FRACTAL GRAPHS AND THEIR PROPERTIES

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The idea of representing urban structure and various communication systems (water and energy supply, telephone and cable TV networks) as fractal objects is not absolutely new. However, known works, devoted to this problem use models and approaches from fractal physics. For example, to simulate urban growth Diffusion Limited Aggregation (DLA) model and Dielectric Breakdown (DB) model are used /1/.

This study introduces a different approach. Net structure of communication system is described by a graph of special type called regular $G(l,r,n)$-graph. Authors provide description of such graph, develop iterative process for its generation and prove its self-similarity, i.e. that every regular graph is a pre-fractal. After the infinite number of steps this process generates a fractal.

The devised algorithm for generation and grathical representation of regular $G(l,r,n)$-graphs with different values of l,r and n has been programmed to receive computer simulations. For optimal graphic presentation of pre-fractals the Optimal Space Ordering method was suggested. It is based on the minimization of the "graph energy" value about vertices' coordinates. The effective procedure for optimization was developed that takes into account specific properties of graph energy as objective function.

For the fractal graph introduced the Hausdorff-Besikovich and similarity dimensions were calculated. It has been shown that "graph energy" is directly related to the graph's fractal properties. For $G(3,3,n)$ and $G(4,4,n)$ graphs fractal dimensions calculated by different methods are the same (D=1,5 and D=2 respectively), while topological dimension is 1.

1. Description of structured $G(l,r,n)$-graph and iterative process of generation

We will describe the graph by using widely known designation $G=(V,E)$. The vertices set $V=\{v\}$ has a capasity $|V|=h=l^n\cdot r$, moreover it is separated on l^n subsets V_ν, $\nu=1,2,\ldots,N, N=l^n$, each of them has capasity $|V|=r$, $\bigcup_{\nu=1}^{N} V_\nu = V$.

The edges set $E=\{e\}$ is separated on $(n+1)$ subsets $E^{(0)},E^{(1)},\ldots,E^{(n)}$, which are defined as follows. Set E^0 is called base set or set of zero rank edges. It is a union $\bigcup_{\nu=1}^{N} E_\nu^0$, where for fixed index ν symbol E_ν^0 - is a set of all C_r^2 edges, which form a complete subgraph $G_\nu=(V_\nu,E_\nu^0)$. That subgraph is built on a base of above represented set of vertices V_ν. The complete pre-fractal $G(l,r,n)$-graph has a meaning $|V_\nu|=|V_1|=r$, and its capasity:

$$\left|E^0\right| = \left|E_1^0\right| \cdot N = C_r^2 \cdot l^n = \frac{r \cdot (r-1)}{2} \cdot l^n . \quad (1)$$

Subgraphs $G_\nu, \nu = 1,2,\ldots,N$ are called base subgraphs.

The construction of $G(l,r,n)$-graph $G = (V,E)$ is fulfilled stage by stage: every stage is numbered by index $s = 1,2,\ldots,n$. On the stage s it is defined a subset $E^{(s)} \subset E$, which is called a subset of edges of a rank $\rho = (n - s + r)$. Let us arrange that $E^{(0)} = E^0$. The subsets of all ranks are partition of the set E: $E = E^0 \cup E^{(1)} \cup \ldots \cup E^{(n)}$. There is a noteworthy speciality of elements of that partition: for any $s \in \{1,2,\ldots,n\}$ the totality of edges $e \in E^{(s)}$ forms a section of graph G, that is removal of edges $e \in E^{(s)}$ extend (not less than in l times) the quality of connection component of graph G.

Let us define now a set $E^{(1)}$ of edges of rank l , $\rho = n$. For the begining let us consider the totality of above defined base subgraphs

$G_\nu = (V_\nu, E_\nu), \nu = 1,2,\ldots,N$,

(2)

which we separate on $N_1 = l^{n-1}$ of subsets with equal capasity

$$\mathsf{G}(k_1) = \{G_\nu : \nu = (k_1 - 1) \cdot l + 1, \quad (k_1 - 1) \cdot l + 2, \quad \ldots, \quad k_1 \cdot l\},$$

(3)

$\left|\mathsf{G}(k_1)\right| = l, \quad k_1 = 1,2,\ldots,N_1$. The totality of subsets of vertices V_ν of graphs $G_\nu \in \mathsf{G}_1(k_1)$ one-to-one correspond to subset $\mathsf{G}_1(k_1)$:

$$W_1(k_1) = \left\{V_\nu : \nu = (k_1 - 1) \cdot l + 1, \quad (k_1 - 1) \cdot l + 2, \quad \ldots, \quad k_1 \cdot l\right\}, \quad k_1 = \overline{1, N_1}$$

(4)

Further let us select a couple of some vertices $v' \in V_{\nu_1}, v'' \in V_{\nu_2}$ $(k_1 - 1) \cdot l + 1 \le \nu_1 \le \nu_2 \le k_1 \cdot l$ from each couple of sets $V_{\nu_1}, V_{\nu_2} \in W_1(k_1)$ and construct edge $e = (v', v'')$ which includes in the set.

