ON SPANNING TREE WITH TOPOLOGICAL CRITERIA

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A multicriterial statement of the above mentioned problem is presented. It differes from the classical statement of Spanning Tree problem. The quality of solution is estimated by vector objective function which contains weight criteria as well as topological criteria (degree and diameter of tree).

1. The formulation of the problem

Let's consider some classes of Spanning Tree problems on a system of subsets (SS). A pair $P=(E,T)$ is called SS, where E - is a set of graph edges and $|E|=Q$, T - is a set of the feasible solutions of the problem and $T=\{t\}$, where t - is the spanning tree. Vector weighting function (VWF) is given on the set E

$$w(e)=(w_1(e),\dots,w_v(e),\dots,w_N(e)), \tag{1}$$

where $w_v(e)\in R \quad \forall v=\overline{1,N} \quad \forall e\in E$.

If elements of set $E=\{e_1,\dots,e_Q\}$ are numbered than one can consider an individual VWF $w(e)$ as matrix $W=\|w_{v\kappa}\|_{N\times Q}$ in space R^{NQ}, where $Q=|E|$.

We determine the vector objective function (VOF) on the set of the solutions T

$$F(t)=(F_1(t),\dots,F_v(t),\dots,F_N(t)), \tag{2}$$

which contains the following criteria

$$F_\nu(t)=\sum w_\nu(e) \;\to\; \underset{T}{opt}, \qquad \nu\in I_1, \tag{3}$$

$$F_\nu(t)=\max_{e\in t} w_\nu(e) \;\to\; \underset{T}{opt}, \quad \nu\in I_2, \tag{4}$$

$$F_\nu(t)=\min_{e\in t} w_\nu(e) \;\to\; \underset{T}{opt}, \quad \nu\in I_3, \tag{5}$$

$$F_\nu(t)=s(t) \;\to\; \underset{T}{opt}, \qquad \nu\in I_4, \tag{6}$$

$$F_\nu(t)=d(t) \;\to\; \underset{T}{opt}, \qquad \nu\in I_5, \tag{7}$$

where I_k, $k=\overline{1,5}$ - are sets of the numbers from $\{1,2,\dots,N\}$. The criteria (3)-(7) are numbered by these numbers respactively; $s(t)=\max_{v\in t}\deg v$ - is the degree of the tree t, $d(t)$ - is the diameter of the tree t [2].

VOF $F(t)$ determines Pareto set (PS) $\tilde{T}$ on the set T. By N - criterial problem we understand the individual problem of finding and presentation of PS $\tilde{T}$ in the explicit form.

Let's designate any individual problem $z_j(W)=(E,T,W,F)$ from class Z_j, $j=\overline{0,3}$ by $z^N(W)$ and PS of this problem - by $\widetilde{T}(W)$.

Remark 1. The criteria (4),(5) belong to nonlinear criteria. The criteria (6),(7) of the degree and diameter are called the topological criteria. If we have different sets of VOF criteria, we obtain the following classes of mass problems:

Z_0^v - N-weighting criterion (3);

Z_1^v - (N-1)-weighting criterion (3) and criterion (6);

Z_2^v - (N-1)-weighting criterion (3) and criterion (7);

Z_3^v - (N-2)-weighting criterion (3) and criteria (6), (7);

Z_0^m - N-criterion (4) or (5);

Z_1^m - (N-1)-criterion (4) or (5) and criterion (6);

Z_2^m - (N-1) - criterion (4) or (5) and criterion (7);

Z_3^m - (N-2)-criterion (4) or (5) and criteria (6), (7).

Later on the symbol **min** is used instead of **opt** in the criteria (3)-(7) .

2. The formulation of the Local Stability Problem

In space R^{NQ} of the matrixes $B=\left\|b_{\nu\kappa}\right\|$ a norm is given [4]: $\|B\|=\max\left\{\left|b_{\nu\kappa}\right|: \nu=\overline{1,N}, k=\overline{1,Q}\right\}$. We denote the set of all matrixes $B=\left\|b_{\nu\kappa}\right\|$ by $\mathrm{B}(\varepsilon)$ such that $\|B\|\le\varepsilon$, $\varepsilon>0$.

