18th International Conference on the Application of Computer
Science and Mathematics in Architecture and Civil Engineering
K. Gürlebeck and C. Könke (eds.)
Weimar, Germany, 07–09 July 2009

HADAMARD GAPS IN WEIGHTED LOGARITHMIC BLOCH SPACE

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Keywords: Hadamards gaps, weighted Bloch space, weighted Dirichlet space

Abstract. *We give a sufficient and a necessary condition for an analytic function f on the unit disc $\mathbf{D}$ with Hadamard gaps, that is, for $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$ where $\frac{n_{k+1}}{n_k} \geq \lambda > 1$ for all $k \in \mathbb{N}$, to belong to the weighted logarithmic Bloch space $\mathcal{B}^{\alpha}_{\omega,\log}$ as well as to the corresponding little weighted logarithmic Bloch space $\mathcal{B}^{\alpha}_{\omega,\log,0}$, under some conditions posed on the weight function ω. Also, we study the relations between the class $\mathcal{B}^{\alpha}_{\omega,\log}$ and some other classes of analytic functions by the help of analytic functions in the Hadamard gap class.*

1 INTRODUCTION

Hadamard gaps are known to study some classes and spaces of holomorphic and hyperholomorphic functions. A wide variety of characterization not only in the type of function spaces, where functions are holomorphic and hyperholomorphic, but also in the coefficients which extend over Taylor or Fourier series expansions. It is one of the important tasks in the study of function spaces to seek for characterizations of functions by the help of their Taylor or Fourier series expansions. In the past few decades both Taylor and Fourier series expansions were studied by the help of Hadamard gap class (see [1, 6, 8, 9, 10] and others).

In the present paper, we shall obtain a sufficient and a necessary condition for an analytic function f on the unit disc with Hadamard gaps to belong to the weighted logarithmic Bloch space $\mathcal{B}^{\alpha}_{\omega,\log}$ as well as to the corresponding little weighted logarithmic Bloch space $\mathcal{B}^{\alpha}_{\omega,\log,0}$, under some conditions posed on the weight function ω.

Let $\mathbf{D} = \{z : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$. Recall that the well known Bloch space (cf. [2]) is defined as follows:

$$\mathcal{B} = \{f : f \text{ analytic in } \mathbf{D} \text{ and } \sup_{z\in\mathbf{D}}(1-|z|^2)|f'(z)| < \infty\}$$

and the little Bloch space $\mathcal{B}_0$ (cf. [2]) is given as follows

$$\lim_{|z|\to 1^-}(1-|z|^2)|f'(z)| = 0.$$

Recently, for given a reasonable function $\omega : (0,1] \to (0,\infty)$, the weighted Bloch space $\mathcal{B}_\omega$ is defined in [5] as the set of all analytic functions f on $\mathbf{D}$ satisfying

$$(1-|z|)|f'(z)| \le C\omega(1-|z|), \quad z \in \mathbf{D},$$

for some fixed $C = C_f > 0$. In the special case where $\omega \equiv 1, \mathcal{B}_\omega$ reduces to the classical Bloch space $\mathcal{B}$.

The Dirichlet space is defined by

$$\mathcal{D} = \{f : f \text{ analytic in } \mathbf{D} \text{ and } \int_{\mathbf{D}} |f'(z)|^2 dA(z) < \infty\},$$

where $dA(z)$ is the Euclidean area element $dxdy$.

Let $0 < q < \infty$. Then the Besov-type spaces consist of analytic functions on $\mathbf{D}$ such that

$$\mathbf{B^q} = \left\{f : f \text{ analytic in } \mathbf{D} \text{ and } \sup_{a\in\mathbf{D}} \int_{\mathbf{D}} |f'(z)|^q (1-|z|^2)^{q-2}(1-|\varphi_a(z)|^2)^2 \, dA(z) < \infty\right\},$$

these classes are introduced and studied intensively Stroethoff (cf. [14]). Here, φ_a stands for the Möbius transformation, where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$. In 1994, Aulaskari and Lappan [2] introduced a class of holomorphic functions, the so called $\mathbf{Q_p}$-spaces as follows:

$$\mathbf{Q_p} = \left\{f : f \text{ analytic in } \mathbf{D} \text{ and } \sup_{a\in\mathbf{D}} \int_{\mathbf{D}} |f'(z)|^2 g^p(z,a) \, dA(z) < \infty\right\},$$

where the weight function

$$g(z,a) = \log\left|\frac{1-\bar{a}z}{a-z}\right|$$

is defined as the composition of the Möbius transformation φ_a and the fundamental solution of the two-dimensional real Laplacian. Now, we give the following definitions:

Definition 1.1 *Let $0 < \alpha < \infty$ and $\omega : (0,1] \to (0,\infty)$. For an analytic function f in $\mathbf{D}$, we define the weighted α-Bloch space $\mathcal{B}_\omega^\alpha$, as follows:*

$$\mathcal{B}_\omega^\alpha = \Big\{ f : f \text{ analytic in } \mathbf{D} \text{ and } \|f\|_{\mathcal{B}_\omega^\alpha} = \sup_{z\in\mathbf{D}} \frac{(1-|z|)^\alpha |f'(z)|}{\omega(1-|z|)} < \infty \Big\}. \tag{1}$$

Also, the little weighted α-Bloch space $\mathcal{B}_{\omega,0}^\alpha$, is a subspace of $\mathcal{B}_\omega^\alpha$ consisting of all $f \in \mathcal{B}_\omega^\alpha$ such that

$$\lim_{|z|\to 1^-} \frac{(1-|z|)^\alpha |f'(z)|}{\omega(1-|z|)} = 0. \tag{2}$$

Miao [10] studied a gap series with Hadamard condition as in the following theorem:

Theorem 1.1 *Let $0 < p < \infty$. If $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$ is analytic on $\mathbf{D}$ and has Hadamard gaps, that is, if*

$$\frac{n_{k+1}}{n_k} \geq \lambda > 1, \qquad (k = 1, 2, \ldots),$$

then the following statements are equivalent:

(I) $f \in \mathbf{B}^p$; $\quad (II)$ $f \in \mathbf{B}_0^p$; $\quad (III)$ $\sum_{k=1}^{\infty} |a_k|^p < \infty$.

Remark 1.1 *The expression $\|f\|_{\mathcal{B}_\omega^\alpha}$ defines a seminorm while the natural norm is given by*

$$\|f\|_{\omega,\alpha} = |f(0)| + \|f\|_{\mathcal{B}_\omega^\alpha}.$$

With this norm the space $\mathcal{B}_\omega^\alpha$ is a Banach space.

Definition 1.2 *Let $0 < \alpha < \infty$ and $\omega : (0,1] \to (0,\infty)$. For an analytic function f in $\mathbf{D}$, we define the weighted logarithmic α-Bloch space $\mathcal{B}_{\omega,\log}^\alpha$ as follows:*

$$\mathcal{B}_{\omega,\log}^\alpha = \Big\{ f : f \text{ analytic in } \mathbf{D} \text{ and } \|f\|_{\mathcal{B}_{\omega,\log}^\alpha} = \sup_{z\in\mathbf{D}} \frac{(1-|z|)^\alpha |f'(z)|}{\omega(1-|z|)} \log\frac{1}{1-|z|} < \infty \Big\}. \tag{3}$$

Moreover, the little weighted logarithmic α-Bloch space $\mathcal{B}_{\omega,\log,0}^\alpha$ is a subspace of $\mathcal{B}_{\omega,\log}^\alpha$ consisting of all $f \in \mathcal{B}_{\omega,\log}^\alpha$ such that

$$\lim_{|z|\to 1^-} \frac{(1-|z|)^\alpha |f'(z)|}{\omega(1-|z|)} \log\frac{1}{1-|z|} = 0. \tag{4}$$

Remark 1.2 *The expression $\|f\|_{\mathcal{B}_{\omega,\log}^\alpha}$ defines a seminorm while the natural norm is given by*

$$\|f\|_{\omega,\log,\alpha} = |f(0)| + \|f\|_{\mathcal{B}_{\omega,\log}^\alpha}.$$

With this norm the space $\mathcal{B}_{\omega,\log}^\alpha$ is a Banach space.

