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ORTHOGONAL DECOMPOSITIONS AND THEIR APPLICATIONS

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Abstract. *It is well known that complex quaternion analysis plays an important role in the study of higher order boundary value problems of mathematical physics. Following the ideas given for real quaternion analysis, the paper deals with certain orthogonal decompositions of the complex quaternion Hilbert space into its subspaces of null solutions of Dirac type operator with an arbitrary complex potential. We then apply them to consider related boundary value problems, and to prove the existence and uniqueness as well as the explicit representation formulae of the underlying solutions.*

1 INTRODUCTION AND STATEMENT OF RESULTS

Complex quaternion analysis is an active research subject by itself and it is thought to play an important role in the treatment of 3D and 4D boundary value problems of mathematical physics. A thorough treatment is listed in the bibliography, e.g. K. Gürlebeck and W. Sprößig [4, 6], V. Kravchenko and M. Shapiro [13], V. Kravchenko [14], M. Shapiro and N. Vasilevski [22, 23], and A. Sudbery [30]. This function theory, which involves the study of complex quaternion functions, may also provide the foundations to generalize the classical theory of holomorphic functions of one complex variable onto the multidimensional situation.

During the last years much effort has been done in the study of orthogonal decompositions of quaternion and Clifford Hilbert spaces, starting for example with the works of S. Bernstein [1, 2], B. Goldschmidt [3], K. Gürlebeck and W. Sprößig [4, 5, 6], V. Kravchenko and M. Shapiro [11, 12], E.I. Obolaschvili [16, 17, 18], J. Ryan [19, 20], M. Shapiro and L.M. Tovar [24], F. Sommen and Z. Xu [25], W. Sprößig [26, 28], I. Stern [29], and Z. Xu [33]. Their investigations provide powerful tools to study certain elliptic boundary value problems of partial differential equations within the framework of quaternion and Clifford analyses. So that this research domain has interacted elegantly in numerous problems of mathematical physics (cf. [9, 27, 32]). Those works include, among others, the Laplace, Helmholtz, Maxwell, Schrödinger, Klein-Gordon, Lamé and Stokes (later Navier-Stokes) equations. However, as far as we know, relatively little effort has been done to establish orthogonal decompositions involving complex quaternion Hilbert spaces. Clearly it would be appropriate for us to explore this connection in detail.

This paper is organized as follows. In Section 2 we describe the fundamental solution of the operators $D_{\pm\alpha} = D \pm \alpha$, where D denotes the classical Dirac operator and α is an arbitrary complex constant. As a first step towards we are able to define the Teodorescu and Cauchy-Fueter operators T_α and F_α, which have the same properties as the operators T and F related to the operator D. Let G be a symmetric domain in $\mathbb{R}^3$ with a piecewise smooth Liapunov boundary Γ. In Section 3 we deduce a proper orthogonal decomposition (with complex potential) of the complex quaternion Hilbert space $L^2(G, \mathbb{CH})$:

$$L^2(G, \mathbb{CH}) = \ker D_\alpha \cap L^2(G, \mathbb{CH}) \oplus_{\mathbb{CH}} D_{\overline{\alpha}} \overset{\circ}{W}{}_2^1 (G, \mathbb{CH}). \tag{1}$$

Here $\overset{\circ}{W}{}_2^1(G, \mathbb{CH})$ is the complex quaternion analogue to the Sobolev space $\overset{\circ}{W}{}_2^1(G)$ of functions that vanish on Γ.

