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APPROXIMATE SOLUTION OF ELASTOPLASTIC PROBLEMS BASED ON THE MOREAU-YOSIDA THEOREM

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Abstract. *We propose a new approach to the numerical solution of quasi-static elastic-plastic problems based on the Moreau-Yosida theorem. After the time discretization, the problem is expressed as an energy minimization problem for unknown displacement and plastic strain fields. The dependency of the minimization functional on the displacement is smooth whereas the dependency on the plastic strain is non-smooth. Besides, there exists an explicit formula, how to calculate the plastic strain from a given displacement field. This allows us to reformulate the original problem as a minimization problem in the displacement only. Using the Moreau-Yosida theorem from the convex analysis, the minimization functional in the displacements turns out to be Fréchet-differentiable, although the hidden dependency on the plastic strain is non-differentiable. The seconds derivative exists everywhere apart from the elastic-plastic interface dividing elastic and plastic zones of the continuum. This motivates to implement a Newton-like method, which converges super-linearly as can be observed in our numerical experiments.*

1 INTRODUCTION

We consider the quasi-static initial-boundary value problem for small strain elastoplasticity with an isotropic hardening [ACZ99]. Starting from the classical formulation, combining the equilibrium of forces with elastoplastic isotropic hardening law under the assumption of small deformations, we can formulate the time-dependent variational inequality. The uniqueness of a solution of such inequality has been for instance proved in [Joh76] utilizing results for general variation inequalities [DL76].

The traditional numerical methods for solving the time-dependent variational inequality were based on the explicit Euler time-discretization with respect to the loading history. In this case the idea of implicit return mapping discretization [SH98] turned out fruitful for calculations. By implicit Euler time-discretization on the other side, the time-dependent inequality is approximated by a sequence of time-independent variational inequalities [KL84] for the unknown displacement u and the plastic strain p. Each of these inequalities is equivalent to a minimization problem $J(u, p) \to \min$ with the convex but non-smooth functional J. We introduce a new algorithm for solving such minimization problem. Our algorithm is of the Newton type and it utilizes the dependence $p = p(\varepsilon(u))$ of the plastic strain on the total strain $\varepsilon(u)$ [ACZ99]. This makes it possible to reformulate the energy minimization problem $J(u) \to$ min for the unknown displacement u only. Since the dependencies of the minimization functional $J(u, p)$ on the plastic strain p, and of the plastic strain p on the total strain $\varepsilon(u)$ are continuous but non-smooth, the Fréchet derivative $D\,J(u)$ seems not to exist. The main theoretical result here is to show that $J(u)$ is in fact differentiable. More precisely, we show that the structure of the functional $J(u)$ satisfies the assumptions of the Moreau-Yosida theorem from convex analysis and therefore $J(u)$ must be (Fréchet) differentiable.

For the space-discretization, the finite element method of the lowest order with the nodal linear displacement and the piece-wise constant plastic strain is used. The unknown discretized displacement $\mathbf{u}$ satisfies the necessary condition $D\,J(\mathbf{u}) = 0$, which represents the system of nonlinear equations. It is shown that the discretized second derivative $D^2 J(\mathbf{u})$ exists everywhere apart from the elastoplastic interface, i.e., apart from the discrete points, which disunion elastic zones from plastic zones. A Newton-like method was implemented in Matlab© and a numerical experiment is documented in Section 4. Observing many numerical experiments, the following conclusions can be made:

- The number of iteration steps is independent of the size of the discretization.
- The Newton-like method converges super-linearly.
- The Newton-like method converges even quadratically after the elastoplastic zones have been identified sufficiently.

The latter remark has also been made independently in the convergence analysis of [Bla97].

The paper is organized as follows. Section 2 recalls the mathematical modeling of elastoplasticity and presents the formula for the explicit calculation of the minimizer p in $J(u, p) \to \min$. Section 3 addresses the Moreau-Yosida Theorem and the Newton-like method for calculating $J(u) \to \min$ in the discrete FE-space. The numerical experiment in Section 4 illustrates the behavior of the Newton-like method.

