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THE EXACT SOLUTION OF THE FREE PRE-STRESSED BAR OSCILLATIONS.

1. INTRODUCTION.

In this paper the results of the investigations of the free oscillations of the pre-stressed flexible structure elements are presented . Two cases of the central preliminary stress are investigated : without intermediate fastening of the tie to the flexible element and with the intermediate fastening in the middle of the element length. The given physical model can be applied to the flexible sloping shells and arches, membranes, large space antenna fields (besides flexible elements).

The peculiarity of these systems is the possibility of the non-adjacent equilibrium form existence at the definite relations of the physical parameters . The transition from one stable equilibrium form to another, non-adjacent form, may be treated as jump. In this case they are called systems with buckling or the systems with two potential «gaps». These systems commenced the new section of the mathematical physics - the theory of chaos and strange attractors.

2. THE DIFFERENTIAL EQUATIONS OF THE FREE OSCILLATIONS.

Suppose the rectangular bar of the length l has the constant cross-section and the ties are fastened in the center of gravity of the end section. The bar is hinged-rested. The free oscillations of such bars are described [1-3] by the non-linear differential equation of Duffing type

$$\ddot{y} - \alpha \cdot y + \beta \cdot y^3 = 0 . \tag{1}$$

Here y - is the generalized coordinate of the cross displacement of the midpoint of the bar length. The coefficient values of this equation α and β are shown in Table 1

.

The values of the parameters α and β.

Table 1.

N system	The tie fastening type	$\alpha , \left[\dfrac{1}{c^2}\right]$	$\beta ,\left[\dfrac{1}{m^2 \cdot c^2}\right]$
I	Without the intermediate fastening point	$\dfrac{1}{m}\left(\dfrac{\pi}{l}\right)^2 \left(N^* - N_n^E\right)$	$\dfrac{EF}{4 \cdot m}\left(\dfrac{\pi}{l}\right)^4$
II	With the intermediate fastening point	$\dfrac{1}{m \cdot l^2} \cdot \dfrac{6\pi^2}{2 \cdot \pi^2 + 3}\left(N^* - 4 \cdot N_n^E\right)$	$\dfrac{18 \cdot \pi^2}{2 \cdot \pi^2 + 3} \cdot \dfrac{EF}{m \cdot l^4}$

The following indications are introduced into Table 1: N^* - the tie tension ; N_n^E - the critical force of n - mode of the bar axis stability loss, equal to $N_n^E = EI \cdot \left(n \cdot \pi/l\right)^2$; EI and EF - the bar rigidities for the bending and compression, correspondingly; m - the mass of the bar linear meter.

Regarding the sign rule at $\alpha \leq 0$ adopted in equation (1) the system possesses one stable equilibrium condition $y=0$. At $\alpha > 0$ the system possess three non-adjacent equilibrium conditions, one of them $y_1=0$ is instable, and two others $y_{2,3}\neq0$ are stable.

As it is shown in Table 1, the critical value of the force in the tie depends on the conditions of its fastening to the bar. Thus, without the intermediate fastening point we see $N_{cr}^* > N_n^E$, and with the intermediate fastening point $N_{cr}^* > 4 \cdot N_n^E$. Therefore, the introduction of the intermediate tie fastening point increases the value of the critical force 4 times more.

3. THE FREE OSCILLATION ANALYSIS.

The non-linear flexible force $R(y) = -\alpha \cdot y + \beta \cdot y^3$ has the symmetry axis in point **a** and two zero values in points **b** and **c** (Fig .1,a). Solving the equation $R(y) = 0$, we find out the conditions of these points :

$$y_a = 0 \quad ; \quad y_b = \sqrt{\frac{\alpha}{\beta}}; \quad y_c = -\sqrt{\frac{\alpha}{\beta}} \quad . \tag{2}$$

The investigation of these condition stability according to Lagrange - Dirichlet theorem shows:
- point y_a=0 corresponds to the instable equilibrium condition - a particular point of the «saddle» type;
- points $y_b = \sqrt{\alpha/\beta}$ and $y_c = -\sqrt{\alpha/\beta}$ corresponds to the stable equilibrium conditions - particular points of the «center» type.