We are at liberty to define and give in an explicit manner the corresponding orthoprojections $\mathbf{P}_\alpha$ and $\mathbf{Q}_\alpha$ onto the subspaces of this decomposition. Further investigation shows a closed connection of such decomposition to the following problem:

$$\begin{aligned} (-\Delta + 2Re(\alpha)D + |\alpha|^2)u &= f \quad \text{in } G \\ u &= g \quad \text{on } \Gamma. \end{aligned}$$

In the case of the unique solvability the solution of these boundary value problem can be represented explicitly. Lastly, Section 3 links decomposition (1) to the following Dirichlet problem:

$$\prod_{i=1}^{n}{}^{(l)} D_{\alpha_i} D_{\overline{\alpha}_i} u = f \quad \text{in } G;$$

$$u = g_0,\ D_{\alpha_1} D_{\overline{\alpha}_1} u = g_1, \ldots, D_{\alpha_{n-1}} D_{\overline{\alpha}_{n-1}} \ldots D_{\alpha_1} D_{\overline{\alpha}_1} u = g_{n-1} \quad \text{on } \Gamma$$

where $\prod^{(l)}$ denotes the left product of the underlying sequences. We shall apply the decomposition (1) to prove the existence and uniqueness, and a representation formula for the solution of this boundary value problem, leaving aside for the moment the question of whether the behaviour of the problem is regular and stable. We will follow mainly the notations introduced in [15]. For more details see [4, 26].

2 PRELIMINARIES

We begin by recalling some basic algebraic facts about real and complex quaternions necessary for the sequel. Let $\{e_0, e_1, e_2, e_3\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^4$ with the (quaternionic) product given according to the multiplication rules: $e_1^2 = e_2^2 = e_3^2 = -1$; $e_1e_2 = e_3$, $e_2e_3 = e_1$, and $e_3e_1 = e_2$. This noncommutative product generates the algebra of real quaternions denoted by $\mathbb{H}$. We put $e_0 = 1$, the latter being the identity element. The real vector space $\mathbb{R}^4$ will be embedded in $\mathbb{H}$ by identifying the element $a = (a_0, a_1, a_2, a_3) \in \mathbb{R}^4$ with the element $a = \sum_{j=0}^{3} a_je_j$ of the algebra. Throughout our presentation, we denote the algebra of quaternions with complex coefficients by $\mathbb{CH}$, where its elements are in the form $a = a^1 + ia^2$, where a^1 and a^2 are real quaternions, and $ie_j = e_ji$, $(j = 1, \ldots, 3)$. In this sense real numbers, complex numbers, and real quaternions can be regarded as special cases of complex quaternions. It is fairly known as the *algebra of complex quaternions* (terminology due to W. Hamilton).

The conjugation corresponding to $\mathbb{CH}$ is readily given by

$$\overline{a}^{\mathbb{CH}} := \overline{a^1} - i\overline{a^2} = \sum_{j=0}^{3} \left(a_j^1\overline{e_j} - ia_j^2\overline{e_j}\right), \quad a_j^1, a_j^2 \in \mathbb{R}.$$

We consider functions defined on G and taking values in the algebra of complex quaternions. A complex quaternion-valued function $f : G \longrightarrow \mathbb{CH}$ or, briefly, an $\mathbb{CH}$-valued function will take the following form

$$f(x) = f_0(x) + f_1(x)e_1 + f_2(x)e_2 + f_3(x)e_3 =: f_0(x) + \mathbf{f}(x),$$

where f_l $(l = 0, 1, 2, 3)$, are complex-valued functions defined on G. The spaces $L^2(G, \mathbb{CH})$, $W_2^k(G, \mathbb{CH})$ and $C^{0,\beta}(\Gamma, \mathbb{CH})$ are defined componentwise respectively as the Lebesgue space of all $\mathbb{CH}$-valued functions whose square is Lebesgue integrable in G, the Sobolev space of k-times differentiable $\mathbb{CH}$-valued functions whose k-th derivative belongs to $L^2(G, \mathbb{CH})$, and the Hölder continous $\mathbb{CH}$-valued function space with the exponent β.

We now turn our attention to some simple considerations that are necessary in our study of orthogonal decompositions in complex quaternion Hilbert spaces. Let us denote by $D = \sum_{k=1}^{3} \partial_ke_k$ the classical Dirac operator, and let G be a symmetric domain relative to the origin with a piecewise smooth Liapunov boundary Γ.