The obtained by the addition of W and $B\in\mathrm{B}(\varepsilon)$ matrixes problem $z^N(W+B)$ is called perturbated and matrix B - perturbating.

Definition . The problem $z^N(W)$ is called ε-stable when the inclusions

$$\widetilde{T}(W+B)\subseteq\widetilde{T}(W) \quad \forall B\in \mathrm{B}(\varepsilon). \tag{8}$$

are fulfilled.

Obviously, the problem $z^N(W)$ is ε-stable for arbitrary $\varepsilon>0$ when $T=\widetilde{T}(W)$. Then this case will be excluded. The problem $z^N(W)$ is called nontrivial, if . $\overline{T(W)}=T\setminus\widetilde{T}(W)\neq\varnothing$.

We define the set

$$\widetilde{T}(W,t^0)=\left\{\widetilde{t}\in\widetilde{T}(W): \tau_\nu^w(t^0,\widetilde{t})\ge 0,\ \nu=\overline{1,N}\right\}, \tag{9}$$

for any $t^0\in\overline{T(W)}$, where $\tau_\nu^w(t^0,\widetilde{t})=F_\nu(t^0,W)-F_\nu(\widetilde{t},W)$, $\nu\in I_k$, $k=\overline{1,3}$.

The conditions of stability for the problems of the class Z_0^v are described in works [1,3]. According to the definition the considerable perturbation of matrix weights does not exert an effect on the value of topological criteria (6),(7). Therefore, the conditions of stability for the problems of the class Z_0^v are transfered on the problems of classes: Z_1^v, Z_2^v, Z_3^v.

Further, we define the conditions of ε- stable for the problems of classes Z_j^m, $j=\overline{0,3}$.

Let's assume that the edge e' is defined by the correlation $M_\nu(t,W)=\underset{t}{opt}\ w_\nu(e)=w_\nu(e')$, $\nu=\overline{1,N}$, i.e. the optimum of criterion ν is attained on the edge e'. The quasioptimum significance of ν-th criterion is called the weight of such edge $e^*\neq e'$, that $w_\nu(e^*,t)=\underset{e\in t\setminus e'}{opt}\ w_\nu(e)$.

Let's introduce the definition of i-th quasioptimum to describe the necessary and sufficient conditions of ε- stable of the class Z_0^m. For this purpose we use the recursion designations on the steps $i=0,1,\dots$, where the value i is limited by length $|t|$ of the solution.

Step 0: $e_\nu(t^0 \cap \tilde{t})$ - the egde $e \in t^0 \cap \tilde{t}$, on which the equation ν-th criteria of pareto and nonpareto solution is obtained, i.e. $M_\nu(t^0, W) = M_\nu(\tilde{t}, W)$. We designate the set of numbers of the criteria for which this equation is fulfilled by $J(t^0, \tilde{t})$;

$M_\nu^1(\tilde{t}, W)$ - the first quasioptimum of solution $\tilde{t} \in \tilde{T}(W, t^0)$ and it is defined by the equation $M_\nu^1(\tilde{t} \setminus e_\nu(t^0 \cap \tilde{t}), W) = M_\nu^1(\tilde{t}, W)$.

Step 1. $e_\nu^1(t^0 \cap \tilde{t})$ - the edge $e \in t^0 \cap \tilde{t}$, on which the first quasioptimums are equal, i.e. $M_\nu^1(t^0, W) = M_\nu^1(\tilde{t}, W) \neq M_\nu(t^0, W)$. We designate the set of numbers of the criteria for which this equation is fulfilled by $J^1(t^0, \tilde{t})$, where $\nu \in J(t^0, \tilde{t})$;

$M_\nu^2(\tilde{t}, W)$ - the second quasioptimum of solution $\tilde{t} \in \tilde{T}(W, t^0)$ and it is defined by the equation $M_\nu^2(\tilde{t} \setminus e_\nu(t^0 \cap \tilde{t}), e_\nu^1(t^0 \cap \tilde{t}), W) = M_\nu^2(\tilde{t}, W)$.

