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THE PROBLEM OF COUPLING BETWEEN ANALYTICAL SOLUTION AND FINITE ELEMENT METHOD

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Abstract. *This paper is focused on the first numerical tests for coupling between analytical solution and finite element method on the example of one problem of fracture mechanics. The calculations were done according to ideas proposed in [1]. The analytical solutions are constructed by using an orthogonal basis of holomorphic and anti-holomorphic functions. For coupling with finite element method the special elements are constructed by using the trigonometric interpolation theorem.*

1 INTRODUCTION

The finite element methods is the most popular numerical method for solving partial differential equations in computational mechanics. For many problems it shows high accuracy of results, but for problems which contain different types of singularities (like for instance cracks, gaps, corners) one should make some adaptation and improvement to get an acceptable result. Another approach is to use analytical method for the near-field of the singularity (crack-tip region) and classical finite element method in the far-field.

One of possible modifications of finite element method for singular problems is the eXtended Finite Element Method (XFEM) [2]. In the XFEM the classical finite element approximation is enriched in certain region of interest (the crack-tip region) by using special set of the enrichment functions. But for many problems we have to refine a lot to get desired accuracy, this increased time costs and reduced the rate of convergence.

In the linear elastic fracture mechanics the analytical solution of the crack-tip has been developed by using methods of complex function theory. Based on the *Formulas of Kolosov* the near-field solution of the crack can be represented by only two holomorphic functions $\Phi(z)$ and $\Psi(z)$, $z \in \mathbb{C}$ [5]. Analytical solution based on the complex function theory gives us a high accuracy of solution in the neighborhood of singularity.

A combination of finite element method with analytical solution was proposed by Piltner in [6] and [7]. He has constructed a special elements containing the crack or the hole by using Formulas of Kolosov and coupled it with finite element mesh by nodes on the boundary of this element. The boundary displacements between two adjacent nodes are chosen as piecewise linear or quadratic functions. This approach gives us advantages of analytical solution near singularity, but coupling with finite elements is realized in such a way, that we have a break on the boundary, because, in general, the analytical solution is not necessary the piecewise linear or quadratic functions on the boundary between two methods.

Our idea is to continue work in direction proposed in [1] for method of coupling between analytical solution and finite element method. The main goal of this approach is to get the continuous coupling between analytical solution and finite element method through the whole interaction interface. For that reason we construct a special element which contain analytical solution and transmission or coupling elements. Advantages of such combination can be high accuracy of solution without special refinement, following investigation of model, like for instance error estimations, rate of convergence evaluations. The purpose of this article is to present first numerical results.

In the following section we introduce the global geometrical settings and explain the structure of the special element. The numerical examples are shown and discussed in the Section 3.

2 GEOMETRICAL SETTINGS AND SPECIAL ELEMENT

2.1 Global settings and notations

We work in the field $\mathbb{C}$ of one complex variable, where we identify each point of the complex plane $\mathbb{C}$ with the ordered pair $z = (x, y) \in \mathbb{R}^2$, $x, y \in \mathbb{R}$ or equaivalently with the complex number $z = x + iy \in \mathbb{C}$, where i denotes the imaginary unit.

For constructing the exact solution to the differential equation we will work with the complex linear Hilbert space of square-integrable $\mathbb{C}$-valued functions defined in Ω, that is denoted by

$L_2(\Omega, \mathbb{C})$, with the corresponding inner product [3]

$$\langle f, g \rangle_{L_2(\Omega,\mathbb{C})} = \int_\Omega f\, \bar{g}\, d\sigma, \quad f, g \in L_2(\Omega, \mathbb{C}), \tag{1}$$

where $d\sigma$ denotes the Lebesgue measure in $\mathbb{R}^2$, the functions f and g are $\mathbb{C}$-valued function such that

$$f(z) = f_0(x, y) + i f_1(x, y), \quad z \in \Omega$$

and the coordinates $f_j : \Omega \to \mathbb{R} \quad (j = 0, 1)$ are real-valued functions defined in Ω.

