Hyperbolic Q_p-scales

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Abstract

The Q_p-scales were first introduced in [1] as interpolation spaces between the Bloch and Dirichlet spaces in the complex space. In '98, they were generalized to $\mathbb{R}^n$ (see [4], [6]) using the automorphisms of the unit disk $\varphi_a(x) = \frac{x-a}{1-\bar{a}x}$, $|a| < 1$, and a modified fundamental solution for the Laplacian.

However, such treatment presents the disadvantage of only considering the Euclidean case. In order to obtain an approach to homogeneous hyperbolic manifolds, the projective model of Gel'fand was retaken in [2]. With the help of a convenient fundamental solution for the hyperbolic (homogeneous of degree α) D_α (see [5]) it was introduced in [7] and [3] equivalent Q_p scales for homogeneous hyperbolic spaces. In this talk we shall present and study some properties of this hyperbolic scale.

1 Introduction

In the last years the study of the so-called Q_p-scales emerged from complex analysis (see e.g. [1]). These function spaces are spaces of holomorphic functions for which

$$\sup_{a \in B_1(0)} \int_{B_1(0)} |f'(z)|^2 g^p(z,a) dxdy < \infty, \tag{1}$$

where $g(z,a) = \ln |\frac{1-\bar{a}z}{a-z}|$ is the Green's function of the two-dimensional real Laplacian, $B_1(0) = \{z : |z| < 1\}$ is the complex unit disk, $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ are automorphisms which map the unit disk onto itself and $0 < p < \infty$, their importance lying in the fact that they form a scale of function spaces invariant under the automorphism group of the complex disk and allowing a "fine-tuning" of the boundary behaviour of functions holomorphic in the unit disk.

A suitable generalization of such a scale to higher dimensions in Euclidean spaces was done, in the framework of Clifford analysis, by K. Gürlebeck, U.

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Kähler, M.V. Shapiro, and L.M. Tovar [6] and by J. Cnops and R. Delanghe [4].

In [2], [7] a Clifford Function Theory for hyperbolic spaces has been introduced, by making use of a homogeneous description of the hyperbolic unit ball whereby points are identified with rays in the future null cone and the isometry group corresponds to the Lorentz group $\mathbf{Spin}(1,n)$. Here we will give a definition of Q_p-scales in the framework of the projective hyperbolic model.

2 Preliminaries

Let $\mathbb{R}^n$ be the n-dimensional vector space, endowed with the usual Euclidean metric. Moreover, we shall provide $\mathbb{R}^n$ with a quadratic form

$$Q(x) = \sum_{i=1}^{p} x_i^2 - \sum_{i=p+1}^{p+q} x_i^2$$

and we will write $\mathbb{R}^{p,q} = (\mathbb{R}^n, Q)$.

We denote by $C\ell_{p,q}$ the universal real Clifford algebra which

i) $x^2 = Q(x),\ \forall x \in \mathbb{R}^n$;

ii) it is generated as an algebra by the above vector space;

iii) it is not generated as an algebra by any proper subspace of $\mathbb{R}^n$.

By $i)$ we obtain as multiplication rules for the orthonormal basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ that

$$\mathbf{e}_i^2 = \begin{cases} +1, & i = 1, \cdots, p \\ -1, & i = 1, \cdots, p+q = n \end{cases}$$

and

$$\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = 0, \quad \text{for } i \neq j.$$

Hence $C\ell_{p,q}$ is a 2^n-dimensional real associative algebra, with basis given by $\mathbf{e}_0 = 1$ and $\mathbf{e}_A = \mathbf{e}_{h_1} \cdots \mathbf{e}_{h_k}$, where $A = \{h_1, \ldots, h_k\} \subset N = \{1, \ldots, n\}$, for $1 \leq h_1 < \cdots < h_k \leq n$.

Each element $x \in C\ell_{p,q}$ can now be written as a linear combination of such elements of the basis. The particular linear combination of basic elements with equal length k is designated a $k-$vector; we shall denote by $[x]_k$ the k-vector part of $x \in C\ell_{p,q}$.

We define an *involutory automorphism* in $C\ell_{p,q}$, which we denoted *conjugation*, as $x \to \overline{x}$, given by its action on the basis elements as $\overline{1} = 1$, $\overline{\mathbf{e}_i} = -\mathbf{e}_i$ and $\overline{ab} = \overline{b}\ \overline{a}$.

