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A MULTIDIMENSIONAL HILBERT TRANSFORM IN ANISOTROPIC CLIFFORD ANALYSIS

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Abstract. *In earlier research, generalized multidimensional Hilbert transforms have been constructed in* $\mathbf{R}^m$ *in the framework of Clifford analysis. Clifford analysis, centred around the notion of monogenic functions, may be regarded as a direct and elegant generalization to higher dimension of the theory of the holomorphic functions in the complex plane. The considered Hilbert transforms, usually obtained as a part of the boundary value of an associated Cauchy transform in* $\mathbf{R}^{m+1}$*, might be characterized as isotropic, since the metric in the underlying space is the standard Euclidean one. This paper adopts the idea developed e.g. in [12] of a so–called anisotropic Clifford setting, which leads to the introduction of a metric dependent Hilbert transform in* $\mathbf{R}^m$*, showing formally the same properties as the isotropic one. As the Hilbert transform is used in signal analysis, this metric dependent setting has the advantage of allowing the adjustment of the co-ordin ate system to possible preferential directions in the signals to be analyzed. A striking result to be mentioned is that the associated anisotropic Cauchy transform in* $\mathbf{R}^{m+1}$ *is no longer uniquely determined, but may stem from a diversity of* $(m+1)$*–dimensional metrics.*

1 INTRODUCTION

In one–dimensional signal analysis, the Hilbert transform has become an indispensable tool for both global and local descriptions of a signal. It may be used to provide information on various independent signal properties, such as the amplitude, phase and frequency spectrum on the one side, and the instantaneous amplitude, phase and frequency on the other. The latter are usually estimated by means of so–called quadrature filters, which allow to distinguish different frequency components and therefore offer the possibility to locally refine the structure analysis. The involved methods are essentially based on the notion of ”analytic signal”, consisting of the linear combination of a bandpass filter, which selects a small part of the spectral information, and its Hilbert transform, which basically results from a phase shift by $\frac{\pi}{2}$ on the original filter (see e.g. [1]).

In mathematical terms, if $f \in L_2(\mathbf{R})$ is a real valued signal of finite energy, and $\mathcal{H}[f]$ denotes its Hilbert transform, i.e.

$$\mathcal{H}[f](x) = \frac{1}{\pi}\, Pv \int_{-\infty}^{+\infty} \frac{f(y)}{x-y}\, dy$$

then the corresponding analytic signal is the function $\frac{1}{2}f + \frac{i}{2}\mathcal{H}[f]$, which belongs to the Hardy space $H^2(\mathbf{R})$ and arises as the L_2 non-tangential boundary value for $y \to 0+$ of the holomorphic Cauchy transform of f in the upper half of the complex plane.

The Hilbert transform has been generalized to higher dimension by embedding $\mathbf{R}^m$ in $\mathbf{R}^{m+1}$ (see e.g. [2], [3]), by considering Lipschitz hypersurfaces in $\mathbf{R}^{m+1}$ (see e.g. [4]) and by considering the boundary of smooth closed surfaces (see [5], [6]). Applications in higher dimension have already been addressed as well, for instance in [7], where the concept of ”analytic signal” was generalized in order to design appropriate quadrature filters for two–dimensional signals. To this end, a generalized two–dimensional Hilbert transform, also referred to as Riesz transform, was used.

Several of these generalizations were established in Clifford analysis, a comprehensive theory offering an elegant and powerful generalization to higher dimension of the theory of holomorphic functions in the complex plane. In its most simple yet still useful setting, flat $(m+1)$-dimensional Euclidean space, Clifford analysis focusses on monogenic functions, i.e. null solutions of the Clifford-vector valued Dirac operator

$$\partial = \sum_{j=0}^{m} e_j \partial_{x_j}$$

where $(e_0, \ldots, e_m)$ forms an orthogonal basis for the quadratic space $\mathbf{R}^{m+1}$ underlying the construction of the Clifford algebra $\mathbf{R}_{0,m+1}$ (see e.g. [8],[9]). Monogenic functions are actually refining the properties of harmonic functions of several variables, since the rotation-invariant Dirac operator factorizes the m-dimensional Laplace operator, as does the Cauchy-Riemann operator in the complex plane.

The above described form of Clifford analysis may be referred to as isotropic, since the metric in the underlying space is the standard Euclidean one. In this paper however, the idea is adopted of a metric dependent (also called anisotropic or metrodynamical) Clifford setting (see e.g. [10], [11], [12]), which offers the possibility of adjusting the co-ordinate system to preferential and not necessarily mutually orthogonal directions in the m–dimensional signal to be analyzed. In this setting a new metric dependent multidimensional Hilbert transform in $\mathbf{R}^m \subset \mathbf{R}^{m+1}$ is defined, which will be seen to show formally the same properties as the, by now classical, Hilbert operator of e.g. [2] and [3]. A special case of such an anisotropic Hilbert transform, fitting in this general framework, was already introduced and used for two–dimensional image processing in [7].

2 THE METRIC DEPENDENT CLIFFORD TOOLBOX

Let $\widetilde{G} = (g_{kl})_{k,l=0,\ldots,m} \in \mathbf{R}^{(m+1)\times(m+1)}$ be a real, symmetric and positive definite tensor, which will be referred to as the metric tensor, and let $G = (g_{kl})_{k,l=1,\ldots,m} \in \mathbf{R}^{m\times m}$, i.e.

$$\widetilde{G} = \begin{pmatrix} g_{00} & \cdots & & g_{0m} \\ \vdots & & & \\ & & G & \\ g_{0m} & & & \end{pmatrix}$$

Notice that G is a metric tensor as well, which is obtained by simply taking the restriction of $\widetilde{G}$ to $\mathbf{R}^m$, the latter being identified with the hyperplane $x^0 = 0$ of $\mathbf{R}^{m+1}$. Furthermore, let $\widetilde{G}^{-1} = (g^{kl})_{k,l=0,\ldots,m}$ denote the reciprocal, or inverse, tensor of $\widetilde{G}$, i.e.

$$\sum_{s=0}^{m} g_{ks}\, g^{sl} = \delta_{kl}$$

In the following lemma, a criterion is given for the specific (and interesting) case where the inverse G^{-1} of G is involved in $\widetilde{G}^{-1}$, i.e. where $(g^{kl})_{k,l=1,\ldots,m} = G^{-1}$. The geometric consequences are discussed in Remark 2.2.

