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Invariance of the essential spectra of operator pencils

H. Gernandt, N. Moalla, F. Philipp, W. Selmi and C. Trunk

Abstract. The essential spectrum of operator pencils with bounded coefficients in a Hilbert space is studied. Sufficient conditions in terms of the operator coefficients of two pencils are derived which guarantee the same essential spectrum. This is done by exploiting a strong relation between an operator pencil and a specific linear subspace (linear relation).

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1. Introduction

Its is a well-known fact that the essential spectrum of a linear operator is invariant under compact perturbations. Here we understand the essential spectrum as the complement of the (semi-) Fredholm domain. More precisely, we investigate four kinds of essential spectra: the Fredholm essential spectrum, the upper and the lower semi Fredholm essential spectrum and the semi Fredholm essential spectrum. For simplicity, we refer to those four kinds just as the "essential spectra".

In many applications, e.g. in mathematical physics or in transport theory, one is interested in the (essential) spectrum of operator pencils, see, e.g., [8, 9] A linear operator pencil is a first order polynomial with bounded operators as coefficients, that is, it is of the form

$$\mathcal{A}_1(\lambda) = \lambda S_1 - T_1,$$

where $\lambda \in \mathbb{C}$ and S_1 and T_1 are bounded operators acting between two normed spaces. By definition (see, e.g., [13, 15]) a complex number λ is in the spectrum of the pencil $\mathcal{A}_1$ if zero is in the spectrum of the operator $\lambda S_1 - T_1$. In the same way the essential spectrum of $\mathcal{A}_1$ is defined as the set of all $\lambda \in \mathbb{C}$ such that the operator $\lambda S_1 - T_1$ is no (semi-) Fredholm operator.

We investigate the question which perturbations of the coefficients do not change the essential spectrum. For this, consider a second operator pencil

$$\mathcal{A}_2(\lambda) = \lambda S_2 - T_2,$$

where S_2 and T_2 are bounded operators acting between the same spaces as S_1 and T_1. If $S_1 - S_2$ and $T_1 - T_2$ are two compact operators, then obviously also the difference

$$\mathcal{A}_1(\lambda) - \mathcal{A}_2(\lambda) = \lambda(S_1 - S_2) - (T_1 - T_2)$$

is compact and, hence, the essential spectra of $\mathcal{A}_1$ and $\mathcal{A}_2$ coincide. But the essential spectrum of two operator pencils may coincide even if the difference of the coefficients is substantial. For example, let M be a bounded and boundedly invertible operator. Then obviously

$$\mathcal{A}_1(\lambda) = \lambda I - T \quad \text{and} \quad \mathcal{A}_2(\lambda) = \lambda M - TM = \mathcal{A}_1(\lambda)M$$

have the same essential spectrum.

Here we make use of the following simple observation: Let $S, T : X \to Y$ be bounded linear operators between two Hilbert spaces X and Y such that the upper semi Fredholm essential spectrum of the pencil $\mathcal{A}(\lambda) := \lambda S - T$ is not $\mathbb{C}$. Then the essential spectra of $\mathcal{A}$ and TS^{-1} coincide (see Corollary 3.5 below). Note, that in general S is not invertible and here S^{-1} and TS^{-1} are understood in the sense of linear relations (or, what is the same, multivalued mappings, see [1, 5, 16]). That is, S^{-1} and TS^{-1} are subspaces of $Y \times X$ and $Y \times Y$, respectively, given by

$$S^{-1} := \{\{Sx, x\} : x \in X\}, \text{ and}$$

$$TS^{-1} := \big\{\{x, z\} : \{x, y\} \in S^{-1},\ \{y, z\} \in T, \text{ for some } y \in X\big\} = \operatorname{ran} \begin{bmatrix} S \\ T \end{bmatrix}.$$

Addition and multiplication of two subspaces are defined in analogy to the addition and multiplication of two linear mappings. In particular, we have for $\lambda \in \mathbb{C}$

$$TS^{-1} - \lambda = \{\{Sx, Tx - \lambda Sx\} : x \in X\}$$

and the notion of (essential) spectrum and resolvent set for linear relations are defined similarly as for linear operators, for details we refer to Section 2 below.

Therefore, the relationship of the essential spectra of two linear operator pencils $\mathcal{A}_1$ and $\mathcal{A}_2$ is the same as the relationship of the essential spectra of the linear relations $T_1S_1^{-1}$ and $T_2S_2^{-1}$. Now one can utilize known results for linear relations (see, e.g., [2]): If the difference of the two orthogonal projections onto the subspaces $T_1S_1^{-1}$ and $T_2S_2^{-1}$ is compact, then the essential spectra of the two pencils coincide. This difference can be expressed with the (pseudo-) inverse Z_j of the operator $S_j^*S_j + T_j^*T_j$, $j = 1, 2$, and it has the form

$$\begin{bmatrix} S_1Z_1S_1^* - S_2Z_2S_2^* & S_1Z_1T_1^* - S_2Z_2T_2^* \\ T_1Z_1S_1^* - T_2Z_2S_2^* & T_1Z_1T_1^* - T_2Z_2T_2^* \end{bmatrix}. \tag{1.1}$$

The first main result (cf. Section 5 below) shows that if (1.1) is compact then the essential spectra of $\mathcal{A}_1$ and $\mathcal{A}_2$ coincide.

The second main result of this paper (cf. Section 5 below) makes use of the so-called singular sequences (cf. Section 2 below). If S_1 and S_2 are Fredholm, then the pseudo-inverses $S_1^\dagger$ and $S_2^\dagger$ exist. If, in addition,

$$(T_2 - T_1)S_2^\dagger S_1, \quad (T_2 - T_1)S_1^\dagger S_2, \quad T_1 S_2^\dagger(S_1 - S_2) \quad \text{and} \quad T_2 S_1^\dagger(S_1 - S_2)$$

are compact, then the upper semi Fredholm essential spectra of $\mathcal{A}_1$ and $\mathcal{A}_2$ coincide. We prove similar results also for the lower semi Fredholm essential spectrum.