Let $\Omega \in \mathbb{R}^n$ be an open domain. We define a Clifford-valued function $f = f(x)$ in Ω as

$$f(x) = \sum_{A \subset N} f_A(x)\mathbf{e}_A,$$

where all $f_A(x)$ stand for real-valued functions. Function spaces of Clifford-valued functions are established as modules over the Clifford algebra $C\ell_{p,q}$, by imposing the coefficient-functions f_A to be in the corresponding real-valued function space. Therefore, we shall use for Clifford-valued function spaces the same notations as in the real-valued case.

We define the Dirac operator on $C\ell_{p,q}$ as

$$\partial f = \sum_{i=1}^{n} \mathbf{e}_i \frac{\partial f}{\partial x_i}, \tag{2}$$

which satisfies $-\partial^2 = \Delta_p - \Delta_q$, where Δ_s is the s-dimensional standard Euclidean Laplace operator. Moreover, a function is said to be *(left) monogenic* if $\partial f = 0$ in Ω.

3 Hyperbolic Clifford Analysis

The following projective model has been introduced in the seventies by Gel'fand as a way of describing the hyperbolic disk in projective coordinates. As main motivation for using this model is the study of the invariance group of null solutions of the Dirac operator, which corresponds to the Lorentz group $\mathbf{Spin}(1, n+1)$ and the fact that the subgroup $\mathbf{Spin}(1, n)$ of Möbius transformations leaving the unit sphere invariant is the isometry group for hyperbolic geometries.

3.1 The Projective Model

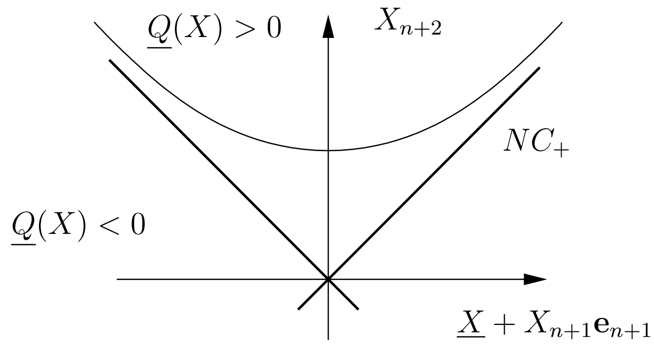
In what follows, we shall consider $\underline{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n$ to be an arbitrary element of the vectorial space $\mathbb{R}^{0,n}$. Each point $\underline{x}$ will be assigned projectively to an element of $\mathbb{R}^{1,n+1}$ by means of

$$\begin{aligned} \underline{x} \mapsto X &= X_1\mathbf{e}_1 + \cdots + X_n\mathbf{e}_n + X_{n+1}\mathbf{e}_{n+1} + X_{n+2}\mathbf{e}_{n+2} \\ &= \underline{X} + X_{n+1}\mathbf{e}_{n+1} + X_{n+2}\mathbf{e}_{n+2} \\ &= \underline{x} + \frac{1-r^2}{2}\mathbf{e}_{n+1} + \frac{1+r^2}{2}\mathbf{e}_{n+2}, \end{aligned} \tag{3}$$

where $r^2 = |\underline{x}|^2 = \sum_{i=1}^{n} x_i^2$ denotes the usual n-dimensional Euclidean metric. This transformation takes the form

$$\left\{ \begin{array}{rcll} X_j &=& x_j & j = 1, \ldots, m \\ X_{n+1} &=& \frac{1-r^2}{2} & \\ X_{n+2} &=& \frac{1+r^2}{2} & \end{array} \right. \tag{4}$$

and, with respect to the standard Euclidean basis, it defines a conic surface NC in the vectorial space $\mathbb{R}^{n+2}$. Moreover, its elements satisfy the quadratic equation $\underline{Q}(X) = -(X_1^2 + \ldots + X_{n+1}^2) + X_{n+2}^2 = 0$. Hence, we have an embedding of $\mathbb{R}^{0,n}$ into $NC_+ = \{X |\ \underline{Q}(X) = 0 \wedge X_{n+2} > 0\} \subset \mathbb{R}^{1,n+1}$ (see figure below).



`fig.1 - Projective model`

Lemma 3.1 *The mapping* $\underline{x} \to X$ *is (up to the sign) an isometry.*

Proof: $ds_E^2 = \sum_{j=1}^{n} dx_j^2 = dX_{n+2}^2 - \sum_{j=1}^{n+1} dX_j^2 = ds_H^2$.