In the complex quaternion Hilbert space $L^2(G, \mathbb{CH})$ we consider the following inner product using the complex quaternion conjugation:

$$(u, v)_{\mathbb{CH}} := \int_G \overline{u(x)}^{\mathbb{CH}} v(x)dG_x, \quad u, v \in L^2(G, \mathbb{CH}).$$

Recall the basic fact that two elements u and v are called *orthogonal* if and only if $(u, v)_{\mathbb{CH}} = 0$. We proceed by finding the orthogonal decomposition for the Dirac type operator $D_\alpha = D + \alpha$ [10], where α is an arbitrary complex constant. First of all, we describe the fundamental solution of this operator from the well known fundamental solution of the Helmholtz type operator $\Delta + \alpha^2 I$. As usual, I denotes the identity operator.

Recall from [8, 10] (cf. [9, 31]) that in case $\alpha \in \mathbb{C}$ the fundamental solutions of the Helmholtz type operator are given, respectively, by

$$\Theta_\alpha(x) := -\frac{1}{4\pi|x|}e^{-i\alpha|x|}, \qquad \widetilde{\Theta}_\alpha(x) := -\frac{1}{4\pi|x|}e^{i\alpha|x|},$$

for $x \in \mathbb{R}^3 \setminus \{(0,0,0)\}$. In greater detail, from the factorization of the Helmholtz operator

$$\Delta + \alpha^2 = (D + \alpha)(-D + \alpha)$$

it follows $(-D+\alpha)\Theta_\alpha \in \ker(D+\alpha)$, and $-(D+\alpha)\Theta_\alpha \in \ker(D-\alpha)$. We have, in effect, the following fundamental solutions of the Dirac type operators $D_{\pm\alpha} := D \pm \alpha$ related to $\Theta_\alpha(x)$:

$$K_\alpha(x) = (\alpha + \frac{x}{|x|^2} + i\alpha\frac{x}{|x|})\Theta_\alpha(x), \qquad K_{-\alpha}(x) = (-\alpha + \frac{x}{|x|^2} + i\alpha\frac{x}{|x|})\Theta_\alpha(x),$$

and $\widetilde{\Theta}_\alpha(x)$:

$$\widetilde{K}_\alpha(x) = (\alpha + \frac{x}{|x|^2} - i\alpha\frac{x}{|x|})\widetilde{\Theta}_\alpha(x), \qquad \widetilde{K}_{-\alpha}(x) = (-\alpha + \frac{x}{|x|^2} - i\alpha\frac{x}{|x|})\widetilde{\Theta}_\alpha(x).$$

We proceed by finding the complex quaternion conjugations of the above functions. Keeping in mind that the functions Θ_α and $\widetilde{\Theta}_\alpha$ contain the scalar variable $|x|$ and complex numbers only, then their complex quaternion conjugations are defined as follows:

$$\overline{\Theta_\alpha(x)}^{\mathbb{CH}} = \widetilde{\Theta}_{\overline{\alpha}}(x), \qquad \overline{\widetilde{\Theta}_\alpha(x)}^{\mathbb{CH}} = \Theta_{\overline{\alpha}}(x).$$

That leads to the complex quaternion conjugations of the fundamental solutions $K_\alpha(x)$ and $\widetilde{K}_\alpha(x)$:

$$\overline{K_\alpha(x)}^{\mathbb{CH}} = \widetilde{K}_{\overline{\alpha}}(-x), \qquad \overline{\widetilde{K}_\alpha(x)}^{\mathbb{CH}} = K_{\overline{\alpha}}(-x).$$

In the sequel, let Γ' be an arbitrary parallel surface to Γ. Following [7, 6], the set of functions $\varphi^l := K_\alpha(y - x^l)$ is complete in $\ker(D + \alpha) \cap L^2(G, \mathbb{CH})$, where $\{x^l\}$ is a dense set on Γ'. We proceed by introducing the operators T_α and F_α with the fundamental solution described beforehand as its kernel:

$$(T_\alpha u)(x) := -\int_G \widetilde{K}_\alpha(x-y)u(y)dG_y, \qquad (F_\alpha u)(x) := \int_\Gamma \widetilde{K}_\alpha(x-y)n(y)u(y)d\Gamma_y.$$

