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# SOLUTION STRATEGIES FOR STOCHASTIC FINITE ELEMENT DISCRETIZATIONS

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**Abstract.** *We consider efficient numerical methods for the solution of partial differential equations with stochastic coefficients or right hand side. The discretization is performed by the stochastic finite element method (SFEM). Separation of spatial and stochastic variables in the random input data is achieved via a Karhunen-Loève expansion or Wiener's polynomial chaos expansion. We discuss solution strategies for the Galerkin system that take advantage of the special structure of the system matrix. For stochastic coefficients linear in a set of independent random variables we employ Krylov subspace recycling techniques after having decoupled the large SFEM stiffness matrix.*

## 1 INTRODUCTION

Physical phenomena occuring in structural mechanics are mathematically modelled by partial differential equations together with appropriate boundary conditions. Their approximate solution can be computed efficiently and accurately for instance by the finite element method. On the other hand, input data such as material parameters, boundary conditions and loads contain measurement or modelling errors that may outweigh discretization errors coming from the numerical simulation.
In recent years it has become popular to take account of this problem in the model itself by treating all input data as spatially dependent random variables, i.e., random fields (RFs) [2]. Consequently the randomness is propagated to the output data, e.g. the displacements, and allows to quantify the uncertainty of the output.

From the mathematical point of view the task is to solve a stochastic boundary value problem. To fix ideas we consider the Lamé-Navier equation on a bounded domain $D \subset \mathbb{R}^n$, $n = 2, 3$. Young's modulus $E$ and the body force density function $F$ are now random fields, whereas Poisson's ratio $\nu$ is deterministic. For simplicity we impose homogeneous deterministic boundary conditons on $\partial D = B_0 \cup B_1$. The task is to find a displacement random field $\boldsymbol{u} : D \times \Omega \to \mathbb{R}$, such that

$$\frac{E(\boldsymbol{x},\omega)}{2(1+\nu)}\left[\nabla^2 \boldsymbol{u}(\boldsymbol{x},\omega) + \frac{1}{1-2\nu}\nabla(\nabla \cdot \boldsymbol{u}(\boldsymbol{x},\omega))\right] + \boldsymbol{F}(\boldsymbol{x},\omega) = \boldsymbol{0} \quad \text{in } D, \tag{1a}$$

$$\boldsymbol{u}(\boldsymbol{x},\omega) = \boldsymbol{0} \quad \text{on } B_0, \tag{1b}$$

$$\boldsymbol{\sigma}(\boldsymbol{u}(\boldsymbol{x},\omega))\boldsymbol{n} = \boldsymbol{0} \quad \text{on } B_1, \tag{1c}$$

is fulfilled $P$-almost surely, where $(\Omega, \mathcal{A}, P)$ is the (complete) probabitity space under consideration.

This paper is concerned with one possibility for the discretization of (1), namely the stochastic finite element method [8].
The basic principle of the SFEM is to treat the spatial and stochastic parts of the problem separately. Ansatz and test functions are elements of tensor product spaces that contain solely deterministic or stochastic shape functions. As a consequence, the discretization of the spatial part is independent of that of the stochastic part.
On the other hand the total number of degrees of freedom (DOFs) is exactly the number of DOFs of the deterministic problem multiplied by the number of stochastic DOFs. Therefore we require efficient solvers for the large linear systems that arise from SFEM.

Following a short review of the basic steps to apply SFEM in Section 2, we discuss strategies for coping with the complexity of SFEM in Section 3. They depend on the representation of the random input data which is addressed in Section 2. Finally we present the results of numerical experiments on a plane strain test problem given in [3].

