SEVERAL KINDS OF STABILITY OF EFFICIENT SOLUTIONS IN VECTOR TRAJECTORIAL DISCRETE OPTIMIZATION PROBLEM[1]

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Let $E = \{e_1, ..., e_m\}$, $m > 1$, be a given set, $2^E \supseteq T = \{t\}$, $\mid T \mid > 1$, be a set of trajectories, i.e. non-empty subsets of the set E.

The vector weight function is ascribed to the elements of E:

$$e_j \longrightarrow (a_1(e_j), a_2(e_j), ..., a_n(e_j)),\ j \in N_m = \{1, 2, ..., m\}.$$

It can be represented as a matrix

$$A = \{a_{ij}\}_{n\times m} \in R^{nm},\ a_{ij} = a_i(e_j),\ n \geq 2.$$

The vector criterion

$$F(t, A) = (F_1(t, A), F_2(t, A), ..., F_n(t, A)),\ F_i(t, A) \longrightarrow \min_T,\ i \in N_n$$

consists of arbitrary combination of partial criteria of the kinds

$$MINSUM \quad F_i(t, A) = \sum_{e\in t} a_i(e) \ \rightarrow \ \min_T,\ i \in I_{SUM},$$

$$MINMAX \quad F_i(t, A) = \max_{e\in t} a_i(e) \ \rightarrow \ \min_T,\ i \in I_{MAX},$$

$$MINMIN \quad F_i(t, A) = \min_{e\in t} a_i(e) \ \rightarrow \ \min_T,\ i \in I_{MIN},$$

where $I_{SUM} \cup I_{MAX} \cup I_{MIN} = N_n$.

We consider
the set $T_1(A) = \{t \in T : \ \forall t' \in T \setminus \{t\}\ \exists i \in N_n \ \ \tau_i(t, t', A) < 0\}$
of the strongly efficient trajectories (Smale set),

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the set $T_2(A) = \{t \in T : \nexists t' \in T \ \forall i \in N_n \ \ (\tau_i(t,t',A) \geq 0 \ \& \ \tau(t,t',A) \neq \mathbf{0})\}$
of the efficient trajectories (Pareto set),
the set $T_3(A) = \{t \in T : \nexists t' \in T \ \forall i \in N_n \ \ \tau_i(t,t',A) > 0\}$
of the weakly efficient trajectories (Slater set), where

$$\tau(t,t',A) = (\tau_1, \tau_2, ..., \tau_n),$$

$$\tau_i = \tau_i(t,t',A) = F_i(t,A) - F_i(t',A),$$

$$\mathbf{0} = (0,0,...,0) \in R^n.$$

The following is evident

$$\forall A \in R^{nm} \ \ T_1(A) \subseteq T_2(A) \subseteq T_3(A) \subseteq T.$$

Different types of stability of the sets $T_1(A)$, $T_2(A)$ and $T_3(A)$ have been investigated in the [1-4].

1. Stability of efficient trajectories to perturbations of the whole matrix A

Let $\mathcal{B}(\varepsilon)$ denote the set $\{B \in R^{nm} : \|B\| < \varepsilon\}$,
where $B = \{b_{ij}\}_{n \times m}$, $\varepsilon > 0$, $\|B\| = \|B\|_\infty = \max\{|b_{ij}| : \ i \in N_n, \ j \in N_m\}$ is the Chebyshev norm in R^{nm}.

A trajectory $t \in T_k(A)$, $k \in N_3$ is said to be stable, if

$$\exists \varepsilon > 0 \ \forall B \in \mathcal{B}(\varepsilon) \quad t \in T_k(A + B).$$

Theorem 1. Any trajectory $t \in T_1(A)$ is stable.

Theorem 2. Let $t \in T_2(A)$. If $t \in T_1(A)$ then t is stable. The converse is true under assumption $I_{SUM} \neq \emptyset$.

Theorem 3. Let $t \in T_3(A)$. If $t \in T_1(A)$ then t is stable. The converse is true under assumption $I_{SUM} = N_n$.

Corollary 1. If $I_{SUM} = N_n$ and $T_1(A) = \emptyset$ then every weakly efficient trajectory isn't stable.

Remark. Since all the norms in R^{nm} are equivalent theorems 1–3 are valid for any norm defined in R^{nm}.

If $t \in T_k(A),\ k \in N_3$, then

$$\rho_k(t,A) = \begin{cases} \sup\{\varepsilon > 0 :\ \forall B \in \mathcal{B}(\varepsilon) \quad t \in T_k(A+B)\} & \text{if t is stable,} \\ 0 & \text{otherwise} \end{cases}$$

is called stability radius of the trajectory t.

The inequalities

$$\forall t \in T_1(A) \quad \rho_1(t,A) \le \rho_2(t,A) \le \rho_3(t,A),$$

$$\forall t \in T_2(A) \quad \rho_2(t,A) \le \rho_3(t,A)$$

are evident.

