HIGH-PRECISION MODELING AND FINITE-ELEMENT INVESTIGATION OF ELASTOPLASTIC DEFORMATION OF NON-ISOTROPIC THICK SANDWICH PLATES AND SHELLS

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1. Introduction

The non-classical, continual theories of elastoplastic deformation of the sloping shells with the isotropic [1] and the transversal-isotropic [2] layers are formed in [1,2]. In the continual theories the general order of differentiation and the structure of the solution equations do not depend on the quantity of layers. A part of the kinematic models [1,2] are the functions, defined from the non-linear solution of the problem through the classic theory. However, solving the problems of deformation of not-thin sandwich plates using the classical theory is absolutely unacceptable [3]. Besides, the determining equations, and therefore, all model correlations [1] are not the particular case of the model [2]. To same extent this contradicts the idea of the continuality and universality of the theory. Below the highly-precise continual model of elastic deformation of not-thin plates with the orthotropic layers [3], that accounts for the peculiarities of flexural and unflexural doformation, is generalized to the development [1,2] for the case of physically non-linear deformation of sloping shells. The modified quadratic criterium of marginal condition of the quasi-fragile orthotropic materials is suggested for receiving of the determining correlations.

2. Problem statement

Physically non-linear stress-strained state (SSS) of a sloping shell having thickness h is being modelled in orthogonal system of co-ordinates x_α (α=1,2,3; $x_3=z$). Orthotropic axes coincide with axes x_α in rigidly connected orthotropic layers of arbitrary $h^{(k)}$ (k=1,...,n) thickness. Axe z and the given surface mechanical loading Y_{30}, Y_{3n} are orthogonal to facial surfaces of the shell $z=a_0$, $z=a_n$. Similarly to [1], for the curvatures k_{ij} (i,j=1,2) of the coordinate surface x_1x_2, we accept that $k_{ij}=const$, $k_{11}k_{22}\approx 0$, $1+zk_{ij}\approx 1$. Under the small bending this makes the deformations $e^{(k)}_{\alpha\beta}$ to depend on the displacements $u^{(k)}_\alpha$ in the form:

$$2e^{(k)}_{\alpha\beta}=u^{(k)}_\alpha,_\beta+u^{(k)}_\beta,_\alpha+2k_{\alpha\beta}u^{(k)}_3;\qquad k_{\alpha 3}=0\,. \tag{1}$$

Here and further the particular derivatives are substituted by lower indexes after a comma. Summarizing of repeated lower indexes of greek alphabet is introduced; at that $\alpha,\beta,\gamma,\delta$=1,2,3. The upper index in brackets is the number of the layer.

For the orthotropic quasi-fragile material with different strength under the single-axe tension and compression there is accepted the quadratic phenomenological yield condition in the form:

$$F(\sigma_{\gamma\delta}) = A^{y}_{1111}[(\sigma_{\alpha\alpha} + h_p a^{y}_{\alpha})^2 / m_{\alpha}{}^2 - \sigma_{\alpha\alpha}\sigma_{\beta\beta} /(m_{\alpha} m_{\beta})] + 2A^{y}_{\alpha\beta\alpha\beta}\sigma_{\alpha\beta}{}^2 - h_p{}^2 = 0; \ \ \alpha \neq \beta, \quad (2)$$

where ($i \neq j$; $i, j = 1,2,3$)

$$A^{g}_{1111} = 4/[\sigma^{g}_{11\,t}\sigma^{g}_{11\,c}(\sigma^{g}_{\alpha\alpha\,t}/\sigma^{g}_{\alpha\alpha\,c} + \sigma^{g}_{\alpha\alpha\,c}/\sigma^{g}_{\alpha\alpha\,t} - 2)]; \quad 4A^{g}_{ijij} = (\sigma^{g}_{ijt})^{-2}; \quad a^{g}_{ii} = (\sigma^{g}_{ii\,c} - \sigma^{g}_{ii\,t})/2.$$

