Finite Difference Approximations of the Cauchy-Riemann Operators and the Solution of Discrete Stokes and Navier-Stokes Problems in the Plane

A. Hommel, K. Gürlebeck   Bauhaus-University of Weimar

The idea to calculate solutions of boundary value problems by using finite differences is very old. Sixty years ago first attempts were made to consider solutions of discrete Cauchy-Riemann equations as a class of discrete analytic functions. We refer for instance to [1], [2], [4], [10], [13] and [14]. In order to establish a discrete function theory some main problems are to overcome: We need a discrete analogue of the Cauchy integral and we are looking for a factorization of the two-dimensional real Laplacian into two adjoint Cauchy-Riemann operators. Furthermore we are confronted with the problem, that discrete analytic functions do not form an algebra with respect to the usual complex multiplication. These are reasons, why there was no essential progress in discrete function theories over a long period of time. A series of work in Clifford analysis has shown, that a commutative algebra is not necessary to adapt function theoretic methods to the solution of boundary value problems. We find a survey and a collection of examples in [5] and [15]. These works were inspired by analogous ideas in the field of discrete potential theory. Main results on this field are published in [16] and [3] and later in [11].

In the following we define difference operators that realise the factorization of the real Laplacian into two adjoint Cauchy-Riemann operators. Based on the existence of a discrete fundamental solution we define a discrete version of the T-operator, that is right-inverse to the discrete Cauchy-Riemann operator. In relation with this operator a discrete Borel-Pompeiu formula is presented. Furthermore a decomposition of the space l_2 into the space of discrete analytic functions and its orthogonal complement is possible. By introducing the orthoprojectors P_h^+ and Q_h^+ we can prove properties that guarantee the existence and uniqueness for the solution of discrete Stokes problems. In addition we state a problem that is equivalent to the Navier-Stokes problem and can be used in an iteration procedure to describe the solution of the discrete Navier-Stokes equation. For a special case of the Navier-Stokes equations we are able to calculate discrete potential and stream functions. The adapted model includes important algebraical properties and can immediately be used for numerical calculations.

1. Approximation of the Cauchy-Riemann Operators in the Complex Plane

Let $\mathbb{R}^2$ be the 2-dimensional Euclidean space with the unit vectors $b_1 = (1,0)$ and $b_2 = (0,1)$ and $x = (x_1, x_2)$ be an arbitrary element of this space. We are looking for a discretization of the Cauchy-Riemann operators $D^1 = (-i)\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)$ and $D^2 = i\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right)$. An equidistant lattice with the mesh width $h > 0$ is defined by $\mathbb{R}_h^2 = \{mh = (m_1 h, m_2 h)\}$ with $m_1, m_2 \in \mathbb{Z}$. We consider complex valued functions $f(mh) = \operatorname{Re} f(mh) + i \operatorname{Im} f(mh) = (f_0(mh), f_1(mh))$ and introduce forward differences $D_h^j f_k(mh) = h^{-1}(f_k(mh + hb_j) - f_k(mh))$ and backward differences $D_h^{-j} f_k(mh) = h^{-1}(f_k(mh) - f_k(mh - hb_j))$ for $j \in \{1,2\}$ and $k \in \{0,1\}$. The Cauchy-Riemann operators can be approximated with the difference operators

$$D_{h,1} = \begin{pmatrix} D_h^{-2} & D_h^1 \\ -D_h^{-1} & D_h^2 \end{pmatrix} \text{ and } D_{h,2} = \begin{pmatrix} D_h^2 & -D_h^1 \\ D_h^{-1} & D_h^{-2} \end{pmatrix}.$$

These difference operators have the important property

$$D_{h,1}\,D_{h,2}=D_{h,2}\,D_{h,1}=I_2\,\Delta_h\,, \qquad (1)$$

where I_2 is the $2x2$ identity matrix and the discrete Laplace operator is defined by $\Delta_h u_h(mh)=\sum_{k\in K} a_k u_h(mh-kh)$ with $K=\{(0,0),(-1,0),(1,0),(0,-1),(0,1)\}$ and $a_k=\begin{cases} 1/h^2 & for\ \ k\in K, k\neq (0,0) \\ -4/h^2 & for\ \ k=(0,0).\end{cases}$ For other approximations of the Cauchy-Riemann operator we refer to [2], [4], [13] and [14].

