transverse strain	Bending with consideration of the transverse strain
a) F_y F_x (1) α 1 y (2) x b) P_y N_1 P_x α 1 N_2	a) y z F F b) y z F N_z	M $d\varphi$ M dx	γ dv Q Q dx γ τ

3   The Automation of the calculation of beam structures

The elaboration of the stiffenss matrix directly in a global coordinate system is complicated. Besides for calculation of beam structures uses the method of the Finiten elements. In this way at beginning the stiffenss matrix would be development in the local coordinate system, where their elaboration is more easy. Afterwards the construction of the global stiffenss matrix will be done (fig.2).

Discretization   **1. Step**

Choice of the elements (beams), definition of the material properties, loads and boundary conditions   **2. Step**

Development of the Elements stiffenss matrix

1 [ ]
2 [ ]   **k**
3 [ ]   **3. Step**

Assembling the the global stiffens matrix   **K**   **4. Step**

Solution of the linear equation system
K·U=F
with
K – stiffenss matrix
U - displacement
F – external force   **5. Step**

Calculation of
strain,
stress,
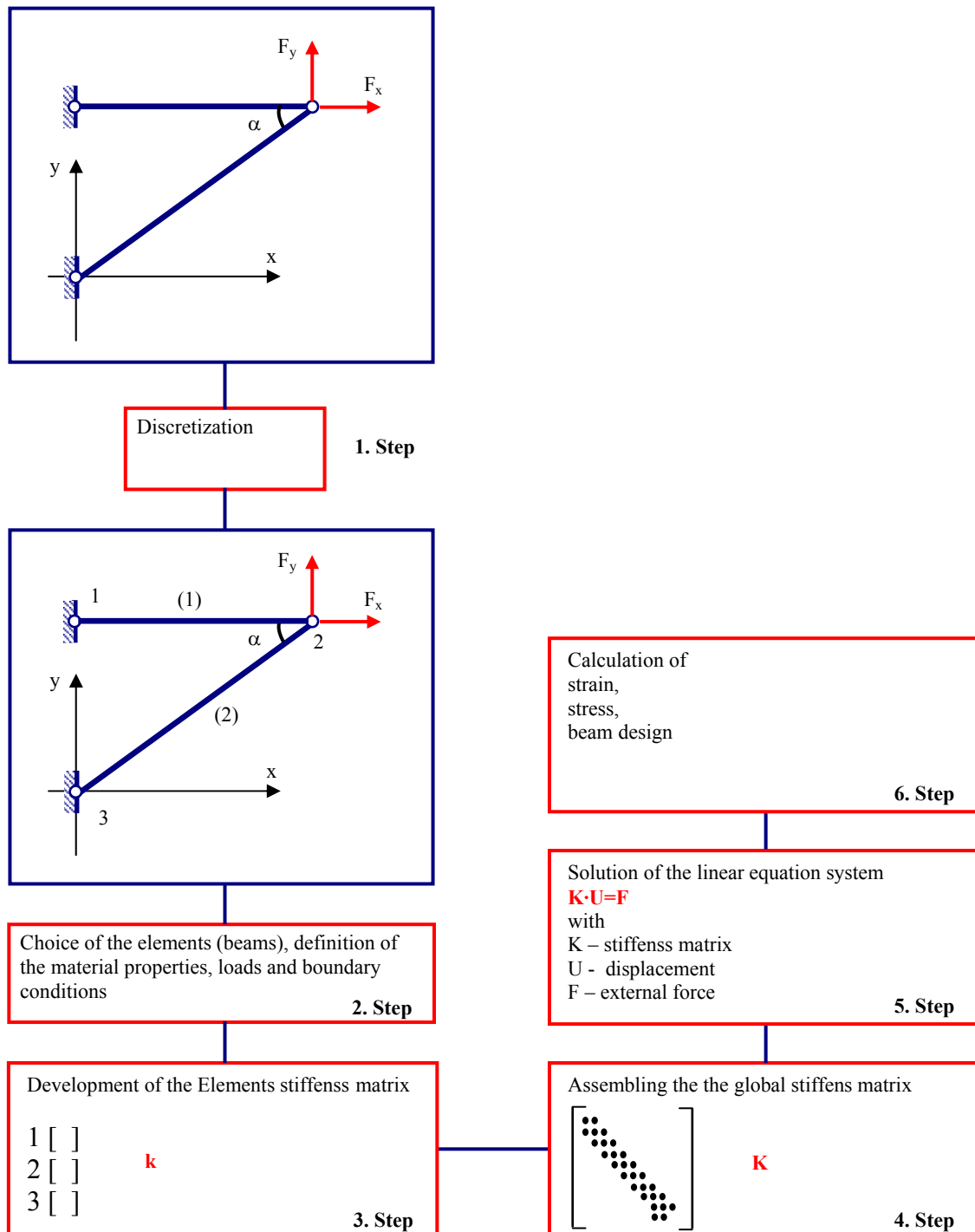beam design   **6. Step**

