Some Results on Resonances for Hyperbolic Surfaces

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Abstract

We prove several results concerning the distribution of resonances for infinite-area hyperbolic surfaces:

- We establish global upper and lower bounds for resonances for covers of Schottky surfaces, which lead to a weak Weyl law for the resonance set in families of covers. This result refines previously known results due to Guillopé–Zworski [34, 35] and complements a result of Borthwick [12]. When applied to congruence covers, we obtain resonance bounds in the level aspect, improving a result of Jakobson–Naud [38]. In the process, we prove a growth estimate on L-functions (twisted Selberg zeta functions) of Schottky groups.

- We give an improved fractal upper bound for resonances for covers of Schottky surfaces in terms of geometric quantities of the cover. This improves and refines the results of Jakobson–Naud [38], Guillopé–Lin–Zworski [33], and Dyatlov [23]. This result can also be interpreted algebraically, in terms of Cayley graphs of the Galois group of the cover. Moreover, it leads to an estimate for the number of eigenvalues for covers of Schottky surfaces, refining a more general result of Ballmann–Matthiesen–Mondal [5].

- We show that the spectral gap of hyperbolic surfaces can be arbitrarily small, by passing to abelian covers. This solves a question raised by Naud and subsumes earlier work on this topic by Randol [75] and Selberg [82]. In the case of Schottky surfaces we give a more precise equidistribution result for resonances of abelian covers close to the 'first' resonance $s = \delta$.

- We establish a fractal upper bound for the Selberg zeta function (and more generally, L-functions) of Hecke triangle groups of the second kind. This implies a fractal Weyl upper bound for hyperbolic surfaces arising from finite-index, torsion-free subgroups of the Hecke triangle family. This result is the first of its kind in the literature for surfaces with cusps, and it complements the result of Guillopé–Lin–Zworski [33].

- We obtain an explicit strip in the complex plane containing infinitely many resonances for hyperbolic surfaces arising from torsion-free, finite-index subgroups of the Hecke triangle family. This follows from a lower bound on the essential spectral gap, which we prove by closely following ideas of Jakobson–Naud [40].

- As a side result we prove that the Selberg zeta function of Hecke triangle groups of the second kind has no zeros in the half-plane $\{\mathrm{Re}(s) > \frac{1}{2}\}$ except for $s = \delta$.

Zusammenfassung

Wir beweisen einige Resultate über die Verteilung von Resonanzen für hyperbolische Flächen unendlichen Volumens:

- Wir beweisen die Existenz globaler oberer und unterer Schranken für Resonanzen für Überlagerungen von Schottkyflächen, die ein schwaches Weylgesetz für Familien von Überlagerungen ergeben. Dieses Ergebnis verfeinert frühere Ergebnisse von Guillopé–Zworski [34, 35] und ergänzt ein Ergebnis von Borthwick [12]. Angewandt auf Kongruenzüberlagerungen, liefert dieses Resultat Schranken für Resonanzen in Abhängigkeit des Levels, die ein Ergebnis von Jakobson–Naud [38] verbessert. Auf dem Weg dahin beweisen wir eine Abschätzung zum Wachstum von L-Funktionen (getwistete Selbergsche Zetafunktionen) von Schottkygruppen.

- Wir zeigen eine verbesserte fraktale obere Schranke für Überlagerungen von Schottkyflächen in Abhängigkeit von geometrischen Größen der Überlagerung. Dies verbessert und verfeinert Resultate von Jakobson–Naud [38], Guillopé–Lin–Zworski [33] und Dyatlov [23]. Dieses Resultat lässt sich auch algebraisch mittels Cayleygraphen der Gruppe der Decktransformationen interpretieren. Außerdem liefert es Abschätzungen zur Anzahl der Eigenwerte von Überlagerungen von Schottkyflächen, welche ein allgemeineres Resultat von Ballmann–Matthiesen–Mondal [5] verfeinern.

- Wir zeigen, dass die spektrale Lücke (spectral gap) von hyperbolischen Flächen beliebig klein sein kann, indem man auf abelsche Überlagerungen übergeht. Dies beantwortet eine Frage von Naud und subsumiert frühere Arbeiten zu diesem Thema von Randol [75] und Selberg [82]. Im Falle der Schottkyflächen liefern wir eine präzisere Gleichverteilungsaussage für Resonanzen abelscher Überlagerungen in der Nähe der "ersten" Resonanz $s = \delta$.

- Wir beweisen eine fraktale obere Schranke für die Selbergsche Zetafunktion (und allgemeiner, für L-Funktionen) von Heckedreiecksgruppen zweiter Ordnung. Dies impliziert eine fraktale obere Weylschranke für hyperbolische Flächen $\widetilde{\Gamma}\backslash\mathbb{H}$, wobei $\widetilde{\Gamma}$ eine torsionsfreie Untergruppe der Heckedreiecksfamilie mit endlichem Index ist. Dieses Ergebnis ist das Erste dieser Art für Flächen mit Spitzen und es ergänzt das Resultat von Guillopé–Lin–Zworski [33].

- Wir geben einen expliziten Streifen in der komplexen Zahlenebene an, der unendlich viele Resonanzen für $\widetilde{\Gamma}\backslash\mathbb{H}$ enthält, wobei $\widetilde{\Gamma}$ eine torsionsfreie Untergruppe der Heckedreicksfamilie mit endlichem Index ist. Dies folgt aus einer unteren Schranke für die essentielle spektrale Lücke (essential spectral gap), die wir wiederum mit den Ideen von Jakobson–Naud [40] beweisen.

- Als Nebenergebnis zeigen wir, dass die Selbergsche Zetafunktion von Heckedreiecksgruppen zweiter Ordnung in der Halbebene $\{\mathrm{Re}(s) > \frac{1}{2}\}$ keine Nullstellen außer $s = \delta$ besitzt.

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Contents

Chapter 1

Introduction

The term ‘resonance’ is often associated with oscillating systems, which are subjected by some outside force having a frequency close to their own natural frequency. Although resonances have their origins in physics, they appear naturally in mathematical subjects such as spectral theory, harmonic analysis, geometry, and number theory.

The resonances considered in this work are complex numbers arising as poles of the resolvent of the Laplace–Beltrami operator Δ_X on some Riemannian manifold X and as such, they may be regarded as generalized eigenvalues of Δ_X. In this setting, the main goal is to understand the interaction between the ‘classical’ side of X (geometry, dynamics, geodesics) and the ‘quantum’ side of X (spectral theory, eigenvalues and resonances).

In the present thesis we focus on the case where X is a *hyperbolic* surface, that is, a two-dimensional manifold of *constant negative* sectional curvature -1. One motivation to study the hyperbolic case stems from the following fact [84, 4]: if X has negative sectional curvature, then the geodesic flow on the tangent bundle $\mathcal{T}X$ is ergodic and in particular chaotic. Hence the study of resonances for hyperbolic surfaces falls into the realm of *quantum chaos*. Quantum chaos seeks to describe chaotic classical dynamical systems in terms of quantum theory. This connection to physics is a valuable source for many conjectures, which are motivated by experimental data.

Another reason to consider hyperbolic surfaces is that one can draw upon more classical subjects of mathematics, such as automorphic forms and Maass cusp forms (which are studied extensively in number theory), and Selberg theory (which is essentially the spectral theory of finite-area hyperbolic surfaces).

One of the standard models for hyperbolic geometry is the Poincaré half-plane

$$\mathbb{H} = \{z = x + iy : x \in \mathbb{R}, y > 0\}$$

endowed with its metric of negative sectional curvature -1,

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

The group of orientation-preserving isometries of the hyperbolic plane is isomorphic to $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm \mathrm{id}\}$. Each element $g \in \mathrm{PSL}_2(\mathbb{R})$ acts on $\mathbb{H}$ by the

Möbius transformation

$$z \mapsto \frac{az+b}{cz+d}, \quad g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Every hyperbolic surface X is isometric to a quotient $\Gamma\backslash\mathbb{H}$ of the hyperbolic plane by some discrete subgroup $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$. Hence, the study of discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$, also called *Fuchsian groups*, goes hand in hand with the study of hyperbolic surfaces. The arguably most prominent example of a Fuchsian group is the modular group $\mathrm{PSL}_2(\mathbb{Z})$.

In this thesis we will mostly consider hyperbolic surfaces of *infinite area* and we will always assume that $X = \Gamma\backslash\mathbb{H}$ is *geometrically finite*. The latter condition is crucial for doing spectral theory on X. It is equivalent with saying that the group Γ is finitely generated, or that the Euler characteristic of X is finite. For the moment we allow X to be of infinite or of finite area.

Let Δ_X denote the positive Laplacian on X. The resolvent of Δ_X is the operator

$$R_X(s) := \big(\Delta_X - s(1-s)\big)^{-1} \colon L^2(X) \to L^2(X).$$

It is well-defined for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1/2$ and $s(1-s)$ not being an L^2-eigenvalue of Δ_X. From the work of Mazzeo–Melrose [53] and Guillopé–Zworski [34] we know that it extends to a meromorphic family

$$R_X(s) \colon C_c^\infty(X) \to C^\infty(X)$$

on $\mathbb{C}$ with poles of finite rank. The poles of the meromorphically continued resolvent are called the *resonances* of X. Each resonance s has a finite multiplicity, which is defined as the rank of the residue of R_X at the pole s. We denote by $\mathcal{R}(X)$ the set of resonances, repeated according to their multiplicities.

Notice that resonances can be thought of as generalized eigenvalues because of their appearance as poles of the resolvent. If $s \in \mathcal{R}(X)$ is a resonance with $\mathrm{Re}(s) > \frac{1}{2}$, then $\lambda = s(1-s)$ is indeed an L^2-eigenvalue. On the other hand, resonances with $\mathrm{Re}(s) < \frac{1}{2}$ are often interpreted as the poles of a scattering operator, called *scattering poles*.

From the point of view of physics, each resonance corresponds to a decaying wave. Indeed, to each resonance $s \in \mathbb{C}$ we can associate a generalized eigenfunction (a so-called purely outgoing eigenstate) $\psi_s \in C^\infty(X)$, which provides a stationary solution of the automorphic wave equation:

$$\varphi(t,x) = e^{(s-\frac{1}{2})t}\psi_s(x), \quad \left(\partial_t^2 + \Delta_X - \frac{1}{4}\right)\varphi(t,x) = 0. \tag{1.1}$$

In light of (1.1), $\mathrm{Re}(s) - \frac{1}{2}$ is the decay rate and $\mathrm{Im}(s)$ is the oscillation frequency of the solution φ.

Whereas resonances for finite-area hyperbolic surfaces are contained in a vertical strip of the complex plane of bounded width, in the infinite-area case they are spread all over the half-plane $\{s \in \mathbb{C} : \mathrm{Re}(s) \leq \delta\}$. Here, the value $\delta = \delta(\Gamma) \in [0,1]$ plays an important role throughout this work. It is the Hausdorff dimension of the limit set $\Lambda(\Gamma) \subseteq \partial\mathbb{H}$ of the group Γ.

Understanding the location, distribution and density of resonances is an extremely difficult task. For instance, the scattering poles s of the modular surface $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$ are directly related to the non-trivial zeros of Riemann zeta function by $\zeta(2s) = 0$. This example shows that even in the finite-area case, it is not a trivial matter to numerically locate resonances, let alone compute them explicitly. Needless to say, there is no analytic formula allowing us to calculate the resonances for general hyperbolic surfaces.

In the last few decades, a great deal of research has been devoted to the study of resonances for hyperbolic surfaces. The principal aim of this work is to further the understanding of their distribution and density. One way to attack any problem related to resonances is through the *Selberg zeta function*, which is defined on the half-plane $\mathrm{Re}(s) > \delta$ by the Euler product

$$Z_\Gamma(s) := \prod_{\ell\in\mathcal{L}(X)} \prod_{k=0}^{\infty} \left(1 - e^{-(s+k)\ell}\right). \tag{1.2}$$

Here, the outer product is taken over the *primitive length spectrum* $\mathcal{L}(X)$ of X, that is, the set of lengths of the primitive periodic geodesics on X. As is well-known, the Selberg zeta function $Z_\Gamma(s)$ extends to a meromorphic function of $s \in \mathbb{C}$. For compact and finite-area hyperbolic surfaces it is a deep fact that the zeros of the (meromorphically continued) Selberg zeta function correspond, apart from some well-understood zeros, to resonances of the Laplacian on $X = \Gamma\backslash\mathbb{H}$. This fact is equivalent to the classical *Selberg trace formula*, which goes back to Selberg [80], and yields a striking relation between the spectral data of X and its closed geodesics. More recently, the same connection between the zeros of Z_Γ and the resonances for $X = \Gamma\backslash\mathbb{H}$ was proven first by Patterson–Perry [68] in the convex cocompact case, and then by Borthwick–Judge–Perry [13] for *all* geometrically finite hyperbolic surfaces.

In view of the above discussion, it is not surprising that the asymptotic behaviour of resonances is intimately connected to the analytic properties of the Selberg zeta function. Consider for instance the problem of estimating the number $N_X(r)$ of resonances inside disks centered at the origin of the complex plane,

$$N_X(r) := \#\{s \in \mathcal{R}(X) \colon\ |s| \le r\},$$

as the radius r tends to infinity. Suppose for a moment that we had a global growth estimate for $Z_\Gamma(s)$ of the type

$$|Z_\Gamma(s)| \le C\exp(C|s|^2), \quad s \in \mathbb{C}, \tag{1.3}$$

where $C > 0$ is some absolute constant. ('Global' because (1.3) applies to every $s \in \mathbb{C}$.) Once such an estimate is established, it is relatively straightforward (using tools from complex analysis) to derive the global upper bound

$$N_X(r) = O(r^2).$$

However, establishing estimates such as (1.3) is by no means easy, given that on the half-plane $\mathrm{Re}(s) \le \delta$, the Selberg zeta function is defined through meromorphic continuation of the infinite product (1.2). Part of this work is concerned with proving refined versions of (1.3), which in turn lead to new upper and lower bounds for the resonance counting function $N_X(r)$ (see Chapter 3).