2. Discrete Fundamental Solution

Each 2x2-matrix $E_h^j(mh)$, which is a solution of the system $D_{h,j}\,E_h^j(mh)=I_2\,\delta(mh)$ with $j\in\{1,2\}$ is called *discrete fundamental solution.* In this notation the discrete Cauchy-Riemann operator acts on each column of $E_h^j(mh)$. We obtain the following representation formulas:

$$E_h^1(mh)=\frac{1}{2\pi}\begin{pmatrix} R_hF(\zeta_{-2}^h/d^2) & R_hF(-\zeta_{-1}^h/d^2) \\ R_hF(-\zeta_1^h/d^2) & R_hF(-\zeta_2^h/d^2)\end{pmatrix} \quad \text{and}$$

$$E_h^2(mh)=\frac{1}{2\pi}\begin{pmatrix} R_hF(-\zeta_2^h/d^2) & R_hF(\zeta_{-1}^h/d^2) \\ R_hF(\zeta_1^h/d^2) & R_hF(\zeta_{-2}^h/d^2)\end{pmatrix},$$

where $R_h u$ is the restriction of a function u to $\mathbb{R}_h^2$, F is the classical Fourier transform, $\zeta_{-j}^h=h^{-1}(1-e^{-ih\zeta_j})$, $\zeta_j^h=h^{-1}(1-e^{ih\zeta_j})$ with $\zeta_j\in\left(-\frac{\pi}{h},\frac{\pi}{h}\right)$ and $d^2=\zeta_{-1}^h\,\zeta_1^h+\zeta_{-2}^h\,\zeta_2^h$. For the details of the proof and the properties of the discrete fundamental solution we refer to [6].

3. Right Inverse Operator and the Discrete Borel-Pompeiu Formula

We consider a bounded domain $G\subset\mathbb{R}^2$ and denote by $G_h=G\cap\mathbb{R}_h^2$ the discrete domain. Using the notation $K=\{k_1=(1,0),k_2=(0,1),k_3=(-1,0)\ and\ k_4=(0,-1)\}$ we define the discrete boundary $\gamma_h^-=\{rh\in\mathbb{R}_h^2\setminus G_h:\exists k_i\ with\ (r+k_i)h\in G_h,\ i=1,\dots,4\}$. Often the boundary γ_h^- is split into the parts $\gamma_{hi}^-=\{rh\in\gamma_h^-:(r+k_i)h\in G_h\}, i=1,\dots,4.$ Furthermore let $\Gamma_{sj}=\{lh\in\mathbb{R}_h^2\setminus(G_h\cup\gamma_h^-):(l+k_j)h\in\gamma_{hs}^-$ and $(l+k_s)h\in\gamma_{hj}^-\}$ with $s,j\in\{1,\dots,4\}$ be outer corners. The boundary values are set to be zero on Γ_{sj} because these corners do not play any role for solving discrete Cauchy-Riemann problems. We only need these outer corners in order to describe discrete tangential derivatives. As a discrete analogue to the T- operator we define $(T_h^1[f_0,f_1])(mh)=((T_{h1}^1[f_0,f_1])(mh),(T_{h2}^1[f_0,f_1])(mh))$. The components of the operator T_h^1 have the structure $(T_{hk}^1[f_0,f_1])(mh)=(T_{hk}^{1,G}[f_0,f_1])(mh)+(T_{hk}^{1,\gamma_h^-}[f_0,f_1])(mh)$ with