They are known as the *Teodorescu transform* and *Cauchy-Fueter type operator*, respectively. As usual, $n(y) = \sum_{k=1}^3 n_k e_k$ is the outer normal on Γ at the point y. We must bear in mind that the *Plemelj-Sokhotzki formulae* (see e.g. [6, 26]) remain true for the operator F_α. As an aside, we may then define the so called Plemelj projections P_α and Q_α [6, 26] onto the space of square integrable functions that have a $D_\alpha-$holomorphic extension into the domains G or $\mathbb{R}^3 \setminus \overline{G}$, and vanish at infinity.

Ultimately, let $u \in C^{0,\beta}(\Gamma, \mathbb{CH})$ $(0 < \beta \leq 1)$, it holds

$$n.t. - \lim_{t \to x,\, t \in G} (F_\alpha u)(t) = (P_\alpha u)(x),$$

and

$$n.t. - \lim_{t \to x,\, t \notin \overline{G}} (F_\alpha u)(t) = -(Q_\alpha u)(x)$$

where the notation $n.t.$-limit means nontangential limit.

Remark 2.1. From the above-mentioned relations of the fundamental solutions K_α and $\widetilde{K}_\alpha$, the kernel of the Teodorescu transform and Cauchy-Fueter type operator can also be chosen by K_α with similar results.

3 AN ORTHOGONAL DECOMPOSITION FORMULA WITH COMPLEX POTENTIAL

Let G denote an arbitrary symmetric domain relative to the origin. We begin by introducing an orthogonal decomposition (Theorems 3.1 below) of the complex quaternion Hilbert space $L^2(G, \mathbb{CH})$ into its subspaces of null solutions of the corresponding Dirac operator invoking orthogonality with complex potential.

Using the classical Hopf maximum principle [21], it follows the result.

Theorem 3.1. *The Hilbert space $L^2(G, \mathbb{CH})$ permits the following orthogonal decomposition:*

$$L^2(G, \mathbb{CH}) = \ker D_\alpha \cap L^2(G, \mathbb{CH}) \oplus_{\mathbb{CH}} D_{\overline{\alpha}} \overset{\circ}{W_2^1} (G, \mathbb{CH}).$$

Proof. Following the ideas given in [6, 26], we set $X_1 := \ker D_\alpha \cap L^2(G, \mathbb{CH})$ and $X_2 := L^2(G, \mathbb{CH}) \ominus_{\mathbb{CH}} X_1$. For each function $u \in X_2$ there exists a function $v \in W_2^1(G, \mathbb{CH})$ so that $u = D_{\overline{\alpha}} v$. For an arbitrary $\varphi \in X_1$ we then have

$$\begin{aligned} 0 &= (u, \varphi)_{\mathbb{CH}} := \int_G \overline{u}^{\mathbb{CH}} \varphi dG_y = \int_G \overline{(D + \overline{\alpha})v}^{\mathbb{CH}} \varphi dG_y = \int_G \overline{Dv}^{\mathbb{CH}} \varphi dG_y + \int_G \overline{v}^{\mathbb{CH}} \alpha \varphi dG_y \\ &= \int_G \overline{D(v^1 + iv^2)}^{\mathbb{CH}} \varphi dG_y + \int_G \overline{v}^{\mathbb{CH}} \alpha \varphi dG_y \\ &= \int_G \sum_{k=1}^3 \sum_{j=0}^3 \overline{\partial_k e_k v_j^1 e_j + i \partial_k e_k v_j^2 e_j}^{\mathbb{CH}} \varphi dG_y + \int_G \overline{v}^{\mathbb{CH}} \alpha \varphi dG_y \\ &= \sum_{k=1}^3 \sum_{j=0}^3 \int_G (\partial_k \overline{e_j} v_j^1 \overline{e_k} - i \partial_k \overline{e_j} v_j^2 \overline{e_k}) \varphi dG_y + \int_G \overline{v}^{\mathbb{CH}} \alpha \varphi dG_y \\ &= -\sum_{k=1}^3 \sum_{j=0}^3 \int_G (\overline{e_j} e_k \partial_k v_j^1 - i \overline{e_j} e_k \partial_k v_j^2) \varphi dG_y + \int_G \overline{v}^{\mathbb{CH}} \alpha \varphi dG_y \end{aligned}$$