2 FORMULATION OF THE PROBLEM

In this section, the basics of mathematical modelling in elastoplasticity are outlined. For a more detailed modelling and a rigorous analysis concerning the existence and uniqueness of the elastoplasticity problem, we refer to [HR99]. We introduce a notation which then will be used throughout the whole article. Let $d \in \mathbb{N}$ be the space dimension, $\Omega \in \mathbb{R}^d$ be an open domain with a Lipschitz-continuous boundary $\Gamma := \partial\Omega$. Let Γ be split into two parts Γ_D (*Dirichlet boundary*) and Γ_N (*Neumann boundary*), such that $\Gamma_D \cup \Gamma_N = \Gamma$ and $\Gamma_D \cap \Gamma_N = \emptyset$. The set Θ be the time interval $\Theta = [0, T]$ for $T \in \mathbb{R}^+$ and $\mathbb{R}^+ := \{x \in \mathbb{R} \mid x > 0\}$. The *matrix-scalar-product* $:$ is defined for two equal size matrices $A = (a_{ij})_{ij}$ and $B = (b_{ij})_{ij}$ as $A : B = \sum_{ij} a_{ij} b_{ij}$. The *Frobenius-norm* of matrix A reads $\|A\|_F := \sqrt{A : A}$. By the use of the *Kronecker symbol* δ_{ij} and the *Lamè constants* $\lambda > 0$ and $\mu > 0$, we define the so called *elasticity tensor* $\mathbb{C}$ with its components $\mathbb{C}_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ and the *elastic norm* of a matrix valued function $v(x)$ in the domain Ω as $\|v\|_{\mathbb{C}} := \left(\int_\Omega v(x) : v(x)\, \mathrm{d}x\right)^{1/2}$. Let I denote the (square) identity matrix. The *trace* and the *deviator* of a matrix $A \in \mathbb{R}^{d\times d}$ are defined by $\operatorname{tr} A := A : I$ and $\operatorname{dev} A := A - \frac{\operatorname{tr} A}{d} I$. Let the *divergence* of a matrix valued function $v(x)$ be defined component wise by the use of Einstein's summation convention as $(\operatorname{div} v)_i := \partial_j v_{ij}$ for $i \in \{1, \ldots, d\}$.

The equilibrium of forces $-\operatorname{div} \sigma(x) = f(x)$ for all $x \in \Omega$ represents the governing law of the quasi-static elastic-plastic problem. Here, σ denotes the *stress* and f the body force density, acting in each material point x of the body Ω. Under the assumption of small deformations, the Green - St. Venant strain tensor $E(u) = \frac{1}{2}\left(\nabla u + \nabla u^T + \nabla u^T \nabla u\right)$ may be replaced by the linear Cauchy strain tensor $\varepsilon(u) := \frac{1}{2}\left(\nabla u + \nabla u^T\right)$. The strain tensor is additively decomposed into the *elastic strain* e and the *plastic strain* p, i.e., $\varepsilon = e + p$, where the elastic strain satisfies *Hook's law* $\sigma = \mathbb{C}e$. We prescribe boundary conditions $\sigma(x)n(x) = g(x)$ for $x \in \Gamma_N$ and $u(x) = u_D(x)$ for $x \in \Gamma_D$.

The time development of p is driven by the *Prandtl-Reuß normality law*. We introduce a parameter α, which is a scalar identifier in case of isotropic hardening [ACZ99]. The tuple (σ, α) is called *generalized stress*. A generalized stress is called *admissible*, if a *dissipation functional* φ with

$$\varphi(\sigma, \alpha) := \begin{cases} 0 & \text{if } \phi(\sigma, \alpha) \leq 0\,, \\ \infty & \text{if } \phi(\sigma, \alpha) > 0\,, \end{cases}$$

satisfies $\varphi(\sigma, \alpha) < \infty$. The function ϕ is convex and called the *yield function*. In case of isotropic hardening it is defined

$$\phi(\sigma, \alpha) := \begin{cases} \|\operatorname{dev} \sigma\|_F - \sigma_y(1 + H\alpha) & \text{if } \alpha \geq 0\,, \\ \infty & \text{if } \alpha < 0\,. \end{cases}$$