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We now consider the homogeneous mapping

$$\underline{x} \mapsto ray(X) = \{Y = \lambda X, \lambda > 0\} \tag{5}$$

which embeds $\mathbb{R}^{0,n}$ into the manifold of rays

$$ray(NC_+) = \{ray(X), X \in NC_+\}. \tag{6}$$

For each $Y \in ray(X)$ we have the recovering formulas

$$\underline{x} \quad = \quad \frac{1}{Y_{n+1} + Y_{n+2}} \underline{Y}, \text{ for } Y_{n+1} + Y_{n+2} \neq 0$$

where X lies in the *paraboloid* $\mathcal{P}$

$$\mathcal{P} := \left\{ \begin{array}{ccc} X_{n+2}^2 & = & X_1^2 + \cdots + X_{n+1}^2 \\ X_{n+2} & > & 0 \\ X_{n+1} + X_{n+2} & = & 1 \end{array} \right. \tag{7}$$

while $ray(\underline{0}, -1, +1)$ is to be identified with the point at infinity ∞.

Remark that the mapping (5) is an injective projection of $S^n = \mathbb{R}^n \cup \{\infty\}$ into the manifold $ray(NC_+)$. Hence, the following identification holds

$$S^n = \mathbb{R}^n \cup \{\infty\} \stackrel{1-1}{\longleftrightarrow} NC_+ \cap \{X : X_{n+2} = 1\}.$$

Moreover, the unitary ball $B_1(0)$ in $\mathbb{R}^n$ is to be identified with the set of points of $\mathcal{P}$ such that $X_{n+1} = \frac{1-r^2}{2} > 0$.

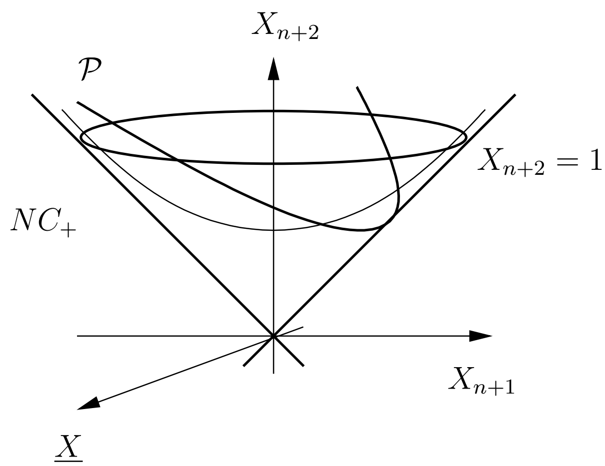

fig.2 - Null cone

Analyzing the invariance group for this model we have

i) the Lorentz group is the angle preserving group of $\mathbb{R}^n$, hence also of $ray(NC_+)$.

ii) the unit sphere S^{n-1} corresponds to $ray(X|\ X_{m+1} = 0)$, while the unit ball $B_1(0)$ is associated to the region $X_{m+1} > 0$, its invariance group being $Spin(1, n)$.

Due to this considerations we shall restrict $\mathbb{R}^{1,n+1}$ to $\mathbb{R}^{1,n}$ by means

$$\underline{X} + X_{n+1}\mathbf{e}_{n+1} + X_{n+2}\mathbf{e}_{n+2} \rightarrowtail \underline{X} + X_{n+2}\mathbf{e}_{n+2}, \tag{8}$$

in order to preserve the hyperbolic structure of the model.

3.2 Hyperbolic Dirac Equation

We denote by *Future Cone* the set

$$FC_+ = \{X|\ \underline{Q}(X) > 0\}. \tag{9}$$

A function $f(\underline{x})$ on $B_1(0)$ may be extended to $\mathcal{P} \cap \mathbb{R}^{1,n}$ by $F(X) = f(\underline{x})$, where $X = \underline{x} + \frac{1+|\underline{x}|^2}{2}\mathbf{e}_{n+2}$.