Note that, If $\alpha = 1$ and $\omega \equiv 1$, then logarithmic α-Bloch space $\mathcal{B}^{\alpha}_{\omega,\log}$ reduces to the logarithmic Bloch space $\mathcal{B}_{\log}$ see [4]. The logarithmic Bloch space $\mathcal{B}_{\log}$ first appeared in the study of boundedness of the Hankle operators on the Bergman and Hardy spaces followed by many authors. For more details of the logarithmic Bloch space we refer to [4, 7, 15, 16] and others.
For a point $a \in \mathbf{D}$ and $0 < r < 1$, the pseudo-hyperbolic disk $D(a, r)$ with pseudo-hyperbolic center a and pseudo-hyperbolic radius r is defined by $D(a, r) = \varphi_a(r\mathbf{D})$.
The pseudo-hyperbolic disk $D(a, r)$ is also an Euclidean disk: its Euclidean center and Euclidean radius are $\frac{(1-r^2)a}{1-r^2|a|^2}$ and $\frac{(1-|a|^2)r}{1-r^2|a|^2}$, respectively (see [14]). Let A denote the normalized Lebesgue area measure on $\mathbf{D}$, and for a Lebesgue measurable set $K_1 \subset \mathbf{D}$, denote by $|K_1|$ the measure of K_1 with respect to A. It follows immediately that:

$$|D(a, r)| = \frac{(1 - |a|^2)^2}{(1 - r^2|a|^2)^2} r^2.$$

Now, we give a few facts about the Möbius function φ_a, which will be used in Section 3. First, the function φ_a is easily seen to be its own inverse under composition:

$$(\varphi_a \circ \varphi_a)(z) = z \quad \text{for all} \quad a, z \in \mathbf{D}\,.$$

The following identity can be obtained by straight forward computation:

$$1 - |\varphi_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \overline{a}z|^2} \quad \text{for all } \; a, z \in \mathbf{D} \; \text{(see} \quad \text{[14])}.$$

A slightly different form in which we will apply the above identity is:

$$\frac{1 - |\varphi_a(z)|^2}{1 - |z|^2} = |\varphi_a'(z)| \quad \text{(see} \quad \text{[14])}.$$

For $a \in \mathbf{D}$, the substitution $z = \varphi_a(w)$ results in the Jacobian change in measure given by

$$dA(w) = |\varphi_a'(z)|^2 dA(z).$$

For a Lebesgue integrable or a non-negative Lebesgue measurable function h on $\mathbf{D}$ we thus have the following change-of-variable formula:

$$\int_{D(0,r)} h(\varphi_a(w)) d(z) = \int_{D(a,r)} h(z) \Big(\frac{1 - |\varphi_a(z)|^2}{1 - |z|^2} \Big)^2 dA(z)\;.$$

Two quantities A_f and B_f, both depending on an analytic function f on $\mathbf{D}$, are said to be equivalent, written as $A_f \approx B_f$, if there exists a finite positive constant C not depending on f such that for every analytic function f on $\mathbf{D}$ we have:

$$\frac{1}{C} B_f \leq A_f \leq C B_f.$$

If the quantities A_f and B_f, are equivalent, then in particular we have $A_f < \infty$ if and only if $B_f < \infty$.

In the present work, we shall obtain a sufficient and a necessary condition for an analytic function f on the unit disc with Hadamard gaps to belong to the weighted logarithmic Bloch space

$\mathcal{B}^{\alpha}_{\omega,\log}$ as well as to the corresponding little weighted logarithmic Bloch space $\mathcal{B}^{\alpha}_{\omega,\log,0}$, under some conditions posed on the weight function ω. Important properties of the weight function ω are also proved, then we give characterization of holomorphic functions in the weighted Dirichlet space in terms of their Taylor coefficients, where the weighted Dirichlet space $\mathcal{D}_\omega$ is the collection of all analytic functions f in $\mathbf{D}$, for which

$$\int_{\mathbf{D}} \frac{|f'(z)|^2}{\omega(1-|z|)}\, dA(z) < \infty. \tag{5}$$

2 LACUNARY SERIES IN $\mathcal{B}^{\alpha}_{\omega,\mathrm{LOG}}$ AND $\mathcal{B}^{\alpha}_{\omega,\mathrm{LOG},0}$

In this section we give some criteria for lacunary series belonging to $\mathcal{B}^{\alpha}_{\omega,\log}$ or $\mathcal{B}^{\alpha}_{\omega,\log,0}$. These will be used to show some strict inclusions between $\mathcal{B}^{\alpha}_{\omega,\log}$ or $\mathcal{B}^{\alpha}_{\omega,\log,0}$ and some other function spaces in the next section. We first give necessary conditions on Taylor coefficients such that an analytic function is in $\mathcal{B}^{\alpha}_{\omega,\log}$ or $\mathcal{B}^{\alpha}_{\omega,\log,0}$. This may be of independent interest.

Proposition 2.1 *Let* $0 < \alpha < \infty,\ \omega : (0,1] \to (0,\infty)$, *and let* $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ *be an analytic function on* $\mathbf{D}$.
$(i)\ If\ f \in \mathcal{B}^{\alpha}_{\omega,\log}$, *then*

$$\lim_{n\to\infty} \sup\, n^{1-\alpha}|a_n|\, \frac{\log n}{\omega(\frac{1}{n})} \le e\|f\|_{\mathcal{B}^{\alpha}_{\omega,\log}}.$$

$(ii)\ If\ f \in \mathcal{B}^{\alpha}_{\omega,\log,0}$, *then*

$$\lim_{n\to\infty} \sup\, n^{1-\alpha}|a_n|\, \frac{\log n}{\omega(\frac{1}{n})} = 0.$$

Proof: (i) It is easy to see that, for $n \ge 1$,

$$a_n = \frac{1}{2\pi n} \int_0^{2\pi} f'(re^{i\theta})\, r^{1-n} e^{i(1-n)\theta} d\theta. \tag{6}$$

Thus,

$$\begin{aligned} |a_n| &\le \frac{1}{2\pi n} \int_0^{2\pi} |f'(re^{i\theta})| r^{1-n} d\theta \\ &\le \frac{\|f\|_{\mathcal{B}^{\alpha}_{\omega,\log}}}{2\pi n} \int_0^{2\pi} \frac{r^{1-n}\omega(1-r)}{(1-r)^{\alpha} \log \frac{1}{1-r}}\, d\theta \\ &= \frac{\|f\|_{\mathcal{B}^{\alpha}_{\omega,\log}} r^{1-n}\omega(1-r)}{n(1-r)^{\alpha}\log\frac{1}{1-r}}. \end{aligned}$$

Let $r = 1 - \frac{1}{n}$, we get for $n > 1$, that

$$|a_n| \le \frac{\|f\|_{\mathcal{B}^{\alpha}_{\omega,\log}} (1-\frac{1}{n})^{1-n}\omega(\frac{1}{n})}{n^{1-\alpha}\log n}.$$

So,

$$\lim_{n\to\infty} \sup\, n^{1-\alpha}|a_n|\, \frac{\log n}{\omega(\frac{1}{n})} \le \|f\|_{\mathcal{B}^{\alpha}_{\omega,\log}} \lim_{n\to\infty}\sup \left(1-\frac{1}{n}\right)^{1-n} = e\ \|f\|_{\mathcal{B}^{\alpha}_{\omega,\log}}\ .$$

(ii) By (6) we have

$$|a_n| \ \leq \frac{1}{2\pi n}\int_0^{2\pi}|f'(re^{i\theta})|r^{1-n}d\theta.$$

Because $f\in\mathcal{B}^{\alpha}_{\omega,\log,0}$, we know that every $\varepsilon>0$, there is a $\delta>0$ such that for every $r\in(0,1)$, and $1-r<\delta$, with

$$\frac{|f'(re^{i\theta})|(1-r)^{\alpha}}{\omega(1-r)}\ \log\frac{1}{1-r}<\varepsilon$$

Now, let $r=1-\frac{1}{n}$. Then $n=\frac{1}{1-r}>\frac{1}{\delta}$, and hence

$$|a_n| \ \leq \frac{\varepsilon}{2\pi n}\int_0^{2\pi}\frac{r^{1-n}\omega(1-r)}{(1-r)^{\alpha}\log\frac{1}{1-r}}d\theta=\frac{\varepsilon(1-\frac{1}{n})\omega(\frac{1}{n})}{n^{1-\alpha}\log n}.$$

So,

$$n^{1-\alpha}|a_n|\frac{\log n}{\omega(\frac{1}{n})}<\varepsilon(1-\frac{1}{n})^{1-n}.$$

Because

$$\lim_{n\to\infty}(1-\frac{1}{n})^{1-n}=e.$$

There exists $N_0>0$ such that for $n>N_0$, $(1-\frac{1}{n})^{1-n}<e+1$. Let $N=\max\{\frac{1}{\delta},N_0\}$. Then, if $n>N$,

$$n^{1-\alpha}|a_n|\frac{\log n}{\omega(\frac{1}{n})}<(e+1)\varepsilon$$

Thus

$$\lim_{n\to\infty}\sup n^{1-\alpha}|a_n|\frac{\log n}{\omega(\frac{1}{n})}=0.$$

The proof is complete.

