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# ON BOUNDARY VALUE PROBLEMS FOR $P$-LAPLACE AND $P$-DIRAC EQUATIONS

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**Abstract.** *The $p$-Laplace equation is a nonlinear generalization of the Laplace equation. This generalization is often used as a model problem for special types of nonlinearities. The $p$-Laplace equation can be seen as a bridge between very general nonlinear equations and the linear Laplace equation. The aim of this paper is to solve the $p$-Laplace equation for $2 < p < 3$ and to find strong solutions. The idea is to apply a hypercomplex integral operator and spatial function theoretic methods to transform the $p$-Laplace equation into the $p$-Dirac equation. This equation will be solved iteratively by using a fixed point theorem.*

## 1 INTRODUCTION

Monogenic functions are defined as null-solutions of a generalized Cauchy-Riemann system or of a Dirac equation. In this sense the theory of monogenic functions has to be seen as a generalization of the complex function theory (of holomorphic functions). Indeed, monogenic functions share a lot of properties with holomorphic functions [9]. First applications to boundary value problems were considered in [10] in 1990. Based on an operator theoretical approach elliptic boundary value problems could be studied (existence, uniqueness, regularity and integral representations). Recently this approach has been extended to parabolic and (partially) to the treatment of hyperbolic equations. In [11] one can find a lot of integral representation formulas for the solutions to elliptic boundary value problems. Some nonlinear cases like the Navier-Stokes equations and similar equations, containing nonlinearities of the type $u \cdot \text{grad}\, u$ or $u^2$ could be also treated by this concept. When looking at more general nonlinear problems one of most important problems is the $p$-Laplace equation. There has been a surge of interest in the $p$-Laplacian in many different contexts, from game theory to mechanics, image processing and non-newtonian fluids. There are many unsettled existence, uniqueness and regularity issues. Currently there are many works proposing weak solutions of the $p$-Laplace equation [1], [2], [3], [5], [6], [7], [8]. This is strongly related to the fact that the $p$-Laplace equation can be introduced by minimizing the $p$-Dirichlet integral $\int_\Omega |\text{grad}\, u|^p \, dx$. This explains on the one hand a lot of similar properties of the solutions, compared with harmonic functions and on the other hand it seems to be natural to apply variational methods for the solution. Due to the fact that the $p$-Laplace equation degenerates at the zeros of $u$ or will have singularities, depending on $p$, there are not so many works on strong solutions of the $p$-Laplace equation. One idea for the treatment of the $p$-Laplace equation is its transformation into a $p$-Dirac equation. This leads to the question to generalize the problem to the $p$-Laplace or the $p$-Dirac equation for vector-valued functions. The idea behind is that for equations for vector-valued functions often function theoretic methods can be applied. The study of such equations in the Clifford analysis framework began with C. A. Nolder and J. Ryan. They introduced non-linear Dirac operators in $\mathbb{R}^N$, associated to the $p$-Laplace equation [14]. The regularity of the $p$-Laplace equation in the plane has been studied for $p \geq 2$, by E. Lindgren and P. Lindqvist [12]. The infinite Dirac operator has been defined and some of its key properties have been explored by T. Bieske and J. Ryan [4]. P. Lindqvist published several surveys on the $p$-Laplace equation (see e.g. [13]). We will solve the $p$-Laplace equation for $2 < p < 3$ and search for strong solutions. The new strategy is to apply a generalized Teodorescu transform to the $p$-Laplace equation and to transform it into a $p$-Dirac equation by applying ideas from Clifford analysis. The obtained $p$-Dirac equation will be solved iteratively by using Banach's fixed point theorem.