Lemma 2.1 *One has that*

$$(g^{kl})_{k,l=1,\ldots,m} \;=\; G^{-1}$$

if and only if the following conditions are simultaneously fulfilled:

(C1) $g_{00}\, g^{00} \;=\; 1$

(C2) $g_{01} \;=\; \ldots \;=\; g_{0m} \;=\; 0$

Proof. It may be clear that conditions (C1)–(C2) directly lead to $(g^{kl})_{k,l=1,\ldots,m} = G^{-1}$. Additionally, (C2) also implies that $g^{01} = \ldots = g^{0m} = 0$. Now the inverse implication is addressed. First, the assumption that $(g^{kl})_{k,l=1,\ldots,m} = G^{-1}$, can be rewritten as

$$\sum_{s=1}^{m} g_{ks}\, g^{sl} = \delta_{kl}\ , \quad k,l = 1,\ldots,m \tag{2.1}$$

This being given, the reciprocity of $\widetilde{G}$ and $\widetilde{G}^{-1}$ may be expressed in a block structure, viz

$$\begin{pmatrix} g_{00} & \underline{u}^T \\ \underline{u} & G \end{pmatrix} \begin{pmatrix} g^{00} & \underline{u}'^T \\ \underline{u}' & G^{-1} \end{pmatrix} \;=\; \begin{pmatrix} 1 & \underline{0}^T \\ \underline{0} & \mathbb{E}_m \end{pmatrix} \tag{2.2}$$

where $\underline{u}^T$, $\underline{u}'^T$ and $\underline{0}^T$ respectively denote the rows $(g_{01} \ \ldots \ g_{0m})$, $(g^{01} \ \ldots \ g^{0m})$ and $(0 \ \ldots \ 0)$ and $\mathbb{E}_m$ is the unity tensor of order m. Explicit calculation of the left–hand side of (2.2) then yields the following equations on the level of the tensor entries:

$$
\begin{aligned}
\sum_{s=0}^{m} g_{0s}\, g^{s0} &= 1, && && (2.3)\\
\sum_{s=0}^{m} g_{0s}\, g^{sl} &= 0, && l = 1, \ldots, m && (2.4)\\
\sum_{s=0}^{m} g_{ks}\, g^{s0} &= 0, && k = 1, \ldots, m && (2.5)\\
\sum_{s=0}^{m} g_{ks}\, g^{sl} &= \delta_{kl}, && k, l = 1, \ldots, m && (2.6)
\end{aligned}
$$

In view of (2.1), the left–hand side of (2.6) may be turned into

$$g_{k0}\, g^{0l} + \sum_{s=1}^{m} g_{ks}\, g^{sl} \;=\; g_{k0}\, g^{0l} + \delta_{kl}\ , \qquad k, l = 1, \ldots, m$$

which leads to the condition

$$g_{k0}\, g^{0l} \;=\; 0 \;=\; g_{0k}\, g^{l0}\ , \qquad k, l = 1, \ldots, m \tag{2.7}$$

seen also the symmetry of $\widetilde{G}$. Combination of (2.7) for $k = l$ with (2.3) then immediately results in condition (C1). Next, condition (C2) can be proven by *reductio ad absurdum*. Assume that there exists an index $\kappa \in \{1, \ldots, m\}$ for which $g_{0\kappa} \neq 0$, then (2.7) implies that $g^{0l} = g^{l0} = 0$ for all $l = 1, \ldots, m$. Hence, for each $k = 1, \ldots, m$, (2.5) reduces to $g_{0k}\, g^{00} = 0$. This leads to a contradiction for $k = \kappa$ since $g_{0\kappa} \neq 0$ and $g^{00} \neq 0$ on account of condition (C1). ■

In $\mathbf{R}^{m+1}$ a covariant basis $(e_k) = (e_0, \ldots, e_m)$ and a contravariant basis $(e^l) = (e^0, \ldots, e^m)$ are considered, corresponding to each other through the metric $\widetilde{G}$, i.e.

$$e_k = \sum_{l=0}^{m} g_{kl}\, e^l\ , \qquad e^l = \sum_{k=0}^{m} g^{lk}\, e_k$$

The universal Clifford algebra $\mathbf{R}_{0,m+1}$ is constructed over $(\mathbf{R}^{m+1}, \widetilde{G})$, with a non–commutative multiplication governed by

$$
\begin{aligned}
e_j\, e_k + e_k\, e_j &= -2\, g_{jk}, && j, k = 0, \ldots, m\\
e^j\, e^k + e^k\, e^j &= -2\, g^{jk}, && j, k = 0, \ldots, m\\
e_j\, e^k + e^k\, e_j &= -2\, \delta_{jk}, && j, k = 0, \ldots, m
\end{aligned}
$$

Remark 2.2 *The above multiplication rules, together with Lemma 2.1, learn that the specific case where $(g^{kl})_{k,l=1,\ldots,m} = G^{-1}$ corresponds to the geometric situation where the e_0–direction in $\mathbf{R}^{m+1}$ will be perpendicular to the $\mathbf{R}^m$–plane spanned by $(e_1, \ldots, e_m)$. Of course, the same holds for the position of e^0–direction with respect to the $\mathbf{R}^m$–plane spanned by $(e^1, \ldots, e^m)$. For $m = 2$ this corresponds to the application considered in [7].*

For a set $A = \{i_1, \ldots, i_h\} \subset \{0, \ldots, m\}$ with $0 \leq i_1 < i_2 < \ldots < i_h \leq m$, one puts $e_A = e_{i_1} e_{i_2} \ldots e_{i_h}$. Moreover, $e_\emptyset = 1$ is the identity element. In this way a covariant basis for the Clifford algebra $\mathbf{R}_{0,m+1}$ is constructed by means of which any $a \in \mathbf{R}_{0,m+1}$ may be written as

$$a = \sum_A a_A \, e_A, \qquad a_A \in \mathbf{R}$$

or still as

$$a = \sum_{k=0}^{m+1} [a]_k, \qquad [a]_k = \sum_{|A|=k} a_A \, e_A$$

where the terms $[a]_k$ correspond to so-called k-vectors $(k = 0, 1, \ldots, m+1)$. Alternatively, also a contravariant basis may be considered for the Clifford algebra.