We need the following lemma in the next

Lemma 2.1 *Let $0<\alpha<\infty$, and $\omega:(0,1]\to(0,\infty)$, continuous and nondecreasing function. If $\frac{n_{k+1}}{n_k}\geq\lambda>1$ for all $k\geq1$, then for sufficiently large k, we have*

$$\frac{\omega(\frac{1}{n_{k+1}})n_{k+1}^{\alpha}\log n_k}{\omega(\frac{1}{n_k})n_k^{\alpha}\log n_{k+1}}\geq\lambda^{\alpha}>1 \tag{7}$$

Proof: Suppose (7) is not true. Then we have

$$\lim_{k\to\infty}\inf\frac{\omega(\frac{1}{n_{k+1}})n_{k+1}^{\alpha}\log n_k}{\omega(\frac{1}{n_k})n_k^{\alpha}\log n_{k+1}}=S<\lambda^{\alpha}. \tag{8}$$

Let $\lambda_k=\frac{n_{k+1}}{n_k}$. Then $\lambda_k\geq\lambda>1$. So,

$$\lim_{k\to\infty}\inf\frac{1}{\left(1+\frac{\log\lambda_k}{\log n_k}\right)\frac{\omega(\frac{1}{n_k})}{\omega(\frac{1}{n_{k+1}})}}\ =\lim_{k\to\infty}\inf\frac{\omega(\frac{1}{n_{k+1}})\log n_k}{\omega(\frac{1}{n_k})\log n_{k+1}}=\lim_{k\to\infty}\inf\frac{\omega(\frac{1}{n_{k+1}})\ n_{k+1}^{\alpha}\ \log n_k}{\omega(\frac{1}{n_k})\ n_k^{\alpha}\ \log n_{k+1}}.\frac{1}{\lambda_k^{\alpha}}$$

$$=\lim_{k\to\infty}\inf\frac{1}{\lambda_k^{\alpha}}.\frac{\omega(\frac{1}{n_{k+1}})\ n_{k+1}^{\alpha}\ \log n_k}{\omega(\frac{1}{n_k})\ n_k^{\alpha}\ \log n_{k+1}}\leq\frac{S}{\lambda^{\alpha}}<1.$$

Hence

$$\lim_{k\to\infty}\sup\Big(1+\frac{\log\lambda_k}{\log n_k}\Big)\frac{\omega(\frac{1}{n_k})}{\omega(\frac{1}{n_{k+1}})}=\frac{\lambda^\alpha}{S}>1,$$

$$\lim_{k\to\infty}\sup\Big(\frac{\omega(\frac{1}{n_k})}{\omega(\frac{1}{n_{k+1}})}+\frac{\omega(\frac{1}{n_k})}{\omega(\frac{1}{n_{k+1}})}\frac{\log\lambda_k}{\log n_k}\Big)=\frac{\lambda^\alpha}{S}>1.$$

Hence ω is continuous, we obtain

$$\frac{\omega(0)}{\omega(0)}+\lim_{k\to\infty}\sup\frac{\omega(\frac{1}{n_k})}{\omega(\frac{1}{n_{k+1}})}\frac{\log\lambda_k}{\log n_k}=\frac{\lambda^\alpha}{S}>1.$$

$$1+\lim_{k\to\infty}\sup\frac{\omega(\frac{1}{n_k})}{\omega(\frac{1}{n_{k+1}})}\frac{\log\lambda_k}{\log n_k}=\frac{\lambda^\alpha}{S}>1.$$

Hence,

$$\lim_{k\to\infty}\sup\frac{\omega(\frac{1}{n_k})}{\omega(\frac{1}{n_{k+1}})}\frac{\log\lambda_k}{\log n_k}=\frac{\lambda^\alpha}{S}-1>0. \tag{9}$$

Let $\tau_k=\frac{\log\lambda_k}{\log n_k}$. By (9) we know that, for $0<\varepsilon<\frac{\lambda^\alpha}{S}-1$, there exists a subsequence of $\{\tau_k\}$, for convergence, we still denote it by $\{\tau_k\}$, such that $\tau_k\geq\varepsilon>0$ for all $k\geq 1$. Now we have

$$\frac{\omega(\frac{1}{n_{k+1}})n_{k+1}^\alpha\log n_k}{\omega(\frac{1}{n_k})n_k^\alpha\log n_{k+1}}=\frac{\lambda_k^\alpha}{\Big(1+\frac{\log\lambda_k}{\log n_k}\Big)\frac{\omega(\frac{1}{n_k})}{\omega(\frac{1}{n_{k+1}})}}=\frac{\Big(n_k^{\frac{\log\lambda_k}{\log n_k}}\Big)^\alpha}{\Big(1+\frac{\log\lambda_k}{\log n_k}\Big)\frac{\omega(\frac{1}{n_k})}{\omega(\frac{1}{n_{k+1}})}}=\frac{n_k^{\alpha\tau_k}}{(1+\tau_k)\frac{\omega(\frac{1}{n_k})}{\omega(\frac{1}{n_{k+1}})}}.$$

Since $\tau_k\geq\varepsilon>0$ and $n_k\to\infty$ as $k\to\infty$, it is obvious that

$$\lim_{k\to\infty}\inf\frac{n_k^{\alpha\tau_k}}{(1+\tau_k)\frac{\omega(\frac{1}{n_k})}{\omega(\frac{1}{n_{k+1}})}}=\infty.$$

Hence

$$\lim_{k\to\infty}\inf\frac{\omega(\frac{1}{n_{k+1}})n_{k+1}^\alpha\log n_k}{\omega(\frac{1}{n_k})n_k^\alpha\log n_{k+1}}=\infty.$$

This is contradiction to (8), and the proof is complete.

Now we give the criteria for lacunary series in $\mathcal{B}^\alpha_{\omega,\log}$ and $\mathcal{B}^\alpha_{\omega,\log,0}$

Theorem 2.1 *Let* $0<\alpha<\infty$, $\omega:(0,1]\to(0,\infty)$, *continuous and nondecreasing function and let* $f(z)=\sum\limits_{k=1}^{\infty}a_kz^{n_k}$ *be an analytic function on* $\mathbf{D}$ *with Hadamard gaps, which means* $\frac{n_{k+1}}{n_k}\geq\lambda>1$ *for all* $k\geq 1$.
(i) $f\in\mathcal{B}^\alpha_{\omega,\log}$, *if and only if*

$$\lim_{k\to\infty}\sup n_k^{1-\alpha}|a_k|\frac{\log n_k}{\omega(\frac{1}{n_k})}<\infty. \tag{10}$$

(ii) $f \in \mathcal{B}^{\alpha}_{\omega,\log,0}$, *if and only if*

$$\lim_{k\to\infty} \sup n_k^{1-\alpha}|a_k|\frac{\log n_k}{\omega(\frac{1}{n_k})} = 0. \tag{11}$$

Proof: (i) By Proposition 2.1 (i), we need only prove " $\Leftarrow$ ". Suppose that f satisfies (10). Without loss of generality, we may assume that $n_k > 1$. Then by (10), there is a constant $M > 0$ such that

$$|a_k| \le M n_k^{\alpha-1}(\log n_k)^{-1}\omega(\frac{1}{n_k}).$$

Hence

$$\begin{aligned}\frac{|zf'(z)|}{1-|z|}\log\frac{1}{1-|z|} &\le \Big(\sum_{k=1}^{\infty} n_k|a_k||z|^{n_k-1}\Big)\Big(\sum_{n=0}^{\infty}|z|^n\Big)\Big(\sum_{n=1}^{\infty}\frac{|z|^n}{n}\Big)\\ &\le M\Big(\sum_{n=1}^{\infty}\Big(\sum_{n_k\le n}\frac{\omega(\frac{1}{n_k})n_k^{\alpha}}{\log n_k}\Big)|z|^n\Big)\Big(\sum_{n=0}^{\infty}\frac{|z|^n}{n}\Big).\end{aligned} \tag{12}$$