For continuously real-differentiable functions $f_j : \Omega \to \mathbb{C}$, the operator

$$D = \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \tag{2}$$

is called *Cauchy-Riemann* operator. The *conjugate Cauchy-Riemann* operator we denote by

$$\overline{D} = \frac{\partial}{\partial x} - i \frac{\partial}{\partial y}. \tag{3}$$

Also we introduce the polar coordinates by $x = r \cos\varphi$, $y = r \sin\varphi$ and arrive to the representation of the complex number

$$z = re^{i\varphi} = r(\cos\varphi + i \sin\varphi), \qquad 0 \le r < \infty, 0 \le \varphi < 2\pi.$$

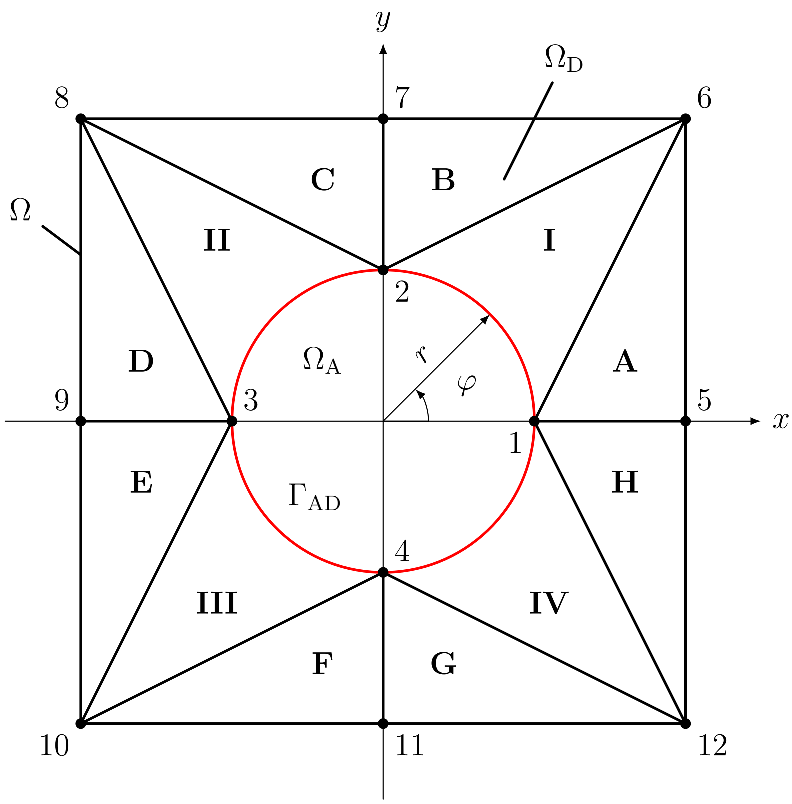

Figure 1: Geometrical setting of special element

Let now $\Omega \subset \mathbb{C}$ be a bounded simply connected domain containing the singularity at the origin (see Figure 1). The domain Ω is decomposed in the two sub-domains $\Omega = \Omega_{\mathrm{A}} \cup \Omega_{\mathrm{D}}$

separated by the fictitious joint interface $\Gamma_{\mathrm{AD}} = \overline{\Omega}_{\mathrm{A}} \cap \overline{\Omega}_{\mathrm{D}}$. The discrete numerical domain, denoted by Ω_{D}, is modeled by two different kinds of elements: the CST-element of class C^0 (in example elements $A-H$) and the Coupling-element of class C^0-C^∞ (in example elements $I-IV$), that couple the discrete domain Ω_{D} with the analytical domain Ω_{A}. We call the sub-domain Ω_{A} analytical in that sense, that the constructed solutions are exact solutions to the differential equation in Ω_{A}. The idea behind this special element is to get the continuous connection through the interface Γ_{AD} by modifying of one side osculating triangles.

2.2 Exact solution to the homogeneous Lamé equation

Inside of Ω_{A} we are going to use the exact polynomial solutions to the homogeneous Lamé by using a basis of holomorphic and anti-holomorphic polynomials. The idea behind is the factorization of the Laplace operator by the Cauchy-Riemann (2) operator and its conjugate (3). First we recall the classical matrix representation of the Lamé equation.