As it is known, the potential energy of $U(y)$ system is equal

$$U(y) = -\frac{\alpha}{2} \cdot y^2 + \frac{\beta}{4} \cdot y^4 + U_0 \quad . \tag{3}$$

The diagram of the $U(y)$ dependence is shown in Fig.1,b. It points to the existence of two potential gaps in the points **c** and **b**. Suppose the initial level of potential energy is U_0=0. In this case two other roots of $U(y)=0$ equation are

$$y_d = \sqrt{\frac{2 \cdot \alpha}{\beta}} \quad ; \quad y_e = -\sqrt{\frac{2 \cdot \alpha}{\beta}} \quad . \tag{4}$$

If the initial level of potential energy $U_1 < U_0$ (Fig.1,b) the trajectories 1-2-3-4 around the particular point **b** or around the particular point **c** (Fig.1.c) correspond to the free oscillations on the phase plane $(y, \dot{y})$. This type of the oscillations was called «small» in [2] for the first time. Their amplitude is in the interval $0 < A_s^2 < 2 \cdot \alpha/\beta$. The initial level of potential energy $U_0 = 0$ conditions the trajectory around the particular points **b** and **c** crossing in particular point **a** (Fig.1,c). In the theory of non-linear oscillations it is called separatrix. If the initial level of potential energy $U_2 > U_0$ (Fig.1,b), the trajectory enveloping all three equilibrium conditions of system **a** , **b** and **c** (Fig.1,c) corresponds to the free oscillations on the phase plane $(y, \dot{y})$. This type of oscillations was called «large» for the first time also in [2]. Their amplitude is $A_l > 2 \cdot \alpha/\beta$.

The whole energy storage of system H is

$$H = U(A) = -\frac{\alpha}{2} \cdot A^2 + \frac{\beta}{4} \cdot A^4 \quad . \tag{5}$$

Therefore, integrating the input equation (1) we find out

$$dt = \frac{dy}{\sqrt{2 \cdot \left[H - U(y) \right]}} \quad . \tag{6}$$

3.1. «SMALL» OSCILLATIONS.

The trajectory 1-2-3-4 corresponds to the «small» oscillations on the phase plane $(y, \dot{y})$, i.e. this trajectory is the motion with two equal half-periods T_{1-3} and T_{3-1}, consisting of two unequal quarters of the periods $T_{1-2} \neq T_{2-3}$ and $T_{3-4} \neq T_{4-1}$. That's why, unlike [3], where $T = 4T_{1-3}$, we determine two quarters of the periods T_{1-2} and T_{2-3}. For that purpose we use the relation (6). Thus,

$$T_{1-2} = \frac{1}{\sqrt{2}} \cdot \int_{A_1}^{y_b} \frac{dy}{\sqrt{\frac{-\alpha}{2} \left(y^2 - A_1^2 \right) + \frac{\beta}{4} \cdot \left(y^4 - A_1^4 \right)}} \quad ; \qquad 0 < y < \sqrt{\frac{\alpha}{\beta}} \quad , \tag{7}$$

where A_1- is the oscillation semispan amplitude on the branch **ab** (Fig.1). The given integral (7) can be easily reduced to the canonical Legendre form

$$T_{1-2} = \frac{F(k_1, \varphi)}{\sqrt{\alpha} \cdot \sqrt{1 - \frac{\tau_1}{2}}} \quad , \tag{8}$$

where τ_1 - is characteristic of the non-linear system equal to $\tau_1 = \beta \cdot A_1^2 / \alpha$; $F(k_1, \varphi)$ - is incomplete elliptic integral of the first kind ; k_1 - is the elliptic integral module , equal to

$$k_1 = \sqrt{2} \cdot \sqrt{\frac{1 - \tau_1}{2 - \tau_1}} \quad . \tag{9}$$

The elliptic integral amplitude is $\varphi = \arccos \left(\sqrt{\tau_1 / 2} \right)$

The time of the motion along the branch 2-3, i.e. the second quarter of the period of «small» oscillations T_{2-3} is determined analogically

$$T_{2-3} = \frac{F \left(k_2, \frac{\pi}{4} \right)}{\sqrt{\alpha} \cdot \sqrt{\frac{\tau_2}{2}}} \quad ; \qquad \sqrt{\frac{\alpha}{\beta}} < y < \sqrt{\frac{2 \cdot \alpha}{\beta}} \quad , \tag{10}$$

where $\tau_2 = \frac{\beta \cdot A_2^2}{\alpha}$; $F \left(k_2, \frac{\pi}{4} \right)$ - is the complete elliptic integral of the first kind; its

amplitude is constant and equal to $\frac{\pi}{4}$, and module is

$$k_2 = \frac{2 \cdot (\tau_2 - 1)}{\tau_2} \quad . \tag{11}$$

The attention should be drawn to the parameters τ_1 and τ_2 change intervals :

$$0 < \tau_1 < 1 \qquad ; \qquad 1 < \tau_2 < 2 \quad . \tag{12}$$

It should be also mentioned that quite definite value τ_2 corresponds to any value τ_1 as a result of the equality [2]

$$\tau_1 + \tau_2 = 2 \ . \tag{13}$$

Therefore, the full period of «small» oscillations is

$$T_s = 2 \cdot \left(T_{1-2} + T_{2-3}\right) \ . \tag{14}$$

3.2 «LARGE» OSCILLATIONS.