2. Preliminaries on linear relations

Let X, Y and Z be Banach spaces. The set of all bounded linear operators from X to Y is denoted by $\mathcal{L}(X,Y)$. As usual, we set $\mathcal{L}(X) := \mathcal{L}(X,X)$. A *linear relation* L from X into Y is a subspace of $X \times Y$ and the set of all linear relations from X into Y is denoted by $LR(X,Y)$. Moreover, $CR(X,Y)$ is the set of all closed linear relations from X into Y. Also here, we set $LR(X) := LR(X,X)$ and $CR(X) := CR(X,X)$. Each $T \in \mathcal{L}(X,Y)$ is identified with an element in $CR(X,Y)$ via its graph.

Given a linear relation $L \in LR(X,Y)$, we introduce the following sets:

$$\begin{aligned}
\operatorname{dom} L &= \{x \in X \,:\, \{x,y\} \in L \text{ for some } y \in Y\},\\
\ker L &= \{x \in X : \{x,0\} \in L\},\\
\operatorname{ran} L &= \{y \in Y \,:\, \{x,y\} \in L \text{ for some } x \in X\},\\
\operatorname{mul} L &= \{y \in Y : \{0,y\} \in L\},
\end{aligned}$$

which are called the *domain*, the *kernel*, the *range* and the *multivalued part* of L, respectively. The *inverse* of the linear relation L is given by

$$L^{-1} := \{\{y,x\} \in Y \times X : \{x,y\} \in L\}. \tag{2.1}$$

The linear relation αL with $\alpha \in \mathbb{C}$ is defined by

$$\alpha L := \{\{x,\alpha y\} \in X \times Y : \{x,y\} \in L\}. \tag{2.2}$$

The (operator-like) sum of two linear relations $L, M \in LR(X,Y)$ is defined as

$$L + M := \{\{x, y + y'\} \in X \times Y : \{x,y\} \in L, \{x,y'\} \in M\}. \tag{2.3}$$

If we assume that $X = Y$ then in view of (2.2) and (2.3) we have

$$L - \lambda = L - \lambda I = \{\{x, y - \lambda x\} : \{x,y\} \in L\}. \tag{2.4}$$

The product of two linear relations $L \in LR(Y,Z)$ and $M \in LR(X,Y)$ is defined by

$$LM := \{\{x,z\} \in X \times Z : \{x,y\} \in M,\ \{y,z\} \in L \text{ for some } y \in Y\}.$$

We recall some basic notions from Fredholm theory for linear relations, see [5].

Definition 2.1. Let $L \in LR(X,Y)$. The *nullity* and the *deficiency* of L are defined as follows

$$\operatorname{nul} L := \dim \ker L, \text{ and}$$
$$\operatorname{def} L := \operatorname{codim} \operatorname{ran} L := \dim Y/\operatorname{ran} L.$$

If either $\operatorname{nul} L < \infty$ or $\operatorname{def} L < \infty$, we define the *index* of a linear relation as follows

$$\operatorname{ind} L := \operatorname{nul} L - \operatorname{def} L,$$

where the value of the difference is taken to be $\operatorname{ind} L := \infty$ if $\operatorname{nul} L$ is infinite and $\operatorname{ind} L := -\infty$ if $\operatorname{def} L$ is infinite.

Furthermore we define the set of *upper* (*lower*) *semi Fredholm* relations, see e.g. [5],

$$\Phi_+(X,Y) := \{L \in CR(X,Y) : \operatorname{nul} L < \infty \text{ and } \operatorname{ran} L \text{ is closed in } Y\},$$
$$\Phi_-(X,Y) := \{L \in CR(X,Y) : \operatorname{def} L < \infty \text{ and } \operatorname{ran} L \text{ is closed in } Y\},$$

and the set of *Fredholm relations* as

$$\Phi(X,Y) := \Phi_+(X,Y) \cap \Phi_-(X,Y).$$

If $X = Y$, we write briefly $\Phi_+(X)$, $\Phi_-(X)$, and $\Phi(X)$, respectively. The following characterization of $\Phi_+(X,Y)$ is based on [5, Theorem V.1.11].

Proposition 2.2. *Let $L \in CR(X,Y)$ where X and Y are Hilbert spaces, then the following are equivalent:*

(i) $L \notin \Phi_+(X,Y)$.
(ii) *There exists a sequence $(\{x_n, y_n\}) \subset L$ such that $\|x_n\| = 1$ for all $n \in \mathbb{N}$, $x_n \rightharpoonup 0$ and $y_n \to 0$.*
(iii) *There exists a sequence $(\{x_n, y_n\}) \subset L$ such that $\|x_n\| = 1$ for all $n \in \mathbb{N}$, $x_n \rightharpoonup 0$ and $\operatorname{dist}(y_n, \operatorname{mul} L) \to 0$.*

Proof. For the proof of (i)⇒(ii), assume first that $\dim \ker L = \infty$ and choose an infinite orthonormal system (x_n) in $\ker L$. Then $\{x_n, 0\} \in L$ is a sequence as required in (ii). Second, assume that $\operatorname{ran} L$ is not closed. Then there exist a sequence $(z_n) \subset \operatorname{ran} L$ and some $z \in Y \backslash \operatorname{ran} L$ such that $z_n \to z$. Choose $u_n \in (\ker L)^\perp$ such that $\{u_n, z_n\} \in L$ for each $n \in \mathbb{N}$. If (u_n) is bounded, then (u_n) has a subsequence (u_{n_k}) such that $u_{n_k} \rightharpoonup u$ for some $u \in X$. Then the closedness of L and $\{u_{n_k}, z_{n_k}\} \rightharpoonup \{u, z\}$ imply that $\{u, z\} \in L$ and thus $z \in \operatorname{ran} L$, which is a contradiction. Hence, (u_n) is unbounded. It is no restriction to assume that $\|u_n\| \to \infty$ as $n \to \infty$. We set $x_n := u_n/\|u_n\| \in (\ker L)^\perp$ and $y_n := z_n/\|u_n\|$. Then $\{x_n, y_n\} \in L$, $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and $y_n \to 0$ as $n \to \infty$. Then a subsequence of (x_n) converges weakly, hence we may assume that $x_n \rightharpoonup x$ for some $x \in (\ker L)^\perp$. As $\{x_n, y_n\} \rightharpoonup \{x, 0\}$ and L is closed, it follows that $x = 0$.