In the last decades a new interpretation for the Selberg zeta function has emerged. In its simplest form, this new point of view can be described as follows: the action of the group Γ on the hyperbolic plane $\mathbb{H}$ induces a discrete dynamical system on its boundary $\partial\mathbb{H}$. (For the modular group this dynamical system is the Gauss map for continued fractions.) To this dynamical system, we associate in a natural way a family of trace class operators $\mathcal{L}_s : \mathcal{H} \to \mathcal{H}$ acting on some suitable Hilbert space $\mathcal{H}$. Then this family of so-called *transfer operators*, which is parametrized by the spectral variable s, leads to the remarkable formula

$$Z_\Gamma(s) = \det(1 - \mathcal{L}_s). \tag{1.4}$$

Here, 'det' is the Fredholm determinant. Details and references concerning the precise way of arriving at (1.4) will be given throughout this work.

The description of Z_Γ in terms of Fredholm determinants of transfer operators has numerous advantages. For instance, (1.4) almost immediately shows the existence of a meromorphic continuation of Z_Γ, which is far from obvious in the purely geometric description as a product over the length spectrum in (1.2). Most of the results in this work are obtained by exploiting transfer operator techniques, commonly referred to as 'thermodynamic formalism'.

There is a long list of authors who have implemented the thermodynamic formalism (in some form or another) to obtain novel results in spectral theory and beyond, which so far are inaccessible by usual spectral methods. It includes Naud [57, 59], Jakobson–Naud [40, 38], Bourgain–Gamburd–Sarnak [17], Oh–Winter [63], just to name a few. For us, the main advantage of transfer operators is that they enable us to control the growth of the Selberg zeta function.

To better explain our motivation, let us have a closer look at one beautiful applicaton of (1.4) due to Guillopé–Lin–Zworski [33]. They proved one of the first rigourous results in the direction of a more general conjecture for open chaotic systems, called the *fractal Weyl law*, by analogy to the classical Weyl law for eigenvalues. This conjecture – motivated by numerical experiments and supported by experimental evidence in the physics literature – was formulated by Sjöstrand [85] and Lu–Sridhar–Zworski [48]. Roughly speaking, this conjecture predicts the distribution of resonances and scattering poles in terms of the fractal dimension of the trapped set of the geodesic flow. When applied to the specific setting of hyperbolic surfaces, it asserts that for all $\sigma \in \mathbb{R}$ negative enough one has

$$N_X(\sigma, T) := \#\{s \in \mathcal{R}(X) : \mathrm{Re}(s) \geq \sigma,\ |\mathrm{Im}(s)| \leq T\} \asymp T^{1+\delta}, \quad \text{as } T \to \infty. \tag{1.5}$$

Recall that δ denotes the Hausdorff dimension of the limit set of Γ. We point out that the trapped set of the geodesic flow (viewed as a subset of the tangent bundle of X) has Hausdorff dimension $2(1+\delta)$, see [14, Chapter 14]. Thus the fractal Weyl conjecture says that the number of resonances in strips parallel to the imaginary axis satisfies a power law, with exponent equal to half of the dimension of the trapped set.

Now suppose that X is an infinite-area geometrically finite hyperbolic surface *without* cusps. In this case it turns out that we can realize X as the quotient $\Gamma\backslash\mathbb{H}$, where Γ is a so-called *Schottky group.* Schottky groups stand out, among other Fuchsian groups, by their particularly simple geometric structure. This structure allows one to deduce a 'natural' transfer operator $\mathcal{L}_s$ satisfying (1.4).

Using (1.4) and some properties of transfer operators in a clever way, Guillopé–Lin–Zworski [33] showed that in all half-planes $\{\mathrm{Re}(s) \geq \sigma\}$, we have

$$|Z_\Gamma(s)| \leq C_\sigma \exp(C_\sigma |\mathrm{Im}(s)|^\delta), \tag{1.6}$$

for some constant C_σ only depending on σ. Notice that in vertical strips, (1.6) vastly improves upon the global bound (1.3). Using this growth estimate and the aforementioned correspondence between zeros of Z_Γ and resonances for $X = \Gamma\backslash\mathbb{H}$, it is only a matter of complex analysis to deduce the following upper bound for all $\sigma \in \mathbb{R}$:

$$N_X(\sigma, T) = O_\sigma(T^{1+\delta}). \tag{1.7}$$

This bound settles the upper bound of the fractal Weyl conjecture for hyperbolic surfaces without cusps. We point out that there are also some known lower bounds (for instance [36, 39]), though these are weaker than predicted by the fractal Weyl conjecture. Sharp lower bounds for $N_X(\sigma, T)$ have remained elusive.

Even though many results supporting the fractal Weyl conjecture (mostly upper bounds) have recently been achieved in different contexts (see for instance [21, 24, 25, 60, 61]), it remains untackled for hyperbolic surfaces with cusps. Nor is it clear in this case how to obtain fractal growth estimates for the Selberg zeta function analogous to (1.6).

In the presence of cusps, the trapped set is *non-compact*, a notoriously hard obstacle to overcome. Another difficulty when dealing with cusps is that there is no obvious way to realize Z_Γ as a Fredholm determinant of a transfer operator family. From Mayer [52], Morita [54], and more recently Pohl [71] and Fedosova–Pohl [26], we know however that this can be done in many cases. A particularly nice class of examples for which a transfer operator is available, is the family of *Hecke triangle groups*. In some sense, Hecke triangle groups generalize the modular group $\mathrm{PSL}_2(\mathbb{Z})$. In Chapter 5, we exploit the properties of this transfer operator to prove a fractal upper bound for the Selberg zeta function of Hecke triangle groups and their subgroups. Our result yields, for the first time, a fractal upper Weyl bound for resonances similar to (1.7) for a large class of surfaces with funnels *and* cusps. This is also the first example in the literature of a fractal Weyl bound in a situation where the trapped set is non-compact.

One further topic we address is 'spectral gap'. Informally, this term refers to the amount by which the bottom of the spectrum of the Laplacian Δ_X is separated from the remaining spectrum. For compact X, the spectrum of Δ_X is a sequence of positive real eigenvalues with a simple zero at the bottom:

$$\lambda_0(X) = 0 < \lambda_1(X) \leq \lambda_2(X) \leq \dots.$$

In the compact case, the spectral gap of X is thus given by $\lambda_1(X) - \lambda_0(X) = \lambda_1(X) > 0$. It turns out that the notion of spectral gap can be extended in a natural way to non-compact surfaces X as well, in terms of the resonances of X. This will be explained at the beginning of Chapter 4. In this setting, Naud [57] and Bourgain–Dyatlov [15] proved the existence of a *positive* spectral gap, a rather non-trivial fact in non-compact situations. The size of the spectral gap directly affects the error term in many counting results, such as the asymptotic distribution of closed geodesics on X. However, the methods used in these proofs do not yield effective bounds on the size of the spectral gap. This raised the question whether this quantity can be arbitrarily small. More precisely, given any $\varepsilon > 0$, can one find a hyperbolic surface whose spectral gap is smaller than ε? In Chapter 4 we show (among other things) that this is indeed the case.

Guide for the reader

Chapter 2 contains the basic material on hyperbolic surfaces and their spectral theory needed to understand the results in the subsequent chapters. Chapters 3, 4, and 5 contain the essential results of the present thesis. These chapters can be read independently of each other and in any possible order, although Chapter 3 begins with a rather elaborate description of some known results on the distribution of resonances. Reading this part first might benefit the reader unfamiliar with the topic at hand. In Chapter 6 a brief outlook for possible future research is presented.

In the following we provide a summary for each of the main chapters.

Chapter 3. The first main result of this chapter is Theorem 3.1, which gives global upper and lower bounds for resonances for covers of Schottky surfaces. In the process of proving Theorem 3.1, we prove Proposition 3.4, which is a growth estimate on L-functions of Schottky groups of independent interest.

The second main result of Chapter 3, Theorem 3.2, is an improved fractal upper bound for resonances for covers of Schottky surfaces in terms of geometric quantities of the cover. Applying Theorem 3.2 to congruence subgroups yields Proposition 3.11. This result can also be interpreted in terms of Cayley graphs, leading to Corollary 3.14.

Chapter 4. The two main statements of Chapter 4 are Theorem 4.1 and Theorem 4.2. Together they show that the spectral gap of hyperbolic surfaces can be arbitrarily small, by passing to abelian covers. Theorem 4.2 gives a much more precise equidistribution result for resonances close to the 'first' resonance δ.

Chapter 5. This chapter deals with the family of Hecke triangle groups Γ_w, which is parametrized by the cusp width w. The modular group $\mathrm{PSL}_2(\mathbb{Z})$ corresponds to the case $w = 1$. Thus we may think of Hecke triangle groups as generalizations of the modular group. We restrict our attention to the non-cofinite Hecke triangle groups, which are precisely those with cusp width $w > 2$. Our primary aim is to prove *fractal* upper bounds for L-functions of Γ_w, Theorem 5.1. This theorem has two immediate corollaries. First, it gives a new growth estimate for the Selberg zeta function of arbitrary finite-index subgroups of the Hecke triangle family Γ_w, see Corollary 5.2. Second, it leads to a fractal Weyl upper bound for hyperbolic surfaces arising from torsion-free, finite-index subgroups of Γ_w, see Corollary 5.3. The latter is the first example in the literature for a fractal Weyl law for hyperbolic surfaces *with cusps*. We provide some explicit examples for torsion-free, finite-index subgroups of Γ_w. As a side result, we show that the Selberg zeta function $Z_{\Gamma_w}(s)$ with $w > 2$ has no zeros in the half-plane $\{\mathrm{Re}(s) > \frac{1}{2}\}$ except at $s = \delta(\Gamma_w)$, see Corollary 5.5.

Another consequence of the newly obtained growth estimate is an explicit strip in the complex plane containing infinitely many resonances for hyperbolic surfaces arising from torsion-free, finite-index subgroups of Γ_w, see Theorem 5.4. This result requires more effort to prove, and follows from an abstract lower bound on the essential spectral gap, see Theorem 5.33.

Chapter 2

Preliminaries

To set the stage for this thesis, we will first review some basic facts about hyperbolic geometry and spectral theory. The goal here is not a complete exposition, but rather a brief overview of the concepts needed to understand the main results of the thesis. All the statements in the present chapter are mentioned without proofs. Most of them can be found in Borthwick's book [14], in which case we do not give a reference.

In this thesis we also make heavy use of concepts of functional analysis (such as trace class operators, Fredholm determinants, and singular values). A brief review of these concepts and some key properties can be found in the Appendix, Section A.1.

2.1 Hyperbolic surfaces

Throughout we use the Poincaré half-plane model of the hyperbolic plane

$$\mathbb{H} = \{z = x + iy : x \in \mathbb{R}, y > 0\}, \quad ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

In these coordinates, the associated (positive) Laplace–Beltrami operator is

$$\Delta_{\mathbb{H}} = -y^2\left(\partial_x^2 + \partial_y^2\right).$$

The group of orientation-preserving isometries of $\mathbb{H}$ is isomorphic to the group $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm\,\mathrm{id}\}$, acting by Möbius transformations on $\mathbb{H}$. This action extends continuously to the geodesic boundary $\partial\mathbb{H}$ of $\mathbb{H}$, which we identify with $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$. More concretely, the action of the element

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{R})$$

on the point $z \in \overline{\mathbb{H}} = \mathbb{H} \cup \partial\mathbb{H}$ is given by the formula

$$g.z := \begin{cases} \infty & \text{if } z = \infty,\ c = 0, \text{ or } z \neq \infty,\ cz + d = 0, \\ \dfrac{a}{c} & \text{if } z = \infty,\ c \neq 0, \\ \dfrac{az + b}{cz + d} & \text{otherwise.} \end{cases}$$

Elements of $\mathrm{PSL}_2(\mathbb{R})$ can be classified in terms of their fixed points. An element $g \in \mathrm{PSL}_2(\mathbb{R})$, $g \neq \mathrm{id}$, is called

- *hyperbolic* if it has precisely two fixed points z_1, z_2 on $\partial\mathbb{H}$ (or equivalently, if $|\operatorname{Tr} g| > 2$),
- *elliptic* if it has a single fixed point in $\mathbb{H}$ ($|\operatorname{Tr} g| < 2$), and
- *parabolic* if it has a single fixed point in $\partial\mathbb{H}$ ($|\operatorname{Tr} g| = 2$).

Notice that $|\operatorname{Tr} g|$ is well-defined in $\mathrm{PSL}_2(\mathbb{R})$.

On $\mathrm{PSL}_2(\mathbb{R})$, we use the standard matrix topology defined by the norm $\|g\|_F = \operatorname{Tr}(g^\top g)^{1/2}$. A *Fuchsian group* is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$.

A surface[1] is called *hyperbolic* if it is equipped with a complete Riemannian metric of constant negative sectional curvature -1. Every hyperbolic surface X is isometric to a quotient $\Gamma\backslash\mathbb{H}$ for some Fuchsian group Γ not containing elliptic elements. If Γ contains at least one elliptic element, then $\Gamma\backslash\mathbb{H}$ has conical singularities, in which case the quotient $\Gamma\backslash\mathbb{H}$ is an *orbifold*, rather than a manifold. Moreover, the fundamental group $\pi_1(X)$ of a hyperbolic surface $X = \Gamma\backslash\mathbb{H}$ is isomorphic to Γ.

Let Γ be a Fuchsian group. An element $g \in \Gamma$, $g \neq \mathrm{id}$, is called *primitive* if for every $h \in \Gamma$ and $n \in \mathbb{N}$, $g = h^n$ implies $n = 1$. We let $[\Gamma]_p$ denote the set of Γ-conjugacy classes of the primitive hyperbolic elements in Γ, and $[\Gamma]$ the set of Γ-conjugacy classes of all hyperbolic elements in Γ.

There is a well-known one-to-one correspondence between the $[\Gamma]_p$ and the set of primitive periodic geodesics on $X = \Gamma\backslash\mathbb{H}$. Similarly, there is a one-to-one correspondence between $[\Gamma]$ and the set of all periodic geodesics on X (allowing multiple passages through the image). Hence, a closed geodesic represented by the element γ is naturally identified with its conjugacy class $[\gamma]$.