We now extend F to the whole Future Cone in such a way that it remains invariant on the manifold of rays $ray(FC_+) = \{ray(X)|\ \underline{Q}(X) > 0\}$. For that, we define F in $ray(FC_+)$ as the α-homogeneous bundle $\mathcal{E}^\alpha$ given by the equivalence relation

$$F(\underline{X} + X_{n+2}\mathbf{e}_{n+2}) \sim \lambda^\alpha F(\underline{X} + X_{n+2}\mathbf{e}_{n+2}), \tag{10}$$

and we obtain a projective identification of $f(\underline{x})$ with

$$F\left(\lambda(\underline{X} + X_{n+2}\mathbf{e}_{n+2})\right) = \lambda^\alpha F(\underline{X} + X_{n+2}\mathbf{e}_{n+2}),\ \lambda > 0. \tag{11}$$

Next, we introduce homogeneous differential operators

$$P(\underline{X}, X_{n+2}, \partial_{\underline{X}}, \partial_{X_{n+2}}) = \sum A_{\underline{l},j}(\underline{X}, X_{n+2}) \partial_{\underline{X}}^{\underline{l}} \partial_{X_{n+2}}^{j} \tag{12}$$

where $\partial_{\underline{X}}^{\underline{\alpha}} = \partial_{X_1}^{\alpha_1} \ldots \partial_{X_n}^{\alpha_n}$ and $A_{\underline{l},j}$ are $\mathbb{R}_{1,n}$-valued polynomials in $\underline{X} + X_{n+2}\mathbf{e}_{n+2}$. We say that the operator (12) has homogeneous degree σ if

$$P(\lambda\underline{X}, \lambda X_{n+2}, \partial_{\lambda\underline{X}}, \partial_{\lambda X_{n+2}}) = \lambda^{\sigma} P(\underline{X}, X_{n+2}, \partial_{\underline{X}}, X_{n+2}).$$

Moreover, $P(\lambda\underline{X}, \lambda X_{n+2}, \partial_{\lambda\underline{X}}, \partial_{\lambda X_{n+2}})$ will map sections of the bundle $\mathcal{E}_\alpha$ into sections of the bundle $\mathcal{E}_\beta$ with $\beta - \alpha = \sigma$.

Definition 3.2 (α-Dirac equation) *We define the α-homogeneous Dirac equation for the section bundle $\mathcal{E}_\alpha$ as*

$$\left\{ \begin{array}{rcl} (-\partial_{\lambda\underline{X}} + \mathbf{e}_{n+2}\partial_{X_{n+2}})F & = & 0 \\ (-\sum_{i=1}^{n} X_i \partial_{X_i} + X_{n+2}\partial_{X_{n+2}})F & = & \alpha F \end{array} \right. \tag{13}$$

for every $F \in \mathcal{E}_\alpha$.

Moreover, the system (13) in $\underline{x}$-coordinates reads

$$D_\alpha F = -\partial_{\underline{x}} + \frac{2}{1 - |\underline{x}|^2}(\underline{x} + \mathbf{e}_{n+2})(\alpha + \sum_{i=1}^{n} x_i \partial_{x_i})F. \tag{14}$$

3.3 Fundamental Solution

We now establish a homogeneous version of the Cauchy-Pompeiu formula. We recall the Léray form

$$L(X, dX) \quad = \quad E_X \rfloor V(dX) \tag{15}$$

arising from the contraction of the Euler operator $E_X = -\sum_{j=1}^{n} X_j \partial_{X_j} + X_{n+2}\partial_{X_{n+2}}$ with the volume $n+1$-form $V(dX) = dX_1 \cdots dX_n dX_{n+2}$.

The oriented surface element is generated by the contraction of the Dirac operator with the volume form $V(dX)$,

$$\begin{aligned} \sigma(dX) &= \partial_X \rfloor dX_1 \cdots dX_n dX_{n+2} \\ &= \sum_{j=1}^{n} (-1)^j \mathbf{e}_j dX_1 \cdots dX_{j-1} dX_{j+1} \cdots dX_{n+2} \\ &\qquad - (-1)^{n+2} \mathbf{e}_{n+2} dX_1 \cdots dX_n \end{aligned}$$

and leads to the homogeneous version of the surface element

$$\begin{aligned} \Sigma(X, dX) &= E_X \rfloor \sigma(dX) \\ &= -\partial_X \rfloor L(X, dX) \\ &= E_X \rfloor \partial_X \rfloor V(dX). \end{aligned} \tag{16}$$

Theorem 3.3 (Cauchy-Pompeiu formula (see [7], [8])) *Let $F \in \mathcal{E}_\alpha$, $G \in \mathcal{E}_\beta$ such that $\alpha + \beta = 1 - n$. Then*

$$\int_{\partial C} F(X)\Sigma(X, dX)G(X) = \int_C [(FD_\alpha)G + F(D_\beta G)]\, L(X, dX). \tag{17}$$

where C represents a $n-1$-chain in the manifold of rays.