Fig.2: Basic steps by the realization of finite element calculations

For the elaboration of the stiffenss matrix for beams the following approach will be suggest:
1. In confirming of the differential equation of the beam the stiffens matrix would be developed in a local coordinate system;
2. Consideration of the character of the connection in the nodes at the end of the beam (stiff oder hinge connection);
3. Extension of the matrix by consideration of additional limbs of the tension and compression.
4. Transforming of the local coordinate system into the global coordinate system.
5. Theelement matrices are assembled in the global stiffnes matrix.
This article will discuss in details only the first point.

3.1. Algorithms for the elaboration of Stiffenss matrix for single beam and beams on elastic foundation with the symbolic computer language Maple

The source parameters of the stiffenss matrix are a differential equation or a system of differential equations (if transverse strain are present), as well as the expressions for the moment (M) and the shear force (Q). In table 1 is shown the systematic approach to the preservation of the differential equation for beam with and without consideration of the transverse strain. In case of a single beam, this is characterized by a differential equation of the fourth order:

$$\frac{d^4 w}{dx^4} = \frac{q}{EI} \tag{1}$$

The formulas of the moment (M) and the shear force (Q) are as follows:

$$M = EI \frac{d^2 w}{dx^2} \quad Q = EI \frac{d^3 w}{dx^3} \tag{2}$$

The differential equation (1) is dissolved in the following type:

$$w = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + \frac{q x^4}{24 EI} \tag{3}$$

Further we write the differential equation to (3) and their derivatives in a matrix form:

$$
\begin{bmatrix} w \\ w' \\ w'' \\ w''' \\ w^{IV} \end{bmatrix} =
\begin{bmatrix}
1 & x & x^2 & x^3 \\
0 & 1 & 2x & 3x^2 \\
0 & 0 & 2 & 6x \\
0 & 0 & 0 & 6 \\
0 & 0 & 0 & 0
\end{bmatrix}
\cdot
\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}
+ \frac{q}{EI} \cdot
\begin{bmatrix} \frac{x^4}{24} \\ \frac{x^3}{6} \\ \frac{x^2}{2} \\ x \\ 1 \end{bmatrix}
\tag{4}
$$

The fifth row is used to check the correctness of differential equation solution. If we substitute in the first two rows of the expression (4) the coordinates for the nodes of the beam with $x = 0$ and $x = l$, follows from it:

$$
\begin{bmatrix} w_i \\ \varphi_i \\ w_j \\ \varphi_j \end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
1 & l & l^2 & l^3 \\
0 & 1 & 2l & 3l^3
\end{bmatrix}
+ \frac{q}{EI} \cdot
\begin{bmatrix} 0 \\ 0 \\ \frac{l^4}{24} \\ \frac{l^3}{6} \end{bmatrix}
\tag{5}
$$