If $g \in \mathrm{PSL}_2(\mathbb{R})$ is hyperbolic, then there is a unique geodesic $\alpha(g)$ connecting its two fixed points $z_1, z_2 \in \partial\mathbb{H}$, called the *axis* of g. On the axis $\alpha(g)$, the element g acts by translation by some fixed length $\ell = \ell(g)$, called the *displacement length* of g. It is given by the formula

$$2\cosh\left(\frac{\ell(g)}{2}\right) = |\operatorname{Tr} g|. \tag{2.1}$$

From (2.1) it follows that the displacement length is invariant under conjugation. Moreover, if a periodic geodesic on X corresponds to the conjugacy class $[g]$, then its length is given by $\ell(g)$.

We denote the length of the shortest closed geodesic on X by $\ell_0(X)$. In other words,

$$\ell_0(X) = \min_{[g]\in[\Gamma]} \ell(g). \tag{2.2}$$

[1]For the purposes of this thesis, a 'surface' is a connected, orientable, two-dimensional smooth manifold.

2.1.1 Cusps and funnels

One way to understand the quotient $\Gamma\backslash\mathbb{H}$ is through a so-called *fundamental domain* for the action of Γ on $\mathbb{H}$, which is an open region $\mathcal{F} \subset \mathbb{H}$ such that

$$\mathbb{H} = \bigcup_{\gamma\in\Gamma} \gamma\overline{\mathcal{F}}$$

and such that $\gamma_1\mathcal{F} \cap \gamma_2\mathcal{F} = \emptyset$ for all $\gamma_1, \gamma_2 \in \Gamma$ with $\gamma_1 \neq \gamma_2$.

Throughout this work, we only consider *geometrically finite* hyperbolic surfaces $X = \Gamma\backslash\mathbb{H}$. This means that there exists a fundamental domain for the action of Γ on $\mathbb{H}$, which is a finite-sided convex[2] polygon. The condition of geometric finiteness has strong geometric implications. It is equivalent to both finite generation of the group Γ, and topologic finiteness of X (i.e. finite Euler characteristic $\chi(X)$). In this case, X can have only two types of endings: *cusps* and *funnels*. More precisely, if the surface $X = \Gamma\backslash\mathbb{H}$ is non-elementary and geometrically finite, then it can be decomposed as

$$X = K \cup C \cup F, \tag{2.3}$$

where K is the *compact core*, and C and F are finite disjoint unions of cusps and funnels, respectively. (Note that F or C might be empty.) For the decomposition in (2.3) to be unique, we need precise definitions for funnels and cusps, since these definitions vary in the literature. Funnels (of diameter ℓ) are isometric to

$$F_\ell := \langle z \mapsto e^\ell z\rangle\backslash\{z \in \mathbb{H} : \mathrm{Re}(z) \geq 0\},$$

while cusps are isometric to

$$C_\infty := \langle z \mapsto z+1\rangle\backslash\{z \in \mathbb{H} : 0 \leq \mathrm{Re}(z) \leq 1,\ \mathrm{Im}(z) \geq 1\},$$

both endowed with the hyperbolic metric. Using the hyperbolic area measure

$$\frac{dxdy}{y^2}$$

one can verify that

$$\mathrm{area}(C_\infty) = 1 \quad \text{and} \quad \mathrm{area}(F_\ell) = \infty.$$

Thus, $X = \Gamma\backslash\mathbb{H}$ has infinite area if and only if it has at least one funnel ending ($F \neq \emptyset$). If X has at least one cusp but no funnels, then it is not compact, but its area is still finite. Algebraically, cusps of $X = \Gamma\backslash\mathbb{H}$ correspond one-to-one to orbits of fixed points of parabolic elements in Γ.

In view of (2.3) we can define the *convex core* of X as the set $K \cup C$, that is, X minus the funnels. Note that if X has no cusps, then its convex and compact cores agree, in which case X is sometimes called 'convex cocompact'. It turns out (but is by no means obvious) that if $X = \Gamma\backslash\mathbb{H}$ is a geometrically finite, infinite-area hyperbolic without cusps, then Γ is a so-called Schottky group. Schottky groups

[2] A subset $U \subset \mathbb{H}$ is 'convex' if for all $z_1, z_2 \in U$, the geodesic arc $[z_1, z_2]$ is contained in U.

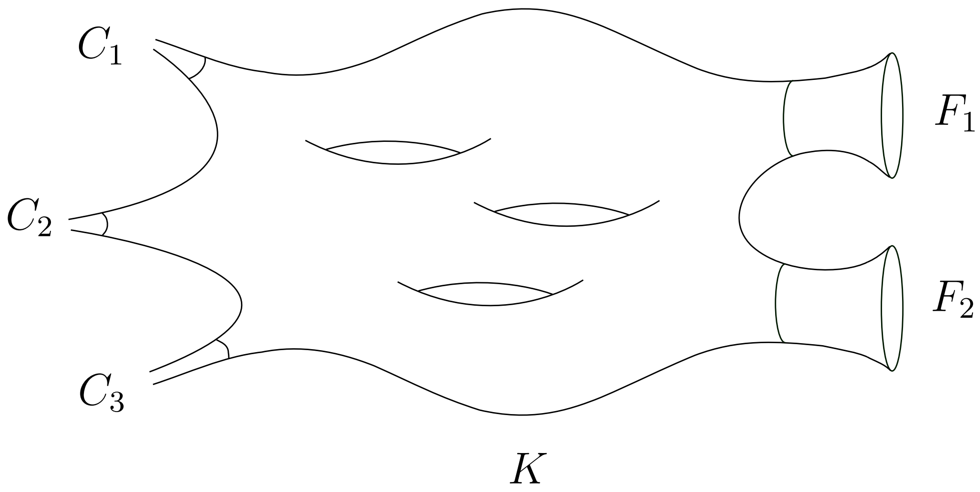

Figure 2.1: Schematic view of a hyperbolic surface with three cusps and two funnels

are Fuchsian groups that arise from a specific geometric construction, which we describe in Subsection 2.2 below.

Given a non-elementary, geometrically finite hyperbolic surface X, we can define its *0-volume* as the volume of its convex core:

$$0\text{-vol}(X) := \text{area}(K \cup C) < \infty.$$

We should point out that the 0-volume is actually defined in terms of a more general '0-integral'. We refer to [14, Page 225] for the details. If X is of finite area it is clear that $0\text{-vol}(X) = \text{vol}(X)$. Thus, the 0-volume can be seen a natural replacement for the volume in the infinite-area case. (Note that we do not distinguish the words 'area' and 'volume' in this context.) Using the Gauss–Bonnet formula, one can show that

$$0\text{-vol}(X) = -2\pi\chi(X), \tag{2.4}$$

where $\chi(X)$ is the Euler characteristic of X.

2.1.2 Limit set

A very important object in the study of hyperbolic surfaces is the *limit set* of a Fuchsian group Γ, usually denoted by $\Lambda(\Gamma)$. It is defined as the set of accumulation points (in the Riemann sphere topology) of all orbits $\Gamma.z$ for $z \in \mathbb{H}$. In fact, one can show that if $z \in \mathbb{H}$ is not a fixed point of an elliptic element in Γ, then the limit set is given by

$$\Lambda(\Gamma) = \overline{\Gamma.z} \cap \partial\mathbb{H}.$$

The limit set is a closed, Γ-invariant subset of $\partial\mathbb{H}$. In general, $\Lambda(\Gamma)$ is a Cantor-like fractal, and its Hausdorff dimension $\delta(\Gamma) := \dim_H \Lambda(\Gamma)$ plays an important role in this work. Whenever Γ or X is fixed, we write δ for $\delta(\Gamma)$ or $\delta(X)$.

For the connection between the limit set of Γ and the trapped set of the geodesic flow on $T^1(\Gamma\backslash\mathbb{H})$ we refer to [14, Chapter 14].

The topologic and measure-theoretic features of $\Lambda(\Gamma)$ reveal a lot of information about the group Γ. Indeed, by the classification of Poincaré and Fricke–Klein, we have the following facts:

- If $\Lambda(\Gamma)$ is finite, then Γ is an elementary group and $\delta(\Gamma) = 0$.
- If $\Lambda(\Gamma) = \partial\mathbb{H}$, then the quotient $\Gamma\backslash\mathbb{H}$ is of finite area and $\delta(\Gamma) = 1$. In this case, Γ is said to be a Fuchsian group of the *first kind*.
- If $\Lambda(\Gamma)$ is a perfect nowhere dense subset of $\partial\mathbb{H}$, then $\Gamma\backslash\mathbb{H}$ has infinite area and $0 < \delta(\Gamma) < 1$. In this case, Γ is said to be a Fuchsian group of the *second kind*.

If the quotient $\Gamma\backslash\mathbb{H}$ has finite-area, then we call Γ 'cofinite'. In particular, it follows from the above characterization that

$$\Gamma \text{ is cofinite} \iff \Lambda(\Gamma) = \partial\mathbb{H} \iff \Gamma\backslash\mathbb{H} \text{ has no funnels.}$$

Another interesting result due to Beardon [8, 9] says that if a non-elementary geometrically finite surface $X = \Gamma\backslash\mathbb{H}$ has at least one cusp, then necessarily $\delta(\Gamma) > \frac{1}{2}$. (The inverse statement is however not true.)

There are several other interesting ways of characterizing the quantity $\delta(\Gamma)$. For instance, $\delta(\Gamma)$ equals both the topological entropy of the geodesic flow on the trapped set (in the convex cocompact case), and the exponent of convergence of the Poincaré series

$$P_\Gamma(s; z, w) := \sum_{\gamma\in\Gamma} e^{-sd_{\mathbb{H}}(z,w)},$$

where $d_{\mathbb{H}}(z, w)$ denotes the hyperbolic distance.

2.2 Schottky groups and Schottky surfaces

In Chapters 3 and 4 of this thesis, Fuchsian Schottky groups Γ play an important role. Classically, Schottky groups are given by a specific geometric construction, which we will take to our advantage. Let us recall this construction now.

Let $m \in \mathbb{N}$ and choose $2m$ open Euclidean disks in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ having mutually disjoint closures, and all of which are centered on $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$. Endow these disks with an ordering, say

$$\mathcal{D}_1, \ldots, \mathcal{D}_{2m}. \tag{2.5}$$

Recall that the action of $\mathrm{PSL}_2(\mathbb{R})$ on $\overline{\mathbb{H}}$ by Möbius transformations extends continuously to the whole Riemann sphere $\overline{\mathbb{C}}$. For $j \in \{1, \ldots, m\}$ let γ_j be an element in $\mathrm{PSL}_2(\mathbb{R})$ that maps the exterior of $\mathcal{D}_j$ to the interior of $\mathcal{D}_{j+m}$. Then the elements $\gamma_1, \ldots, \gamma_m$ and its inverses freely generate a Fuchsian group Γ. A Fuchsian group Γ is called *Schottky* if it arises from such a construction.

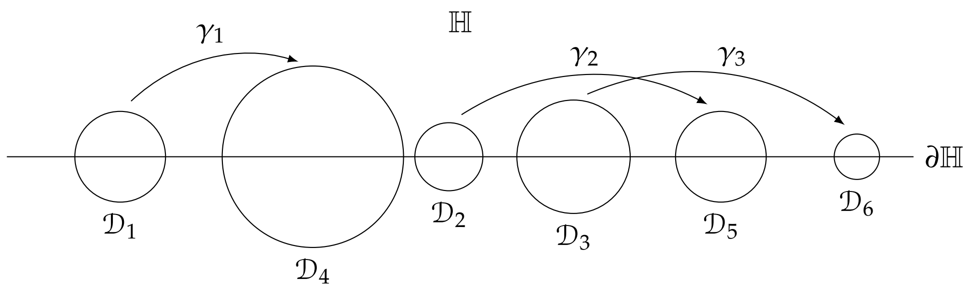

Figure 2.2: A configuration of Schottky disks and isometries with $m = 3$

A hyperbolic surface X is said to be a *Schottky surface* if there exists a Schottky group Γ such that X is isometric to $\Gamma\backslash\mathbb{H}$.

By switching to a conjugate $g^{-1}\Gamma g$ of Γ for some suitable isometry $g \in \mathrm{PSL}_2(\mathbb{R})$, we can always assume that the disks $\mathcal{D}_j$ in (2.5) are Euclidean disks in $\mathbb{C}$ (rather than $\overline{\mathbb{C}}$). This restriction does not lead to a loss of generality and does not affect our results, since we are mostly interested in Schottky surfaces $X = \Gamma\backslash\mathbb{H}$ (in which case we are free to choose Γ such that no such conjugation is needed) and since our results are invariant under conjugations.

For convenience, given a Schottky group Γ, we omit throughout any reference to a possibly necessary conjugation. Whenever we deal with a Schottky group Γ, we assume that there exists $m \in \mathbb{N}$ and a configuration of disks $\mathcal{D}_1, \ldots, \mathcal{D}_{2m} \subset \mathbb{C}$ and corresponding isometries $\gamma_1, \ldots, \gamma_m$ as above such that $\Gamma = \langle \gamma_1^{\pm}, \ldots, \gamma_m^{\pm} \rangle$. Moreover, for $j \in \{m+1, \ldots, 2m\}$, it is extremely helpful to define

$$\gamma_j := \gamma_{j-m}^{-1},$$

and to extend this definition cyclically to $\mathbb{Z}$ by defining

$$\gamma_k := \gamma_{k \bmod 2m} \quad \text{for } k \in \mathbb{Z}.$$

Further, we let

$$\mathcal{D} := \bigcup_{j=1}^{2m} \mathcal{D}_j \tag{2.6}$$

be the union of the disks (2.5) used in the construction.

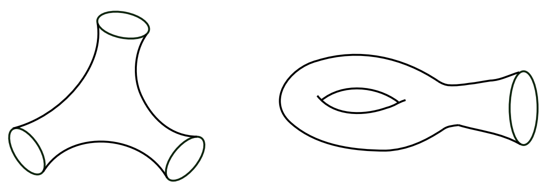

Figure 2.3: Two Schottky surfaces: a three-funnel surface and a funneled torus

The importance of Schottky groups stems from a result of Button [20], which asserts that (Fuchsian) Schottky groups are precisely those Fuchsian groups that are geometrically finite, not cofinite, and have no elliptic nor parabolic elements. In other words, the class of Schottky surfaces coincides with the class of convex cocompact hyperbolic surfaces of infinite area (no cusps, no conical singularities, just funnels!).