Here $\sigma^{g}_{jj\,t}$, $\sigma^{g}_{jj\,c}$ and $\sigma^{g}_{ij\,t}$ - are the limits of yield (g=y) or the limits of strength (g=s) under the single-axe tension, compression and pure shift, accordingly; h_p - function of the isotropic strengthening ($1 \le h_p \le h^s$). The hypothesis about the similarity of all the diagrams of the non-linear deformations and about proportionality of hardening to a single parameter, based in [4] for isotropic materials is introduced by the function h_p. Then $\sigma^{s}_{jk\,v}/\sigma^{y}_{jk\,v}$=$C_0$=*const* ($i,k = 1,2,3$ $v = t;c$) which decreases the number of basis experiments. Coefficients m_i are introduced hypothetically and correspond to the correlations:

$$m_i^2 = A^{g}_{1111}/A^{g}_{iiii} = \sigma^{g}_{ii\,t}\sigma^{g}_{ii\,c}/(\sigma^{g}_{11\,t}\sigma^{g}_{11\,c}); \quad m_i \succ 0; \quad 1/(m_i m_j) = -2A_{iijj}/A_{1111}; \ \ i \neq j; \ \ i, j = 1,2,3.$$

This also decreases the number of basis experiments, necessary for determining of all components tensors of the yield (strength) $A^{g}_{\alpha\beta\gamma\delta}$ in the quadratic condition of yield with the anisotropic strengthening of Mises-Hill type $A^{g}_{\alpha\beta\gamma\delta}(\sigma_{\alpha\beta} + a^{g}_{\alpha\beta})(\sigma_{\gamma\delta} + a^{g}_{\gamma\delta}) = 1$.

Limiting surface in the shape of a paraboloid with an axis bend over hydrostatic axis corresponds to the condition (2). This axis declination is defined by the accepted hypotheses for the coefficients m_i.

In case of quasi-fragile isotropic material with σ_{iit}=σ_t, σ_{iic}=σ_c, $\sigma_t \neq \sigma_c$ the condition of yield beginning (2) under the three-axes loading takes the shape of the well-known criterion of Balandin [5] (here $\delta_{\alpha\beta}$ are Kroneker's symbol):

$$\delta_{\alpha\alpha}[\sigma^2_{\alpha\alpha} + \sigma_{\alpha\alpha}(\sigma_c - \sigma_t) - 0{,}5\sigma_{\alpha\alpha}\sigma_{\beta\beta}] = \sigma_t\sigma_c. \quad (3)$$

For plastic orthotropic materials with σ_{iit}=σ_{iic}, we have in (2) $a^{g}_{i} = 0$ and condition (2) is the following:

$$\sigma^2_{\alpha\alpha}/\sigma^2_{\alpha\alpha t} - \sigma_{\alpha\alpha}\sigma_{\beta\beta}/(\sigma_{\alpha\alpha t}\sigma_{\beta\beta t}) + \sigma^2_{\alpha\beta}/\sigma^2_{\alpha\beta t} - h^2_p = 0, \quad (4)$$

from which in the flat stressed state the widely-used J.Marin and L.W.Hu's criterium is derived. The latter comes to a better agreement with the experiment data [6] on the yield and strength of the non-isotropic metals and organic materials than the Mises-Hill's criterium.

We will present the plastic (non-linear) components of active deformation $e^{p}_{\alpha\beta}$ in the determining correlations of the deformational plasticity theory in the form, associated with the introduced surface yield (2):

$$e^{p}_{ii} - e^{f}_{ii} = RA^{y}_{1111}[(1/m_i^2)\sigma_{ii} - 1/(2m_i m_j)\sigma_{jj} - 1/(2m_i m_s)\sigma_{ss}]; \quad (5)$$

$$2e^{p}_{ij} = 2R\sigma_{ij}/(\sigma^{y}_{ij\,t})^2; \quad e^{f}_{ii} = h_p RA^{y}_{1111}(\sigma^{y}_{ii\,c} - \sigma^{y}_{ii\,t})/(2m_i^2); \quad i \neq j \neq s; \quad i, j, s = 1,2,3.$$

Here $R = qe^{p}_{eq}/\sigma_{eq}$ - non-linear (if $F>0$) function, invariant to the variety of stressed state; and $R = 0$ if $F \le 0$, and also $\sigma_{eq} = (qF(\sigma_{ij}))^{1/2}$ and $e^{p}_{eq} = (F(e^{p}_{ij}))^{1/2}$ - accordingly, are the equivalent tension and the equivalent plastic deformation; $q = 3/(A_{1111}\delta_{\alpha\alpha}/(m^2_{\alpha}))$. Components

e_{ii}^{f} - beginning plastic deformations for plastic materials with the previous anisotropic strengthening, or the fictitious functions addings for the non-strengthened quasi-fragile materials.