Let $\gamma_i(t,t',A) = \begin{cases} -\frac{\tau_i(t,t',A)}{2} & \text{if } i \in I_{MAX} \cup I_{MIN}, \\ -\frac{\tau_i(t,t',A)}{\Delta(t,t')} & \text{if } i \in I_{SUM}, \end{cases}$

where $\Delta(t,t') = \mid (t \cup t') \setminus (t \cap t') \mid$.

Theorem 4. For any trajectory $t \in T_k(A),\ k \in N_3$, and any sets $I_{SUM},\ I_{MAX},\ I_{MIN},\ \ I_{SUM} \cup I_{MAX} \cup I_{MIN} = N_n$, we have

$$\rho_k(t,A) \ge \varphi(t,A) = \min_{t' \in T \setminus \{t\}} \ \max_{i \in N_n} \gamma_i(t,t',A),$$

moreover

$$\forall t \in T_1(A) \quad \rho_1(t,A) = \varphi(t,A),$$

$$if\ \ I_{SUM} \neq \emptyset\ then \ \ \forall t \in T_2(A) \quad \rho_2(t,A) = \varphi(t,A),$$

$$if\ \ I_{SUM} = N_n\ then \ \ \forall t \in T_3(A) \quad \rho_3(t,A) = \varphi(t,A).$$

2. Stability of efficient trajectories to perturbations of the s-th string of the matrix A (s-stability)

Let $s \in N_n,\ \varepsilon > 0$,

$\mathcal{B}^s(\varepsilon) = \{B = \{b_{ij}\}_{n \times m} : \forall j \in N_m \quad \mid b_{sj} \mid < \varepsilon, \quad \forall i \neq s \quad b_{ij} = 0\}$.

It is evident that $\mathcal{B}^s(\varepsilon) \subseteq \mathcal{B}(\varepsilon)$.

For any trajectory $t \in T_k(A), k \in N_3$, the number

$$\rho_k^s(t,A) = \begin{cases} \sup \Lambda_k^s(t,A) & \text{if } \Lambda_k^s(t,A) \neq \emptyset, \\ 0 & \text{if } \Lambda_k^s(t,A) = \emptyset, \end{cases}$$

where $\Lambda_k^s(t,A) = \{\varepsilon > 0 : \forall B \in \mathcal{B}^s(\varepsilon) \quad t \in T_k(A+B)\}$, is called s-stability radius of the trajectory t.

The inequalities

$$\forall t \in T_1(A)\ \forall s \in N_n \quad \rho_1^s(t,A) \leq \rho_2^s(t,A) \leq \rho_3^s(t,A),$$

$$\forall t \in T_2(A)\ \forall s \in N_n \quad \rho_2^s(t,A) \leq \rho_3^s(t,A)$$

are evident.

Let $P^s(t,A) = \{t' \in T \setminus \{t\} : \forall i \in N_n \setminus \{s\} \quad \tau_i(t,t',A) \geq 0\}$,

$$\varphi^s(t,A) = \begin{cases} \min\{\gamma_s(t,t',A) :\ t' \in P^s(t,A)\} & \text{if } P^s(t,A) \neq \emptyset, \\ +\infty, & \text{if } P^s(t,A) = \emptyset. \end{cases}$$

Theorem 5. For any $A \in R^{nm},\ t \in T_1(A),\ s \in I_{SUM} \cup I_{MAX} \cup I_{MIN}$ we have

$$\rho_1^s(t,A) = \varphi^s(t,A).$$

Theorem 6. For any $A \in R^{nm},\ t \in T_2(A),\ s \in I_{SUM} \cup I_{MAX} \cup I_{MIN}$ we have

$$\rho_2^s(t,A) \geq \varphi^s(t,A),$$

moreover if $s \in I_{SUM}$ then $\rho_2^s(t,A) = \varphi^s(t,A)$.

Let $R^s(t,A) = \{t' \in T : \forall i \in N_n \setminus \{s\} \quad \tau_i(t,t',A) > 0\}$,

$$\psi^s(t,A) = \begin{cases} \min\{\gamma_s(t,t',A) :\ t' \in R^s(t,A)\} & \text{if } R^s(t,A) \neq \emptyset, \\ +\infty & \text{if } R^s(t,A) = \emptyset. \end{cases}$$

Theorem 7. For any $A \in R^{nm},\ t \in T_3(A),\ s \in I_{SUM} \cup I_{MAX} \cup I_{MIN}$ we have

$$\rho_3^s(t,A) \geq \psi^s(t,A),$$

moreover if $s \in I_{SUM}$ then $\rho_3^s(t,A) = \psi^s(t,A)$.

