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MODELS FOR THE BUS HEADWAY DISTRIBUTION IN THE FLOW BEHIND A TRAFFIC SIGNAL

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Abstract. *Several results concerning the distribution of the headway of buses in the flow behind a traffic signal are presented. In the main focus of interest is the description of analytical models, which are verified by the results of Monte-Carlo-Methods. The advantage of analytical models (verified, but not derived by simulation methods) is their flexibility with respect to possible generalizations. For instance, several random distributions of the flow incoming to the traffic signal can be compared. The attention will be directed at the question, how the primary headway H (analyzed in front of the traffic signal) is mapped to the headway H', analyzed behind of the traffic signal and how the random distribution of H is mapped to that of H'.*

1 INTRODUCTION

The aim of this paper is the derivation of results concerning the distribution of the headway of buses in the flow behind a traffic signal. In the main focus of interest is the description of analytical models, which are verified by the results of Monte-Carlo-Methods. Usually, simulation methods are applied solely to derive corresponding results. The advantage of analytical models is their flexibility with respect to possible generalizations. For instance, several random distributions of the flow incoming to the traffic signal can be compared.

Several models for the traffic behaviour of buses in front of the traffic signal are considered. The first case corresponds to a saturated traffic flow, where buses share one lane with the motorized individual traffic. In the second case, a separated bus lane is considered and finally a certain mixed situation is analyzed.

In the paper [6] some case of filtration process of a bus flow by fixed traffic signal was investigated. Entering bus headways has been described as Gamma distribution. Buses operate together with other vehicles at one lane only. The vehicle flow is saturated and its nature is deterministic with even headways. Linear mapping of the headway before traffic signal in the headway behind traffic signal has been considered. In the present paper, the results of former investigations have been verified and moreover the current analysis has been gone deeply and extended, including the case with separated bus lane.

2 GENERAL ASSUMPTIONS

We consider a traffic signal, whose random incoming bus flow is described by the following random model.

Assumption 2.1 The headway H of two incoming buses is assumed to follow a continuous random distribution with probability density $f_H(h)$.

Example 2.2 A very simply model could assume the exponential distribution for H:

$$f_H(h) = \lambda \exp(-\lambda h)\,,$$

where $\lambda > 0$ is a parameter describing the intensity of bus flow. The expected value of bus headway (average headway) and the variance of headways of the incoming flow are then given by

$$\mathbb{E}H = \frac{1}{\lambda} \quad \text{and} \quad \mathbb{D}^2 H = \frac{1}{\lambda^2}\,,$$

respectively. λ can be estimated from a given sample $(H^1, \ldots, H^n)$ of bus headways by

$$\widehat{\lambda} = \frac{1}{\overline{H}}$$

with

$$\overline{H} = \frac{1}{n} \sum_{i=1}^{n} H^i\,.$$

For suitable tests, whether the assumption of an exponential distribution is legitimate reference is made for instance to [2].

Example 2.3 A more realistic model for the headway H (c.f. recommendation in [3]) is the Gamma distribution. This kind of distribution in very relevant way expresses clashing of deterministic and random effects on vehicle flow in public transport operation. Gamma distribution for bus headways has been approved by numerous own investigations, c.f. [5]. Clearly, exponential distribution as described in Example 2.2 is a special case of Gamma distribution. The probability density of H is given by

$$f_H(h) = \frac{(ak)^k}{\Gamma(k)} h^{k-1} \exp(-akh)\,,$$

where $\Gamma(\cdot)$ denotes the Gamma function, the parameters a and k characterize the random character in the way, that the expected value of the bus headway and the variance of headways of the incoming flow are given by

$$\mathbb{E}H = \frac{1}{a} \quad \text{and} \quad \mathbb{D}^2 H = \frac{1}{a^2 k}\,, \quad \text{respectively.}$$

Consequently, these parameters can be estimated from a given sample $(H^1, \ldots, H^n)$ of bus headways by

$$\widehat{a} = \frac{1}{\overline{H}} \quad \text{and} \quad \widehat{k} = \frac{\overline{H}^2}{S^2}$$

with

$$\overline{H} = \frac{1}{n} \sum_{i=1}^{n} H^i \quad \text{and} \quad S^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left(H^i - \overline{H}\right)^2 .$$

Assumption 2.4 The traffic signal has a constant cycle length C and an effective green time G, the ratio of both times is denoted by r,

$$r = \frac{G}{C}\,.$$

Starting point of time t ($t = 0$) is the beginning of a red signal.

Assumption 2.5 The first bus occurs in the random time T of the first cycle, which is assumed to be uniformly distributed on the time interval $[0, C)$. The headway of the first two incoming buses in front of the traffic signal is denoted by H_1. It is assumed to be independent of T.

Remark 2.6 In order to consider more general situations for bus headways, especially the distribution of bus headways H_i, $i \geq 2$ between buses after the first two buses, it would be necessary to consider the distribution of the fraction of the departure time of the i-th bus with respect to the cycle length. However, uniform distribution can serve as a good approximation for the exact distribution also in these cases. On the other hand, there is no difficulty to relax Assumption 2.5 from the theoretical point of view, however, the calculations become much more complicated.

For the traffic flow in front of the traffic signal several models shall be discussed.

Assumption 2.7
Buses operate on one lane only, the traffic flow is saturated (c.f. Section 3),
– or alternatively –
Buses operate on a separated bus lane (c.f. Section 4),
– or alternatively –
Situations between these cases (c.f. Section 5).

Let us now investigate how the primary headway H_1 (analyzed in front of the traffic signal) is mapped to the headway H_1', analyzed behind of the traffic signal and how the random distribution of H_1 is mapped to that of H_1'.

3 BUSES OPERATE ON ONE LANE ONLY, THE TRAFFIC FLOW IS SATURATED

In the case of a saturated traffic flow, the first bus passes the traffic signal at the random moment

$$T_1 = C - G + rT\,.$$

The headway to the next bus is H_1 and there exists a certain cycle number M_1 with

$$M_1 C \le H_1 < (M_1 + 1)C\,.$$

Obviously, this random cycle number is nothing else than

$$M_1 = \left[\frac{H_1}{C}\right],$$

where the symbol $[\cdot]$ denotes the next nearest integer towards 0. Because of $0 \le T < C$ two cases are possible,