$$(T_{hk}^{1,G}[f_0,f_1])(mh)=\sum_{lh\in G_h} h^2\begin{pmatrix} E_{hk1}^1(mh-lh) \\ E_{hk2}^1(mh-lh)\end{pmatrix}^T\begin{pmatrix} f_0(lh) \\ f_1(lh)\end{pmatrix} \quad \text{and}$$

$$(T_{hk}^{1,\gamma_h^-}[f_0,f_1])(mh) = \sum_{lh\in\gamma_{h1}^-\cup\gamma_{h4}^-\cup\Gamma_{14}} h^2 \begin{pmatrix} E_{hk1}^1(mh-lh) \\ E_{hk2}^1(mh-lh) \end{pmatrix}^T \begin{pmatrix} f_0(lh) \\ 0 \end{pmatrix} + \sum_{lh\in\gamma_{h2}^-\cup\gamma_{h3}^-\cup\Gamma_{23}} h^2 \begin{pmatrix} E_{hk1}^1(mh-lh) \\ E_{hk2}^1(mh-lh) \end{pmatrix}^T \begin{pmatrix} 0 \\ f_1(lh) \end{pmatrix}$$

where in the union of boundary parts γ_{hi}^-, $i=1,\dots,4$ inner corners are counted only once and E_{hkj}^1 are the matrix components of E_h^1.

Theorem 1: *For functions* $f(mh)=(f_0(mh),f_1(mh))$ *with* $mh\in G_h$ *it can be proved that*

$$D_{h,1}(T_h^1[f_0,f_1])(mh)=f(mh).$$

For the proof we refer to [12]. We remark, that the summand $T_{hk}^{1,\gamma_h^-}[f_0,f_1]$ is only added in order to get a special structure of the discrete Borel-Pompeiu formula. The summation runs over boundary points and the factor h^2 causes that this summand of the operator T_h^1 tends more quickly to zero as $h\to 0$ than $T_{hk}^{1,G}[f_0,f_1]$. In a similar way we can define an operator $T_h^2[f_0,f_1](mh)$ such that $D_{h,2}(T_h^2[f_0,f_1])(mh)=f(mh)$. For the details we refer to [7].

Now we present a discrete version of the Borel-Pompeiu formula. In order to describe normal unit vectors we use the homeomorphism between complex numbers $a+ib$ and matrices $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. On the boundary parts γ_{hj}^-, $j=1,\dots,4$ we define $\begin{pmatrix} n_1^j & n_2^j \\ n_3^j & n_4^j \end{pmatrix}$ by

$$\begin{pmatrix} n_1^1 & n_2^1 \\ n_3^1 & n_4^1 \end{pmatrix} = -\begin{pmatrix} n_1^3 & n_2^3 \\ n_3^3 & n_4^3 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad\text{and}\quad \begin{pmatrix} n_1^2 & n_2^2 \\ n_3^2 & n_4^2 \end{pmatrix} = -\begin{pmatrix} n_1^4 & n_2^4 \\ n_3^4 & n_4^4 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We introduce the boundary operator $(F_h^1[f_0,f_1])(mh)=((F_{h1}^1[f_0,f_1])(mh),(F_{h2}^1[f_0,f_1])(mh))$ with the components $(F_{hk}^1[f_0,f_1])(mh)=(F_{hk}^{1,\gamma_h^-}[f_0,f_1])(mh)+(F_{hk}^{1,\gamma_I^-}[f_0,f_1])(mh)$. In detail the summands are the terms

$$(F_{hk}^{1,\gamma_h^-}[f_0,f_1])(mh) = i\sum_{j=1}^{4}\sum_{lh\in\gamma_{hj}^-} h \begin{pmatrix} E_{hk1}^1(mh-lh) \\ E_{hk2}^1(mh-lh) \end{pmatrix}^T \begin{pmatrix} n_1^j & n_2^j \\ n_3^j & n_4^j \end{pmatrix} \begin{pmatrix} f_0(lh) \\ f_1(lh) \end{pmatrix}$$