In linear elasticity theory the pysical state of each continuum model is described by three fundamental equations: the equilibrium equations, the constutive equations and the strain-displacement relations. Solving these three equations with respect to unknown displacement vector $\underline{\mathbf{u}} = [u_1(x,y), u_2(x,y)]^T$ we get the Lamé (or Navier) equation in vector form [4]:

$$(\lambda+\mu)\nabla(\nabla\cdot\underline{\mathbf{u}}) + \mu\nabla^2\underline{\mathbf{u}} = -\underline{p}, \tag{4}$$

where λ and μ are material constants (Lamé constants). Also, we can rewrite equation (4) in the classical matrix formulation

$$-\underline{p} = \underline{D}_e\,\underline{\mathbf{E}}\,\underline{D}_k\underline{\mathbf{u}} \tag{5}$$

where $\underline{p} = [p_x, p_y]^T$ denotes the vector of the outer forces and

$$\underline{D}_e = (\underline{D}_k)^T = \begin{bmatrix} \partial_{,x} & 0 & \partial_{,y} \\ 0 & \partial_{,y} & \partial_{,x} \end{bmatrix}$$

are the adjoint differential operators of equilibrium and kinematics respectively. The matrix

$$\underline{\mathbf{E}} = G\begin{bmatrix} \frac{\kappa+1}{\kappa-1} & -\frac{\kappa-3}{\kappa-1} & 0 \\ -\frac{\kappa-3}{\kappa-1} & \frac{\kappa+1}{\kappa-1} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{where} \quad \kappa = \begin{cases} 3-4\nu & \text{plane strain state} \\ \dfrac{3-\nu}{1+\nu} & \text{plane stress state} \end{cases}$$

includes material parameters for a linear elastic, homogeneous and isotropic material in the usual notations.

For complex representation of the Lamé equation we identify the displacement vector $\underline{u}$ with the complex valued function $\mathbf{u} = u_1(x,y) + iu_2(x,y) \in \Omega$ in the bounded simply connected domain $\Omega \subset \mathbb{C}$. A purely complex representation of the homogeneous form of (5) is given by

$$0 = D\tilde{M}^{-1}\overline{D}\mathbf{u}, \tag{6}$$

where D denotes the Cauchy-Riemann operator and $\tilde{M}$ a multiplication operator which is, acting on a function $u = u_0 + iu_1$, defined by

$$\tilde{M}u = \frac{\kappa+1}{\kappa-1}u_0 + iu_1.$$

Proposition 1 *For a fixed $n \in \mathbb{N}_0$ the $4n+2$ polynomials of the system*

$$\begin{array}{rcl} \{f_k(z)\}_{k=0,\dots,n,} & = & \left\{\varphi_k(z) + \frac{1}{2}(M_0-1)D(\mathbf{M}_p\varphi_k(z))\right\}_{k=0,\dots,n,} \\ \left\{\hat{f}_l(z)\right\}_{l=0,\dots,n,} & = & \left\{\psi_l(z) + \frac{1}{2}(M_0-1)D(\mathbf{M}_p\psi_l(z))\right\}_{l=0,\dots,n,} \end{array} \tag{7}$$

are exact solutions to the homogeneous Lamé equation (6). The operator $\mathbf{M}_p$ is defined by

$$\mathbf{M}_p h(z) = x\mathbf{Sc}\, h(z) + y\mathbf{Vec}\, h(z) = xh_0(x,y) + yh_1(x,y).$$

In the formulas (7) systems $\varphi_k(z)$ and $\psi_k(z)$ are the holomorphic and anti-holomorphic polynomials

$$\begin{array}{rcl} \{\varphi_k(z)\}_{k=0,\dots,n,} & = & \left\{\sqrt{\frac{k+1}{r_a^{k+1}\pi}}z^k\right\}_{k=0,\dots,n,} \\ \{\psi_l(z)\}_{l=0,\dots,n,} & = & \left\{\sqrt{\frac{l+1}{r_a^{l+1}\pi}}\overline{z}^l\right\}_{l=0,\dots,n,} \end{array} \tag{8}$$

2.3 Construction of the coupling element

The numerical domain Ω_{D} consists of two different kinds of elements. In the far-field of the analytical inclusion we use CST-elements of class C^0, where the primary variables are linearly interpolated and the secondary variables are constantly represented. The second kind of the used elements are the so called *Coupling* elements, that connect the discrete domain Ω_{D} modeled by CST-elements with the analytical domain Ω_{A}. For these special curved triangles we have following restrictions:

- C^0 continuity on two boundaries with CST-elements;
- at least C^0 through the joint interface Γ_{AD} with the analytical domain.