## 2 REVIEW OF THE STOCHASTIC FINITE ELEMENT METHOD

In analogy to the case of deterministic input data we recast the stochastic boundary value problem (1) to an equivalent variational formulation in the tensor product space $H^1_\Gamma(D) \otimes L^2_P(\Omega)$, where

$$H^1_\Gamma(D) := \{\boldsymbol{w} \in H^1(D)^n : \boldsymbol{w} = \boldsymbol{0} \text{ on } B_1\}. \tag{2}$$

is the variational space for the deterministic companion piece to (1). We seek a function $\boldsymbol{u} \in H^1_\Gamma(D) \otimes L^2_P(\Omega)$ such that there holds

$$\frac{1}{1+\nu}\left\langle \int_D E\,\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})d\boldsymbol{x} + \frac{\nu}{1-2\nu}\int_D E\,(\nabla\cdot\boldsymbol{u})\,(\nabla\cdot\boldsymbol{v})d\boldsymbol{x}\right\rangle = \left\langle \int_D \boldsymbol{F}\cdot\boldsymbol{v}d\boldsymbol{x}\right\rangle, \tag{3}$$

for all test functions $\boldsymbol{v} \in H^1_\Gamma(D) \otimes L^2_P(\Omega)$. Here $\langle\cdot\rangle$ denotes the expectation operator w.r.t. the probability space.

### 2.1 Discretization steps

For practical applications it is realistic to assume that the input RFs depend on a finite number of mutually independent random variables $\{\xi_m\}_{m=1}^M$ only. These random variables are required to have a probability density function $\rho_m : \Gamma_m \to [0,\infty)$. Since we assume the $\{\xi_m\}_{m=1}^M$ to be independent, their joint probability density function is given by

$$\rho(\boldsymbol{\xi}) := \rho_1(\xi_1)\cdots\rho_M(\xi_M), \qquad \boldsymbol{\xi} \in \Gamma := \Gamma_1 \times \cdots \times \Gamma_M.$$

We can now identify $L^2_P(\Omega)$ with $L^2_\rho(\Gamma)$ and reformulate (3) in terms of the random vector $\boldsymbol{\xi}$ to obtain the problem of finding a function $\boldsymbol{u} \in H^1_\Gamma(D) \otimes L^2_\rho(\Gamma)$ such that

$$\frac{1}{1+\nu}\int_\Gamma \rho(\boldsymbol{\xi})\left[\int_D E\,\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})d\boldsymbol{x} + \frac{\nu}{1-2\nu}\int_D E\,(\nabla\cdot\boldsymbol{u})\,(\nabla\cdot\boldsymbol{v})d\boldsymbol{x}\right]d\boldsymbol{\xi} = \int_\Gamma \rho(\boldsymbol{\xi})\int_D \boldsymbol{F}\cdot\boldsymbol{v}d\boldsymbol{x}d\boldsymbol{\xi}, \tag{4}$$

for all test functions $\boldsymbol{v} \in H^1_\Gamma(D) \otimes L^2_\rho(\Gamma)$.

For the discretization of (4) we form the tensor product space $X^h \otimes W^h$, where

$$X^h := \text{span}\{\boldsymbol{\phi}_1(\boldsymbol{x}), \boldsymbol{\phi}_2(\boldsymbol{x}), \ldots, \boldsymbol{\phi}_{N_{\boldsymbol{x}}}(\boldsymbol{x})\} \subset H^1_\Gamma(D) \tag{5}$$

is any suitable finite dimensional subspace of the deterministic variational space and

$$W^h := \text{span}\{\psi_1(\boldsymbol{\xi}), \psi_2(\boldsymbol{\xi}), \ldots, \psi_{N_{\boldsymbol{\xi}}}(\boldsymbol{\xi})\} \subset L^2_\rho(\Gamma) \tag{6}$$

is a finite dimensional subspace of the stochastic one. A basis of this discrete variational space consists of all functions $\boldsymbol{\phi}_i(\boldsymbol{x})\psi_j(\boldsymbol{\xi})$ where $\boldsymbol{\phi}_i$ is a basis function of $X^h$ and $\psi_j$ belongs to a basis of $W^h$. Thus Ansatz and test functions have the form

$$\boldsymbol{u}^h(\boldsymbol{x},\boldsymbol{\xi}) = \sum_{i,j} u_{i,j}\boldsymbol{\phi}_i(\boldsymbol{x})\psi_j(\boldsymbol{\xi})$$