The implication (ii)⇒(iii) is trivial. Thus, let us prove (iii)⇒(i). For this, let $(\{x_n, y_n\}) \subset L$ be a sequence as in (iii). Suppose that $\dim \ker L < \infty$ and that $\operatorname{ran} L$ is closed. Consider the linear relation

$$M := L \cap \left[(\ker L)^\perp \times (\operatorname{mul} L)^\perp\right].$$

Then M is obviously closed and (the graph of) an operator. Moreover, $\ker M = \{0\}$ and $\operatorname{ran} M = \operatorname{ran} L$ is closed. Hence M, considered as an operator from $\operatorname{dom} M$, equipped with the graph norm, is a bounded upper semi Fredholm operator. Let $x_n = u_n + v_n$ and $y_n = w_n + z_n$, where $u_n \in \ker L$, $v_n \in (\ker L)^\perp$, $w_n \in \operatorname{mul} L$, and $z_n \in (\operatorname{mul} L)^\perp$, $n \in \mathbb{N}$. Then $x_n \rightharpoonup 0$ and $\dim \ker L < \infty$ imply $u_n \to 0$ and $\|v_n\| \to 1$. Also, $\|z_n\| = \operatorname{dist}(y_n, \operatorname{mul} L) \to 0$. We have $\{v_n, z_n\} \in M$, that is, $v_n \in \operatorname{dom} M$ and $Mv_n = z_n \to 0$, which is a contradiction to the fact that M is an upper semi Fredholm operator (cf. [4, XI Theorem 2.5]). □

In what follows, we introduce the adjoint of a linear relation. For this we assume in addition that the spaces X and Y are Hilbert spaces equipped with inner products $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_Y$, respectively. If no confusion arises, we use for simplicity just the notion $(\cdot,\cdot)$. The adjoint L^* of $L \in LR(X,Y)$ is a linear relation from Y to X, defined by

$$L^* = \{\{y, x\} \in Y \times X : (y, v)_Y = (x, u)_X \text{ for all } \{u, v\} \in L\}.$$

Note that always $L^* \in CR(Y, X)$. The following identities for $L \in LR(X,Y)$ are straightforward (see also [16, Section 14.1], [3, Proposition 2.4], and [12])

$$(L^*)^{-1} = (L^{-1})^*,$$

$$(\lambda L)^* = \overline{\lambda} L^*, \quad \overline{\lambda} \neq 0,$$

$$\ker L^* = (\operatorname{ran} L)^\perp, \tag{2.5}$$

$$(\operatorname{ran} L^*)^\perp = \ker \overline{L}, \tag{2.6}$$

$$L^* = -(L^\perp)^{-1}. \tag{2.7}$$

The range of L is closed if and only if the range of L^* is closed, see, e.g. [3, Proposition 2.5]. This together with (2.5) and (2.6) implies that for all $L \in CR(X,Y)$

$$L \in \Phi_\pm(X,Y) \quad \text{if and only if} \quad L^* \in \Phi_\mp(Y,X). \tag{2.8}$$

Next, we define the spectrum of a linear relation and introduce different types of essential spectra as in [17], see also [6] for the operator case.

Definition 2.3. Let $L \in LR(X)$. The *spectrum* and the *resolvent set* of L are defined by

$$\sigma(L) := \{\lambda \in \mathbb{C} : (L-\lambda)^{-1} \in \mathcal{L}(X)\} \qquad \text{and} \qquad \rho(L) := \mathbb{C} \setminus \sigma(L),$$

respectively. The essential spectra of L are defined as

$$\sigma_{e1}(L) := \{\lambda \in \mathbb{C} \ : \ L - \lambda \notin \Phi_+(X) \cup \Phi_-(X)\},$$

$$\sigma_{e2}^{\pm}(L) := \{\lambda \in \mathbb{C} \ : \ L - \lambda \notin \Phi_\pm(X)\},$$

$$\sigma_{e3}(L) := \{\lambda \in \mathbb{C} \ : \ L - \lambda \notin \Phi(X)\}.$$

Note that $L-\lambda\in\Phi_\pm(X)$ requires $L-\lambda$ (and thus L) to be closed. Hence, if L is not closed, we have $\sigma(L)=\sigma_{e1}(L)=\sigma_{e2}^{\pm}(L)=\sigma_{e3}(L)=\mathbb{C}$. Also, we obviously have

$$\sigma_{e1}(L)=\sigma_{e2}^{+}(L)\cap\sigma_{e2}^{-}(L) \qquad \text{and} \qquad \sigma_{e3}(L)=\sigma_{e2}^{+}(L)\cup\sigma_{e2}^{-}(L).$$

In particular,

$$\sigma_{e1}(L)\subset\sigma_{e2}^{\pm}(L)\subset\sigma_{e3}(L).$$

3. Essential spectra of the operator pencil $\lambda S-T$ and the linear relation TS^{-1}

Throughout this section let X and Y be Banach spaces. Given $S,T\in\mathcal{L}(X,Y)$, we will establish a relationship between the (essential) spectra of the operator pencil $\mathcal{A}(\lambda)=\lambda S-T$ and the associated linear relation

$$TS^{-1}\in LR(Y).$$

Note that S^{-1} is the inverse of the graph of S viewed as a linear relation. Then it follows from (2.1) and (2.4) that

$$\begin{aligned} TS^{-1} &= \big\{\{y,z\}:\{y,x\}\in S^{-1},\ \{x,z\}\in T \text{ for some } x\in X\big\} \\ &= \{\{Sx,Tx\}:x\in X\} \qquad (3.1)\\ &= \operatorname{ran}\begin{bmatrix} S\\ T\end{bmatrix}. \qquad (3.2)\end{aligned}$$

From this it is immediate that

$$\begin{aligned} \operatorname{dom}(TS^{-1}) &= \operatorname{ran}S, \quad \ker(TS^{-1})=S\ker T,\\ \operatorname{ran}(TS^{-1}) &= \operatorname{ran}T, \quad \operatorname{mul}(TS^{-1})=T\ker S.\end{aligned}$$

The spectrum and the essential spectra for a linear operator pencil are defined similarly as for linear relations.

