17th International Conference on the Application of Computer
Science and Mathematics in Architecture and Civil Engineering
K. Gürlebeck and C. Könke (eds.)
Weimar, Germany, 12–14 July 2006

A COUPLED RITZ-GALERKIN APPROACH USING HOLOMORPHIC AND ANTI-HOLOMORPHIC FUNCTIONS

S. Bock* **and K. Gürlebeck**

**Institute of Mathematics/Physics, Bauhaus-University, Weimar, Germany*
E-mail: sebastian.bock@bauing.uni-weimar.de

Keywords: FEM, holomorphic functions, complete orthonormal systems, coupling.

Abstract. *The contribution focuses on the development of a basic computational scheme that provides a suitable calculation environment for the coupling of analytical near-field solutions with numerical standard procedures in the far-field of the singularity. The proposed calculation scheme uses classical methods of complex function theory, which can be generalized to 3-dimensional problems by using the framework of hypercomplex analysis. The adapted approach is mainly based on the factorization of the Laplace operator* $\Delta = D\overline{D}$ *by the Cauchy-Riemann operator* D*, where exact solutions of the respective differential equation are constructed by using an orthonormal basis of holomorphic and anti-holomorphic functions.*

1 INTRODUCTION

The Finite Element Method takes a central position in computational mechanics. Adaptivity and flexibility are the main reasons for their wide application in engineering practice. However a significant drawback of the classical FE-Method lies in the modeling and computation of boundary value problems with singularities, like for instance in fracture mechanics. Sufficiently accurate results in the near-field of the crack tip can mostly just obtained by high expenses in net-refinement and computational resources. To overcome these numerical problems one strategy adapted is to use analytical approaches in the near-field of the singularity and numerical standard procedures (e.g. FEM) in the far-field of the crack, where the influence of the tension singularity is negligible small. The underlying idea, to increase the quality of the solution in localized regions, coincides with other adaptive methods, like for instance the hp-method or multi-grid-methods.

In the linear plain crack theory efficient solution representations in the neighborhood of the crack-tip have been developed by using analytical methods of complex function theory [6, 4]. Based on the *Formulas of Kolosov* the near-field solution of the crack can be represented by only two holomorphic functions $\Phi(z)$ and $\Psi(z)$, $z \in \mathbb{C}$. The evaluation of $\Phi(z)$ and $\Psi(z)$ for a given plain elasticity problem is generally done by conformal mapping techniques of the exterior crack region onto the interior of the unit disc or by using complex power series to approximate the solution in the neighborhood of the crack-tip [7].

So the motivation in using coupled hybrid numeric-analytical approaches is to utilize the advantages of both methods and combine them in one stable calculation method. The numerical procedures provide a high flexibility and adaptivity in net generation in particular for arbitrary bounded domains, fast and efficient solvers for large systems with high degrees of freedom and possess further a high level of standardization concerning for instance finite element models or solution algorithms. In contrast to that, analytical methods are more convenient for such problems where a high accuracy of the solution in the neighborhood of singular areas are needed, because mechanical conservation laws and invariance properties are exactly reflected. Furthermore, one remarkable advantage in using especially function theoretical or generalized hypercomplex methods is the fact, that there exist various fundamental methods and utilities to evaluate and classify the analytical solution concerning, like for instance, error estimations, rate of convergence evaluations or completeness considerations.

In the following we are going to introduce some basic notations and the global geometrical setting of the computational scheme. In Section 3 exact polynomial solutions of the homogeneous Lamé equation are constructed by using a complete orthonormal system of holomorphic and anti-holomorphic polynomials. These almost orthonormal polynomial solutions to the Lamé equation will be used in Section 4 to establish an abstract calculation scheme for the proposed hybrid numeric-analytical method, which preserves the claimed conditions.

2 PRELIMINARIES AND NOTATIONS

We work in the field $\mathbb{C}$ of one complex variable, where we identify each point of the complex plane $\mathbb{C}$ with the ordered pair $z = (x, y) \in \mathbb{R}^2$ of the real numbers $x, y \in \mathbb{R}$ or equivalently with the complex number $z = x + iy \in \mathbb{C}$, where i denotes the imaginary unit of z. The addition of two complex numbers z_1, z_2 is defined coordinate-wisely

$$z_1 + z_2 = (x_1 + i\,y_1) + (x_2 + i\,y_2) = (x_1 + x_2) + i\,(y_1 + y_2)$$

and coincides with the real vector addition in $\mathbb{R}^2$. The complex multiplication is commutative and associative

$$z_1 z_2 = (x_1 + i\, y_1)\,(x_2 + i\, y_2) = (x_1\, x_2 - y_1\, y_2) + i\,(x_1\, y_2 + x_2\, y_1)\,,$$

where the relation $\sqrt{-1} = i$ holds. The complex number $\overline{z} = x - i\, y$ is called *conjugate* of z, such that the norm of z can be introduced by $|z| = \sqrt{z\,\overline{z}} = \sqrt{x^2 + y^2}$, which is analogous to the corresponding Euclidian norm of z, as a vector in $\mathbb{R}^2$. Note that $|z|^2 = z\,\overline{z}$ includes the complex division given by

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2}\,, z \neq 0\,.$$

Furthermore we call $x = \mathbf{Sc}(z) = \frac{1}{2}(z + \overline{z})$ the real or scalar part and $y = \mathbf{Im}(z) = -\frac{i}{2}(z - \overline{z})$ the imaginary part of z.

Let Ω be an open subset of $\mathbb{R}^2$ with a piecewise smooth boundary. The mapping $f : \Omega \longrightarrow \mathbb{C}$ defines a $\mathbb{C}$-valued function such that

$$f(z) = f_0(x, y) + i\, f_1(x, y),\; z \in \Omega$$

and the coordinates $f_j : \Omega \longrightarrow \mathbb{R}\,(j = 0, 1)$ are real-valued functions defined in Ω. Continuity, differentiability and integrability are defined coordinate-wisely. According to [3], the inner product for the linear Hilbert space $L_2(\Omega, \mathbb{C})$ is defined by

$$< f, g >_{L_2(\Omega,\mathbb{C})} = \int_\Omega f\,\overline{g}\, d\sigma,\;\; f, g \in L_2(\Omega, \mathbb{C})\,, \tag{1}$$

where $d\sigma$ denotes the Lebesgue measure in $\mathbb{R}^2$. Due to the condition that linear combinations of real solutions to a given differential equation are again exact solutions, we will work with the real linear Hilbert space of square-integrable $\mathbb{C}$-valued functions defined in Ω, that is denoted by $L_2(\Omega, \mathbb{R})$. The corresponding inner product is defined by

$$< f, g >_{L_2(\Omega,\mathbb{R})} = \int_\Omega \mathbf{Sc}(f\,\overline{g})\, d\sigma,\;\; f, g \in L_2(\Omega, \mathbb{R})\,. \tag{2}$$