A point $(x^0, \ldots, x^m) \in \mathbf{R}^{m+1}$ will be identified with the Clifford (1-)vector $\sum_{k=0}^{m} e_k x^k$. The above multiplication rules then lead to the decomposition of the Clifford product of two Clifford-vectors $x = \sum_k e_k x^k$ and $y = \sum_k e_k y^k$ as

$$xy \;=\; -\langle x, y \rangle_g + x \wedge y$$

with

$$\langle x, y \rangle_g \;=\; \sum_{k=0}^{m} \sum_{l=0}^{m} g_{kl} \, x^k \, y^l \tag{2.8}$$

a scalar, symmetric bilinear form associated to the metric and replacing the classical scalar product

$$\langle x, y \rangle \;=\; \sum_{k=0}^{m} x^k \, y^k \tag{2.9}$$

and with

$$x \wedge y \;=\; \frac{1}{2} x^k y^l (e_k e_l - e_l e_k)$$

a bivector. The norm of a vector x then is given by

$$|x| \;=\; \sqrt{\langle x, x \rangle_g}$$

Obviously, when $\widetilde{G} = \mathbb{E}_{m+1}$, one recovers the traditional Clifford algebra stemming from the standard Euclidean metric, and (2.8) reduces to (2.9).

In this metric dependent context, the anisotropic Dirac operator

$$\partial_g = \sum_{k=0}^{m} e^k \, \partial_{x^k}$$

is introduced, with fundamental solution

$$E_g(x) = \frac{1}{a_{m+1}} \frac{\overline{x}}{|x|^{m+1}}$$

as well as the anisotropic Laplace operator

$$\Delta_g = -\partial_g\,\partial_g = \sum_{k=0}^{m}\sum_{l=0}^{m} g^{kl}\partial_{x^k}\partial_{x^l}$$

with fundamental solution

$$F_g(x) = -\frac{1}{(m-1)a_{m+1}}\,\frac{1}{|x|^{m-1}}$$

In the above, $\overline{\cdot}$ denotes the usual conjugation in $\mathbf{R}_{0,m+1}$, defined as the main anti–involution for which $\overline{e}_k = -e_k$ (and thus also $\overline{e}^k = -e^k$), $k = 0,\ldots,m$. In particular for a vector x one has $\overline{x} = -x$. Additionally, a_{m+1} stands for the area of the unit sphere S^m in $\mathbf{R}^{m+1}$.

A function defined on $\mathbf{R}^{m+1}$ and taking values in $\mathbf{R}_{0,m+1}$, is called g–monogenic in the open region Ω of $\mathbf{R}^{m+1}$ if f is continuously differentiable in Ω and satisfies the equation

$$\partial_g\, f = 0 \qquad \text{in } \Omega$$

As the Dirac operator factorizes the Laplace operator Δ_g, a g–monogenic function in Ω is g–harmonic, and so are its components.

In what follows, also the anisotropic Cauchy–Riemann operator will be considered, which is defined by

$$D_g = \overline{e}_0\,\partial_g = \partial_{x^0} + \overline{e}_0\,\underline{\partial}_g$$

with fundamental solution

$$C_g(x) \;=\; \frac{1}{a_{m+1}}\,\frac{\overline{x}\,e^0}{|x|^{m+1}} \;=\; (g^{00})^{\frac{m+1}{2}}\,\frac{1}{a_{m+1}}\,\frac{x^0 + e^0\,\underline{x}}{|x^0 + e^0\underline{x}|^{m+1}}$$

where, in an obvious notation

$$\underline{x} = \sum_{k=1}^{m} x^k e_k$$

is a vector in $\mathbf{R}^m$ ($x^0 = 0$) and

$$\underline{\partial}_g = \sum_{k=1}^{m} e^k \partial_{x^k}$$

stands for the m–dimensional Dirac operator. Note that, since $D_g = \overline{e}_0\,\partial_g$ and $\partial_g = e^0 D_g$, g–monogenicity may equivalently be expressed w.r.t. the Cauchy–Riemann operator.

3 AN ANISOTROPIC MULTIDIMENSIONAL HILBERT TRANSFORM

The fundamental solution $C_g(x)$ of the Cauchy–Riemann operator is easily seen to decompose as

$$C_g(x) = \frac{1}{2}\left(P_g(x) + \overline{e}^0\,Q_g(x)\right)\,, \qquad x^0 \neq 0$$

where

$$P_g(x) = P_g(x^0,\underline{x}) \;=\; \frac{2}{a_{m+1}}\,\frac{x^0}{|x|^{m+1}} \;=\; \frac{2}{a_{m+1}}\,(g^{00})^{\frac{m+1}{2}}\,\frac{x^0}{|x^0 + e^0\underline{x}|^{m+1}}\,, \qquad x^0 \neq 0$$

is scalar valued and

$$Q_g(x) = Q_g(x^0, \underline{x}) \; = \; \frac{2}{a_{m+1}} \frac{\overline{x}}{|x|^{m+1}} \; = \; \frac{2}{a_{m+1}} \left(g^{00}\right)^{\frac{m+1}{2}} \frac{\overline{\underline{x}}}{|x^0 + e^0 \underline{x}|^{m+1}} \, , \qquad x^0 \neq 0$$

is bivector valued. It then readily follows from the g–monogenicity of $C_g(x)$ in $\mathbf{R}^{m+1}_+$ that $P_g(x)$ and $Q_g(x)$ are g–harmonic in $\mathbf{R}^{m+1}_+$ (and similarly in $\mathbf{R}^{m+1}_-$). In accordance with previous definitions (see e.g. [13]) they will be called g–harmonic conjugates.