Definition 3.1. For an operator pencil $\mathcal{A}(\lambda)=\lambda S-T$ with $S,T\in\mathcal{L}(X,Y)$ the *spectrum* $\sigma(\mathcal{A})$ and the *resolvent set* $\rho(\mathcal{A})$ are defined as

$$\begin{aligned} \sigma(\mathcal{A}) &:= \{\lambda\in\mathbb{C} \ :\ \lambda S-T \text{ is not boundedly invertible}\},\\ \rho(\mathcal{A}) &:= \mathbb{C}\setminus\sigma(\mathcal{A}).\end{aligned}$$

The essential spectra of $\mathcal{A}$ are given by

$$\begin{aligned} \sigma_{e1}(\mathcal{A}) &:= \{\lambda\in\mathbb{C} \ :\ \lambda S-T\notin\Phi_+(X,Y)\cup\Phi_-(X,Y)\},\\ \sigma_{e2}^{\pm}(\mathcal{A}) &:= \{\lambda\in\mathbb{C} \ :\ \lambda S-T\notin\Phi_\pm(X,Y)\},\\ \sigma_{e3}(\mathcal{A}) &:= \{\lambda\in\mathbb{C} \ :\ \lambda S-T\notin\Phi(X,Y)\}.\end{aligned}$$

The next proposition shows how the spectra of $\mathcal{A}$ and TS^{-1} are related to each other.

Proposition 3.2. *Let $\mathcal{A}(\lambda)=\lambda S-T$ be an operator pencil with $S,T\in\mathcal{L}(X,Y)$ and $\lambda\in\mathbb{C}$ then the following holds.*

(a) $\ker(TS^{-1}-\lambda)=S\ker\mathcal{A}(\lambda)$.

(b) $\operatorname{ran}(TS^{-1} - \lambda) = \operatorname{ran}\mathcal{A}(\lambda)$.
(c) *We have*

$$\dim\ker(TS^{-1} - \lambda) = \dim\frac{\ker\mathcal{A}(\lambda)}{\ker S \cap \ker T}.$$

(d) *If* $\sigma_{e2}^{+}(\mathcal{A}) \neq \mathbb{C}$, *then* TS^{-1} *is closed, i.e.,* $TS^{-1} \in CR(Y)$. *This is in particular the case if* $\rho(\mathcal{A}) \neq \emptyset$.
(e) *We have* $\sigma(TS^{-1}) \subset \sigma(\mathcal{A})$.
(f) *If* $\ker S \cap \ker T = \{0\}$, *then*

$$\sigma(TS^{-1}) = \sigma(\mathcal{A}).$$

Proof. From (2.3) and (3.1) it is easy to see

$$TS^{-1} - \lambda = \{\{Sx, Tx - \lambda Sx\} : x \in X\}$$

which implies (a) and (b). Observe that the map $[x] \mapsto Sx$ from $\frac{\ker(\lambda S - T)}{\ker S \cap \ker T}$ to $S\ker(\lambda S - T)$ is bijective which proves (c).

In order to prove (d) set $N_0 := \ker S \cap \ker T$ and let $\lambda \in \mathbb{C}$ such that $\mathcal{A}(\lambda) \in \Phi_+(X,Y)$. Then $\ker\mathcal{A}(\lambda)$ is finite dimensional and, hence, closed. It has a complementary subspace and we have

$$\ker\mathcal{A}(\lambda) = N_0 \dotplus N_1 \qquad \text{and} \qquad X = \ker\mathcal{A}(\lambda) \dotplus M$$

with closed subspaces $N_1 \subset \ker\mathcal{A}(\lambda)$ and $M \subset X$. Let $\{y_n, z_n\}$ be a sequence in TS^{-1} which converges to $\{y, z\} \in Y \times Y$. Then, by (3.1), we find a sequence (x_n) in X with

$$y_n = Sx_n \quad \text{and} \quad z_n = Tx_n.$$

We have to prove that there exists some $x \in X$ such that $Sx_n \to Sx$ and $Tx_n \to Tx$. To this end, we write $x_n = u_n + v_n + w_n$ with $u_n \in N_0$, $v_n \in N_1$ and $w_n \in M$. Since $\mathcal{A}(\lambda)$ maps M bijectively onto its (closed) range and $A(\lambda)w_n = A(\lambda)x_n = \lambda Sx_n - Tx_n \to \lambda y - z$, it follows that (w_n) converges to some $w \in M$. Hence, (Sw_n) and (Tw_n) converge and therefore (Sv_n) converges. Since $\ker(S|_{N_1}) = \{0\}$, (v_n) converges to some $v \in N_1$ and we obtain $Sx_n = S(v_n + w_n) \to S(v + w)$ and $Tx_n = T(v_n + w_n) \to T(v + w)$.

For the proof of (e) let $\lambda \in \rho(\mathcal{A})$. Then TS^{-1} is closed by (d) and $\ker(TS^{-1} - \lambda) = \{0\}$, $\operatorname{ran}(TS^{-1} - \lambda) = Y$ by (a) and (b). Hence,

$$\operatorname{mul}(TS^{-1} - \lambda)^{-1} = \ker(TS^{-1} - \lambda) = \{0\}$$

and $(TS^{-1} - \lambda)^{-1}$ is a closed operator in Y with domain Y. By the closed graph theorem, it is an element of $\mathcal{L}(Y)$. This proves (e). For (f), assume that $\lambda \in \rho(TS^{-1})$ and, in addition, that $\ker S \cap \ker T = \{0\}$. Then $\operatorname{ran}\mathcal{A}(\lambda) = Y$ by (b) and $\ker\mathcal{A}(\lambda) = \ker S \cap \ker T = \{0\}$ by (c). □

Remark 3.3. Note that the condition $\ker S \cap \ker T = \{0\}$ in (f) is necessary for $\rho(\mathcal{A})$ to be non-empty. In fact, if $x \in \ker S \cap \ker T$, $x \neq 0$, then $x \in \ker\mathcal{A}(\lambda)$ for all $\lambda \in \mathbb{C}$ and thus $\rho(\mathcal{A}) = \emptyset$.

The following proposition shows that also the essential spectra of the pencil $\lambda S - T$ and the linear relation TS^{-1} are intimately connected to each other.