The material constants $\sigma_y > 0$ and $H > 0$ are called *yield stress* and *modulus of hardening*. All admissible generalized stresses are characterized by $\phi(\sigma, \alpha) \leq 0$. The Prandtl-Reuß normality law states, that for all generalized stresses (τ, β) there holds

$$\dot{p} : (\tau - \sigma) - \dot{\alpha}\,(\beta - \alpha) \;\leq\; \varphi(\tau, \beta) - \varphi(\sigma, \alpha)\,,$$

where $\dot{p}$ and $\dot{\alpha}$ denote the first time derivative of p and α, and are being replaced by the backward difference as follows. Let $\tau > 0$ and $\Theta_\tau := \{t_k \mid t_k = k\,\tau,\; k \in \mathbb{N}\} \cap \Theta$ be a uniform time decomposition. We abbreviate $u_k := u(t_k)$, $p_k := p(t_k)$ and approximate $\dot{p}(t_k) \approx (p_k - p_{k-1})/\tau$, where the initial conditions $u_0 = 0$ and $p_0 = 0$ are prescribed.

The elastic-plastic one-time step problem at $t_k \in \Theta_\tau$ may be formulated as the minimization of the *energy functional* J with respect to the displacement $u_k \in V := [H^1(\Omega)]^d$ and the plastic strain $p_k \in Q := [L_2(\Omega)]^{d\times d}_{\text{sym}}$.

Problem 1 (one-time-step)**.** *Let $k \in \mathbb{N}$ and (p_{k-1}, α_{k-1}) be given. Find $(u_k, p_k) \in V \times Q$, such that $J(u_k, p_k) = \min_{(v,q)\in V\times Q} J(v, q)$, where*

$$J(v, q) := \frac{1}{2}\|\varepsilon(v) - q\|^2_{\mathbb{C}} + \Psi(q) - \langle F, v\rangle\,, \tag{1}$$

with

$$\Psi(q) := \frac{1}{2}\int_\Omega (\alpha_{k-1} + \sigma_y H\|q - p_{k-1}\|_F)^2 \,\mathrm{d}x + \int_\Omega \sigma_y\|q - p_{k-1}\|_F \,\mathrm{d}x\,, \tag{2}$$

and

$$\langle F, v\rangle := \int_\Omega f \cdot v \,\mathrm{d}x + \int_{\Gamma_N} g \cdot v \,\mathrm{d}S(x)\,. \tag{3}$$

Note, that the functional $J = J_k$ in (1) depends on the initial values p_{k-1} and α_{k-1}, and thus on the time step k. For easier reading, this dependency will not be mentioned explicitly here and in the sequal. In order to obtain the minimization functional $J = J_{k+1}$ of the next time step,

$$\alpha_k = \alpha_{k-1} + \sigma_y H\|p_k - p_{k-1}\|_F \tag{4}$$

has to be calculated.

There exists an explicit formula [ACZ99] for calculating p_k s.t. $J(v, p_k) = \min_{q\in Q} J(v, q)$:

$$p_k(\varepsilon(v)) = \frac{\max(0, \|\operatorname{dev} A\|_F - \beta)}{2\mu + \sigma_y^2 H^2}\frac{\operatorname{dev} A}{\|\operatorname{dev} A\|_F} + p_{k-1}\,, \tag{5}$$

where $A := \mathbb{C}\,(\varepsilon(v) + p_{k-1})$ and $\beta := \sigma_y\,(1 + \alpha_{k-1}H)$.

Thus, Problem 1 is equivalent to the following minimization problem, which depends on the displacement u_k only.

Problem 2 (minimization in the displacement only)**.** *Let (p_{k-1}, α_{k-1}) be given. Find $u_k \in V$, such that $J(u_k) = \min_{v\in V} J(v)$, where*

$$J(v) := \frac{1}{2}\|\varepsilon(v) - p_k(\varepsilon(v))\|^2_{\mathbb{C}} + \Psi(p_k(\varepsilon(v))) - \langle F, v\rangle\,,$$

with Ψ, F and $p_k(\varepsilon(v))$ as in (2), (3) and (5).

Note, that again the functional J depends on the time step k and that we have to calculate α_k as in (4) in order to obtain the minimization functional $J = J_{k+1}$ of the next time step.

3 THE NEW APPROACH

Minimization of the functional J can be done by finding the root of its first derivative $D\,J(v)$. The next theorem states, that $J(v)$ is indeed smooth, no matter the dependencies $\Psi(p_k)$ and $p_k(\varepsilon(v))$ are non-smooth.