For continuously real-differentiable functions $f : \Omega \longrightarrow \mathbb{C}$, the operator

$$D = \frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y} \tag{3}$$

is called *Cauchy-Riemann* operator. The *conjugate Cauchy-Riemann* operator we denote by

$$\overline{D} = \frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\,.$$

Remark. By reason of a later on generalization of the approach to higher dimensions, we remark, that the notation of the Cauchy-Riemann operator and the conjugate Cauchy-Riemann operator introduced in this article are different to the commonly used notations. ($\overline{D} \,\widehat{=}\, \partial_z,\; D \,\widehat{=}\, \overline{\partial}_z$)

A function $f : \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{C}$ is called holomorphic in Ω (resp., anti-holomorphic in Ω) if

$$Df = 0 \quad \text{in} \quad \Omega \quad (\text{resp.,}\ \overline{D}f = 0 \quad \text{in} \quad \Omega\,).$$

The class of all holomorphic functions in Ω will be denoted, as usual, by $\mathcal{O}(\Omega)$ and in the same manner the class of all anti-holomorphic functions in Ω by $\overline{\mathcal{O}}(\Omega)$. Moreover we recall the well known factorization of the *Laplace* operator by the Cauchy-Riemann operator and its conjugate that is given by $\Delta = D\,\overline{D} = \overline{D}\,D$, which decomposes the class $\mathcal{H}(\Omega)$ of all harmonic functions in Ω into the direct sum $\mathcal{H}(\Omega) = \mathcal{O}(\Omega) \oplus \overline{\mathcal{O}}(\Omega)$ and naturally implies that any holomorphic function (resp., anti-holomorphic function) is also a harmonic function. Since we will work with polynomial systems, we identify the space of all holomorphic (resp. anti-holomorphic) polynomials up to a maximum polynomial degree of $n \in \mathbb{N}_0$ with $\mathcal{O}_n(\Omega)$ (resp. $\overline{\mathcal{O}}_n(\Omega)$). In the same way, the subspace of all harmonic polynomials up to a maximum polynomial degree of $n \in \mathbb{N}_0$ is denoted by $\mathcal{H}_n(\Omega)$.

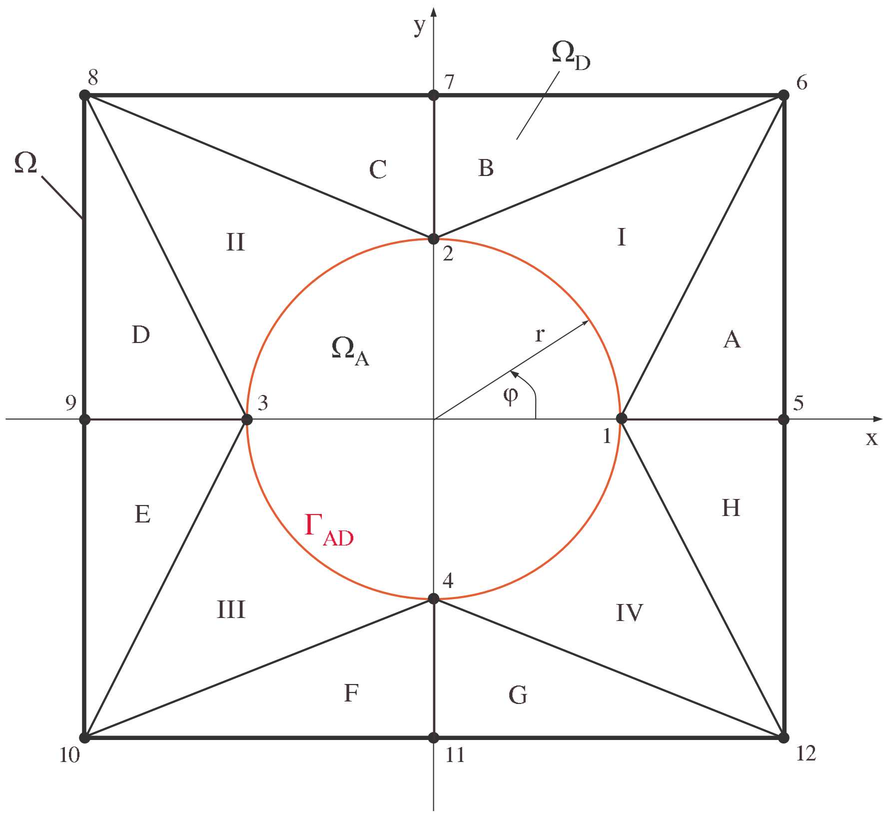

Figure 1: Global setting and notation

Finally we introduce polar coordinates by $x = r\cos\varphi,\ y = r\sin\varphi$ and arrive to the representation

$$z = r\,e^{i\varphi} = r\,(\cos\varphi + i\,\sin\varphi), \quad 0 \le r < \infty,\ 0 \le \varphi < 2\pi\,.$$

Let now $\Omega \subset \mathbb{C}$ be a bounded simply connected domain that is decomposed in the two sub-domains $\Omega = \Omega_A \cup \Omega_D$ separated by the fictitious joint interface $\Gamma_{AD} = \overline{\Omega}_A \cap \overline{\Omega}_D$ (see Figure 1). The discrete numerical domain, denoted by Ω_D, is modeled by two different kinds of elements: the CST-element of class C^0 (in example elements $A - H$) and the Coupling-element of class $C^0 - C^\infty$ (in example elements $I - IV$), that couples the discrete domain Ω_D with the analytical domain Ω_A. We call the sub-domain Ω_A analytical in that sense, that the constructed solutions are exact solutions to the differential equation in Ω_A and thus represent a possible mechanical state.

3 CONSTRUCTION OF EXACT SOLUTIONS TO THE LAMÉ EQUATION

In this section we construct exact polynomial solutions to the homogeneous Lamé equation by using a basis of holomorphic and anti-holomorphic polynomials. The idea behind is the already mentioned factorization of the Laplace operator by the Cauchy-Riemann operator and its conjugate. First we recall the classical well known matrix formulation of the Lamé equation and proof the analogy to their complex representation by help of the *Cauchy-Riemann* operator. Complete $\mathbb{R}$-linear orthonormal systems of holomorphic and anti-holomorphic polynomials are introduced and finally used to construct complete almost orthonormal systems that are exact solutions to the Lamé equation.