1. $M_1 C \le T + H_1 < (M_1 + 1)C$, in this case the departure of the second bus occurs in the moment

   $$T_2 = (M_1 + 1)(C - G) + r(T + H_1)\,.$$

   The headway H_1' in this case calculates as

   $$H_1' = T_2 - T_1 = M_1(C - G) + rH_1\,.$$

   For a given headway $H_1 = h$ the probability, that this case occurs can with the help of

   $$\left[\frac{h}{C}\right] C \le T + h < \left(\left[\frac{h}{C}\right] + 1\right) C$$
   $$\Leftrightarrow$$
   $$\left[\frac{h}{C}\right] - \frac{h}{C} \le \frac{T}{C} < \left[\frac{h}{C}\right] + 1 - \frac{h}{C}$$
   $$\Leftrightarrow$$
   $$0 \le \frac{T}{C} < \left[\frac{h}{C}\right] - \frac{h}{C} + 1$$

   easily be obtained by $\left[\frac{h}{C}\right] - \frac{h}{C} + 1\,.$

2. $(M_1 + 1)C \le T + H_1 < (M_1 + 2)C$, in this case the departure of the second bus occurs in the moment

   $$T_2 = (M_1 + 2)(C - G) + r(T + H_1)\,.$$

   The headway H_1' in this case calculates as

   $$H_1' = T_2 - T_1 = (M_1 + 1)(C - G) + rH_1\,.$$

Clearly, for given headway $H_1 = h$ that this case occurs with probability

$$1 - \left(\left[\frac{h}{C} \right] - \frac{h}{C} + 1 \right) = \frac{h}{C} - \left[\frac{h}{C} \right] .$$

The distribution function $F_{H_1'}$ of H_1' can be calculated in a straightforward manner. It holds

$$\begin{aligned} F_{H_1'}(x) = \mathbb{P}(H_1' \leq x) = & \int_0^{kC} f_{H_1}(h)\, dh \\ & + \mathbb{1}_{\{kC \leq x < kC+G\}} \int_{kC}^{\frac{1}{r}(x-k(C-G))} \left(k + 1 - \frac{h}{C} \right) f_{H_1}(h)\, dh \\ & + \mathbb{1}_{\{kC+G \leq x < (k+1)C-G\}} \int_{kC}^{(k+1)C} \left(k + 1 - \frac{h}{C} \right) f_{H_1}(h)\, dh \\ & + \mathbb{1}_{\{(k+1)C-G \leq x < (k+1)C\}} \left\{ \int_{kC}^{\frac{1}{r}(x-((k+1)(C-G))} f_{H_1}(h)\, dh \right. \\ & \left. + \int_{\frac{1}{r}(x-((k+1)(C-G))}^{(k+1)C} \left(k + 1 - \frac{h}{C} \right) f_{H_1}(h)\, dh \right\} , \end{aligned}$$

where the notation $k = \left[\frac{x}{C}\right]$ has been used. $F_{H_1'}(x)$ turns out to be absolutely continuous, the corresponding probability density function $f_{H_1'}(x)$ can be obtained by differentiation with respect to x. It is given by

$$\begin{aligned} f_{H_1'}(x) = \frac{1}{r} \Bigg[& \mathbb{1}_{\{kC \leq x < kC+G\}} \left(k + 1 - \frac{1}{G} (x - k(C-G)) \right) f_{H_1} \left(\frac{1}{r} (x - k(C-G)) \right) \\ & + \mathbb{1}_{\{(k+1)C-G \leq x < (k+1)C\}} \left(\frac{1}{G} (x - (k+1)(C-G)) - k \right) * \\ & * f_{H_1} \left(\frac{1}{r} (x - (k+1)(C-G)) \right) \Bigg] . \end{aligned}$$

For a detailed derivation of these formulas reference is made to Appendix A.

Example 3.1 In case of exponential distribution of H_1' (c.f. Example 2.2) the density $f_{H_1'}(x)$ is given by

$$\begin{aligned} f_{H_1'}(x) = \frac{\lambda}{r} \exp\left(-\frac{\lambda}{r} x \right) \Bigg[& \mathbb{1}_{\{kC \leq x < kC+G\}} \left(k + 1 - \frac{1}{G} (x - k(C-G)) \right) * \\ & * \exp\left(\frac{\lambda}{r} k(C-G) \right) \\ & + \mathbb{1}_{\{(k+1)C-G \leq x < (k+1)C\}} \left(\frac{1}{G} (x - (k+1)(C-G)) - k \right) * \\ & * \exp\left(\frac{\lambda}{r} (k+1)(C-G) \right) \Bigg] . \end{aligned}$$

Example 3.2 Gamma distribution (c.f. Example 2.3) is a more realistic way to describe the headways in the incoming flow. Fitting of empirical data to this model was made in [5] for a narrow (one lane) street crossing of railway. Undercrossing is regulated by a traffic signal with $C = 90$ s $= 1.5$ min and $G = 40$ s $= 2/3$ min, which obviously corresponds to the considered case $G \leq C/2$. Average headway was $\overline{h} = 1.75$ min, consequently the parameter a of Gamma distribution was estimated with $a = 1/1.75$. Moreover, the second parameter of Gamma distribution was estimated as $k = 1.33$. The analytic expression of the density $f_{H_1'}(x)$ derived by the above formula reads as

$$f_{H_1'}(x) = \begin{cases} 2.28(1-1.5x)x^{0.33}\exp(-1.71x) & 0 \leq x < 0.67 \\ 1.75(-1.25+1.5x)(2.25x-1.88)^{0.33}\exp(-1.71x+1.43) & 0.83 \leq x < 1.5 \\ 1.75(3.25-1.5x)(2.25x-1.88)^{0.33}\exp(-1.71x+1.43) & 1.5 \leq x < 2.17 \\ 1.75(-3.5+1.5x)(2.25x-3.75)^{0.33}\exp(-1.71x+2.85) & 2.33 \leq x < 3 \\ 1.75(5.5-1.5x)(2.25x-3.75)^{0.33}\exp(-1.71x+2.85) & 3 \leq x < 3.67 \\ \ldots & \end{cases}$$

Figure 1 shows the density of H_1' as well as the density of H_1 (Gamma distribution) of headways behind and in front of the traffic signal calculated with the above equation in comparison with simulation results.