We insert the approximation $\boldsymbol{u}^h$ into equation (4) together with a test function $\boldsymbol{v}(\boldsymbol{x},\boldsymbol{\xi}) = \boldsymbol{\phi}_k(\boldsymbol{x})\psi_\ell(\boldsymbol{\xi})$ and obtain a system of equations in $N_{\boldsymbol{x}} \cdot N_{\boldsymbol{\xi}}$ unknowns:

$$AU = F. \tag{7}$$

### 2.2 Structure of Galerkin equations

The stiffness matrix $A$ as well as the solution $U$ and load vector $F$ possess a special block structure. For further explanation we define the matrices

$$A_{\ell,j} = \int_\Gamma \rho(\boldsymbol{\xi})\psi_j(\boldsymbol{\xi})\psi_\ell(\boldsymbol{\xi})K(\boldsymbol{\xi})d\boldsymbol{\xi} \in \mathbb{R}^{N_{\boldsymbol{x}}\times N_{\boldsymbol{x}}}, \quad \ell, j = 1, \ldots, N_{\boldsymbol{\xi}}, \tag{8}$$

each of them the size of a deterministic problem, along with the vectors

$$\boldsymbol{f}_j = \int_\Gamma \rho(\boldsymbol{\xi})\psi_j(\boldsymbol{\xi})\boldsymbol{f}(\boldsymbol{\xi})d\boldsymbol{\xi} \in \mathbb{R}^{N_{\boldsymbol{x}}}, \quad j = 1, \ldots, N_{\boldsymbol{\xi}}. \tag{9}$$

In (8) and (9) we introduced a matrix $K(\boldsymbol{\xi}) \in \mathbb{R}^{N_{\boldsymbol{x}}\times N_{\boldsymbol{x}}}$ and a vector $\boldsymbol{f}(\boldsymbol{\xi}) \in \mathbb{R}^{N_{\boldsymbol{x}}}$ that depend on the random vector $\boldsymbol{\xi}$:

$$[K(\boldsymbol{\xi})]_{i,k} = \tfrac{1}{1+\nu}\int_D E(\boldsymbol{x},\boldsymbol{\xi})\,\boldsymbol{\varepsilon}(\boldsymbol{\phi}_i(\boldsymbol{x})) : \boldsymbol{\varepsilon}(\boldsymbol{\phi}_k(\boldsymbol{x}))d\boldsymbol{x} + \tfrac{\nu}{1-\nu-2\nu^2}\int_D E(\boldsymbol{x},\boldsymbol{\xi})\,\nabla\cdot\boldsymbol{\phi}_i(\boldsymbol{x})\,\nabla\cdot\boldsymbol{\phi}_k(\boldsymbol{x})d\boldsymbol{x}, \tag{10}$$

$$[\boldsymbol{f}]_k = \int_D F(\boldsymbol{x},\boldsymbol{\xi})\cdot\boldsymbol{\phi}_k(\boldsymbol{x})d\boldsymbol{x}, \quad i,k = 1,\ldots,N_{\boldsymbol{x}}. \tag{11}$$

Finally we obtain stiffness matrix $A$ and load vector $F$

$$A = \begin{bmatrix} A_{1,1} & \ldots & A_{1,N_{\boldsymbol{\xi}}} \\ \vdots & & \vdots \\ A_{N_{\boldsymbol{\xi}},1} & \ldots & A_{N_{\boldsymbol{\xi}},N_{\boldsymbol{\xi}}} \end{bmatrix}, \qquad F = \begin{bmatrix} \boldsymbol{f}_1 \\ \vdots \\ \boldsymbol{f}_{N_{\boldsymbol{\xi}}} \end{bmatrix},$$

together with the solution vector

$$U = \begin{bmatrix} \boldsymbol{u}_1 \\ \vdots \\ \boldsymbol{u}_{N_{\boldsymbol{\xi}}} \end{bmatrix}, \quad \boldsymbol{u}_j = \begin{bmatrix} u_{1,j} \\ \vdots \\ u_{N_{\boldsymbol{x}},j} \end{bmatrix} \in \mathbb{R}^{N_{\boldsymbol{x}}}, \quad j = 1,\ldots,N_{\boldsymbol{\xi}}.$$

So far we cannot solve equation (7), because the connection of the input random fields to the set of random variables $\{\xi\}_{m=1}^M$ is not yet specified. The last ingredient for the application of SFEM is the specification of the RF model and the construction of the stochastic variational space $W^h$ in (6) that strongly depends on the modeler's knowledge of the randomness in the input data.

