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TOOL TO CHECK TOPOLOGY AND GEOMETRY FOR SPATIAL STRUCTURES ON BASIS OF THE EXTENDED MAXWELL'S RULE

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Abstract. *One of the simplest principle in the design of light-weight structures is to avoid bending. This can be achieved by dissolving girders into members acting purely in axial tension or compression. The employment of cables for the tensioned members leads to even lighter structures which are called cable-strut structures. They constitute a subclass of spatial structures. To give fast information about the general feasibility of an architectural concept employing cable-strut structures is a challenging task due to their sophisticated mechanical behavior. In this regard it is essential to control if the structure is stable and if pre-stress can be applied. This paper presents a tool using the spreadsheet software Microsoft (MS) Excel which can give such information. Therefore it is not necessary to purchase special software and the according time consuming training is much lower. The tool was developed on basis of the extended Maxwell's rule, which besides topology also considers the geometry of the structure. For this the rank of the node equilibrium matrix is crucial. Significance and determination of the rank and the implementation of the corresponding algorithms in MS Excel are described in the following. The presented tool is able to support the structural designer in an early stage of the project in finding a feasible architectural concept for cable-strut structures. As examples for the application of the software tool two special cable-strut structures, so called tensegrity structures, were examined for their mechanical behavior.*

1 THEORETICAL BASIS

1.1 Maxwell's rule

The geometry and topology design concept of spatial structures comprises fundamental information about the load bearing behavior. The questions if the structure is stable and if pre-stress can be applied can be answered based on the extension of the well known Maxwell's rule. The number of bars, which are at least necessary to stabilize a space frame with frictionless joints, can be determined by Maxwell's rule.

The rule for the construction of rigid two-dimensional frameworks with b bars and j frictionless joints is:

$$b = 2\,j - 3$$

and:

$$b = 3\,j - 6$$

for rigid three-dimensional frameworks.

Maxwell himself assumed exceptions of his rule. He anticipated that stiff structures may be possible with a smaller number of bars and also cases occur, where the structure is free to move even if his rule is satisfied [1].

1.2 The extended Maxwell's rule

In the seventies of the 20th century Calladine [2] went back to the exceptions of Maxwell's rule. The result from his study is the extended version of Maxwell's rule, which includes all possible cases:

$$3\,j - b - c = m - s$$

Therein c is the number of kinematic constraints ($c \geqq 6$ in three dimensions), m the number of internal mechanisms and s the number of self-stress states.

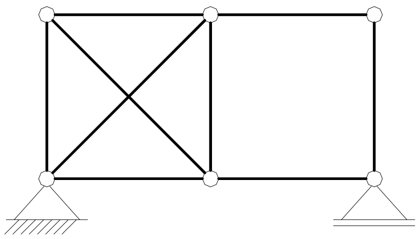


Figure 1 Two dimensional structure which satisfies Maxwell's rule but includes a mechanism

The two dimensional example in Figure 1 illustrates the advancement of the rule. The structure has six joints, three cinematic constraints and nine bars, so Maxwell's rule is exactly fulfilled. Despite this fact the structure is obviously not stable and includes one mechanism m and one self-stress state s. The self-stress state can not stabilize the whole structure. Configurations like this can be checked with the presented tool as described in the following.

2 ALGORITHMS

2.1 Determination of m and s

The key value for the calculation of m and s is the rank of the equilibrium matrix of the nodes. The largest number of linear independent column vectors and the largest number of linear independent row vectors are always equal. This number r_A is called rank of the matrix. The equilibrium matrix A is a (m x n) matrix, where m is the number of joints (j * 3) and n is the number of unknown element forces. The Matrix A contains the direction cosines, in the x, y, z directions, of each element. One feasible method to determine the rank of a matrix is the singular value decomposition (SVD). For this the number of non-zero singular values of a matrix is identical with the rank of this matrix. Detailed information on this operation and on its application to the equilibrium matrix is given in [3].

After the rank determination, m and s can be calculated with the following formulas:

$$s = b - r_A$$

$$m = 3\,j - c - r_A$$

The values of m and s depend not only on the number of bars and joints, nor even on the topology of the structure, but essentially on the complete geometry [4].