3. Stability of efficient trajectories to perturbations of one element of the matrix A ((s,p)-stability)

Let $(s,p) \in N_n \times N_m, \quad \varepsilon > 0,$
$\mathcal{B}^{sp}(\varepsilon) = \{B = \{b_{ij}\}_{n\times m} : \forall (i,j) \neq (s,p) \ \ (b_{ij} = 0, \ | b_{sp} | < \varepsilon)\}.$

It is evident that $\mathcal{B}^{sp}(\varepsilon) \subseteq \mathcal{B}(\varepsilon)$.

For any trajectory $t \in T_k(A), k \in N_3$, the number

$$\rho_k^{sp}(t,A) = \begin{cases} \sup \Lambda_k^{sp}(t,A) & \text{if } \Lambda_k^{sp}(t,A) \neq \emptyset, \\ 0 & \text{if } \Lambda_k^{sp}(t,A) = \emptyset, \end{cases}$$

where $\Lambda_k^{sp}(t,A) = \{\varepsilon > 0 : \forall B \in \mathcal{B}^{sp}(\varepsilon) \ \ t \in T_k(A+B)\}$, is called (s,p)-stability radius of the trajectory t.

The inequalities

$$\forall t \in T_1(A) \ \ \rho_1^{sp}(t,A) \leq \rho_2^{sp}(t,A) \leq \rho_3^{sp}(t,A),$$

$$\forall t \in T_2(A) \ \ \rho_2^{sp}(t,A) \leq \rho_3^{sp}(t,A)$$

are evident.

Let $t \neq t', \ (s,p) \in N_n \times N_m,$

$\Gamma^{sp}(t,t',A) = \{\varepsilon > 0 : \forall B \in \mathcal{B}^{sp}(\varepsilon) \quad \tau_s(t,t',A+B) < 0\},$

$\lambda^{sp}(t,t'A) = \begin{cases} \sup \Gamma^{sp}(t,t',A) & \text{if } \Gamma^{sp}(t,t',A) \neq \emptyset, \\ 0 & \text{if } \Gamma^{sp}(t,t',A) = \emptyset, \end{cases}$

$\varphi^{sp}(t,A) = \begin{cases} \min\{\lambda^{sp}(t,t',A) : \ t' \in P^s(t,A)\} & \text{if } P^s(t,A) \neq \emptyset, \\ +\infty, & \text{if } P^s(t,A) = \emptyset, \end{cases}$

$Q^s(t,A) = \{t' \in P^s(t,A) : \exists i \in N_n \setminus \{s\} \ \ \tau_i(t,t',A) \geq 0\},$

$\Omega^{sp}(t,t',A) = \{\varepsilon > 0 : \forall B \in \mathcal{B}^{sp}(\varepsilon) \quad \tau_s(t,t',A+B) \leq 0\},$

$\delta^{sp}(t,t'A) = \begin{cases} \sup \Omega^{sp}(t,t',A) & \text{if } \Omega^{sp}(t,t',A) \neq \emptyset, \\ 0 & \text{if } \Omega^{sp}(t,t',A) = \emptyset, \end{cases}$

$\omega^{sp}(t,t',A) = \begin{cases} \lambda^{sp}(t,t',A) & \text{if } t' \in Q^s(t,A), \\ \delta^{sp}(t,t',A) & \text{if } t' \notin Q^s(t,A), \end{cases}$

$\psi^{sp}(t,A) = \begin{cases} \min\{\omega^{sp}(t,t',A) : \ t' \in P^s(t,A)\} & \text{if } P^s(t,A) \neq \emptyset, \\ +\infty, & \text{if } P^s(t,A) = \emptyset, \end{cases}$

$\pi^{sp}(t,A) = \begin{cases} \min\{\delta^{sp}(t,t',A) : \ t' \in R^s(t,A)\} & \text{if } R^s(t,A) \neq \emptyset, \\ +\infty, & \text{if } R^s(t,A) = \emptyset. \end{cases}$

Theorem 8. For any $A \in R^{nm}, \ p \in N_m, \ s \in I_{SUM} \cup I_{MAX} \cup I_{MIN}$ we have

$$\forall t \in T_1(A) \ \ \rho_1^{sp}(t,A) = \varphi^{sp}(t,A),$$

$$\forall t \in T_2(A) \quad \rho_2^{sp}(t, A) = \psi^{sp}(t, A),$$

$$\forall t \in T_3(A) \quad \rho_3^{sp}(t, A) = \pi^{sp}(t, A).$$

Simple procedures for values $\lambda^{sp}(t, t', A)$ and $\delta^{sp}(t, t', A)$ calculation have been developed.

For example: if $s \in I_{MAX}$ then

$$\lambda^{sp}(t, t', A) = \begin{cases} F_s(t', A) - a_{sp} & \text{if } e_p \in t, \\ a_{sp} - F_s(t, A) & \text{if } e_p \in t' \setminus t, \tau_s(t, t' \setminus \{e_p\}, A) \geq 0, \\ +\infty & \text{otherwise.} \end{cases}$$

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