2.3 Spectral theory and resonances

Let $X = \Gamma\backslash\mathbb{H}$ be a geometrically finite hyperbolic surface, let Δ_X denote the Laplacian on X, and let

$$\delta := \delta(X) := \dim_H \Lambda(\Gamma)$$

denote the Hausdorff dimension of the limit set $\Lambda(\Gamma)$ of Γ. In this section we assume that X is of infinite-area.

In the infinite-area case the spectrum of Δ_X on $L^2(X)$ is rather sparse. By Lax–Phillips [47] and Patterson [66], the spectrum satisfies the following basic properties, assuming $\operatorname{vol}(X) = \infty$:

- The (absolutely) continuous spectrum is $[1/4, \infty)$.
- The pure point spectrum is finite and contained in $(0, 1/4)$. In particular, there are no eigenvalues embedded in the continuous spectrum.
- If $\delta < 1/2$ then the pure point spectrum is empty. If $\delta > 1/2$ then $\delta(1-\delta)$ is the smallest eigenvalue.

The resolvent

$$R_X(s) := \big(\Delta_X - s(1-s)\big)^{-1} \colon L^2(X) \to L^2(X)$$

of Δ_X is defined for $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1/2$ and $s(1-s)$ not being an L^2-eigenvalue of Δ_X. By [53, 34], it extends to a meromorphic family

$$R_X(s) \colon C_c^\infty(X) \to C^\infty(X)$$

on $\mathbb{C}$ with poles of finite rank. The *resonances* of X are the poles of R_X. We denote the set of resonances, repeated according to multiplicities, by

$$\mathcal{R}(X).$$

The set $\mathcal{R}(X)$ of resonances is contained in the half-plane

$$\{s \in \mathbb{C} : \operatorname{Re}(s) \leq \delta\},$$

and, obviously, each L^2-eigenvalue gives rise to a (pair of) resonance(s). All resonances with $\operatorname{Re}(s) > \frac{1}{2}$ correspond to L^2-eigenvalues (pure point spectrum), and there are only finitely many of them. On the other hand, there are infinitely many resonances in the half-plane $\{\operatorname{Re}(s) < \frac{1}{2}\}$. Moreover, there are no resonances on the critical line $\operatorname{Re}(s) = \frac{1}{2}$, except possibly at $s = \frac{1}{2}$. The set of resonances $\mathcal{R}(X)$ furnishes the natural replacement for the L^2-eigenvalues of Δ_X.

We refer to the introduction of Chapter 3 for some known results on the finer structure of the set $\mathcal{R}(X)$. When it comes to understanding the distribution of resonances, we are mainly interested in the following two resonance counting functions:

$$N_X(r) := \#\{s \in \mathcal{R}(X) \colon |s| \leq r\}, \quad r > 0,$$

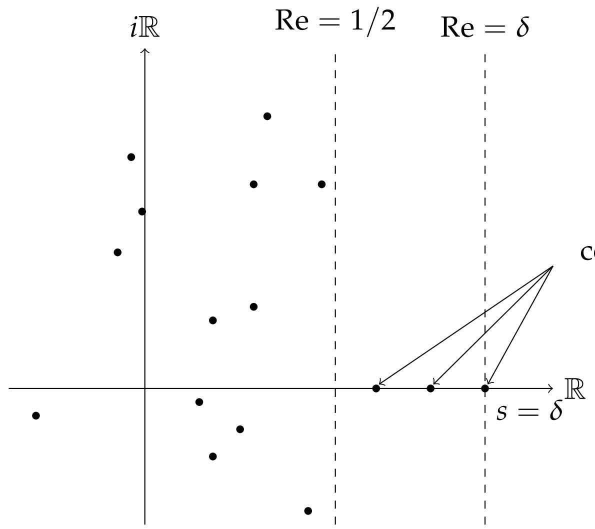

Figure 2.4: Distribution of resonances for infinite-area $\Gamma\backslash\mathbb{H}$ in the case $\delta > \frac{1}{2}$
There are no resonances on the critical line $\mathrm{Re}(s) = \frac{1}{2}$, except possibly at $s = \frac{1}{2}$

and

$$M_X(\sigma, T) := \#\left\{s \in \mathcal{R}(X) : \mathrm{Re}(s) \geq \sigma,\ |\mathrm{Im}(s) - T| \leq 1\right\}, \quad \sigma, T \in \mathbb{R}.$$

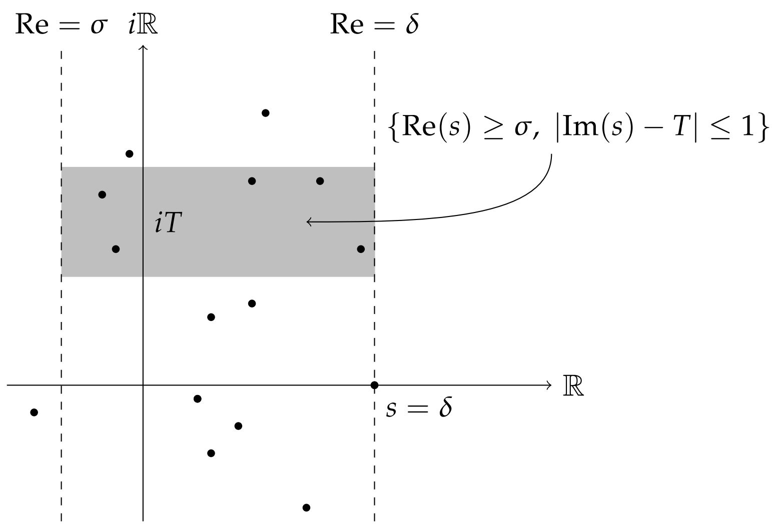

Figure 2.5: Resonances in a box parallel to the imaginary axis

2.4 Representation

For any Fuchsian group Γ and any finite-dimensional representation $\rho\colon \Gamma \to U(V)$ of Γ on a finite-dimensional unitary space V, we denote the inner product on V by $\langle\cdot,\cdot\rangle_V$ and its associated norm by $\|\cdot\|_V$. We drop the subscript V from the notation, whenever the vector space V is fixed.

We define the *dimension* of ρ to be the dimension of V:

$$\dim \rho := \dim V.$$

Further, we denote by $\mathbf{1}_V$ the trivial representation of Γ on V if Γ is understood implicitly, and by $\mathbf{1}_\Gamma$ the trivial representation of Γ on V if V is understood implicitly. The *character* of a representation (ρ, Γ) is the function $\chi\colon \Gamma \to \mathbb{C}$ defined by

$$\chi(g) := \operatorname{Tr} \rho(g)$$

for all $g \in \Gamma$. Clearly, χ is constant on conjugacy classes of Γ.

2.4.1 Induced representation

Let $\widetilde{\Gamma}$ be a subgroup of Γ with finite index $n = [\Gamma : \widetilde{\Gamma}]$. Let $\widetilde{\rho}\colon \widetilde{\Gamma} \to U(\widetilde{V})$ be a representation of $\widetilde{\Gamma}$. The *induced representation* $\rho = \operatorname{Ind}_{\widetilde{\Gamma}}^{\Gamma}(\widetilde{\rho})$ is the 'natural' extension of $\widetilde{\rho}$ to the larger group Γ, and can be described as follows. Let $R = \{g_1, \ldots, g_n\}$ be a complete set of representatives in Γ of the left cosets in $\Gamma/\widetilde{\Gamma}$. The induced representation can be thought of as acting on the vector space

$$V := \bigoplus_{i=1}^{n} g_i \widetilde{V}.$$

The elements of V can be written as

$$v = \sum_{i=1}^{n} g_i \widetilde{v}_i,$$

where $\widetilde{v}_1, \ldots, \widetilde{v}_n$ belong to $\widetilde{V}$. For every $g \in \Gamma$ and $g_i \in R$ there exists $h_i \in \widetilde{\Gamma}$ and $g_{j(i)} \in R$ such that $gg_i = g_{j(i)} h_i$. The induced representation is then given by the following action of Γ:

$$\rho(g) v := \sum_{i=1}^{n} g_{j(i)} \widetilde{\rho}(h_i) \widetilde{v}_i.$$

The *Frobenius formula* (also called *Mackey formula*) enables us to compute the character $\chi = \operatorname{Tr} \rho$ of the induced representation in terms of the character $\widetilde{\chi} = \operatorname{Tr} \widetilde{\rho}$ by

$$\chi(g) = \sum_{\substack{x \in R \\ x^{-1} g x \in \widetilde{\Gamma}}} \widetilde{\chi}(x^{-1} g x). \tag{2.7}$$

If there exists no $x \in R$ such that $x^{-1} g x \in \widetilde{\Gamma}$, then the sum in (2.7) is empty, in which case we set $\chi(g) := 0$.

2.5 Selberg zeta function and L-function

Let $X = \Gamma \backslash \mathbb{H}$ be a geometrically finite hyperbolic surface and $\rho\colon \Gamma \to U(V)$ a unitary representation of Γ on a finite-dimensional unitary space V.

The L-function (twisted Selberg zeta function) associated to (Γ, ρ) is determined by the initially only formal Euler product

$$L_\Gamma(s, \rho) = \prod_{[\gamma] \in [\Gamma]_p} \prod_{k=0}^{\infty} \det\left(1 - \rho(\gamma) e^{-(s+k)\ell(\gamma)}\right). \tag{2.8}$$

Recall that $\ell(\gamma)$ is the displacement length of γ given by (2.1). The infinite product (2.8) converges compactly on $\{s \in \mathbb{C} : \mathrm{Re}(s) > \delta(X)\}$. In many cases, a meromorphic extension to all of $\mathbb{C}$ can be established, see for instance [26, 41]. If Γ is Schottky group, then $L_\Gamma(s,\rho)$ extends to an entire function, as we explain in Section 2.6 below.

For the one-dimensional trivial representation $\mathbf{1}_\Gamma$, the L-function reduces to the classical Selberg zeta function:

$$Z_\Gamma(s) = L_\Gamma(s, \mathbf{1}_\Gamma) = \prod_{[\gamma]\in[\Gamma]_p} \prod_{k=0}^{\infty} \left(1 - e^{-(s+k)\ell(\gamma)}\right).$$

Let us now assume that $X = \Gamma\backslash\mathbb{H}$ is a non-elementary, geometrically finite hyperbolic surface of *infinite area*. By the work of Guillopé–Zworski [34, 35], we know that $N_X(r) = O(r^2)$ as $r \to \infty$. This in turn allows us to define the following Weierstrass product over resonances:

$$P_X(s) := s^{m(0)} \prod_{\zeta\in\mathcal{R}(X)\smallsetminus\{0\}} \left(1 - \frac{s}{\zeta}\right) \exp\left(\frac{s}{\zeta} + \frac{s^2}{2\zeta^2}\right), \tag{2.9}$$

where $m(0)$ is the multiplicity of $s = 0$ as a resonance of X. P_X is an entire function and its set of zeros, with multiplicities, is equal to $\mathcal{R}(X)$.

With this definition in place, we can now state the following result on the structure of $Z_\Gamma(s)$, which is due to Borthwick–Judge–Perry [13]. The Selberg zeta function $Z_\Gamma(s)$ extends to a meromorphic function of $s \in \mathbb{C}$ and admits the factorization

$$Z_\Gamma(s) = e^{q(s)} G_\infty(s)^{-\chi(X)} \Gamma\left(s - \frac{1}{2}\right)^{n_c} P_X(s), \tag{2.10}$$

where $n_c \geq 0$ is the number of cusps of X, q is a polynomial of degree at most 2, and the function G_∞ is

$$G_\infty(s) = (2\pi)^{-s}\Gamma(s)G(s)^2,$$

G being the Barnes G-function. The zeros of G_∞ are precisely $s = -n$ for $n \in \mathbb{N}_0$, with multiplicity $2n+1$.

It follows immediately that outside the set $\frac{1}{2}(1-\mathbb{N}_0)$, the zeros of $Z_\Gamma(s)$ are (including multiplicities) equal to resonances for X. Therefore, counting zeros of $Z_\Gamma(s)$ and counting resonances for X in a given domain of the complex plane are essentially the same problems. It is precisely for this reason that we use the Selberg zeta function as a tool for understanding the distribution of resonances.

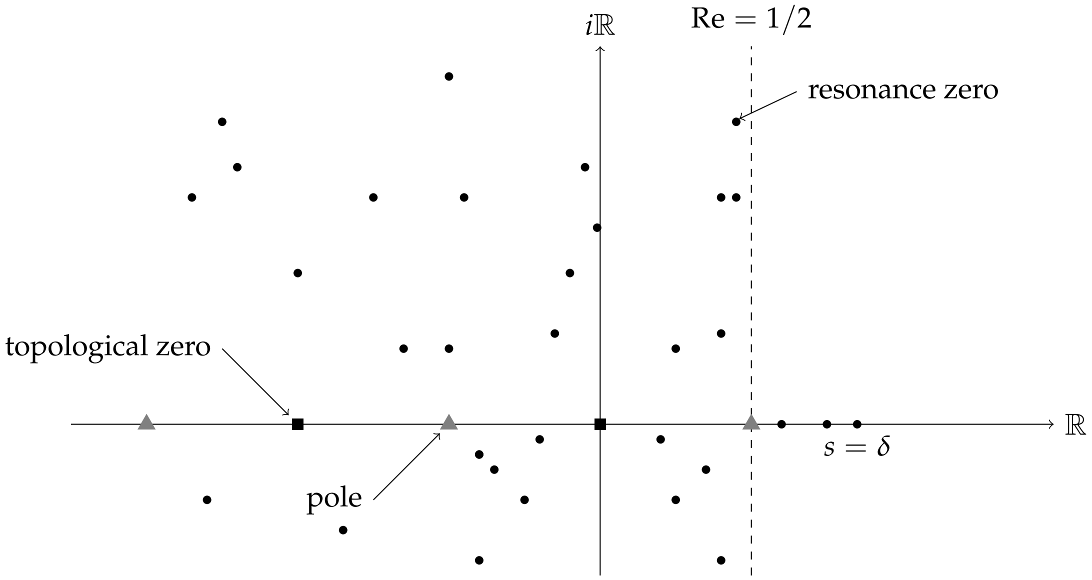

Figure 2.6: Zeros and poles of the Selberg zeta function

2.6 Transfer operators for Schottky groups

The thermodynamic formalism allows us to represent the Selberg zeta function (and more generally, L-functions) as Fredholm determinants of well chosen transfer operators. In this section we consider the specific case of Schottky groups and we introduce the transfer operators needed in Chapters 3 and 4.

