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THE HOWE DUALITY FOR HODGE SYSTEMS

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Abstract. *As is well-known, classical Clifford analysis is a refinement of harmonic analysis. In [4], it is shown that analysis of Hodge systems can be viewed even as a refinement of Clifford analysis. In this note, we recall the Howe duality for harmonic analysis and Clifford analysis and, moreover, we describe quite explicitly the Howe duality for Hodge systems. Our main aim is to illustrate relations between these theories.*

1 INTRODUCTION

In this note, we describe quite explicitly the Howe duality for Hodge systems and connect it with the well-known facts of harmonic analysis and Clifford analysis.

In Section 2, we recall briefly the Fisher decomposition and the Howe duality for harmonic analysis. In Section 3, the well-known fact that Clifford analysis is a real refinement of harmonic analysis is illustrated by the Fisher decomposition and the Howe duality for the space of spinor-valued polynomials in the Euclidean space $\mathbb{R}^m$ under the so-called L-action.

On the other hand, for Clifford algebra valued polynomials in $\mathbb{R}^m$, we can consider another action, called in Clifford analysis the H-action. In the last section, we state the Fisher decomposition for the H-action obtained recently in [4]. As in Clifford analysis the prominent role plays the Dirac equation in this case the basic set of equations is formed by the Hodge system. According to [4], analysis of Hodge systems can be viewed even as a refinement of Clifford analysis. In this note, we describe the Howe duality for the H-action. In particular, in Proposition 1, we recognize the Howe dual partner of the orthogonal group $O(m)$ in this case as the Lie superalgebra $\mathfrak{sl}(2|1)$. Furthermore, Theorem 2 gives the corresponding multiplicity free decomposition with an explicit description of irreducible pieces.

2 HARMONIC ANALYSIS

In this section, we recall briefly the Howe duality for the space $\mathcal{P}$ of complex-valued polynomials in the Euclidean space $\mathbb{R}^m$. We consider the space $\mathcal{P}$ as an module over the full orthogonal group $O(m)$ of $\mathbb{R}^m$. The action of the group $O(m)$ on the space $\mathcal{P}$ is given by

$$[g \cdot P](\underline{x}) = P(g^{-1}\underline{x}),\ g \in O(m),\ P \in \mathcal{P} \text{ and } \underline{x} \in \mathbb{R}^m.$$

It is easily seen that the multiplication by the polynomial

$$r^2 = x_1^2 + \cdots + x_m^2,\ \underline{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$$

and the Laplace operator

$$\Delta = \sum_{j=1}^{m} \partial_{x_j}^2$$

are both $O(m)$-invariant linear operators on the space $\mathcal{P}$. In fact, by the so-called Fisher duality, the operators r^2 and Δ correspond to each other. Let $\mathcal{P}_k$ be the space of k-homogeneous polynomials of $\mathcal{P}$ and

$$\mathcal{H}_k = \{P \in \mathcal{P}_k;\ \Delta P = 0\}.$$

Then the well-known Fisher decomposition of the space $\mathcal{P}$ reads as follows:

$$\mathcal{P} = \bigoplus_{k=0}^{\infty} \mathcal{P}_{(k)} \quad \text{with} \quad \mathcal{P}_{(k)} = \bigoplus_{p=0}^{\infty} r^{2p}\mathcal{H}_k. \tag{1}$$

In (1), all $O(m)$-modules $\mathcal{H}_k$ are irreducible and mutually inequivalent and $\mathcal{P}_{(k)}$ are corresponding $O(m)$-isotypic components of $\mathcal{P}$.

Now let us recall (see e.g. [7, 10]) that the space $\mathcal{P}$ has the so-called hidden symmetry given by a Lie algebra of differential operators commuting with the $O(m)$-action. To be explicit, let

$\mathcal{W}$ be the Weyl algebra of differential operators with polynomial coefficients in $\mathbb{R}^m$. This is the subalgebra of $\mathrm{End}(\mathcal{P})$ generated as an associative algebra by the elements

$$x_1, \ldots, x_m \text{ and } \partial_{x_1}, \ldots, \partial_{x_m}.$$

Polynomials of $\mathcal{P}$ are contained in $\mathcal{W}$ as polynomial differential operators of order zero and, in a natural way, the space $\mathcal{P}$ can be viewed as a module over $\mathcal{W}$. Furthermore, the Weyl algebra $\mathcal{W}$ endowed with the commutator

$$[T_1, T_2] = T_1T_2 - T_2T_1$$

forms a Lie algebra. Then it is easy to see that $E^+ = -\frac{1}{2}\Delta$ and $E^- = \frac{1}{2}r^2$ generate in the Weyl algebra $\mathcal{W}$ a copy of $\mathfrak{sl}(2)$. Indeed, let

$$E = \sum_{j=1}^{m} x_j \partial_{x_j}$$

be the Euler operator and $H = -\frac{1}{2}(E + \frac{m}{2})$. Then the following relations hold true:

$$[H, E^{\pm}] = \pm E^{\pm} \quad \text{and} \quad [E^+, E^-] = 2H.$$

Moreover, let $\mathbb{I}_k$ be the Verma module for $\mathfrak{sl}(2)$ with the highest weight $-(k + \frac{m}{2})$. Then a realization of $\mathbb{I}_k$ is a module

$$\mathbb{I}_k(P) = \mathrm{span}\{r^{2p}P;\ p = 0, 1, 2, \ldots\} \tag{2}$$

for any non-zero $P \in \mathcal{H}_k$, cf. [2]. Let $\mathbb{H}_k$ be an irreducible $O(m)$-module isomorphic to the module $\mathcal{H}_k$. Under the joint action of the Howe dual pair $O(m) \times \mathfrak{sl}(2)$, the component $\mathcal{P}_{(k)}$ is then isomorphic to $\mathbb{H}_k \otimes \mathbb{I}_k$ and the space $\mathcal{P}$ is isomorphic to the multiplicity free direct sum

$$\mathcal{P} \simeq \bigoplus_{k=0}^{\infty} \mathbb{H}_k \otimes \mathbb{I}_k.$$

See e.g. [10] for details.

3 CLIFFORD ANALYSIS

In this section, following [2], we recall the Howe duality for spinor valued polynomials. Let $\mathbb{C}_m$ be the complex Clifford algebra generated by vectors $e_1, \ldots, e_m$ of the standard basis of $\mathbb{R}^m$. We consider $\mathbb{R}^m$ as a part of $\mathbb{C}_m$ as usual. Namely, we identify the vector $\underline{x} = (x_1, \ldots, x_m)$ of $\mathbb{R}^m$ with the element $e_1x_1 + \cdots + e_mx_m$ of $\mathbb{C}_m$. Recall that the Pin group $Pin(m)$ can be realized inside $\mathbb{C}_m$ as

$$Pin(m) = \{s_1s_2\cdots s_k;\ k \in \mathbb{N},\ s_j \in S^{m-1}\}$$

where S^{m-1} is the unit sphere in $\mathbb{R}^m$. Moreover, $Pin(m)$ is a double cover of the group $O(m)$. Indeed,

$$\rho(s)\underline{x} = s^{-1}\underline{x}s, \quad s \in Pin(m) \text{ and } \underline{x} \in \mathbb{R}^m$$

is a two-fold covering homomorphism of $Pin(m)$ onto $O(m)$. Denote by $\mathbb{S}$ a basic spinor representation for the Pin group $Pin(m)$. Such a representation $\mathbb{S}$ is called a spinor space and is usually realized as a subspace of the Clifford algebra $\mathbb{C}_m$. Let $\mathcal{P}(\mathbb{S}) = \mathcal{P} \otimes \mathbb{S}$ be the $Pin(m)$-module of spinor valued polynomials in $\mathbb{R}^m$. The so-called L-action on the space $\mathcal{P}(\mathbb{S})$ is given by

$$[L(s)(P)](\underline{x}) = s\,P(s^{-1}\underline{x}s),\ s \in Pin(m),\ P \in \mathcal{P}(\mathbb{S}) \text{ and } \underline{x} \in \mathbb{R}^m. \tag{3}$$

It is easily seen that the multiplication by $\underline{x} = e_1x_1 + \cdots + e_mx_m$ and the Dirac operator

$$\underline{D} = e_1\partial_{x_1} + \cdots + e_m\partial_{x_m}$$

are both $Pin(m)$-invariant linear operators on the space $\mathcal{P}(\mathbb{S})$. Actually, by the Fisher duality, the operators $-\underline{x}$ and $\underline{D}$ correspond to each other. Denote by $\mathcal{M}_k(\mathbb{S})$ the space of k-homogeneous polynomials $P \in \mathcal{P}(\mathbb{S})$ which are solutions of the Dirac equation $\underline{D}P = 0$. In [2], the following Fisher decomposition for this case is given:

$$\mathcal{P}(\mathbb{S}) = \bigoplus_{k=0}^{\infty} \mathcal{P}(\mathbb{S})_{(k)} \quad \text{with} \quad \mathcal{P}(\mathbb{S})_{(k)} = \bigoplus_{p=0}^{\infty} \underline{x}^p \mathcal{M}_k(\mathbb{S}). \tag{4}$$

In (4), all $Pin(m)$-modules $\mathcal{M}_k(\mathbb{S})$ are irreducible and mutually inequivalent and $\mathcal{P}(\mathbb{S})_{(k)}$ are corresponding $Pin(m)$-isotypic components of $\mathcal{P}(\mathbb{S})$. Moreover, realizing that

$$\mathcal{H}_k \otimes \mathbb{S} = \mathcal{M}_k(\mathbb{S}) \oplus \underline{x}\mathcal{M}_{k-1}(\mathbb{S})$$

and $\underline{x}^2 = -r^2$, it is easy to see that the decomposition (4) is a real refinement of (1).