Proposition 3.4. *Let $\mathcal{A}(\lambda) = \lambda S - T$ be an operator pencil with $S, T \in \mathcal{L}(X, Y)$ and $\lambda \in \mathbb{C}$. Then we have*

$$\sigma_{e2}^{+}(TS^{-1}) \subset \sigma_{e2}^{+}(\mathcal{A}) \qquad \text{and} \qquad \sigma_{e2}^{-}(TS^{-1}) \supset \sigma_{e2}^{-}(\mathcal{A}). \tag{3.3}$$

If TS^{-1} is closed, then

$$\sigma_{e2}^{-}(TS^{-1}) = \sigma_{e2}^{-}(\mathcal{A}). \tag{3.4}$$

If $\dim(\ker S \cap \ker T) < \infty$, *then*

$$\sigma_{e2}^{+}(TS^{-1}) = \sigma_{e2}^{+}(\mathcal{A}). \tag{3.5}$$

Hence, if TS^{-1} is closed and $\dim(\ker S \cap \ker T) < \infty$, *then*

$$\sigma_{e1}(TS^{-1}) = \sigma_{e1}(\mathcal{A}) \qquad \text{and} \qquad \sigma_{e3}(TS^{-1}) = \sigma_{e3}(\mathcal{A}).$$

Proof. From Proposition 3.2 (b) it follows that $\operatorname{ran}(TS^{-1} - \lambda)$ is closed if and only if $\operatorname{ran}\mathcal{A}(\lambda)$ is closed and $\operatorname{def}(TS^{-1} - \lambda) = \operatorname{def}\mathcal{A}(\lambda)$. This proves the second relation in (3.3). If $\mathcal{A}(\lambda) \in \Phi_+(X, Y)$ for some $\lambda \in \mathbb{C}$, then TS^{-1} is closed by Proposition 3.2 (d) and from Proposition 3.2 (a) we conclude $\operatorname{nul}(TS^{-1} - \lambda) \leq \operatorname{nul}(\mathcal{A}(\lambda))$. Hence, $TS^{-1} - \lambda \in \Phi_+(Y)$ and (3.3) is proved.

If TS^{-1} is closed, then obviously $\mathcal{A}(\lambda) \in \Phi_-(X, Y)$ implies $TS^{-1} - \lambda \in \Phi_-(Y)$, which shows (3.4). If $\dim(\ker S \cap \ker T) < \infty$, then $TS^{-1} - \lambda \in \Phi_+(Y)$ implies $\dim\ker\mathcal{A}(\lambda) < \infty$ (see Proposition 3.2 (c)) and therefore $\mathcal{A}(\lambda) \in \Phi_+(X, Y)$. □

The following corollary follows from Proposition 3.2 (d) and the fact that $\mathcal{A}(\lambda) \in \Phi_+(X, Y)$ implies $\dim(\ker S \cap \ker T) < \infty$.

Corollary 3.5. *If $\sigma_{e2}^{+}(\mathcal{A}) \neq \mathbb{C}$ (in particular, if $\rho(\mathcal{A}) \neq \emptyset$), then*

$$\sigma_{e2}^{+}(TS^{-1}) = \sigma_{e2}^{+}(\mathcal{A}) \qquad \text{and} \qquad \sigma_{e2}^{-}(TS^{-1}) = \sigma_{e2}^{-}(\mathcal{A}),$$

and therefore also

$$\sigma_{e1}(TS^{-1}) = \sigma_{e1}(\mathcal{A}) \qquad \text{and} \qquad \sigma_{e3}(TS^{-1}) = \sigma_{e3}(\mathcal{A}).$$

4. Essential spectrum of linear relations under perturbations

In this section we let X and Y be Hilbert spaces. We say that $L, M \in CR(X, Y)$ are *compact perturbations of each other* if $P_L - P_M$ is compact. Here, P_L denotes the orthogonal projection onto the closed subspace L. If $\rho(L) \cap \rho(M) \neq \emptyset$, this is equivalent to $(L - \mu)^{-1} - (M - \mu)^{-1}$ being compact for some (and hence for all) $\mu \in \rho(L) \cap \rho(M)$ (see [2]).

Lemma 4.1. *Two linear relations $L, M \in CR(X, Y)$ in the Hilbert spaces X, Y are compact perturbations of each other if and only if L^* and M^* are compact perturbations of each other.*

Proof. Relation (2.7) and the unitary mapping $U : X \times Y \to Y \times X$ which is given by

$$U(x, y) := (y, -x)$$

yield $L^* = UL^\perp$. Therefore

$$P_{L^*} - P_{M^*} = P_{UL^\perp} - P_{UM^\perp} = U(P_{L^\perp} - P_{M^\perp})U^* = U(P_L - P_M)U^*.$$

Hence, $P_{L^*} - P_{M^*}$ is compact if and only if $P_L - P_M$ is compact. □

Proposition 4.2. *Let X, Y be Hilbert spaces and let $L, M \in CR(X,Y)$ be compact perturbations of each other. Then $L \in \Phi_\pm(X,Y)$ if and only if $M \in \Phi_\pm(X,Y)$. In particular,*

$$\sigma_{e2}^+(L) = \sigma_{e2}^+(M) \qquad \textit{and} \qquad \sigma_{e2}^-(L) = \sigma_{e2}^-(M),$$

and therefore also

$$\sigma_{e1}(L) = \sigma_{e1}(M) \qquad \textit{and} \qquad \sigma_{e3}(L) = \sigma_{e3}(M).$$

Proof. Let $L \notin \Phi_+(X,Y)$. Due to Proposition 2.2 there exists a sequence $(\{x_n, y_n\}) \subset L$ with $\|x_n\| = 1$ for all $n \in \mathbb{N}$, $x_n \rightharpoonup 0$, and $y_n \to 0$. Set $\{x_n', y_n'\} := P_M\{x_n, y_n\} \in M$, $n \in \mathbb{N}$. Since $\{x_n, y_n\} \rightharpoonup 0$, we conclude from

$$\{x_n', y_n'\} := (P_M - P_L)\{x_n, y_n\} + \{x_n, y_n\}$$

and the compactness of $P_M - P_L$ that $\|x_n'\| \to 1$, $y_n' \to 0$ as $n \to \infty$, and $x_n' \rightharpoonup 0$. Setting $x_n'' := x_n'/\|x_n'\|$ and $y_n'' := y_n'/\|x_n'\|$, we obtain $\{x_n'', y_n''\} \in L$ with $\|x_n''\| = 1$ for all $n \in \mathbb{N}$, $x_n'' \rightharpoonup 0$, and $y_n'' \to 0$. Hence, Proposition 2.2 implies that $M \notin \Phi_+(X,Y)$. This shows that $L \in \Phi_+(X,Y)$ if and only if $M \in \Phi_+(X,Y)$. Using this, Lemma 4.1, and (2.8), we obtain the same statement with $\Phi_+(X,Y)$ replaced by $\Phi_-(X,Y)$.