and

$$\begin{aligned}(F_{hk}^{1,\gamma_I^-}[f_0,f_1])(mh) = &-i\sum_{lh\in\gamma_{h1}^-\cap\gamma_{h4}^-} h \begin{pmatrix} E_{hk1}^1(mh-lh) \\ 0 \end{pmatrix}^T \begin{pmatrix} n_1^1+n_1^4 & n_2^1+n_2^4 \\ n_3^1+n_3^4 & n_4^1+n_4^4 \end{pmatrix} \begin{pmatrix} f_0(lh) \\ f_1(lh) \end{pmatrix} \\ &-i\sum_{lh\in\gamma_{h2}^-\cap\gamma_{h3}^-} h \begin{pmatrix} 0 \\ E_{hk2}^1(mh-lh) \end{pmatrix}^T \begin{pmatrix} n_1^2+n_1^3 & n_2^2+n_2^3 \\ n_3^2+n_3^3 & n_4^2+n_4^3 \end{pmatrix} \begin{pmatrix} f_0(lh) \\ f_1(lh) \end{pmatrix} \\ &-i\sum_{lh\in\gamma_{h1}^-\cap\gamma_{h2}^-} h \begin{pmatrix} E_{hk1}^1(mh-lh) \\ E_{hk2}^1(mh-lh) \end{pmatrix}^T \begin{pmatrix} n_1^1+n_1^2 & n_2^1+n_2^2 \\ n_3^1+n_3^2 & n_4^1+n_4^2 \end{pmatrix} \begin{pmatrix} f_0(lh) \\ 0 \end{pmatrix} \\ &-i\sum_{lh\in\gamma_{h3}^-\cap\gamma_{h4}^-} h \begin{pmatrix} E_{hk1}^1(mh-lh) \\ E_{hk2}^1(mh-lh) \end{pmatrix}^T \begin{pmatrix} n_1^3+n_1^4 & n_2^3+n_2^4 \\ n_3^3+n_3^4 & n_4^3+n_4^4 \end{pmatrix} \begin{pmatrix} 0 \\ f_1(lh) \end{pmatrix}.\end{aligned}$$

Theorem 2: *The Borel-Pompeiu formula has the coordinate-wise structure*

$$(T_{hk}^1[D_h^{-2}f_0+D_h^1 f_1, -D_h^{-1}f_0+D_h^2 f_1])(mh)+(F_{hk}^1[f_0,f_1])(mh)=f_{k-1}(mh)\chi_{k-1}, \quad k=1,2$$

with $\chi_0=\begin{cases}1 & \forall mh\in G_h\cup\gamma_{h1}^-\cup\gamma_{h2}^- \\ 0 & else\end{cases}$ *and* $\chi_1=\begin{cases}1 & \forall mh\in G_h\cup\gamma_{h3}^-\cup\gamma_{h4}^- \\ 0 & else.\end{cases}$

In a similar way a Borel-Pompeiu formula can be proved, which is based on the operators T_h^2 and F_h^2. For the details we refer to [7].

4. Orthogonal Decomposition of the Space $l_2(G_h)$

Let $l_2(G_h)$ be the space of functions $w(mh)=(w_0(mh),w_1(mh))$ and $v(mh)=(v_0(mh),v_1(mh))$ with the scalar product $<w,v>=\sum_{mh\in G_h} h^2 \begin{pmatrix} w_0(mh) \\ w_1(mh) \end{pmatrix}^T \begin{pmatrix} v_0(mh) \\ v_1(mh) \end{pmatrix}$. If for fixed mesh width h in all points $mh\in G_h$ not only the values $w(mh)$ and $v(mh)$ are defined but also the difference quotients $D_h^{\pm j} w_i(mh)$ and $D_h^{\pm j} v_i(mh)$ for $j\in\{1,2\}$ and $i\in\{0,1\}$ and the condition $w(rh)= v(rh)=(0,0)$ is fulfilled for all $rh\in\gamma_h^-$ then we denote this space of functions by $\overset{\circ}{w}{}_2^1(G_h)$.