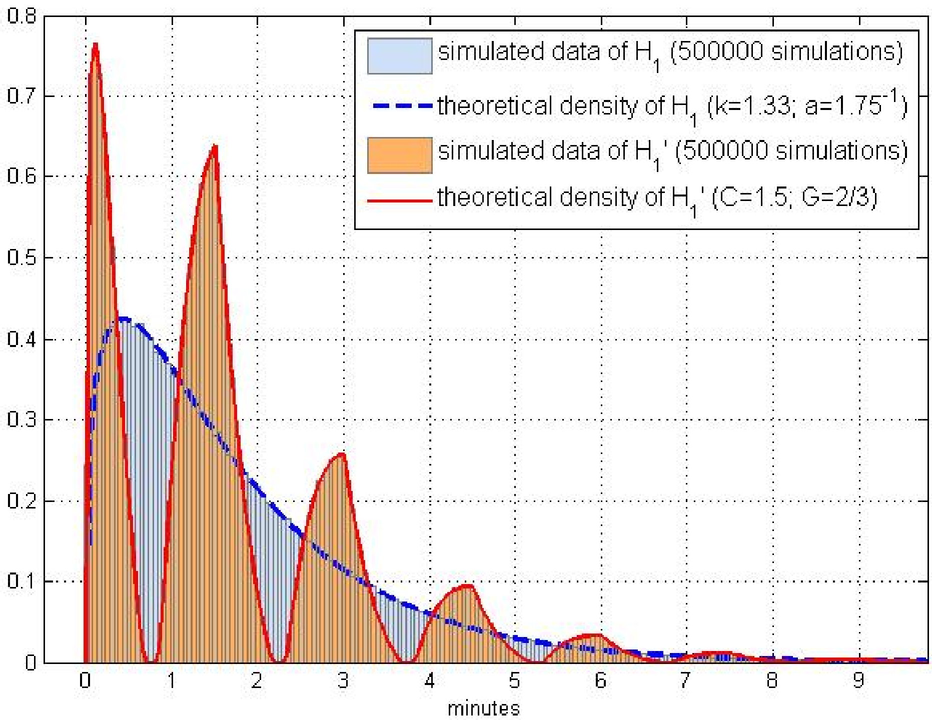


Figure 1: Results of Example 3.2

In [6] corresponding results are compared with empirical data obtained by extensive measurements and verified by suitable statistical tests.

Example 3.3 Using the same parameters as in Example 3.2, but setting $G = 50$ s $= 0.83$ min instead of $G = 40$ s, Figure 2 shows the density of H_1' as well as the density of H_1 (Gamma distribution) of headways behind and in front of the traffic signal calculated with the above equation in comparison with simulation results. It should be noted, that in the case $G > C/2$ in general there are no areas where $f_{H_1'}(x)$ vanishes.

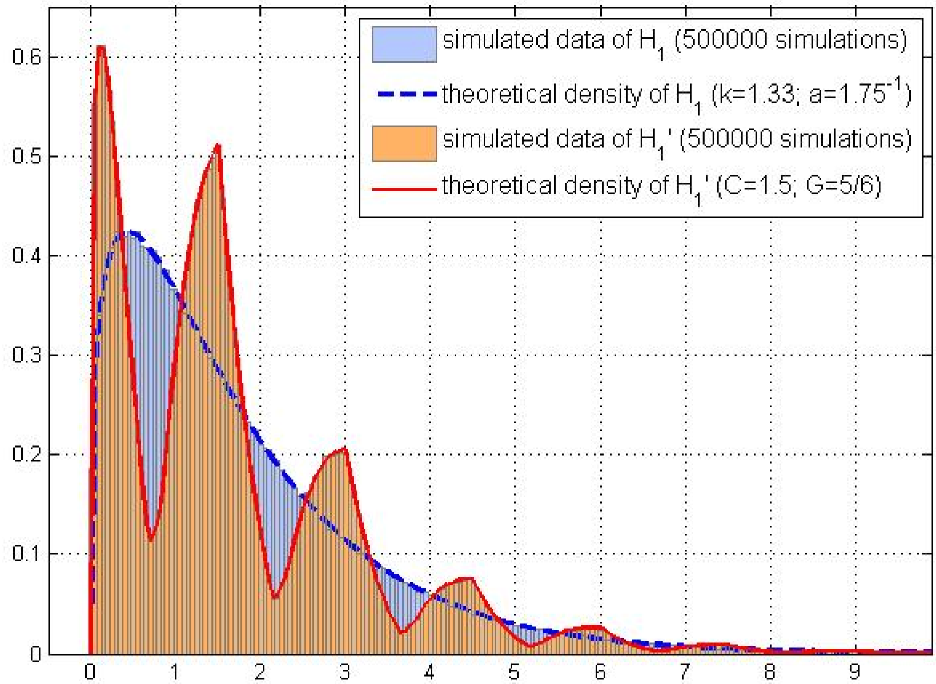


Figure 2: Results of Example 3.3

4 BUSES OPERATE ON A SEPARATED BUS LANE

In the case of a separated bus lane, the first bus passes the traffic signal as soon as the green signal appears (or in case that the bus arrives during the green time, immediately), that means at the random time

$$T_1 = \max\{C - G, T\} \ .$$

As in Section 3, the headway to the next bus is H_1 and there exists a certain cycle number M_1 with

$$M_1 C \leq H_1 < (M_1 + 1)C \, ,$$

calculated by

$$M_1 = \left[\frac{H_1}{C} \right] \ .$$

Again two cases are possible,

1. $M_1 C \le T + H_1 < (M_1+1)C$, in this case the departure of the second bus occurs in the moment
$$T_2 = \max\{(M_1+1)C - G, T + H_1\}\,.$$
2. $(M_1+1)C \le T + H_1 < (M_1+2)C$, here it holds
$$T_2 = \max\{(M_1+2)C - G, T + H_1\}\,.$$

In difference to the model in Section 3, here in both cases the random variable T occurs in the formula for $H_1' = T_2 - T_1$, which makes the calculations much more difficult. The detailed derivation of the distribution function $F_{H_1'}$ can be found in Appendix B. With $k = \left[\frac{x}{C}\right]$ it holds

$$\begin{aligned}
F_{H_1'}(x) = \mathbb{P}(H_1' \le x) = &\int_0^{kC} f_{H_1}(h)\,dh \\
&+ \mathbb{1}_{\{kC \le x < kC+G\}} \left\{ \int_{kC}^{(k+1)C-G} \left(k+1-\frac{G}{C}-\frac{h}{C}\right) f_{H_1}(h)\,dh \right. \\
&\quad + \int_{kC}^{x} \frac{G}{C} f_{H_1}(h)\,dh + \int_{x}^{(k+1)C-G} \left(\frac{x}{C}-k\right) f_{H_1}(h)\,dh \\
&\quad \left. + \int_{(k+1)C-G}^{x+C-G} \left(\frac{x}{C}+1-\frac{G}{C}-\frac{h}{C}\right) f_{H_1}(h)\,dh \right\} \\
&+ \mathbb{1}_{\{kC+G \le x < (k+1)C-G\}} \int_{kC}^{(k+1)C} \left(k+1-\frac{h}{C}\right) f_{H_1}(h)\,dh \\
&+ \mathbb{1}_{\{(k+1)C-G \le x < (k+1)C\}} \left\{ \int_{kC}^{(k+1)C} \left(k+1-\frac{h}{C}\right) f_{H_1}(h)\,dh \right. \\
&\quad \left. + \int_{kC}^{G-C+x} \left(\frac{h}{C}-k\right) f_{H_1}(h)\,dh + \int_{G-C+x}^{x} \left(\frac{G}{C}+\frac{x}{C}-(k+1)\right) f_{H_1}(h)\,dh \right\}.
\end{aligned}$$

A deeper analysis of this formula shows, that in the situation of this section the headway H_1' has no continuous distribution, there are discrete parts (namely jumps in the points kC, $k = 0, 1, \ldots$). Therefore it is not possible to specify a distribution density function.