## 3 REPRESENTATION OF RANDOM FIELDS

A random field $\kappa$ is a mapping $\kappa : D \times \Omega \to \mathbb{R}$, such that for every fixed point $\boldsymbol{x} \in D$ in the spatial domain, $\kappa(\boldsymbol{x},\cdot)$ is a random variable with respect to the probability space $(\Omega, \mathcal{A}, P)$. The random field $\kappa$ is completely determined by all finite-dimensional distribution functions $F_{\boldsymbol{x}}(\boldsymbol{a}) = P(\kappa(x_1) < a_1 \wedge \kappa(x_2) < a_2 \wedge \cdots \wedge \kappa(x_d) < a_d)$.
In most applications this information is not available. A widely used modelling assumption is for the random field to depend on second order random variables, i.e., $\kappa(\boldsymbol{x},\cdot)$ is a random variable with finite mean and variance. Therefore it suffices to characterize $\kappa$ by means of its

statistical moments up to order two: the mean value $\langle \kappa(\boldsymbol{x}) \rangle$, $\boldsymbol{x} \in D$, and the covariance function $\mathrm{Cov}_\kappa : D \times D \to \mathbb{R}$,

$$\mathrm{Cov}_\kappa(\boldsymbol{x}, \boldsymbol{y}) := \langle (\kappa(\boldsymbol{x}, \cdot) - \langle \kappa(\boldsymbol{x}) \rangle)(\kappa(\boldsymbol{y}, \cdot) - \langle \kappa(\boldsymbol{y}) \rangle) \rangle, \quad \boldsymbol{x}, \boldsymbol{y} \in D \times D. \tag{12}$$

In view of the solution strategies discussed in Section 3, we distinguish two types of RF models: (a) random fields that are *linear* and (b) random fields *nonlinear* in $M$ independent random variables $\{\xi_m\}_{m=1}^M$. Furthermore we consider only RF models that separate the dependencies on the spatial variable $\boldsymbol{x}$ and the stochastic variable $\omega$.

### 3.1 Karhunen-Loève expansion

Gaussian RFs can be represented by their (truncated) Karhunen-Loève expansion (KL expansion) (cf. [16]) that is linear in a set of mutually uncorrelated and hence independent Gaussian random variables. In general, every random field $\kappa$ with given mean value and bounded, continuous covariance function posseses an $L_P^2$-convergent expansion

$$\kappa(\boldsymbol{x}, \omega) = \langle \kappa(\boldsymbol{x}) \rangle + \sum_{i=1}^{\infty} \sqrt{\lambda_i} k_i(\boldsymbol{x}) \xi_i(\omega), \tag{13}$$

where $\{\xi_i\}_{i \geq 1}$ is a sequence of mutually uncorrelated centered random variables of unit variance. The deterministic functions $k_i(\boldsymbol{x})$ solve the integral equation

$$\int_D \mathrm{Cov}_\kappa(\boldsymbol{x}, \boldsymbol{y}) k_i(\boldsymbol{y}) d\boldsymbol{y} = \lambda_i k_i(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in D.$$

For non-Gaussian fields the KL expansion can be employed with independence of the underlying variables as an additional modelling assumption. In numerical simulations one truncates expansion (13) after $M + 1$ terms.

### 3.2 Wiener's polynomial chaos expansion

Type (b) contains RFs that are represented with the help of Wiener's polynomial chaos expansion, where the total degree of polynomials is greater than one. One may think of lognormal fields, that are nonlinear transformations of Gaussian RFs.
For the construction of Wiener's polynomial chaos let $\{h_j\}_{j=0}^{\infty}$ denote the Hermite polynomials, $\mathcal{I}$ the set of multi-indices $\alpha \in \mathbb{N}_0^{\mathbb{N}}$, $|\alpha| < \infty$, and

$$H_\alpha(\boldsymbol{\eta}(\omega)) := \prod_{j=1}^{\infty} h_{\alpha_j}(\eta_j(\omega)),$$

where $\boldsymbol{\eta}(\omega) := [\eta_1(\omega), \ldots, \eta_j(\omega), \ldots]^T$ is a random vector. Every random variable $X : \Omega \to \mathbb{R}$ with finite variance, i.e., $X \in L_P^2(\Omega)$ posseses an $L_P^2$-convergent approximation in multivariate Hermite polynomials of uncorrelated standard Gaussian random variables named Wiener chaos expansion (cf. [12]):

$$X(\omega) = \sum_{\alpha \in \mathcal{I}} c_\alpha H_\alpha(\boldsymbol{\eta}(\omega)).$$