Theorem 3: W*e get the orthogonal decomposition* $l_2(G_h)=\ker D_{h,1}(G_h)\oplus D_{h,2}(\overset{\circ}{w}{}_2^1(G_h))$.

For the proof we refer to [8]. Similar to the continuous case we call functions in the kernel of the operator $D_{h,1}$ *discrete analytic functions.* Based on the orthogonal decomposition in Theorem 3 we denote the orthoprojectors on $\ker D_{h,1}(G_h)$ or $D_{h,2}(\overset{\circ}{w}{}_2^1(G_h))$ by P_h^+ or Q_h^+, respectively. For all $mh\in G_h$ we write $Q_h^+[f_0,f_1](mh)=((Q_{h1}^+[f_0,f_1])(mh),(Q_{h2}^+[f_0,f_1])(mh))$.

5. Discrete Stokes- and Navier-Stokes Problems in the Plane

We consider the boundary value problem

$$\begin{aligned} -\Delta_h u_0(mh)+\frac{1}{\mu}D_h^1 p(mh) &= \frac{\rho}{\mu} f_0(mh) && \forall mh\in G_h \\ -\Delta_h u_1(mh)+\frac{1}{\mu}D_h^2 p(mh) &= \frac{\rho}{\mu} f_1(mh) && \forall mh\in G_h \\ D_h^{-1}u_0(mh)+D_h^{-2}u_1(mh) &= \varphi(mh) && \forall mh\in G_h \\ u(rh)=(u_0(rh),u_1(rh)) &= (\psi_0(rh),\psi_1(rh)) && \forall rh\in\gamma_h^-, \end{aligned} \qquad (2)$$

where ρ is the density, μ the viscosity, p the pressure of the fluid, f_0 and f_1 are the vector components of the exterior forces, u_0 and u_1 are the velocity components of the fluid inside the domain, $\varphi(mh)$ is a measure for the compressibility of the fluid and ψ_0 and ψ_1 are the velocity components on the boundary.

Theorem 4: *The boundary value problem (2) with* $\varphi(mh)=0\ \forall mh\in G_h$ *and* $\psi_0(rh)=\psi_1(rh)=0\ \ \forall rh\in\gamma_h^-$ *has for each right-hand side* $f(mh)=(f_0(mh),f_1(mh))\in l_2(G_h)$ *a unique solution* $u(mh)=(u_0(mh),u_1(mh))$. *The pressure* $p(mh)\in(l_2(G_h)\cup\gamma_{h3}^-\cup\gamma_{h4}^-)$ *is unique up to a constant.*

For the proof we refer to [8]. We remark that the system (2) is only solvable, if a necessary condition between $\varphi(mh)$ and $\psi(rh)$ is fulfilled.

Based on the Stokes problem we will present a possibility to solve the Navier-Stokes equations

$$\begin{aligned} -\Delta_h u_0(mh)+\frac{1}{\mu}D_h^1 p(mh)+\frac{\rho}{\mu}(u_0(mh)D_h^{-1}u_0(mh)+u_1(mh)D_h^{-2}u_0(mh) &= \frac{\rho}{\mu}f_0(mh) \\ -\Delta_h u_1(mh)+\frac{1}{\mu}D_h^2 p(mh)+\frac{\rho}{\mu}(u_0(mh)D_h^{-1}u_1(mh)+u_1(mh)D_h^{-2}u_1(mh) &= \frac{\rho}{\mu}f_1(mh) \qquad (3) \\ D_h^{-1}u_0(mh)+D_h^{-2}u_1(mh) &= 0 \\ u(rh) &= (0,0) \end{aligned}$$

for all $mh \in G_h$ and $rh \in \gamma_h^-$. In order to simplify the notation we substitute

$$M_{h0}(mh)=\frac{\rho}{\mu}(u_0(mh)D_h^{-1}u_0(mh)+u_1(mh)D_h^{-2}u_0(mh)-\frac{\rho}{\mu}f_0(mh)$$

$$M_{h1}(mh)=\frac{\rho}{\mu}(u_0(mh)D_h^{-1}u_1(mh)+u_1(mh)D_h^{-2}u_1(mh)-\frac{\rho}{\mu}f_1(mh).$$