The above functions may be used as the kernels for metric dependent counterparts of well–known integral transforms. Indeed, for an appropriate function, belonging to $L_2(\mathbf{R}^m)$, or a tempered distribution f, its Cauchy integral may be defined by

$$\mathcal{C}_g[f] = C_g * f$$

which is monogenic in $\mathbf{R}^{m+1}_+$ (and in $\mathbf{R}^{m+1}_-$). Analogously its Poisson and conjugate Poisson transforms are introduced as the g–harmonic functions

$$\begin{aligned} \mathcal{P}_g[f] &= P_g * f \\ \mathcal{Q}_g[f] &= Q_g * f \end{aligned}$$

so that

$$\mathcal{C}_g[f] = \frac{1}{2} \mathcal{P}_g[f] + \frac{\overline{e}^0}{2} \mathcal{Q}_g[f]$$

either in $\mathbf{R}^{m+1}_+$ or in $\mathbf{R}^{m+1}_-$.

As a final preparatory step for the introduction of the desired new anisotropic Hilbert kernel, one needs to calculate the distributional limits of $P_g(x^0, \underline{x})$ and $Q_g(x^0, \underline{x})$ for $x^0 \to 0+$. The outcome of those limits is presented in Proposition 3.3, which is preceded by the following two auxiliary lemmas.

Lemma 3.1 *Let $\widetilde{G} = (g_{kl})_{k,l=0,\ldots,m}$ be a metric tensor, and let $G = (g_{kl})_{k,l=1,\ldots,m}$, then*

$$\det \widetilde{G} \; = \; \det G \left(g_{00} - \underline{u}^T \, G^{-1} \, \underline{u}\right) \tag{3.1}$$

where $\underline{u}^T$ denotes the row $(g_{01} \; \ldots \; g_{0m})$.

Proof. As the tensor G is positive definite, one can write $G = B^T B$, with $B \in \mathrm{GL}(m; \mathbf{R})$. Defining $\underline{v} = (B^T)^{-1} \underline{u}$, the tensor $\widetilde{G}$ can then be factorized as

$$\widetilde{G} \; = \; \begin{pmatrix} g_{00} & \underline{u}^T \\ \underline{u} & B^T B \end{pmatrix} \; = \; \begin{pmatrix} 1 & \underline{0}^T \\ \underline{0} & B^T \end{pmatrix} \begin{pmatrix} g_{00} & \underline{v}^T \\ \underline{v} & \mathbb{E}_m \end{pmatrix} \begin{pmatrix} 1 & \underline{0}^T \\ \underline{0} & B \end{pmatrix}$$

from which it follows that

$$\begin{aligned} \det \widetilde{G} \; &= \; \det(B^T) \det \begin{pmatrix} g_{00} & \underline{v}^T \\ \underline{v} & \mathbb{E}_m \end{pmatrix} \det(B) \\ &= \; \det(B^T B) \left(g_{00} - \underline{v}^T \underline{v}\right) \; = \; \det G \left(g_{00} - \underline{u}^T \, G^{-1} \, \underline{u}\right) \end{aligned}$$

∎

Lemma 3.2 *Let $\widetilde{G} = (g_{kl})_{k,l=0,\dots,m}$ be a metric tensor, and let $\hat{x} = e_0 + \underline{x}$, then*

$$\int_{\mathbf{R}^m} \frac{dV(\underline{x})}{|\hat{x}|^{m+1}} = \frac{a_{m+1}}{2\sqrt{\det \widetilde{G}}}$$

Proof. As the restriction G of the metric tensor $\widetilde{G}$ to $\mathbf{R}^m$ is a metric tensor as well, there exists an orthogonal $m \times m$ matrix A such that

$$A^T G A = \text{diag}(\mu_1^2, \dots, \mu_m^2)$$

with $\mu_1^2, \dots, \mu_m^2$ the strictly positive (not necessarily different) eigenvalues of G. Then, introducing a new integration variable $\underline{x}'$ by means of the transformation

$$\underline{x} = A\underline{x}' - G^{-1}\underline{u}$$

and interpreting the vectors $\underline{x}$ and $\underline{x}'$ as column matrices, one has

$$\begin{aligned} |\hat{x}|^2 = \langle \hat{x}, \hat{x} \rangle_g &= \begin{pmatrix} 1 & \underline{x}^T \end{pmatrix} \widetilde{G} \begin{pmatrix} 1 \\ \underline{x} \end{pmatrix} \\ &= \begin{pmatrix} 1 & \underline{x}'^T A^T - \underline{u}^T G^{-1} \end{pmatrix} \begin{pmatrix} g_{00} & \underline{u}^T \\ \underline{u} & G \end{pmatrix} \begin{pmatrix} 1 \\ A\underline{x}' - G^{-1}\underline{u} \end{pmatrix} \\ &= \sum_{j=1}^m \left(\mu_j x'^j\right)^2 + \left(g_{00} - \underline{u}^T G^{-1} \underline{u}\right) \end{aligned}$$

Once again introducing a new variable, now through the transformation

$$\underline{x}' = \sqrt{\frac{\det \widetilde{G}}{\det G}} \, \text{diag}(\mu_1^{-1}, \dots, \mu_m^{-1}) \, \underline{x}''$$

and moreover invoking (3.1), one arrives at

$$|\hat{x}|^2 = \langle \hat{x}, \hat{x} \rangle_g = \frac{\det \widetilde{G}}{\det G} \left(\sum_{j=1}^m \left(x''^j\right)^2 + 1 \right)$$