As $\kappa(\boldsymbol{x}, \cdot)$ is a random variable for every $\boldsymbol{x} \in D$ let the chaos coefficients depend on the spatial variable $\boldsymbol{x}$ to obtain the formal expansion of a random field:

$$\kappa(\boldsymbol{x}, \omega) = \sum_{\alpha \in \mathcal{I}} \kappa_\alpha(\boldsymbol{x}) H_\alpha(\boldsymbol{\eta}(\omega)). \tag{14}$$

One truncates expansion (14) by using only multivariate Hermite polynomials in $M$ variables such that the total degree sums to at most $d$.

## 4 SOLUTION STRATEGIES FOR THE DISCRETE SYSTEM

Having a representation of the random fields in problem (1) at hand we can now turn to the discussion of solution strategies for the discrete stochastic variational problem (7). There are several possibilities for breaking the complexity of this large system, for instance block iterative solvers combined with clever data management [10] or multilevel methods and hierarchical approaches are employed [13, 15, 10]. In any case it is important to analyze the structure of the stiffness matrix $A$ and load vector $F$ depending on the the RF type of the input data.
For simplicity we assume the RF's $E$ and $\boldsymbol{F}$ to be fully cross-correlated, i.e. their random behaviour underlies the same set of independent random variables. In any case we have to expand the input RFs in the given stochastic basis (6), that is compute expansions of form

$$E(\boldsymbol{x},\boldsymbol{\xi}) = \sum_{m=1}^{N_{\boldsymbol{\xi}}} E_m(\boldsymbol{x})\psi_m(\boldsymbol{\xi}), \qquad \boldsymbol{F}(\boldsymbol{x},\boldsymbol{\xi}) = \sum_{m=1}^{N_{\boldsymbol{\xi}}} \boldsymbol{F}_m(\boldsymbol{x})\psi_m(\boldsymbol{\xi}). \tag{15}$$

Then the matrix $K(\boldsymbol{\xi})$ and vector $\boldsymbol{f}(\boldsymbol{\xi})$ from (10) take on the form

$$K(\boldsymbol{\xi}) = \sum_{m=1}^{N_{\boldsymbol{\xi}}} K_m\psi_m(\boldsymbol{\xi}), \qquad f(\boldsymbol{\xi}) = \sum_{m=1}^{N_{\boldsymbol{\xi}}} f_m\psi_m(\boldsymbol{\xi}),$$

with matrices $K_m$ and $f_m$ given by

$$\begin{aligned}
[K_m]_{i,k} &= \tfrac{1}{1+\nu}\int_D E_m(\boldsymbol{x})\,\boldsymbol{\varepsilon}(\boldsymbol{\phi}_i(\boldsymbol{x})) : \boldsymbol{\varepsilon}(\boldsymbol{\phi}_k(\boldsymbol{x}))d\boldsymbol{x} + \tfrac{\nu}{1-\nu-2\nu^2}\int_D E_m(\boldsymbol{x})\,\nabla\cdot\boldsymbol{\phi}_i(\boldsymbol{x})\,\nabla\cdot\boldsymbol{\phi}_k(\boldsymbol{x})d\boldsymbol{x}, \\
[f_m]_k &= \int_D F_m(\boldsymbol{x})\cdot\boldsymbol{\phi}_k(\boldsymbol{x})d\boldsymbol{x}, \quad i,k = 1,\ldots,N_{\boldsymbol{x}}, \quad m = 1\ldots,N_{\boldsymbol{\xi}}.
\end{aligned} \tag{16}$$

As a consequence the stiffness matrix $A$ and load vector $F$ from (7) possess the form of sums of Kronecker tensors

$$A = \sum_{m=1}^{N_{\boldsymbol{\xi}}} G_m \otimes K_m, \qquad F = \sum_{m=1}^{N_{\boldsymbol{\xi}}} g_m \otimes f_m. \tag{17}$$