To satisfy these condtions we introduce the special coupling element of polynomial degree $n = c(m+1)-1$ on Γ_{AD}. The parameters c and m define a discrete point grid for the identity coupling element $\mathbb{T}_n^{c,m}$, where $c \in \mathbb{N}, c \geq 2$ denotes the number of coupling elements used to discretize the joint coupling interface Γ_{AD}. The second parameter $m \in \mathbb{N}_0$ is concerned to the number of nodes used additionally on the boundary Γ_{AD}.

To construct the coupling element we are going to use the *Discrete Fourier Analysis* in $\mathbb{C}$, which based on the following trigonometric interpolation theorem.

Theorem 1 *For given observations $\mathrm{Y}_0, \mathrm{Y}_1, \dots, \mathrm{Y}_n$ exist a unique function*

$$t_n(\varphi) = \sum_{k=0}^{n} c_k e^{ik\varphi} \tag{9}$$

that satisfies the interpolation conditions $t_n(\varphi_j) = \mathrm{Y}_j$, $j = 0,\dots,n$. The coefficients c_k are given by

$$c_k = \frac{1}{n+1}\sum_{j=0}^{n} \mathrm{Y}_j e^{-ij\varphi_k}, \tag{10}$$

$\varphi_k = k\frac{2\pi}{n+1}$ denote the $n+1$ equidistant interpolation nodes on the interval $\varphi \in [0, 2\pi)$.

Using (9) and (10) we rearrange the trigonometric interpolation formula and obtain

$$t_n(\varphi) = \sum_{k=0}^{n} S_k(\varphi)\mathrm{Y}_k, \qquad S_k(\varphi) = \frac{1}{n+1}\sum_{j=0}^{n} e^{-i(j\varphi - k\varphi_j)}, \qquad S_k(\varphi), \mathrm{Y}_k \in \mathbb{C}. \tag{11}$$

Proposition 2 *For a fixed discretization* (c, m) *and* $n = c(m+1)-1 \in \mathbb{N}$ *the geometry mapping* $\mathbb{X}_n^{c,m}(\xi, \eta) \in \mathbb{C}$ *for the* $(n+2)$*-node identity coupling element* $\mathbb{T}_n^{c,m}$ *is given by*

$$\mathbb{X}_n^{c,m}(\xi, \eta) = \sum_{q=0}^{n+1} N_q(\xi, \eta) Y_q, \tag{12}$$

where

$$N_q(\xi, \eta) = \begin{cases} \frac{\eta}{n+1} \sum_{j=0}^{n} e^{2\pi i j \frac{(m+1)\xi - q}{c(m+1)}} & : \ q < n+1 \\ 1 - \eta & : \ q = n+1 \end{cases}$$

The orientation of the axes ξ and η of local coordinate system of the coupling element is shown in figure 2.