## 2 PRELIMINARIES

Mostly, the $p$-Poisson equation is studied in the form

$$\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u) = f,$$

where $u$ and $f$ are scalar-valued functions, defined in a domain $G \subset \mathbb{R}^N$, $N \geq 1$ with sufficiently smooth boundary $\partial G$ and $1 \leq p \leq \infty$. If $f = 0$ then the equation is called $p$-Laplace equation. Substituting $w := \nabla u$, the $p$-Dirac equation will be obtained

$$\text{div}\, (|w|^{p-2} w) = f. \tag{1}$$

Here $w$ is a vector-valued function. This leads to the idea to work from the very beginning with vector-valued functions $u$. Replacing the application of the divergence operator and the gradient by the application of a generalized Dirac operator one obtains a $p$-Laplace (Poisson) equation for vector-valued functions and a corresponding $p$-Dirac equation. The $p$-Poisson and the $p$-Dirac equation will be studied in the scale of Sobolev spaces. The structure of the differential equations shows that some knowledge about the regularity of powers and products of functions belonging to certain Sobolev spaces is necessary. For the values $2 < p < 3$ the Sobolev spaces do not form an algebra and these questions must be studied in detail. Consider $m \geq 0$ and $p \geq 1$. The norm $\|.\|_{m,p}$ is defined by $\|u\|_{m,p} = \left( \sum\limits_{|\alpha| \leq m} \|\partial^\alpha u\|_{0,p}^p \right)^{1,p}$, where $\|.\|_{0,p}$ is the usual norm of $L_p(G)$.
Let $N \geq 1$ and an orthonormal basis $e_0, e_1, \ldots, e_N$ of $\mathbb{R}^{N+1}$ generates the Clifford algebra $C\ell_{0,N}$. For more information we refer to [11]. We recall some basic facts and formulae from spatial function theory:

- A Dirac-type operator is defined by

$$Du = \sum_{i=1}^{N} e_i \frac{\partial}{\partial x_i} u.$$

The solutions of $Du = 0$ will be called $C\ell$-holomorphic functions.

- The Teodorescu transform over $G$ is defined by

$$(T_G u)(x) = -\int_G \mathbf{e}(x-y) u(y) dy,$$

with the Cauchy kernel $\mathbf{e}(x) = \frac{1}{\sigma_N} \frac{\bar{\omega}(x)}{|x|^{N-1}}$, $\omega(x) = \frac{x}{|x|}$, $x = \sum\limits_{i=1}^{N} x_i e_i$. Here $\sigma_N$ denotes the surface area of the unit ball in $\mathbb{R}^N$. The Teodorescu transform is a weakly singular operator and the most important algebraic property is that it is a right inverse of the Dirac operator $D$.

- The Cauchy-Bitsadze operator or Cauchy-Fueter operator is given by

$$(F_\Gamma u)(x) = \int_\Gamma \mathbf{e}(x-y) \alpha(y) u(y) d\Gamma_y,$$

where $\alpha$ stands for the outward pointing unit normal vector at $y \in \Gamma$. A sufficient smoothness of the boundary $\Gamma = \partial G$ has to be assumed.

- The identity $I$, the Cauchy operator $F$, the Dirac operator $D$ and its right inverse $T$ are connected through the Borel-Pompeiu formula

$$F_\Gamma u + T_G D u = u,$$

where $u$ is an arbitrary sufficiently smooth function. If $u$ is a $C\ell$-holomorphic function it means that $F_\Gamma u = u$ and the Cauchy integral reconstructs the $C\ell$-holomorphic function from its boundary values.

## 3 THE MAIN RESULT

We will use Banach's fixed-point theorem for proving existence and uniqueness of solutions to the $p$-Dirac equation. Consider a (Neumann-type) boundary value problem for the $p$-Poisson equation and rewrite it in terms of the Dirac operator $D$

$$\Delta_p u = D\left(|Du|^{p-2} Du\right) = f \quad \text{in } G, \tag{2}$$
$$Du = 0 \quad \text{on } \Gamma. \tag{3}$$

Let $u \in W^{1,p}(G)$, $1 < p < \infty$. Note that $D^2 = -\Delta$ is the Laplacian in $\mathbb{R}^N$. Substituting $Du$ by $w$ we obtain the $p$-Dirac equation

$$D\left(|w|^{p-2} w\right) = f. \tag{4}$$

Let $G$ be a bounded domain with smooth boundary $\partial G = \Gamma$ in $\mathbb{R}^N,\ N \geq 3$. We search for $w \in L_p(G), 2 < p < 3$, satisfying

$$D\left(|w|^{p-2} w\right) = f \qquad \text{in } G \tag{5}$$
$$w = 0 \qquad \text{on } \partial G. \tag{6}$$