$$= \sum_{k=1}^{3}\sum_{j=0}^{3}\left[\int_G (\overline{e_j}v_j^1\partial_k e_k\varphi - i\overline{e_j}v_j^2\partial_k e_k\varphi)dG_y - \int_\Gamma (\overline{e_j}e_k v_j^1 n_k\varphi - i\overline{e_j}e_k v_j^2 n_k\varphi)d\Gamma_y\right]$$

$$+\int_G \overline{v}^{\mathbb{CH}}\alpha\varphi dG_y$$

$$= \int_G \overline{v}^{\mathbb{CH}} D\varphi dG_y + \int_G \overline{v}^{\mathbb{CH}}\alpha\varphi dG_y - \int_\Gamma \overline{v}^{\mathbb{CH}} n(y)\varphi(y)d\Gamma_y$$

$$= \int_G \overline{v}^{\mathbb{CH}} D_\alpha\varphi dG_y + \int_\Gamma \overline{v}^{\mathbb{CH}}\overline{n(y)}^{\mathbb{CH}}\varphi(y)d\Gamma_y = \overline{\int_\Gamma \overline{\varphi}^{\mathbb{CH}} n(y)v(y)d\Gamma_y}^{\mathbb{CH}} .$$

If we substitute $\varphi := K_\alpha(y-x^l)$, and use the relation $\overline{K_\alpha(y-x^l)}^{\mathbb{CH}} = \widetilde{K}_{\overline{\alpha}}(x^l-y)$ then $(F_{\overline{\alpha}}v)(x^l) = 0$, $x^l \in \Gamma'$. That means $tr_\Gamma v \in imP_{\overline{\alpha}} \cap W_2^{\frac{1}{2}}(\Gamma,\mathbb{CH})$. Hence, there exists a function $h \in \ker D_{\overline{\alpha}} \cap W_2^1(G,\mathbb{CH})$ so that $tr_\Gamma h = tr_\Gamma v$. So far, let $w := v-h \in \overset{\circ}{W}{}_2^1(G,\mathbb{CH})$ then $u = D_{\overline{\alpha}}v = D_{\overline{\alpha}}w \in D_{\overline{\alpha}}\overset{\circ}{W}{}_2^1(G,\mathbb{CH})$. □

Many results that follow, and in particular the following theorem for the existence of two orthoprojections onto the occurring subspaces, are related in one way or another to the previous orthogonal decomposition.

Theorem 3.2. *There exist the orthoprojections*

$$\mathbf{P}_\alpha : L^2(G,\mathbb{CH}) \longmapsto \ker D_\alpha \cap L^2(G,\mathbb{CH}),$$
$$\mathbf{Q}_\alpha : L^2(G,\mathbb{CH}) \longmapsto D_{\overline{\alpha}}\overset{\circ}{W}{}_2^1(G,\mathbb{CH}) \cap L^2(G,\mathbb{CH})$$

with $\mathbf{Q}_\alpha = I - \mathbf{P}_\alpha$. *Furthermore we have*

$$\mathbf{P}_\alpha = F_\alpha(tr_\Gamma T_{\overline{\alpha}}F_\alpha)^{-1}tr_\Gamma T_{\overline{\alpha}}$$

with $\ker\mathbf{P}_\alpha = D_{\overline{\alpha}}\overset{\circ}{W}{}_2^1(G,\mathbb{CH})$, *and* $im\mathbf{P}_\alpha = \ker D_\alpha \cap L^2(G,\mathbb{CH})$.