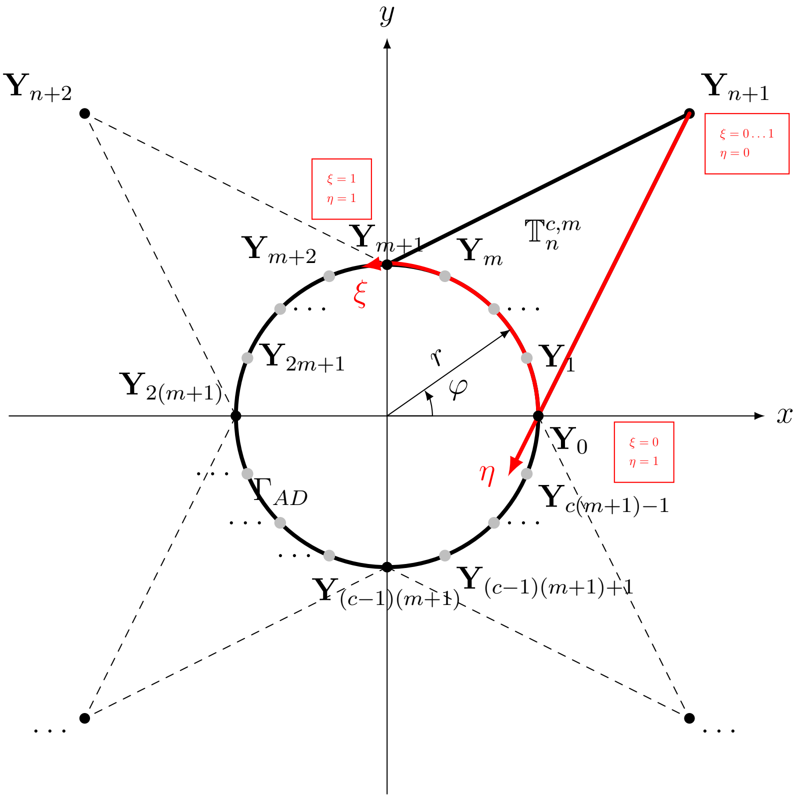

Figure 2: $(n + 2)$-node identity coupling element $\mathbb{T}_n^{c,m}$

2.4 Coupling of Ω_{A} and Ω_{D}

The solution of the original boundary value problem in Ω determines the values of unknown displacements $\mathrm{U}_0, \ldots, \mathrm{U}_n$ at the nodes on the fictitious joint interface Γ_{AD}. To interpolate these unknown values at the nodes we use the interpolation formula (9). The point is that then the displacement are continuous also along the arcs between the nodes.

3 FIRST NUMERICAL EXAMPLES

Based on the theory which was presented in the previous section, now we would like to show first numerical tests. A general design model is shown in figure 3.

We have an arbitrary square domain Ω and we discretize it accordantly with scheme proposed in the Section 2. We start our tests with 37 Finite Elements (32 CST-elements, 4 Coupling elements and 1 analytical domain), and after we make gradually refinment of our finite element mesh up to 973 Elements. The special elements are always located at the origin, but we would like to underline, that is not a restriction. We can use several special elements in one finite element mesh. Also is not restricted to use the special element only of square shape.

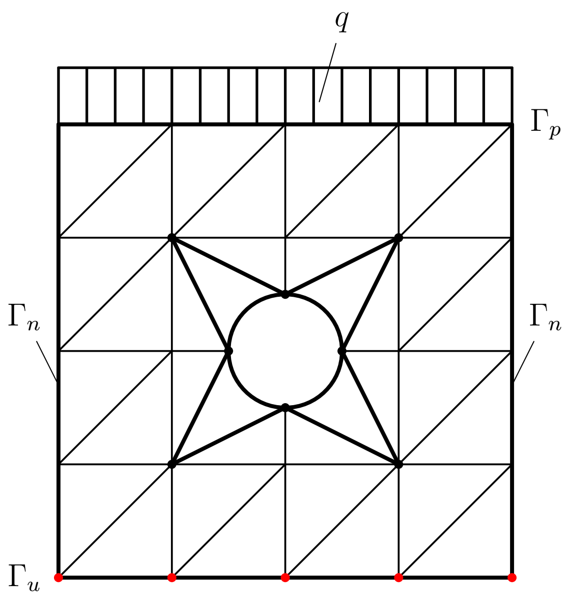

Figure 3: a general design model

All calculation are performed with the boundary condtions:

$$\begin{array}{rcll} u & = & 0 & \text{on } \Gamma_u, \\ p_y & = & -q & \text{on } \Gamma_p, \\ \sigma_{xx} & = & 0 & \text{on } \Gamma_n. \end{array} \tag{13}$$

The material is supposed to be linear isotropic and the problem is considered under plain strain state. For values of material constant and applying loads we used synthetic data, because at the present moment the goal is to investigate general behaviour of the method and to understand possible ways for improvement.