Furthermore, the volume elements $dV(\underline{x})$ and $dV(\underline{x}'')$ are then seen to correspond as follows:

$$\begin{aligned} dV(\underline{x}) &= |\det A| \; dV(\underline{x}') \\ &= \left(\sqrt{\frac{\det \widetilde{G}}{\det G}} \right)^m \left(\prod_{j=1}^m \mu_j^{-1} \right) dV(\underline{x}'') \\ &= \frac{\left(\sqrt{\det \widetilde{G}}\right)^m}{\left(\sqrt{\det G}\right)^{m+1}} \, dV(\underline{x}'') \end{aligned}$$

So eventually one has

$$\begin{aligned} \int_{\mathbf{R}^m} \frac{dV(\underline{x})}{|\hat{x}|^{m+1}} &= \frac{1}{\sqrt{\det \widetilde{G}}} \int_{\mathbf{R}^m} \frac{dV(\underline{x}'')}{\left[1 + \sum_{j=1}^m (x''^j)^2\right]^{\frac{m+1}{2}}} \\ &= \frac{a_{m+1}}{2\sqrt{\det \widetilde{G}}} \end{aligned}$$

the last equality leaning on the classical result

$$\int_{\mathbf{R}^m} \frac{dV(\underline{x}'')}{\left[1+\sum_{j=1}^m (x''^j)^2\right]^{\frac{m+1}{2}}} \;=\; \frac{a_{m+1}}{2}$$

■

Proposition 3.3 *In distributional sense one has*

$$\begin{aligned}\lim_{x^0\to 0+} P_g(x^0,\underline{x}) &= \frac{1}{\sqrt{\det \widetilde{G}}}\,\delta(\underline{x})\\ \lim_{x^0\to 0+} Q_g(x^0,\underline{x}) &= \frac{1}{\sqrt{\det \widetilde{G}}}\,H_g(\underline{x})\end{aligned}$$

with

$$H_g(\underline{x}) \;=\; \sqrt{\det \widetilde{G}}\left(\frac{2}{a_{m+1}}\,\mathrm{Pv}\,\frac{\overline{x}}{|\underline{x}|^{m+1}}\right)$$

Proof. First consider the distributional limit of $P_g(x^0,\underline{x})$. It is well–known that, if a real valued integrable function h defined on $\mathbf{R}^m$ satisfies the property

$$\int_{\mathbf{R}^m} h(\underline{x})\,dV(\underline{x}) \;=\; 1 \tag{3.2}$$

then one has in distributional sense

$$\lim_{x^0\to 0+} \widetilde{h}(x^0,\underline{x}) \;=\; \delta(\underline{x})$$

where

$$\widetilde{h}(x^0,\underline{x}) \;=\; \frac{1}{(x^0)^m}\,h(\frac{\underline{x}}{x^0})\,, \quad x^0>0$$

As Lemma 3.2 implies that the specific integrable function

$$h(\underline{x}) \;=\; \frac{2\sqrt{\det \widetilde{G}}}{a_{m+1}}\,\frac{1}{|\hat{x}|^{m+1}}$$

satisfies property (3.2), it follows that

$$\lim_{x^0\to 0+} P_g(x^0,\underline{x}) \;=\; \frac{1}{\sqrt{\det \widetilde{G}}}\lim_{x^0\to 0+} \widetilde{h}(x^0,\underline{x}) \;=\; \frac{1}{\sqrt{\det \widetilde{G}}}\,\delta(\underline{x})$$

Next, since $Q_g(x^0,\underline{x}) \in L_1^{\mathrm{loc}}(\mathbf{R}^m)$ for each $x^0>0$, it defines a regular distribution whose action on a testing function ϕ with compact support in $\mathbf{R}^m$ is given by

$$\langle Q_g(x^0,\underline{x}),\phi(\underline{x})\rangle \;=\; \int_{\mathbf{R}^m} Q_g(x^0,\underline{x})\,\phi(\underline{x})\,dV(\underline{x})$$

Taking limits for $x^0 \to 0+$ results in

$$\begin{aligned}
\langle \lim_{x^0 \to 0+} Q_g(x^0, \underline{x}), \phi(\underline{x}) \rangle &= \int_{\mathbf{R}^m} \lim_{x^0 \to 0+} Q_g(x^0, \underline{x})\, \phi(\underline{x})\, dV(\underline{x}) \\
&= \lim_{\varepsilon \to 0+} \int_{\mathbf{R}^m \setminus B(\underline{0};\varepsilon)} \lim_{x^0 \to 0+} Q_g(x^0, \underline{x})\, \phi(\underline{x})\, dV(\underline{x}) \\
&= \mathrm{Pv} \int_{\mathbf{R}^m} \frac{2}{a_{m+1}} \frac{\overline{\underline{x}}}{|\underline{x}|^{m+1}}\, \phi(\underline{x})\, dV(\underline{x}) \;=\; \langle \frac{1}{\sqrt{\det \widetilde{G}}}\, H_g(\underline{x}), \phi(\underline{x}) \rangle
\end{aligned}$$

which completes the proof. ∎

The previous proposition directly leads us to the distributional limits of the Poisson transform and its (g–)conjugate, viz

$$\begin{aligned}
\lim_{x^0 \to 0+} \mathcal{P}_g[f] &= \frac{1}{\sqrt{\det \widetilde{G}}}\, f \\
\lim_{x^0 \to 0+} \mathcal{Q}_g[f] &= \frac{1}{\sqrt{\det \widetilde{G}}}\, H_g * f
\end{aligned}$$

whence

$$\lim_{x^0 \to 0+} \mathcal{C}_g[f] = \frac{1}{\sqrt{\det \widetilde{G}}} \left(\frac{1}{2}\, f + \frac{1}{2}\, \overline{e}^0\, H_g * f \right)$$

Similarly, for $x^0 \to 0-$, one obtains

$$\lim_{x^0 \to 0-} \mathcal{C}_g[f] = \frac{1}{\sqrt{\det \widetilde{G}}} \left(-\frac{1}{2}\, f + \frac{1}{2}\, \overline{e}^0\, H_g * f \right)$$

The above results are known in the isotropic case as the Plemelj–Sokhotski formulae and give rise to the definition of the Hilbert transform.