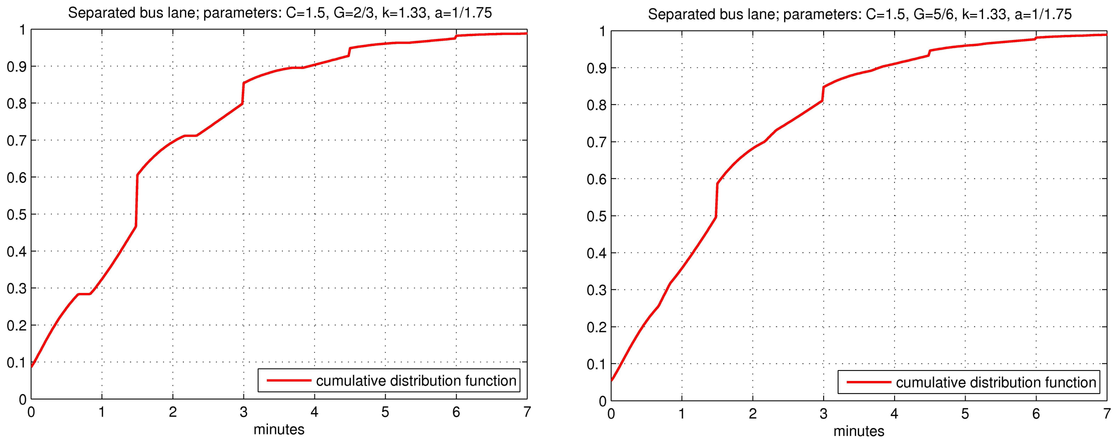
Example 4.1 Using the parameters as in Example 3.2 and Example 3.3, Figure 3 shows the cumulative distribution functions of H_1' in these two cases.

From Figure 3 it can be seen for instance, that the case $H_1' = 0$ occurs with a positive probability.

5 MIXED MODEL

In this section, the models for the traffic flow presented in Section 3 and 4 are combined in the following way. The first bus passes the traffic signal at the random moment

$$T_1 = \max\left\{C - G, \frac{G}{C-W}T + C\left(1 - \frac{G}{C-W}\right)\right\}.$$



Case 1: $G \leq \frac{C}{2}$ Case 2: $G > \frac{C}{2}$

Figure 3: Cumulative distribution functions of Example 4.1

Here, additionally to the variables introduced in the sections above a variable W is considered, which connects the models of a saturated traffic flow and a separated bus lane. $W \in [0, C - G]$ can be either a fixed quantity or alternatively a uniformly distributed random variable on $[0, C - G)$, which should be independent of T and H in that case. Figure 4 shows the function T_1 in dependency on T.

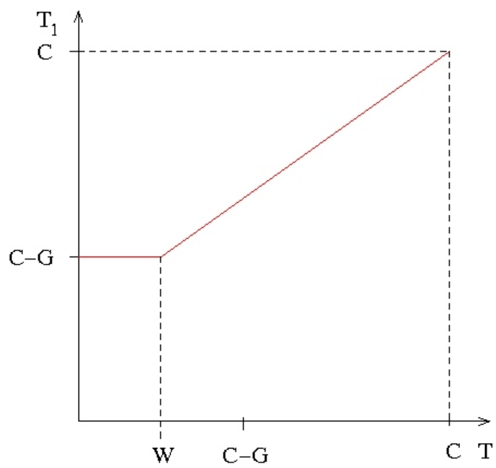


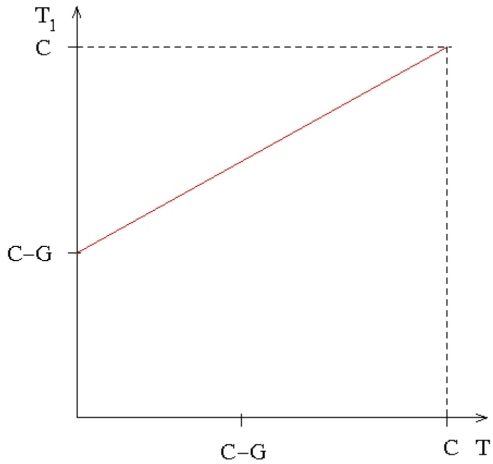
Figure 4: Mixed Model

Remark 5.1 Obviously, the cases considered above are special cases of the mixed model. The case of a saturated traffic flow can be obtained by setting $W \equiv 0$. The case of a separated bus lane corresponds to $W \equiv C - G$. Figure 5 shows T_1 as function of T in these cases.

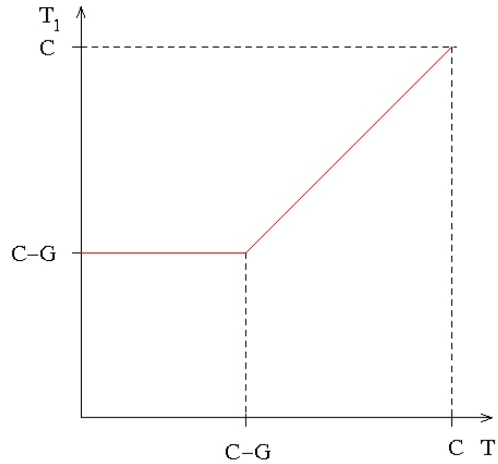
M_1 denotes again the cycle number with $M_1 C \leq H_1 < (M_1 + 1)C$. In order to get the expression for the headway behind the traffic signal again we have to consider to cases:

1. $M_1 C \leq T + H_1 < (M_1 + 1)C$; in this case the departure of the second bus occurs in the moment

$$T_2 = \max\left\{(M_1 + 1)C - G, \frac{G}{C - W}(T + H_1) + (M_1 + 1)C\left(1 - \frac{G}{C - W}\right)\right\}$$