By Lemma 2.1, there is a constant $K > 0$ such that for every $k \ge K$,

$$\frac{\omega(\frac{1}{n_{k+1}})\ \log n_k\ n_{k+1}^{\alpha}}{\omega(\frac{1}{n_k})\ \log n_{k+1}\ n_k^{\alpha}} \ge \lambda^{\alpha} > 1.$$

We may assume that $K \ge 3$. Hence

$$\frac{n_k^{\alpha}\omega(\frac{1}{n_k})}{\log n_k} \le \frac{n_{k+1}^{\alpha}\omega(\frac{1}{n_{k+1}})}{\lambda^{\alpha}\log n_{k+1}}. \tag{13}$$

Let $n_{k_0} \le n \le n_{k_0} + 1$. Without loss of generality, we may assume that $k_0 > K$. Then, for $K \le k < k_0$, by (13) and the fact that $\frac{x}{\log x}$ is increasing with respect to x for $x \ge 3$, we get

$$\frac{n_k^{\alpha}\omega(\frac{1}{n_k})}{\log n_k} \le \frac{n_{k_0}^{\alpha}\omega(\frac{1}{n_{k_0}})}{\lambda^{\alpha(k_0-k)}\log n_{k_0}} \le \frac{n^{\alpha}\omega(\frac{1}{n})}{\lambda^{\alpha(k_0-k)}\log n}.$$

Therefore, we have

$$\sum_{k=K}^{k_0}\frac{n_k^{\alpha}\omega(\frac{1}{n_k})}{\log n_k} \le \frac{n^{\alpha}\omega(\frac{1}{n})}{\log n}\sum_{k=K}^{k_0}\frac{1}{\lambda^{\alpha(k_0-k)}} \le \frac{n^{\alpha}\omega(\frac{1}{n})}{(1-\lambda^{\alpha})\log n}. \tag{14}$$

On the other hand, for $1 \le k < K$, since we have only finite terms, we can find a constant $\bar{M} > 0$ such that

$$\frac{n_k^{\alpha}\omega(\frac{1}{n_k})}{\log n_k} \le \bar{M}\frac{n^{\alpha}\omega(\frac{1}{n})}{\log n}.$$

Thus

$$\sum_{k=1}^{K-1}\frac{n_k^{\alpha}\omega(\frac{1}{n_k})}{\log n_k} \le \bar{M}K\frac{n^{\alpha}\omega(\frac{1}{n})}{\log n}. \tag{15}$$

Combining (14) and (15) we know that there is a constant $R > 0$ such that

$$\sum_{n_k\le n}\frac{n_k^{\alpha}\omega(\frac{1}{n_k})}{\log n_k} \le R\frac{n^{\alpha}\omega(\frac{1}{n})}{\log n}. \tag{16}$$

From (12) and (16), notice that $n_k > 1$, we have

$$
\begin{aligned}
\frac{|zf'(z)|}{1-|z|}\log\frac{1}{1-|z|} &\le MR\Big(\sum_{n=2}^{\infty}\frac{n^\alpha\omega(\frac{1}{n})}{\log n}|z|^n\Big)\Big(\sum_{n=1}^{\infty}\frac{|z|^n}{n}\Big)\\
&\le MR\Big(\sum_{n=3}^{\infty}\Big(\sum_{k=2}^{n-1}\frac{k^\alpha\omega(\frac{1}{k})}{(n-k)\log k}\Big)|z|^n\Big). \qquad (17)
\end{aligned}
$$

It is easy to see that, for $2\le k<n$, $\frac{k^\alpha\omega(\frac{1}{k})}{\log k}\le\frac{Cn^\alpha\omega(\frac{1}{n})}{\log n}$, where $C>0$ is a constant. Then

$$
\frac{k^\alpha\omega(\frac{1}{k})}{(n-k)\log k}\le\frac{Cn^\alpha\omega(\frac{1}{n})}{(n-k)\log n}.
$$

Hence

$$
\sum_{k=2}^{n-1}\frac{k^\alpha\omega(\frac{1}{k})}{(n-k)\log k}\le\frac{C\omega(\frac{1}{n})n^\alpha}{\log n}\sum_{k=2}^{n-1}\frac{1}{n-k}\le\frac{C_1\omega(\frac{1}{n})n^\alpha\log n}{\log n}=C_1\omega(\frac{1}{n})n^\alpha. \qquad (18)
$$

Thus, by (17) and (18), we have

$$
\frac{|zf'(z)|}{1-|z|}\log\frac{1}{1-|z|}\le C_1MR\sum_{n=3}^{\infty}n^\alpha\omega(\frac{1}{n})|z|^n\le C_2\frac{|z|\omega(1-|z|)}{(1-|z|)^{1+\alpha}}.
$$

Hence,

$$
\sup_{z\in\mathbf{D}}|f'(z)|\frac{(1-|z|^2)^\alpha}{\omega(1-|z|)}\log\frac{1}{1-|z|}\le 2^\alpha\sup_{z\in D}|f'(z)|\frac{(1-|z|)^\alpha}{\omega(1-|z|)}\log\frac{1}{1-|z|}\le 2^\alpha C_2<\infty.
$$

So $f\in\mathcal{B}^\alpha_{\omega,\log}$.

(ii) By Theorem 2.1 (ii), we need only prove " $\Leftarrow$ ". Suppose that f satisfies (11). Without loss of generality, we may assume that $n_k>1$. Denote $A=\sup_{k\ge1}|a_k|n_k^{1-\alpha}\omega^{-1}(\frac{1}{n_k})\log n_k$. Then by (11), $A<+\infty$, and for every $\varepsilon>0$, there is a constant $K=K(\varepsilon)>0$ such that for every $k>K$, $|a_k|n_k^{1-\alpha}\omega^{-1}(\frac{1}{n_k})\log n_k<\varepsilon$. Denote

$$
M=\sum_{k=1}^{K}\frac{n_k^\alpha\omega(\frac{1}{n_k})}{\log n_k}.
$$

Since the series has only finite terms, $M<+\infty$. Thus

$$
\begin{aligned}
\frac{|zf'(z)|}{1-|z|}\log\frac{1}{1-|z|} &\le \Big(\sum_{k=1}^{\infty}n_k|a_k||z|^{n_k-1}\Big)\Big(\sum_{n=0}^{\infty}|z|^n\Big)\Big(\sum_{n=1}^{\infty}\frac{|z|^n}{n}\Big)\\
&\le A\Big(\sum_{k=1}^{K}\frac{n_k^\alpha\omega(\frac{1}{n_k})}{\log n_k}|z|^{n_k-1}\Big)\Big(\sum_{n=0}^{\infty}|z|^n\Big)\Big(\sum_{n=1}^{\infty}\frac{|z|^n}{n}\Big)\\
&+ \varepsilon\Big(\sum_{k=K+1}^{\infty}\frac{n_k^\alpha\omega(\frac{1}{n_k})}{\log n_k}|z|^{n_k-1}\Big)\Big(\sum_{n=0}^{\infty}|z|^n\Big)\Big(\sum_{n=1}^{\infty}\frac{|z|^n}{n}\Big)\\
&\le \frac{AM|z|}{1-|z|}\log\frac{1}{1-|z|}+\varepsilon\, C_2\frac{|z|}{(1-|z|)^{1+\alpha}}.
\end{aligned}
$$

The second term of the right hand side is from the proof of (i). Hence

$$|f'(z)|\frac{(1-|z|)^\alpha}{\omega(1-|z|)}\log\frac{1}{1-|z|} \le AM\frac{(1-|z|)^\alpha}{\omega(1-|z|)}\log\frac{1}{1-|z|} + C_2\varepsilon.$$