The remaining statements on the essential spectra follow from Proposition 4.3 in [2] which implies that L and M are compact perturbations of each other if and only if $L - \lambda$ and $M - \lambda$ are compact perturbations of each other. □

5. Essential spectrum of operator pencils under perturbations

In this section we give sufficient conditions for the equality of the essential spectra of two operator pencils $\mathcal{A}_1$ and $\mathcal{A}_2$

$$\mathcal{A}_1(\lambda) = \lambda S_1 - T_1 \qquad \text{and} \qquad \mathcal{A}_2(\lambda) = \lambda S_2 - T_2$$

in terms of their coefficients $S_1, S_2, T_1, T_2 \in \mathcal{L}(X,Y)$. In the proofs of our main theorems we use the above-established concept of the relationship between operator pencils and linear relations.

The first statement is obvious and follows from the well-known fact that $\mathcal{L}(X,Y) \cap \Phi_\pm(X,Y)$ is invariant under compact perturbations.

Proposition 5.1. *Assume that $T_2 - T_1$ and $S_2 - S_1$ are compact. Then*

$$\sigma_{e1}(\mathcal{A}_1) = \sigma_{e1}(\mathcal{A}_2), \quad \sigma_{e2}^\pm(\mathcal{A}_1) = \sigma_{e2}^\pm(\mathcal{A}_2), \quad \textit{and} \quad \sigma_{e3}(\mathcal{A}_1) = \sigma_{e3}(\mathcal{A}_2).$$

Let $A \in \mathcal{L}(X,Y)$. It follows from $\ker A = \ker A^*A$ and the closed range theorem that A has closed range if and only if the same is true for A^*A. In this case, $X = \ker A \oplus \operatorname{ran} A^*$, $Y = \ker A^* \oplus \operatorname{ran} A$ and the restriction $A_0 = A|_{\operatorname{ran} A^*} : \operatorname{ran} A^* \to \operatorname{ran} A$ is boundedly invertible. Recall that the *pseudo-inverse* $A^\dagger$ of A is then defined by

$$A^\dagger := A_0^{-1} P_{\operatorname{ran} A}.$$

For an overview of equivalent definitions of the pseudo-inverse of linear operators we refer to [7, Chapter II]. It is immediate that

$$P_{\operatorname{ran} A} = AA^\dagger \tag{5.1}$$

and one can show, see e.g. [11, Theorem 4], that

$$(A^\dagger)^* = (A^*)^\dagger. \tag{5.2}$$

Moreover we have from [7, Theorem 2.1.5] that

$$A^\dagger = (A^*A)^\dagger A^* = A^*(AA^*)^\dagger. \tag{5.3}$$

Our first main theorem is the following.

Theorem 5.2. *Let X, Y be Hilbert spaces and $S_1, S_2, T_1, T_2 \in \mathcal{L}(X,Y)$ with corresponding pencils*

$$\mathcal{A}_1(\lambda) = \lambda S_1 - T_1 \quad \textit{and} \quad \mathcal{A}_2(\lambda) = \lambda S_2 - T_2.$$

Assume that for both $j = 1,2$ the operator $S_j^*S_j + T_j^*T_j \in \mathcal{L}(X)$ has closed range and that the operator

$$\begin{bmatrix} S_1Z_1S_1^* - S_2Z_2S_2^* & S_1Z_1T_1^* - S_2Z_2T_2^* \\ T_1Z_1S_1^* - T_2Z_2S_2^* & T_1Z_1T_1^* - T_2Z_2T_2^* \end{bmatrix} \in \mathcal{L}(Y \times Y) \tag{5.4}$$

is compact, where

$$Z_j := (S_j^*S_j + T_j^*T_j)^\dagger, \quad j = 1,2.$$

Then

$$\sigma_{e2}^-(\mathcal{A}_1) = \sigma_{e2}^-(\mathcal{A}_2).$$

If, in addition, $S_j^*S_j + T_j^*T_j \in \Phi_+(X)$ for $j = 1,2$, then

$$\sigma_{e2}^+(\mathcal{A}_1) = \sigma_{e2}^+(\mathcal{A}_2).$$

Proof. Let $j = 1,2$ and set $A_j := \left[\begin{smallmatrix} S_j \\ T_j \end{smallmatrix}\right]$. Then $A_j^*A_j = S_j^*S_j + T_j^*T_j$ implies that A_j has closed range which means that the relation $T_jS_j^{-1}$ is closed. As discussed before, we find with (5.3) that

$$A_jA_j^\dagger = A_j(A_j^*A_j)^\dagger A_j^* = A_jZ_jA_j^* = \begin{bmatrix} S_j \\ T_j \end{bmatrix} Z_j \begin{bmatrix} S_j^* & T_j^* \end{bmatrix} = \begin{bmatrix} S_jZ_jS_j^* & S_jZ_jT_j^* \\ T_jZ_jS_j^* & T_jZ_jT_j^* \end{bmatrix}$$

is the orthogonal projection onto $\operatorname{ran} A_j = T_jS_j^{-1}$. Hence, the operator in (5.4) is the difference of the orthogonal projections onto the closed subspaces $T_1S_1^{-1}$ and $T_2S_2^{-1}$ of $Y \times Y$. Also note that $\ker S_j \cap \ker T_j = \ker A_j = \ker A_j^*A_j$. Now, the statements of Theorem 5.2 follow from Proposition 4.2 and Proposition 3.4. □

Example. (a) Let us consider the example from the introduction, where $X = Y$ and $\mathcal{A}_1(\lambda) = \lambda I - T$ and $\mathcal{A}_2(\lambda) = (\lambda I - T)M$ with $T, M \in \mathcal{L}(X)$ and M boundedly invertible. Clearly, all the essential spectra of $\mathcal{A}_1$ and $\mathcal{A}_2$ coincide, respectively. We have $S_1 = I$, $T_1 = T$, $S_2 = M$ and $T_2 = TM$. Then both $S_1^* S_1 + T_1^* T_1 = I + T^*T$ and $S_2^* S_2 + T_2^* T_2 = M^*(I + T^*T)M$ are boundedly invertible and the operator matrix in (5.4) is the zero matrix. Indeed, we have

$$T_2 S_2^{-1} = \operatorname{ran} \begin{bmatrix} M \\ TM \end{bmatrix} = \operatorname{ran} \begin{bmatrix} I \\ T \end{bmatrix} = T_1 S_1^{-1}.$$