3.1 Matrix representation of the Lamé equation

In linear elastostatics the physical state of each continuums model is described by the 3 fundamental equations of linear elasticity theory, which are the equilibrium equations, the constitutive equations and the strain-displacement relations. Solving the equations with respect to the unknown displacement vector $\underline{\mathbf{u}} = [\, u(x,y)\, ,\, v(x,y)\,]^T$ we obtain the Lamé equation in the classical matrix formulation

$$-\underline{p} \;=\; \underline{D}_e\, \underline{\mathbf{E}}\, \underline{D}_k\, \underline{\mathbf{u}} \tag{4}$$

where $\underline{p} = [\, p_x\, ,\, p_y\,]^T$ denotes the vector of the outer forces and

$$\underline{D}_e \;=\; (\underline{D}_k)^T \;=\; \begin{bmatrix} \partial_{,x} & 0 & \partial_{,y} \\ 0 & \partial_{,y} & \partial_{,x} \end{bmatrix}$$

are the adjoint differential operators of equilibrium and kinematics respectively. The matrix

$$\underline{\mathbf{E}} = G \begin{bmatrix} \frac{\kappa+1}{\kappa-1} & -\frac{\kappa-3}{\kappa-1} & 0 \\ -\frac{\kappa-3}{\kappa-1} & \frac{\kappa+1}{\kappa-1} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ where } \kappa = \begin{cases} 3-4\nu & : \quad \text{plane strain state} \\ \dfrac{3-\nu}{1+\nu} & : \quad \text{plane stress state} \end{cases}$$

includes material parameters for a linear elastic, homogeneous and isotropic material in the usual notations.

Evaluating equation (4) we obtain

$$\underline{D}_k\, \underline{\mathbf{u}} = \left[u_{,x}\ ,\quad v_{,y}\ ,\quad u_{,y} + v_{,x}\right]^T$$

and

$$\underline{\mathbf{E}}\, \underline{D}_k\, \underline{\mathbf{u}} = G \begin{bmatrix} \frac{\kappa+1}{\kappa-1} u_{,x} - \frac{\kappa-3}{\kappa-1} v_{,y} \\ \frac{\kappa+1}{\kappa-1} v_{,y} - \frac{\kappa-3}{\kappa-1} u_{,x} \\ u_{,y} + v_{,x} \end{bmatrix}$$

and finally

$$-\underline{p} \;=\; \underline{D}_e\, \underline{\mathbf{E}}\, \underline{D}_k\, \underline{\mathbf{u}} \;=\; G \begin{bmatrix} \frac{\kappa+1}{\kappa-1} u_{,xx} + u_{,yy} + \left(1 - \frac{\kappa-3}{\kappa-1}\right) v_{,xy} \\ \frac{\kappa+1}{\kappa-1} v_{,yy} + v_{,xx} + \left(1 - \frac{\kappa-3}{\kappa-1}\right) u_{,xy} \end{bmatrix}. \tag{5}$$

3.2 Complex representation of the Lamé equation

Proposition 3.1 *Let* $\mathbf{u} = u(x,y) + i\,v(x,y) \in \Omega$ *be the displacement function in the bounded simply connected domain* $\Omega \subset \mathbb{C}$. *A purely complex representation of (4) is given by*

$$-p \,=\, D\,M^{-1}\,\overline{D}\,\mathbf{u} \tag{6}$$

where D *denotes the* Cauchy-Riemann *operator (3) and* M^{-1} *a multiplication operator which is, acting on a function* $w \,=\, w_0 \,+\, i\,w_1$, *defined by*

$$M^{-1}\,w \,=\, G\frac{\kappa+1}{\kappa-1}\,w_0 \,+\, i\,G\,w_1\,.$$

Proof. We show the analogy to (5) by a straightforward computation. Applying $\overline{D}$ to $\mathbf{u}$ we get

$$\overline{D}\,\mathbf{u} \,=\, (\partial_{,x} - i\,\partial_{,y}\,)(u \,+\, i\,v) \,=\, (u_{,x} + v_{,y}\,) \,+\, i\,(v_{,x} - u_{,y}\,).$$

Further

$$M^{-1}\,\overline{D}\,\mathbf{u} \,=\, G\frac{\kappa+1}{\kappa-1}(u_{,x} + v_{,y}\,) \,+\, i\,G\,(v_{,x} - u_{,y}\,)$$

and

$$\begin{aligned} D\,M^{-1}\,\overline{D}\,\mathbf{u} \;&=\; \left[G\frac{\kappa+1}{\kappa-1}(u_{,xx} + v_{,xy}\,) \,-\, G\,(v_{,xy} - u_{,yy}\,)\right] \\ &\quad + i\,\left[G\frac{\kappa+1}{\kappa-1}(u_{,xy} + v_{,yy}\,) \,+\, G\,(v_{,xx} - u_{,xy}\,)\right] \\ &=\; \left[G\frac{\kappa+1}{\kappa-1}\,u_{,xx} + G\,u_{,yy} + G\left(\frac{\kappa+1}{\kappa-1} - 1\right)\,v_{,xy}\right] \\ &\quad + i\,\left[G\frac{\kappa+1}{\kappa-1}\,v_{,yy} + G\,v_{,xx} + G\left(\frac{\kappa+1}{\kappa-1} - 1\right)\,u_{,xy}\right]\,. \end{aligned}$$

By comparison of the coefficients (5) follows, that the first two conditions are directly fulfilled. Extending the third coefficient by $G\left(\frac{(\kappa-1)-(\kappa-1)}{\kappa-1}\right)$ we get

$$G\left(\frac{\kappa+1}{\kappa-1} - 1\right) \,=\, G\left(\frac{(\kappa+1)-(\kappa-1)+(\kappa-1)-(\kappa-1)}{\kappa-1}\right) \,=\, G\left(1 - \frac{\kappa-3}{\kappa-1}\right).$$

and the proposition is proofed. ■

Remark. Referring to [5], we point out that equation (6) is analogously defined as in their corresponding hypercomplex representation in the skew field of real quaternions $\mathbb{H}$.

3.3 An $\mathbb{R}$-linear complete orthonormal system of holomorphic and anti-holomorphic polynomials

To prepare the construction of exact polynomial solutions to the Lamé equation, we introduce complete orthonormal systems of holomorphic and anti-holomorphic functions. This is mainly motivated by the fact, that the resulting contribution to the global stiffness matrix is sparse and well-conditioned.