For a function $f \in L_2(\mathbf{R}^m)$ (or a tempered distribution), its anisotropic Hilbert transform is defined as

$$\mathcal{H}_g[f] = \overline{e}^0\, H_g * f$$

by means of which the Plemelj-Sokhotski formulae can be rewritten as

$$\lim_{x^0 \to 0\pm} \mathcal{C}_g[f] = \frac{1}{\sqrt{\det \widetilde{G}}} \left(\pm\frac{1}{2}\, f + \frac{1}{2}\, \mathcal{H}_g[f] \right) \tag{3.3}$$

For $m = 2$, such an anisotropic Hilbert transform was considered in [7], however for the special case where the e_0–direction in $\mathbf{R}^3$ is chosen perpendicular to the $\mathbf{R}^2$–plane spanned by (e_1, e_2). This corresponds to a $\widetilde{G}$–matrix of order 3 in which $g_{01} = g_{02} = 0$ (see also Remark 2.2).

The properties of the newly introduced linear operator $\mathcal{H}_g$ will also be studied in the Fourier domain, so a proper definition for the Fourier transform on $\mathbf{R}^m$ in the present metric dependent setting is needed. In the isotropic case one has

$$\mathcal{F}[f](\underline{x}) = \int_{\mathbf{R}^m} \exp(-2\pi i \langle \underline{x}, \underline{y} \rangle)\, f(\underline{y})\, dV(\underline{y}) = \int_{\mathbf{R}^m} \exp(-2\pi i\; \underline{x}^T \underline{y})\, f(\underline{y})\, dV(\underline{y}) \tag{3.4}$$

where $\langle \underline{x}, \underline{y} \rangle$ denotes the restriction of the classical scalar product (2.9) to $\mathbf{R}^m$ (identified with $x^0 = 0$) and, in the last equality, the vectors $\underline{x}$ and $\underline{y}$ are interpreted as column matrices. In a natural way, this leads to the following anisotropic Fourier transform:

$$\mathcal{F}_g[f](\underline{x}) = \int_{\mathbf{R}^m} \exp(-2\pi i \langle \underline{x}, \underline{y} \rangle_g)\, f(\underline{y})\, dV(\underline{y}) = \int_{\mathbf{R}^m} \exp\left(-2\pi i\ \underline{x}^T G \underline{y}\right)\ f(\underline{y})\, dV(\underline{y}) \quad (3.5)$$

where the restriction of the scalar product (2.8) to $\mathbf{R}^m$ comes into play. One immediately finds

$$\mathcal{F}_g[f](\underline{x}) = \mathcal{F}[f](G\underline{x}) \quad (3.6)$$

due to the symmetric character of G.

The following properties of the Hilbert transform $\mathcal{H}_g$ may then be proven:

Proposition 3.4

(P1) $\mathcal{H}_g$ *is translation invariant, i.e.*

$$\mathcal{H}_g[f(\underline{y} - \underline{t})](\underline{x}) = \mathcal{H}_g[f](\underline{x} - \underline{t})$$

(P2) $\mathcal{H}_g$ *is dilation invariant, i.e.*

$$\mathcal{H}_g[f(a\underline{y})](\underline{x}) = \mathcal{H}_g[f](a\underline{x}), \qquad \forall a > 0$$

which is equivalent to its kernel H_g *being a homogeneous distribution of degree* $-m$

(P3) $\mathcal{H}_g$ *is a bounded operator on* $L_2(\mathbf{R}^m)$*, which is equivalent to its Fourier symbol*

$$\mathcal{F}_g[H_g](\underline{x}) = \sqrt{\frac{\det \widetilde{G}}{\det G}}\ i\ \frac{\underline{x}}{|\underline{x}|} \quad (3.7)$$

being a bounded function

(P4) Up to a metric related constant, $\mathcal{H}_g$ *squares to unity, i.e.*

$$(\mathcal{H}_g)^2 = g^{00}\, \frac{\det \widetilde{G}}{\det G}\, \mathbf{1}$$

(P5) $\mathcal{H}_g$ *is selfadjoint, or* $\mathcal{H}_g^* = \mathcal{H}_g$*, i.e.*

$$(\mathcal{H}_g[f], g)_{L_2} = (f, \mathcal{H}_g[g])_{L_2}$$

(P6) $\mathcal{H}_g$ *arises in a natural way by considering non-tangential boundary values of the Cauchy transform* $\mathcal{C}_g$ *in* $\mathbf{R}^{m+1}$ *of an appropriate function or distribution in* $\mathbf{R}^m$*.*

Proof. The proof of properties (P1), (P2) and (P5) is rather straightforward, starting from the definition of $\mathcal{H}_g$ and taking into account the anisotropic setting. Furthermore, (P6) is a direct consequence of the results in previous section and was in fact already contained in (3.3).