We now introduce the following simplification of notation: we shall denote by t the last coordinate, X_{n+2}. Hence, we write $X = \underline{X} + t\mathbf{e}_{n+2}$.

Definition 3.4 (Eelbode, see [5]) *The fundamental solution for D_α is given by*

$$\begin{aligned} E(\underline{X}, t) \;\; = \;\; & \frac{(-1)^{\frac{n-1}{2}}}{\alpha 2^n \pi^{\frac{n-1}{2}} \Gamma(\frac{n-1}{2})} \\ & \left(-\partial_{\underline{X}} + \mathbf{e}_{n+2}\partial_t\right) \Delta_{\underline{X}}^{\frac{n-1}{2}} \int_0^1 (t - ry)^\alpha (1 - y^2)^{\frac{n-3}{2}} dy \end{aligned} \tag{18}$$

with $r = |\underline{X}|$.

Theorem 3.5 (Cauchy's Integral Formula) *For every $G \in \mathcal{E}_\beta$ such that $\alpha + \beta = 1 - n$ we have*

$$\begin{aligned} \int_{\partial C} E(Y - X)\Sigma(Y, dY)G(Y) = G(X) + \\ \int_C [E(Y - X)(D_\beta G)]\, L(Y, dY). \end{aligned} \tag{19}$$

Proof: This theorem is an immediate consequence of the Cauchy-Pompeiu formula (17), assuming $F(Y) = E(Y - X) \in \mathcal{E}_\alpha$.

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4 $Q_{\mathcal{L},p}$-scales on the Hyperbolic Space

In the complex case, the Q_p-spaces

$$\{f | f \;\; hol. \;\, in \;\, B_1(0), \quad \sup_{a \in B_1(0)} \int_{B_1(0)} |f'(z)|^2 g^p(z, a) dxdy < \infty\}$$

represents a continuous scale of spaces linking the Dirichlet space

$$\mathbf{D} = \{f | f \;\; anal. \;\, in \;\, B_1(0), \mathcal{D}(f) = \int_\Delta |f'(z)|^2 dxdy < \infty\} \tag{20}$$

to the Bloch space

$$\mathbf{B} = \{f | f \text{ } anal. \text{ } in \text{ } B_1(0), \mathcal{B}(f) = \sup_{z \in B_1(0)} (1 - |z|^2)|f'(z)| < \infty\} \qquad (21)$$

by strict inclusions. However, in higher dimensions the Green function $g(z,a)$ of the original definition (1) has a singularity which only allows to consider Q_p-spaces for $p < n/(n-2)$. Hence, we follow [6] by replacing the term $g^p(z,a)$ by $(1 - |\varphi_a(z)|^2)^p$, with φ_a an automorphism of the unit ball.

4.1 General Definitions

Definition 4.1 *Let $\mathcal{L}$ be a first order differential operator. Moreover, assume that, for every $F \in Ker(D_\alpha) \cap \mathcal{E}_\alpha$ we have $\mathcal{L}F \in Ker(D_\beta) \cap \mathcal{E}_\beta$ (hence $\alpha + \beta = -n$). Then we define the $Q_{\mathcal{L},p}$-space as the set of all $F \in Ker(D_\alpha) \cap \mathcal{E}_\alpha$ satisfying*

$$\sup_{\varphi \in \mathbf{Spin}(1,n)} \int_{B_1(0)} |\mathcal{L}F(X)|^2 S_H^p(\varphi(X)) L(X, dX) < \infty \qquad (22)$$

where $0 < p < \infty$, $S_H(X) = |X_{n+2}^2 - \sum_{j=1}^n X_j^2|$ and $B_1(0)$ denotes the identification of the unitary ball as subset of $\{X | 2X_{n+2} - \sum_{j=1}^n X_j^2 = 1 \wedge |\underline{X}|^2 \leq 1\}$.

Moreover, we define the corresponding *Dirichlet space* $\mathbf{D}_\mathcal{L}$ as the set of all $F \in Ker(D_\alpha) \cap \mathcal{E}_\alpha$ satisfying to

$$\sup_{\varphi \in \mathbf{Spin}(1,n)} \int_{B(1)} |\mathcal{L}F(X)|^2 L(X, dX) < \infty. \qquad (23)$$

Then it is easy to see that

i) the $Q_{\mathcal{L},p}$-spaces form a scale of Banach Clifford-modules.

ii) they satisfy the usual inclusion property, that is, $Q_{\mathcal{L},p} \subset Q_{\mathcal{L},q}$, for $0 < p < q$.

iii) $Q_{\mathcal{L},0} = \mathbf{D}_\mathcal{L}$.