The period of «large» oscillations in symmetric non-linear system (1) consists of four equal quarters of the period. It is determined by analogy with the above-mentioned method for the quarter of the period of «small» oscillations. In particular ,

$$\frac{1}{4} \cdot T_l = \frac{1}{\sqrt{2}} \cdot \int_0^{A_l} \frac{dy}{\sqrt{-\dfrac{\alpha}{2} \cdot \left(A_l^2 - y^2\right) + \dfrac{\beta}{4} \cdot \left(A_l^4 - y^4\right)}} \ , \tag{15}$$

where A_l- is the amplitude of «large» oscillations.

After the reducing of this elliptic integral to the canonical Legendre form for the period of «large» oscillations we receive

$$T_l = \frac{4 \cdot K(k)}{\sqrt{\alpha} \cdot \sqrt{(\tau - 1)}} \quad ; \quad y > \sqrt{\frac{2 \cdot \alpha}{\beta}} \ , \tag{16}$$

where τ - is the parameters of non-linear system equal to $\tau = \beta \cdot A_l^2 / \alpha$; $K(k)$ - is the complete elliptic integral of the first kind with module k equal to

$$k = \sqrt{\frac{\tau}{2 \cdot (\tau - 1)}} \ . \tag{17}$$

The received here exact solutions (8),(10),(14) and (16) for the free oscillation periods of the system (1) are graphically shown in Fig 1,d as the amplitude-frequency dependence $A(\omega)$, where ω - is the circular natural frequency of the system (1).
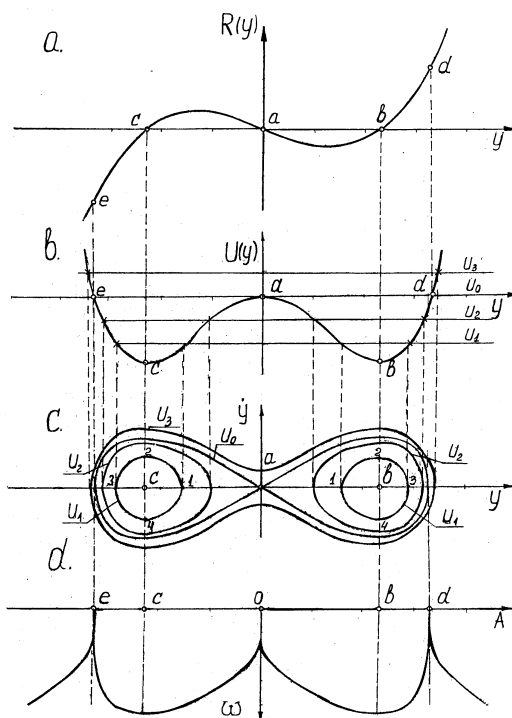

Fig.1. The graphics of the system (1) parameter dependence on the oscillation amplitude: a-elastic force $R(y)$; b- potential energy of the system $U(y)$; c - oscillation velocities $\dot y$; d - oscillations frequency ω .

3.CONCLUSION.

(i) The analysis of the solutions (8),(10) and (16) confirms the received for the first time by the author and given in [1],[2] effect of the oscillation period doubling of the system (1) during the transition from the «small» oscillations relatively center **b** (or **c**) to the «large» relatively all three equilibrium conditions **a** ,**b** and **c** .

(ii) The character of the frequency (period) dependence on the free oscillation amplitudes of the non-linear system (1) (Fig.1,d) also confirms the received earlier [1],[2] result of the duality of the system (1) behaviour :

-«small» oscillations possess the qualities of soft system;

-«large» oscillations possess the qualities of rigid system.

(iii) The «small» oscillation natural frequency changing, depending on the oscillation amplitudes, is in the internal $0 \leq \omega_s \leq \sqrt{2 \cdot \alpha}$. Here the frequency takes zero value at the amplitude values A_a and A_d (or A_a and A_e); the frequency takes maximum value $\omega_s = \sqrt{2 \cdot \alpha}$ at the amplitude value $A \rightarrow 0$ near point **b** .The «large» oscillation natural frequency changes in the interval $0 < \omega_l \leq \infty$. Here $|A_l| > \sqrt{2 \cdot \alpha / \beta}$ is also observed .

(iiii) The influence of the tie intermediate fastening doesn't introduce qualitative changes in the behaviour of the investigated system (1). It only increases (four times) the critical value of the preliminary tension force.

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