We refer to [68, 33, 14] for details regarding the representation of the Selberg zeta function, and to [26] for the extension to twisted transfer operators and L-functions.

Let Γ be a Schottky group, let $(\mathcal{D}_j)_{j=1}^{2m}$ be the family of open disks in $\mathbb{C}$, and $(\gamma_j)_{j=1}^{2m}$ the family of elements in $\mathrm{PSL}_2(\mathbb{R})$ used in a (fixed) geometric construction of Γ (see Section 2.2), and recall that

$$\Gamma = \langle \gamma_1^{\pm 1}, \ldots, \gamma_m^{\pm 1} \rangle$$

is freely presented as a group (thus, the only (omitted) relations are of the form $\gamma\gamma^{-1} = \mathrm{id}$). Set

$$\mathcal{D} := \bigcup_{j=1}^{2m} \mathcal{D}_j.$$

Let $\rho \colon \Gamma \to U(V)$ be a finite-dimensional unitary representation of Γ. The transfer operator $\mathcal{L}_{s,\rho}$ with parameter $s \in \mathbb{C}$ associated to (Γ, ρ) is (initially only formally) given by

$$\mathcal{L}_{s,\rho} := \sum_{j=1}^{2m} 1_{\mathcal{D}_j} \sum_{\substack{i=1 \\ i \neq j+m}}^{2m} \nu_s(\gamma_i), \tag{2.11}$$

where $1_{\mathcal{D}_j}$ denotes the characteristic function of $\mathcal{D}_j$, and for $g \in \mathrm{PSL}_2(\mathbb{R})$, $U \subseteq \mathbb{C}$,

$f\colon U \to \mathbb{C}$ we set

$$\nu_{s,\rho}(g^{-1})f(z) := \big(g'(z)\big)^s \rho(g^{-1}) f(g.z), \tag{2.12}$$

whenever this is well-defined. For the complex powers in (2.11) we use the standard complex logarithm on $\mathbb{C} \setminus \mathbb{R}^-$ (principal arc). A straightforward calculation shows well-definedness on $\mathcal{D}$ (which is what we need in this work).

More concretely, if $z \in \mathcal{D}_j$ and $f\colon \mathcal{D} \to V$, then the transfer operator is given by

$$\mathcal{L}_{s,\rho} f(z) = \sum_{\substack{i=1\\ i\neq j+m}}^{2m} \rho(\gamma_i) \left[(\gamma_i^{-1})'(z)\right]^s f(\gamma_i^{-1}.z). \tag{2.13}$$

So far we have omitted to specify the domain of definition of the transfer operator family $\mathcal{L}_{s,\rho}$, since there is some freedom in the choice of the function space on which $\mathcal{L}_{s,\rho}$ can act. In fact, the precise choice of this space is a crucial ingredient in some proofs. For the sake of definiteness, we will now define a concrete Hilbert space $\mathcal{H}$, which we will take to be the 'default' space for $\mathcal{L}_{s,\rho}$. For each $j \in \{1, \ldots, 2m\}$, let

$$\mathcal{H}_j := H^2(\mathcal{D}_j; V) := \left\{ f\colon \mathcal{D}_j \to V \text{ holomorphic} \;\middle|\; \int_{\mathcal{D}_j} \|f\|_V^2 \,\mathrm{dvol}_V < \infty \right\}$$

denote the space of holomorphic square-integrable V-valued functions on $\mathcal{D}_j$. Here, 'vol' is the standard Lebesgue measure on $\mathbb{C}$. Endowed with the inner product

$$\langle f, g\rangle := \int_{\mathcal{D}_j} \langle f(z), g(z)\rangle_V \,\mathrm{dvol}(z),$$

the space $\mathcal{H}_j$ is a Hilbert space, called the *(Hilbert) Bergman space* on $\mathcal{D}_j$. Let

$$\mathcal{H} := \bigoplus_{j=1}^{2m} \mathcal{H}_j$$

denote the direct sum of the Hilbert spaces $\mathcal{H}_j$, $j = 1, \ldots, 2m$. As usual, we identify tacitly functions

$$f \in \mathcal{H}, \quad f = \bigoplus_{j=1}^{2m} f_j \qquad (f_j \in \mathcal{H}_j)$$

with functions on $\mathcal{D}$.

Note that for all $i, j \in \{1, \ldots, 2m\}$, $i \neq j + m \mod 2m$, we have $\overline{\gamma_i^{-1}(\mathcal{D}_j)} \subset \mathcal{D}_i$. Hence $\gamma_i^{-1}\colon \mathcal{D}_j \to \mathcal{D}_i$ is a holomorphic contraction, the transfer operator $\mathcal{L}_{s,\rho}$ is well-defined as an operator

$$\mathcal{L}_{s,\rho}\colon \mathcal{H} \to \mathcal{H},$$

and as such it is compact and of *trace class*.

The crucial property for all our applications (and the reason why we are interested in thermodynamic formalism) is that the Fredholm determinant of the transfer operators $\mathcal{L}_{s,\rho}$ represents the L-function. Indeed, for all $s \in \mathbb{C}$ we have the identity

$$L_\Gamma(s,\rho) = \det(1 - \mathcal{L}_{s,\rho}). \tag{2.14}$$

In particular, (2.14) immediately implies that $L_\Gamma(s,\rho)$ extends to a holomorphic function on the entire complex plane. We postpone the proof of (2.14) to the Appendix, Subsection A.2. For more details and references regarding the functional-analytic aspects of transfer operators, we refer to Subsection A.1.

2.7 Further Notation

For $z \in \mathbb{C}$ we set $\langle z \rangle := \sqrt{1+|z|^2}$. For any $z \in \mathbb{C}$ and $R > 0$ we let $D(z,R)$ denote the open Euclidean ball in $\mathbb{C}$ with center z and radius R, and we let $\overline{D(z,R)}$ denote its closure. We write $\mathbb{N} = \{1,2,\dots\}$ for the natural numbers starting with 1 and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

We use the standard symbols from analytic number theory $\ll$, $\gg$, $\asymp$, and the O-notation. In particular, we write $f(x) \ll g(x)$ (respectively $f(x) \gg g(x)$) if there exists a constant $C > 0$ such that $|f(x)| \leq C|g(x)|$ (respectively $|f(x)| \geq C|g(x)|$) for all x under consideration. Furthermore, $f(x) \asymp g(x)$ means that $f(x) \ll g(x)$ and $f(x) \gg g(x)$. We frequently use the notations $O_{a,b,c,\dots}$, $\ll_{a,b,c,\dots}$, $\gg_{a,b,c,\dots}$, and $\asymp_{a,b,c,\dots}$ to mean that the implied constant C may depend on the variables $a, b, c, \dots$. We use the latter convention whenever we have to keep track of the variables $a, b, c \dots$.

In a statement or in a proof involving the 'key' variables $X_1, X_2, \dots, X_N$, we write $C = C(X_{i_1}, \dots, X_{i_n})$ to mean that the constant C may depend on $X_{i_1}, \dots, X_{i_n}$, but does not depend on the other variables. Without further specification, C is an absolute constant not depending on any of the key variables. The same convention applies to all the other constants (not just C) such as $\alpha, \alpha_0, \alpha_1, C_0, C_1, c_1, c_2, \dots$.

Chapter 3

Density of resonances for covers of Schottky surfaces

3.1 Introduction and statement of main results

The distribution, localization, and asymptotics of resonances and eigenvalues of the Laplacian of hyperbolic surfaces are of interest in several different areas of research, in particular in spectral theory, harmonic analysis, number theory and mathematical physics. Classically, these questions were studied for compact and finite-area hyperbolic surfaces, where the spectrum of the Laplacian is more directly linked to automorphic forms and Maass cusp forms.

Over the last decade, understanding the distribution of resonances for *infinite-area* surfaces has attracted some attention and continues to be a topic of ongoing research.

For certain applications, an understanding of the distribution of resonances is mandatory. The generalization of Selberg's $\frac{3}{16}$ Theorem by Bourgain–Gamburd–Sarnak [17] and Oh–Winter [64] as well as the progress towards the Zaremba conjecture by Bourgain–Kontorovich [18] are number-theoretic applications in which resonances are indispensable.

The asymptotic distribution of resonances in the finite-area case is fairly well understood by now. However, in the infinite-area case even the most basic questions regarding these asymptotics have remained elusive. With the results that we have obtained in this work, we contribute to the understanding of the distribution of the resonances of hyperbolic surfaces of infinite area.

The main interest in this chapter is to understand how resonance counting functions behave when one passes from a hyperbolic surface X to a finite cover $\widetilde{X}$ of X. Put differently, given a family $(X_j)_j$ of covers of X (a tower of coverings), how does the asymptotic behaviour for resonances vary among members of this family as $j \to \infty$?

To explain our motivation and results in more detail, let us review some of the known results in the literature. A good reference on the spectral theory of infinite-area hyperbolic surfaces is Borthwick's book [14]. For a rather general review of the current knowledge on counting results for resonances, we refer to the recent exhaustive survey of Zworski [92]. We will focus on the hyperbolic case.

Let X be a geometrically finite hyperbolic surface (of finite or infinite area), and let Δ_X denote the (positive) Laplacian on X. The resolvent

$$R_X(s) := \big(\Delta_X - s(1-s)\big)^{-1} \colon L^2(X) \to L^2(X)$$

of Δ_X is defined for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1/2$ and $s(1-s)$ not being an L^2-eigenvalue of Δ_X. From the works of Mazzeo–Melrose [53] and Guillopé–Zworski [34], we know that it extends to a meromorphic family

$$R_X(s) \colon C_c^\infty(X) \to C^\infty(X)$$

on $\mathbb{C}$ with poles of finite rank. The resonances of X are the poles of this meromorphic continuation. The multiplicity of a resonance s is given by the rank of the residue operator, that is,

$$m(s) := \operatorname{rank} \operatorname{res}_{t=s}\big(R_X(t)\big),$$

where $\operatorname{res}_{t=s}$ denotes the residue at s. In the sequel, $\mathcal{R}(X)$ will denote the set of resonances, repeated according to their multiplicities.

We are interested in the asymptotics of two resonance counting functions. The first one counts the number of resonances in growing disks centered at the origin $0 \in \mathbb{C}$:

$$N_X(r) := \#\{s \in \mathcal{R}(X) \colon |s| \le r\}, \qquad r > 0.$$

In the literature, some authors prefer to consider the number of resonances in disks centered at $1/2$:

$$\widetilde{N}_X(r) := \#\left\{s \in \mathcal{R}(X) \colon \left|s - \tfrac{1}{2}\right| \le r\right\}, \qquad r > 0.$$

However, since

$$\widetilde{N}_X(r) \le N_X\left(r + \tfrac{1}{2}\right) \le \widetilde{N}_X(r+1),$$

all counting results considered in this chapter are identical for N_X and $\widetilde{N}_X$ (up to the values of some implied or unspecified constants). It is slightly more convenient for us to work with N_X.

If X is a *compact* hyperbolic surface then all resonances arise from L^2-eigenvalues of Δ_X, and the famous Weyl law gives us the precise asymptotics

$$\frac{1}{2}N_X(r) \sim \#\left\{\lambda < r^2 : \lambda \text{ is } L^2\text{-eigenvalue}\right\} \sim \frac{\operatorname{vol}(X)}{4\pi} r^2 \qquad \text{as } r \to \infty. \tag{3.1}$$

If X is not compact, but still of *finite-area* (i.e, X has at least one cusp but no funnels), the situation is more subtle. In this case, resonances arise not only from L^2-eigenvalues but also as *scattering poles*. By taking into account the contribution of the scattering poles, Selberg [79] managed to establish an analog of the Weyl law for infinite-area hyperbolic surfaces, which reads as

$$\#\left\{\lambda < r^2 : \lambda \text{ is } L^2\text{-eigenvalue}\right\} - \frac{1}{4\pi}\int_{-r}^{r} \frac{\phi'}{\phi}\left(\tfrac{1}{2} + it\right)\, dt \sim \frac{\operatorname{vol}(X)}{4\pi} r^2 \tag{3.2}$$

as $r \to \infty$. Here, ϕ denotes the determinant of the scattering matrix of X.

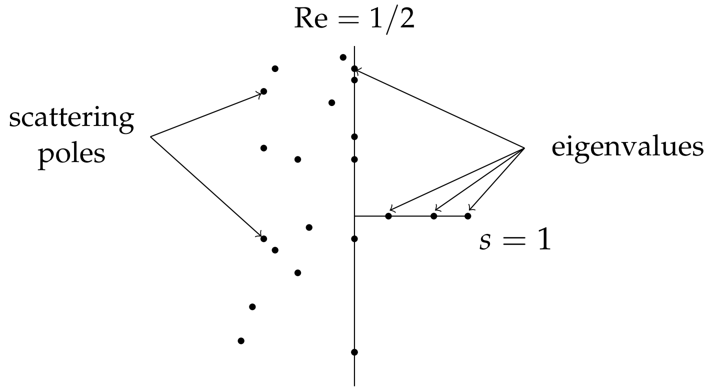

Figure 3.1: Resonances for generic finite-area $X = \Gamma\backslash\mathbb{H}$

Furthermore, Selberg realized that for certain arithmetic surfaces and orbifolds (such as the modular surface $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$ and its congruence covers), the corresponding scattering matrix can be expressed in terms of well-known functions from analytic number theory (such as the Riemann zeta function ζ and Dirichlet L-functions). For instance, the resonances for the Laplacian on $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$ which do not arise from L^2-eigenvalues, are directly related to the non-trivial zeros of ζ by the equation $\zeta(2s) = 0$.