Since

$$\lim_{|z|\to 1^-}\frac{(1-|z|)^\alpha}{\omega(1-|z|)}\log\frac{1}{1-|z|} = 0,$$

we can choose $r,\ 0<r<1$, such that for $|z|>r$,

$$AM\frac{(1-|z|^2)^\alpha}{\omega(1-|z|)}\log\frac{1}{1-|z|} < \varepsilon.$$

Hence, for $|z|>r$

$$|f'(z)|\frac{(1-|z|^2)^\alpha}{\omega(1-|z|)}\log\frac{1}{1-|z|} < 2^\alpha(1+C_2)\varepsilon,$$

which means that

$$\lim_{|z|\to 1_-}|f'(z)|\frac{(1-|z|^2)^\alpha}{\omega(1-|z|)}\log\frac{1}{1-|z|} = 0.$$

Thus $f\in\mathcal{B}^\alpha_{\omega,\log,0}$.
In the end of Section 1, we have seen that

$$M(\mathcal{B}) = M(\mathcal{B}_0) = H^\infty\bigcap\mathcal{B}_{\omega,\log}.$$

It is natural to ask that if it is true that H^∞ and $\mathcal{B}_{\omega,\log}$ are not included in each other. Since it is well known that $H^\infty\not\subset\mathcal{B}_\omega$, and by Lemma 2.1, $\mathcal{B}_{\omega,\log}\subset\mathcal{B}_\omega$, we see that $H^\infty\not\subset\mathcal{B}_{\omega,\log}$. The following result show that $\mathcal{B}_{\omega,\log}\not\subset H^\infty$ (note that $\mathcal{B}_{\omega,\log,0}\subset\mathcal{B}_{\omega,\log}$), and so H^∞ and $\mathcal{B}_{\omega,\log}$ are indeed not included in each other.

Corollary 2.1 *Let* $\omega:(0,1]\to(0,\infty)$, *so* $\mathcal{B}_{\omega,\log,0}\not\subset H^\infty$.

Proof: Let $f(z)=\sum\limits_{k=2}^{\infty}a_kz^{2^k}=\sum\limits_{k=2}^{\infty}(k\log k)^{-1}z^{2^k}$. Then

$$\lim_{k\to\infty}\sup|a_k|\frac{\log 2^k}{\omega(\frac{1}{2^k})} = \lim_{k\to\infty}\sup\frac{k\log 2}{\omega(\frac{1}{2^k})k\log k} = 0.$$

Thus, by Theorem 2.1, $f\in\mathcal{B}_{\omega,\log,0}$. Since

$$\lim_{r\to 1}|f(r)| = \lim_{r\to 1}\sum_{k=2}^{\infty}\frac{r^{2^k}}{k\log k} = \infty.$$

We know that $f\notin H^\infty$, and the proof is complete.

Corollary 2.2 *Let* $\omega:(0,1]\to(0,\infty)$, *so* $\mathcal{B}_{\omega,0}\not\subset H^\infty$.

Proof: The prove is very similar as Corollary 2.1.

3 STRICT INCLUSIONS

In this section we study the relations between the space $\mathcal{B}^{\alpha}_{\omega,\log}$ and some other spaces. The following result shows that the space $\mathcal{B}^{\alpha}_{\omega,\log}$ is quite close to the α-Bloch space $\mathcal{B}^{\alpha}$. Anyhow, they are not same.

The following lemma is useful in our study

Lemma 3.1 *[3] For $0 < p \leq 1,\ a \in D$ and $z = re^{i\theta}$ in $\mathbf{D}$,*

$$\int_0^{2\pi} \frac{d\theta}{|1 - \bar{a}re^{i\theta}|^{2p}} \leq \frac{C}{(1-|a|r)^p}\ ,$$

where $C > 0$ is a constant.

Theorem 3.1 *Let $\omega : (0,1] \to (0,\infty)$ and $0 < \alpha < \infty$. Then*

$$\bigcup_{0<\alpha^*<\alpha} \mathcal{B}^{\alpha^*}_{\omega} \not\subseteq \mathcal{B}^{\alpha}_{\omega,\log,0} \not\subseteq \mathcal{B}^{\alpha}_{\omega,\log} \not\subseteq \mathcal{B}^{\alpha}_{\omega,0}.$$

Proof: The inclusion $\mathcal{B}^{\alpha}_{\omega,\log} \subset \mathcal{B}^{\alpha}_{\omega,0}$ is proved in (see [12] Lemma 2.2). It is obvious that $\mathcal{B}^{\alpha}_{\omega,\log,0} \subset \mathcal{B}^{\alpha}_{\omega,\log}$. To prove the left hand side inclusion, let $0 < \alpha^* < \alpha$, and let $f \in \mathcal{B}^{\alpha^*}_{\omega}$. Then

$$\|f\|_{\mathcal{B}^{\alpha^*}_{\omega}} = \sup_{z\in D} |f'(z)|\frac{(1-|z|)^{\alpha}}{\omega(1-|z|)} < \infty.$$

Hence,

$$\lim_{|z|\to 1^-} |f'(z)|\frac{(1-|z|^2)^{\alpha}}{\omega(1-|z|)} \log\frac{1}{1-|z|} = \|f\|_{\mathcal{B}^{\alpha^*}_{\omega}} \lim_{|z|\to 1^-} (1-|z|^2)^{\alpha-\alpha^*} \log\frac{1}{1-|z|} = 0.$$

Thus $f \in \mathcal{B}^{\alpha}_{\omega,\log,0}$.

Now we prove the strictness of the inclusions. First, let $f_1(z) = \sum_{k=1}^{\infty} a_k z^{2^k}$, where $a_k = 2^{-k(1-\alpha)}(k\log k)^{-1}\omega(\frac{1}{2^k})$. Then

$$\lim_{k\to\infty} \sup |a_k|(2^k)^{1-\alpha}\frac{\log 2^k}{\omega(\frac{1}{2^k})} = \lim_{k\to\infty} \sup \frac{\log 2}{\log k} = 0.$$

Thus, by Theorem 2.1, $f_1 \in \mathcal{B}^{\alpha}_{\omega,\log,0}$. But for every α^*, $0 < \alpha^* < \alpha$,

$$\lim_{k\to\infty} \sup |a_k|\frac{(2^k)^{1-\alpha^*}}{\omega(\frac{1}{2^k})} = \lim_{k\to\infty} \sup 2^{k(\alpha-\alpha^*)}(k\log 2)^{-1} = \infty.$$

So, by Corollary 2.2, $f_1 \notin \mathcal{B}^{\alpha^*}_{\omega}$.

Next, let $f_2(z) = \sum_{k=1}^{\infty} a_k z^{2^k}$, where $a_k = 2^{-k(1-\alpha)}(k\log 2)^{-1}\omega(\frac{1}{2^k})$. Then

$$\lim_{k\to\infty} \sup |a_k|(2^k)^{1-\alpha}\frac{\log 2^k}{\omega(\frac{1}{2^k})} = 1 < \infty.$$

Thus, by Theorem 2.1, $f_2 \in \mathcal{B}^{\alpha}_{\omega,\log} \backslash \mathcal{B}^{\alpha}_{\omega,\log,0}$.

Finally, let $f_3(z)=\sum\limits_{k=1}^{\infty}a_k z^{2^k}$, where $a_k=2^{-k(1-\alpha)}(\log k)^{-1}\omega(\frac{1}{2^k})$. Then

$$\lim_{k\to\infty}\sup|a_k|\frac{(2^k)^{1-\alpha}}{\omega(\frac{1}{2^k})}=0.$$

So, we have $f_3\in\mathcal{B}^{\alpha}_{\omega,0}$. But

$$\lim_{k\to\infty}\sup|a_k|(2^k)^{1-\alpha}\frac{\log 2^k}{\omega(\frac{1}{2^k})}=\lim_{k\to\infty}\sup\frac{k\log 2}{\log 2}=\infty.$$

So by Theorem 2.1, $f_3\notin\mathcal{B}^{\alpha}_{\omega,\log}$. The proof is complete.

The following is a direct consequence of Theorem 3.1.

Corollary 3.1 *Let* $\omega:(0,1]\to(0,\infty)$ *and* $0<\alpha_1<\alpha_2<\infty$. *Then* $\mathcal{B}^{\alpha_1}_{\omega,\log}\not\subset\mathcal{B}^{\alpha_2}_{\omega,\log}$, $\mathcal{B}^{\alpha_1}_{\omega,\log,0}\not\subset\mathcal{B}^{\alpha_2}_{\omega,\log,0}$.