Proof. We refer to [4, 6, 15] for the proof. □

We are now able to consider the related boundary value problem expressed as follows:

$$\begin{aligned} D_\alpha D_{\overline{\alpha}}u &= f \quad \text{in } G,\\ u &= g \quad \text{on } \Gamma \end{aligned}$$

where $D_\alpha D_{\overline{\alpha}} = -\Delta + 2Re(\alpha)D + |\alpha|^2$ does not contain any complex term. The following theorem can be proved.

Theorem 3.3. *Let* $f \in W_2^k(G,\mathbb{CH})$ *and* $g \in W_2^{k+\frac{3}{2}}(\Gamma,\mathbb{CH})$, *then the Dirichlet problem*

$$\begin{aligned} (-\Delta + 2Re(\alpha)D + |\alpha|^2)u &= f \quad \text{in } G,\\ u &= g \quad \text{on } \Gamma \end{aligned}$$

has the unique solution

$$u = F_{\overline{\alpha}} g + T_{\overline{\alpha}} \mathbf{P}_{\alpha} D_{\overline{\alpha}} h + T_{\overline{\alpha}} \mathbf{Q}_{\alpha} T_{\alpha} f,$$

where h denotes a $W_2^{k+2}(G, \mathbb{CH})$-extension of g.

Proof. We seek to show that this function satisfies the Dirichlet problem. The first equation can be rewritten as $D_{\alpha} D_{\overline{\alpha}} u = f$. In addition, notice that

$$D_{\alpha} T_{\alpha} = I, \quad D_{\alpha} \mathbf{Q}_{\alpha} = D_{\alpha}, \quad D_{\alpha} \mathbf{P}_{\alpha} = 0, \quad D_{\alpha} F_{\alpha} = 0.$$

A direct computation shows that

$$\begin{aligned} D_{\alpha} D_{\overline{\alpha}} u &= D_{\alpha} D_{\overline{\alpha}} F_{\overline{\alpha}} g + D_{\alpha} D_{\overline{\alpha}} T_{\overline{\alpha}} \mathbf{P}_{\alpha} D_{\overline{\alpha}} h + D_{\alpha} D_{\overline{\alpha}} T_{\overline{\alpha}} \mathbf{Q}_{\alpha} T_{\alpha} f \\ &= D_{\alpha} T_{\alpha} f = f. \end{aligned}$$

This function satisfies also the boundary condition. The proof of the uniqueness may be found in [6].

□

In particular, if α is a pure complex number that means $\alpha := i\lambda$ $(\lambda \in \mathbb{R})$ a similar result as in [15] can be obtained:

$$L^2(G, \mathbb{CH}) = \ker D_{i\lambda} \cap L^2(G, \mathbb{CH}) \oplus_{\mathbb{CH}} D_{-i\lambda} \overset{\circ}{W}{}_2^1 (G, \mathbb{CH}).$$

Following the ideas given in [6, 26] we can now consider more general boundary value problems (of order $2n$) in $\mathbb{R}^3$ involving complex potentials. For the convenience, we denote the *left*- and *right-products of the sequences* respectively by

$$\prod_{i=k}^{m}{}^{(l)} A_i := A_m A_{m-1} \dots A_k \,, \qquad \prod_{i=k}^{m}{}^{(r)} A_i := A_k A_{k+1} \dots A_m \,.$$

In the sequel, let α_i $(i = 1, \dots, n)$ be arbitrary complex numbers, then we have:

Theorem 3.4. *Let $f \in L^2(G, \mathbb{CH})$ and $g_i \in W_2^{2n - \frac{4i+1}{2}}(\Gamma, \mathbb{CH})$ $(i = 0, \dots, n-1)$ then the Dirichlet problem*

$$\begin{aligned} &\prod_{i=1}^{n}{}^{(l)} D_{\alpha_i} D_{\overline{\alpha}_i} u = f \quad \text{in } G; \\ &u = g_0, \; D_{\alpha_1} D_{\overline{\alpha}_1} u = g_1, \; \dots, D_{\alpha_{n-1}} D_{\overline{\alpha}_{n-1}} \dots D_{\alpha_1} D_{\overline{\alpha}_1} u = g_{n-1} \quad \text{on } \Gamma \end{aligned}$$

has the unique solution $u \in W_2^{2n}(G, \mathbb{CH})$ given explicitly by the formula

$$u = r_1(g_0) + T_{\overline{\alpha}_1} \mathbf{Q}_{\alpha_1} T_{\alpha_1} r_2(g_1) + \dots + \prod_{i=1}^{n-1}{}^{(r)} T_{\overline{\alpha}_i} \mathbf{Q}_{\alpha_i} T_{\alpha_i} r_n(g_{n-1}) + \prod_{i=1}^{n}{}^{(r)} T_{\overline{\alpha}_i} \mathbf{Q}_{\alpha_i} T_{\alpha_i} f,$$

where with $k = 1, \dots, n$

$$r_k(g_{k-1}) := F_{\overline{\alpha}_k} g_{k-1} + T_{\overline{\alpha}_k} F_{\alpha_k} \left(tr_{\Gamma} T_{\overline{\alpha}_k} F_{\alpha_k} \right)^{-1} Q_{\overline{\alpha}_k} g_{k-1}.$$

In general, if we assume that $(\alpha_k,\ \beta_k)$ $(k = 1, \ldots, n)$ are pairs of complex numbers, which are chosen such that the boundary value problems

$$\begin{aligned} D_{\alpha_k} D_{\beta_k} u &= f_k \quad \text{in}\, G, \\ u &= g_{k-1} \quad \text{on}\, \Gamma \quad (k = 1, \ldots, n), \end{aligned}$$

are uniquely solvable, and $\mathbf{P}_{\alpha_k\beta_k}, \mathbf{Q}_{\alpha_k\beta_k}$ are projections defined by

$$\begin{aligned} \mathbf{P}_{\alpha_k\beta_k} &= F_{\alpha_k} \left(tr_\Gamma T_{\beta_k} F_{\alpha_k}\right)^{-1} tr_\Gamma T_{\beta_k}, \\ \mathbf{Q}_{\alpha_k\beta_k} &= I - \mathbf{P}_{\alpha_k\beta_k}. \end{aligned}$$

In this sense the previous theorem can be generalized and stated as follows:

Theorem 3.5. *Let* $f \in L^2(G, \mathbb{CH})$ *and* $g_i \in W_2^{2n-\frac{4i+1}{2}}(\Gamma, \mathbb{CH})$ $(i = 0, \ldots, n-1)$*, then the unique solution of the Dirichlet problem*

$$\begin{aligned} &\prod_{i=1}^{n}{}^{(l)} D_{\alpha_i} D_{\beta_i} u = f \quad \text{in } G; \\ &u = g_0,\ D_{\alpha_1} D_{\beta_1} u = g_1,\ \ldots, D_{\alpha_{n-1}} D_{\beta_{n-1}} \ldots D_{\alpha_1} D_{\beta_1} u = g_{n-1} \quad \text{on } \Gamma \end{aligned}$$

has the explicit representation

$$u = r_1(g_0) + T_{\beta_1} \mathbf{Q}_{\alpha_1\beta_1} T_{\alpha_1} r_2(g_1) + \ldots + \prod_{i=1}^{n-1}{}^{(r)} T_{\beta_i} \mathbf{Q}_{\alpha_i\beta_i} T_{\alpha_i} r_n(g_{n-1}) + \prod_{i=1}^{n}{}^{(r)} T_{\beta_i} \mathbf{Q}_{\alpha_i\beta_i} T_{\alpha_i} f,$$

where

$$r_k(g_{k-1}) := F_{\beta_k} g_{k-1} + T_{\beta_k} F_{\alpha_k} \left(tr_\Gamma T_{\beta_k} F_{\alpha_k}\right)^{-1} Q_{\beta_k} g_{k-1}.$$

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