The system (5) in simplistic form:

$$\vec{z}=\begin{bmatrix} \vec{z}_i \\ \vec{z}_j \end{bmatrix} = L \cdot \vec{c} + \frac{q}{EI} \cdot \vec{L} \tag{6}$$

We express the constant of integration by the displacement of the beam nodes:

$$\vec{c}=L^{-1} \cdot \vec{z} - \frac{q}{EI} \cdot L^{-1} \cdot \vec{L} \tag{7}$$

Putting in the formula (2) the derived expression (4) (the second two rows (4)) we get:

$$\begin{bmatrix} M \\ Q \end{bmatrix} = EI \cdot \begin{bmatrix} 0 & 0 & 2 & 6x \\ 0 & 0 & 0 & 6 \end{bmatrix} \cdot \vec{c} + q \begin{bmatrix} \frac{x^2}{2} \\ x \end{bmatrix} \tag{8}$$

The shear force and moments at the ends of the beam are Q_i, M_i, Q_j, M_j and correspond with the reactions $r_{wi}, r_{\varphi i}, r_{wj}, r_{\varphi j}$.

$$\begin{bmatrix} r_{wi} \\ r_{\varphi i} \\ r_{wj} \\ r_{\varphi j} \end{bmatrix} = \begin{bmatrix} Q_i \\ -M_i \\ -Q_j \\ M_j \end{bmatrix} = EI \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 2 & 6l \end{bmatrix} \cdot \vec{c} + q \cdot \begin{bmatrix} 0 \\ 0 \\ -l \\ \frac{l^2}{2} \end{bmatrix} \tag{9}$$

or $\qquad \vec{r} = EI \cdot L_1 \cdot \vec{c} + q \cdot \vec{L}_1$ \hfill (10)

By using of formula (7) in the expression (10) we receive:

$$\vec{r} = EI \cdot L_1 \cdot (L^{-1} \cdot \vec{z} - \frac{q}{EI} \cdot L^{-1} \cdot \vec{L}) + q \cdot \vec{L}_1 = EI \cdot L_1 \cdot L^{-1} \cdot \vec{z} - q \cdot L_1 \cdot L^{-1} \cdot \vec{L} + q \cdot \vec{L}_1 =$$

$$= EI \cdot L_1 \cdot L^{-1} \cdot \vec{z} - q \cdot (L_1 \cdot L^{-1} \cdot \vec{L} - \vec{L}_1)$$

At the end: $\vec{r} = EI \cdot r \cdot \vec{z} - q \cdot \vec{r}_q$

with $r = L_1 \cdot L^{-1}$ - the relative stiffenss matrix with $EI = 1$, $\vec{r}_q = -L_1 \cdot L^{-1} \cdot \vec{L} + \vec{L}_1$ - the relative load column with $q = 1$.

The finally matrix for the single beam is:

$$\begin{bmatrix} 12\frac{EJ}{l^3} & 6\frac{EJ}{l^2} & -12\frac{EJ}{l^3} & 6\frac{EJ}{l^2} & -\frac{1}{2}ql \\[2mm] 6\frac{EJ}{l^2} & 4\frac{EJ}{l} & -6\frac{EJ}{l^2} & 2\frac{EJ}{l} & -\frac{1}{12}ql^2 \\[2mm] -12\frac{EJ}{l^3} & -6\frac{EJ}{l^2} & 12\frac{EJ}{l^3} & -6\frac{EJ}{l^2} & -\frac{1}{2}ql \\[2mm] 6\frac{EJ}{l^2} & 2\frac{EJ}{l} & -6\frac{EJ}{l^2} & 4\frac{EJ}{l} & \frac{1}{12}ql^2 \end{bmatrix}$$

In analogy would be developed the stiffenss matrix with more difficult differential equations - beam on elastic foundation:

$$\frac{d^4 w}{dx^4} + 4 \cdot n^4 \cdot w = \frac{q}{EI}, \text{ mit } n = \sqrt[4]{\frac{kb}{4EI}}$$