Let $K_{r_a} = \left\{ z \,\middle|\, |z| \leq r_a \right\} \subset \mathbb{C}$ be the disc with radius r_a and centered in the origin. Following [3] we consider the holomorphic

$$\left\{ \sqrt{\frac{k+1}{r_a^{k+1}\,\pi}}\, z^k \right\}_{k=0,\ldots,n}, \tag{7}$$

and the anti-holomorphic powers

$$\left\{ \sqrt{\frac{k+1}{r_a^{k+1}\,\pi}}\, \overline{z}^l \right\}_{l=0,\ldots,n}. \tag{8}$$

As a matter of fact, system (7) (resp. system (8)) is a basis in $\mathcal{O}_n(K_{r_a}, \mathbb{C})$ (resp. in $\overline{\mathcal{O}}_n(K_{r_a}, \mathbb{C})$) concerning the inner product (1). Since $\dim \mathcal{H}_n(\Omega, \mathbb{C}) = 2n+1$, the dimension of the class of all harmonic polynomials in $\mathcal{H}_n(\Omega, \mathbb{R})$ has to be $\dim \mathcal{H}_n(\Omega, \mathbb{R}) = 4n+2$. To obtain a basis in $\mathcal{H}_n(K_{r_a}, \mathbb{R})$ in terms of holomorphic and anti-holomorphic polynomials we multiply the systems (7) and (8) by i and remove the linearly dependent complex constants of polynomial degree $n = 0$.

Proposition 3.2 *(Decomposition) We have the orthogonal decomposition*

$$\mathcal{H}_n(K_{r_a}, \mathbb{R}) = \mathcal{O}_n(K_{r_a}, \mathbb{R}) \oplus \overline{\mathcal{O}}_n(K_{r_a}, \mathbb{R}),$$

where the system of holomorphic polynomials

$$\left\{ \phi_k(z) \right\}_{k=0,\ldots,2n} = \left\{ \sqrt{\frac{l+1}{r_a^{l+1}\,\pi}}\, z^l \,,\, \sqrt{\frac{m+1}{r_a^{m+1}\,\pi}}\, i z^m \right\}_{(l=0,\ldots,n,m=0,\ldots,n-1)} \tag{9}$$

and the system of anti-holomorphic polynomials

$$\left\{ \psi_k(z) \right\}_{k=0,\ldots,2n} = \left\{ \sqrt{\frac{m+1}{r_a^{m+1}\,\pi}}\, \overline{z}^m \,,\, \sqrt{\frac{l+1}{r_a^{l+1}\,\pi}}\, i\, \overline{z}^l \right\}_{(l=0,\ldots,n,m=0,\ldots,n-1)} \tag{10}$$

are a basis in $\mathcal{O}_n(K_{r_a}, \mathbb{R})$ and $\overline{\mathcal{O}}_n(K_{r_a}, \mathbb{R})$, respectively.

Proposition 3.3 *(Completeness) The systems $\left\{ \phi_k(z) \right\}_{k=0,\ldots,\infty}$ and $\left\{ \psi_k(z) \right\}_{k=0,\ldots,\infty}$ are complete in $L_2(K_{r_a}, \mathbb{R}) \bigcap \ker D\overline{D}$.*

Finally, we summarize some properties of the subspaces (9) and (10). Let $f(z) = f_0(x,y) + i\, f_1(x,y) \in \left\{ \phi_k(z) \right\}_{k=0,\ldots,2n}$ and $\hat{f}(z) = \hat{f}_0(x,y) + i\, \hat{f}_1(x,y) \in \left\{ \psi_k(z) \right\}_{k=0,\ldots,2n}$ we have

$$\begin{aligned} (i) \quad & f(z) \in ker\, D \subset \ker D\overline{D}, \\ (ii) \quad & \hat{f}(z) \in \ker \overline{D} \subset \ker D\overline{D}, \\ (iii) \quad & f_j(x,y), \hat{f}_j(x,y) \in \ker D\overline{D},\ j = 0,1. \end{aligned} \tag{11}$$

3.4 Exact polynomial solutions to Lamé equation

Consider now the coupling of the numerical domain Ω_{D} and the analytical domain Ω_{A} on the joint interface Γ_{AD}. We are going to construct exact solutions to the homogeneous Lamé equation

$$0 \,=\, D\,\tilde{M}^{-1}\,\overline{D}\,\mathbf{u}\,, \tag{12}$$

where the multiplication operator M^{-1} can be reduced to an operator $\tilde{M}^{-1}$ that is only acting on the scalar part of the function. Dividing (12) by G, we obtain consequently

$$\tilde{M}\,\mathbf{u} \,=\, M_0\,\mathbf{Sc}\,\mathbf{u} + \mathbf{Vec}\,\mathbf{u} \,=\, \frac{\kappa-1}{\kappa+1}\,u \,+\, i\,v\,.$$

Proposition 3.4 *For a fixed $n \in \mathbb{N}_0$ the 4n+2 polynomials of the system*

$$\begin{aligned}
(\mathbf{I}) \quad \Big\{\mathbf{f}_k(z)\Big\}_{k=0,\ldots,2n} \quad &= \quad \Big\{\phi_k(z) \,+\, \frac{1}{2}(M_0-1)\,D\,\big(\mathbf{M}_p\,\phi_k(z)\big)\Big\}_{k=0,\ldots,2n} \\
(\mathbf{II}) \quad \Big\{\hat{\mathbf{f}}_l(z)\Big\}_{l=0,\ldots,2n} \quad &= \quad \Big\{\psi_l(z) \,+\, \frac{1}{2}(M_0-1)\,D\,\big(\mathbf{M}_p\,\psi_l(z)\big)\Big\}_{l=0,\ldots,2n}
\end{aligned} \tag{13}$$

are exact solutions to the homogeneous Lamé equation (12). The operator $\mathbf{M}_p$ is defined by

$$\mathbf{M}_p\,h(z) = x\,\mathbf{Sc}\,h(z) \,+\, y\,\mathbf{Vec}\,h(z) = x\,h_0(x,y) + y\,h_1(x,y)\;. \tag{14}$$

Proof. We prove that the system (13) is an exact solution to (12) in a generalized way.