Theorem 1 (Moreau-Yosida). *Let H be a Hilbert space with scalar product $\langle\cdot,\cdot\rangle_H$, H^* its dual space, $\psi : H \to \mathbb{R}$ convex, and function f be defined as*

$$f : H \to \mathbb{R},\, y \mapsto \min_{x\in H}\left[\frac{1}{2}\|y-x\|_H^2 + \psi(x)\right] .$$

Further let $\tilde{x}(y)$ denote the (unique) function, such that for all $y \in H$

$$f(y) = \frac{1}{2}\|y-\tilde{x}(y)\|_H^2 + \psi(\tilde{x}(y)) .$$

Then there holds: f is convex and Fréchet-differentiable with $D\,f(y) = y - \tilde{x}(y) \;\in H^*$.

Proof. See [Mor65]. □

Theorem 1 says that the first derivative $D\,J(v)$ exists and is continuous,

$$D\,J(v,w) = \int_\Omega \mathbb{C}\left(\varepsilon(v) - p_k(\varepsilon(v))\right) : \varepsilon(w)\ \mathrm{d}x - \langle F, w\rangle . \tag{6}$$

The second derivative $D^2 J(v)$ exists everywhere apart from the elastoplastic interface. For more details, see [GV06]. Discretization in space is realized by linear nodal FE-ansatz functions. The discrete form of u is denoted by $\mathbf{u}$. The Newton-like method is applied for the calculation of $\mathbf{u}$ such that $D\,J(\mathbf{u}) = 0$ and $\mathbf{u}$ satisfies the Dirichlet boundary condition:

$$\mathbf{u}_i = \mathbf{u}_{i-1} + \mathbf{w}_i \qquad (\forall i \in \mathbb{N}) , \tag{7}$$

where $\mathbf{w}_i$ solves $-D^2 J(\mathbf{u}_{i-1})\ \mathbf{w}_i = D\,J(\mathbf{u}_{i-1})$. Note, that $\mathbf{u}_i$ must satisfy (generally inhomogeneous) Dirichlet boundary conditions for all $i \in \mathbb{N}$. Therefore it is sufficient for the initial approximation $\mathbf{u}_0$ to fulfill the inhomogeneous Dirichlet condition, and for $\mathbf{w}_i$ to fulfill the homogeneous Dirichlet condition. For the termination of the Newton-like method we check, whether the relative error of the displacement $u_i \in V_h$,

$$\frac{|u_i - u_{i-1}|_\varepsilon}{|u_i|_\varepsilon + |u_{i-1}|_\varepsilon} , \tag{8}$$

is smaller than a given prescribed bound $\epsilon \in \mathbb{R}^+$, where $|\cdot|_\varepsilon := (\int_\Omega ||\varepsilon(\cdot)||^2\ \mathrm{d}x)^{1/2}$. Note that the semi-norm $|\cdot|_\varepsilon$ is easier computable than the equivalent H_1 norm.

4 NUMERICAL EXPERIMENT

The following test was calculated on a computer with Intel(R) Xeon(TM) CPU 2.80GHz, 4 Gb RAM using Matlab© version 7.0 on Linux OS. We define 'DOF' as the short form of *degrees of freedom*, and 'VPZ' to be the short form of *variation in plastic zones* which is calculated as follows: In the i-th iteration step the boolean vector $\mathbf{w}^i$ stores the information about which elements are plastic and which are not by defining its components

$$w_j^i := \begin{cases} 1 & \text{if } T_j \text{ is a plastic element,} \\ 0 & \text{else.} \end{cases}$$