It turns out that the hidden symmetry of $\mathcal{P}(\mathbb{S})$ is given by the Lie superalgebra $\mathfrak{osp}(1|2)$ generated in the twisted Weyl algebra $\mathcal{W} \otimes \mathbb{S}$ by the odd elements $\underline{D}$ and $\underline{x}$. Indeed, by [2], there is an infinite-dimensional irreducible $\mathfrak{osp}(1|2)$-module $\tilde{\mathbb{I}}_k$ which is isomorphic to

$$\tilde{\mathbb{I}}_k(P) = \operatorname{span}\{\underline{x}^p P;\ p = 0, 1, 2, \ldots\} \tag{5}$$

for any non-zero $P \in \mathcal{M}_k(\mathbb{S})$. Let $\mathbb{M}_k$ be an irreducible $Pin(m)$-module equivalent to $\mathcal{M}_k(\mathbb{S})$. Under the joint action of the Howe dual pair $Pin(m) \times \mathfrak{osp}(1|2)$, the component $\mathcal{P}(\mathbb{S})_{(k)}$ is then isomorphic to $\mathbb{M}_k \otimes \tilde{\mathbb{I}}_k$ and the space $\mathcal{P}(\mathbb{S})$ is isomorphic to the multiplicity free direct sum

$$\mathcal{P}(\mathbb{S}) \simeq \bigoplus_{k=0}^{\infty} \mathbb{M}_k \otimes \tilde{\mathbb{I}}_k,$$

see [2] for details. For an account of Lie superalgebras, we refer to [11] or [5].

We conclude this section by showing the following result.

Lemma 1. *For each non-negative integer k, the module $\tilde{\mathbb{I}}_k$ is the Verma module for $\mathfrak{osp}(1|2)$ with the highest weight $-(k + \frac{m}{2})$.*

Remark 1. Following [11], we recall Verma modules for a basic classical Lie superalgebra $\mathfrak{g}$. In this note, $\mathfrak{g} = \mathfrak{osp}(1|2)$ or $\mathfrak{g} = \mathfrak{sl}(2|1)$. Let $\mathfrak{h}$ be a Cartan subalgebra of the even part $\mathfrak{g}_0$ of $\mathfrak{g}$ and $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ be a Borel subalgebra of $\mathfrak{g}$. For $\eta \in \mathfrak{h}^*$ we define the Verma module $\tilde{M}(\eta)$ for $\mathfrak{g}$ by

$$\tilde{M}(\eta) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}v_\eta$$

where $\mathbb{C}v_\eta$ is the one dimensional $\mathfrak{b}$-module with $tv_\eta = \eta(t)v_\eta$ for $t \in \mathfrak{h}$ and $\mathfrak{n}^+ v_\eta = 0$. Here $U(\mathfrak{g})$ and $U(\mathfrak{b})$ is the universal enveloping algebra of $\mathfrak{g}$ and $\mathfrak{b}$, respectively. Moreover, we call v_η a maximal vector of the weight η and $\tilde{M}(\eta)$ the Verma module with the highest weight η. Finally, denote by $\tilde{L}(\eta)$ a unique irreducible factor module of $\tilde{M}(\eta)$. We call $\tilde{L}(\eta)$ an irreducible $\mathfrak{g}$-module with the highest weight η. If the Verma module $\tilde{M}(\eta)$ is itself irreducible then, of course, $\tilde{M}(\eta) = \tilde{L}(\eta)$.

For any graded subspace $\mathfrak{t}$ of $\mathfrak{g}$ write $\mathfrak{t} = \mathfrak{t}_0 \oplus \mathfrak{t}_1$ where $\mathfrak{t}_0$ (resp. $\mathfrak{t}_1$) is the even (resp. odd) part of $\mathfrak{t}$. For $\eta \in \mathfrak{h}^*$ we define the Verma module $M(\eta)$ for $\mathfrak{g}_0$ and an irreducible $\mathfrak{g}_0$-module $L(\eta)$ with the highest weight η analogously but in the definition we replace $\mathfrak{g}$, $\mathfrak{b}$ and $\mathfrak{n}^+$ with $\mathfrak{g}_0$, $\mathfrak{b}_0$ and $\mathfrak{n}_0^+$.

Proof of Lemma 1. First let us fix the notation. Denote by $\mathfrak{g}$ the Lie superalgebra generated in $\mathcal{W} \otimes \mathbb{S}$ by the odd elements $\underline{D}$ and $\underline{x}$. Denoting

$$H = -\frac{1}{2}(E + \frac{m}{2}),\ \ E^+ = -\frac{1}{2}\Delta,\ \ E^- = \frac{1}{2}r^2,\ \ F^+ = -\frac{1}{2\sqrt{2}}\,\underline{D}\ \text{ and }\ F^- = -\frac{1}{2\sqrt{2}}\,\underline{x},$$

it is easy to see that $\mathfrak{g}$ is isomorphic to the Lie superalgebra $\mathfrak{osp}(1|2)$ and $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ where $\mathfrak{g}_0 = \mathrm{span}\{H, E^+, E^-\}$ and $\mathfrak{g}_1 = \mathrm{span}\{F^+, F^-\}$ is the even and the odd part of $\mathfrak{g}$, respectively. Moreover, the even part $\mathfrak{g}_0$ is isomorphic to $\mathfrak{sl}(2)$. Indeed, it is sufficient to verify the following relations:

$$\begin{aligned} &[H, E^\pm] = \pm E^\pm, \quad [E^+, E^-] = 2H, \quad [H, F^\pm] = \pm\tfrac{1}{2}F^\pm, \\ &[E^\pm, F^\mp] = -F^\pm, \quad \{F^+, F^-\} = \tfrac{1}{2}H, \quad \{F^\pm, F^\pm\} = \pm\tfrac{1}{2}E^\pm. \end{aligned}$$

Here $\{T, S\} = TS + ST$ is the anticommutator of operators T and S.