First to prove group $(\mathbf{I})$ of (13) we show that for any holomorphic function $f(z) \in \ker D$ the function $\mathbf{f}(z) = f(z) \,+\, \frac{1}{2}(M_0-1)\,D\,\big(\mathbf{M}_p\,f(z)\big)$ is an exact solution to the homogeneous Lamé equation (12). We have

$$\begin{aligned}
\mathbf{f}(z) \quad = \quad f_0 + i\,f_1 \quad + \quad \frac{1}{2}(M_0-1)\,\Big[\,&(f_0 \,+\, x\,f_{0,_x} \,+\, y\,f_{1,_x}) \\
&+ (f_1 \,+\, x\,f_{0,_y} \,+\, y\,f_{1,_y})\,i\Big]\,.
\end{aligned} \tag{15}$$

Applying $\overline{D}$ to (15) we get

$$\begin{aligned}
\overline{D}\,\mathbf{f}(z) \quad = \quad & (f_{0,_x} \,+\, f_{1,_y}) \,+\, i\,(f_{1,_x} \,-\, f_{0,_y}) \\
& + \frac{1}{2}(M_0-1)\,\Big[\,\Big(2\,f_{0,_x} \,+\, x\,f_{0,_{xx}} \,+\, y\,f_{1,_{xx}} \\
& \qquad + 2\,f_{1,_y} \,+\, x\,f_{0,_{yy}} \,+\, y\,f_{1,_{yy}}\Big) \\
& \qquad + i\,\Big(f_{1,_x} \,+\, f_{0,_y} \,+\, x\,f_{0,_{yx}} \,+\, y\,f_{1,_{yx}} \\
& \qquad\quad - f_{1,_x} \,-\, f_{0,_y} \,-\, x\,f_{0,_{xy}} \,-\, y\,f_{1,_{xy}}\Big)\,.
\end{aligned} \tag{16}$$

Using the properties (11), equality (16) reduces to

$$\begin{aligned}
\overline{D}\,\mathbf{f}(z) \quad &= \quad (f_{0,_x} \,+\, f_{1,_y}) \,+\, i\,(f_{1,_x} \,-\, f_{0,_y}) \\
&\qquad + \frac{1}{2}(M_0-1)\,\big[\,2\,(f_{0,_x} \,+\, f_{1,_y}) + x\,(f_{0,_{xx}} \,+\, f_{0,_{yy}}) + y\,(f_{1,_{xx}} \,+\, f_{1,_{yy}}) \\
&= \quad M_0\,(f_{0,_x} \,+\, f_{1,_y}) \,+\, i\,(f_{1,_x} \,-\, f_{0,_y}) \\
&= \quad \tilde{M}\,\overline{D}\,f(z)\,.
\end{aligned} \tag{17}$$

Since $\overline{D}\, f(z) \,\in\, ker\, D$, we get finally

$$D\,\tilde{M}^{-1}\,\overline{D}\,\mathbf{f}(z) \;\;=\;\; D\,\tilde{M}^{-1}\,\tilde{M}\,\overline{D}\,f(z) \;=\; 0$$

and the first system $(\mathbf{I})$ of the proposition is proved.

To prove system $(\mathbf{II})$ we show in analogy to (15), that for any anti-holomorphic function $\hat{f}(z) \in \ker \overline{D}$ the function $\hat{\mathbf{f}}(z) = \hat{f}(z) + \frac{1}{2}(M_0 - 1)\, D\left(\mathbf{M}_p\, \hat{f}(z)\right)$ is an exact solution to (12). Hence, we obtain

$$\begin{aligned}\hat{\mathbf{f}}(z) \;\;=\;\; \hat{f}_0 + i\,\hat{f}_1 \;\;+\;\; \frac{1}{2}(M_0 - 1)\Big[\, & (\hat{f}_0 \,+\, x\,\hat{f}_{0,_x} \,+\, y\,\hat{f}_{1,_x}\,) \\ & + (\hat{f}_1 \,+\, x\,\hat{f}_{0,_y} \,+\, y\,\hat{f}_{1,_y})\, i\,\Big]\,.\end{aligned}$$

Due do the fact that $\hat{f}(z) \in \ker \overline{D}$ we end up directly with

$$\begin{aligned}\overline{D}\,\hat{\mathbf{f}}(z) \;\;&=\;\; (\hat{f}_{0,_x} \,+\, \hat{f}_{1,_y}) \,+\, i\,(\hat{f}_{1,_x} \,-\, \hat{f}_{0,_y}) \\ &\quad + \frac{1}{2}(M_0 - 1)\left[\, 2\,(\hat{f}_{0,_x} \,+\, \hat{f}_{1,_y}) + x\,(\hat{f}_{0,_{xx}} \,+\, \hat{f}_{0,_{yy}}) + y\,(\hat{f}_{1,_{xx}} \,+\, \hat{f}_{1,_{yy}})\,\right] \\ &=\;\; 0\end{aligned}$$

and therefore

$$D\,\tilde{M}^{-1}\,\overline{D}\,\hat{\mathbf{f}}(z) \;\;=\;\; 0\,,$$

which proves the proposition for system $(\mathbf{II})$. ■

Remark. Besides the properties defined in (11), we can state the following additional ones. Because of (17) the condition $\dfrac{1}{2}\,D\left(\mathbf{M}_p\,\phi_k(z)\right) \,\in\, \ker \overline{D}\,D\,\overline{D}$ holds, which implies that the result of applying the multiplication operator (14) to a holomorphic function $\phi_k(z) \in \ker D\,(k = 0, \ldots, 2n)$ yields to a bi-harmonic function

$$\frac{1}{2}\,\mathbf{M}_p\,\phi_k(z) \,\in\, \ker\, \overline{D}\,D\,\overline{D}\,D \;=\; \ker\, \triangle\,\triangle\,.$$

In contrast we obtain by applying (14) to an anti-holomorphic function $\psi_l(z) \,\in\, \ker \overline{D}\,(l = 0, \ldots, 2n)$ a harmonic function. Since $\dfrac{1}{2}\,D\left(\mathbf{M}_p\,\psi_l(z)\right) \,\in\, \ker \overline{D}$, we get consequently

$$\frac{1}{2}\,\mathbf{M}_p\,\psi_l(z) \,\in\, \ker \overline{D}\,D \;=\; \ker\, \triangle\,.$$

4 ABSTRACT CALCULATION SCHEME

4.1 Description of the analytical domain Ω_{A}

The exact polynomial solutions (13) to the homogeneous *Lamé* equation (12) constructed in Section 3.4 are used to approximate exact solutions for the analytical domain Ω_{A}. Let u be the

$\mathbb{C}$-valued displacement field in Ω_{A}. The associated boundary value problem of Dirichlet type in linear elastostatics is given by

$$\begin{aligned} D\tilde{M}^{-1}\overline{D}\,\mathbf{u} &= 0 \ \text{ in } \Omega_{\text{A}}\,, \\ \mathbf{u} &= g \ \text{ on } \Gamma_{\text{AD}}\,, \end{aligned} \tag{18}$$

where $g(z) \in \mathbb{C}$ is a given displacement function on Γ_{AD}.