The matrices $G_m$ and vectors $g_m$ depend on the stochastic basis functions from (6) only:

$$[G_m]_{\ell,k} = \langle\psi_m\psi_k\psi_\ell\rangle, \qquad [g_m]_k = \langle\psi_m\psi_k\rangle, \qquad m,\ell,k = 1,\ldots,N_{\boldsymbol{\xi}}. \tag{18}$$

### 4.1 Random fields linear in $\boldsymbol{\xi}$

For input RFs of type (a) we will use the discrete stochastic variational space

$$W^h = W_1^h \otimes \ldots W_M^h, \quad W_m^h = \mathcal{P}_{\alpha_m}(\xi_m) \subset L^2_{\rho_m}(\Gamma_m),\ \alpha_m \in \mathbb{N}_0,\ m = 1,\ldots,M. \tag{19}$$

It consists of products of global polynomials of maximum degree $\alpha_m$ in variable $\xi_m$. Consequently we have $N_{\boldsymbol{\xi}} = \prod_{m=1}^{M}(1+\alpha_m)$ DOFs in the stochastic space. There are other constructions, for instance spaces based on piecewise polynomials on $\Gamma$ [1], but our choice is due to the

solution strategy for linear input RFs.
If $E$ is of form (13) there are only $M+1$ terms in sum (17), where $G_m$ and $g_m$ are given by

$$[G_0]_{\ell,k} = \langle \psi_k \psi_\ell \rangle, \qquad [g_0]_\ell = \langle \psi_\ell \rangle,$$
$$[G_m]_{\ell,k} = \langle \xi_m \psi_k \psi_\ell \rangle, \qquad [g_m]_\ell = \langle \xi_m \psi_\ell \rangle, \qquad \ell, k = 1, \ldots, N_{\boldsymbol{\xi}}, \quad m = 1, \ldots, M.$$

Using appropriate stochastic shape functions the matrices $G_0, \ldots, G_m$ become diagonal [1, 7]. Due to the tensor product structure of (19) it suffices to construct double orthogonal polynomials in each random variable $\xi_m$. For that one has to solve $M$ tridiagonal eigenproblems each of them of size $\alpha_m$.
Hence the global stiffness matrix $A$ in (7) is block diagonal, so the large linear system in (7) decouples into a sequence of independent lower dimensional problems in $N_{\boldsymbol{x}}$ unknowns. Of course this sequence of systems can be solved on a parallel machine, but we employ iterative methods for the sequential solution of these systems.
Recently de Sturler et al. [6] proposed two algorithms, a modification of GCROT [4] and a new one named GCRO-DR, that recycle Krylov subspaces when solving a sequence of linear systems. Both methods are based on the GCRO algorithm [5]. They recycle the outer approximation and correction spaces but the construction of these spaces follows different approaches. The modified GRCOT simply recycles the subspaces created by ordinary GRCOT, whereas GCRO-DR recycles spaces that contain approximate eigenvectors of the system matrix.
Since all system matrices in the sequence are related to one finite element stiffness matrix $A$, recycling of information generated during the solution process of one system in the sequence could be useful for the solution of the next one in order to reduce the total cost of the solution process.

### 4.2 Random fields nonlinear in $\boldsymbol{\xi}$

In connection with RF of type (b) that are represented by Wiener's polynomial chaos expansion we make use of the discrete stochastic variational space

$$W^h = \{H_\alpha(\boldsymbol{\xi}),\ \alpha \in \mathcal{I}_N\}\,, \tag{20}$$

where $\mathcal{I}_N := \{\alpha \in \mathbb{N}_0^M,\ |\alpha| \le d\}$. Hence there are $N_{\boldsymbol{\xi}} = |\mathcal{I}_N| = \binom{M+d}{d}$ DOFs in the stochastic space. For input RFs nonlinear in $\boldsymbol{\xi}$ one has to allow for higher order terms in (17). In contrast to type (a), no suitable stochastic shape functions have yet been found to decouple the large stiffness matrix $A$. Nevertheless there is some hope to exploit the special structure of both the system matrix and right hand side in (17) to make computations feasible.
In recent investigations Hackbusch et al. [11] provide a general framework for truncated iterations on structured matrices. They use an appropriate truncation operator after each iteration step to preserve the special structure of all intermediate matrices as well as the convergence rate of the iterative process.
It is still an open question whether we can make use of an truncated iteration to evaluate $A^{-1}$, for instance by an approximate Newton-Schulz iteration, because it is not known whether the desired result, i.e., the inverse of a sum of Kronecker tensors posseses the same structure or can at least be approximated by such a structured matrix.