Function R is approximated according to the experimental diagram $\sigma_{eq} = \Phi(e_{eq}^{p})$.

Full deformations are equal to $e_{\alpha\beta} = e_{\alpha\beta}^{e} + e_{\alpha\beta}^{p} = D_{\alpha\beta\gamma\delta}\sigma_{\gamma\delta} + e_{\alpha\beta}^{f}$, where $e_{\alpha\beta}^{e} = C_{\alpha\beta\gamma\delta}\sigma_{\gamma\delta}$ - their elastic part, and $D_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} + a_{\alpha\beta\gamma\delta}R$, if (5) is given as $e_{\alpha\beta}^{p} = a_{\alpha\beta\gamma\delta}R\sigma_{\gamma\delta} + e_{\alpha\beta}^{f}$. The equations $\sigma_{\alpha\beta} = B_{\alpha\beta\gamma\delta}(e_{\gamma\delta} - e_{\gamma\delta}^{f})$ are easily derived from here; tensor $[B] = [D]^{-1}$.

The transversal shear and normal layer pliabilities $(e_{\alpha 3}^{(k)} \neq 0)$ are accounted for in the continual (along thickness) non-classical model of deformation of sandwich shell. In contradistinction to [1,2], the hypotheses for tensions of the transverse shear $\sigma_{i3}^{(k)}$ (i=1,2) in layer k in the non-linear problem are accepted similarly to their transverse approximations along the z coordinate in the linear elastic problem of plate's deformation [3]. Besides, the approximations of not only flexural (polynoms of z of maximal even powers), but also unflexural (polynom of z of the third power) deformations are introduced in $\sigma_{i3}^{(k)}$ [3]. The influence on $\sigma_{i3}^{(k)}$ of the present in (1) curvatures k_{ij} is formally accounted for by the unflexural components in the approximation $\sigma_{i3}^{(k)}$.

The hypoteses look like:

$$\sigma_{i3}^{(k)} = f_{i\alpha}(z^{\alpha+1};R)\beta_{i\alpha}; \qquad i=1,2, \tag{6}$$

where the given functions $f_{i\alpha}$ are showed in [3], and in the brackets is indicated the maximal degree of the approximative polynoms along z in the linear problem, that now depends on function R. The elastic characteristics $C_{\alpha\beta\gamma\delta}^{(k)}$ of the layer material are formally substituted for the components of the non-linear tensors $D_{\alpha\beta\gamma\delta}^{(k)}$ and $B_{\alpha\beta\gamma\delta}^{(k)}$. The base for this substitution is the possibility of linearization of the problem along the z coordinate by means of discretization of layers along thickness for the purpose that in the sublayer $D_{\alpha\beta\gamma\delta}^{(k)}(x_1, x_2, z) \approx D_{\alpha\beta\gamma\delta}^{(k)}(x_1, x_2)$. This doesn't complicate the continual model because its solution equations do not depend on the number of layers.

The necessity of the introduction of the flexural higher approximations and the unflexural approximations of the transverse shear and also the necessity of taking into consideration the transverse compression while calculating linear elastic deformation of the non-thin non-isotropic plates is numerically based in [3].

The hypothesis for the transversal deformation $e_{33}^{(k)}$ is also introduced in the form:

$$e_{33}^{(k)} = (\sigma_{33}^{a} + \sigma_{33}^{b} + B_{33\alpha\alpha}^{(k)} e_{\alpha\alpha}^{f}) / B_{3333}^{(k)} + e_{v}^{a} + e_{v}^{b}, \tag{7}$$

where both $\sigma_{i3}^{(k)}$ in (6) and σ_{33}^{a}, e_{v}^{a} are accepted similarly to the correlations of the linear theory:

$$\sigma_{33}^{a} = F_0(z^3;R)q_3 + p_3 + F_{\chi}(z^{2+\chi};R)\gamma_{\chi}; \qquad q_3 = 0,5(Y_{30} + Y_{3n}); \quad p_3 = Y_{3n} - q_3;$$