(a) Model of saturated traffic flow



(b) Model of separated bus lane

Figure 5: Models of Section 3 and 4 as special case of the mixed model

and we can calculate the headway

$$\begin{aligned} H_1' &= T_2 - T_1 \\ &= \max\left\{(M_1+1)C - G, \frac{G}{C-W}(T+H_1) + (M_1+1)C\left(1-\frac{G}{C-W}\right)\right\} \\ &\quad - \max\left\{C-G, \frac{G}{C-W}T + C\left(1-\frac{G}{C-W}\right)\right\} \\ &= M_1C + \max\left\{-G, \frac{G}{C-W}\left(T+H_1-(M_1+1)C\right)\right\} \\ &\quad - \max\left\{-G, \frac{G}{C-W}\left(T-C\right)\right\} . \end{aligned}$$

2. $(M_1+1)C \le T+H_1 < (M_1+2)C$; in this case the departure of the second bus occurs in the moment

$$T_2 = \max\left\{(M_1+2)C - G, \frac{G}{C-W}(T+H_1) + (M_1+2)C\left(1-\frac{G}{C-W}\right)\right\}$$

and we can calculate the headway

$$\begin{aligned} H_1' &= T_2 - T_1 \\ &= \max\left\{(M_1+2)C - G, \frac{G}{C-W}(T+H_1) + (M_1+2)C\left(1-\frac{G}{C-W}\right)\right\} \\ &\quad - \max\left\{C-G, \frac{G}{C-W}T + C\left(1-\frac{G}{C-W}\right)\right\} \\ &= (M_1+1)C + \max\left\{-G, \frac{G}{C-W}\left(T+H_1-(M_1+2)C\right)\right\} \\ &\quad - \max\left\{-G, \frac{G}{C-W}\left(T-C\right)\right\} . \end{aligned}$$

The distribution function of H_1' in the case of a uniformly distributed random variable W is given by

$$\begin{aligned}
F_{H_1'}(x) &= \mathbb{P}(H_1' \leq x) \\
&= \int_0^\infty \int_0^C \int_0^{C-G} \mathbb{P}(H_1' \leq x | H_1 = h, T = t, W = w) f_{H_1}(h) f_T(t) f_W(w) dw\, dt\, dh \\
&= \frac{1}{C(C-G)} \int_0^\infty \int_0^C \int_0^{C-G} \mathbb{P}(H_1' \leq x | H_1 = h, T = t, W = w) f_{H_1}(h) dw\, dt\, dh \\
&= \frac{1}{C(C-G)} \sum_k \int_{kC}^{(k+1)C} \int_0^C \int_0^{C-G} \mathbb{P}(H_1' \leq x | H_1 = h, T = t, W = w) f_{H_1}(h) dw\, dt\, dh.
\end{aligned}$$

Finally for an explicit calculation we consider the case of a deterministic variable $W = w$ with $w \in [0, C - G]$ in the case $G \leq \frac{C}{2}$. In this case we derive with the help of a similar calculation as in the above situations the following distribution function of H_1'. The derivation of this result is available from the authors on request.

$$\begin{aligned}
F_{H_1'}(x) = \mathbb{P}(H_1' \leq x) = &\int_0^{kC} f_{H_1}(h)\, dh \\
&+ \mathbb{1}_{\{kC \leq x < kC+G\}} \Bigg\{ \int_{kC}^{kC+w} \left(\frac{w}{C} + k - \frac{h}{C} \right) f_{H_1}(h)\, dh \\
&\quad + \int_{kC+w}^{x+w} \left(\frac{x}{C} - \frac{G}{C-w} \frac{h}{C} - k - \frac{G}{C} + \frac{(k+1)G}{C-w} \right) \frac{C-w}{G} f_{H_1}(h)\, dh \\
&\quad + \int_x^{kC+w} \left(\left(\frac{x}{C} - k - \frac{G}{C} \right) \frac{C-w}{G} - \frac{w}{C} + 1 \right) f_{H_1}(h)\, dh \\
&\quad + \int_{kC}^{x} \left(\frac{h}{C} - k \right) f_{H_1}(h)\, dh + \int_{kC}^{\left(x - k\left(C - \frac{CG}{C-w}\right)\frac{C-w}{G}\right)} \left(k + 1 - \frac{h}{C} - \frac{w}{C} \right) f_{H_1}(h)\, dh \Bigg\} \\
&+ \mathbb{1}_{\{kC+G \leq x < (k+1)C-G\}} \Bigg\{ \int_{kC}^{kC+w} \left(\frac{w}{C} + k - \frac{h}{C} \right) f_{H_1}(h)\, dh \\
&\quad + \int_{kC+w}^{kC+G+w} \left(k + 1 - \frac{h}{C} \right) f_{H_1}(h)\, dh + \int_{kC+G}^{kC+w} \left(1 - \frac{w}{C} \right) f_{H_1}(h)\, dh \\
&\quad + \int_{kC}^{kC+G} \left(\frac{h}{C} - k \right) f_{H_1}(h)\, dh + \int_{kC}^{(k+1)C-w} \left(k + 1 - \frac{h}{C} - \frac{w}{C} \right) f_{H_1}(h)\, dh \Bigg\} \\
&+ \mathbb{1}_{\{(k+1)C-G \leq x < (k+1)C\}} \Bigg\{ \int_{kC}^{kC+w} \left(\frac{w}{C} + k - \frac{h}{C} \right) f_{H_1}(h)\, dh \\
&\quad + \int_{kC+w}^{kC+G+w} \left(k + 1 - \frac{h}{C} \right) f_{H_1}(h)\, dh + \int_{kC+G}^{kC+w} \left(1 - \frac{w}{C} \right) f_{H_1}(h)\, dh \\
&\quad + \int_{kC}^{kC+G} \left(\frac{h}{C} - k \right) f_{H_1}(h)\, dh + \int_{kC}^{(k+1)C-w} \left(k + 1 - \frac{h}{C} - \frac{w}{C} \right) f_{H_1}(h)\, dh
\end{aligned}$$

$$+ \int_{kC}^{(x-(k+1)C+G)\frac{C-w}{G}+kC} \left(\frac{h}{C} - k\right) f_{H_1}(h)\, dh$$

$$+ \int_{(x-(k+1)C+G)\frac{C-w}{G}+kC}^{kC+w} (G + x - (k+1)C)\, \frac{C-w}{CG} f_{H_1}(h)\, dh$$

$$+ \int_{kC+w}^{(x-(k+1)C)\frac{C-w}{G}+(k+1)C} \left(k + \frac{w}{C} - \frac{h}{C} - ((k+1)C - G - x)\, \frac{C-w}{CG}\right) f_{H_1}(h)\, dh$$

$$+ \int_{kC+w}^{(x-(k+1)C)\frac{C-w}{G}+(k+1)C} \left(\frac{h}{C} - \frac{w}{C} - k\right) f_{H_1}(h)\, dh \Bigg\}$$

Again it is set $k = \left[\frac{x}{C}\right]$. In the case $w = 0$ this distribution function coincides with the result of the model of a saturated traffic flow and in the case $w = C - G$ we get the result of the model of a separated bus lane.