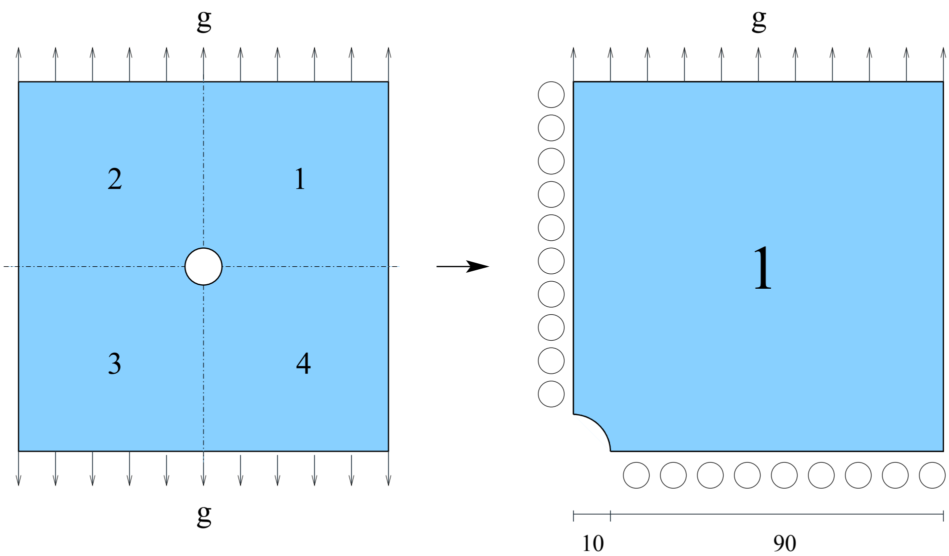

Figure 1: Problem geometry in the numerical experiment.

Let the starting vector $\mathbf{w}^0 = 0$. Variation in plastic zones VPZ^i_{i-1} from the $(i-1)$-st to the i-th iteration step is defined by

$$\mathrm{VPZ}^i_{i-1} = \frac{100}{\mathrm{N}_{\mathcal{T}}} \sum_{j=1}^{n} |w^i_j - w^{i-1}_j| \,. \qquad (9)$$

The example domain is a thin plate represented by the square $(-10, 10) \times (-10, 10)$ with a circular hole of the radius $r = 1$ in the middle, as can be seen in Figure 1. A surface load g is applied on the plate's upper and lower edge. Due to symmetric geometry only the right upper quarter of the domain is discretized. Therefore it is necessary to incorporate homogeneous Dirichlet boundary conditions in the normal direction (gliding conditions) to both symmetric edges. The material parameters are now set

$$E = 206900, \ \nu = 0.29, \ \sigma_Y = \sqrt{\frac{2}{3}}\,450, \ H = \frac{1}{2}\,.$$

Figure 2 shows the yield function (right) and the elastic-plastic zones, where purely elastic zones are colored green (light gray in case of a non-color print respectively), and elastic-plastic zones are colored pink (dark grey respectively). The displacement is multiplied by 100. Table 1 reports on the convergence of the Newton-like method for graduated uniform meshes. The termination bound $\epsilon = 1e - 12$ was used for the comparison with the relative error of u_i (8).

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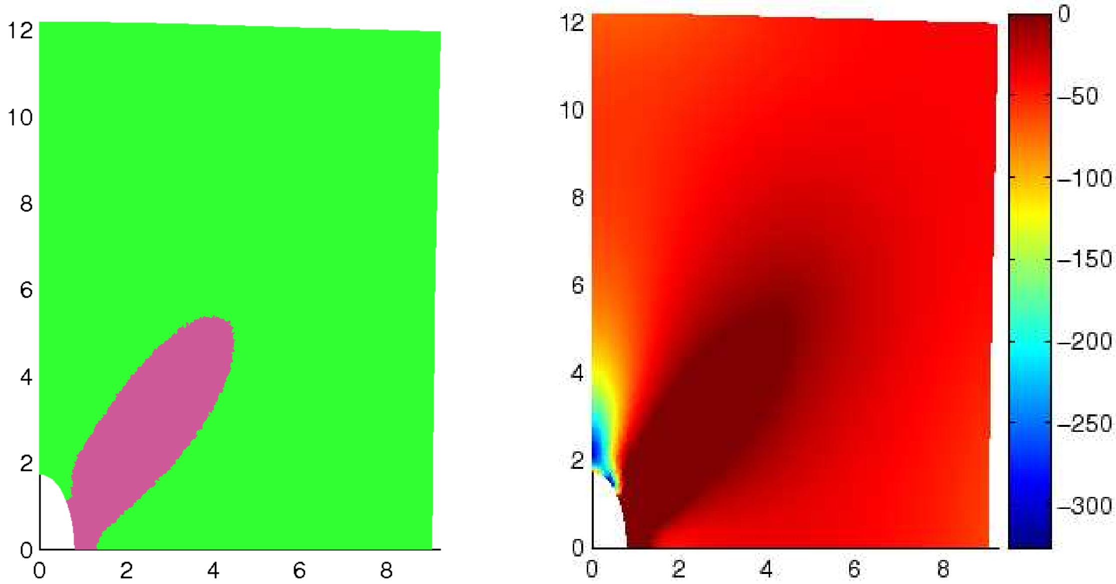

Figure 2: The two plots show the elastoplastic zones (left) and the yield function (right) of the deformed domain. The displacement is magnified by the factor 100.