Theorem 5: *The boundary value problem (3) is equivalent to the problem*

$$\begin{aligned} u(mh) = &\quad (T_h^2 Q_h^+ T_h^1[M_{h0},M_{h1}])(mh)+\frac{1}{\mu}(T_h^2 Q_h^+[0,p])(mh) \\ -(Q_{h2}^+ T_h^1[M_{h0},M_{h1}])(mh) = &\quad \frac{1}{\mu}(Q_{h2}^+[0,p])(mh). \end{aligned}$$

The proof is published in [8]. Based on Theorem 5 an iteration procedure can be established in order to calculate the solution of the problem (3):

Theorem 6: *Let* $(u_0^0(mh),u_1^0(mh)) \in \overset{\circ}{w}{}_2^1(G_h) \cap \ker div_h^-$ *with* $div_h^- u(mh) = D_h^{-1}u_0(mh)+D_h^{-2}u_1(mh)$. *The iteration procedure*

$$u^n(mh)=(T_h^2 Q_h^+ T_h^1[M_{h0}^{n-1},M_{h1}^{n-1}])(mh)+\frac{1}{\mu}(T_h^2 Q_h^+[0,p^n])(mh)$$

$$-(Q_{h2}^+ T_h^1[M_{h0}^{n-1},M_{h1}^{n-1}])(mh)=\frac{1}{\mu}(Q_{h2}^+[0,p^n])(mh) \qquad n=1,2,3,\ldots$$

with $M_{hj}^{n-1}(mh)=\frac{\rho}{\mu}(u_0^{n-1}(mh)D_h^{-1}u_j^{n-1}(mh)+u_1^{n-1}(mh)D_h^{-2}u_j^{n-1}(mh))-\frac{\rho}{\mu}f_j(mh)$ *for* $j\in\{0,1\}$ *converges to the solution of the problem (3).*

For the proof we refer to [8]. We remark, that in each step n the approximate solution of (3) is expressed by the solution (u^n,p^n) of a Stokes problem.

6. Potential- and Stream Functions

We consider now a special case of the stationary Navier-Stokes equations. We write the classical equations in the form

$$\rho\left(u_0\frac{\partial u_0}{\partial x_1}+u_1\frac{\partial u_0}{\partial x_2}\right)=f_0(x)+\mu\left(\frac{\partial^2 u_0}{\partial x_1^2}+\frac{\partial^2 u_0}{\partial x_2^2}\right)-\frac{\partial p}{\partial x_1}$$

$$\rho\left(u_0\frac{\partial u_1}{\partial x_1}+u_1\frac{\partial u_1}{\partial x_2}\right)=f_1(x)+\mu\left(\frac{\partial^2 u_1}{\partial x_1^2}+\frac{\partial^2 u_1}{\partial x_2^2}\right)-\frac{\partial p}{\partial x_2}$$

and approximate them by

$$\begin{aligned}\rho(u_0^{lk}D_h^1u_0^{lk-1}+u_1^{lk}D_h^2u_0^{lk-1})&=f_0^{lk}+\mu(D_h^{-1}D_h^1u_0^{lk}+D_h^{-2}D_h^2u_0^{lk})-D_h^1p^{lk}\\ \rho(u_0^{lk}D_h^1u_1^{l-1k}+u_1^{lk}D_h^2u_1^{l-1k})&=f_1^{lk}+\mu(D_h^{-1}D_h^1u_1^{lk}+D_h^{-2}D_h^2u_1^{lk})-D_h^2p^{lk}\end{aligned} \tag{4}$$