2.2 Implementing in MS Excel

The Excel add-in was developed using visual basic for applications (VBA) as programming language. The add-in can be employed to calculate the above described values m and s for general plane and spatial structures. An Excel worksheet works as user interface. The required input data are the node coordinates, the kinematic constraints and the topology of the connections. Using a virtual reality modeling language (VRML) plug in, it is possible to visualize the input data of the structure in any internet browser. The VBA algorithm generates the node equilibrium matrix based on the input data. Subsequently the SVD of this matrix is carried out and the number of non-zero singular values can be achieved. One has to note that none of this values is actually equal to zero, but some are much smaller than others. Pellegrino [3] recommends to treat values as zero if they are smaller than 10^{-3} after being multiplied with the largest singular value. This procedure is valid for the most structural assemblies and therefore basis for the rank determination in the VBA algorithm. The output of the values m and s will be presented in an worksheet.

2.3 Relevance of m and s

A summary of the information about the fundamental mechanical properties of spatial structures, that can be derived from the values m and s, is given in Table 1.

Table 1 Information about the fundamental mechanical properties on basis of m and s

1	m = 0 s = 0	The structure is stable and can not be pre-stressed. → useable
2	m > 0 s = 0	The structure is not stable and can not be pre-stressed. → unusable
3	m = 0 s > 0	The structure is stable and can be pre-stressed. → useable
4	m > 0 s > 0	The structure has mechanisms and can be pre-stressed. → conditionally useable (further investigation necessary)

Structures which belong to the fourth row of table 1 are considered as kinematically indeterminate ($m > 0$) and statically indeterminate ($s > 0$). In this case further investigations of the vector of the element pre-stress forces (t_0) are necessary.

2.4 Check of t_0

For classical buildings, structures with internal mechanisms are not desirable. Within a few special constructions, for instance cable nets or tensegrity structures, these mechanisms are accepted and part of the concept. Condition for the load bearing capacity of such structures is, that the pre-stress stabilizes all mechanisms. This can be clarified with the investigation of t_0. The vector t_0 is filled with the element forces for the loading case pre-stress and must be a solution of the following homogeneous equation system.

$$A\, t_0 = 0$$

It has to be examined whether element forces in t_0 are equal to or approximately zero. If this is not the case, the available mechanisms can be stabilised by pre-stress.

The two dimensional example in Figure 1 includes one mechanism and one state of self stress. But the self-stress state can not stabilize the whole structure, which is therefore not stable and unusable. To make this visible in Figure 2 the normalised pre-stress forces (contained in t_0) are assigned to the elements, the diagonal elements are the bars.

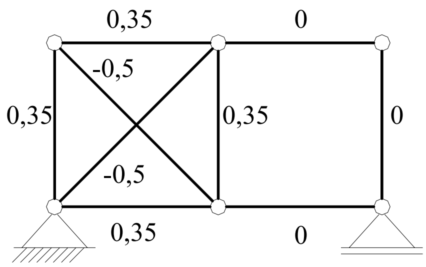


Figure 2 Two dimensional structure with normalised element pre-stress forces.

When using cables, because of their unilateral rigidity, it must be checked additionally if the cables are tensioned and the bars are subjected to compression.

2.5 Determination of t_0

As mentioned above, the SVD of the equilibrium matrix in the described Excel add-in is used for the rank determination. This decomposes the matrix A to:

$$A = U\, W\, V^T$$

For a (m x n) matrix the m left singular vectors are returned in U and the n right singular vectors are returned in V. The matrix W is diagonal and contains the singular values of A.

The right singular vectors are of special interest. All right singular vectors with a corresponding singular value equal to zero are self-stress states [4]. The self-stress state is identical with t_0 only for the case $s = 1$. In the Excel tool, the self-stress state (s_1) is plotted to a worksheet.

To determine t_0 in the case of $s > 1$ either an equation system or a system of inequalities, depending on the requested qualitative distribution of pre-stress, has to be solved. For instance, if for symmetrical structures the pres-tress forces shall be the same in certain elements, an equation system has to be used.

But if within cable-strut structures all cables shall be tensioned and all bars shall be subjected to compression, a system of inequalities has to be solved to determine a feasible domain of coefficients α.