To solve (18) we use a classical *Ritz* approach and calculate the best approximation in $L_2(\partial\Omega_{\text{A}}, \mathbb{R})$. Using the system (13), we obtain for a fixed polynomial degree $n \in \mathbb{N}_0$ the associated extreme value problem

$$\left\| \sum_{j=0}^{2n} \mathbf{f}_j(z)\,\beta_j + \sum_{k=0}^{2n} \hat{\mathbf{f}}_k(z)\,\gamma_k - g(z) \right\|^2_{L_2(\partial\Omega_{\text{A}},\mathbb{R})} \longrightarrow \min\,, \tag{19}$$

that leads to a linear system of $4n+2$ equations in the unknown coefficients $\beta_j, \gamma_k \in \mathbb{R}$ ($j, k = 0, \ldots, 2n$).

4.2 Description of the numerical domain Ω_{D}

The numerical domain Ω_{D} consists of two different kinds of elements. In the far-field of the analytical inclusion we use *Constant-Strain* elements of class C^0, where the primary variables are linearly interpolated and the secondary variables are constantly represented. We remark that this approach is not restricted to CST-elements, whereby elements of higher order can be easily adapted as well. The second kind of the used elements are the so called *Coupling* elements, that connect the discrete domain Ω_{D} modeled by CST-elements with the analytical domain Ω_{A}. These special Coupling elements are interpolating the primary variables on two boundaries with C^0 continuity and with an trigonometric C^∞ function on Γ_{AD}. The idea behind was mainly motivated by the fact, that a high accuracy of the solution in the interior of Ω_{A} can just be obtained by a shape preserving and conformal coupling of Ω_{D} and Ω_{A} on Γ_{AD}.

To obtain a shape preserving or at least a nearly shape preserving coupling, spline approaches were considered. But using splines to describe Γ_{AD} coming from Ω_{D} and holomorphic/anti-holomorphic functions on Γ_{AD} coming from Ω_{A} has a significant drawback. Let us consider a fixed interval on Γ_{AD}, the holomorphic polynomials z^n are highly oscillating if the polynomial degree n is increasing. Subsequently, a nearly shape preserving interpolation can just be obtained by using higher order splines and by decreasing the interpolation interval at the same time. Therefore spline approaches are not suitable for this setting. To solve these problems we are going to use the *Discrete Fourier Analysis* in $\mathbb{C}$, which is based on the following trigonometric interpolation theorem.

Theorem 4.1 *For given observations* $\mathrm{Y}_0, \mathrm{Y}_1, \ldots, \mathrm{Y}_n \in \mathbb{C}$ *exists a unique function*

$$t_n(\varphi) = \sum_{k=0}^{n} c_k\,\mathbf{e}^{ik\varphi} \tag{20}$$

that satisfies the interpolation conditions $t_n(\varphi_j) = \mathrm{Y}_j,\ j = 0, \ldots, n$. *The coefficients* c_k *are given by*

$$c_k = \frac{1}{n+1} \sum_{j=0}^{n} \mathrm{Y}_j\,\mathbf{e}^{-ij\varphi_k}\,, \tag{21}$$

where $\varphi_k = k\dfrac{2\pi}{n+1}$ *denotes the* $n+1$ *equidistant interpolation nodes on the interval* $\varphi \in [0, 2\pi)$.

Using (20) and (21) we rearrange the trigonometric interpolation formula as follows

$$\begin{aligned} t_n(\varphi) &= \sum_{k=0}^{n} \left[\tfrac{1}{n+1} \sum_{j=0}^{n} \mathrm{Y}_j\, \mathbf{e}^{-ij\varphi_k}\right] \mathbf{e}^{ik\varphi} \\ &= \sum_{k=0}^{n} \left[\tfrac{1}{n+1} \sum_{j=0}^{n} \mathbf{e}^{ij\varphi}\, \mathbf{e}^{-ik\varphi_j}\right] \mathrm{Y}_k \end{aligned}$$

and obtain consequently

$$t_n(\varphi) = \sum_{k=0}^{n} S_k(\varphi)\mathrm{Y}_k, \quad S_k(\varphi) = \frac{1}{n+1}\sum_{j=0}^{n} \mathbf{e}^{i(j\varphi - k\varphi_j)}, \quad S_k(\varphi), \mathrm{Y}_k \in \mathbb{C}\,. \tag{22}$$

4.2.1 Construction of a $(n+2)$-node identity coupling element $\mathbb{T}_n^{c,m}$

On the basis of (22) we construct a $(n+2)$-node coupling element that can be conformally coupled to the adjacent CST-elements of the numerical domain Ω_{D} and piecewisely to the analytical domain Ω_{A}. For a fixed discretization (c, m), we denote $\mathbb{T}_n^{c,m}$ as the identity coupling

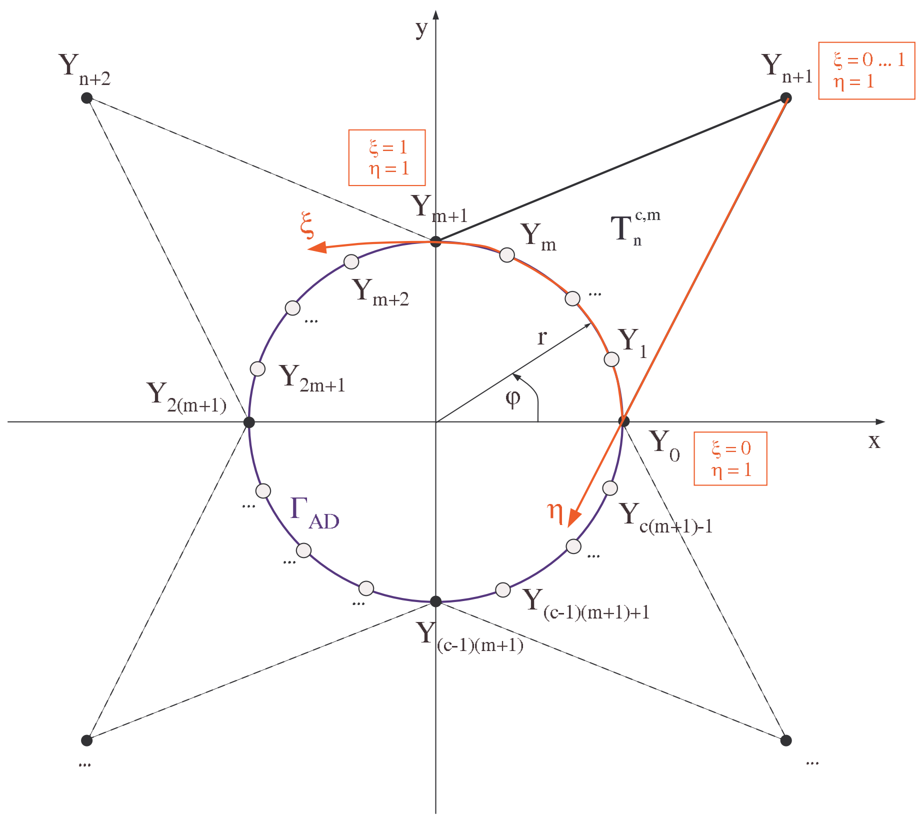

Figure 2: $(n+2)$-node identity coupling element $\mathbb{T}_n^{c,m}$

element of polynomial degree $n = c(m+1) - 1$ on Γ_{AD}. The parameters c and m define a discrete point grid for the identity coupling element $\mathbb{T}_n^{c,m}$, where $c \in \mathbb{N}, c \geq 2$ denotes the number of coupling elements used to discretize the joint coupling interface Γ_{AD}. The second parameter $m \in \mathbb{N}_0$ is concerned to the number of nodes used additionally on the boundary