We assume additionally

$$f \in L_{\tilde{p}}(G) \cap L_q^{loc}(G), \text{ with } \tilde{p} = \frac{p}{p-1}, q > N \text{ and } \mathrm{tr}_\Gamma T_G f = 0.$$

The condition $q > N$ is needed for technical reasons because it guarantees that $T_G f \in C(G)$. We can now apply the operator $T_G$ to equation (5), resulting in

$$T_G(D(|w|^{p-2} w)) = T_G f.$$

By applying the Borel-Pompeiu formula we have

$$T_G(D(|w|^{p-2} w)) = |w|^{p-2} w - F_\Gamma(|w|^{p-2} w) = T_G f.$$

From the boundary condition $w|_\Gamma = 0$ one obtains $F_\Gamma(|w|^{p-2} w) = 0$. Then we have

$$|w|^{p-2} w = T_G f.$$

From now on we assume that $|w|^{p-2} \neq 0$ in $G$. Dividing the previous equation by $|w|^{p-2}$ we obtain the equation

$$w = \frac{T_G f}{|w|^{p-2}}. \tag{7}$$

This equation will be iterated. Starting from an initial guess $w_0$ we define for $n = 1, 2, ....$

$$w_n = \frac{T_G f}{|w_{n-1}|^{p-2}}. \tag{8}$$

We consider the subsequence with even indices and introduce $w_0 = g_0$, $w_2 = g_1$, ..., and $w_{2n} = g_n$. In terms of $g_n$ we get finally the equations

$$g_n = \frac{T_G f}{|T_G f|^{p-2}} \left|g_{n-1}\right|^{(p-2)^2}, n = 1, 2, \ldots. \tag{9}$$

Taking the modulus on both sides of equation (9), one obtains

$$|g_n| = |T_G f|^{3-p} \, |g_{n-1}|^{(p-2)^2} . \tag{10}$$

We can express all $g_n$ by using only $g_0$ as follows:

$$g_n = T_G f \, |T_G f|^{\sum_{k=0}^{n-2}(1-(p-2))(p-2)^{2(k+1)}-(p-2)} \, |g_0|^{(p-2)^{2n}} .$$

It can be seen from the properties of $T_G f$ that $g_n = w_{2n}$ satisfies the boundary condition if $g_0$ satisfies it. We study at first the mapping properties of the mapping defined by equation (10). Because of our basic assumption for the right hand side $f \in L_{\tilde{p}}(G)$, where $\tilde{p} = \frac{p}{p-1}$, we can conclude that $T_G f \in W^{1,\tilde{p}}(G)$. For a detailed description of the mapping properties of $T$ we refer to [11]. For the purpose of the paper we need the property $T : W^{p,m}(G) \mapsto W^{p,m+1}$(G). With the results from [16] we obtain that $|T_G f|^{3-p} \in W^{1,\frac{\tilde{p}}{3-p}}(G)$.
We know that $|g_{n-1}|^{(p-2)^2} \in L_{\frac{p}{(p-2)^2}}(G)$. Using a theorem on products of functions belonging to certain Sobolev spaces (see [15]), we conclude that $|T_G f|^{3-p} \, |g|^{(p-2)^2} \in L_r(G)$, for $p < r < \frac{Np}{N-p}$. By using embedding theorems we have $L_r(G) \subset L_p(G)$. All together means $g_n \in L_p(G)$, where $2 < p < 3$.
Now, we are ready to study the iteration procedure. To avoid zeros of the iterates in $G$ let us assume that the initial guess satisfies $k_1 |T_G f(x)|^{\frac{3-p}{1-(p-2)^2}} \leq |g_0(x)| \leq |T_G f(x)|^{\frac{3-p}{1-(p-2)^2}}$ , for all $x \in G$ with $0 < k_1 < 1$.
Suppose that $|g_k(x)| \leq |T_G f(x)|^{\frac{3-p}{1-(p-2)^2}}$ for $k = 0, \ldots, n-1$ and we prove that it holds also for $k = n$. Recalling equation (10) for $k = n$ and substituting $|g_{k-1}(x)| \leq |T_G f(x)|^{\frac{3-p}{1-(p-2)^2}}$ we obtain

$$|g_k(x)| \leq |T_G f(x)|^{3-p} \left( |T_G f(x)|^{\frac{3-p}{1-(p-2)^2}} \right)^{(p-2)^2} = |T_G f(x)|^{3-p} \, |T_G f(x)|^{\frac{(3-p)(p-2)^2}{1-(p-2)^2}} .$$