The number of inequalities is equal to the number of elements n and the number of unknowns is equal to the number of states of self-stress s, as shown below.

$$
\begin{array}{cl}
\alpha_{11}\ s_{11} + \alpha_{21}\ s_{21} + \ldots\ \alpha_{s1}\ s_{s1} \leq 0 & \\
\alpha_{12}\ s_{12} + \alpha_{22}\ s_{22} + \ldots\ \alpha_{s2}\ s_{s2} \leq 0 & \text{b - bars} \\
\ldots & \\
\alpha_{1b}\ s_{1b} + \alpha_{2b}\ s_{2b} + \ldots\ \alpha_{sb}\ s_{sb} \leq 0 & \\
\hline
\alpha_{1b+1}\ s_{1b+1} + \alpha_{2b+1}\ s_{2b+1} + \ldots\ \alpha_{sb+1}\ s_{sb+1} \geq 0 & \\
\ldots & \text{c - cables} \\
\alpha_{1b+c}\ s_{1b+c} + \alpha_{2b+c}\ s_{2b+c} + \ldots\ \alpha_{sb+c}\ s_{sb+c} \geq 0 &
\end{array}
$$

Using the calculated coefficients α_1 to α_s the element pre-stress vector t_0 can be determined with the following equation.

$$t_0 = \alpha_1\ s_1 + \alpha_2\ s_2 + \ldots\ \alpha_s\ s_s$$

On the left side of Figure 3 there is an example for a two dimensional structure with s = 2. The diagonals are the bars, all other elements are cables. On the right side the feasible domain of α_1 and α_2 is shown.

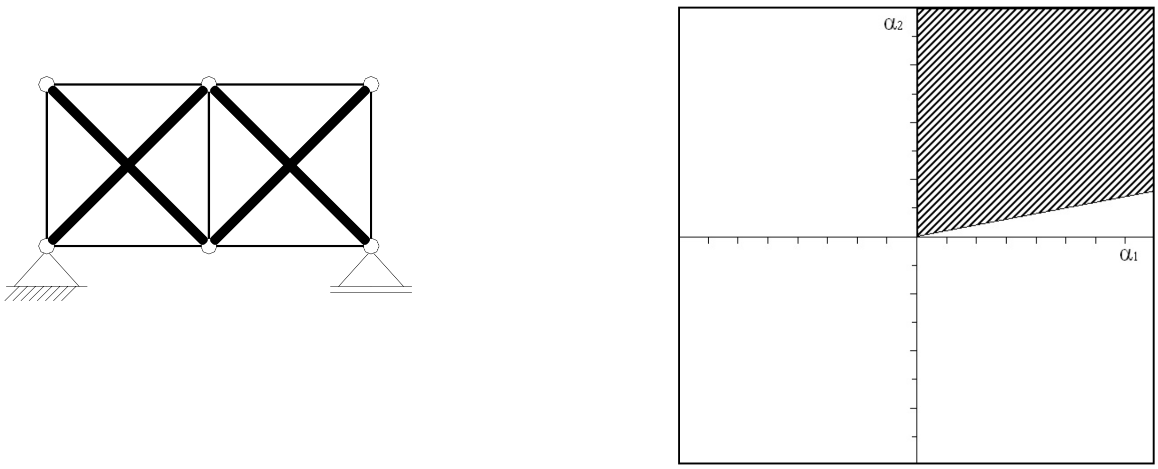


Figure 3 Two dimensional structure with s = 2 and feasible domain of α_1 and α_2

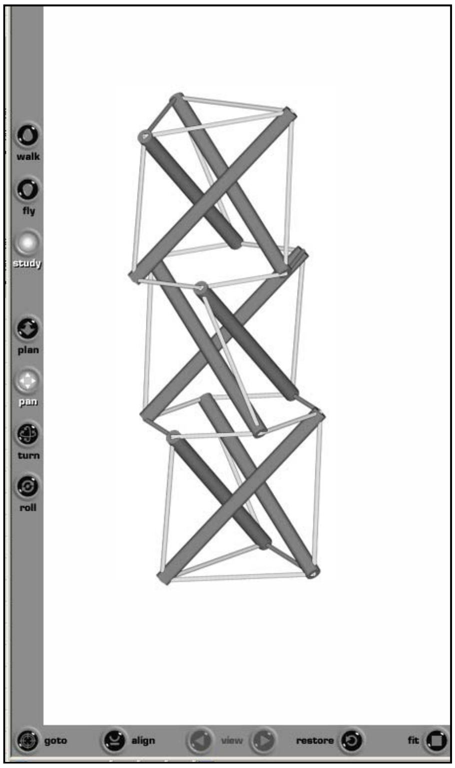
3 APPLICATION EXAMPLE

As examples of use two spatial tensegrity structures are to be examined. Tensegrity structures are a subclass of cable-strut structures containing internal mechanisms, which can be stabilised by pre-stress. Richard Buckminster Fuller created the neologism from tens(ion) and (int)egrity 60 years ago.