$(\xi,\eta) \in (0,1)\times\{1\}$. Figure 2 indicates, for instance, a discretization that uses $c=4$ coupling elements on Γ_{AD} and $m \geq 2$ nodes on $(\xi,\eta) \in (0,1)\times\{1\}$. Hence, each identity coupling element $\mathbb{T}_n^{c,m}$ is uniquely defined by the ordered set

$$[\mathrm{Y}_0, \mathrm{Y}_1, \ldots, \mathrm{Y}_n, \mathrm{Y}_{n+1}]\,,\ \mathrm{Y}_p \in \mathbb{C}\,,\ p = 0,\ldots,n+1\,,$$

which implies that for a fixed discretization (c,m) with a polynomial degree of $n = c(m+1)-1$ on Γ_{AD} are $n+2$ discrete nodes required. Caused by the uniqueness of the boundary polynomial defined by the subset of $n+1$ points $\{\mathrm{Y}_q\}_{(q=0,\ldots,n)}$ on Γ_{AD}, all other coupling elements that belong to the same discretization (c,m) can be obtained by cyclic permutation of the nodes, like for instance:

$$\begin{aligned}
\mathbb{T}_n^{c,m} = \mathbb{T}_{n,0}^{c,m} \quad &:= \quad [\mathrm{Y}_0, \mathrm{Y}_1, \ldots, \mathrm{Y}_{n-1}, \mathrm{Y}_n, \mathrm{Y}_{n+1}] \\
\mathbb{T}_{n,1}^{c,m} \quad &:= \quad [\mathrm{Y}_{m+1}, \ldots, \mathrm{Y}_n, \mathrm{Y}_0, \ldots, \mathrm{Y}_m, \mathrm{Y}_{n+2}] \\
&\ \vdots
\end{aligned}$$

Adapting the general trigonometric polynomial (22) which is defined on $\varphi \in [0,2\pi)$ to the given geometrical setting, we have to restrict the function to the sub interval $\tilde{\varphi} \in [\varphi_0, \varphi_{m+1}]$ by the linear mapping to $\xi \in [0,1]$. With $\varphi(0) = \varphi_0$ and $\varphi(1) = \varphi_{m+1}$ we obtain the linear transformation

$$\varphi(\xi) = \frac{2\pi(m+1)}{n+1}\,\xi = \frac{2\pi(m+1)}{c(m+1)-1+1}\,\xi = \frac{2\pi}{c}\,\xi\,,$$

which is substituted in (22)

$$S_k\big(\varphi(\xi)\big) = \frac{1}{n+1}\sum_{j=0}^{n} \mathbf{e}^{i(j\varphi(\xi)-k\varphi_j)} = \frac{1}{n+1}\sum_{j=0}^{n} \mathbf{e}^{i(j\frac{2\pi}{c}\xi - kj\frac{2\pi}{n+1})}$$

and consequently

$$S_k(\xi) = \frac{1}{n+1}\sum_{j=0}^{n} \mathbf{e}^{2\pi i j\frac{(m+1)\,\xi-k}{c(m+1)}}\,, \quad \xi \in [0,1]\,. \tag{23}$$

We identify (23) with the k-th node shape function of the identity coupling element $\mathbb{T}_n^{c,m}$ which piecewisely interpolates the curvilinear boundary of Γ_{AD}. Finally we introduce the coordinate $\eta \in [0,1]$ that is used as a linear interpolator between the subinterval $\tilde{\varphi} \in [\varphi_0, \varphi_{m+1}]$ of Γ_{AD} and the node Y_{n+1} and end up by the following proposition.

Proposition 4.1 *For a fixed discretization (c,m) and $n = c(m+1)-1 \in \mathbb{N}$ the characteristic interpolation polynomial (geometry mapping) $\mathbb{X}_n^{c,m}(\xi,\eta) \in \mathbb{C}$ for the $(n+2)$-node identity coupling element $\mathbb{T}_n^{c,m}$ is given by*

$$\mathbb{X}_n^{c,m}(\xi,\eta) = \sum_{q=0}^{n+1} \mathrm{N}_q(\xi,\eta)\,\mathrm{Y}_q\,, \tag{24}$$

where

$$\mathrm{N}_q(\xi,\eta) = \begin{cases} \dfrac{\eta}{n+1}\displaystyle\sum_{j=0}^{n} \mathbf{e}^{2\pi i j\frac{(m+1)\,\xi-q}{c(m+1)}} & : \ q < n+1 \\[2ex] 1-\eta & : \ q = n+1 \end{cases}\,. \tag{25}$$

4.3 Coupling of Ω_{A} and Ω_{D}

The coupling of the analytical domain Ω_{A} and the numerical domain Ω_{D} is mainly motivated by the trigonometric interpolation theorem 4.1. Due to the fact that every trigonometric polynomial of degree n on Γ_{AD} is uniquely defined through $n+1$ discrete points which are in particular the unknown displacements $\text{U}_0, \ldots, \text{U}_n$ on Γ_{AD}, we can use the relation (22) to represent the boundary displacement function $g(z)$ as well. Thus, the extreme value problem (19) can be reformulated by

$$\left\| \sum_{j=0}^{2n} \mathbf{f}_j(z)\, \beta_j + \sum_{k=0}^{2n} \hat{\mathbf{f}}_k(z)\, \gamma_k - \sum_{l=0}^{N} S_l(\varphi) \text{U}_l \right\|^2_{L_2(\partial\Omega_{\text{A}}, \mathbb{R})} \longrightarrow \min ,$$

where N denotes the number of polynomials used to describe the given boundary displacement function $g(z)$. Taking into consideration that the polynomial degree of the approximated solution in the interior of $\overline{\Omega}_{\text{A}}$ can be at most the degree of the displacement function on its boundary Γ_{AD}, we adapt N and end up with

$$\left\| \sum_{j=0}^{2n} \mathbf{f}_j(z)\, \beta_j + \sum_{k=0}^{2n} \hat{\mathbf{f}}_k(z)\, \gamma_k - \sum_{l=0}^{n} S_l(\varphi) \text{U}_l \right\|^2_{L_2(\partial\Omega_{\text{A}}, \mathbb{R})} \longrightarrow \min . \tag{26}$$

.