6 CONCLUSIONS

In the previous sections, several models for the traffic behaviour of buses were considered. Continuing the considerations in [6], the main focus was on an analytic description of the transformation of the headway of buses in front of a traffic signal into that behind the signal.

Figure 6 shows the comparison of the empirical cumulative distribution functions in the three considered models on the basis of the parameters chosen in Example 3.2 and 4.1, respectively (case $G = 40$ s $= 2/3$ min).

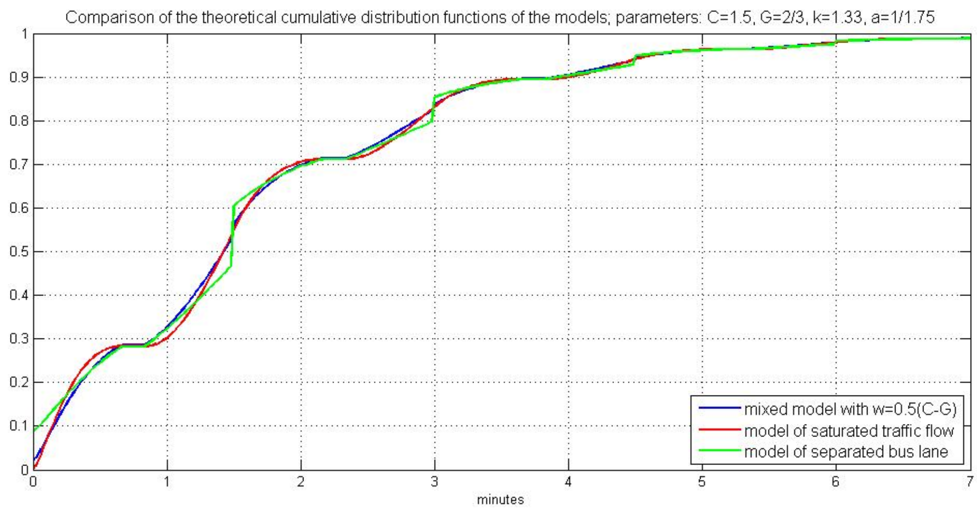


Figure 6: Comparison of the empirical cumulative distribution functions of the models

For the sake of clarity in Figure 7 a zoom in the time interval $[0, 1.6]$ is shown.

All considered situations show similarities, the principal shape of the distributions is relatively unaffected of the traffic behaviour in front of the traffic signal. Nevertheless, there are differences (for instance a positive probability for selective headways in the model of a separated bus line and in the mixed model).

The considerations open a lot of perspectives with respect to generalizations of the input distributions as well as for solving optimization tasks (for instance with respect to the parameters

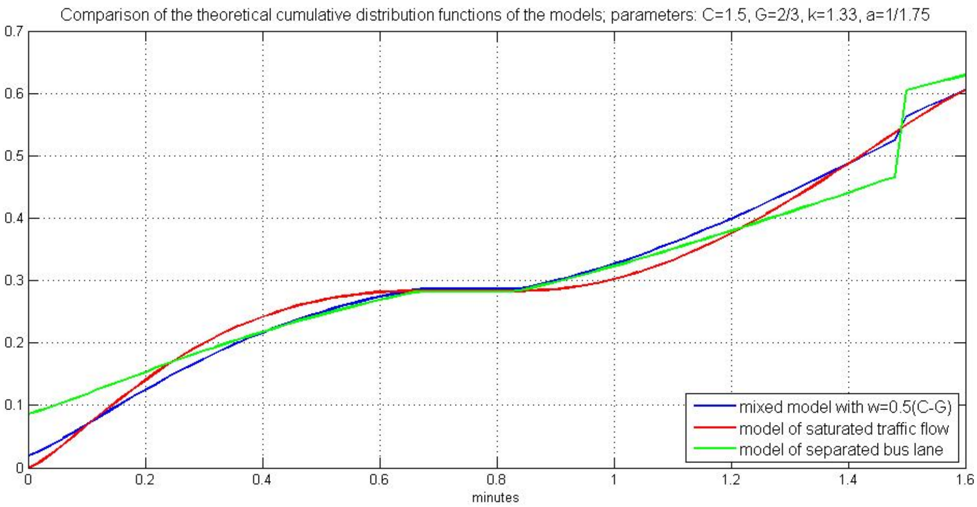


Figure 7: Comparison of the empirical cumulative distribution functions of the models (zoomed)

of the traffic signal). Further research has to be done with respect to the generalization of the assumption of uniform distribution of the variable T (and therefore the exact description of the distribution of all headways). Better representation of vehicle flow (e.g. to consider Hyperlang probability distribution as generalized traffic headway model [1]) and cases of undersaturated and oversaturated flow may be well to take into account. Likely, an analytical solution would be difficult obtain. On the other hand, using mixed methods (analytical and simulation, e.g. VISSIM software [4]) it could be possible to gain effective and realistic solutions. VISSIM applies the vehicle following behaviour model as improved version of Wiedemann's 1999 following model. Finally, an iterative process would be of interest, where the output distribution could serve as input distribution for the next traffic signal. Moreover, a comprehensive empirical study should be done.

APPENDIX

A DERIVATION OF THE RESULTS OF SECTION 3

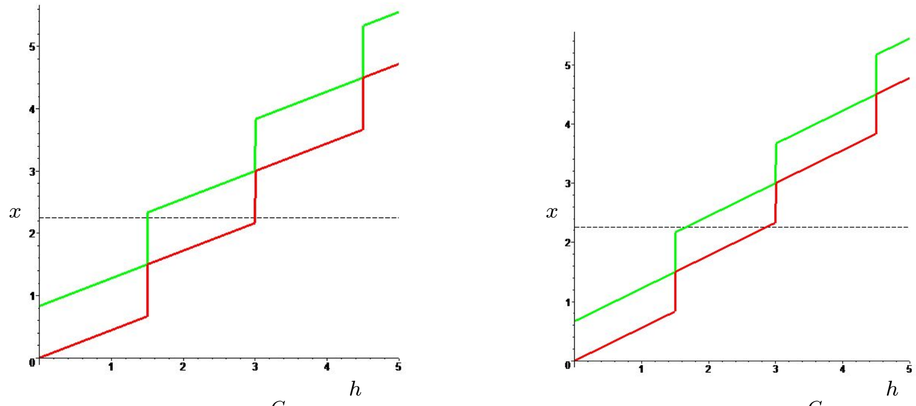
There are several ways for obtaining the random distribution of H_1'. We choose the method of the calculation of the distribution function $F_{H_1'}$ and the subsequent differentiation to obtain the probability density $f_{H_1'}$. From the considerations in Section 3 it follows for $x \in [0, \infty)$

$$\begin{aligned}&\mathbb{P}(H_1' \le x | H_1 = h) \\ &= \begin{cases} 0 & x < \left[\frac{h}{C}\right](C-G) + rh \\ \left[\frac{h}{C}\right] - \frac{h}{C} + 1 & \left[\frac{h}{C}\right](C-G) + rh \le x < \left(\left[\frac{h}{C}\right] + 1\right)(C-G) + rh \\ 1 & x \ge \left(\left[\frac{h}{C}\right] + 1\right)(C-G) + rh\,. \end{cases}\end{aligned}$$