Now we fix the notation for roots of the Lie superalgebra $\mathfrak{g}$. Denote by $\mathfrak{h} = \mathrm{span}\{H\}$ a Cartan subalgebra of $\mathfrak{g}$. For $\eta \in \mathfrak{h}^*$ we write $\eta = a$ where $a = \eta(2H)$. The set of roots of $\mathfrak{g}$ is $\Phi = \{\pm 1,\ \pm 2\}$. Moreover, let $\mathfrak{n}^+ = \mathrm{span}\{F^+, E^+\}$ and $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ be a Borel subalgebra of $\mathfrak{g}$.

For a given non-zero $P \in \mathcal{M}_k(\mathbb{S})$, let $\tilde{\mathbb{I}}_k(P)$ be as in (5). It is easily seen that $P \in \tilde{\mathbb{I}}_k(P)$ is a maximal vector for the weight $\eta_k = -(k + \frac{m}{2})$. By [13] or [9, Corollary 4.4], the Verma module $\tilde{M}(\eta_k)$ is irreducible and hence $\tilde{\mathbb{I}}_k(P) \simeq \tilde{M}(\eta_k)$, which completes the proof. $\square$

Remark 2. As an $\mathfrak{sl}(2)$-module, it is easily seen that $\tilde{\mathbb{I}}_k \simeq \mathbb{I}_k \oplus \mathbb{I}_{k+1}$, see [2]. Here $\mathbb{I}_k$ is the Verma module for $\mathfrak{sl}(2)$ as in (2). Indeed, for a given non-zero $P \in \mathcal{M}_k(\mathbb{S})$, we have that

$$\tilde{\mathbb{I}}_k(P) = \mathrm{span}\{r^{2p}P;\ p = 0, 1, 2, \ldots\} \oplus \mathrm{span}\{r^{2p}\underline{x}P;\ p = 0, 1, 2, \ldots\} \simeq \mathbb{I}_k \oplus \mathbb{I}_{k+1}.$$

4 HODGE SYSTEMS

In the previous section, we dealt with the space of spinor valued polynomials under the L-action. But, for Clifford algebra valued functions, we can consider another action, given by the adjoint action of $O(m)$ on values of functions. Namely, for $P \in \mathcal{P} \otimes \mathbb{C}_m$, the so-called H-action is given by

$$[H(s)(P)](\underline{x}) = s\,P(s^{-1}\underline{x}s)s^{-1},\ s \in Pin(m)\ \text{ and }\ \underline{x} \in \mathbb{R}^m. \tag{6}$$

In what follows, we use the language of differential forms. When we identify the Clifford algebra $\mathbb{C}_m$ with the Grassmann algebra $\Lambda^*(\mathbb{C}^m)$ the space $\mathcal{P} \otimes \mathbb{C}_m$ corresponds namely to the space

$\mathcal{P}^* = \mathcal{P} \otimes \Lambda^*(\mathbb{C}^m)$ of polynomial (differential) forms and the H-action to a natural action of the group $O(m)$ on the space $\mathcal{P}^*$, see [6, p. 153]. Furthermore, under this identification, the Dirac operator $\underline{D}$ coincides with $d + d^*$ where d and d^* is the de Rham differential and codifferential, respectively. Denoting by $dx_j \wedge$ (resp. $dx_j \rfloor$) the exterior (resp. interior) multiplication by dx_j, we have that

$$d = \sum_{j=1}^{m} \partial_{x_j} dx_j \wedge \quad \text{and} \quad d^* = -\sum_{j=1}^{m} \partial_{x_j} dx_j \rfloor.$$

By the Fisher duality, d^* and d correspond to the operators

$$x = -\sum_{j=1}^{m} x_j \, dx_j \wedge \quad \text{and} \quad x^* = \sum_{j=1}^{m} x_j \, dx_j \rfloor,$$

and, under the identification, the Clifford multiplication by a vector $-\underline{x}$ coincides with $x + x^*$. See [1] for details.

Before we state the Fisher decomposition for this case we need more notation. Let $\Lambda^s(\mathbb{C}^m)$ be the space of s-vectors over $\mathbb{C}^m$ and $\mathcal{P}_k^s = \mathcal{P}_k \otimes \Lambda^s(\mathbb{C}^m)$. Of course, we have that

$$\Lambda^*(\mathbb{C}^m) = \bigoplus_{s=0}^{m} \Lambda^s(\mathbb{C}^m) \;\; \text{and} \;\; \mathcal{P}^* = \bigoplus_{s=0}^{m} \bigoplus_{k=0}^{\infty} \mathcal{P}_k^s.$$

Denote by Ω the set of all non-trivial words in the letters x and x^*, that is,

$$\Omega = \{1, x, x^*, xx^*, x^*x, xx^*x, x^*xx^*, \ldots\}. \tag{7}$$

In this connection, let us note that $x^2 = 0$ and $(x^*)^2 = 0$. Denote by H_k^s the set of polynomial forms $P \in \mathcal{P}_k^s$ satisfying the Hodge system

$$dP = 0, \;\; d^*P = 0.$$

Then the Fisher decomposition for this case is given in the following theorem, see [4].