Following [13], We say that an analytic function f belongs to the space $Q_{p,\omega}$, $0<p<\infty$, if

$$\sup_{a\in\mathbf{D}}\int_{\mathbf{D}}|f'(z)|^2\frac{g^p(z,a)}{\omega(1-|z|)}dA(z)<\infty,$$

where $g(z,a)=\log|\frac{1-\bar{a}z}{a-z}|$ is the Green's function of $\mathbf{D}$ with singularity at a. Similarly, an analytic function f belongs to the space $Q_{p,\omega,0}$, $0<p<\infty$, if

$$\lim_{|a|\to1^-}\int_{\mathbf{D}}|f'(z)|^2\frac{g^p(z,a)}{\omega(1-|z|)}dA(z)=0.$$

We also recall that the weighted Dirichlet space $\mathcal{D}_\omega$ is the collection of the analytic functions f for which

$$\int_{\mathbf{D}}|f'(z)|^2\frac{1}{\omega(1-|z|)}dA(z)<\infty.$$

An important tool in the study of $\mathcal{D}_\omega$ space is the auxiliary function Ψ_ω defined by

$$\Psi_\omega(s)=\sup_{0<t<1}\frac{\omega(st)}{\omega(t)},\quad 0<s<1.$$

The following condition has played a crucial role in the study of $\mathcal{D}_\omega$ space:

$$\int_1^{\frac{1}{t}}\Psi_\omega(s)\frac{ds}{s^2}<\infty. \tag{19}$$

The function theory of $\mathcal{D}_\omega$ obviously depends on the properties of ω. Given two weight functions ω_1 and ω_2, we are going to write $\omega_1\lesssim\omega_2$ if there exists a constant $C>0$, independent of t, such that $\omega_1(t)\le C\omega_2(t)$ for all t. The notation $\omega_1\gtrsim\omega_2$ is used in a similar fashion. When $\omega_1\lesssim\omega_2\lesssim\omega_2$ we write $\omega_1\approx\omega_1$.

Lemma 3.2 *If* ω *satisfies condition* (19)*, then the function*

$$\omega^*(t)=t\int_t^1\frac{\omega(s)}{s^2}ds\quad (\textit{where, } 0<t<1),$$

has the following properties :

(a) ω^* *is nondecreasing on* $(0,1)$.

(b) $\frac{\omega^*(t)}{t}$ *is nonincreasing on* $(0,1)$.

(c) $\omega^*(t) \geq \omega(t)$ *for all* $t \in (0,1)$.

(d) $\omega^* \lesssim \omega$ *on* $(0,1)$.

If $\omega(t) = \omega(1)$ *for* $t \geq 1$, *then we also have*

(e) $\omega^*(t) = \omega^*(1) = \omega(1)$ *for* $t \geq 1$, *so* $\omega^* \approx \omega$ *on* $(0,1)$.

Proof: If $t \in (0,1)$, then a change of variables gives

$$\begin{aligned} \omega^*(t) &= t\int_t^1 \omega(s)\frac{ds}{s^2} = \int_1^{\frac{1}{t}} \omega(ts)\frac{ds}{s^2} \\ &= \omega(t)\int_1^{\frac{1}{t}} \frac{\omega(ts)}{\omega(t)}\frac{ds}{s^2} \leq \omega(t)\int_1^{\frac{1}{t}} \Psi_\omega(s)\frac{ds}{s^2}. \end{aligned}$$

So condition (19) implies that $\omega^*(t) \lesssim \omega(t)$ for $t \in (0,1)$. This yields property (e) and shows that $\omega^*(t)$ is well defined for all $0 < t < 1$.
Since

$$\frac{\omega^*(t)}{t} = \int_t^1 \frac{\omega(s)}{s^2}ds$$

and ω is nonnegative, we see that the function $\frac{\omega^*(t)}{t}$ is decreasing. This proves (b). Property (e) follows from a direct calculation.
Using the assumption that ω is nondecreasing again, we obtain

$$\omega^*(t) = t\int_t^1 \frac{\omega(s)}{s^2}ds \geq t\,\omega(t)\int_t^1 \frac{ds}{s^2} = (1-t)\omega(t)$$

for all $0 < t < 1$. This proves property (c).

It remains for us to show that ω^* is nondecreasing. To this end, we fix $0 \leq t_1 < t < 1$ and consider the difference

$$S_1 = \omega^*(t_1) - \omega^*(t) = t_1\int_{t_1}^1 \frac{\omega(s)}{s^2}ds - t\int_t^1 \frac{\omega(s)}{s^2}ds$$

Since ω is nondecreasing and nonnegative, we have

$$S_1 \geq t_1\omega(1)\int_{t_1}^1 \frac{ds}{s^2} - t\omega(1)\int_t^1 \frac{ds}{s^2} = (t-t_1)\omega(1) \geq 0.$$

This proves property (a) and completes the proof of the lemma.

Corollary 3.2 *If* ω *satisfies condition* (19), *then there exists a constant* $C > 0$ *such that* $\omega(2t) \leq C\,\omega(t)$ *for all* $0 < 2t < 1$.

Proof: For any $t > 0$, we have

$$\frac{\omega^*(2t)}{\omega^*(t)} = \frac{2\int\limits_{2t}^{1} \frac{\omega(s)}{s^2}\, ds}{\int\limits_{t}^{1} \frac{\omega(s)}{s^2}\, ds} \leq 2.$$

The desired estimate now follows from parts (c) and (d) of Lemma 3.1.

We begin with an estimate of the weighted Dirichlet integral interms of Taylor coefficients

Lemma 3.3 *Let* $\omega : (0, 1] \to (0, \infty)$, *then for any* $s \geq 1,\ \alpha \geq 1$ *and* $0 \leq \beta < 1$, *we have*

$$\int_0^1 r^{\alpha-1}(1-r)^{-\beta} \frac{dr}{\omega(1-r)} \approx \frac{\frac{1}{\alpha}}{\omega(\frac{1}{\alpha})} \sum_{s=1}^{\infty} \frac{(1-\alpha)_s (\frac{1}{\alpha})^{s-\beta}}{s!(s-\beta)},$$

where

$$\begin{aligned} &(1-\alpha)_s = (1-\alpha)(2-\alpha)(3-\alpha)\dots(s-\alpha), \quad s \geq 1, \\ &(1-\alpha)_0 = 1, \quad 1-\alpha \neq 0. \end{aligned}$$

Proof: Let

$$I = \int_0^1 r^{\alpha-1}(1-r)^{-\beta} \frac{dr}{\omega(1-r)} \ .$$

By a change of variables we have

$$I = \int_0^1 (1-t)^{\alpha-1} t^{-\beta} \frac{dt}{\omega(t)}.$$

We write $I = I_1 + I_2$, where

$$I_1 = \int_0^{\frac{1}{\alpha}} (1-t)^{\alpha-1} t^{-\beta} \frac{dt}{\omega(t)}.$$

and

$$I_2 = \int_{\frac{1}{\alpha}}^1 (1-t)^{\alpha-1} t^{-\beta} \frac{dt}{\omega(t)}.$$

By part (c) of Lemma 3.1, we have

$$I_1 \leq \int_0^{\frac{1}{\alpha}} (1-t)^{\alpha-1} t^{-\beta} \frac{dt}{\omega^*(t)}.$$

According to part (b) of Lemma 3.1, the function $\frac{\omega^*(t)}{t}$ is decreasing on $(0, 1)$, so

$$I_1 \leq \frac{\frac{1}{\alpha}}{\omega^*(\frac{1}{\alpha})} \int_0^{\frac{1}{\alpha}} (1-t)^{\alpha-1} t^{-(\beta+1)} dt.$$

This together with part (d) of Lemma 3.1 shows that

$$I_1 \leq \frac{\frac{1}{\alpha}}{\omega(\frac{1}{\alpha})} \int_0^{\frac{1}{\alpha}} (1-t)^{\alpha-1} t^{-(\beta+1)} dt.$$

From [11], the binomial theorem states that

$$(1-t)^{-(1-\alpha)} = \sum_{s=0}^{\infty} \frac{(\alpha-1)(\alpha-2)\dots(\alpha-s)(-1)^s t^s}{s!},$$

which may be written

$$(1-t)^{-(1-\alpha)} = \sum_{s=0}^{\infty} \frac{(1-\alpha)(2-\alpha)\dots(s-\alpha)t^s}{s!}.$$