The displacement field for 973 Elements is presented in figure 4

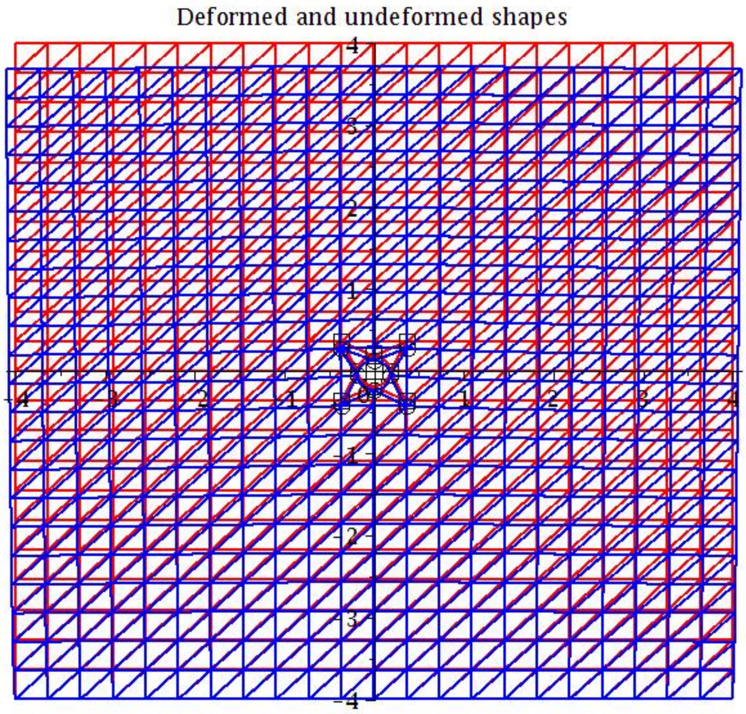

Figure 4: Displacement field for 973 elements

The figure 4 shows the approximation of the displacement field for given boundary conditions. Another important result is that we can easily see how the special element can be integrated to the global finite element mesh. Also, we would like to underline that location and number of special elements are not restricted: we can construct a mesh with several special elements, for instance in case of multiple cracks. The changing of the condition number of the global stiffness matrix with refinement is shown in figure 5

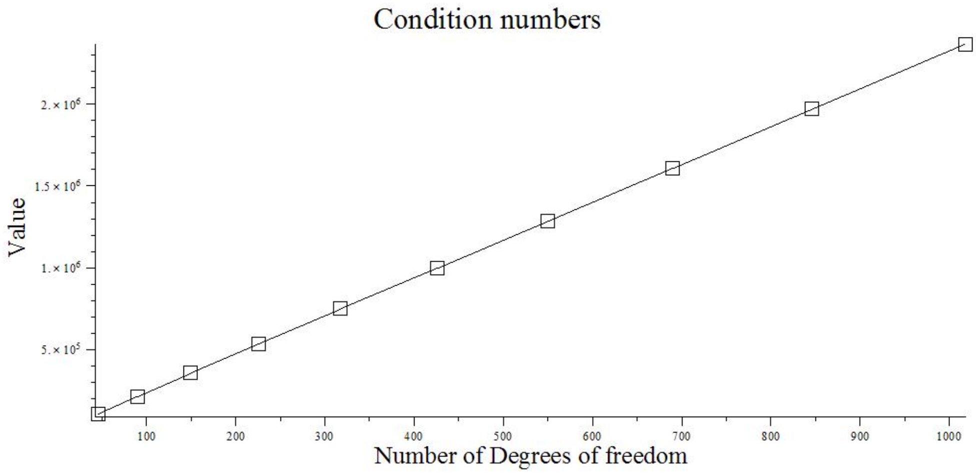

Figure 5: Condition number

We see that the condition number is growing as linear function

To study the flexibility of the method we would like to change a shape of the special element from square to rectangle. The displacement field for 973 Elements is presented in figure 6

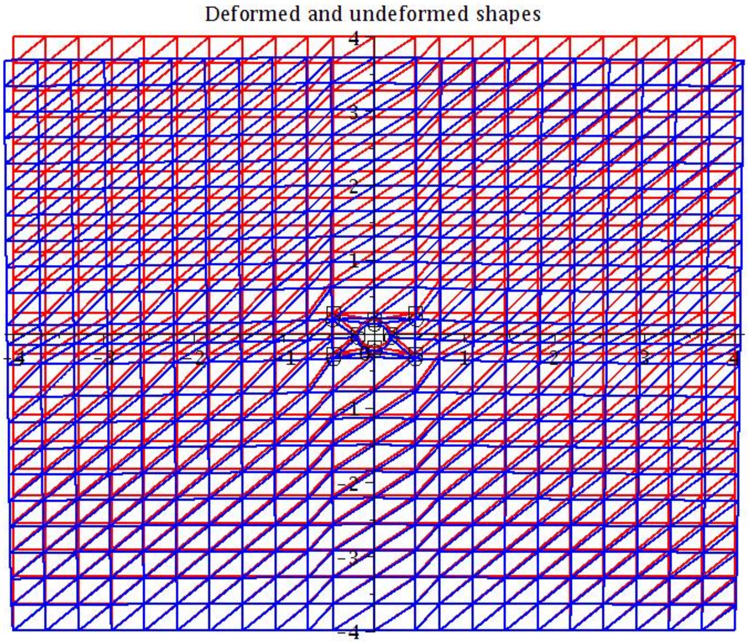

Figure 6: Displacement field for 973 elements for rectangle

The figure 6 shows the approximation of the displacement field for given boundary conditions. The changing of the condition number of the global stiffness matrix with refinement is shown in figure 7