$$e_{v}^{a} = \mu_0(z^0;R)\gamma_0 + \mu_{\chi}(z^{2+\chi};R)\gamma_{\chi};$$

$$\mu_m = \mu_{m\chi}^{(k)} B_{33\chi\chi}^{(k)} / B_{3333}^{(k)}; \quad \mu_{ij}^{(k)} = \int_{\delta_j}^{z} D_{j3j3}^{(r)} f_{ji}^{(r)} dz; \quad \mu_{0j}^{(k)} = 1; \quad m = 0,1,2; \quad \chi, i, j = 1,2,$$

at that σ_{33}^{a} corresponds to [3], and the components of the unflexural deformation in σ_{33}^{a} and e_{v}^{a} indirectly account for the curvatures k_{ij}, which are the part of (1) and of the equilibrium equation $\sigma_{\alpha 3}^{(k)},_{\alpha} - k_{\alpha\beta}\sigma_{\alpha\beta}^{(k)} = 0$.

Components $\sigma_{\alpha 3}^{b}$ and e_{ν}^{b} are the consequence of the physical non-linearity. For the quantity of unknown functions in the linear and non-linear models to be equal, $\sigma_{\alpha 3}^{b}$ and e_{ν}^{b} are accepted as given from the previous iterational step in the solution of the non-linear problem and look like:

$$\sigma_{33}^{b} = F_{\chi\alpha}^{b}(z^{\alpha+2}; R,_{\chi})\beta_{\chi\alpha}^{b}; \quad F_{\chi\alpha}^{b} = -\int_{a_0}^{z}(f_{\chi\alpha}^{(r)},_{\chi} - \delta_{3}^{\chi\alpha})dz; \quad F_{\chi\alpha}^{b}(z = a_0) = 0; \quad \chi = 1,2;$$

$$\sigma_{i3}^{b} = f(e_{jj}^{f}; B_{jj\alpha\beta}^{(k)}); \; e_{33}^{b} = -(B_{33\chi\chi}^{(k)} / B_{3333}^{(k)})(\int_{\delta_\chi}^{z} D_{\chi 3\chi 3}^{(s)} f_{\chi\alpha}^{(s)},_{\chi} dz)\beta_{\chi\alpha}^{b}; \; r = \overline{1,k}; \; s = \overline{l_\alpha, k}. \quad (8)$$

Here the sum of integrals in the quantity of z from the non-continuous function is marked as one integral of the given function [1-3].

Hypotheses (6) and (7) allow to determine the displacement vector and to formulate the problem relative to the unknown functions of the coordinate surface
(x_1x_2): tangential $v_i(x_j)$ and normal $v_3(x_j)$ displacements to the coordinate surface of the shell on the voluntary surfaces $z=\delta_\alpha$, and also functions of the transverse shear $\beta_{i\alpha}(x_j)$ and of the transverse compression $\gamma_m(x_j)$ ($m = 0,1,2$):

$$u_3^{(k)} = v_3 + \int_{\delta_3}^{z} e_{33}^{(s)} dz; \qquad s = \overline{l_\alpha, k}; \qquad j, i = 1,2; \qquad (9)$$

$$u_i^{(k)} = v_i - \int_{\delta_i}^{z} u_3^{(s)},_i dz + \int_{\delta_i}^{z} D_{i3i3}^{(s)} \sigma_{i3}^{(s)} dz.$$

The expressions for the SSS components may be obtained by filling (9) into the Cauchy's correlation (1) and then into $\sigma_{\alpha\beta} = B_{\alpha\beta\gamma\delta}(e_{\gamma\delta} - e_{\gamma\delta}^{f})$. This leads to the unfulfilling in $\sigma_{33}^{(k)}$ of static conditions on the facial surfaces of layers , which has a little influence on the accuracy of computation. However, the definition $\sigma_{33}^{(k)}$ from the law of Hooke and not from the hypothes, allows to get a symmetric matrix of coefficients in the solutable system of differential equations of the linear elastic problem.