## 5 EXPERIMENTS

### 5.1 Model problem

We consider a plane strain problem given in Part II, Chapter 4 of [3], that is we solve the Lamé-Navier equation (1) for the case $n = 2$. It is a model for the settlement of a foundation on an elastic soil mass, where Young's modulus $E$ is subject to random fluctuations resulting from measurement errors or the lack of knowlegde on the exact material properties of the soil.

Let $D = [0, 60] \times [0, -30] \subset \mathbb{R}^2$, $\partial D = B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5$ as shown in Figure 1. We look for the components $u$ and $v$ of the displacement vector $\boldsymbol{u} = [u, v]^T$. Poisson's ratio $\nu = 0.3$ is fixed, as well as the body force density function, $\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{0}$ for all $\boldsymbol{x} \in D$. The boundary conditions are listed below:

$$
\begin{aligned}
v = 0, \qquad & (\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n})_1 = 0 && \text{on } B_1, \\
u = 0, \qquad & (\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n})_2 = 0 && \text{on } B_2 \cup B_5, \\
& \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \boldsymbol{0} && \text{on } B_3, \\
& \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = [0, -0.2 \cdot 10^6]^T && \text{on } B_4.
\end{aligned}
$$

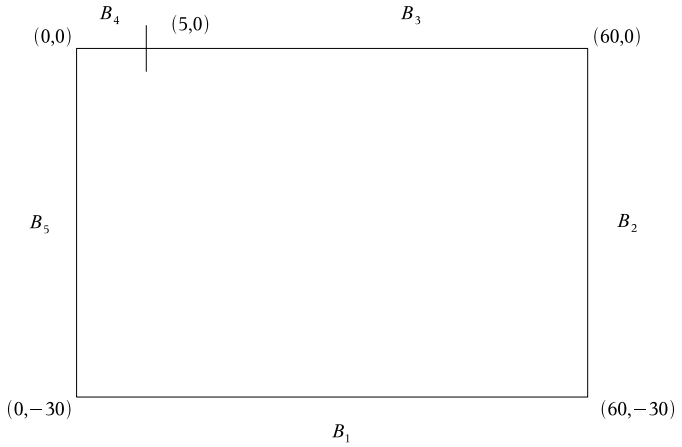


Figure 1: Domain and boundary for plane strain problem.

#### 5.1.1 Representation of random input

Young's modulus $E$ is modeled as a lognormal RF,

$$E = E(\boldsymbol{x}, \omega) = \exp(a + b\, \kappa(\boldsymbol{x}, \omega)). \tag{21}$$

It results from a nonlinear transformation of a centered homogeneous Gaussian RF $\kappa$ with covariance function

$$\mathrm{Cov}_\kappa(\boldsymbol{x}, \boldsymbol{y}) = \exp(-|\boldsymbol{x} - \boldsymbol{y}|^2/\ell^2).$$

The (approximate) KL expansion of $E$ is possible but cannot be used without further computations, because the probability density functions of the random variables therein are not known.

Additionally one has to perform a combined KL and Wiener chaos expansion (cf. [14]). For simplicity we compute a truncated KL expansion of the Gaussian RF $\kappa$ and insert it in (21). Thereafter the Wiener chaos coefficients $E_m$ can be computed in closed form, see [9].
The constants $a$ and $b$ are used to prescribe the mean value $\mu_E$ and the ratio of standard deviation and mean value $\delta_E := \frac{\sigma_E}{\mu_E}$, via

$$a = \log \mu_E - 1/2 \log (1 + \delta_E^2), \qquad b = \sqrt{\log(1 + \delta_E^2)}.$$

In the computations we use $\ell = 30,\ \mu_E = 50 \cdot 10^6$ and $\delta_E \in [0, 0.25]$.