Next, the calculation of the Fourier symbol in (P3) is established by invoking the factorization of the positive definite tensor G as

$$G = B^T B, \qquad \text{with } \ B \in \mathrm{GL}(m, \mathbf{R})$$

We then have

$$\begin{aligned}
\mathcal{F}_g[H_g](\underline{x}) &= \sqrt{\det \widetilde{G}}\, \frac{2}{a_{m+1}} \int_{\mathbf{R}^m} \exp(-2\pi i \underline{x}^T G \underline{y})\ \mathrm{Pv}\, \frac{\overline{\underline{y}}}{\left[\underline{y}^T G \underline{y}\right]^{\frac{m+1}{2}}}\, dV(\underline{y}) \\
&= \sqrt{\det \widetilde{G}}\, \frac{2}{a_{m+1}} \int_{\mathbf{R}^m} \exp(-2\pi i ((B\underline{x})^T B\underline{y})\ \mathrm{Pv}\, \frac{\overline{\underline{y}}}{\left[(B\underline{y})^T B\underline{y}\right]^{\frac{m+1}{2}}}\, dV(\underline{y})
\end{aligned}$$

Putting $\underline{y}' = B\underline{y}$ one arrives at

$$\mathcal{F}_g[H_g](\underline{x}) = \sqrt{\frac{\det \widetilde{G}}{\det G}}\, B^{-1}\, \frac{2}{a_{m+1}} \int_{\mathbf{R}^m} \exp(-2\pi i (B\underline{x})^T \underline{y}')\ \mathrm{Pv}\, \frac{\overline{\underline{y}}'}{\left[\underline{y}'^T \underline{y}'\right]^{\frac{m+1}{2}}}\, dV(\underline{y}')$$

such that the anisotropic Fourier transform of the anisotropic Hilbert kernel can be rewritten in terms of the isotropic Fourier transform of the isotropic Hilbert kernel, i.e.

$$\mathcal{F}_g[H_g](\underline{x}) \;=\; \sqrt{\frac{\det \widetilde{G}}{\det G}}\, B^{-1}\, \mathcal{F}[H](B\underline{x})$$

with the isotropic Hilbert kernel being given by

$$H(\underline{x}) \;=\; \frac{2}{a_{m+1}}\ \mathrm{Pv}\, \frac{\overline{\underline{x}}}{\left(\langle \underline{x}, \underline{x}\rangle\right)^{\frac{m+1}{2}}}$$

Its Fourier symbol $\mathcal{F}[H]$ is well-known (see e.g. [14]) and reads

$$\mathcal{F}[H](\underline{x}) \;=\; i\, \frac{\underline{x}}{\sqrt{\langle \underline{x}, \underline{x}\rangle}}$$

yielding

$$\mathcal{F}_g[H_g](\underline{x}) \;=\; \sqrt{\frac{\det \widetilde{G}}{\det G}}\, B^{-1}\, i\, \frac{B\underline{x}}{\sqrt{\langle B\underline{x}, B\underline{x}\rangle}} \;=\; \sqrt{\frac{\det \widetilde{G}}{\det G}}\, i\, \frac{\underline{x}}{|\underline{x}|}$$

Finally, property (P4) then results from a conversion to the Fourier domain. Indeed,

$$\begin{aligned}
\mathcal{F}_g\left[\mathcal{H}_g^2[f]\right] &= \mathcal{F}_g\left[\mathcal{H}_g\left[\mathcal{H}_g[f]\right]\right] \;=\; \mathcal{F}_g\left[\overline{e}^0\, H_g * \mathcal{H}_g[f]\right] \;=\; \mathcal{F}_g\left[\overline{e}^0\, H_g\right]\ \mathcal{F}_g\left[\mathcal{H}_g[f]\right] \\
&= \mathcal{F}_g\left[\overline{e}^0\, H_g\right]\ \mathcal{F}_g\left[\overline{e}^0\, H_g * f\right] \;=\; \mathcal{F}_g\left[\overline{e}^0 H_g\right]^2\ \mathcal{F}_g[f]
\end{aligned}$$

whence

$$\mathcal{F}_g\left[\mathcal{H}_g^2[f]\right] = -\frac{\det \widetilde{G}}{\det G}\, g^{00}\, \frac{\underline{x}^2}{|\underline{x}^2|}\, \mathcal{F}_g[f] \;=\; g^{00}\, \frac{\det \widetilde{G}}{\det G}\, \mathcal{F}_g[f]$$

■

Notice that, due to the properties (P4)–(P5), the operator

$$\widetilde{\mathcal{H}}_g = \sqrt{\frac{\det G}{g^{00}\, \det \widetilde{G}}}\, \mathcal{H}_g$$

is unitary.

4 EXAMPLE

Consider in $\mathbf{R}^m$ the tempered distribution

$$f(\underline{x}) = \exp\left(2\pi i \ \langle \underline{a}, \underline{x} \rangle\right)$$

where $\underline{a}$ is a given, nonzero vector. Both the isotropic Hilbert transform of f and its metric dependent counterpart, defined respectively by

$$\mathcal{H}[f] = \overline{e}_0 \ H * f = \overline{e}_0 \ \frac{2}{a_{m+1}} \ \mathrm{Pv} \ \frac{\overline{\underline{x}}}{(\underline{x}^T \underline{x})^{\frac{m+1}{2}}} * f$$

and

$$\mathcal{H}_g[f] = \overline{e}^0 \ H_g * f = \overline{e}^0 \ \sqrt{\det \widetilde{G}} \ \frac{2}{a_{m+1}} \ \mathrm{Pv} \ \frac{\overline{\underline{x}}}{(\underline{x}^T \, G \, \underline{x})^{\frac{m+1}{2}}} * f$$

will be calculated, in order to illustrate the differences between both cases on a concrete example. Note that the above formulae show once more how $\mathcal{H}_g$ reduces to $\mathcal{H}$ when $\widetilde{G} = \mathbb{E}_{m+1}$, seen also the fact that $e_0 = e^0$ in that case.