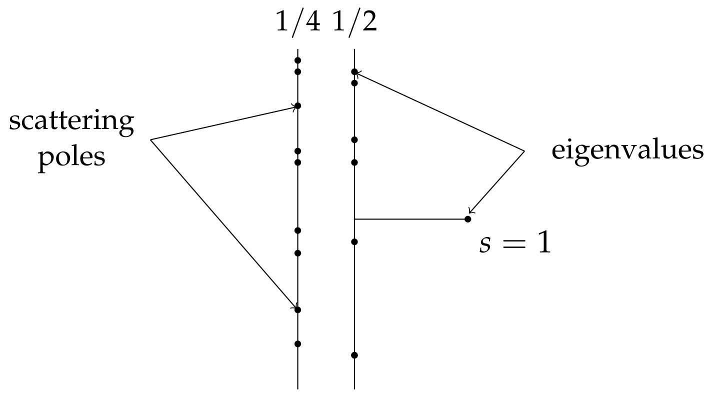

Figure 3.2: Resonances for $X = \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$ under the assumption of the Riemann Hypothesis

Exploiting well-known zero-density results for these number-theoretic functions, Selberg proved that the contribution coming from the scattering matrix is negligible ($\ll r \log r$) compared to the main term. This yields a Weyl law for these arithmetic surfaces, thus showing that they possess an infinitude of eigenvalues (and corresponding Maass cusp forms). For generic (that is, non-arithmetic) finite-area hyperbolic surfaces it is not clear whether the analogue of the Weyl law holds true for the set of L^2-eigenvalues. In fact, according to the Phillips–Sarnak conjecture [70], (3.1) is false for generic surfaces $X = \Gamma\backslash\mathbb{H}$ with cusps. On the other hand, Müller [55] proved that (3.2) yields a *Weyl law for the resonance set*:

$$N_X(r) \sim \frac{\mathrm{vol}(X)}{2\pi} r^2. \tag{3.3}$$

Let us now turn to hyperbolic surfaces X of *infinite area*, which will be the case of interest for the rest of this chapter. For a schematic view on the distribution of resonances for infinite-area hyperbolic surfaces, see Figure 2.4.

In the infinite-area case there is only a finite number of eigenvalues (or possibly none), all of which are contained in the interval $(0, \frac{1}{4})$, see Section 2.3. Moreover, there are no resonances on the critical line $\mathrm{Re}(s) = \frac{1}{2}$, except possibly at $s = \frac{1}{2}$.

Hence, in the infinite-are case, resonances constitute the main spectral data and much less is known about their distribution. For instance, an analogue of the Weyl law for the resonance set such as (3.3) is not known yet and perhaps not even to be expected. Obviously, in any infinite-area analogue of (3.3), one would have to replace $\mathrm{vol}(X) = \infty$ by some other geometric quantity of X.

So far, the only hyperbolic surfaces for which all resonances can be computed explicitly, are the *elementary* hyperbolic surfaces: the hyperbolic plane $\mathbb{H}$, the hyperbolic cylinders

$$C_\ell := \langle z \mapsto e^\ell z \rangle \backslash \mathbb{H},$$

and the parabolic cylinders

$$C_w := \langle z \mapsto z + w \rangle \backslash \mathbb{H}.$$

Their resonance counting functions satisfy

$$N_{\mathbb{H}}(r) \sim r^2, \qquad N_{C_\ell}(r) \sim \frac{\ell}{2} r^2, \qquad N_{C_w}(r) = 1. \tag{3.4}$$

The order of growth of the resonance counting function $N_X(r)$ is well-understood and matches with the Weyl law. Indeed, for any geometrically finite hyperbolic surface X, Guillopé and Zworski [34, 35] showed that

$$N_X(r) \asymp r^2 \tag{3.5}$$

(which subsumes the elementary hyperbolic surfaces by noticing that the lower bound is allowed to be zero). Unfortunately, the methods used in these proofs yield only ineffective constants, with no clear dependence on the surface X.

Using methods from spectral theory, Borthwick [12] proved the bounds (valid for finite and infinite-area hyperbolic surfaces)

$$N_X(r) \leq \left(\frac{0\text{-vol}(X)}{2\pi} + \sum_{j=1}^{n_f} \frac{\ell_j}{4} \right) \exp(1) r^2 + o(r^2) \tag{3.6}$$

and

$$c_k \frac{0\text{-vol}(X)}{2\pi} \left(1 + \frac{2\pi}{0\text{-vol}(X)} \sum_{j=1}^{n_f} \frac{\ell_j}{4} \right)^{-\frac{2}{k}} r^2 \leq N_X(r), \tag{3.7}$$

where $0\text{-vol}(X)$ denotes the 0-volume of X, n_f is the number of funnels, $\ell_1, \ldots, \ell_{n_f}$ are the diameters of the geodesic boundary of the funnels, k is any element of $\mathbb{N}$, and c_k is a constant depending only on k. The limit for the o-term is $r \to \infty$, its speed of convergence may depend on X. Obviously, these bounds have a clear geometric content and the upper bound (3.6) resembles the classical Weyl law when $n_f = 0$ (that is to say, for hyperbolic surfaces of finite area).

The upper bound is sharp in the sense that it agrees (up to some absolute constants) with the asymptotics for the hyperbolic cylinder N_{C_ℓ}. It is not clear whether it is sharp in general.

A more algebraic approach to produce estimates for resonance counting functions was pursued by Jakobson and Naud [38]. They considered convex cocompact hyperbolic surfaces X (no singularities!) of infinite area, thus, geometrically finite hyperbolic surfaces without cusps and with at least one funnel. In this work we refer to such surfaces as *Schottky surfaces*, for reasons explained in Section 2.2.

They restricted further to those Schottky surfaces X for which the fundamental group Γ is conjugate to some subgroup of $\mathrm{PSL}_2(\mathbb{Z})$. These are essentially convex cocompact subroups of arithmetic groups. We call such Schottky groups *integral*. Given such an integral Schottky surface $X = \Gamma\backslash\mathbb{H}$, Jakobson and Naud considered the sequence of finite covers

$$X_q = \Gamma(q)\backslash\mathbb{H}, \qquad q \in \mathbb{N} \text{ prime}$$

where

$$\Gamma(q) := \{g \in \Gamma : g \equiv \mathrm{id} \mod q\}$$

is the 'principal congruence subgroup' of Γ of level q. They showed that

- there exist constants $C_1 > 0$, $q_0 \in \mathbb{N}$ (possibly depending on X) such that for all $q \geq q_0$, q prime, and all $r \geq 1$ we have
$$N_{X_q}(r) \leq C_1[\Gamma : \Gamma(q)] \log(q) r^2, \tag{3.8}$$
- and there exist constants $C_2, r_0 > 0$ (possibly depending on X) such that for all $\varepsilon > 0$ there exists $q_0 \in \mathbb{N}$ such that for all $q \geq q_0$, q prime, and all $r > r_0$ we have
$$N_{X_q}\left(r \cdot (\log q)^{\varepsilon}\right) \geq C_2[\Gamma : \Gamma(q)] r^2. \tag{3.9}$$

We note that if $Y = \Gamma\backslash\mathbb{H}$, $\widetilde{Y} = \widetilde{\Gamma}\backslash\mathbb{H}$ are hyperbolic surfaces of finite area with $\widetilde{\Gamma} \subseteq \Gamma$ then the constant in the Weyl law (3.3) scales by $[\Gamma : \widetilde{\Gamma}]$ when passing from the asymptotics of N_Y to those of $N_{\widetilde{Y}}$:

$$\frac{\mathrm{vol}(\widetilde{Y})}{2\pi} r^2 = [\Gamma : \widetilde{\Gamma}] \frac{\mathrm{vol}(Y)}{2\pi} r^2.$$

Consequently, the estimates (3.9) and (3.8) can be seen as weak versions of the Weyl law for large prime levels q.

Our first main result is a generalization and improvement of this result. To make the statement of our result more compact, let us introduce the notation

$$\mathrm{dcov}(\widetilde{X}, X) := \text{degree of } \widetilde{X} \text{ as a cover of } X.$$

Theorem 3.1. *Let X be a Schottky surface (not necessarily integral).*

(i) *There exists a constant* $C_1 > 0$ *such that for each finite cover* $\widetilde{X}$ *of* X *and all* $r \geq 1$ *we have*
$$N_{\widetilde{X}}(r) \leq C_1 \operatorname{dcov}(\widetilde{X}, X) r^2.$$

(ii) *There exist constants* $C_2, r_0 > 0$ *such that for each finite cover* $\widetilde{X}$ *of* X *and all* $r \geq r_0$ *we have*
$$N_{\widetilde{X}}(r) \geq C_2 \operatorname{dcov}(\widetilde{X}, X) r^2.$$

We list a few remarks about Theorem 3.1 and its relation to the counting results mentioned above.

- If $X = \Gamma\backslash\mathbb{H}$ and $\widetilde{X} = \widetilde{\Gamma}\backslash\mathbb{H}$ then
$$\operatorname{dcov}(\widetilde{X}, X) = [\Gamma : \widetilde{\Gamma}].$$
Thus, Theorem 3.1 shows that the bounding constants for the resonance counting function can be chosen such that they scale exactly as the constant in the Weyl law (3.3) when passing to covers.

- The hyperbolic cylinders C_ℓ are Schottky surfaces. For these, Theorem 3.1 follows easily from (3.4), see the detailed discussion in Section 3.3 below.

- Theorem 3.1 obviously applies to sequences of principal congruence covers of integral Schottky surfaces, and it improves upon the result in [38].

- For any finite cover $\widetilde{X}$ of a Schottky surface X we have the relation
$$\operatorname{dcov}(\widetilde{X}, X) \cdot 0\text{-vol}(X) = 0\text{-vol}(\widetilde{X}).$$
For non-elementary Schottky surfaces (in which case $0\text{-vol}(X) \neq 0$) this relation can be read as
$$\operatorname{dcov}(\widetilde{X}, X) = \frac{0\text{-vol}(\widetilde{X})}{0\text{-vol}(X)}. \tag{3.10}$$
Using (3.10) in Theorem 3.1 and merging the term $0\text{-vol}(X)$ into the constants C_1, C_2 (which are allowed to depend on X) gives
$$C_2\, 0\text{-vol}(\widetilde{X}) r^2 \leq N_{\widetilde{X}}(r) \leq C_1\, 0\text{-vol}(\widetilde{X}) r^2, \tag{3.11}$$
which is reminiscent of (3.3). Consequently, Theorem 3.1 shows that for any sequence of finite covers $(X_j)_j$ of a non-elementary Schottky surface X we have a weak Weyl law
$$N_{X_j}(r) \asymp 0\text{-vol}(X_j) r^2,$$
with implied constants only depending on the base surface X. Unfortunately, we cannot provide any further insight about the unspecified constants C_1, C_2 in Theorem 3.1.

- Theorem 3.1 is stated with $\mathrm{dcov}(\widetilde{X}, X)$ instead of using (the arguably more intriguing variants) (3.10) and (3.11) in order to be able to subsume the hyperbolic cylinders into the statement (note that 0-vol$(C_\ell) = 0$).
- It would certainly be interesting to understand the relation between Theorem 3.1 and Borthwick's bounds (3.6)-(3.7). This is however out of the scope of this work.

The second resonance counting function we investigate is

$$M_X(\sigma, T) := \#\{s \in \mathcal{R}(X) : \mathrm{Re}(s) \geq \sigma,\ |\mathrm{Im}(s) - T| \leq 1\} \tag{3.12}$$

for $\sigma, T \in \mathbb{R}$. For any hyperbolic surface $X = \Gamma\backslash\mathbb{H}$ it is known that the right half plane $\{s \in \mathbb{C} \colon \mathrm{Re}(s) > \delta\}$ does not contain any resonances, where

$$\delta := \delta(X) := \dim_H \Lambda(\Gamma)$$

is the Hausdorff dimension of the limit set of Γ. Thus, $M_X(\sigma, T)$ counts the number of resonances inside the rectangle

$$[\sigma, \delta] + i[T - 1, T + 1].$$

The counting function $M_X(\sigma, T)$ is closely related to

$$N_X(\sigma, T) := \#\{s \in \mathcal{R}(X) : \mathrm{Re}(s) \geq \sigma,\ |\mathrm{Im}(s)| \leq T\},$$

which counts the number of resonances in the vertical strip of the complex plane, parallel to the imaginary axis. According to the fractal Weyl law conjecture [85, 48] for hyperbolic surfaces we should have, as $T \to \infty$,

$$N_X(\sigma, T) \asymp T^{1+\delta}, \tag{3.13}$$

for all $\sigma \in \mathbb{R}$ negative enough. For hyperbolic surfaces of finite area (in this case $\delta = 1$), (3.13) follows from (3.3). For hyperbolic surfaces of infinite area, it is still under investigation.

Clearly, every asymptotics for $M_X(\sigma, T)$ yields one for $N_X(\sigma, T)$. For Schottky surfaces X, Guillopé–Lin–Zworski [33] showed that for any $\sigma \in \mathbb{R}$ there exists a constant $C > 0$ such that for $T > 1$ we have the upper fractal Weyl bound

$$M_X(\sigma, T) \leq C T^{\delta}. \tag{3.14}$$

An improved upper fractal Weyl bound was recently provided by Dyatlov [23], showing that for σ near δ, the exponent in (3.14) can be improved.

In [38] (prior to [23], and with different techniques), Jakobson and Naud also studied the behaviour of the function $M_X(\sigma, T)$ for principal congruence covers of level q of integral Schottky surface in the large q regime. They found functions $\alpha, \beta \colon \mathbb{R} \to \mathbb{R}$ that are strictly concave, increasing, and positive on $(\delta/2, \delta]$ such that for each $\sigma > \delta/2$ there exists $C > 0$ and $q_0 \in \mathbb{N}$ such that for all $T \geq 1$ and all levels $q \geq q_0$, q prime, we have

$$M_{X_q}(\sigma, T) \leq C[\Gamma : \Gamma(q)]^{1-\alpha(\sigma)} \langle T \rangle^{\delta - \beta(\sigma)}. \tag{3.15}$$

For $\sigma \in (\delta/2, \delta]$, the bound (3.15) simultaneously improves upon the fractal Weyl upper bound (3.14) and shows how the bounding constants behave in the level aspect.

Our second main result is a generalization of (3.15) to arbitrary Schottky surfaces and arbitrary finite covers.