Therefore, in factorial function notation,

$$(1-t)^{\alpha-1} = (1-t)^{-(1-\alpha)} = \sum_{s=0}^{\infty} \frac{(1-\alpha)_s t^s}{s!}$$

It is easy to see that

$$\begin{aligned} \int_0^{\frac{1}{\alpha}} (1-t)^{\alpha-1} t^{-(\beta+1)} dt &= \int_0^{\frac{1}{\alpha}} \sum_{s=0}^{\infty} \frac{(1-\alpha)_s t^{s-\beta-1}}{s!} dt = \sum_{s=0}^{\infty} \frac{(1-\alpha)_s}{s!} \int_0^{\frac{1}{\alpha}} t^{s-\beta-1} dt \\ &= \sum_{s=0}^{\infty} \frac{(1-\alpha)_s (\frac{1}{\alpha})^{s-\beta}}{s!(s-\beta)} \end{aligned}$$

$$I_1 \leq \frac{\frac{1}{\alpha}}{\omega(\frac{1}{\alpha})} \sum_{s=1}^{\infty} \frac{(1-\alpha)_s (\frac{1}{\alpha})^{s-\beta}}{s!(s-\beta)}$$

Since $\omega(t)$ is increasing function, we have

$$\begin{aligned} I_2 &= \int_{\frac{1}{\alpha}}^{1} (1-t)^{\alpha-1} t^{-(\beta+1)} \frac{t}{\omega(\frac{1}{t})} dt. \\ &\leq \frac{\frac{1}{\alpha}}{\omega(\frac{1}{\alpha})} \int_{\frac{1}{\alpha}}^{1} (1-t)^{\alpha-1} t^{-\beta} dt. \\ &= \frac{\frac{1}{\alpha}}{\omega(\frac{1}{\alpha})} \sum_{s=1}^{\infty} \frac{(1-\alpha)_s}{s!(s-\beta)} \left[1 - (\frac{1}{\alpha})^{s-\beta}\right] \end{aligned}$$

Combining this with what was proved in previous paragraph, we have

$$I \lesssim \frac{\frac{1}{\alpha}}{\omega(\frac{1}{\alpha})} \sum_{s=1}^{\infty} \frac{(1-\alpha)_s (\frac{1}{\alpha})^{s-\beta}}{s!(s-\beta)}.$$

On the other hand, we have

$$I \geq \int_{\frac{1}{\alpha}}^{1} (1-t)^{\alpha-1} t^{-\beta} \frac{dt}{\omega(t)}$$

The assumption that $\omega(t)$ is increasing gives

$$I \geq \frac{\frac{1}{\alpha}}{\omega(\frac{1}{\alpha})} \sum_{s=1}^{\infty} \frac{(1-\alpha)_s (\frac{1}{\alpha})^{s-\beta}}{s!(s-\beta)}.$$

This complete the prove of the theorem.

Theorem 3.2 *Let* $\omega : (0,1] \to (0,\infty)$ *and*

$$f(z) = \sum_{n=1}^{\infty} a_n z^n,$$

then

$$\int_{\mathbf{D}} |f'(z)|^2 \frac{dA(z)}{\omega(1-|z|)} \approx \sum_{n=1}^{\infty} \frac{n|a_n|^2}{\omega(\frac{1}{n})} \sum_{s=1}^{\infty} \frac{(1-2n)_s(\frac{1}{2n})^s}{s!s}.$$

Proof: Let

$$I(f) = \int_{\mathbf{D}} |f'(z)|^2 \frac{dA(z)}{\omega(1-|z|)}$$

Using polar coordinates one gets

$$\begin{aligned} I(f) &= \int_0^1 \int_0^{2\pi} |\sum_{n=1}^{\infty} n a_n z^{n-1}|^2 \frac{d\theta dr}{\omega(1-r)} \\ &= \int_0^1 \int_0^{2\pi} \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-1} \frac{d\theta dr}{\omega(1-r)} . \\ &\leq 2\pi \sum_{n=1}^{\infty} n^2 |a_n|^2 \int_0^1 r^{2n-1} \frac{dr}{\omega(1-r)} \end{aligned}$$

We apply Lemma 3.2 with $\beta = 0$ and $\alpha = 2n$ to obtain

$$I(f) \leq 2\pi \sum_{n=1}^{\infty} n^2 |a_n|^2 \frac{\frac{1}{2n}}{\omega(\frac{1}{2n})} \sum_{s=1}^{\infty} \frac{(1-2n)_s(\frac{1}{2n})^s}{s!s}.$$

$$\begin{aligned} I(f) &\approx 2\pi \sum_{n=1}^{\infty} \frac{n^2|a_n|^2}{\omega(\frac{1}{2n})} \frac{1}{2n} \sum_{s=1}^{\infty} \frac{(1-2n)_s(\frac{1}{2n})^s}{s!s} \\ &\approx \sum_{n=1}^{\infty} \frac{n|a_n|^2}{\omega(\frac{1}{n})} \sum_{s=1}^{\infty} \frac{(1-2n)_s(\frac{1}{2n})^s}{s!s} \end{aligned}$$

It is well known that an analytic function $f(z) = \sum_{n=1}^{\infty} a_n z^n$ belongs to the weighted Dirichlet space if and only if $\sum_{n=1}^{\infty} \frac{n|a_n|^2}{\omega(\frac{1}{n})} \sum_{s=1}^{\infty} \frac{(1-2n)_s(\frac{1}{2n})^s}{s!s} < \infty$.
It is easy to prove that $Q_{p,\omega} = \mathcal{B}_\omega$ and $Q_{p,\omega,0} = \mathcal{B}_{\omega,0}$ for all $p,\ 1 < p < \infty$.

We give the relation between $\mathcal{B}^\alpha_{\omega,\log}$ and the $Q_{p,\omega,0}$ spaces.

Theorem 3.3 *Let* $M_\omega = \sup_{z \in D_{\frac{1}{2}}} \frac{|f'(z)|^2}{\omega(1-|z|)} < \infty$, *we obtain that*

(i) If $0 < \alpha < \frac{1}{2}$, *then*

$$\mathcal{B}^\alpha_{\omega,\log} \not\subseteq \mathcal{D}_\omega \subset \bigcap_{p>0} Q_{p,\omega,0}.$$

(ii) If $\frac{1}{2} < \alpha \leq 1$, *then*

$$\mathcal{B}^\alpha_{\omega,\log} \not\subseteq Q_{\omega,2\alpha-1,0}.$$

Proof: (i) In the view of Corollary 3.1, we need only prove $\mathcal{B}^{\frac{1}{2}}_{\omega,\log}\not\subset\mathcal{D}_\omega$. Let $f\in\mathcal{B}^{\frac{1}{2}}_{\omega,\log}$. Then

$$\|f\|_{\mathcal{B}^{\frac{1}{2}}_{\omega,\log}}=\sup_{z\in D}|f'(z)|\frac{(1-|z|^2)^{\frac{1}{2}}}{\omega(1-|z|)}\log\frac{1}{1-|z|}<\infty.$$

Let

$$I=\int_{\mathbf{D}}|f'(z)|^2\frac{1}{\omega(1-|z|)}dA(z)=\Big(\int_{\mathbf{D}_{\frac{1}{2}}}+\int_{\mathbf{D}\backslash\mathbf{D}_{\frac{1}{2}}}\Big)|f'(z)|^2\frac{dA(z)}{\omega(1-|z|)}=I_1+I_2,$$

where $\mathbf{D}_{\frac{1}{2}}=\{z:|z|<\frac{1}{2}\}$. It is obvious that $I_1\le M<\infty$. For I_2 we have

$$\begin{aligned}I_2 &\le \|f\|_{\mathcal{B}^{\frac{1}{2}}_{\omega,\log}}\int_{D\backslash\mathbf{D}_{\frac{1}{2}}}\frac{dA(z)}{(1-|z|^2)(\log\frac{1}{1-|z|})^2}\le 2\pi\|f\|_{\mathcal{B}^{\frac{1}{2}}_{\omega,\log}}\int_{\frac{1}{2}}^1\frac{rdr}{(1-r)(\log\frac{1}{1-r})^2}\\ &\le 2\pi\|f\|_{\mathcal{B}^{\frac{1}{2}}_{\omega,\log}}\int_{\log 2}^\infty\frac{dt}{t^2}=\frac{2\pi}{\log 2}\|f\|_{\mathcal{B}^{\frac{1}{2}}_{\omega,\log}}<\infty.\end{aligned}$$

So $I=I_1+I_2<\infty$ and then $f\in\mathcal{D}_\omega$.