(b) Let X, Y be Hilbert spaces and let $M_1, M_2 \in \mathcal{L}(X, Y)$ be boundedly invertible. Let $K_S, K_T \in \mathcal{L}(Y)$ be compact such that $-1 \notin \sigma(K_S) \cap \sigma(K_T)$. Then the operator $R := (I + K_S)^*(I + K_S) + (I + K_T)^*(I + K_T)$ is boundedly invertible. Indeed, R is a compact perturbation of $2I$ and therefore Fredholm with index zero and the condition $-1 \notin \sigma(K_S) \cap \sigma(K_T)$ guarantees that $\ker R = \{0\}$. Consider

$$S_1 = T_1 = M_1, \quad \text{and} \quad S_2 = (I + K_S)M_2, \quad T_2 = (I + K_T)M_2.$$

Using the invertibility of M_1, M_2, we note

$$T_1 S_1^{-1} = \operatorname{ran} \begin{bmatrix} S_1 \\ T_1 \end{bmatrix} = \operatorname{ran} \begin{bmatrix} M_1 \\ M_1 \end{bmatrix} = \operatorname{ran} \begin{bmatrix} I \\ I \end{bmatrix}$$

and

$$T_2 S_2^{-1} = \operatorname{ran} \begin{bmatrix} S_2 \\ T_2 \end{bmatrix} = \operatorname{ran} \begin{bmatrix} (I + K_S)M_2 \\ (I + K_T)M_2 \end{bmatrix} = \operatorname{ran} \begin{bmatrix} I + K_S \\ I + K_T \end{bmatrix}.$$

Set $Z_2 := ((I + K_S)^*(I + K_S) + (I + K_T)^*(I + K_T))^{-1}$. In this case, the operator in (5.4) reads as

$$\begin{bmatrix} \frac{1}{2}I - (I + K_S)Z_2(I + K_S)^* & \frac{1}{2}I - (I + K_S)Z_2(I + K_T)^* \\ \frac{1}{2}I - (I + K_T)Z_2(I + K_S)^* & \frac{1}{2}I - (I + K_T)Z_2(I + K_T)^* \end{bmatrix}.$$

Obviously, this operator is compact as

$$\frac{1}{2}I - Z_2$$

is compact. Hence, the conditions in Theorem 5.2 are satisfied and all essential spectra of the two pencils

$$\mathcal{A}_1(\lambda) = \lambda S_1 - T_1 \qquad \text{and} \qquad \mathcal{A}_2(\lambda) = \lambda S_2 - T_2$$

coincide.

Lemma 5.3. *Let X, Y be Hilbert spaces, $S, T \in \mathcal{L}(X, Y)$, $S \in \Phi_+(X, Y)$, and $\lambda \in \mathbb{C}$. Assume furthermore that TS^{-1} is closed. Then we have $\lambda \in \sigma_{e2}^+(TS^{-1})$ if and only if there exists a sequence $(y_n) \subset (\ker S)^\perp$ such that $\|Sy_n\| \to 1$, $y_n \rightharpoonup 0$, and $(\lambda S - T)y_n \to 0$ as $n \to \infty$.*

Proof. Assume that $TS^{-1} - \lambda \notin \Phi_+(X,Y)$. By Proposition 2.2 there exists a sequence $\{x_n, z_n\} \in TS^{-1} - \lambda$ with $\|x_n\| = 1$ for all $n \in \mathbb{N}$, $x_n \rightharpoonup 0$, and $z_n \to 0$ as $n \to \infty$. As $TS^{-1} - \lambda = \{\{Sx, Tx - \lambda Sx\} : x \in X\}$ (see (2.3) and (3.1)), there exists a sequence $(v_n) \subset X$ such that $\|Sv_n\| = 1$ for all $n \in \mathbb{N}$, $Sv_n \rightharpoonup 0$, and $Tv_n - \lambda Sv_n \to 0$ as $n \to \infty$. For $n \in \mathbb{N}$ let $v_n = u_n + y_n$ with $u_n \in \ker S$ and $y_n \in (\ker S)^\perp$. Then $\|Sy_n\| = 1$ and $Sy_n \rightharpoonup 0$. Since S maps $(\ker S)^\perp$ bijectively onto the closed subspace $\operatorname{ran} S$, it follows that $y_n \rightharpoonup 0$. Hence, $Ty_n - \lambda Sy_n \rightharpoonup 0$ so that $Tv_n - \lambda Sv_n \to 0$ implies that $Tu_n \rightharpoonup 0$. But (Tu_n) is contained in the finite-dimensional subspace $T \ker S$ and thus $Tu_n \to 0$ as $n \to \infty$, which implies $(\lambda S - T)y_n \to 0$.

Conversely, let $(y_n) \subset (\ker S)^\perp$ be a sequence as in the lemma. Set $y_n' := \|Sy_n\|^{-1} y_n$ and $x_n := Sy_n'$ as well as $z_n := \lambda S y_n' - T y_n'$. Then $\{x_n, z_n\} \in TS^{-1} - \lambda$, $\|x_n\| = 1$ for all $n \in \mathbb{N}$, $x_n \rightharpoonup 0$, and $z_n \to 0$ as $n \to \infty$. Hence, $TS^{-1} - \lambda \notin \Phi_+(X,Y)$ by Proposition 2.2. □

The following proposition is the second main result of this paper.

Proposition 5.4. *Let X, Y be Hilbert spaces and $S_1, S_2, T_1, T_2 \in \mathcal{L}(X,Y)$. Assume that the following assumptions are satisfied.*

1. $S_1 \in \Phi_+(X,Y)$.
2. $S_2 \in \Phi(X,Y)$.
3. $(T_2 - T_1)S_2^\dagger S_1$ *is a compact operator.*
4. $T_1 S_2^\dagger (S_1 - S_2)$ *is a compact operator.*

Then $T_1 S_1^{-1}$ and $T_2 S_2^{-1}$ both are closed and

$$\sigma_{e2}^+(T_1 S_1^{-1}) \subset \sigma_{e2}^+(T_2 S_2^{-1}).$$

Proof. Let $j \in \{1,2\}$. For $\lambda \in \mathbb{C}$ we have $\mathcal{A}_j(\lambda) = \lambda S_j - T_j = \lambda(S_j - \frac{T_j}{\lambda})$. Since $S_j \in \Phi_+(X,Y)$, for $|\lambda|$ sufficiently large we have that $\mathcal{A}_j(\lambda) \in \Phi_+(X,Y)$ (see [10, Theorem IV-5.31]). Therefore, $T_j S_j^{-1}$ is closed by Proposition 3.2 (d).