with $u_i^{lk}=u_i(lh,kh)$, $f_i^{lk}=f_i(lh,kh)$ for $i\in\{0,1\}$ and $p^{lk}=p(lh,kh)$. We consider the special case $\mu=0$ as well as $f_0^{lk}=f_1^{lk}=0$ and eliminate the pressure in (4). From the equation $-D_h^2(u_0^{lk}D_h^1u_0^{lk-1}+u_1^{lk}D_h^2u_0^{lk-1})+D_h^1(u_0^{lk}D_h^1u_1^{l-1k}+u_1^{lk}D_h^2u_1^{l-1k})=0$ it follows

$$-D_h^2u_0^{lk}(D_h^1u_0^{lk}+D_h^2u_1^{lk})+D_h^1u_1^{lk}(D_h^1u_0^{lk}+D_h^2u_1^{lk})+u_0^{lk}D_h^1(D_h^{-1}u_1^{lk}-D_h^{-2}u_0^{lk})+u_1^{lk}D_h^2(D_h^{-1}u_1^{lk}-D_h^{-2}u_0^{lk})=0,$$

where $D_h^1u_0^{lk}+D_h^2u_1^{lk}=0$ approximates the continuity equation $\frac{\partial u_0}{\partial x_1}+\frac{\partial u_1}{\partial x_2}=0$ and $D_h^{-1}u_1^{lk}-D_h^{-2}u_0^{lk}=0$ approximate the equation $\frac{\partial u_1}{\partial x_1}-\frac{\partial u_0}{\partial x_2}=0$. In the following we neglect the index lk in order to simplify the notation. By using the ansatz $u_0=D_h^{-1}\Phi_h=D_h^2\Psi_h$ and $u_1=D_h^{-2}\Phi_h=-D_h^1\Psi_h$ we can prove the following properties of the discrete potential function Φ_h:

$$\begin{aligned}D_h^{-1}u_1-D_h^{-2}u_0&=D_h^{-1}D_h^{-2}\Phi_h-D_h^{-2}D_h^{-1}\Phi_h=0\\ D_h^1u_0+D_h^2u_1&=D_h^1D_h^{-1}\Phi_h+D_h^2D_h^{-2}\Phi_h=\Delta_h\Phi_h=0.\end{aligned}$$

For the discrete stream function Ψ_h we obtain

$$\begin{aligned}D_h^{-1}u_1-D_h^{-2}u_0&=-D_h^{-1}D_h^1\Psi_h-D_h^{-2}D_h^2\Psi_h=-\Delta_h\Psi_h=0\\ D_h^1u_0+D_h^2u_1&=D_h^1D_h^2\Psi_h-D_h^2D_h^1\Psi_h=0.\end{aligned}$$

The above ansatz can be also written in the form $\begin{pmatrix}D_h^{-2} & D_h^1\\ -D_h^{-1} & D_h^2\end{pmatrix}\begin{pmatrix}\Phi_h\\ \Psi_h\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$, such that the relation to the discrete Cauchy-Riemann operator becomes obviously.

At the end we present a numerical example. From the continuous case we know, that in case of a stream with the potential function $\Phi(x_1,x_2)=\frac{1}{2}(x_1^2-x_2^2)$ the stream lines have the structure $x_1\cdot x_2=const.$ We have calculated the solution of the following discrete problem in a square with the corners $(0,0),(1.5,0),(1.5,1.5)$ and $(0,1.5)$:

$$\begin{pmatrix}D_h^{-2} & D_h^1\\ -D_h^{-1} & D_h^2\end{pmatrix}\begin{pmatrix}\Phi_h(mh)\\ \Psi_h(mh)\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\quad \forall mh\in G_h$$

$$\Phi_h(rh)=\frac{1}{2}\left((r_1h)^2-(r_2h)^2\right)\quad \forall rh\in\gamma_h^-$$

$$\Psi_h(m^*h)=(m_1^*h)\cdot(m_2^*h)\quad \textit{with}\quad m_1^*=m_2^*=1\quad (\textit{in order to get uniqueness}).$$

For the details we refer to [9]. The behaviour of the calculated stream lines is presented in the following picture:

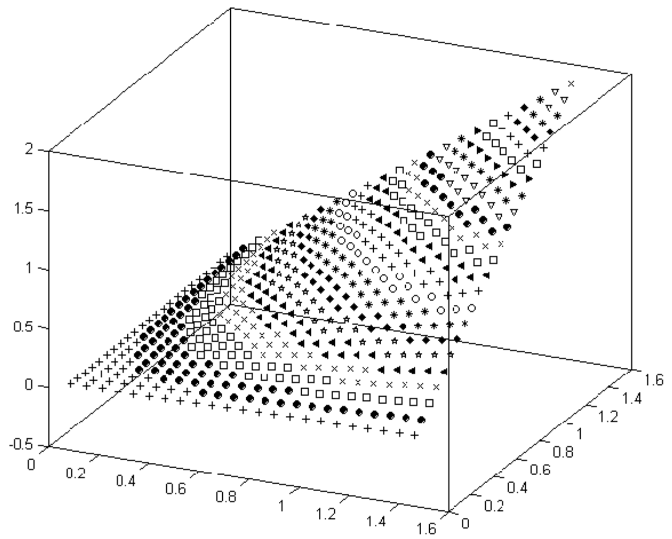

Figure 1: Behaviour of the stream lines in the discrete case

References

[1] Deeter, C.R.;Lord,M.E.: Further theory of operational calculus on discrete analytic functions. J.Math.Anal.Appl. 26 (1969), 92-113.

[2] Duffin, R.J.: Basic properties of discrete analytic functions. Duke Math. J. 23 (1956), 335-363.

[3] Duffin, R.J.: Discrete potential theory. Duke Math. J. 20 (1953), 233-251.

[4] Ferrand,J.: Fonctions preharmonique et fonctions preholomorphes. Bulletin des Sciences Mathematique, sec. series 68 (1944), 152-180.

[5] Gürlebeck,K.;Sprößig,W.:Quaternionic and Clifford Calculus for Engineers and Physicists. John Wiley & Sons,Chichester, 1997.

[6] Gürlebeck,K.; Hommel,A.: Finite Difference Cauchy-Riemann Operators and Their Fundamental Solutions in the Complex Case. This paper will be published in Operator Theory-Advances and Applications.

[7] Gürlebeck,K.; Hommel,A.: A Discrete Analogue of the Complex T-Operator and a Proof of the Discrete Borel-Pompeiu Formula. Preprint of the Bauhaus-University Weimar, 2003.

[8] Gürlebeck,K.; Hommel,A.: On Discrete Stokes and Navier-Stokes Equations in the Plane. accepted paper for the proceedings of the 6th Conference on Clifford Algebras in Cookeville, Tennessee, 2002.

[9] Gürlebeck,K.; Hommel,A.: A Discretization of the Navier-Stokes Equations. submitted paper of the proceedings of the converence Clifford Analysis and Its Applications, Macau, August 2002 .

[10] Hayabara,S.: Operational calculus on the discrete analytic functions. Math. Japon. 11 (1966), 35-65.

[11] Hommel,A.: Fundamentallösungen partieller Differenzenoperatoren und die Lösung diskreter Randwertprobleme mit Hilfe von Differenzenpotentialen. Dissertation, Bauhaus-Universität Weimar, 1998.

[12] Hommel,A.: Construction of a Right Inverse Operator to the Discrete Cauchy-Riemann Operator. submitted paper of the proceedings of the 3th international ISAAC congress in Berlin, August 2001.

[13] Isaacs,R.: Monodiffric Functions. National Bureau of Standards Applied Mathematics Series 18 (1952), 257-266.

[14] Isaacs, R.P.: A finite difference function theory. Universidad Nacional Tucoman, Revista 2 (1941), 177-201.

[15] Kravchenko,V.V.;Shapiro,M.V.: Integral representations for spatial models of mathematical physics. Research Notes in Mathematics 351, Pitman Advanced Publishing Program, London, 1996.

[16] Ryabenkij,V.S.: The method of difference potentials for some problems of continuum mechanics. Moscow, Nauka 1987 (Russian).