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Step i **:** $e_\nu^i(t^0 \cap \tilde{t})$ - the edge $e \in t^0 \cap \tilde{t}$, on which the i-th quasioptimums of solutions $t^0, \tilde{t}$ are equal and the following sequence of equations and inequalities is fulfilled:

$$\left[M_\nu(t^0, W) = M_\nu(\tilde{t}, W)\right] < \left[M_\nu^1(\tilde{t} \setminus e_\nu(t^0 \cap \tilde{t}), W) = M_\nu^1(t^0 \setminus e_\nu(\tilde{t} \cap t^o), W\right] <$$

$$.<...< \left[M_\nu^i(\tilde{t} \setminus E_\nu^{i-1}(t^o \cap \tilde{t}), W) = M_\nu^i(t^o \setminus E_\nu^{i-1}(t^o \cap \tilde{t}), W)\right],$$

where $E_\nu^{i-1}(t^0 \cap \tilde{t}) = \left\{e_\nu(t^0 \cap \tilde{t}), e_\nu^1(t^o \cap \tilde{t}), ..., e_\nu^{i-1}(t^o \cap \tilde{t})\right\}$. (10)

We designate the set of numbers of the criteria for which this equation is fulfilled by $J^i(t^o, \tilde{t})$, where $\nu \in J^{i-1}(t^o, \tilde{t})$. An meaningful interpretation of the sequence (10) consists of :

1) the first i-th quasioptimums are defined;
2) these quasioptimums form the monotonically strictly increasing sequence;
3) the final equation of sequence (10) is attained on i-th step.

Now we formulate the conditions of ε-stable for the problems of the class Z_0^m.

Theorem. For ε-stability of the nontrivial problem $z^N(W)$ it is necessary and sufficiently, that the solution $\tilde{t} \in \tilde{T}(W, t^0)$ exists for any solution $t^0 \in \overline{T(W)}$ with the system of the inequalities

$$\begin{cases} M_\nu(t^o, W) - M_\nu^i(\tilde{t}, W) \geq 2\varepsilon & \forall \nu \in J^i(t^0, \tilde{t}) \subseteq J(t^0, \tilde{t}), \\ \tau_\nu^w(t^o, \tilde{t}) \geq 2\varepsilon & \forall \nu \in \{1, 2, ..., N\} \setminus J(t^0, \tilde{t}). \end{cases} \quad (11)$$

3. The stability estimation by means of the quantity method

We use the number $\rho(W) = \sup\left\{\varepsilon : \tilde{T}(W + B) \subseteq \tilde{T}(W) \quad \forall B \in \mathrm{B}(\varepsilon)\right\}$ for the quantitative measure of the stability. It is called the radius of the stability. The stability radius of the problem $z^N(W)$ is defined by the limit of the perturbation elements of the matrix W, when a new PO do not arise.

By means of the above mentioned theorem the formulas for the stability radius of the problems connected with Z_j^v and Z_j^m, $j = \overline{0,3}$ classes are obtained:

1) for the problems of the class Z_0^v we have :

$$\rho(W) = \min_{t^O} \max_{\tilde{t}} \min_{\nu} \frac{\tau_\nu^w(t^o, \tilde{t})}{c(t^o, \tilde{t})}, \quad \nu \in \overline{1, N}, \quad (12)$$

where $c(t^0, \tilde{t}) = 2(n - 1 - \left|t^0 \cap \tilde{t}\right|)$.

Remark 2. The formula (12) is fulfilled for the problems Z_1^v, Z_2^v under the condition that $\nu \in \left\{\overline{1, N-1}\right\}$ and formula (12) is fulfilled for the problem Z_3^v, if $\nu \in \left\{\overline{1, N-2}\right\}$.