Lagrange's variational principle allows us to get the non-linear solution system of the differentional equations

$$(L^e + L^p)[V] = Q + S^p \qquad (10)$$

relative to the functions $[V]^T =[\, v_\alpha; \beta_{i\alpha}; \gamma_m]$, where the matrix of the non-linear differentive operators L^p contains R and is a subject to linearization, and S^p reflects the influence of σ_{33}^{b} and e_{ν}^{b} , e_{ii}^{f} playing a part of additional loading to the matrix of the functions depending on the given loading. The general order of differentiation (10) does not depend on the quantity of layers (continual model), and the part of each of the functions v_i, $\beta_{i\alpha}$ in it makes 2, and functions v_3, γ_m - 4.

Using the discrete-continual scheme of the finite-element method, suggested in [1], and linked with this method the ideas of piece linearization of the initial non-linear problem in the limits of the elementary layer (sublayer), allow us to receive the solution (10) as a sequence of non-classical linear solutions. At that, the first approximation is a specified linear solution under the initial elastic parameters $C_{\alpha\beta\gamma\delta}^{(k)}$. Next step, in contradistinction to [1,2], are also the non-classical solutions, that are correcting R, L^p and S^p in the equations (10) in correspondence with the defined from (2), (5) h_p ($1 \le h_p \le h^s$)..

From the point of view of the influence of the error FEM on the coincidence of the iteration process while solwing non-linear equations using of step-by-step method of higher accuracy is recommended [1].

The absence of the specifying factors in the model (6)-(10), linked with the linear and non-linear solutions of the problem on classical theory and also using defining equations with one function of hardening R insteated of the equations of the simplified theory of B.E.Pobedria with the tree functions like that, makes this model different from the known non-classical model of deformation of orthotropic shells and plates [3]. Let's note that in the linear-elastic problems of deformation of thick rectangular plates with the relative thickness $b_i/h<10$ and with the transversal normal and shear hardness of the bearing layer the sequence less then its tangential hardness, the solutions for the function of flexures according classical theory, used in [1,2] are not reliable quantitatively and qualitatively [3,7].

The approach is approbated on the test problem [1] of the curving of three-layer rectangular plate (b_i/h=10) with rigid face diaphragms on the contour, loaded in the center with the concentrated load P. The outer layers are made of aluminium and and the filling is foam plastic. The net of laying out to the finite elements is 8×8. The law of the intensity of tensions is taken as:

$$\sigma_i^{(k)} = \sigma_{11c}^{s\ (k)}(1 - e^{-\xi^{(k)} e_i^{(k)}}); \qquad \xi^{(k)} = 3G^{(k)} / \sigma_{11c}^{s\ (k)}.$$

For isotropic layers k=1,2,3 correspondingly, the conditional limits of yield are $\sigma_{11c}^{y\ (k)}=$ =100; 1; 100 MPa; the limits of strength are $\sigma_{11c}^{s\ (k)}$=230; 1; 230 MPa; thickness $h^{(k)}$=1; 18; 1 mm; the initial modules of the elasticity are $E^{(k)}$=7×10^4; 3×10^2; 7×10^4 MPa; Pyasson's coefficients $\nu^{(k)}$=0,32; 0,4; 0,32. The load was applied by steps (gradually) ΔP=0,5 kN. In the outside layers the limit of yield $\sigma_{11c}^{y\ (k)}$ is obtained under P^y=2 kN and $\sigma_{11c}^{s\ (k)}$ - under the destructing ruining load P^s=11 kN, while in [1] P^s=10 kN is got. It's interesting to note, that under $h^{(k)}$=1; 17; 2 mm and under $h^{(k)}$=1; 16; 3 mm the ruining load is P^s=11,5 kN. For the this plate under consideration the differences in P^s from the model [1,2] become essential under $b_i/h\leq4$.

Table 1. Exreme normal stresses $\bar{\sigma}_{11} = \sigma_{11\ \max}^{(3)} / q_0$ in the exfoliated isotropic plate and shear stresses $\bar{\sigma}_{13}$ in the modulated low-module-interlayer of «sliding» according to the three-dimensional [7, 8]and specified pure-shift S_1 [1] solution.

