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ON BLOCK MATRICES OF PASCAL TYPE IN CLIFFORD ANALYSIS

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Abstract. *Since the 90-ties the Pascal matrix, its generalizations and applications have been in focus of a great amount of publications. As it is well known, the Pascal matrix, the symmetric Pascal matrix and other special matrices of Pascal type play an important role in many scientific areas, among them Numerical Analysis, Combinatorics, Number Theory, Probability, Image processing, Sinal processing, Electrical enginneering, etc. We present a unified approach to matrix representations of special polynomials in several hypercomplex variables (new Bernoulli, Euler etc. polynomials), extending results of H. Malonek, G.Tomaz: Bernoulli polynomials and Pascal matrices in the context of Clifford Analysis, Discrete Appl. Math. 157(4) (2009) 838-847.*
The hypercomplex version of a new Pascal matrix with block structure, which resembles the ordinary one for polynomials of one variable will be discussed in detail.

1 INTRODUCTION

The role of the Bernoulli and Euler polynomials in several areas of pure and applied mathematics is widely known. They appear, for instance, in Differential Topology, Number Theory, and Numerical Analysis. One of the most well known result that involves Bernoulli numbers is the Euler-Maclaurin summation formula which allows to accelerate the convergence of series. The connection between Bernoulli, Euler and other special polynomials with the Pascal matrix are also well known. This relation has been explored in various publications [1, 4, 6, 7]. The Pascal matrix stands out when we need to deal with polynomials in a more friendly way, especially, if we are interested in using them in applications. As a general tool of dealing with polynomials in several variables and their matrix representation, we introduce a block Pascal matrix.

2 HYPERCOMPLEX BERNOULLI AND EULER POLYNOMIALS

2.1 (Classical) Bernoulli and Euler polynomials

Let

$$g(x,t) = \frac{te^{xt}}{e^t - 1}.$$

Developing $g(x,t)$ in a formal series of powers of t by

$$g(x,t) = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \tag{1}$$

the coefficients $B_n(x)$ are called Bernoulli polynomials and $g(x,t)$ is the generating function for these polynomials.

The Bernoulli numbers are simply the values of $B_n(x)$ in $x = 0$, i.e.,

$$B_n := B_n(0), \quad n = 0, 1, \ldots.$$

Let now

$$h(x,t) = \frac{2e^{xt}}{e^t + 1}.$$

The Euler polynomials are implicitly given by

$$h(x,t) = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \tag{2}$$

The Euler numbers, E_n, are also related to the Euler polynomials by

$$E_n = 2^n E_n(\frac{1}{2}).$$

2.2 Construction of hypercomplex Bernoulli and Euler polynomials

Analyzing the several generalizations of Bernoulli and Euler polynomials which have appeared in the last years, we can realize as common idea the modification of their generating functions [2, 3, 8]. Following an analogous reasoning, and to overcome the problem of the possible loss of monogenicity when forming a quotient of monogenic functions, we start by (1) written in the form

$$e^{xt} = \left(\sum_{r=0}^{\infty} \frac{t^r}{(r+1)!} \right) \left(\sum_{k=0}^{\infty} \frac{1}{k!} B_k(x) t^k \right). \tag{3}$$

Considering the hypercomplex structure for $\mathbb{R}^{n+1}$, proposed in [5], based on an isomorphism between this vector space and

$$\mathcal{H}^n = \{\vec{z} : \vec{z} = (z_1, \ldots, z_n), z_k = x_k - x_0 e_k, \quad x_0, x_k \in \mathbb{R}, k = 1, \ldots, n\},$$

we define a hypercomplex exponential function by the formal power series

$$\mathbf{Exp}(\vec{t}, \vec{z}) := \exp(t_1 z_1 + \cdots + t_n z_n) = \sum_{k=0}^{\infty} \frac{1}{k!} (t_1 z_1 + \cdots + t_n z_n)^k.$$