### 5.1.2 Spatial discretization

We make use of FEMLAB's structural mechanics application mode "Plane Strain". The shape functions of the two components of the displacement vector are quadratic polynomials on a mesh of 735 triangles, so there are $N_{\boldsymbol{x}} = 2985$ spatial degrees of freedom.

### 5.1.3 Stochastic discretization

We use a Wiener polynomial chaos expansion of the input and output random fields in $M = 4$ standard Gaussian random variables. The total degree of multivariate Hermite polynomials in the expansion of $E$ is limited to $d = 1$ to obtain a RF linear in the underlying variables.
The chaos coefficients of the input random field $E(\boldsymbol{x}, \omega)$ are computed at prescribed grid points of an equidistant hp finite element mesh that consists of $7 \times 5$ rectangles. The shape functions are tensor product polynomials of degree at most $8$ in each spatial direction on a rectangle. FEMLAB then performs bilinear interpolation to compute the stiffness matrices $K_m$ in (16).

## 5.2 Results

We present the results of some numerical experiments when the sequence of $N_{\boldsymbol{\xi}}$ systems in $N_{\boldsymbol{x}}$ unknowns that comes from the discretizaton of the model problem stated above is solved by Krylov subspace recycling methods.
We used an incomplete Cholesky factorization with drop tolerance $10^{-2}$ as preconditioner for all solvers. The stopping criterion was a reduction of the Eucledian residual norm by a factor of $10^6$. For GCROT the parameters were (see [4]) $m = 15$, $s = 10$, $p_1 = 0$, $p_2 = 2$, $k_{thresh} = 20$, $k_{max} = 25$ and $k_{new} = 5$. For GCRO-DR we used $m = 40$ and $k = 25$. All parameters are chosen in a way that at most $m = 40$ vectors must be stored during the iteration and up to $k = 25$ vectors are recycled.
Table 1 shows the average iteration count per block system of the decoupled stiffness matrix $A$ when the total polynomial degree in the output RF is $d = 3$, hence $N_{\boldsymbol{\xi}} = 256$ linear systems were to be solved. We observe a considerable reduction of average iteration counts for GCRO-DR while GCROT with recycling performs worse. The iteration count increases while $\delta_E$ does.
In Table 2 we vary the polynomial degree in the stochastic space resulting in a larger number linear systems in the sequence to be solved, which seems to have little influence on the average iteration count. This may be due to the fact that a higher polynomial degree does not lead to a better approximation of the random displacement field. Again GRCOT with recycling performs worse than GCRO-DR a better preconditioner might repair.
All calculations were performed in MATLAB 7 SP3 together with FEMLAB 3.1.

| $\delta_E$ | GMRES | GCROT | GCROT-rec. | GCRO-DR-rec. |
|---|---|---|---|---|
| 0.05 | 31 | 35 | 19 | 13 |
| 0.10 | 31 | 35 | 21 | 13 |
| 0.15 | 32 | 36 | 23 | 14 |
| 0.20 | 33 | 37 | 25 | 15 |
| 0.25 | 35 | 39 | 28 | 18 |

Table 1: Average iteration counts per linear system using full GMRES, GCROT without and with recycling and GCRO-DR with recycling for $M = 4$ and $d = 3$.

| $d$ | $N_{\boldsymbol{\xi}}$ | GMRES | GCROT | GCROT-rec. | GCRO-DR-rec. |
|---|---|---|---|---|---|
| 1 | 16 | 30 | 35 | 23 | 14 |
| 2 | 81 | 31 | 35 | 23 | 13 |
| 3 | 256 | 31 | 35 | 21 | 13 |
| 4 | 625 | 31 | 35 | 21 | 13 |
| 5 | 1296 | 32 | 36 | 21 | 13 |

Table 2: Average iteration counts per linear system using full GMRES, GCROT without and with recycling and GCRO-DR with recycling for $\delta_E = 0.1$ and $M = 4$.

## 6 CONCLUSIONS

We have discussed the special structure of linear systems resulting from a Galerkin finite element discretization of a stochastic partial differential equation. We have shown how this structure can be exploited to solve the large systems efficiently. For random fields linear in the underlying variables we demonstrated the usage of Krylov subspace recycling techniques that can contribute to reduce the overall cost of the solution process, but much work remains to be done.

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