This proves that for all $n$

$$|g_n(x)| \leq |T_G f(x)|^{\frac{3-p}{1-(p-2)^2}} . \tag{11}$$

Taking the $L_p$ norm in (11) leads to

$$\|g_n\|_{0,p} \leq \left\| |T_G f|^{\frac{3-p}{1-(p-2)^2}} \right\|_{0,p} . \tag{12}$$

Then the sequence of iterations $w_{2n}$ or $g_n$ is bounded from above by $|T_G f|^{\frac{3-p}{1-(p-2)^2}}$. This bound depends only on $p$ and $f$.
Additionally, it can be shown that the sequence $|g_n(x)|$ is increasing far all $x \in G$. Recalling equation (11) and taking the $(1-(p-2)^2)$-th power on both sides we get $|g_{n-1}|^{1-(p-2)^2} \leq |T_G f|^{3-p}$ .
Multiplying both sides of our inequality by $|g_{n-1}|^{(p-2)^2}$ results in

$$|g_{n-1}|^{1-(p-2)^2} \, |g_{n-1}|^{(p-2)^2} \leq |T_G f|^{3-p} \, |g_{n-1}|^{(p-2)^2} .$$

This means for all $n$ and $2 < p < 3$, we have proved

$$|g_{n-1}| \leq |g_n| . \tag{13}$$

By taking the $L_p$ norm in inequality (13) we get that also that the sequence of the norms of $w_{2n}$ or $g_n$ is an increasing sequence

$$\|g_{n-1}\|_{0,p} \leq \|g_n\|_{0,p}\,. \tag{14}$$

We study now the contractivity of the mapping we will collect all obtained estimates for $2 < p < 3$. We have

$$g_n = \frac{T_G f}{|T_G f|^{p-2}} |g_{n-1}|^{(p-2)^2}, |g_{n-1}| \leq |g_n| \leq |T_G f|^{\frac{3-p}{1-(p-2)^2}}, \|g_{n-1}\|_{0,p} \leq \|g_n\|_{0,p} \leq \left\| |T_G f|^{\frac{3-p}{1-(p-2)^2}} \right\|_{0,p}.$$

Now we can conclude that

$$|g_n - g_{n-1}| = \left| \frac{T_G f}{|T_G f|^{p-2}} \left( |g_{n-1}|^{(p-2)^2} - |g_{n-2}|^{(p-2)^2} \right) \right| \leq |T_G f|^{3-p} \left| |g_{n-1}|^{(p-2)^2} - |g_{n-2}|^{(p-2)^2} \right|. \tag{15}$$

Consider the function $y(s) = s^{(p-2)^2}$ for $s > 0$. Fix $x \in G$ and apply the mean value theorem to $y(|g_{n-1}(x)|) = |g_{n-1}(x)|^{(p-2)^2}$ and $y(|g_{n-2}(x)|) = |g_{n-2}(x)|^{(p-2)^2}$. With $|\zeta| \in [|g_{n-2}(x)|, |g_{n-1}(x)|]$, and $|y'(|\zeta|)| = \frac{(p-2)^2}{|\zeta|^{1-(p-2)^2}}$ we have

$$\left| |g_{n-1}(x)|^{(p-2)^2} - |g_{n-2}(x)|^{(p-2)^2} \right| = \frac{(p-2)^2}{|\zeta|^{1-(p-2)^2}} \left| |g_{n-1}(x)| - |g_{n-2}(x)| \right|.$$

We use $|\zeta| \in [|g_{n-2}(x)|, |g_{n-1}(x)|]$, $|g_{n-2}(x)| \leq |g_{n-1}(x)|$ and $|g_0| \geq k_1 |T_G f|^{\frac{3-p}{1-(p-2)^2}}$. Then

$$|\zeta|^{1-(p-2)^2} \geq |g_{n-2}|^{1-(p-2)^2} \geq |g_0|^{1-(p-2)^2} \geq {k_1}^{1-(p-2)^2} |T_G f|^{3-p}.$$