With this function and founded on (3), we establish the definition of hypercomplex Bernoulli polynomials [7], $B_{j_1,\ldots,j_n}(z_1, \ldots, z_n)$, $j_k \in \mathbb{N}_0, k = 1, \ldots, n$, as coefficients of a multiple power series

$$\mathbf{Exp}(\vec{t}, \vec{z}) = \left(\sum_{r=0}^{\infty} \frac{1}{(r+1)!} (t_1 + \cdots + t_n)^r \right) \left(\sum_{|j|=0}^{\infty} \frac{1}{j!} B_{j_1,\ldots,j_n}(z_1, \ldots, z_n) t_1^{j_1} \ldots t_n^{j_n} \right).$$

Analogously, using the same generating function of the generalized powers, $\mathbf{Exp}(\vec{t}, \vec{z})$, and (2) written in the form

$$2e^{xt} = \left(1 + \sum_{r=0}^{\infty} \frac{t^r}{r!} \right) \left(\sum_{k=0}^{\infty} \frac{1}{k!} E_k(x) t^k \right), \tag{4}$$

we arrive to the definition of hypercomplex Euler polynomials [6], $E_{j_1,\ldots,j_n}(z_1, \ldots, z_n)$, $j_k \in \mathbb{N}_0, k = 1, \ldots, n$, as coefficients of a multiple power series

$$2\mathbf{Exp}(\vec{t}, \vec{z}) = \left(1 + \sum_{r=0}^{\infty} \frac{1}{(r+1)!} (t_1 + \cdots + t_n)^r \right) \left(\sum_{|j|=0}^{\infty} \frac{1}{j!} E_{j_1,\ldots,j_n}(z_1, \ldots, z_n) t_1^{j_1} \ldots t_n^{j_n} \right).$$

3 BLOCK PASCAL MATRIX

3.1 (Classical) Pascal matrix

The (classical) Pascal matrix of order $n+1$, P, has the following structure:

$$P = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 2 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{n} \end{bmatrix},$$

that is, $P = [P_{ij}]$, where

$$P_{ij} = \begin{cases} \binom{i}{j} & , i \geq j \\ 0 & , \text{otherwise}, \end{cases} \quad i, j = 0, \ldots, n.$$

In the literature we can find a large number of modifications and generalizations of the classical Pascal matrix according to the applications. One of those is the matrix $P[x] = [P_{ij}[x]]$, where

$$P_{ij}[x] = \begin{cases} \binom{i}{j} x^{i-j} & , i \geq j \\ 0 & , \text{otherwise}, \quad i, j = 0, \ldots, n. \end{cases}$$

This matrix appears involved in the solution of the initial value problem

$$\begin{cases} \frac{d}{dx} y(x) = H y(x) \\ y(0) = y_0, \end{cases}$$

being $H = [H_{ij}]$ such that

$$H_{ij} = \begin{cases} i & , i = j + 1 \\ 0 & , \text{otherwise}, \quad i, j = 0, \ldots, n, \end{cases}$$

called creation matrix. As it is well known, the unique solution of this problem is $y(x) = e^{Hx} y_0$ and $P[x] := e^{Hx}$, i.e.,

$$P[x] = \sum_{k=0}^{\infty} \frac{(Hx)^k}{k!}$$

(cf.[1, 4]). Actually, the sum can be written

$$P[x] = \sum_{k=0}^{n} \frac{(Hx)^k}{k!}$$

because $H^n = 0, k > n$.

Obviously, if $x = 1$ we obtain $P := e^H = \sum_{k=0}^{n} \frac{H^k}{k!}$, that is, the classical Pascal matrix can be regarded as an exponential matrix.

The generalized matrix, $P[x]$, is also related to the Bernoulli polynomial matrix $\mathscr{B}(x) = [B_{ij}(x)]$:

$$B_{ij}(x) = \begin{cases} \binom{i}{j} B_{i-j}(x) & , i \geq j \\ 0 & , \text{otherwise}, \quad i, j = 0, \ldots, n, \end{cases}$$

through

$$\mathscr{B}(x) = P[x] \mathscr{B}$$

(cf. [10]).