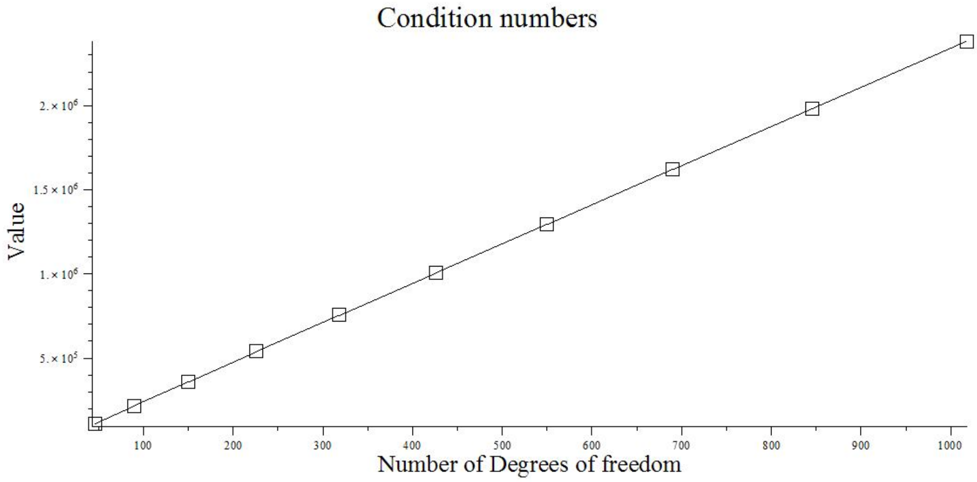

Figure 7: Condition number

Again we see, that the condition number is growing as linear function. Finally, figure 8 shows the comparison of condition numbers between square and rectangle shapes.

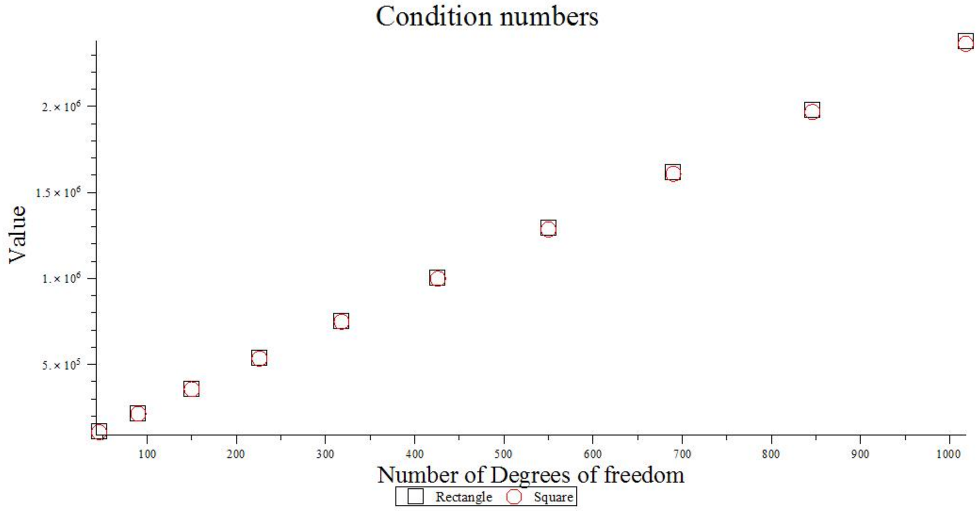

Figure 8: Comparison of condition numbers

As we can see, the condition numbers are growing always as a linear function of total amout degrees of freedom, and, in fact, doesn't depend on the shape of the super element.

4 CONCLUSIONS

The first numerical results for coupling between analytical solution and finite element method were presented. For the shape preserving geometry mapping and continuous coupling of the displacement field the special element was constructed. The behaviour of the condition number during refinement procedure was shown. Flexibility and possible points for improvement were discussed.

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