Theorem 1. *The space $\mathcal{P}^* = \mathcal{P} \otimes \Lambda^*(\mathbb{C}^m)$ decomposes as follows:*

$$\mathcal{P}^* = \mathcal{P}^*_{(0,0)} \oplus \left(\bigoplus_{s=1}^{m-1} \bigoplus_{k=0}^{\infty} \mathcal{P}^*_{(s,k)} \right) \oplus \mathcal{P}^*_{(m,0)} \quad \textit{with} \quad \mathcal{P}^*_{(s,k)} = \bigoplus_{w \in \Omega} w H_k^s. \tag{8}$$

In addition, in (8), all $O(m)$-modules H_k^s are non-trivial, irreducible and mutually inequivalent and $\mathcal{P}^*_{(s,k)}$ are corresponding $O(m)$-isotypic components of $\mathcal{P}^*$.

Remark 3. As is shown in [4], the decomposition (8) can be understood as a refinement of (4).

By [10], to find the hidden symmetry of the space $\mathcal{P}^*$ we should replace the Weyl algebra in this case with $\mathcal{W}(\Lambda^*)$, the associative subalgebra of $\mathrm{End}(\mathcal{P}^*)$ generated by the elements

$$x_1, \ldots, x_m, \quad \partial_{x_1}, \ldots, \partial_{x_m}, \quad dx_1 \wedge, \ldots, dx_m \wedge \quad \text{and} \quad dx_1 \rfloor, \ldots, dx_m \rfloor.$$

In particular, the operators x, x^*, d and d^* are $O(m)$-invariant elements of $\mathcal{W}(\Lambda^*)$. Furthermore, the algebra $\mathcal{W}(\Lambda^*)$ has a natural $\mathbb{Z}_2$-gradation such that the elements $x_1, \ldots, x_m$ and $\partial_{x_1}, \ldots, \partial_{x_m}$ are even and the elements $dx_1 \wedge, \ldots, dx_m \wedge$ and $dx_1 \rfloor, \ldots, dx_m \rfloor$ are odd. Thus the superalgebra $\mathcal{W}(\Lambda^*)$ endowed with the supercommutator forms a Lie superalgebra which has the operators x, x^*, d and d^* as odd elements. Even we can prove the following result.

Proposition 1. *The odd elements x, x^*, d and d^* of the Lie superalgebra $\mathcal{W}(\Lambda^*)$ generate the Lie superalgebra isomorphic to $\mathfrak{sl}(2|1)$.*

Remark 4. Our case is a part of the general theory of the Howe duality developed in [10]. But, in [10], the Howe dual partner of $O(m)$ in this case is not explicitly described as the Lie superalgebra $\mathfrak{sl}(2|1)$.

Proposition 1 follows easily from the next well-known result, see e.g. [1].

Lemma 2. *Let E be the Euler operator and $\hat{E}$ be the skew Euler operator, that is,*

$$E = \sum_{j=1}^{m} x_j \partial_{x_j} \quad \textit{and} \quad \hat{E} = \sum_{j=1}^{m} (dx_j \wedge)(dx_j \rfloor).$$

Then we have that $EP = kP$ and $\hat{E}P = sP$ for each $P \in \mathcal{P}_k^s$. Furthermore, the following relations hold true:

$$\begin{array}{lll} \{x, x\} = 0, & \{x^*, x^*\} = 0, & \{x, x^*\} = -r^2 = -\sum_{j=1}^m x_j^2, \\ \{d, d\} = 0, & \{d^*, d^*\} = 0, & \{d, d^*\} = -\Delta = -\sum_{j=1}^m \partial_{x_j}^2, \\ \{x^*, d\} = E + \hat{E}, & \{x, d^*\} = E - \hat{E} + m, & \{x^*, d^*\} = 0 = \{x, d\}. \end{array}$$

Proof of Proposition 1. Denote by $\mathfrak{g}$ the Lie superalgebra generated by the elements x, x^*, d and d^*. By Lemma 2, it is easy to see that the even part $\mathfrak{g}_0$ of $\mathfrak{g}$ is isomorphic to $\mathfrak{gl}(2)$. Indeed, denoting

$$H = -\frac{1}{2}(E + \frac{m}{2}), \;\; E^+ = -\frac{1}{2}\Delta, \;\; E^- = \frac{1}{2}r^2 \text{ and } Z = \frac{1}{2}(\hat{E} - \frac{m}{2}),$$

we have that

$$\mathfrak{g}_0 = \operatorname{span}\{H, E^+, E^-\} \oplus \mathbb{R}(2Z) \simeq \mathfrak{sl}(2) \oplus \mathfrak{sl}(1) \simeq \mathfrak{gl}(2).$$