3.2 Pascal matrix with a block structure

As a general tool of dealing with polynomials in several variables and their matrix representation, we introduce a block Pascal matrix, $\mathcal{P}$. The global structure of this matrix simulates the structure of the classical Pascal matrix. In fact we consider $\mathcal{P} = [\mathcal{P}_{sr}]$:

$$\mathcal{P}_{sr} = \begin{cases} \binom{s}{r} P & , s \geq r \\ O & , \text{otherwise}, \quad s, r = 0, \ldots, n, \end{cases} \tag{5}$$

(O is the null matrix of order $n + 1$).

Analogously we define the block creation matrix $\mathbb{H} = [\mathbb{H}_{sr}]$ by:

$$\mathbb{H}_{sr} = \begin{cases} H & , s = r \\ sI & , s = r + 1 \\ O & , \text{otherwise}, \quad s, r = 0, \ldots, n, \end{cases}$$

(I is the identity matrix of order $n+1$).

This matrix possesses similar properties of those of H, namely,

$$\mathbb{H}^k = O, \quad k > 2n.$$

Similarly to the classical case the block Pascal matrix can be viewed as an exponential matrix:

$$\mathcal{P} = e^{\mathbb{H}}, \text{i.e.}, \mathcal{P} = \sum_{k=0}^{2n} \frac{\mathbb{H}^k}{k!}. \tag{6}$$

4 APPLICATIONS OF HYPERCOMPLEX MATRICES

In this section we restrict our study to the 3-dimensional real Euclidean space, which implies the use of two hypercomplex variables.

First of all, we will see how to transform a vector of hypercomplex Bernoulli polynomials into a vector of multiple powers of z_1 and z_2. Secondly, taking into account this result we'll use it to transform the Taylor expansion of a function of two hypercomplex variables into an expansion in terms of hypercomplex Bernoulli polynomials.

The hypercomplex Pascal matrix, $\mathcal{P}(z_1, z_2) = [\mathcal{P}_{sr}(z_1, z_2)]$, was introduced in [7]. It is such that

$$(\mathcal{P}(z_1, z_2))_{sr} = \begin{cases} \binom{s}{r} P(z_1) \times z_2^{s-r} & , s \geq r \\ 0 & , \text{otherwise}, \quad s, r = 0, \ldots, n, \end{cases}$$

where

$$(P(z_1))_{ij} = \begin{cases} \binom{i}{j} z_1^{i-j} & , i \geq j \\ 0 & , \text{otherwise}, \quad i, j = 0, \ldots, n. \end{cases}$$

The notation $P(z_1) \times z_2^{s-r}$ means that we use the symmetric "$\times$"-product, introduced in [5], between each entry of the matrix $P(z_1)$ and z_2^{s-r}.

Notice that $\mathcal{P}(1,1) = \mathcal{P}$.

In order to establish the mentioned results, let us introduce the following definitions similar to the ordinary but now suitable in the Clifford Analysis context:

Definition 4.1 *The Kronecker "$\times$"-product of two matrices, A and B, of type $m \times n$ and $p \times q$, respectively, is the $mp \times nq$ matrix defined as*

$$A \circledast B = \begin{bmatrix} A_{11} \times B & A_{12} \times B & \cdots & A_{1n} \times B \\ A_{21} \times B & A_{22} \times B & \cdots & A_{2n} \times B \\ \cdots & \cdots & \cdots & \cdots \\ A_{m1} \times B & A_{m2} \times B & \cdots & A_{mn} \times B \end{bmatrix},$$

where

$$A_{ij} \times B = \begin{bmatrix} A_{ij} \times B_{11} & A_{ij} \times B_{12} & \cdots & A_{ij} \times B_{1q} \\ A_{ij} \times B_{21} & A_{ij} \times B_{22} & \cdots & A_{ij} \times B_{2q} \\ \cdots & \cdots & \cdots & \cdots \\ A_{ij} \times B_{p1} & A_{ij} \times B_{p2} & \cdots & A_{ij} \times B_{pq} \end{bmatrix}.$$

Definition 4.2 *The Hadamard* " $\times$ "*-product of two matrices, A and B, both of order* $m \times n$*, is the* $m \times n$ *matrix defined as*

$$A \odot B = \begin{bmatrix} A_{11} \times B_{11} & A_{12} \times B_{12} & \cdots & A_{1n} \times B_{1n} \\ A_{21} \times B_{21} & A_{22} \times B_{22} & \cdots & A_{2n} \times B_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{m1} \times B_{m1} & A_{m2} \times B_{m2} & \cdots & A_{mn} \times B_{mn} \end{bmatrix}.$$

For our purpose we also use the vectorization of matrices that, like usually in matrix calculus, is a linear transformation which converts an $m \times n$ matrix A into an $mn \times 1$ column vector by stacking the columns of A on the top of each other, and it is denoted by $vec(A)$.