An example program for the elaboration of the stiffenss matrix for an single beam with the symbolic computer language MAPLE

> restart;
> with(linalg): with(LinearAlgebra):

The differential equation of the fourch order:

> diff(w(x),x$4)=q/(E*J);

$$\frac{d^4}{dx^4}\,w(x) = \frac{q}{E\,J}$$

Die formuls for moment (M) and shear force (Q)

> Q:=(EJ)(d^3*w/dx^3); M:=(EJ)(d^2*w/dx^2);

$$Q := \mathrm{EJ}\left(\frac{d^3\,w}{dx^3}\right) \quad M := \mathrm{EJ}\left(\frac{d^2\,w}{dx^2}\right)$$

Dissolving of the differential equation

> vv:=dsolve(diff(w(x),x$4)=q/(E*J),w(x));

$$vv := w(x) = \frac{q\,x^4}{24\,E\,J} + \frac{\_C1\,x^3}{6} + \frac{\_C2\,x^2}{2} + \_C3\,x + \_C4$$

> prav_fun_vv:=rhs(vv):

Derived from the deflection

> v1:=diff(prav_fun_vv,x):
> v2:=diff(v1,x):
> v3:=diff(v2,x):
> v4:=diff(v3,x):

Dissolving in Matrix form and their derived

> ur:=[prav_fun_vv,v1,v2,v3,v4]:
> Dmatrix:=mulcol(genmatrix(ur,[_C1,_C2,_C3,_C4],flag),5,-1);

$$Dmatrix := \begin{bmatrix} \dfrac{x^3}{6} & \dfrac{x^2}{2} & x & 1 & \dfrac{q\,x^4}{24\,E\,J} \\[2ex] \dfrac{x^2}{2} & x & 1 & 0 & \dfrac{q\,x^3}{6\,E\,J} \\[2ex] x & 1 & 0 & 0 & \dfrac{q\,x^2}{2\,E\,J} \\[2ex] 1 & 0 & 0 & 0 & \dfrac{q\,x}{E\,J} \\[2ex] 0 & 0 & 0 & 0 & \dfrac{q}{E\,J} \end{bmatrix}$$

Check of the correctness of the solved differential equation

> simplify(diff(vv,x$4)=q/(E*J));

$$\left(\frac{d^4}{dx^4}\,w(x) = \frac{q}{E\,J}\right) = \frac{q}{E\,J}$$

Elaboration of the matrix L and the vector L

> str_12:=convert(submatrix(Dmatrix,1..2,1..5),Matrix) :
>
matrvectL:=Matrix(4,5,[[subs(x=0,row(str_12,1))],[subs(x=0,row(str_12,2))],[subs(x=l,row(str_12,1))],[subs(x=l,row(str_12,2))]]):
> mat_L:=convert(submatrix(matrvectL,1..4,1..4),Matrix) :
> mat_L_obrat:=MatrixInverse(mat_L):
> vec_L:=convert(submatrix(matrvectL,1..4,5..5),Matrix) :

Elaboration of the matrix L1and the vector L1

> str_34:=convert(submatrix(Dmatrix,3..4,1..5),Matrix) :
>
matrvectL1:=scalarmul(mulrow(mulrow(Matrix(4,5,[[subs(x=0,row(str_34,2))],[subs(x=0,row(str_34,1))],[subs(x=l,row(str_34,2))],[subs(x=l,row(str_34,1))]]),2,-1),3,-1),E*J):
> mat_L1:=convert(submatrix(matrvectL1,1..4,1..4),Matrix) :
> vec_L1:=convert(submatrix(matrvectL1,1..4,5..5),Matrix) :

Elaboration of the relative stiffnes matrix with EJ=1

> matr_r:=multiply(mat_L1/E/J,mat_L_obrat):