By the law of total probability it follows

$$F_{H_1'}(x) = \mathbb{P}(H_1' \le x) = \int_0^\infty \mathbb{P}(H_1' \le x | H_1 = h) f_{H_1}(h)\, dh\,.$$

To perform this integration, it is necessary to distinguish two cases: $G \le C/2$ and $G > C/2$. In Figure 8 the domain of integration is illustrated in both cases. For fixed x, one has to integrate between 0 and the green line (integrand $f_{H_1}(h)$) as well as between the green and the red line (integrand $\left(\left[\frac{h}{C}\right] - \frac{h}{C} + 1\right) f_{H_1}(h)$). Between the red line and ∞ the integrand equals 0.



Case 1: $G \le \frac{C}{2}$ Case 2: $G > \frac{C}{2}$

Figure 8: Domain of integration

Starting with the first case ($G \le C/2$) and introducing the notation $k = \left[\frac{x}{C}\right]$ it follows

$$\begin{aligned}\mathbb{P}(H_1' \le x) &= \int_0^{kC} f_{H_1}(h)\,dh \\ &+ \mathbf{1}_{\{kC \le x < kC+G\}} \int_{kC}^{\frac{1}{r}(x-k(C-G))} \left(k+1-\frac{h}{C}\right) f_{H_1}(h)\,dh \\ &+ \mathbf{1}_{\{kC+G \le x < (k+1)C-G\}} \int_{kC}^{(k+1)C} \left(k+1-\frac{h}{C}\right) f_{H_1}(h)\,dh \\ &+ \mathbf{1}_{\{(k+1)C-G \le x < (k+1)C\}} \left\{ \int_{kC}^{\frac{1}{r}(x-((k+1)(C-G))} f_{H_1}(h)\,dh \right. \\ &\qquad \left. + \int_{\frac{1}{r}(x-((k+1)(C-G))}^{(k+1)C} \left(k+1-\frac{h}{C}\right) f_{H_1}(h)\,dh \right\}.\end{aligned}$$

Differentiation with respect to x gives the probability density function $f_{H_1'}(x)$,

$$\begin{aligned}f_{H_1'}(x) = \frac{1}{r}\Bigg[&\mathbf{1}_{\{kC \le x < kC+G\}} \left(k+1-\frac{1}{G}(x-k(C-G))\right) f_{H_1}\left(\frac{1}{r}(x-k(C-G))\right) \\ &+ \mathbf{1}_{\{(k+1)C-G \le x < (k+1)C\}} \left\{ f_{H_1}\left(\frac{1}{r}(x-(k+1)(C-G))\right) \right. \\ &\left. - \left(k+1-\frac{1}{G}(x-(k+1)(C-G))\right) f_{H_1}\left(\frac{1}{r}(x-(k+1)(C-G))\right) \right\}\Bigg],\end{aligned}$$

which results in the formula given in Section 3.

The second case, $G > C/2$ can be considered analogously. For splitting up the domain of integration in

$$\int_0^{\infty} \mathbb{P}(H_1' \le x | H_1 = h) f_{H_1}(h)\,dh$$

see the right part of Figure 8. It follows

$$\begin{aligned}
\mathbb{P}(H_1' \leq x) &= \int_0^{kC} f_{H_1}(h)\,dh \\
&+ \mathbf{1}_{\{kC \leq x < kC+C-G\}} \int_{kC}^{\frac{1}{r}(x-k(C-G))} \left(k+1-\frac{h}{C}\right) f_{H_1}(h)\,dh \\
&+ \mathbf{1}_{\{kC+C-G \leq x < kC+G\}} \left\{ \int_{\frac{1}{r}(x-(k+1)(C-G))}^{\frac{1}{r}(x-k(C-G))} \left(k+1-\frac{h}{C}\right) f_{H_1}(h)\,dh \right. \\
&\qquad \left. + \int_{kC}^{\frac{1}{r}(x-(k+1)(C-G))} f_{H_1}(h)\,dh \right\} \\
&+ \mathbf{1}_{\{kC+G \leq x < (k+1)C\}} \left\{ \int_{\frac{1}{r}(x-(k+1)(C-G))}^{(k+1)C} \left(k+1-\frac{h}{C}\right) f_{H_1}(h)\,dh \right. \\
&\qquad \left. + \int_{kC}^{\frac{1}{r}(x-(k+1)(C-G))} f_{H_1}(h)\,dh\,. \right\}
\end{aligned}$$

It can be seen easily, that differentiation with respect to x gives the same result for the probability density function $f_{H_1'}(x)$ as before, however it should be noted, that the indicator functions in the case $G > C/2$ do overlap, there are no areas where $f_{H_1'}(x)$ vanishes.

B DERIVATION OF THE RESULTS OF SECTION 4

Because of the dependence on T of

$$H_1' = T_2 - T_1\,,$$

which reads as

$$H_1' = \max\{(M_1+1)C - G, T + H_1\} - \max\{C - G, T\}$$

in the first case considered in Section 4 and

$$H_1' = \max\{(M_1+2)C - G, T + H_1\} - \max\{C - G, T\}$$

in the second case, respectively, the law of total probability has to be used in the form

$$F_{H_1'}(x) = \mathbb{P}(H_1' \leq x) = \int_0^\infty \int_0^\infty \mathbb{P}(H_1' \leq x | H_1 = h, T = t) f_T(t) f_{H_1}(h)\,dt\,dh\,,$$

where

$$f_T(t) = \begin{cases} \frac{1}{C} & 0 \leq t \leq C \\ 0 & \text{else} \end{cases}$$

is the density of the uniformly distributed variable T. For the sake of simplicity here only the case $G \leq C/2$ is considered, however as in Section 3 the result will hold in general.