4.2 A First Estimate

From (19) we have

Lemma 4.2 *For every G such that $\mathcal{L}G \in Ker(D_\beta) \cap \mathcal{E}_\beta$ it holds*

$$\int_{\partial C} E(Y - X) \Sigma(Y, dY) \mathcal{L}G(Y) = \mathcal{L}G(X). \qquad (24)$$

We now obtain an estimative for the modulus of $\mathcal{L}G$ in the origin.

Theorem 4.3 *Let G be such that $\mathcal{L}G \in Ker(D_\beta) \cap \mathcal{E}_\beta$. Then there exists a real constant $K(\alpha) > 0$ such that*

$$|\mathcal{L}G(0)| \leq K(\alpha) \int_{\partial C} |\Sigma(X, dX)||\mathcal{L}G(X)| \tag{25}$$

for every $n-1$-chain C in $\{X | 2t - \sum_{j=1}^{n} X_j^2 = 1 \wedge |\underline{X}|^2 \leq 1\}$.

Proof: Indeed

$$\begin{aligned} |\mathcal{L}G(0)| &\leq \left| \int_{\partial C} E(X)\Sigma(X, dX)\mathcal{L}G(X) \right| \\ &\leq \int_{\partial C} |E(X)||\Sigma(X, dX)|\mathcal{L}G(X)| \end{aligned}$$

The fundamental solution (18) of the α-homogeneous Dirac operator can be written as

$$\begin{aligned} E(\underline{X}, t) &= C(\alpha)(-\partial_{\underline{X}} + \mathbf{e}_{n+2}\partial_t)\partial_{\underline{X}}^{n-1} \int_0^1 (t - ry)^\alpha (1 - y^2)^{\frac{n-3}{2}} dy \\ &= \sum_{j=0}^{\alpha} \left\{ C(\alpha)(-1)^j \binom{\alpha}{j} \int_0^1 [y^j (1 - y^2)^{\frac{n-3}{2}} dy](-\partial_{\underline{X}} + \mathbf{e}_{n+2}\partial_t)\partial_{\underline{X}}^{n-1} \; t^{\alpha - j} r^j \right\} \\ &= \sum_{j=0}^{\alpha} \left\{ C'(\alpha, j)(-\partial_{\underline{X}} + \mathbf{e}_{n+2}\partial_t)\partial_{\underline{X}}^{n-1} \; t^{\alpha - j} r^j \right\} \end{aligned}$$

where $C(\alpha)$ is, up to the sign, $\frac{(-1)^{\frac{n-1}{2}}}{\alpha 2^n \pi^{\frac{n-1}{2}} \Gamma(\frac{n-1}{2})}$ and

$$C'(\alpha, j) = (-1)^j C(\alpha) \binom{\alpha}{j} \int_0^1 [y^j (1 - y^2)^{\frac{n-3}{2}} dy] \neq 0,$$

for all $j = 0, \cdots, \alpha$.

Hence, for every $n-1$-chain C in $\{X | 2t - \sum_{j=1}^{n} X_j^2 = 1 \wedge |\underline{X}|^2 \leq 1\}$ (which implies $\frac{1}{2} \leq |t| \leq 1$) we have

$$\begin{aligned} |E(\underline{X}, t)| &\leq \sum_{j=0}^{\alpha} \left| C'(\alpha, j)(-\partial_{\underline{X}} + \mathbf{e}_{n+2}\partial_t)\partial_{\underline{X}}^{n-1} \; t^{\alpha - j} r^j \right| \\ &\leq \sum_{j=0}^{\alpha} \left(\left| C'(\alpha, j)\partial_{\underline{X}}^{n} \; t^{\alpha - j} r^j \right| + \left| C'(\alpha, j)\partial_t \partial_{\underline{X}}^{n-1} \; t^{\alpha - j} r^j \right| \right), \end{aligned}$$

which ensures the existence of a positive constant $K(\alpha)$ given by

$$\sum_{j=0}^{\alpha} \left(\left| C'(\alpha, j)\partial_{\underline{X}}^{n} \; t^{\alpha - j} r^j \right| + \left| C'(\alpha, j)\partial_t \partial_{\underline{X}}^{n-1} \; t^{\alpha - j} r^j \right| \right).$$

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