Theorem 3.2. *Let X be a Schottky surface, and let $\delta := \delta(X)$ denote the Hausdorff dimension of its limit set. Then there exist functions $\tau_1, \tau_2 \colon \mathbb{R} \to \mathbb{R}$ that are strictly concave, strictly increasing and positive on $(\delta/2, \delta]$ such that for every $\sigma > \delta/2$ there exists $C > 0$ such that for each finite cover $\widetilde{X}$ of X and all $T \in \mathbb{R}$ we have*

$$M_{\widetilde{X}}(\sigma, T) \leq C\, 0\text{-vol}(\widetilde{X}) e^{-\tau_1(\sigma)\ell_0(\widetilde{X})} \langle T \rangle^{\delta - \tau_2(\sigma)}, \tag{3.16}$$

where $\langle T \rangle := \sqrt{1 + |T|^2}$ and $\ell_0(\widetilde{X})$ denotes the minimal length of a periodic geodesic on $\widetilde{X}$.

For hyperbolic cylinders, Theorem 3.2 is vacuously true, since both sides of the estimate (3.16) vanish. The functions τ_1 and τ_2 can be determined in terms of the *topological pressure* of the natural dynamical system on $\Lambda(\Gamma)$ induced by Schottky groups. Details will follow in Section 3.4.

Integrating the bound along T yields the following (weaker) statement, which should be seen as an extension of [59, Theorem 1.1]. It allows us to understand the behaviour of the multiplicative constants in families of covers $(X_j)_j$ of a fixed base surface X.

Corollary 3.3. *With hypotheses and notation as in Theorem 3.2, we have*

$$N_{\widetilde{X}}(\sigma, T) \leq C\, 0\text{-vol}(\widetilde{X}) e^{-\tau_1(\sigma)\ell_0(\widetilde{X})} \langle T \rangle^{1+\delta - \tau_2(\sigma)}.$$

Without the aspect of the transition to covers, Corollary 3.3 is the same as [59, Theorem 1.1], although it is slighlty weaker with respect to τ_2. In [59], Naud showed further properties of the function τ_2. Both [23] and [59, Theorem 1.1] (which is older than [23] and uses different techniques) show that the exponent in the upper fractal Weyl bound (3.14) can be improved near δ. In other words, there exists a function $\sigma \mapsto \tau_2(\sigma)$ as in Theorem 3.2 such that

$$\lim_{T \to \infty} \frac{\log N_X(\sigma, T)}{\log T} \leq 1 + \delta - \tau_2(\sigma).$$

Our result in Corollary 3.3 shows that this improvement ($= \tau_2(\sigma)$) is uniform along a family of covers $(X_j)_j$. We point out that this does not contradict the fractal Weyl conjecture (3.13), since the function τ_2 is only known to be positive on the interval $(\delta/2, \delta]$. According to the fractal Weyl conjecture, we should have

$$\lim_{T \to \infty} \frac{\log N_X(\sigma, T)}{\log T} = 1 + \delta$$

for all $\sigma \in \mathbb{R}$ *negative enough*.

If we assume furthermore that $\widetilde{X} = \widetilde{\Gamma}\backslash\mathbb{H}$ is a *regular* cover of $X = \Gamma\backslash\mathbb{H}$ ('regular' means that $\widetilde{\Gamma}$ is normal in Γ), then Theorem 3.2 can be rephrased in terms of the girth of the Cayley graph of $\Gamma/\widetilde{\Gamma}$. This gives an algebraic reformulation of our second main result, which we discuss in Section 3.6 below.

It turns out that our second main theorem of this chapter, Theorem 3.2, has new implications for the number of L^2-eigenvalues of the Laplacian on Schottky surfaces. As already observed by Jakobson–Naud [38], along sequences of principal congruence covers $(X_q)_q$ the bound (3.15) implies the growth estimate

$$\#\{\lambda \text{ Laplace } L^2\text{-eigenvalue of } X_q\} = O(0\text{-vol}(X_q)^{1-\varepsilon}) \quad \text{as } q \to \infty \tag{3.17}$$

for some $\varepsilon > 0$. A similar estimate (in more generality) was recently shown by Oh [62], and the same conclusion can be deduced from (3.16). In fact, a similar statement holds for more general families of congruence covers, see Proposition 3.11 below.

These estimates complement the recent bounds by Ballmann, Matthiesen and Mondal [5]. Let $\Omega(X)$ denote the set of L^2-eigenvalues of the Laplacian of X inside the interval $(0, 1/4)$ (the so called 'small' eigenvalues of Δ_X). Then their result states that for any geometrically finite hyperbolic surface, one has

$$\#\Omega(X) \leq -\chi(X), \tag{3.18}$$

where $\chi(X)$ denotes the Euler characteristic of X. Using the relation between 0-volume and Euler characteristic (see (2.4)), the result of Ballmann–Matthiesen–Mondal can be rewritten as

$$\#\Omega(X) \leq \frac{0\text{-vol}(X)}{2\pi}. \tag{3.19}$$

By applying Theorem 3.2 to $T = 0$ we obtain the following refinement of (3.19) for Schottky surfaces: there exist constants $C, \tau_1 > 0$ such that for every finite cover $\widetilde{X}$ of X we have

$$\#\Omega(\widetilde{X}) \leq C\, 0\text{-vol}(\widetilde{X}) e^{-\tau_1 \ell_0(\widetilde{X})}.$$

In particular, if $(X_j)_j$ is a sequence of finite covers of X such that $\ell_0(X_j) \to \infty$ as $j \to \infty$, then

$$\frac{\#\Omega(X_j)}{0\text{-vol}(X_j)} \to 0 \quad \text{as } j \to \infty. \tag{3.20}$$

If $\delta(X) < 1/2$ then (3.20) is useless because $\#\Omega(X_j) = 0$ in this case. However, for $\delta(X) > 1/2$, Laplace eigenvalues are known to exist, and they are all contained in $(0, 1/4)$. In Section 3.5 below we provide examples of families $(X_j)_j$ for which the minimal length $\ell_0(X_j)$ of closed geodesics on X_j grows to infinity as $j \to 0$.

Let us provide a brief overview of the structure of this chapter. The proofs of Theorem 3.1 and 3.2 are based on thermodynamic formalism and transfer operator techniques. In particular, we make use of the standard transfer operator $\mathcal{L}_{s,\rho}$ for Schottky surfaces $X = \Gamma\backslash\mathbb{H}$ that are twisted with finite-dimensional unitary representations $\rho\colon \Gamma \to U(V)$, which we introduced in Section 2.6. Recall that the Fredholm determinant of $\mathcal{L}_{s,\rho}$ is known to be equal to the L-function (twisted Selberg zeta function)

$$L_\Gamma(s,\rho) = \prod_{[\gamma]\in[\Gamma]_p} \prod_{k=0}^{\infty} \det\left(1 - \rho(\gamma)e^{-(s+k)\ell(\gamma)}\right), \qquad \mathrm{Re}(s) \gg 1$$

and its analytic continuation to all of $\mathbb{C}$ (see Section 2.6 for notation and Section A.2 for a proof). Thus,

$$L_\Gamma(s,\rho) = \det\left(1 - \mathcal{L}_{s,\rho}\right). \tag{3.21}$$

The special nature of Schottky groups makes it possible to decouple the representation $\rho : \Gamma \to U(V)$ from the action of Γ (see Section 3.3 below). Combining this *decoupling argument* with the known growth estimates on the singular values of $\mathcal{L}_{s,1_{\mathbb{C}}}$, enables us to establish the following result on the growth of L_Γ, which is a key ingredient for the proof of Theorem 3.1.

Proposition 3.4. *Let Γ be a Schottky group. Then there exists $C > 0$ such that for every finite-dimensional unitary representation ρ of Γ and all $s \in \mathbb{C}$ we have*

$$\log\left|L_\Gamma(s,\rho)\right| \le C \cdot \dim\rho \cdot \langle s\rangle^2.$$

For the necessary background knowledge on Schottky surfaces and transfer operators, we refer to Section 2.2 and 2.6, respectively. Sections 3.2 and 3.3 are devoted to the proofs of Proposition 3.4 and Theorem 3.1, respectively. In Section 3.4 we provide a proof of Theorem 3.2. The final two Sections 3.5 and 3.6 discuss examples for (3.20) and a relation of Theorem 3.2 to Cayley graphs, respectively.

3.2 Proof of Proposition 3.4

In this section we provide a proof of Proposition 3.4. We note that if $\rho = \mathbf{1}_{\mathbb{C}}$ is the trivial one-dimensional representation, then Proposition 3.4 coincides with [33, Proposition 3.2]. Using a simple decoupling argument, we can reduce the proof of Proposition 3.4 to an already known estimate on the singular values of the one-dimensional trivial transfer operator. We must carefully show that all the estimates for the proof of Proposition 3.4 are uniform for all finite-dimensional unitary representations.

Throughout this section let Γ be a Schottky group. We use the notation from Section 2.2. In particular, we let $\mathcal{D}_1, \ldots, \mathcal{D}_{2m}$ denote the open disks in $\mathbb{C}$ and $\gamma_1, \ldots, \gamma_{2m}$ the generators (already including the inverses) of Γ used in the geometric construction of Γ, and we use the Hilbert Bergman space from Section 2.6.

Proof of Proposition 3.4. Let V be a finite-dimensional unitary space, $\rho\colon \Gamma \to U(V)$ a unitary representation of Γ, and let $\mathcal{L}_{s,\rho}$ denote the transfer operator associated to (Γ,ρ) (see (2.11)). We consider $\mathcal{L}_{s,\rho}$ as an operator on the Hilbert Bergman space $\mathcal{H}$ defined in Section 2.6. Recall from (2.14) that

$$L_\Gamma(s,\rho) = \det\left(1 - \mathcal{L}_{s,\rho}\right).$$

For all $s \in \mathbb{C}$ the Weyl inequality (see (A.4)) implies that

$$|L_\Gamma(s,\rho)| = |\det\left(1 - \mathcal{L}_{s,\rho}\right)| \leq \det(1 + |\mathcal{L}_{s,\rho}|).$$

In the following we estimate the right hand side further from above. Let us now introduce the decoupling argument, which is the following observation. Roughly speaking, we can separate the unitary action $\rho\colon \Gamma \to \mathrm{U}(V)$ from the action of Γ on $\mathcal{D}$. To materialize this idea, consider the operator

$$U := \bigoplus_{j=1}^{2m} \rho(\gamma_j)$$

which acts on $\mathcal{H}$ by

$$Uf = \bigoplus_{j=1}^{2m} \rho(\gamma_j) f_j$$

for all $f = \bigoplus_{j=1}^{2m} f_j \in \mathcal{H}$. With this notation in place, we can write

$$\mathcal{L}_{s,\rho} = \mathcal{L}_{s,\mathbf{1}_V} \circ U.$$

Now notice that U is unitary, meaning that $U^* = U^{-1}$, since ρ is unitary. Unitarity of U now implies that

$$|\mathcal{L}_{s,\rho}| = U^{-1} \circ |\mathcal{L}_{s,\mathbf{1}_V}| \circ U.$$

In other words, the spectra of the absolute values $|\mathcal{L}_{s,\rho}|$ and $|\mathcal{L}_{s,\mathbf{1}_V}|$ coincide, which leads to (see (A.3))

$$\det(1 + |\mathcal{L}_{s,\rho}|) = \det(1 + |\mathcal{L}_{s,\mathbf{1}_V}|).$$

Let I_V denote the identity operator on V. From $\mathcal{L}_{s,\mathbf{1}_V} = \mathcal{L}_{s,\mathbf{1}_\mathbb{C}} \otimes I_V$ it follows that

$$|\mathcal{L}_{s,\mathbf{1}_V}| = |\mathcal{L}_{s,\mathbf{1}_\mathbb{C}}| \otimes I_V.$$

Therefore, we have

$$\det\left(1 + |\mathcal{L}_{s,\mathbf{1}_V}|\right) = \det(1 + |\mathcal{L}_{s,\mathbf{1}_\mathbb{C}}|)^{\dim V}.$$

On the other hand, by (A.3), we have

$$\det(1 + |\mathcal{L}_{s,\mathbf{1}_\mathbb{C}}|) = \prod_{k=1}^{\infty} (1 + \mu_k(\mathcal{L}_{s,\mathbf{1}_\mathbb{C}})).$$

Thus, we have reduced all matters to an estimate on the singular values $\mu_k(\mathcal{L}_{s,\mathbf{1}_\mathbb{C}})$ of the one-dimensional operator $\mathcal{L}_{s,\mathbf{1}_\mathbb{C}}$. Fourtunately, such an estimate already exists in the literature. By [33, Proof of Proposition 3.2] there exist constants $c_1, c_2 > 0$ (only depending on Γ) such that for all $s \in \mathbb{C}$ and all $k \in \mathbb{N}$ we have

$$\mu_k(\mathcal{L}_{s,\mathbf{1}_\mathbb{C}}) \leq c_1 e^{c_1|s| - c_2 k}.$$

Thus,

$$\prod_{k=1}^{\infty} (1 + \mu_k(\mathcal{L}_{s,\mathbf{1}_\mathbb{C}})) \leq \prod_{k=1}^{\infty} \left(1 + c_1 e^{c_1|s| - c_2 k}\right)$$

Let

$$\ell(s) := \left\lceil \frac{1}{c_2} \left(\log c_1 + c_1 |s| \right) \right\rceil .$$

Then

$$\prod_{k=\ell(s)+1}^{\infty} \left(1 + c_1 e^{c_1|s| - c_2 k} \right) \leq \prod_{m=1}^{\infty} \left(1 + e^{-c_2 m} \right),$$

which is convergent and bounded independently of s.

Further (note that $e^{c_1|s|} \geq 1$ for the second inequality)

$$\begin{aligned}
\prod_{k=1}^{\ell(s)} \left(1 + c_1 e^{c_1|s| - c_2 k} \right) &\leq \left(1 + c_1 e^{c_1|s|} \right)^{\ell(s)} \\
&\leq \left(c_3 e^{c_1|s|} \right)^{\ell(s)} \\
&\leq \exp\left(c_3 + c_4|s| + c_5|s|^2 \right) \\
&\leq \exp\left(c_6 + c_7|s|^2 \right)
\end{aligned}$$

with appropriate constants $c_3, \ldots, c_7 > 0$ (again, only depending on Γ). Thus, there exists $c_8 > 0$ such that

$$|L_\Gamma(s, \rho)| \leq \left(c_8 e^{c_6 + c_7|s|^2} \right)^{\dim V} .$$

It follows that

$$\log |L_\Gamma(s, \rho)| \ll \dim V \cdot \langle s \rangle^2,$$

completing the proof of Proposition 3.4. □

3.3 Proof of Theorem 3.1

In this section we prove Theorem 3.1. Throughout let X be a Schottky surface and let Γ be a Schottky group such that $X = \Gamma \backslash \mathbb{H}$.