Elaboration the relative load column with q=1

> matr_rq:=simplify(matadd(-multiply(matr_r,E*J*vec_L),vec_L1)):

The finally stiffenss matrix and load column for the single Beam is:

> resultmatr:=augment(E*J*matr_r,matr_rq);

$$resultmatr := \begin{bmatrix} \dfrac{12\,E\,J}{l^3} & \dfrac{6\,E\,J}{l^2} & -\dfrac{12\,E\,J}{l^3} & \dfrac{6\,E\,J}{l^2} & -\dfrac{q\,l}{2} \\[2ex] \dfrac{6\,E\,J}{l^2} & \dfrac{4\,E\,J}{l} & -\dfrac{6\,E\,J}{l^2} & \dfrac{2\,E\,J}{l} & -\dfrac{q\,l^2}{12} \\[2ex] -\dfrac{12\,E\,J}{l^3} & -\dfrac{6\,E\,J}{l^2} & \dfrac{12\,E\,J}{l^3} & -\dfrac{6\,E\,J}{l^2} & -\dfrac{q\,l}{2} \\[2ex] \dfrac{6\,E\,J}{l^2} & \dfrac{2\,E\,J}{l} & -\dfrac{6\,E\,J}{l^2} & \dfrac{4\,E\,J}{l} & \dfrac{q\,l^2}{12} \end{bmatrix}$$

Table 2 shows all models of differential equations. For these differential equations were developed the stiffness matrices with unfolding algorithm in the symbolic computer language Maple.

The Stiffness matrices which were get by the use of MAPLE are heavy to estimate (a matrix about some typed pages sometimes applies). Therefore, the control of the matrices is a very important stage in the production process. There are two types of the control: either to count back by calculation of the limit value crossing to the differential equations or by duplication of the length of the elements. The algorithm to the control of the matrix was also written in the symbolic language MAPLE.

Table 2

Differential equations and formulas for M and Q

	Bending without consideration of the transverse strain		Bending with consideration of the transverse strain	
	Differential equation	Formula for M and Q	Differential equation	Formula for M and Q
Single beam	$\dfrac{d^4w}{dx^4} = \dfrac{q}{EI}$	$M = EI \cdot \dfrac{d^2w}{dx^2}$ $Q = EI \cdot \dfrac{d^3w}{dx^2}$	$\left.\begin{array}{l} \dfrac{d^3\varphi}{dx^3} = \dfrac{q}{EI} \\[1ex] \dfrac{dw}{dx} = \varphi - \dfrac{EI}{GF} \cdot \dfrac{d^2\varphi}{dx^2} \end{array}\right\}$	$M = EI \cdot \dfrac{d\varphi}{dx}$ $Q = EI \cdot \dfrac{d^2\varphi}{dx^2}$
Biaxial bending	$\dfrac{d^4w}{dx^4} \pm N \cdot \dfrac{d^2w}{dx^2} = \dfrac{q}{EI}$	$M = EI \cdot \dfrac{d^2w}{dx^2}$ $Q = EI \cdot \dfrac{d^3w}{dx^2} \pm N\dfrac{dw}{dx}$	$\left.\begin{array}{l} EI \cdot \dfrac{d^3\varphi}{dx^3} \pm N\dfrac{d^2w}{dx^2} = q \\[1ex] \varphi = \dfrac{dw}{dx} + \dfrac{EI}{GF} \cdot \dfrac{d^2\varphi}{dx^2} \pm \dfrac{1}{GF} \cdot \dfrac{dw}{dx} \end{array}\right\}$	$M = EI \cdot \dfrac{d\varphi}{dx}$ $Q = EI \cdot \dfrac{d^2\varphi}{dx^2} \pm N \cdot \dfrac{dw}{dx}$

The use of the computer language MAPLE permits the precise calculation of the stiffness matrices of complicated sbeam structures in symbolic form.