In accord with the definition (4) the quantity of all possible couples $V_{\nu_1}, V_{\nu_2} \in W_1(k_1)$ is equal to number of combinations $m = C_l^2$. Consequently the capasity is $\left|E_{k_1}^{(1)}\right| = m = C_l^2$. The set of first rank edges $E^{(1)}$ obtains in the results of union of sets $E^{(1)}(k_1)$:

$$E^{(1)} = \bigcup_{k_1=1}^{N} E^{(1)}(k_1) \quad (5)$$

Capasity is $\left|E^{(1)}\right| = N_1 \cdot m = \frac{l-1}{2} \cdot l^n \quad (6)$

With regard to the relations (1) and (6) the equality of capasities $\left|E^{(1)}\right| = \left|E^0\right|$ achieves only for $l = r^2 - r + 1$.

The union $(E^0 \cup E^{(1)})$ forms graph $G^{(1)} = (V, E^0 \cup E^{(1)})$, which consists of $N_1 = l^{n-1}$ connected component, $G^{(1)}(k_1), 1 \le k_1 \le N_1$. The component $G^{(1)}(k_1)$ is formed in the result of mutual complexion of l subgraphs, which contains subset (3), by edges $e \in E_{k_1}^{(1)}$. Let us designate the set of vertices and the set of edges of component $G^{(1)}(k_1)$ accodingly as $V^{(1)}(k_1)$ and $E^{(1)}(k_1)$: $G^{(1)}(k_1) = (V^{(1)}(k_1), E^{(1)}(k_1))$; where $V^{(1)}(k_1)$ is obtained as the result of union of elements of the set (4) by $k_1 = 1,2,\ldots,N_1$.

Let us define the set $E^{(s)}$ of edges of rank $\rho = n - s + 1$ in the assumption that there were $(s-1)$ stages, and there were built the sets of edges $E^0, E^{(1)}, E^{(2)}, \ldots, E^{(s-1)}$ of ranks $0,\ n,\ n-1,\ \ldots,\ n-s+2$ consequently. That union forms graph $G^{(s-1)} = (V, E^0 \cup E^{(1)} \cup \ldots \cup E^{(s-1)})$, which has $N_{s-1} = l^{n-s+1}$ connected components.

$$G^{(s-1)}(k_{s-1}), k_{s-1} = 1,2,\ldots, N_{s-1} \qquad (7)$$

Let us divide all totality (7) on $N_s = \frac{1}{T} N_{s-1} = l^{n-s}$ subsets of equal capasity

$$\mathbb{G}_s(k_s) = \{G^{(s-1)}(t): t = (k_s - 1)l + 1, (k_s - 1)l + 2, \ldots, k_s l\},\ |\mathbb{G}_s(k_s)| = l, k_s = 1,2,\ldots, N_s \qquad (8)$$

The totality of sets of vertices $V^{(s-1)}(k_{s-1})$,which belong to the components $G^{(s-1)}(k_s - 1) \in \mathbb{G}_s(k_s)$, mutually corresponds to the subset $\mathbb{G}_s(k_s)$:

$$W_s(k_s) = \{V^{(s-1)}(t):\ \ t = (k_s - 1)\cdot l + 1,\ \ (k_s - 1)\cdot l + 2,\ \ \ldots,\ \ k_s \cdot l\},\ k_s = 1,2,\ldots, N_s \qquad (9)$$

Further let us choose in every couple of sets $V^{(s-1)}_{k'_{s-1}}, V^{(s-1)}_{k''_{s-1}} \in W_s(k_s)$ accordingly a couple of some vertices $v' \in V^{(s-1)}_{k'_{s-1}}, v'' \in V^{(s-1)}_{k''_{s-1}}$, $(k_s - 1)\cdot l + 1 \le k'_{s-1} < k''_{s-1} \le k_s \cdot l$ and construct the edge $e = (v',\ v'')$, which is including in set $E^{(s)}_{k_s}$. As capasity is $|W_s(k_s)| = l$, then the amount of all possible ϖouples $V^{(s-1)}_{k'_{s-1}}, V^{(s-1)}_{k''_{s-1}} \in W_s(k_s)$ is equal to the number of combinations $m = C^2_l$.