Stresses	t=0	t=2·10^{-4}; V_1; V_2		t=2·10^{-3}; V_2		t=0,7; V_2		t=2·10^{-3}; V_3	
	[8]	[7]	S_1	[7]	S_1	[7]	S_1	[7]	S_1
$\bar{\sigma}_{11}$	38, 35	28, 62	27, 29	36, 46	34, 16	385, 4	380, 0	38, 47	35, 42
$\bar{\sigma}_{13}$	0	4,00	3,67	0,83	0,68	0,13	0,09	0,37	0,29

Under physically non-linear deformation a partial destruction can occur because of the exfoliation or loosening of the structure of the plate. In the limits of the finite element there is a possibility of modelling it by introducing of the thin light-hard layers of sliding or the layers of loosening. However, the layer should not be too thin, to give the transverse shear, for example, the opportunity to develop. It goes out of the comparison of the exact 3-dimensional linear solution in which the boundary conditions of interlinear sliding without friction [8] on the level of the middle surface in isotropic (ν=0,3; E=2,6×10^5 MPa) rectangular (b_i/h=3) plate are satisfied exactly and the exact 3-dimensional linear solution under the method [7] for 3-layer plate with highly pliable layer of "sliding" $h^{(2)}$ (table 1). Relative thickness $t = h^{(2)}/h$ and relative hardness $n_1 = G_{i3}^{(2)}/G$, $n_2 = E_i^{(2)}/E$, $n_3 = E_3^{(2)}/E$ (i=1,2), which for variant V_1 is equal to n_1=10^{-4}; n_2=n_3=10^{-2}; for

V_2: n_1=10^{-4}; n_2=10^{-2}; n_3=1; for V_3: n_1=10^{-6}; n_2=10^{-3}; n_3=1 were varied in the layer k=2. It is evident that control after the condition $\overline{\sigma}_{13} = \sigma^{(2)}_{13\ \max} / q_0 \to 0$ only is unsufficient for fixing t. Thus, $t > 10\, n_1$ can be recommended for the «sliding» layers. Boundary conditions of Navier - type [7,8] were modelled in the given problem, under the load $Y_{3n} = q_0 \sin(\pi x_1 / b_1) \sin(\pi x_2 / b_2)$.

3. Conclusions

Kinematic hypotheses accepted for the whole package of orthotropic layers underlied construction of the refined nonclassical theory of elastoplastic strain of sandwich sloping shells with allowance for transversal shear and reduction strains. The peculiarities of unflexural deformation are accounted for in the model. A scheme for numerical realization of the suggested theoty is presented in frames of the discrete-continual scheme of the finite-element method.

References

[1] Piskunov, V.G., Verijenko, V.E. "Linear and non-linear problems of computation laminated constructions". Kiev: Budivelnik, 176 p. (1986).

[2] Verijenko, V.E. "Nonclassical theory of elastoplastic strain in laminated transverse-isotropic sloping shells". Problems of Strength, Nat.Ac.Sci. of Ukraine, N 4, pp. 89-94 (1987).

[3] Gurtovy, O.G. "High-precision modelling of deformation of laminated structures". Mechanics of Composite Materials, N 1, pp. 13-28 (1999).

[4] Panferov, V.M. "The theoty of elasticity and the deformation theory of plasticity for solids with various properties as regards compression, tension and torsion". Doclady Ac. Sci. of USSR, V.180, N 1, pp. 41-44 (1968).

[5] Pisarenko, G.S., Lebedev, A.A. "Deformation and strength of materials at a complex stress state". Kiev: Naukova dumka, 415 p. (1976).

[6] Lebedev, A.A., Kovalchuk, B.I., Giginyak, F.F., Lamashevsky, V.P. "Mechanical properties of structural materials at a complex stress state. Handbook.". Kiev: Naukova dumka, 367 p. (1983).

[7] Piskunov, V.G., Sipetov, V.S., Tuimetov, Sh.Sh. "Solution of the problems of statics for laminated orthotropic plates in the three-dimensional statement". Prickladnaia mekhanika, Nat.Ac.Sci. of Ukraine, V.26, N 2, pp.41-49 (1990).

[8] Marchuk, A.V. Марчук А.В. " Three-dimensional analytical solution for laminated plates with allowance for the layers sliding". Prickladnaia mekhanika, Nat.Ac.Sci. of Ukraine, V.33, N 9, pp.10-14 (1997).