The elaboration of the Stiffness matrices occurs automatically and directly after the differential equations of the task. The application of this symbolic language facilitates the development of big and difficult tasks and, in addition, simplifies the programming. Furthermore Maple permits the issue of the results in a code of another computer language (C ++, VB).

4   Continuing use of provided algorithms

The developed algorithms for beam structures should serve not only for the pure calculation, but also be available to the teaching company at the university in suitable form. The students should use not only ready algorithms, but recognize solution steps and to be enabled to work on different application cases. On the one hand, special solution libraries can be used in Maple. With it the solution way is better understandable for the students or usable as a gradually control for their own calculation. It is important in this connection, that it is possible to create a pleasing

optical-interactive presentation, by which the learners can fully concentrate upon the task, based on the GUI development. With it the service is substantially simplified and takes from many users the fear of possible "programming expenditure" (Fig.3).

On the other hand exists the variant to create an complete tutors course or practise block. So theory and practical examples as well as practise tasks are tied together directly with each other. In this case pleasing formed working surroundings are compellingly necessary, i.e. Internet pages similar construction.

However, this is possible with Maple. In addition, the examples must be fitted out with learning-supporting help functions and mistake analyses, e.g. hyperlinks to auxiliary texts, procedures for the examination of interresults or mistakes.

These program blocks can be explained either locally or Intranet-basing or are put on a server with suitable Java servlet-interface also for availability of distant accesses.
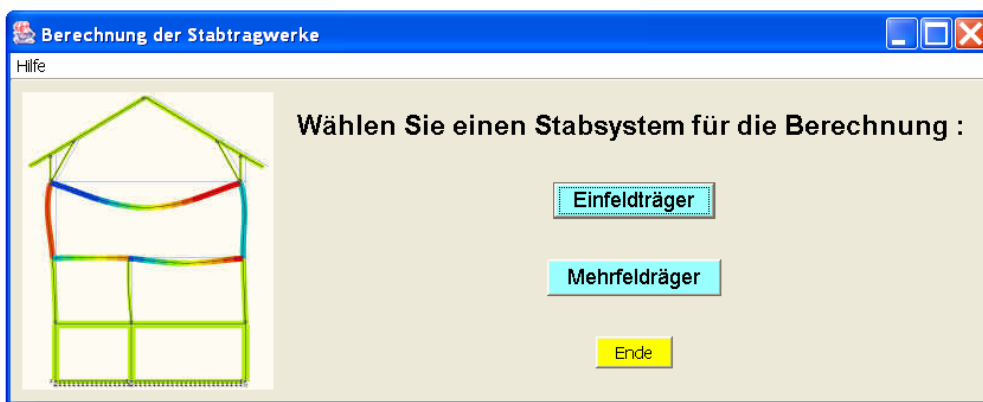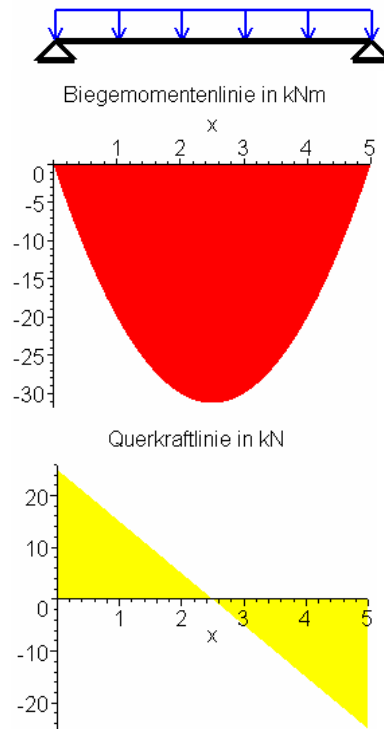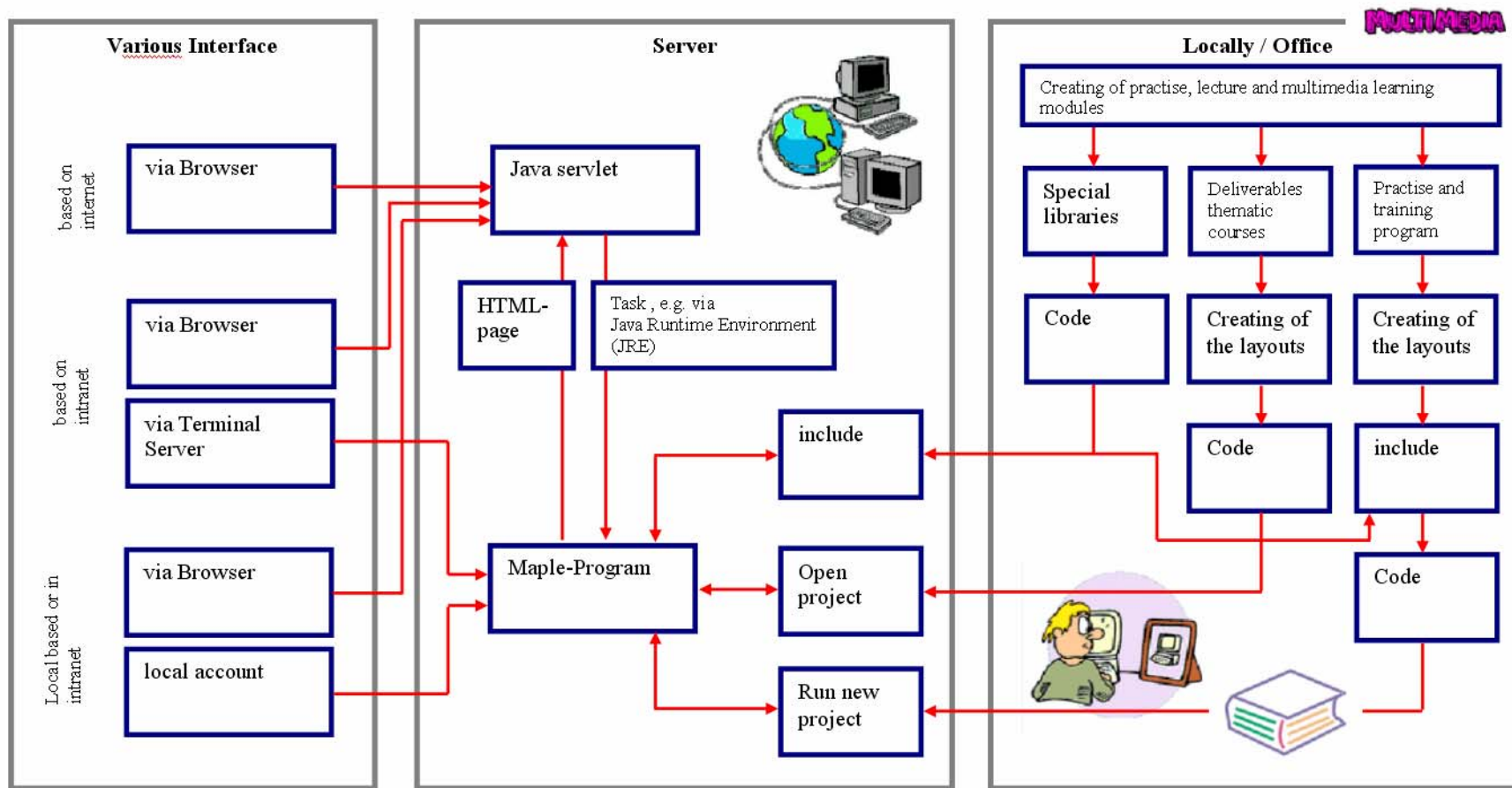

Fig.3: Example of Graphical user Interface (GUI)

Fig.4: Example result visualization with Maple

Scheme 1: System concept for the interaktive use of netbased Maple applications

5 Reference

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