Assume that $\lambda \in \sigma_{e2}^+(T_1 S_1^{-1})$. Then by Lemma 5.3 there exists $y_n \in (\ker S_1)^\perp$ such that $\|S_1 y_n\| \to 1$, $y_n \rightharpoonup 0$, and $(\lambda S_1 - T_1) y_n \to 0$ as $n \to \infty$. We set $y_n' := S_2^\dagger S_1 y_n \in \operatorname{ran} S_2^* = (\ker S_2)^\perp$, $n \in \mathbb{N}$. Obviously, $y_n' \rightharpoonup 0$. Since $\dim \ker S_2^* < \infty$ and $y_n \rightharpoonup 0$, it follows from (5.1)

$$\|S_2 y_n'\| = \|P_{\operatorname{ran} S_2} S_1 y_n\| = \|S_1 y_n - P_{\ker S_2^*} S_1 y_n\| \to 1$$

as $n \to \infty$. Also, setting $K := T_2 - T_1$,

$$\begin{aligned} T_2 y_n' - \lambda S_2 y_n' &= T_2 S_2^\dagger S_1 y_n - \lambda(S_1 y_n - P_{\ker S_2^*} S_1 y_n) \\ &= K S_2^\dagger S_1 y_n + T_1 S_2^\dagger S_1 y_n - \lambda S_1 y_n + \lambda P_{\ker S_2^*} S_1 y_n \\ &= K S_2^\dagger S_1 y_n + T_1 (S_2^\dagger S_1 - I) y_n + \lambda P_{\ker S_2^*} S_1 y_n - (\lambda S_1 - T_1) y_n. \end{aligned}$$

Now, the claim follows from Lemma 5.3, the compactness of K and $P_{\ker S_2^*}$ and the fact that $S_2^\dagger S_1 - I = S_2^\dagger (S_1 - S_2) - P_{\ker S_2} S_2^*$. □

Theorem 5.5. *Let X, Y be Hilbert spaces and $S_1, S_2, T_1, T_2 \in \mathcal{L}(X,Y)$ and let $S_1, S_2 \in \Phi(X,Y)$.*

(i) *If* $(T_2-T_1)S_2^\dagger S_1$, $(T_2-T_1)S_1^\dagger S_2$, $T_1 S_2^\dagger(S_1-S_2)$, *and* $T_2 S_1^\dagger(S_1-S_2)$ *are compact, then*

$$\sigma_{e2}^+(\mathcal{A}_1)=\sigma_{e2}^+(\mathcal{A}_2). \tag{5.5}$$

(ii) *If* $S_1 S_2^\dagger(T_2-T_1)$, $S_2 S_1^\dagger(T_2-T_1)$, $(S_1-S_2)S_2^\dagger T_1$ *and* $(S_1-S_2)S_1^\dagger T_2$ *are compact, then*

$$\sigma_{e2}^-(\mathcal{A}_1)=\sigma_{e2}^-(\mathcal{A}_2). \tag{5.6}$$

Proof. From Proposition 5.4 we obtain $\sigma_{e2}^+(T_1S_1^{-1})=\sigma_{e2}^+(T_2S_2^{-1})$ and (5.5) is a consequence of Proposition 3.4.

By assumption (cf. (2.8)) we have $S_1^*,S_2^*\in\Phi(Y,X)$ and $T_2^*-T_1^*$ is compact. The assumptions in (ii) and (5.2) imply the compactness of $T_1^*(S_2^*)^\dagger(S_1^*-S_2^*)$ and of $T_2^*(S_1^*)^\dagger(S_1^*-S_2^*)$. Proposition 5.4 yields

$$\sigma_{e2}^+(T_1^*(S_1^*)^{-1})=\sigma_{e2}^+(T_2^*(S_2^*)^{-1}).$$

Hence we have together with Corollary 3.5 that

$$\sigma_{e2}^+(\mathcal{A}_1^*)=\sigma_{e2}^+(T_1^*(S_1^*)^{-1})=\sigma_{e2}^+(T_2^*(S_2^*)^{-1})=\sigma_{e2}^+(\mathcal{A}_2^*)$$

with $\mathcal{A}_i^*(\lambda):=\lambda S_i^*-T_i^*$ for $i=1,2$. Therefore, $\overline{\lambda}\in\sigma_{e2}^+(\mathcal{A}_1^*)$ if and only if $\overline{\lambda}\in\sigma_{e2}^+(\mathcal{A}_2^*)$. Now, (5.6) follows from (2.8) applied to the operators $\mathcal{A}_1^*(\overline{\lambda})$ and $\mathcal{A}_2^*(\overline{\lambda})$. □

Remark 5.6. Let $S\in\mathcal{L}(X,Y)$ and let T be a densely defined closed linear operator in X. Set $\mathcal{A}(\lambda):=\lambda S-T$. Assume that $\mu\in\rho(\mathcal{A})$. Then we have by definition

$$(TS^{-1}-\mu)^{-1}=\{\{Tx-\mu Sx,Sx\}:x\in\operatorname{dom}T\}=\{\{y,S(T-\mu S)^{-1}y\}:y\in X\}.$$

Using compactness of the perturbation of the corresponding linear relations we obtain the following result: For $i=1,2$ let $\mathcal{A}_i(\lambda)=\lambda S_i-T_i$ with $S_i\in\mathcal{L}(X,Y)$ bounded and T_i closed and densely defined from X to Y and let $\mu\in\rho(\mathcal{A}_1)\cap\rho(\mathcal{A}_2)$ with

$$S_1(T_1-\mu S_1)^{-1}-S_2(T_2-\mu S_2)^{-1}\quad\text{compact}$$

then $\sigma_{e2}^\pm(\mathcal{A}_1)=\sigma_{e2}^\pm(\mathcal{A}_2)$ (cf. Proposition 3.4 and Proposition 4.2). Note that the compactness of the resolvent difference does not depend on the choice of μ. Furthermore, we have no inclusion assumption on the multivalued parts as in [17].

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