Furthermore, the odd part $\mathfrak{g}_1$ is generated by the operators

$$F^+ = \frac{1}{\sqrt{2}}\, d, \;\; F^- = -\frac{1}{\sqrt{2}}\, x, \;\; \bar{F}^+ = \frac{1}{\sqrt{2}}\, d^* \text{ and } \bar{F}^- = \frac{1}{\sqrt{2}}\, x^*.$$

To conclude that the Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is isomorphic to $\mathfrak{sl}(2|1)$ it is sufficient to verify the following relations:

$$\begin{array}{lll} [H, E^{\pm}] = \pm E^{\pm}, & [H, F^{\pm}] = \pm\frac{1}{2}F^{\pm}, & [H, \bar{F}^{\pm}] = \pm\frac{1}{2}\bar{F}^{\pm}, \\ [Z, E^{\pm}] = 0, & [Z, F^{\pm}] = \frac{1}{2}F^{\pm}, & [Z, \bar{F}^{\pm}] = -\frac{1}{2}\bar{F}^{\pm}, \\ [E^{\pm}, F^{\pm}] = 0, & [E^{\pm}, F^{\mp}] = -F^{\pm}, & [E^{\pm}, \bar{F}^{\mp}] = \bar{F}^{\pm}, \\ [E^{\pm}, \bar{F}^{\pm}] = 0, & [E^+, E^-] = 2H, & [Z, H] = 0, \\ \{F^{\pm}, F^{\pm}\} = 0, & \{\bar{F}^{\pm}, \bar{F}^{\pm}\} = 0, & \{F^{\pm}, \bar{F}^{\pm}\} = E^{\pm}, \\ \{F^{\pm}, F^{\mp}\} = 0, & \{\bar{F}^{\pm}, \bar{F}^{\mp}\} = 0, & \{F^{\pm}, \bar{F}^{\mp}\} = Z \mp H. \end{array} \tag{9}$$

These relations follow easily from Lemma 2. $\square$

Following [12], we fix the notation of roots for the Lie superalgebra $\mathfrak{sl}(2|1)$. Denote by $\mathfrak{h} = \mathrm{span}\{H, Z\}$ a Cartan subalgebra of the even part $\mathfrak{g}_0 = \mathfrak{gl}(2)$ of $\mathfrak{g} = \mathfrak{sl}(2|1)$. Here we use the notation of generators of $\mathfrak{g}$ from the proof of Proposition 1. For $\eta \in \mathfrak{h}^*$ we write $\eta = (a, b)$ where $a = \eta(2H)$ and $b = \eta(2Z)$. Letting $\alpha = (2, 0)$ and $\beta = (-1, -1)$, the set of roots of $\mathfrak{g}$ is

$$\Phi = \{\pm\alpha,\ \pm\beta,\ \pm(\alpha + \beta)\}.$$

Set $e_\alpha = E^+$, $e_\beta = \bar{F}^-$, $e_{\alpha+\beta} = \bar{F}^+$, $e_{-\alpha} = E^-$, $e_{-\beta} = F^+$ and $e_{-(\alpha+\beta)} = F^-$. Moreover, let

$$\mathfrak{n}^+ = \mathrm{span}\{e_\alpha, e_\beta, e_{\alpha+\beta}\}$$

and $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ be a Borel subalgebra of $\mathfrak{g}$. For $\eta \in \mathfrak{h}^*$ we define the Verma modules $\tilde{M}(\eta)$ and $M(\eta)$ and irreducible modules $\tilde{L}(\eta)$ and $L(\eta)$ as in Remark 1. Recall that, in this case, $\tilde{M}(\eta)$ is irreducible if and only if $\eta = (a, b)$ is typical and $a \notin \mathbb{N}$, see [12, Theorem 1.6]. Moreover, $\eta = (a, b)$ is typical if $a - b \neq 0$ and $a + b + 2 \neq 0$.

Before stating our main theorem we need the next lemma.

Lemma 3. *Denote by $\mathbb{N}_0$ the set of non-negative integers and let $(s, k) \in \{1, \dots, m-1\} \times \mathbb{N}_0$ or $(s, k) \in \{(0, 0), (m, 0)\}$. For a non-zero $P \in H^s_k$, set*

$$\tilde{\mathbb{V}}^s_k(P) = \mathrm{span}\{wP;\ w \in \Omega\}$$

where Ω is as in (7). Then the following statements hold true:

(i) The space $\tilde{\mathbb{V}}^s_k(P)$ is an infinite-dimensional irreducible $\mathfrak{sl}(2|1)$-module.

(ii) For $1 \leq s \leq m-1$ and $k \in \mathbb{N}_0$, the module $\tilde{\mathbb{V}}^s_k(P)$ is the Verma module for $\mathfrak{sl}(2|1)$ with the highest weight

$$\eta^s_k = (-k - \frac{m}{2} - 1,\ s - \frac{m}{2} - 1)$$

and a maximal vector x^*P.