Consequently, the capasity is $|E^{(s)}_{k_s}| = m$. The set $E^{(s)}$ of edges of rank $(n-s+1)$ is obtained in the result of union by all indices $k_s :: E^{(s)} = \sum_{k_s=1}^{N_s} E^{(s)}_{k_s}$

(10)

The capasity is $|E^{(s)}| = m \cdot N_s = \frac{l-1}{2} \cdot l^{n-s+1}$ (11)

Taking into account the expressions (6) and (9) we obtain the capasity relation $|E^{(s-1)}| = |E^{(s)}| \cdot l$.

The union $(E^0 \cup E^{(1)} \cup E^{(2)} \cup \ldots \cup E^{(s-1)})$ forms the graph $G^{(s)} = (V, E^0 \cup E^{(1)} \cup E^{(2)} \cup \ldots \cup E^{(s-1)})$, which consists of $N_s = l^{n-s}$ connected components $G^{(s)}(k_s), 1 \le k_s \le N_s$. The component $G^{(s)}(k_s)$ is formed in the result of mutual complexion of l subgraphs which contains subset (8) by edges $e \in E^{(s)}_{k_s}$. Let us denote by $V^{(s)}(k_s)$ and $E^{(s)}(k_s)$ the sets of vertices and edges of the component $G^{(s)}(k_s)$.

From the inductive definition of the term *"the stage s '* with regard to the notations (8) and (9) it follows that on completion the stage $s = n - 1$ we obtain the graph which consists of the l component $G^{(n-1)}(k_n - 1), k_n = 1,2,\ldots, l$

(12)

Further on the final stage $s = n$ let us select a couple of vertices v', v'' from each couple of these components and construct edge $e = (v', v'')$.The totality of all C^2_l edges $e = (v', v'')$ forms the set of first rank E^n, that is of rank $\rho = 1$. The edges $e \in E^{(n)}$ unite the l component $G^{(n-1)}(k_n - 1)$ in the liaison graph $G(V, E), E = E^0 \cup E^{(1)} \cup E^{(2)} \cup \ldots \cup E^{(n)}$.

The complete fractal graph is obtained from the pre-fractal $G(l, r, n)$-graph by realization of infinite number of stages, that is by $n \to \infty$.

2. Properties of the complete pre-fractal $G(l,r,n)$ **-graph**

In accord (1), (6), (10) the capacity of set of its edges makes

$$|E| = \left|E^0\right| + \sum_{s=1}^{n}\left|E^{(s)}\right| = \frac{1}{2}r(r-1)l^n + \frac{1}{2}(l-1)\sum_{s=1}^{n} l^{n-s+1} = \frac{1}{2}\left(\left[r(r-1)+l\right]l^n - l\right) \tag{13}$$

For any $l \geq 2, \quad r \geq 1$ the diameter of $G(l,r,n)$*-graph* depends only of its rank n and is calculated by formula

$$d(G) = \sum_{s=0}^{n} 2^s = 2^{n+1} - 1 \tag{14}$$

The degree of any vertex $v \in V$ of graph $G = (V,E)$ satisfies to the inequalities

$$r - 1 \leq \deg v \leq r - 1 + n \tag{15}$$

and for any $l \geq 2, \ r \geq 1, \quad n \geq 1$ there exist *(r,l,h)*-graphs, for which upper estimation is reacheable, that is $\max\limits_{v \in V} \deg v = r - 1 + n$ (16)

The complete pre-fractal *(r,l,h)*-graph has the property of self-similarity ("a part is similar to a whole"), the main point of which is the following. Let us designate as $\mathbb{G}(l,r,n) = \{G\}$ the set of all complete regular *(r,l,h)*-graphs for a fixed three parameter meanings l, r, n. Further discussion is true for both forms of graph's reprezentations $G = (V,E) \in \mathbb{G}(l,r,n)$: vertices and edges of graph can be numbered or not.

The definition of self-similarity property is based on the procedure, which is called *extention of base subgraph* (EBS). Let us choose some graph $G = (V,E) \in \mathbb{G}(l,r,n)$ and consider one of it's component (12), for example, the component $G^{(n-1)}$ (1), which we designate as $G' = (V',E')$. According to the construction graph G' consists of exactly $N_1 = l^{n-1}$ base subgraphs $G_v = (V_v, E_v), \ v = 1,2,\ldots,N_1$. Let fix any index $v \in \{1,2,\ldots,N_1\}$ and realize procedure EBS to the G_v.