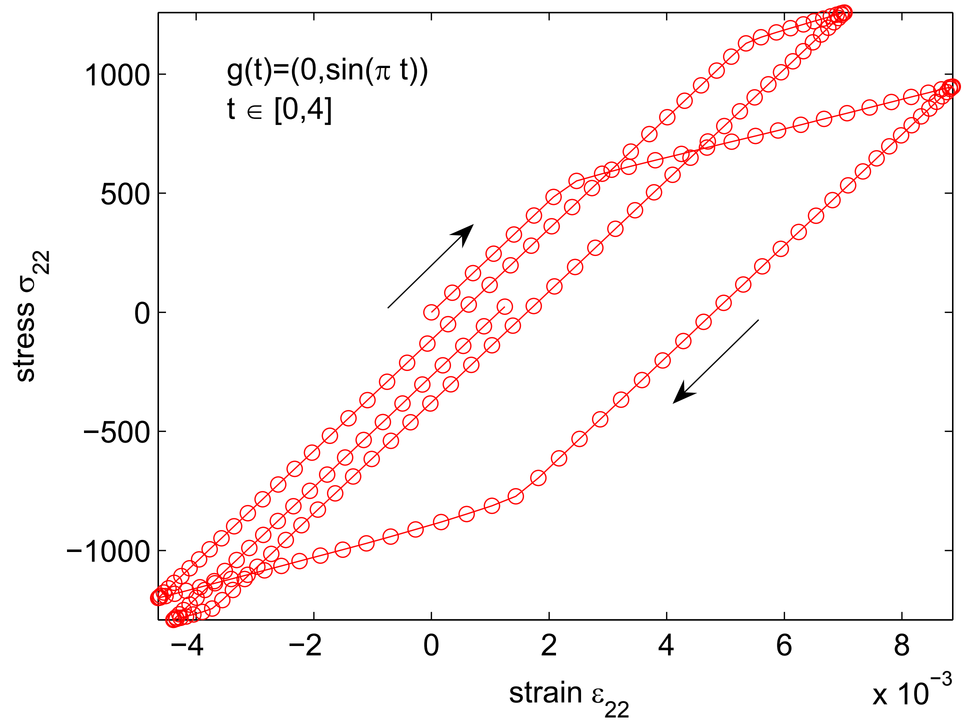

Figure 3: Hysteresis curve with respect to the time dependent surface load $g(t) = (0, sin(\pi t))$ for $t \in [0, 4]$. The stress component σ_{22} is plotted versus the strain component ε_{22} which both are set to zero at $t = 0$, as is implied from the initial conditions in Section 2. The time development takes place in direction of the arrows.

Level	0	1	...	3	4	5
DOF	245	940	...	14560	57920	231040
relative error:						
step 1	2.1826e-02	3.5365e-02	...	4.5238e-02	4.6300e-02	4.6603e-02
step 2	2.2225e-03	5.8553e-03	...	8.0839e-03	8.3886e-03	8.5454e-03
step 3	1.0478e-04	1.6539e-04	...	3.4440e-04	4.0032e-04	4.1602e-04
step 4	1.4404e-08	3.9755e-08	...	1.5206e-05	1.2050e-05	1.3944e-05
step 5	7.2634e-16	6.9728e-15	...	2.4947e-07	7.2972e-07	3.2631e-07
step 6			...	3.5062e-13	5.3972e-12	1.6473e-12
step 7			...		7.2441e-15	1.4518e-14
VPZ (%):						
step 0-1	4.889	5.889	...	7.042	7.116	7.129
step 1-2	1.333	4.222	...	5.444	5.549	5.546
step 2-3	0.8889	1.222	...	1.056	1.125	1.098
step 3-4	0	0	...	0.1597	0.1233	0.1215
step 4-5	0	0	...	0.01389	0.01042	0.008247
step 5-6			...	0	0	0
step 6-7			...		0	0
Time (sec.)	2	4.6	...	64	286	1195

Table 1: The convergence table displays the relative error in displacements (8) and the variation of plastic zones VPZ (9) per iteration step for various uniformly refined meshes.

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