Standardization problem: Resource Allocation in a Network[1]

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1. Mathematical model of the standardization problem.

Among problems of operation research standardization problems take important place, i.e. problems of choosing optimal parameters and the number of items (products) belonging to an uniform collection. The term "uniform collection" means a set of items applying for the same goal, but having different parameters. For example, for the uniform collection of electrical measuring devices one could choose current power, accuracy, sensitivity as such parameters; for construction materials -- mass, width, length, solidity and so on.

Let each item of some uniform collection be characterized by *k* parameters. Let *A* and *B* denote the sets of the all parameters vectors $(\beta_1,\beta_2,\ldots,\beta_k)$ for items supply and items demand, respectively. Let us consider a mathematical model for the standardization problem. Let the sets *A* and *B* consist of *m* and *n* elements, respectively. A qualification matrix $Q = (q_{ij})_{m\times n}$ contains information about the possibilities of replacement of some items type by another. One item of the *i*th type$(i\in[m]=\{1,2,\ldots,m\})$ can replace q_{ij} items of the *j*th type$(j\in[n])$. The replacement of items of the *j*th type by items of *i*th type is impossible, if q_{ij}=0. Taking into account the replacement possibilities it is necessary for each type $i\in[m]$ to determine the amount y_i of items that should be produced in order to satisfy items demand $(b_1,b_2,\ldots,b_n)$and to maximize total profit (or to minimize production costs).

Thus the mathematical model of the standardization problem has the following structure.

$$\text{SP:}\qquad \sum_{i=1}^{m} f_i(y_i) \rightarrow \max$$

(1)

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$$\sum_{j=1}^{n} x_{ij} = y_i \quad i \in [m] \tag{2}$$

$$\sum_{i=1}^{m} q_{ij} x_{ij} = b_j \quad j \in [n] \tag{3}$$

$$x_{ij} \in Z_+ \quad i \in [m], j \in [n] \tag{4}$$

where $f_i(y_i)$ is a concave nondecreasing function of profit for all $i \in [m]$, x_{ij} is the amount of items of the *i*th type produced to replace items of *j*th type. In real life standardization problems (1)-(4) the objective function can be chosen also among the following functions

$$\min f(y_1, y_2, \dots, y_m)$$

$$\min \sum_{i=1}^{m} f_i(y_i)$$

$$\min \sum_{i=1}^{m} f_i(y_i) + \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \ ,$$

where c_{ij} is additional costs concerning with the replacement of one item of the *j*th type by one item of the *i*th type, $f, f_i (i \in [m])$ are concave nondecreasing functions. Solution algorithms for standardization problems with such objective functions was proposed in [4-6]. We call the set of feasible vectors $y \in Z_+^m$ defined by (2)-(4) the standardization polytope. Sometimes one adds to the conditions (2)-(4) the condition

$$\sum_{i=1}^{m} sign\, y_i \le s, \quad s < m, s \in Z_+ , \tag{5}$$

where $sign\, y_i = 1, if\, y_i > 0$ and $sign\, y_i = 0\ if\, y_i = 0$. Condition (5) guarantees that the total number of produced types of items does not exceed *s.* We shall consider only the standardization problems with (0,1)-matrix *Q,* i.e. one item can replace not more than one item. The matrix

$$Q_1 = \left(q_{ij}^{(1)}\right)_{m \times n}, q_{ij}^{(1)} = 1, if\ i \ge j\ and\ q_{ij}^{(1)} = 0, otherwise$$

is the qualification matrix of the one-parameter standardization problem. In this case an item with greater parameter value can replace items with smaller parameter value, and greater index means greater parameter value.

In problem SP items demand was specified by the vector $(b_1, b_2, \dots, b_n)$. In general items demand can be given as the set of admissible demand vectors in the following way. Let $d{:}2^{[n]} \to Z_+$ be a set function such that *d(J)* represents the total demand in items of the types forming the set *J*. Then condition (3) can be rewritten in the following general form

$$\sum_{j\in J}\sum_{i=1}^{m} q_{ij}x_{ij} \ge d(J) \qquad \forall J \subseteq [n] \tag{3'}$$

$$\sum_{j=1}^{n}\sum_{i=1}^{m} q_{ij}x_{ij} = d([n]) \tag{3''}$$

Conditions (3'),(3'') are called general demand conditions. Let us recall that function $r: 2^{[n]} \to R$ is called submodular (supermodular, modular), if

$$r(I) + r(J) \ge \ (\le, =)\ r(I \cap J) + r(I \cup J) \qquad \forall I, J \subseteq [n].$$

The function d is usually supermodular. Problem SP with general demand conditions is called Generalized Standardization Problem (GSP) and its feasible solutions set is called the generalized standardization polytope. In problem SP the function d is determined as follows : $d(\{j\}) = b_j \quad \forall j \in [n]$ and $d(I) = \sum_{j\in J} d(\{j\}) \quad \forall J \subseteq [n]$, i.e. d is modular.

2.Polymatroidal characterization of the standardization polytope.

In this section we give some basic facts from the theory of polymatroids and show that the standardization polytope is the base polyhedron of a network polymatroid.

Given a function $r: 2^{[n]} \to R_+$, which is normalized $(r(\emptyset) = 0)$, nondecreasing and submodular the polytope $P(r) = \{x \in R_+^{[n]} \mid \sum_{i\in I} x_i \le r(I) \quad \forall I \subseteq [n]\}$ is called a polymatroid and r is called its rank function [2]. We denote $\sum_{i\in I} x_i$ by $x(I)$ in the sequel. A polymatroid $P(r)$ is integer (that is, it has only integer vertices) if and only if the rank function r is integer valued [2]. For a given polymatroid $P(r)$ the set $B(r) = \{x \in P(r) \mid x([n]) = r([n])\}$ is usually called the base polyhedron of $P(r)$.