At the begining we connects a set $\overline{V_v}$, which called complementary set, to the set V_v, $|V_v| = r$. Set $\overline{V_v}$ consits of $\left|\overline{V_v}\right| = (l-1)\cdot r$ new vertices. We divide the union set $\left(V_v \cup \overline{V_v}\right)$ arbitrary by l capasity equal subsets $V_{v,i}$, $\left|V_{v,i}\right| = r, \quad i = 1,2,\ldots,l$. Let us build the set of edges, which called extending set for the united set of vertices $\left(V_v \cup \overline{V_v}\right)$, keeping all edges of considering graph G', including all edges, which are incident to the vertices $v \in V_v$, by following. Every from C_r^2 vertices couples $v', v'' \in V_{v,i}$ will be connected by edge in every set $V_{v,i}, i = 1,2,\ldots,l$. In the result we obtain l complete r-vertices subgraphs

$$G_{v,i} = \left(V_{v,i}, E_{v,i}^0\right) \tag{17}$$

Further let us choose in every couple of subgraphs G_{v,i_1}, G_{v,i_2}, $1 \leq i_1 \leq i_2 \leq l$ a couple of vertices $v' \in V_{v,i_1}$, $v'' \in V_{v,i_2}$ and connect that couple by edge $e = (v',v'')$. The totality of all C_l^2 such couples forms the set of edges E_v^*.

Above the discribed procedure of extention of one base subgraph G_v generates l base subgraphs $G_{v,i}, \quad i = 1,2,\ldots,l$. Fullfiling of that procedure for all base subgraphs G_v of graph $G', \quad v = 1,2,\ldots,N_1$ generates $N = l^n$ base subgraphs $G_{v,i}$. Analogically, the union of

all sets E_v^* by all $i = 1,2,\ldots,N_1$ forms the set of edges of rank n with respect to generated base subgraphs $G_{v,i}$. In the result of fullfiling of procedure EBS to all base subgraphs of componenet G' we obtain the graph, which we call "generated" and designate as $G^* = (V, E^*)$. The following statements are true:

- Generated graph G^* belongs to the set $\mathbb{G}(l,r,n)$.

- For fixed meanings l, r let us consider sequence of sets $\mathbb{G}(l,r,n)$, $n = 1,2,\ldots$. If we make the procedure of extention for each base subgraph $G \in \mathbb{G}(l,r,n)$, we obtain in the result complete pre-fractal *(r,l,h+1)*-graph, which belongs to the set $\mathbb{G}(l,r,n+1)$.

That is true the assumption about self-similarity of *(r,l,h)*-graph.

3. Calculation of the complete fractal graph dimension

Let us calculate a fractal dimension of complete fractal graph for arbitrary initial parameters and scaling coefficient $1/x$, $x > 1$. The lehgths of edges on the stage k is equal:

$$L(k) = L(k-1) + r^{k-1} \cdot l \cdot (l-1) \cdot \left(\frac{1}{x}\right)^k \Big/ 2, \tag{18}$$

where $L(k-1) = \dfrac{r(r-1)\cdot\left(\left(\dfrac{r}{x}\right)^{k-1} - 1\right)}{\dfrac{r}{x} - 1}$ (19)

If we use the Hausdorff-Besikovich dimension /1/, we obtain

$$\left[\frac{l\cdot(l-1)\cdot\left(\left(\dfrac{l}{x}\right)^{k-1}\right)}{2\cdot\delta\cdot\left(\dfrac{l}{x}-1\right)} + \frac{r\cdot(r-1)}{2\cdot\delta}\left(\frac{l}{x}\right)^{k-1}\right]\cdot\delta^D = a,\ \delta = \left(\frac{1}{x}\right)^{k-1} \tag{20}$$

Let us express k in terms of δ: $k = 1 - \log_x \delta$ (21)

In (20) let us express D in terms of the rest parameters:

$$D = \frac{\ln\left(2a\delta\cdot(r-x)\right) - \ln x}{\ln\delta} - \frac{\ln\left[r(r-1)\delta\Big/\left(l^{\log_x\delta}\right) - 1 + l(l-1)\left(\dfrac{r}{x}-1\right)\delta\Big/\left(r^{\log_x\delta}\right)\right]}{\ln\delta} \tag{22}$$

Now we fullfil a passing to the limit $\lim\limits_{\delta\to 0} D$, and take into accont that $\lim\limits_{\delta\to 0}(c/\delta) = 0$, $\lim\limits_{\delta\to 0}\left(\ln(c\cdot\delta)/\ln\delta\right) = 1$, where c - some constant and $r^{\log_x\delta} = \delta^{\log_x r}$:

$$\lim_{\delta\to 0} D = 1 - 1 + \lim_{\delta\to 0}\left[\frac{\log_x\delta\cdot\ln r}{\ln\delta}\right] = \log_x r \tag{23}$$

Therefore we obtain the common formula for calculation a fractal dimension of complete fractal graph for arbitrary initial parameters:

$$D = \log_x r \tag{24}$$

4. Construction of the optimal geometric image of pre-fractal graph

As the graph contains a structural information only and by itself doesn't have a certain geometric image,it seems to be useful to formulate the principle which should allow to get some standard geometric representation of any given graph. This problem is especially actual for graphs with fractal structure. For its decision authors propose method of optimal space ordering (OSO-method).