The two examples have similar shape but differ considerably in their mechanical behavior. This can be shown with the presented Excel tool.

The basic module of the first example structure is a three-bar module with triangular base polygons. This is the most simple spatial tensegrity module. It consists of three bars and nine cables and is characterised by a twist-angle of 30° between upper and lower base polygons. The example comprises of three stages of three-bar modules. Here the modules are installed on top of each other in a way that the bars of the upper module are placed in the middle of the upper polygon cable of the lower unit. This creates regular hexagons in the intermediate planes and results in a “pure” tensegrity in which none of the bars are in direct contact with each other. A topology check shows that the example has 18 nodes and 6 kinematic constraints. In order to be stable it must have (3 * 18 – 6 = 48) elements according to Maxwell’s rule. But, in fact, the structure only owns 36 elements. A geometry check shows the structure is nevertheless stable, but contains (m = 13) internal mechanisms. The state of self-stress (s = 1) stabilises the structure. However, how model tests show, the structure is not stiff enough to carry significant loads besides own weight. That’s why, such topologies are often only being used for Tensegrity sculptures [5].

Figure 4 shows on the left a screenshot of the structure view with the VRML plug in, and on the right a screenshot of the worksheet with the output data. For the case s = 1, s_1 is identical with the normalised t_0. In the table of topology the bars are in the first nine rows, followed by the horizontal cables and the vertical cables in the end.



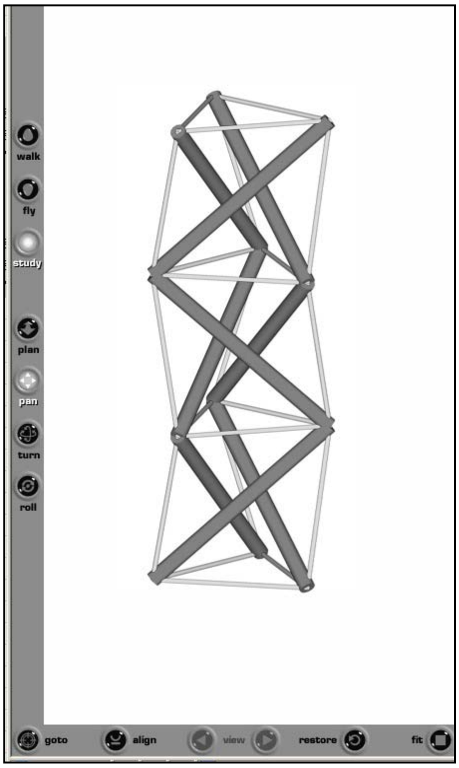
Element	from node	to node	S_1
1	1	6	-0,225833
2	2	4	-0,225833
3	3	5	-0,225833
4	7	12	-0,225833
5	8	10	-0,225833
6	9	11	-0,225833
7	13	18	-0,225833
8	14	16	-0,225833
9	15	17	-0,225833
10	1	2	0,0870388
11	2	3	0,0870388
12	3	1	0,0870388
13	4	8	0,1507557
14	8	5	0,1507557
15	5	9	0,1507557
16	9	6	0,1507557
17	6	7	0,1507557
18	7	4	0,1507557
19	10	14	0,1507557
20	14	12	0,1507557
21	12	13	0,1507557
22	13	11	0,1507557
23	11	15	0,1507557
24	15	10	0,1507557
25	16	17	0,0870388
26	17	18	0,0870388
27	18	16	0,0870388
28	1	5	0,1573442
29	2	6	0,1573442
30	3	4	0,1573442
31	7	10	0,1573442
32	8	11	0,1573442
33	9	12	0,1573442
34	13	17	0,1573442
35	14	18	0,1573442
36	15	16	0,1573442

Figure 4 Left: Structure view of the first example; Right: Table of Topology and computed state of self-stress s_1.