To prove the strictness, let $f_4(z)=\sum\limits_{k=1}^\infty a_kz^{2^k}$, where $a_k=k^{\frac{-1-\varepsilon}{2}}\ 2^{\frac{-k}{2}}\omega(\frac{1}{2^k})$, and $0<\varepsilon<1$. Then

$$\lim_{k\to\infty}\sup|a_k|2^{k(1-\frac{1}{2})}\frac{\log 2^k}{\omega(\frac{1}{2^k})}=\lim_{k\to\infty}\sup k^{\frac{(1-\varepsilon)}{2}}\log 2=\infty.$$

Hence, by Theorem 2.1, $f_4\notin\mathcal{B}^{\frac{1}{2}}_{\log,\omega}$. On the other hand,

$$\sum_{n=1}^\infty\frac{2^k|a_k|^2}{\omega(\frac{1}{2^k})}\sum_{s=1}^\infty\frac{(1-2n)_s(\frac{1}{2n})^s}{s!s}=\sum_{k=1}^\infty k^{-(1+\varepsilon)}\omega(\frac{1}{2^k})\sum_{s=1}^\infty\frac{(1-2n)_s(\frac{1}{2n})^s}{s!s}<\infty.$$

So $f\in\mathcal{D}_\omega$.

(ii) Let $f\in\|f\|_{\mathcal{B}^\alpha_{\omega,\log}}$, $\frac{1}{2}<\alpha\le 1$. Then

$$\|f\|_{\mathcal{B}^\alpha_{\omega,\log}}=\sup_{z\in D}|f'(z)|\frac{(1-|z|^2)^\alpha}{\omega(1-|z|)}\log\frac{1}{1-|z|}<\infty.$$

Let

$$\begin{aligned}I(a) &= \int_{\mathbf{D}}|f'(z)|^2\frac{(1-|\varphi_a(z)|^2)^{2\alpha-1}}{\omega(1-|z|)}\,dA(z)\\ &= \Big(\int_{\mathbf{D}_{\frac{1}{2}}}+\int_{\mathbf{D}\backslash\mathbf{D}_{\frac{1}{2}}}\Big)|f'(z)|^2\frac{(1-|\varphi_a(z)|^2)^{2\alpha-1}}{\omega(1-|z|)}dA(z)=I_1(a)+I_2(a).\end{aligned}$$

Because $2\alpha-1>0$, we have

$$\lim_{|a|\to 1^-}I_1(a)\le\lim_{|a|\to 1^-}M_\omega\int_{\mathbf{D}_{\frac{1}{2}}}(1-|\varphi_a(z)|^2)^{2\alpha-1}dA(z)=0.$$

On the other hand,

$$\begin{aligned} I_2(a) &\le \|f\|_{\mathcal{B}^\alpha_{\omega,\log}} \int_{\mathbf{D}\setminus\mathbf{D}_{\frac{1}{2}}} \frac{(1-|\varphi_a(z)|^2)^{2\alpha-1} dA(z)}{(1-|z|^2)^{2\alpha}(\log\frac{1}{1-|z|})^2} \\ &= \|f\|_{\mathcal{B}^\alpha_{\omega,\log}} (1-|a|^2)^{2\alpha-1} \int_{\mathbf{D}\setminus\mathbf{D}_{\frac{1}{2}}} \frac{dA(z)}{(1-|z|^2)(\log\frac{1}{1-|z|})^2 |1-\bar{a}z|^{4\alpha-2}} \\ &\le \|f\|_{\mathcal{B}^\alpha_{\omega,\log}} (1-|a|^2)^{2\alpha-1} \int_{\frac{1}{2}}^1 \frac{rdr}{(1-r)(\log\frac{1}{1-r})^2} \int_0^{2\pi} \frac{d\theta}{|1-\bar{a}re^{i\theta}|^{2(2\alpha-1)}} . \end{aligned}$$

By using Lemma 3.1 when $p = 2\alpha-1$, we obtain

$$\int_0^{2\pi} \frac{d\theta}{|1-\bar{a}re^{i\theta}|^{2(2\alpha-1)}} \le \frac{C}{(1-|a|r)^{2\alpha-1}} .$$

So

$$I_2(a) \le C\|f\|_{\mathcal{B}^\alpha_{\omega,\log}} (1-|a|^2)^{2\alpha-1} \int_{\frac{1}{2}}^1 \frac{rdr}{(1-r)(\log\frac{1}{1-r})^2 (1-|a|r)^{2\alpha-1}} .$$

Because

$$\int_{\frac{1}{2}}^1 \frac{dr}{(1-r)(\log\frac{1}{1-r})^2} = \int_{\frac{1}{2}}^\infty \frac{dt}{t^2} < \infty,$$

for every $\varepsilon > 0$, there is a δ, $\frac{1}{2} < \delta < 1$, such that

$$\int_\delta^1 \frac{dr}{(1-r)(\log\frac{1}{1-r})^2} < \varepsilon. \tag{20}$$

Now

$$\begin{aligned} I_2(a) &\le 2^{2\alpha-1} C\|f\|_{\mathcal{B}^\alpha_{\omega,\log}} \int_{\frac{1}{2}}^1 \Big(\frac{1-|a|}{(1-|a|r)}\Big)^{2\alpha-1} \frac{dr}{(1-r)(\log\frac{1}{1-r})^2} \\ &\le 2^{2\alpha-1} C\|f\|_{\mathcal{B}^\alpha_{\omega,\log}} \Big(\int_{\frac{1}{2}}^\delta + \int_\delta^1\Big)\Big(\frac{1-|a|}{(1-|a|r)}\Big)^{2\alpha-1} \frac{dr}{(1-r)(\log\frac{1}{1-r})^2} \\ &= 2^{2\alpha-1} C\|f\|_{\mathcal{B}^\alpha_{\omega,\log}} (J_1(a) + J_2(a)). \end{aligned}$$

Then by (20),

$$J_2(a) = \int_\delta^1 \Big(\frac{1-|a|}{(1-|a|r)}\Big)^{2\alpha-1} \frac{dr}{(1-r)(\log\frac{1}{1-r})^2} \le \int_\delta^1 \frac{dr}{(1-r)(\log\frac{1}{1-r})^2} < \varepsilon.$$

On the other hand,

$$J_1(a) = \int_{\frac{1}{2}}^\delta \Big(\frac{1-|a|}{(1-|a|r)}\Big)^{2\alpha-1} \frac{dr}{(1-r)(\log\frac{1}{1-r})^2} \le \frac{1}{\log 2}\Big(\frac{1-|a|}{(1-\delta|a|)}\Big)^{2\alpha-1}$$

Hence, if "a" is sufficiently close to 1, then $J_1(a) < \varepsilon$. Therefore,

$$I_2(a) \le 2^{2\alpha} C\|f\|_{\mathcal{B}^\alpha_{\omega,\log}} \varepsilon.$$

Thus $\lim\limits_{|a|\to 1^-} I_2(a) = 0$, and so

$$\lim_{|a|\to 1^-} I(a) = I_1(a) + I_2(a) = 0.$$

Thus $f \in Q_{2\alpha-1,0}$.

To prove the strictness, let $f_5(z) = \sum\limits_{k=1}^{\infty} a_k z^{2^k}$, where $a_k = k^{\frac{-1-\varepsilon}{2}} \; 2^{-k(1-\alpha)}\omega(\frac{1}{2^k})$, and $0 < \varepsilon < 1$. Then

$$\sum_{k=1}^{\infty} |a_k| 2^{k(1-(2\alpha-1))} = \sum_{k=1}^{\infty} k^{-(1+\varepsilon)} < \infty.$$

So $f_5 \in Q_{2\alpha-1,0}$. On the other hand,

$$\lim_{k\to\infty} \sup |a_k| 2^{k(1-\alpha)} \frac{\log 2^k}{\omega(\frac{1}{2^k})} = \lim_{k\to\infty} \sup k^{\frac{(1-\varepsilon)}{2}} \log 2 = \infty.$$

So, by Theorem 2.1 , $f_5 \notin \mathcal{B}^{\alpha}_{\omega,\log}$. The proof is complete.

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