We first consider the isotropic case. Using definition (3.4), the isotropic Fourier transform of f reads

$$\mathcal{F}[f(\underline{x})](\underline{y}) = \delta(\underline{y} - \underline{a})$$

leading to

$$\mathcal{F}\left[\mathcal{H}[f](\underline{x})\right](\underline{y}) = i \ \overline{e}_0 \ \frac{\underline{y}}{|\underline{y}|} \ \delta(\underline{y} - \underline{a}) = i \ \overline{e}_0 \ \frac{\underline{a}}{|\underline{a}|} \ \delta(\underline{y} - \underline{a})$$

and eventually to

$$\mathcal{H}[f(\underline{x})](\underline{y}) = i \ \overline{e}_0 \ \frac{\underline{a}}{|\underline{a}|} \ \exp\left(2\pi i \ \langle \underline{a}, \underline{x} \rangle\right)$$

In the anisotropic case, the Fourier transform is defined by (3.5) or the equivalent form (3.6), so that

$$\mathcal{F}_g[f(\underline{x})](\underline{y}) = \mathcal{F}[f(\underline{x})](G\underline{y}) = \delta(G\underline{y} - \underline{a})$$

and thus

$$\mathcal{F}_g\left[\mathcal{H}_g[f](\underline{x})\right](\underline{y}) = \overline{e}^0 \ i \ \sqrt{\frac{\det \widetilde{G}}{\det G}} \ \frac{G^{-1}\underline{a}}{|G^{-1}\underline{a}|} \ \delta(G\underline{y} - \underline{a})$$

with

$$|G^{-1}\underline{a}| = \left[\left(G^{-1}\underline{a}\right)^T \ G \ \left(G^{-1}\underline{a}\right)\right]^{\frac{1}{2}} = \left[\underline{a}^T \ G^{-1} \ \underline{a}\right]^{\frac{1}{2}}$$

Subsequent calculations learn that

$$\begin{aligned}
\mathcal{F}_g^{-1}\left[\delta(G\underline{y} - \underline{a})\right](\underline{x}) &= \int_{\mathbf{R}^m} \exp\left(2\pi i \ \underline{x}^T G \underline{y}\right) \delta(G\underline{y} - \underline{a}) \ dV(\underline{y}) \\
&= \frac{1}{\det G} \int_{\mathbf{R}^m} \exp\left(2\pi i \ \underline{x}^T \underline{y}'\right) \delta(\underline{y}' - \underline{a}) \ dV(\underline{y}') \\
&= \frac{1}{\det G} \ \exp\left(2\pi i \ \langle \underline{a}, \underline{x} \rangle\right)
\end{aligned}$$

Hence

$$\mathcal{H}_g[f(\underline{x})](\underline{y}) = i \ \overline{e}^0 \ \sqrt{\frac{\det \widetilde{G}}{(\det G)^3}} \ \frac{G^{-1}\underline{a}}{|G^{-1}\underline{a}|} \ \exp\left(2\pi i \ \langle \underline{a}, \underline{x} \rangle\right)$$

5 CONCLUDING REMARKS

As is the case for the definition in the isotropic setting, the present Hilbert kernel $H_g(\underline{x})$ has been obtained in a constructive way, by taking distributional limits of a g–harmonic function in $\mathbf{R}_+^{m+1}$, which is one of the two conjugate harmonic parts in which the g–monogenic Cauchy kernel $C_g(x)$ splits. The resulting Hilbert transform $\mathcal{H}_g[f] = H_g * f$ depends on the underlying metric in two different ways:

(1) the determinant of the "mother" metric $\widetilde{G}$ on $\mathbf{R}^{m+1}$ arises as an explicit factor in the expression for the kernel,

and

(2) the induced metric G on $\mathbf{R}^m$ implicitly comes into play through both the numerator and the denominator of the kernel, since the vector $\overline{\underline{x}}$ contains the (skew) basis vectors $(e_k)_{k=1}^m$, and $|\underline{x}|^{m+1}$ can be rewritten as $\left[\underline{x}^T\, G\underline{x}\right]^{\frac{m+1}{2}}$

The particularity of this metric dependence may also be seen in the Fourier domain, where the metric G not only arises in the Fourier symbol (3.7) of $\mathcal{H}_g$, but is also hidden in the definition of the Fourier transform itself, while the "mother" metric $\widetilde{G}$ again only pops up through its determinant.

The above observations raise the question whether there exists a one–to–one correspondence between a given Hilbert transform on $(\mathbf{R}^m, G)$ and the associated Cauchy transform on $(\mathbf{R}^{m+1}, \widetilde{G})$ from which it originates, or in other words: does the Hilbert transform contain enough geometrical information to completely determine the "mother" metric $\widetilde{G}$? One may already intuitively feel that the answer is negative, since only the induced metric G and the determinant $\det \widetilde{G}$ seem to be involved.

To answer this question properly, we consider, for a given G and $\det \widetilde{G}$, the equation

$$g_{00} - \underline{u}^T\, G^{-1}\, \underline{u} \;=\; \frac{\det \widetilde{G}}{\det G}$$

derived from (3.1). If we want $\widetilde{G}$ to be uniquely determined, then this equation should have a unique solution $(g_{00}, \underline{u}^T)$, which clearly is not the case, since we directly see that

$$g_{00} \;=\; \frac{\det \widetilde{G}}{\det G}, \quad \underline{u}^T \;=\; \underline{0} \qquad \text{and} \qquad g_{00} \;=\; \frac{\det \widetilde{G}}{\det G} + (G^{-1})_{11}, \quad \underline{u}^T \;=\; (1\,0\,\ldots\,0)$$

already constitute two different solutions, and others may be found straightaway.

We conclude that, given a Hilbert kernel

$$H_g = c\, \left(\frac{2}{a_{m+1}}\; \mathrm{Pv}\, \frac{\overline{\underline{x}}}{|\underline{x}|^{m+1}} \right)$$

which depends on the m–dimensional metric G and on the strictly positive constant c, it is part of the boundary value of a Cauchy kernel in $(\mathbf{R}^{m+1}, \widetilde{G})$, with

$$\widetilde{G} = \begin{pmatrix} g_{00} & \underline{u}^T \\ \underline{u} & G \end{pmatrix}$$

where $(g_{00}, \underline{u}^T)$ are characterized, but not uniquely determined, by the equation

$$g_{00} - \underline{u}^T \, G^{-1} \, \underline{u} = \frac{c}{\det G}$$

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