Then we have

$$\frac{1}{|\zeta|^{1-(p-2)^2}} \leq \frac{1}{|g_0|^{1-(p-2)^2}} \leq \frac{1}{{k_1}^{1-(p-2)^2} |T_G f|^{3-p}}.$$

By substituting the latter estimates in equation (15) one obtains

$$|g_n - g_{n-1}| \leq \frac{(p-2)^2 |T_G f|^{3-p}}{{k_1}^{1-(p-2)^2} |T_G f|^{3-p}} \left| |g_{n-1}| - |g_{n-2}| \right| \leq \frac{(p-2)^2}{{k_1}^{1-(p-2)^2}} |g_{n-1} - g_{n-2}|. \tag{16}$$

It has been shown that with $c_1 = \frac{(p-2)^2}{{k_1}^{1-(p-2)^2}}$

$$|g_n - g_{n-1}| \leq c_1 |g_{n-1} - g_{n-2}|. \tag{17}$$

The last step is to take the $L_p$ norm in the inequality (17). We obtain

$$\|g_n - g_{n-1}\|_{0,p} \leq c_1 \|g_{n-1} - g_{n-2}\|_{0,p}.$$

That means we have proved for $k_1 > (p-2)^{\frac{2}{1-(p-2)^2}}$, dependent on $p \in (2,3)$ there is contractivity coefficient $c_1 < 1$. All conditions for the application of Banach's fixed-point theorem are now fulfilled.

Before we are going to state the next theorem, remember that the formula for $g_n$ is given by $g_0 = w_0$, $g_1 = w_2$, ..., $g_n = w_{2n}$. That means we state the theorem in terms of $w_{2n}$.

**Theorem 1** *Let $G$ be a bounded domain with smooth boundary $\partial G = \Gamma$ in $\mathbb{R}^N$, $N \geq 3$. Let $f \in L_{\tilde{p}}(G) \cap L_q^{loc}(G)$, with $\tilde{p} = \frac{p}{p-1}$, $q > N$, $tr_\Gamma T_G f = 0$ and $2 < p < 3$. Then the boundary value problem*

$$\begin{aligned} D\left(|w|^{p-2} w\right) &= f \quad \text{in} \quad G & (18)\\ w &= 0 \quad \text{on} \quad \partial G = \Gamma, & (19)\end{aligned}$$

*has a unique solution $w \in L_p(G)$. Under the condition*

$$k_1 \left|T_G f\right|^{\frac{3-p}{1-(p-2)^2}} \leq |w_0(x)| \leq \left|T_G f\right|^{\frac{3-p}{1-(p-2)^2}},$$

*with $(p-2)^{\frac{2}{1-(p-2)^2}} < k_1 < 1$ the sequence defined by $w_{2n} = \frac{T_G f}{|T_G f|^{p-2}} \left|w_{2(n-1)}\right|^{(p-2)^2}$ for $n \in \mathbb{N} \setminus \{0\}$ converges in $L_p(G)$ to the unique solution.*

## 4 CONCLUSIONS

It could be shown in the paper that the $p$-Poisson equation with certain Neumann-type boundary conditions can be transferred to a Dirichlet boundary value problem for the $p$-Dirac equation. Main tool was an operator calculus taken from hypercomplex function theory. The obtained $p$-Dirac equation could be solved iteratively by a fixed-point iteration. This was possible for all $p \in (2, 3)$ and as an estimate for the contractivity constant the value $c_1 = \frac{(p-2)^2}{{k_1}^{1-(p-2)^2}}$, for $(p-2)^{\frac{2}{1-(p-2)^2}} < k_1 < 1$ was obtained. These results support the idea to reduce as in the linear case the study of the second order differential equation to two equations of first order, where we can apply function theoretic methods. For the first step, discussed in this paper this is visible. The second task, to calculate $u$ from $Du = w$ and to improve the result on the regularity of the solution has still to be solved.

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