The minimization of (26) with respect to the *Ritz-coefficients* $\beta_j, \gamma_k \in \mathbb{R}$ is done on the coupling boundary Γ_{AD} and yields to the following linear system of $4n + 2$ equations

$$\begin{bmatrix} \langle \mathbf{f}_j, \mathbf{f}_k \rangle_{L_2(\partial\Omega_{\text{A}}, \mathbb{R})} & \langle \mathbf{f}_j, \hat{\mathbf{f}}_k \rangle_{L_2(\partial\Omega_{\text{A}}, \mathbb{R})} \\ \langle \hat{\mathbf{f}}_k, \mathbf{f}_j \rangle_{L_2(\partial\Omega_{\text{A}}, \mathbb{R})} & \langle \hat{\mathbf{f}}_k, \hat{\mathbf{f}}_j \rangle_{L_2(\partial\Omega_{\text{A}}, \mathbb{R})} \end{bmatrix} \begin{bmatrix} \beta_j \\ \gamma_k \end{bmatrix} = \begin{bmatrix} \langle \mathbf{f}_j, \sum_{l=0}^{n} S_l(\varphi)\, \text{U}_l \rangle_{L_2(\partial\Omega_{\text{A}}, \mathbb{R})} \\ \langle \hat{\mathbf{f}}_k, \sum_{l=0}^{n} S_l(\varphi)\, \text{U}_l \rangle_{L_2(\partial\Omega_{\text{A}}, \mathbb{R})} \end{bmatrix}, \tag{27}$$

where $j, k = 0, \ldots, 2n$. Solving (27), we obtain the unknown coefficients $\beta_j = f(\text{U}_0, \ldots, \text{U}_n)$ and $\gamma_k = f(\text{U}_0, \ldots, \text{U}_n)$ as linear functions of the $n+1$ discrete node displacements $\text{U}_0, \ldots, \text{U}_n$ on Γ_{AD}. Thus, we can calculate the field of displacements $\mathbf{u}(z, \text{U}_0, \ldots, \text{U}_n) \in \mathbb{C}$ in Ω_{A} depending on the unknown discrete node displacements

$$\mathbf{u}(z, \text{U}_0, \ldots, \text{U}_n) \;=\; \sum_{j=0}^{2n} \mathbf{f}_j(z)\; \beta_j(\text{U}_0, \ldots, \text{U}_n) + \sum_{k=0}^{2n} \hat{\mathbf{f}}_k(z)\; \gamma_k(\text{U}_0, \ldots, \text{U}_n) .$$

If we identify now each polynomial coefficient

$$\begin{aligned} \mathbf{u}(z, 1, 0, \ldots, 0) &=: \text{N}_0^{\text{lamé}}(z) \\ \mathbf{u}(z, 0, 1, \ldots, 0) &=: \text{N}_1^{\text{lamé}}(z) \\ &\vdots \\ \mathbf{u}(z, 0, 0, \ldots, 1) &=: \text{N}_n^{\text{lamé}}(z) \end{aligned}$$

of the l-th node displacement U_l with the shape function $\text{N}_l^{\text{lamé}}(z)$ ($l = 0, \ldots, n$), we finally obtain

$$\begin{aligned} \mathbf{u}(z, \text{U}_0, \ldots, \text{U}_n) &= \textstyle\sum_{j=0}^{2n} \mathbf{f}_j(z)\; \beta_j(\text{U}_0, \ldots, \text{U}_n) + \sum_{k=0}^{2n} \hat{\mathbf{f}}_k(z)\; \gamma_k(\text{U}_0, \ldots, \text{U}_n) \\ &= \textstyle\sum_{l=0}^{n} \text{N}_l^{\text{lamé}}(z)\; \text{U}_l . \end{aligned} \tag{28}$$

We remark that the constructed shape functions $\text{N}_l^{\text{lamé}}(z)$ ($l = 0, \ldots, n$) are still exact solutions to the homogeneous Lamé equation (12).

4.4 Structure of the global stiffness matrix

Following the iso-parametric concept, the equations (25) and (28) are used to obtain the respective contributions of the coupling elements $\mathbb{T}_n^{c,m}$ and the analytical domain Ω_{A} to the global stiffness matrix. Hence, to obtain the stiffness relations it is advantageous to apply a purely complex variational scheme by using the Cauchy-Riemann operator and adapted numerical integration methods as well.

5 CONCLUSIONS

A strategy for the coupling of an classical Ritz approach with analytical basis functions and numerical standard procedures using the framework of complex function theory was presented. One notable advantage of the developed approach is the shape preserving geometry mapping and hence the conformal coupling of the displacement field by adopting a special class of Coupling elements. The analytical solution of Ω_{A} yields in every approximation step to an exact solution of the Lamé equation, which naturally implies that each approximation step is representing a possible mechanical state of the deformed structure. The proposed calculation scheme is furthermore appropriate for the generalization to the 3-dimensional case by using the skew field of real quaternions $\mathbb{H}$. In this connection complete orthonormal systems of spherical monogenics [1, 2] were developed, which can be understood as generalizations of the holomorphic functions in the complex theory.

REFERENCES

[1] I. Cacao, K. Gürlebeck, and S. Bock, *On Derivatives of Spherical Monogenics*, submitted for being published in the Journal of Complex Variables (03/2005).

[2] ______, *Complete Orthonormal Systems of Spherical Monogenics-A Constructive Approach*, (**eds.**) Le Hung Son, W. Tutschke, S. Jain, (**in**) *Methods of Complex and Clifford Analysis (Proceedings of ICAM Hanoi 2004)*, SAS International Publications (ISBN: 81-88296-01-5).

[3] D. Gaier, *Lectures on complex approximation*, Birkhäuser, 1985.

[4] D. Gross, *Bruchmechanik*, Springer-Verlag Berlin Heidelberg New York, 1996.

[5] K. Guerlebeck and W. Sprössig, *Quaternionic Analysis and Elliptic Boundary Value Problems*, Akademie Verlag, Berlin, 1989.

[6] H.G. Hahn, *Bruchmechanik: Einführung in die theoretischen Grundlagen*, B. G. Teubner Stuttgart, 1976.

[7] N.I. Mußchelischwili, *Einige Grundaufgaben der mathematischen Elastizitätstheorie*, VEB Fachbuchverlag Leipzig, 1971.