Considering the shift matrix [4], $K = [K_{ij}]$, with

$$K_{ij} = \begin{cases} 1 & , i = j+1 \\ 0 & , \text{otherwise}, \quad i, j = 0, \dots, n, \end{cases}$$

we constructed the auxiliary block matrix $M = [M_{sr}]$ as follows:

$$M_{sr} = \begin{cases} z_1 K & , s = r \\ z_2 I & , s = r+1 \\ O & , \text{otherwise}, \quad s, r = 0, \dots, n. \end{cases}$$

Using this matrix it is possible conclude that the hypercomplex Pascal matrix is also an exponential matrix:

$$\mathcal{P}(z_1, z_2) = e^{(M \odot \mathbb{H})}, \text{i.e.}, \mathcal{P}(z_1, z_2) = \sum_{k=0}^{2n} \frac{(M \odot \mathbb{H})^k}{k!}. \tag{7}$$

It is worth noting that, the matrix $M \odot \mathbb{H} = [(M \odot \mathbb{H})_{sr}]$ is such that

$$(M \odot \mathbb{H})_{sr} = \begin{cases} z_1 H & , s = r \\ s z_2 I & , s = r+1 \\ O & , \text{otherwise}, \quad s, r = 0, \dots, n \end{cases}$$

and taking $z_1 = z_2 = 1$ in (7) we obtain (6).

The final goal of our paper is to find a matrix that allows to transform the Taylor expansion of a function into an expansion containing Bernoulli polynomials.

With this intention, we recall that the hypercomplex polynomial Bernoulli matrix is defined as the $(n+1) \times (n+1)$-block matrix, $\mathcal{B}(z_1, z_2) = [\mathcal{B}^{sr}_{ij}(z_1, z_2)]$, such that

$$\mathcal{B}^{sr}_{ij}(z_1, z_2) = \begin{cases} \binom{i}{j}\binom{s}{r} B_{i-j, s-r}(z_1, z_2) & , i \geq j \wedge s \geq r \\ 0 & , \text{otherwise}, \quad i, j, s, r = 0, \dots, n, \end{cases}$$

($B_{i-j,s-r}(z_1, z_2)$ are hypercomplex Bernoulli polynomials). The matrix $\mathcal{B} := \mathcal{B}(0,0)$ is called Bernoulli matrix (cf. [7]).

Hypercomplex Bernoulli polynomials have many properties similar to the ordinary (real and complex) case [7]. One of them is

$$B_{j_1, j_2}(1, 1) = (-1)^{|j|} B_{j_1, j_2}, \quad j_k \in \mathbb{N}_0, k = 1, 2,$$

using the notation $B_{j_1,j_2} := B_{j_1,j_2}(0,0)$ for the values of the generalized Bernoulli polynomials in the origin. This property can be written in the matrix form by

$$\mathcal{B}(1,1) - \mathcal{B} = \mathbb{H}. \tag{8}$$

Theorem 4.1

$$(\mathcal{P}-\mathcal{I})\mathcal{B}=\mathbb{H}.$$

Proof

By the known result $\mathcal{B}(z_1,z_2)=\mathcal{P}(z_1,z_2)\mathcal{B}$ (Theorem 3.2 [7]) and (8) becomes

$$(\mathcal{P}-\mathcal{I})\mathcal{B}=\mathcal{P}\mathcal{B}-\mathcal{B}=\mathcal{B}(1,1)-\mathcal{B}=\mathbb{H}.\ \square$$