(iii) For $s = 0$, the module $\tilde{\mathbb{V}}^0_0(P)$ is an irreducible $\mathfrak{sl}(2|1)$-module with the highest weight

$$\eta^0_0 = (-\frac{m}{2}, -\frac{m}{2})$$

and a maximal vector P.

(iv) For $s = m$, the module $\tilde{\mathbb{V}}^m_0(P)$ is an irreducible $\mathfrak{sl}(2|1)$-module with the highest weight

$$\eta^m_0 = (-\frac{m}{2} - 1, \frac{m}{2} - 1)$$

and a maximal vector x^*P.

(v) In particular, the $\mathfrak{sl}(2|1)$-module $\tilde{\mathbb{V}}^s_k(P)$ is isomorphic to a module $\tilde{\mathbb{V}}^{s'}_{k'}(P')$ if and only if $s = s'$ and $k = k'$.

(vi) Let $\tilde{\mathbb{V}}^s_k$ be an irreducible $\mathfrak{sl}(2|1)$-module with the highest weight η^s_k and $\mathbb{V}^s_k$ be an irreducible $\mathfrak{gl}(2)$-module with the highest weight $(-k - \frac{m}{2}, s - \frac{m}{2})$. As a $\mathfrak{gl}(2)$-module, the module $\tilde{\mathbb{V}}^s_k$ decomposes as

$$\tilde{\mathbb{V}}^s_k \simeq \mathbb{V}^s_k \oplus \mathbb{V}^{s+1}_{k+1} \oplus \mathbb{V}^{s-1}_{k+1} \oplus \mathbb{V}^s_{k+2}$$

if $1 \leq s \leq m-1$; moreover, $\tilde{\mathbb{V}}^0_0 \simeq \mathbb{V}^0_0 \oplus \mathbb{V}^1_1$ and $\tilde{\mathbb{V}}^m_0 \simeq \mathbb{V}^m_0 \oplus \mathbb{V}^{m-1}_1$.

Remark 5. Let $\mathbb{V}_k^s$ be an irreducible $\mathfrak{gl}(2)$-module as in Lemma 3 (vi). Obviously, we have that

$$\mathbb{V}_k^s \simeq \mathbb{I}_k \otimes \mathbb{C}_s$$

where $\mathbb{I}_k$ is the Verma module for $\mathfrak{sl}(2)$ as in (2) and $\mathbb{C}_s$ is a representation of $\mathfrak{sl}(1)$ with the highest weight $(s - \frac{m}{2})$ (that is, the generator $2Z$ of $\mathfrak{sl}(1)$ has $(s - \frac{m}{2})$ as the eigenvalue).

Proof of Lemma 3. Let $0 \neq P \in H_k^s$ and $\mathbb{V} = \tilde{\mathbb{V}}_k^s(P)$.

(a) To show that $\mathbb{V}$ is an $\mathfrak{sl}(2|1)$-module it is sufficient to verify that, for any word $w \in \Omega$, $d(wP) \in \mathbb{V}$ and $d^*(wP) \in \mathbb{V}$. But it is easy to show by induction on the length of the word w using the relations of Lemma 2.

(b) The $\mathfrak{sl}(2|1)$-module $\mathbb{V}$ is infinite-dimensional. Indeed, by Theorem 1, the elements wP with $w \in \Omega$ form a vector space basis of $\mathbb{V}$ when $1 \leq s \leq m-1$. On the other hand, if $s = 0$ (resp. $s = m$) then $wP = 0$ for words $w \in \Omega$ with the last letter x^* (resp. x). Thus, in the case when $s = 0$ (resp. $s = m$), the elements wP for words $w \in \Omega$ with the last letter x (resp. x^*) form a vector space basis of $\mathbb{V}$.

(c) The $\mathfrak{sl}(2|1)$-module $\mathbb{V}$ is irreducible. Indeed, let $0 \neq \mathbb{W} \subset \mathbb{V}$ be a submodule. We can assume that there is a non-zero $P' \in \mathbb{W}$ such that $P' = wP + P''$ for some word $w \in \Omega$ of the length $2p+1$ and

$$P'' \in \bigoplus_{j=k}^{2p+k} \mathcal{P}_j \otimes \Lambda^*(\mathbb{C}^m).$$

Otherwise, apply the operators x and x^* alternatively to P'. Moreover, for the sake of explicitness, let $0 \leq s < m$ and $w = r^{2p}x$. Denoting $U_{k+1}^{s+1} = xH_k^s$, it is easy to see that mappings

$$\Delta : r^{2p}U_{k+1}^{s+1} \to r^{2(p-1)}U_{k+1}^{s+1}$$

and $d^* : U_{k+1}^{s+1} \to H_k^s$ are both one-to-one and onto. Here we use the relations $[\Delta, r^2] = 4E + 2m$, $[\Delta, x] = -2d$ and $\{x, d^*\} = E - \hat{E} + m$, see (9). Hence we have that

$$0 \neq d^*\Delta^p(P') = d^*\Delta^p(wP) \in H_k^s \cap \mathbb{W},$$

which implies $\mathbb{W} = \mathbb{V}$.