Let G -common (non-oriented, without loops and multiple edges) connected finite n-vertex graph. Geometric image (or $\Re$-representation) of graph G in space R^p is the tuple of vectors $\Re = \{\rho_1, \rho_2, ..., \rho_n\}, \rho_i \in R^p, i = 1, ..., n$ where ρ_i is a radius-vector of vertex with number i in R^p.

Let us choose a couple of functions $\varphi_0(\rho), \varphi_1(\rho)$, where $\rho = |\rho_i - \rho_j|$ is the distance between vertices i and j. Let function $\varphi_0(\rho)$ have the meaning of interaction energy of a couple of non-adjacent vertices and $\varphi_1(\rho)$ be a similar function for adjacent vertices. Energy of interaction of vertex with index i with the rest vertices will be equal to the sum:

$$\Psi(\rho_i) = \sum_{j=1, j\neq i}^{n} \{(1 - s_{ij})\varphi_0(|\rho_i - \rho_j|) + s_{ij}\varphi_1(|\rho_i - \rho_j|) \quad (25)$$

where s_{ij} is an element of adjacency matrix of graph.

Energy of graph in $\Re$-representation will be represented by sum of energies of all it's vertices:

$$\Phi(\Re) = \sum_{i=1}^{n} \Psi(\rho_i) \quad (26)$$

Let us consider $\Re$-representation is standard if appropriate energy of graph is minimum on the set of all possible its representations when the dimension of space p is fixed. Therefore the problem of construction of standard geometric image of graph is reduced to the optimization problem with objective function in form (26) and with $p \cdot n$ varied parameters.

For solving that problem authors propose to use the algorithm, which is based on the process of gradient-step group relaxation. That algorithm is based on the following property of function $\Phi(\Re)$.Let us pretend we have some image $\Re(\rho_1, ..., \rho_i, ... \rho_n)$ of graph G. Let us consider we have another image $\Re^*(\rho_1, ..., \rho_i^*, ... \rho_n)$, which was obtained by changing co-ordinate ρ_i of vertex with number i to the ρ_i^*. It follows from the definition of function $\Phi(\Re)$ that true following statement:

$$\forall i = \overline{1,n}: \quad \Psi(\rho_i^*) < \Psi(\rho_i) \Rightarrow \Phi(\Re^*) < \Phi(\Re) \quad (27)$$

For describing the whole scheme of algorithm we will use following designation: λ - admissible error for vectors in $\Re$-representation of graph for present step; λ_0 - initial meaning of λ; ε - the meaning of that error for final result; $\Re^0$ - initial representation of graph ; i - number of vertex.

Algorithm consists of following steps:

1. To let $\lambda = \lambda_0$, $\Re = \Re^0$.
2. To let $i = 1$.
3. To calculate ρ_i^* by formula: $\rho_i^* = \rho_i + \lambda \cdot (F_i / |F_i|)$, where vector F_i is found by rule: $F_i = -\nabla^p \Psi(\rho_i)$.
4. To test the condition $\Psi(\rho_i^*) < \Psi(\rho_i)$. If it fullfils, then choose for ρ_i new meaning ρ_i^*.
5. To increase i on 1 and to pass to the step 3.

6. If while realisating steps 2-5 for $i = \overline{1,n}$ not one of the vectors ρ_i has changed its value, we should test the condition: $\lambda < \varepsilon$. If the condition fullfils then present meaning $\Re$ is needed result, otherwise - we should divide λ by 2 and pass to step 2.

In correspondence with proposed method the construction of optimal images for $G(4,4,4)$ and $G(3,3,4)$ graphs was fulfilled.At the same time as the potentials of vertices' interaction the following functions were used:

$\varphi_0(\rho) = k \cdot e^{-\alpha_0(\rho-\rho_0)}$, $\varphi_1(\rho) = e^{-2\alpha_1(\rho-\rho_0)} - 2 \cdot e^{-\alpha_1(\rho-\rho_0)}$.

The meaning of parameters are: $\alpha_0 = 6.0$, $\alpha_1 = 2.0$, $\rho_0 = 0.12$, $k = 0.03$.

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