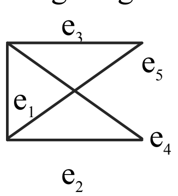
2) for the problems of the class Z_0^m we have :

$$\rho(W) = \min_{t^o \in T(W)} \max_{\tilde{t} \in \tilde{T}(W,t^o)} \min \frac{1}{2} \left\{ \min_{\nu \in J^i(t^o,\tilde{t})} \left\{ (M_\nu(t^0,W) - M_\nu^{i+1}(\tilde{t},W)), (M_\nu(t^0,W) - M_\nu^{i+1}(t^0,W)) \right\}, \min_{\nu \in \overline{1,N} \setminus J(t^o,\tilde{t})} \tau_\nu^w(t^0,\tilde{t}) \right\}. \quad (13)$$

Remark 3. The formula (13) is fulfilled for the problems Z_1^m, Z_2^m under the condition that $\nu \in \{\overline{1, N-1}\}$ and formula (13) is fulfilled for the problem Z_3^m, if $\nu \in \{\overline{1, N-2}\}$.

4. An example of the stability radius calculation for Spanning Tree Problem

Let's calculate the stability radius for individual 3-criterial problem of the class Z_2^m on the weighting 4-vertex graph G:



The following matrix of the weights W is given :

N\E(t)	e_1	e_2	e_3	e_4	e_5
1	17	23	19	26	20
2	22	21	24	18	25

VOF: $F(t) = (F_1(t), F_2(t), F_3(t))$, where

$$F_{1,2} = \max_{e \in t} w_\nu(e) \to \min_T \quad (14)$$

$$F_3(t) = d(t) \quad \to \quad \min_T \quad (15)$$

For conveniente let's place the results of the stability radius calculation with the help of the formula (13) into the following table:

T	E(t)	F_1	F_2	F_3	$\tilde{T}$		τ_1	τ_2	min	max	min/2
t_1	$e_1e_2e_3$	23	24	3	*						
t_2	$e_2e_3e_4$	26	24	3	t_1,t_4	3,9	3,7 2,2	3,2 2,2	3,2 2	3 2/2	3/2
t_3	$e_2e_3e_5$	23	25	3	t_1,t_5	4,3	3,3 1,3	1,4 1,3	1,3 3	3	
t_4	$e_1e_2e_4$	26	22	2	*						
t_5	$e_1e_3e_5$	20	25	2	*						
t_6	$e_3e_4e_5$	26	25	3	t_1,t_4,t_5	3,6,6	3,6,6 1,3,3	1,3,7 1,3,3	1,3,6 3	6	
t_7	$e_1e_4e_5$	26	25	3	t_1,t_4,t_5	3,6,6	3,6,6 1,3,1	1,3,3 1,3,1	1,3,3 3	3	
t_8	$e_2e_4e_5$	26	25	3	t_1,t_4,t_5		3,6,6 3,9,6	1,3,4 1,3,1	1,3,4 1,3,1	4 3	

Each column of the table is given below:
Column 1 - the set of the feasible solutions , which presents all spanning trees of the graph G.
Column 2 - the edge structure of each spanning tree.
Columns 3 - 5 - the significances of VOF criteria, which are calculated by formulas (14),(15).

Column 6 - PO are maked by the symbol *. PO are enumerated in the lines which correspond to the nonpareto solutions t^0 and these PO dominate the solution t^0 under consideration.
Columns 7 - 8 - the difference of significances on criteria $F_1(F_2)$ between pareto and nonpareto solutions. It is calculated by formula (9). Here, the upper row of numbers is the difference $M_\nu(t^o,W)-M_\nu^i(t^0,W)$ and the lower one of numbers is - $M_\nu(t^o,W)-M_\nu^i(\tilde{t},W)$.
Column 9 - the choice of minimum significances among τ_1 and τ_2; the first number is obtained by means of the choice of the minimum among the first significances τ_1 and τ_2; the second one is obtained by means of the second significances τ_1 and τ_2 and so on.
Column 10 - the maximum of the values which were taken before from the column 9.
Column 11 - values which represent the minimum among significances of the column 10.

According to formula (13) we get:

$$\rho(W)=\min\left\{\frac{3}{2},\frac{2}{2}\right\}=1.$$

Conclusion

Many real processes are not determined yet. And that is why the investigation of the stability is very important. Many errors are connected with calculations. The stability analysis of vector combinatorial problems allows to discover the value of changes in the initial data for which the optimal solution is not changed.

Furthermore, the investigation of the stability allows to construct the class of the problems on base of the one problem by means of the parameter variations. Analysis of the problems with belong to this class allows to obtaine axact and adecuate discription of model.

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