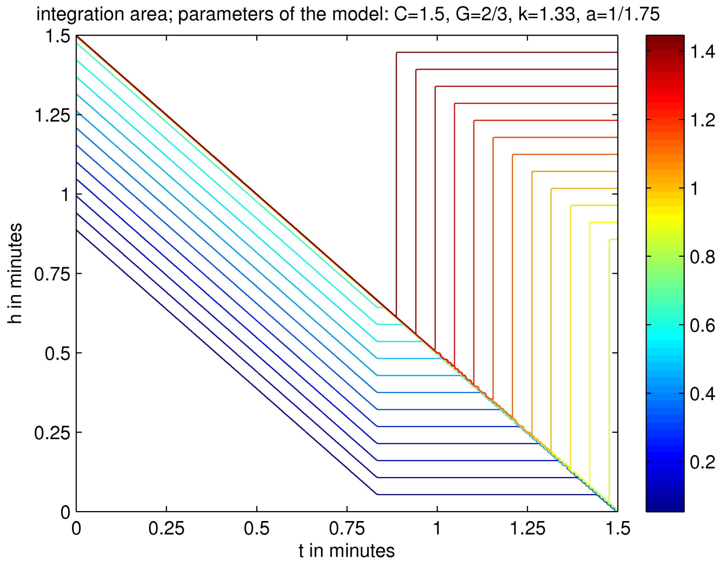


Figure 9: Domain of integration and contour lines of H_1'

Figure 9 shows the domain of integration for the above integral. The abscissa stands for the variable t, at the ordinate the variable h is shown. It should be noted, that a periodic behaviour of h (period length C) occurs, which is suppressed in the figure.

For fixed $H_1 = h$ and $T = t \in [0, C)$ the value of H_1' calculates as

$$H_1' = \begin{cases} \left[\frac{h}{C}\right] C & h < \left(\left[\frac{h}{C}\right]+1\right) C - G - t \\ h & t \geq C - G\,,\; h < \left(\left[\frac{h}{C}\right]+1\right) C - t \\ t + h - C + G & t < C - G\,,\; \left(\left[\frac{h}{C}\right]+1\right) C - G - t \leq h < \left(\left[\frac{h}{C}\right]+1\right) C - t \\ \left(\left[\frac{h}{C}\right]+1\right) C & t < C - G\,,\; h \geq \left(\left[\frac{h}{C}\right]+1\right) C - t \\ \left(\left[\frac{h}{C}\right]+2\right) C - G - t & t \geq C - G\,,\; \left(\left[\frac{h}{C}\right]+1\right) C - t \leq h < \left(\left[\frac{h}{C}\right]+2\right) C - G - t \\ h & h \geq \left(\left[\frac{h}{C}\right]+2\right) C - G - t\,. \end{cases}$$

It follows with the notation $k = \left[\frac{x}{C}\right]$

$$\begin{aligned} F_{H_1'}(x) = \mathbb{P}(H_1' \leq x) = & \int_0^{kC} f_{H_1}(h)\,dh \\ & + \mathbb{1}_{\{kC \leq x < kC+G\}} \left\{ \int_{kC}^{(k+1)C-G} \int_0^{(k+1)C-G-h} \frac{1}{C} f_{H_1}(h)\,dt\,dh \right. \\ & \quad + \int_{kC}^{x} \int_{C-G}^{(k+1)C-h} \frac{1}{C} f_{H_1}(h)\,dt\,dh + \int_{kC}^{x} \int_{(k+1)C-G-h}^{C-G} \frac{1}{C} f_{H_1}(h)\,dt\,dh \\ & \quad \left. + \int_{x}^{(k+1)C-G} \int_{(k+1)C-G-h}^{x+C-G-h} \frac{1}{C} f_{H_1}(h)\,dt\,dh + \int_{(k+1)C-G}^{x+C-G} \int_0^{x+C-G-h} \frac{1}{C} f_{H_1}(h)\,dt\,dh \right\} \\ & + \mathbb{1}_{\{kC+G \leq x < (k+1)C-G\}} \left\{ \int_{kC}^{(k+1)C-G} \int_0^{(k+1)C-G-h} \frac{1}{C} f_{H_1}(h)\,dt\,dh \right. \end{aligned}$$

$$+\int_{kC}^{kC+G}\int_{C-G}^{(k+1)C-h}\frac{1}{C}\,f_{H_1}(h)\,dt\,dh+\int_{kC}^{kC+G}\int_{(k+1)C-G-h}^{C-G}\frac{1}{C}\,f_{H_1}(h)\,dt\,dh$$

$$+\int_{kC+G}^{(k+1)C-G}\int_{(k+1)C-G-h}^{(k+1)C-h}\frac{1}{C}\,f_{H_1}(h)\,dt\,dh+\int_{(k+1)C-G}^{(k+1)C}\int_{0}^{(k+1)C-h}\frac{1}{C}\,f_{H_1}(h)\,dt\,dh\Bigg\}$$

$$+\mathbb{1}_{\{(k+1)C-G\le x<(k+1)C\}}\Bigg\{\int_{kC}^{(k+1)C-G}\int_{0}^{(k+1)C-G-h}\frac{1}{C}\,f_{H_1}(h)\,dt\,dh$$

$$+\int_{kC}^{kC+G}\int_{C-G}^{(k+1)C-h}\frac{1}{C}\,f_{H_1}(h)\,dt\,dh+\int_{kC}^{kC+G}\int_{(k+1)C-G-h}^{C-G}\frac{1}{C}\,f_{H_1}(h)\,dt\,dh$$

$$+\int_{kC+G}^{(k+1)C-G}\int_{(k+1)C-G-h}^{(k+1)C-h}\frac{1}{C}\,f_{H_1}(h)\,dt\,dh+\int_{(k+1)C-G}^{(k+1)C}\int_{0}^{(k+1)C-h}\frac{1}{C}\,f_{H_1}(h)\,dt\,dh$$

$$+\int_{kC}^{G-C+x}\int_{(k+1)C-h}^{C}\frac{1}{C}\,f_{H_1}(h)\,dt\,dh+\int_{G-C+x}^{(k+1)C-G}\int_{(k+2)C-G-x}^{C}\frac{1}{C}\,f_{H_1}(h)\,dt\,dh$$

$$+\int_{(k+1)C-G}^{x}\int_{(k+2)C-G-x}^{(k+2)C-G-h}\frac{1}{C}\,f_{H_1}(h)\,dt\,dh+\int_{(k+1)C-G}^{x}\int_{(k+2)C-G-h}^{C}\frac{1}{C}\,f_{H_1}(h)\,dt\,dh\Bigg\}\,.$$

A straightforward calculation gives the result shown in Section 4.

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