Let $G = (V, E, c)$ be a capacited network, i.e. V is the set of nodes, $E = V \times V$ is a collection of arcs. The network has the source s, the sink q, and the capacity vector $c \in R_+^E$. Let $\delta_v^+ (\delta_v^-)$ be the set of arcs directed into (from) the node v. Denote $E_1 = \{(u,v) \in E \mid u = s\}$, $E_2 = \{(u,v) \in E \mid v = q\}$. Let $c(e) = +\infty \ \forall\, e \in E_1 \cup E_2$ and $P(\bar{r})$ be a polymatroid defined in $R_+^{E_1}$. A flow $f: E \to R_+$ is said to be feasible if

$$f(\delta_v^+) = f(\delta_v^-) \ \forall\, v \in V \setminus \{s,q\}$$
$$f(e) \le c(e) \ \forall\, e \in E$$
$$f(I) \le \bar{r}(I) \ \forall\, I \subseteq E_1.$$

Let $P(r)$ be the set of the all $x \in R_+^{E_2}$ such that there exists a feasible flow *f* in *G* with $f(e) = x(e)\ \forall\ e \in E_2$. Then $P(r)$ is a polymatroid and its rank function *r* is given by $r(I) = max\{f(I) \mid f$ is a feasible flow in $G\}\ \forall\ I \subseteq E_2$. Such a polymatroid is called a network polymatroid. We say that the polymatroid $P(r)$ is induced from $P(\bar{r})$ by flow in *G* and we write $P(r) = Q(\bar{r}, G)$. Denote $V_1 = \{v \in V \mid (s,v) \in E\}$, $V_2 = \{v \in V \mid (v,q) \in E\}$.Assume now that $E \setminus (E_1 \cup E_2) = \{(i,j) \mid i \in V_1, j \in V_2, q_{ji} = 1\}$, $\tilde{n}(e) = +\infty\ \forall\ e \in E, \bar{r}(I) = d([n]) - d([n] \setminus I)\ \forall\ I \subseteq E_1$. We denote this network by *N*. In the network *N* the set E_1 corresponds to the set of given item types, and the set E_2 corresponds to the set of required item types.

Theorem 1. *The base polyhedron of the polymatroid* $P(r) = Q(\bar{r}, N)$ *is the generalized standardization polytope.*

Corollary 1. *The standardization polytope is the base polyhedron of the polymatroid* $P(r)$*, where* $r(I) = b(u(I)), u(I) = \{j \in V_1 \mid \sum_{i \in I} q_{ji} \geq 1\}$.

Corollary 2. *Suppose* $A = B$*, i.e.* $m = n$ *holds. Then the feasible solutions set of the one-parameter standardization problem is defined by the following inequalities*

$$y([i]) \leq b([i])\ \forall\ i \in [m].$$

3. Solution algorithm.

It follows from the previous section that GSP can be rewritten in the following form

$$\text{AP:} \qquad \sum_{i=1}^{m} f_i(y_i) \rightarrow \max$$

$$y \in B(r) \cap Z_+^m,$$

where $B(r)$ is the base polyhedron of the polymatroid $P(r) = Q(d^*, G)$, $d^*(I) = d([n]) - d([n] \setminus I)$.

Problem AP is usually called a discrete resource allocation problem. Many papers have been devoted to its solution. Some of them deal with the case when the underlying polymatroid is generated by well known structure (uniform polymatroid [3,9], tree-structured polymatroid [1,9], generalized symmetric polymatroid [8]; other with the case when polymatroid is defined only by its rank function [7,8,9]. For the solution of GSP we adapt here Dichotomic Greedy Algorithm (DGA) developed in [7] for problem AP. At each step *t* of DGA it is necessary to find point $z^t = max\{y \in Z_+^m \mid y \in P(r), y \leq x^t\}$ and set $I^t = \{i \in [n] \mid z^t + e_i \in P(r)\}$, where x^t is the known current point. We show how one can efficiently find z^t, I^t in our case. We set $c(i,t) = x_i^t \ \forall \ i \in [m]$ in the network *N*. Using any algorithm for maximal flow computation we find the maximal flow *f* in the obtained network. Then $z_i^t = f(i,q) \ \forall \ i \in [m]$. Testing the feasibility of $z^t + e_i$ is equivalent to testing whether in a given residual network an increment of a unit of flow from source to the node $i \in V_2$ is feasible. For a network on *l* arcs this can be implemented in *O(l)*. Thus we can find the set I^t in *O(|E|m).* The complexity of the other steps of DGA is less than the complexity of finding z^t, I^t. The total number of steps of DGA is $O(m \ \log(\sum_{i=1}^{m} r(\{i\}) \ / \ m))$. It implies the following theorem.

Theorem 2. *The algorithm DGA solves SP in* $O(nm^3 \ \log(\sum_{i=1}^{m} r(\{i\}) \ / \ m))$ *and GSP in* $O(Knm^3 \ \log(\sum_{i=1}^{m} r(\{i\}) \ / \ m))$*, where K is the complexity of the membership test for the polymatroid* $P(d^*)$ *.*

Remark. *Suppose that m=n. Then the one-parameter standardization problem can be solved by DGA in* $O(m^3 \ \log(\sum_{i=1}^{m} r(\{i\}) \ / \ m))$*, since its feasible solutions set is a nested polymatroid (for details, see [7]).*

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