In accordance with this Theorem and since $\mathcal{B}$ is invertible, we have

$$\mathcal{P}-\mathcal{I}=\mathbb{H}\mathcal{B}^{-1}.$$

Since,

$$\mathcal{P}-\mathcal{I} \;=\; \sum_{k=1}^{2n}\frac{\mathbb{H}^k}{k!}=\sum_{k=0}^{2n}\frac{\mathbb{H}^{k+1}}{(k+1)!}=\mathbb{H}\sum_{k=0}^{2n}\frac{\mathbb{H}^k}{(k+1)!}$$

we conclude that

$$\mathcal{B}^{-1}=\sum_{k=0}^{2n}\frac{\mathbb{H}^k}{(k+1)!}.$$

In [1, 4] the problem of the expression of the Taylor series of real functions in terms of series involving Bernoulli polynomials was studied. There it was achieved the matrix,

$$L=\sum_{k=0}^{n}\frac{H^k}{(k+1)!}$$

satisfying $Lb(x)=\xi(x), x\in\mathbb{R}$, where $b(x)=(B_0(x)\quad B_1(x)\quad\cdots\quad B_n(x))^T$ and $\xi(x)=(1\quad x\quad\ldots\quad x^n)^T$, i.e., L is the transformation matrix between the Taylor expansion and an expansion in terms of Bernoulli polynomials. Like in those papers, we shortly represent $\mathcal{B}^{-1}$ by $\mathbb{L}$, and due to $\mathcal{B}(z_1,z_2)=\mathcal{P}(z_1,z_2)\mathcal{B}$, we arrive to

$$\mathbb{L}\mathcal{B}(z_1,z_2)=\mathbb{L}\mathcal{B}\mathcal{P}(z_1,z_2)=\mathcal{P}(z_1,z_2).$$

Thus, the matrix $\mathbb{L}$ transforms the hypercomplex polynomial Bernoulli matrix, $\mathcal{B}(z_1,z_2)$, into the hypercomplex Pascal matrix, $\mathcal{P}(z_1,z_2)$.

Assigning by $\mathbf{b}(z_1,z_2)$ the first column of $\mathcal{B}(z_1,z_2)$ and by $\mathbf{p}(z_1,z_2)$ the first column of $\mathcal{P}(z_1,z_2)$, we have

$$\mathbb{L}\mathbf{b}(z_1,z_2)=\mathbf{p}(z_1,z_2).$$

Let the vector $\xi(z_i)=(1\quad z_i\quad\cdots\quad z_i^n)^T,\quad i=1,2$ and $F=[f_{ij}]_{i,j=0,\ldots,n}$ the matrix containing the coefficients of the Taylor expansion of the function $f(z_1,z_2)$, in this way, f_{ij} is the coefficient of $z_1^i\times z_2^j$.

Suppose that the Taylor expansion of the function $f(z_1,z_2)$ is

$$f(z_1,z_2)=(vec(F))^T vec(\xi(z_1)\circledast\xi(z_2)^T)+\cdots.$$

The vector $vec(\xi(z_1)\circledast\xi(z_2)^T)$ is the first column of the hypercomplex Pascal matrix, i.e., $vec(\xi(z_1)\circledast\xi(z_2)^T)=\mathbf{p}(z_1,z_2)$.

Using the matrix $\mathbb{L}$, it is possible to transform the Taylor expansion of $f(z_1, z_2)$ into the Bernoulli expansion, that is, into the expansion in terms of hypercomplex Bernoulli polynomials:

$$\begin{aligned} f(z_1, z_2) &= (vec(F))^T \mathbf{p}(z_1, z_2) + \cdots \\ &= (vec(F))^T \mathbb{L}\mathbf{b}(z_1, z_2) + \cdots . \end{aligned}$$

5 CONCLUSION AND ACKNOWLEDGEMENT

In this paper we have mainly referred to results concerning hypercomplex Bernoulli polynomials, nevertheless, a similar approach can be discussed for hypercomplex Euler polynomials.

We have seen that block matrices can be useful to work with polynomials in several variables and, taking into account the last result, it can be an important tool to achieve an interpolation formula for functions in several variables similarly to that obtained by Tauber [9].

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