(d) Let $1 \leq s \leq m-1$. Then it is easy to see that $x^*P \in \mathbb{V}$ is a maximal vector of the weight η_k^s and $\mathbb{V} \subset \tilde{M}(\eta_k^s)$. Moreover, the highest weight η_k^s is typical. By [12, Theorem 1.6], the Verma module $\tilde{M}(\eta_k^s)$ is thus irreducible, which completes the proof of (ii).

(e) Let $k = 0$. If $s = 0$ then $P \in \mathbb{V}$ is a maximal vector of the weight η_0^0. On the other hand, if $s = m$ then $x^*P \in \mathbb{V}$ is a maximal vector of the weight η_0^m. In both these cases, the Verma module $\tilde{M}(\eta_k^s)$ is not irreducible and the module $\mathbb{V}$ is instead isomorphic to $\tilde{L}(\eta_k^s)$, see [12, Theorem 1.6].

(f) The statement (v) is obvious.

(g) Now we prove the statement (vi). Denote

$$\mathbb{V}_1 = \operatorname{span}\{r^{2p}P;\ p \in \mathbb{N}_0\},\ \ \mathbb{V}_2 = \operatorname{span}\{r^{2p}xP;\ p \in \mathbb{N}_0\},\ \ \mathbb{V}_3 = \operatorname{span}\{r^{2p}x^*P;\ p \in \mathbb{N}_0\}$$

$$\text{and}\ \ \mathbb{V}_4 = \operatorname{span}\{r^{2p}((k+m-s)xx^* - (k+s)x^*x)P;\ p \in \mathbb{N}_0\}.$$

By [4], it is easy to see that the space $\mathbb{V}$ decomposes as

$$
\begin{aligned}
\mathbb{V} &= \mathbb{V}_1 \oplus \mathbb{V}_2 \oplus \mathbb{V}_3 \oplus \mathbb{V}_4 \quad \text{if } 1 \leq s \leq m-1, \\
&= \mathbb{V}_1 \oplus \mathbb{V}_2 \quad \text{if } s = 0, \\
&= \mathbb{V}_1 \oplus \mathbb{V}_3 \quad \text{if } s = m.
\end{aligned}
$$

Obviously, as $\mathfrak{gl}(2)$-modules, $\mathbb{V}_1 \simeq \mathbb{V}_k^s$, $\mathbb{V}_2 \simeq \mathbb{V}_{k+1}^{s+1}$, $\mathbb{V}_3 \simeq \mathbb{V}_{k+1}^{s-1}$ and $\mathbb{V}_4 \simeq \mathbb{V}_{k+2}^s$, which completes the proof. $\square$

Now we are ready to state the following theorem.

Theorem 2. *Under the joint action of the pair $O(m) \times \mathfrak{sl}(2|1)$, the space $\mathcal{P}^* = \mathcal{P} \otimes \Lambda^*(\mathbb{C}^m)$ is isomorphic to the multiplicity free direct sum*

$$
\mathcal{P}^* \simeq (\mathbb{H}_0^0 \otimes \tilde{\mathbb{V}}_0^0) \oplus \left(\bigoplus_{s=1}^{m-1} \bigoplus_{k=0}^{\infty} \mathbb{H}_k^s \otimes \tilde{\mathbb{V}}_k^s \right) \oplus (\mathbb{H}_0^m \otimes \tilde{\mathbb{V}}_0^m) \tag{10}
$$

where $\mathbb{H}_k^s$ is an irreducible $O(m)$-module isomorphic to H_k^s and $\tilde{\mathbb{V}}_k^s$ is an infinite-dimensional irreducible $\mathfrak{sl}(2|1)$-module with the highest weight η_k^s (defined in Lemma 3).

Remark 6. Theorem 2 is a special case of [10, Theorem 8] but, in addition, it gives an explicit description of irreducible pieces of the decomposition (10). In addition, in [3], irreducible $O(m)$-modules H_k^s are characterized in terms of the highest weights for the corresponding $SO(m)$-modules.

Proof. For the sake of completeness, we give a direct proof without referring to the general theory developed in [10]. To that end, let $\mathcal{B}_k^s$ be a vector space basis of the space H_k^s. For each $P \in \mathcal{B}_k^s$, by Lemma 3, the $\mathfrak{sl}(2|1)$-module $\tilde{\mathbb{V}}_k^s(P)$ is isomorphic to an irreducible module with the highest weight η_k^s we denote by $\tilde{\mathbb{V}}_k^s$. If $\mathbb{H}_k^s$ stands for an irreducible $O(m)$-module isomorphic to H_k^s, then it is not difficult to see that

$$
\mathcal{P}^*_{(s,k)} = \bigoplus_{w \in \Omega}^{\infty} w H_k^s
$$

is isomorphic to the irreducible $O(m) \times \mathfrak{sl}(2|1)$-module $\mathbb{H}_k^s \otimes \tilde{\mathbb{V}}_k^s$. Finally, using Theorem 1 and Lemma 3, we conclude that the whole space $\mathcal{P}^*$ is isomorphic to the multiplicity free direct sum (10), which completes the proof. $\square$

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