The second example structure is to be assembled using the same basic module as in example one. Different to the first one is the vertical addition of the modules in such a way, that the bar ends of two adjacent stages are directly in contact. Consequently, the whole tower contains only 12 nodes and needs, according to Maxwell's rule, at least (3 * 12 – 6 = 30) elements to be stable. This condition is exactly fulfilled. The geometry check shows, that the structure nevertheless contains (m = 3) internal mechanisms, but can be stabilised by (s = 3) independent states of self-stress.

Such a topology was built as a landmark on the trade area of Rostock [6].

In Figure 5 the second example is plotted and a part of the MS Excel worksheet shows (s = 3) independent states of self-stress and a determined vector of element pre-stress forces t_0. Here the pre-stress force shall be the same in each stage.



Element	from node	to node	S1	S2	S3	t0
1	1	6	-0,433957	0	0	-1
2	2	4	-0,433957	-8,33E-16	1,049E-15	-1
3	3	5	-0,433957	9,497E-17	1,401E-15	-1
4	6	7	0,0364159	-0,279719	0,3317627	-1
5	4	8	0,0364159	-0,279719	0,3317627	-1
6	5	9	0,0364159	-0,279719	0,3317627	-1
7	7	12	-0,003035	-0,307299	-0,306393	-1
8	8	10	-0,003035	-0,307299	-0,306393	-1
9	9	11	-0,003035	-0,307299	-0,306393	-1
10	1	2	0,1672525	3,714E-16	-6,37E-18	0,385413
11	2	3	0,1672525	1,919E-17	-7,87E-16	0,385413
12	3	1	0,1672525	-2,36E-16	-3,89E-16	0,385413
13	4	5	0,1532173	0,1078073	-0,127866	0,7708259
14	5	6	0,1532173	0,1078073	-0,127866	0,7708259
15	6	4	0,1532173	0,1078073	-0,127866	0,7708259
16	7	8	-0,012866	0,2262442	-0,009778	0,7708259
17	8	9	-0,012866	0,2262442	-0,009778	0,7708259
18	9	7	-0,012866	0,2262442	-0,009778	0,7708259
19	10	11	0,0011696	0,1184369	0,1180878	0,385413
20	11	12	0,0011696	0,1184369	0,1180878	0,385413
21	12	10	0,0011696	0,1184369	0,1180878	0,385413
22	1	5	0,3023502	-7,25E-16	-9,48E-16	0,6967291
23	2	6	0,3023502	5,087E-17	1,084E-15	0,6967291
24	3	4	0,3023502	1,077E-15	-1,41E-15	0,6967291
25	5	7	-0,025372	0,1948883	-0,231149	0,6967291
26	6	8	-0,025372	0,1948883	-0,231149	0,6967291
27	4	9	-0,025372	0,1948883	-0,231149	0,6967291
28	7	11	0,0021143	0,214104	0,2134728	0,6967291
29	8	12	0,0021143	0,214104	0,2134728	0,6967291
30	9	10	0,0021143	0,214104	0,2134728	0,6967291

Figure 5 Left: Structure view of the second example; Right: Table of Topology and computed states of self-stress s_1 to s_3 and an feasible t_0.

4 CONCLUSION / OUTLOOK

The question of stability is certainly the first to be solved when cable-strut structures are to be designed. A software tool was developed, for this purpose, that gives information about the stability while using only standard software and freeware. The theoretical and mathematical basics for the used algorithms and the implementing in MS Excel where described.

The use of the tool was demonstrated on two special cable-strut structures, so called tensegrity structures. It can be shown, that in an early stage of design different concepts with regard to their principle load bearing behavior can be evaluated.

An interesting additional feature of the described tool could be to survey the robustness of the structure. Robust design of structures considers the sensitivity to unavoidable variations of input parameters. That means a less variability of the system response, and so a better predictability of this. Robustness can be measured e.g. by means of standard deviations of certain input and response quantities. More detailed information about robust design of structures can be read in [7] and [8].

The scattered input parameter on an investigation of robustness of pre-stressed cable-strut structures could be the element length. So it would be good to know if the structure is more ore less sensitive to manufacturing tolerances. To minimise the standard deviation of the element pre-stress force could be the objective of the optimisation.

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