Let us first discuss the much easier case when X is elementary, hence a hyperbolic cylinder. Then Γ is generated by a single hyperbolic element, say $\Gamma = \langle \gamma \rangle$. In this case the resonances of X can be computed explicitly (see for instance [14, Proposition 5.2]). The counting function satisfies the asymptotic formula

$$N_X(r) \sim \frac{\ell(\gamma)}{2} r^2.$$

If $\widetilde{X}$ is a cover of X of degree k then $\widetilde{X} = \langle \gamma^k \rangle \setminus \mathbb{H}$. Hence

$$N_X(r) \sim \frac{\ell(\gamma^k)}{2} r^2 = k \frac{\ell(\gamma)}{2} r^2,$$

which establishes an even stronger result than Theorem 3.1. Unfortunately, the same argument does not apply in the non-elementary case.

From now on, we assume that X is non-elementary. We first prove the upper bound stated in Theorem 3.1(i). This proof relies on the following two key ingredients: Suppose that $\widetilde{X} = \widetilde{\Gamma}\backslash\mathbb{H}$ is a finite cover of X, or equivalently, suppose that $\widetilde{\Gamma} \subseteq \Gamma$ is a subgroup of finite index. As explained in Section 2.5, the resonance counting function $N_{\widetilde{X}}(r)$ can be bounded from above by the number of zeros of the Selberg zeta function $Z_{\widetilde{\Gamma}}$ in $\{|s| \leq r\}$. By the Venkov–Zograf factorization formula (Theorem A.3), the Selberg zeta function $Z_{\widetilde{\Gamma}}$ of $\widetilde{X}$ is identical to the L-function of (Γ, λ), where

$$\lambda = \mathrm{Ind}_{\widetilde{\Gamma}}^{\Gamma} \mathbf{1}_{\widetilde{\Gamma}}$$

is the representation of Γ obtained from the induction of the one-dimensional trivial representation of $\widetilde{\Gamma}$. Formally,

$$Z_{\widetilde{\Gamma}}(s) = L_{\Gamma}(s, \lambda).$$

Proposition 3.4 allows us to bound L_Γ (and hence $Z_{\widetilde{\Gamma}}$) in terms of

$$\dim \lambda = [\Gamma : \widetilde{\Gamma}] = \mathrm{dcov}(\widetilde{X}, X)$$

and additional factors that are independent of $\widetilde{\Gamma}$. These estimates result in an upper bound for $N_{\widetilde{X}}(r)$.

The lower bound stated in Theorem 3.1(ii) is then shown by using the upper bound in combination with the so-called Guillopé–Zworski argument [35, 36].

Throughout we assume without loss of generality that the Schottky group Γ is chosen such that the disks (2.5) used in the geometric construction of Γ are contained in $\mathbb{C}$. Moreover, for any finite cover $\widetilde{X}$ of X we choose a representative $\widetilde{\Gamma}$ of its fundamental group such that $\widetilde{\Gamma}$ is a subgroup of Γ.

Before we proceed, let us recall Titchmarsh's Number of Zeros Theorem, which is consequence of the more prominent Jensen's formula, see [89, Chapter III]. This is a standard device from complex analysis which allows us to convert growth estimates for the Selberg zeta function into estimates on the number of its zeros. Recall that

$$D(z_0, R) := \{z \in \mathbb{C} : |z - z_0| < R\}$$

denotes the open Euclidean disk of radius R around z_0.

Lemma 3.5 (Titchmarsh's Number of Zeros Theorem). *Fix $z_0 \in \mathbb{C}$ and $R > 0$. Let $f\colon \overline{D(z_0, R)} \to \mathbb{C}$ be a bounded function, which is holomorphic on $D(z_0, T)$. Assume furthermore that $f(z_0) \neq 0$. Then for all $0 < \eta < 1$, the number of zeros of f inside $D(\eta R, z_0)$ (counted with multiplicities) is bounded from above by*

$$\frac{1}{\log(\eta^{-1})} \left(\max_{|z-z_0|=R} \log|f(z)| - \log|f(z_0)| \right).$$

Proof of Theorem 3.1(i) (upper bound). All constants c_n with $n \in \{1, 2, \dots\}$ that appear during the proof are positive and may depend on Γ (or equivalently, on X). None of these constants depend on any finite cover of X.

Let $\widetilde{X} = \widetilde{\Gamma}\backslash\mathbb{H}$ be a finite cover of X, let $\mathbf{1}_{\widetilde{\Gamma}}\colon \widetilde{\Gamma} \to \mathbb{S}^1$ denote the trivial character of $\widetilde{\Gamma}$, and let

$$\lambda := \operatorname{Ind}_{\widetilde{\Gamma}}^{\Gamma} \mathbf{1}_{\widetilde{\Gamma}}$$

denote its induction to a representation of Γ. Let $s \in \mathbb{C}$. By the Venkov–Zograf factorization formula (Theorem A.3) we have

$$Z_{\widetilde{\Gamma}}(s) = L_{\widetilde{\Gamma}}(s, \mathbf{1}_{\widetilde{\Gamma}}) = L_{\Gamma}(s, \lambda).$$

Recall that

$$\dim \lambda = [\Gamma : \widetilde{\Gamma}] = \operatorname{dcov}(\widetilde{X}, X).$$

From Proposition 3.4 it follows that

$$\log \left| Z_{\widetilde{\Gamma}}(s) \right| \leq c_1 \operatorname{dcov}(\widetilde{X}, X) \langle s \rangle^2. \tag{3.22}$$

In order to convert the growth estimate for $Z_{\widetilde{\Gamma}}$ into an upper bound for the number of resonances, we note that

$$\begin{aligned} N_{\widetilde{X}}(r) &\leq \#\{s \in \mathbb{C}\colon |s| \leq r,\ Z_{\widetilde{\Gamma}}(s) = 0\} \\ &\leq \#\{s \in \mathbb{C}\colon |s-1| \leq r+1,\ Z_{\widetilde{\Gamma}}(s) = 0\}. \end{aligned}$$

Since $L_\Gamma(\cdot, \lambda) = Z_{\widetilde{\Gamma}}$ is analytic on all of $\mathbb{C}$, and $Z_{\widetilde{\Gamma}}(1) > 0$, Titchmarsh's Number of Zeros Theorem (Lemma 3.5) with $z_0 = 1$, $T = 2(r+1)$ and $\eta = 1/2$ yields

$$\begin{aligned} N_{\widetilde{X}}(r) &\leq \frac{1}{\log 2} \left(\log \max \left\{ |Z_{\widetilde{\Gamma}}(s)| : |s-1| = 2(r+1) \right\} - \log Z_{\widetilde{\Gamma}}(1) \right) \\ &\leq \frac{1}{\log 2} \left(c_2 \operatorname{dcov}(\widetilde{X}, X) \langle 2(r+1) \rangle^2 - \log Z_{\widetilde{\Gamma}}(1) \right). \end{aligned}$$

Since $r \geq 1$, we have $\langle 2(r+1) \rangle^2 \ll \langle r \rangle^2 \ll r^2$, and hence

$$N_{\widetilde{X}}(r) \leq c_3 \left(\operatorname{dcov}(\widetilde{X}, X) r^2 - \log Z_{\widetilde{\Gamma}}(1) \right). \tag{3.23}$$

We claim that

$$Z_{\widetilde{\Gamma}}(1) \geq Z_\Gamma(1)^{\dim \lambda}. \tag{3.24}$$

Indeed, since $1 > \delta$, the expression of L-functions as Euler products applies and yields

$$\begin{aligned} Z_{\widetilde{\Gamma}}(1) &= L_\Gamma(1, \lambda) \\ &= \prod_{[g] \in [\Gamma]_p} \prod_{k=0}^{\infty} \det \left(1 - \lambda(g) e^{-(1+k)\ell(\gamma)} \right) \\ &\geq \prod_{[g] \in [\Gamma]_p} \prod_{k=0}^{\infty} \left(1 - e^{-(1+k)\ell(\gamma)} \right)^{\dim \lambda} \\ &= L_\Gamma(1, \mathbf{1})^{\dim \lambda} \\ &= Z_\Gamma(1)^{\dim \lambda}, \end{aligned}$$

which shows (3.24). Thus,

$$-\log Z_{\widetilde{\Gamma}}(1) \leq -\dim \lambda \cdot \log Z_\Gamma(1) = -\operatorname{dcov}(\widetilde{X}, X) \log Z_\Gamma(1) \tag{3.25}$$

Clearly, $-\log Z_\Gamma(1)$ is a positive constant only depending on Γ. Combining (3.23) and 3.25, we conclude that there exists $c_4 > 0$ such that

$$N_{\widetilde{X}}(r) \leq c_4 \operatorname{dcov}(\widetilde{X}, X) r^2$$

for all $r \geq 1$. □

Taking advantage of the already established upper bound for the resonance counting function, we can now prove the lower bound.

Proof of Theorem 3.1(ii) (lower bound). As in the proof of Theorem 3.1(i), the constants c_n with $n \in \{1, 2, \dots\}$ are all positive and may depend on X, but are independent of any finite cover of X.

We take advantage of the following *wave 0-trace formula* provided by Guillopé–Zworski [36], which we recall now. For any function $\varphi \in C_c^\infty\big((0,\infty)\big)$ let

$$\widehat{\varphi}(z) := \int_{-\infty}^{\infty} e^{-ixz} \varphi(x)\, dx.$$

be its Fourier transform. Then, for any non-elementary Schottky surface Y and all test functions $\varphi \in C_c^\infty\big((0,\infty)\big)$ we have

$$\begin{aligned} \sum_{s \in \mathcal{R}(Y)} \widehat{\varphi}\left(i\left(s - \frac{1}{2}\right)\right) &= -\frac{0\text{-vol}(Y)}{4\pi} \int_{-\infty}^{\infty} \frac{\cosh \frac{t}{2}}{\sinh^2 \frac{t}{2}} \varphi(t)\, dt \\ &\quad + \sum_{\ell \in \mathcal{L}(Y)} \sum_{k=1}^{\infty} \frac{\ell}{2 \sinh \frac{k\ell}{2}} \varphi(k\ell), \end{aligned} \tag{3.26}$$

where $\mathcal{L}(Y)$ is the *primitive length spectrum* of Y, that is, the set of lengths of the primitive periodic geodesics on Y (with multiplicities).

Let $\widetilde{X}$ be a finite cover of X. Pick $\varphi_1 \in C_c^\infty\big((0,\infty)\big)$ such that φ_1 is non-negative and $\operatorname{supp} \varphi_1 \subseteq (0, \ell_0(X))$. For $T \in \mathbb{R}$, $T > 0$, we define $\varphi_T \in C_c^\infty\big((0,\infty)\big)$ by

$$\varphi_T(x) := T\varphi_1(Tx).$$

We want to apply the wave 0-trace formula to $\widetilde{X}$ and φ_T with $T \geq 1$. Note that $\operatorname{supp} \varphi_T \subseteq (0, \ell_0(X)/T)$.

Since $\ell_0(\widetilde{X}) \geq \ell_0(X)$, the sum on the right hand side of (3.26) vanishes for all $T \geq 1$:

$$\sum_{\ell \in \mathcal{L}(\widetilde{X})} \sum_{k=1}^{\infty} \frac{\ell}{2 \sinh \frac{k\ell}{2}} \varphi_T(k\ell) = 0.$$

Thus, we arrive at

$$\left| \sum_{s \in \mathcal{R}(\widetilde{X})} \widehat{\varphi}_T\left(i\left(s - \frac{1}{2}\right)\right) \right| = \frac{0\text{-vol}(\widetilde{X})}{4\pi} \int_{-\infty}^{\infty} \frac{\cosh \frac{t}{2}}{\sinh^2 \frac{t}{2}} \varphi_T(t)\, dt. \tag{3.27}$$

The strategy now is to estimate the left hand side of (3.27) from above and below. For the lower bound we note that

$$\int_0^\infty \frac{\cosh(t/2)}{\sinh(t/2)^2}\varphi_T(t)dt = \int_0^{\ell_0(X)} \frac{\cosh(t/2T)}{\sinh(t/2T)^2}\varphi_1(t)dt.$$

From $\cosh(t/2T) \geq 1$ and

$$\sinh(t/2T) = \sum_{k=1}^{\infty} \frac{(t/2T)^{2k+1}}{(2k+1)!} \leq \frac{1}{T}\sum_{k=1}^{\infty} \frac{(t/2)^{2k+1}}{(2k+1)!} = \frac{1}{T}\sinh(t/2)$$

for all $t > 0$ (recall that $T \geq 1$) it follows that

$$\int_0^\infty \frac{\cosh(t/2)}{\sinh(t/2)^2}\varphi_T(t)dt \geq T^2 \int_0^{\ell_0(X)} \frac{1}{\sinh(t/2)^2}\varphi_1(t)dt.$$

Thus, (3.27) can be bounded from below by

$$\left| \sum_{s\in\mathcal{R}(\widetilde{X})} \widehat{\varphi_T}\left(i(s-\frac{1}{2})\right) \right| \geq c_1 0\text{-vol}(\widetilde{X})T^2, \tag{3.28}$$

with

$$c_1 := \frac{1}{4\pi}\int_0^{\ell_0(X)} \frac{1}{\sinh(t/2)^2}\varphi_1(t)dt \in (0,\infty).$$

For an upper bound of (3.27) we let $r \geq 1$, split the sum in the left hand side of (3.27) at r, and estimate

$$\left| \sum_{s\in\mathcal{R}(\widetilde{X})} \widehat{\varphi}_T\left(i\left(s-\frac{1}{2}\right)\right) \right| \leq \sum_{\substack{s\in\mathcal{R}(\widetilde{X})\\ |s|\leq r}} \left|\widehat{\varphi}_T\left(i\left(s-\frac{1}{2}\right)\right)\right| + \sum_{\substack{s\in\mathcal{R}(\widetilde{X})\\ |s|> r}} \left|\widehat{\varphi}_T\left(i\left(s-\frac{1}{2}\right)\right)\right|.$$

We estimate both sums on the right hand side separately.

Since $\varphi_1 \in C_c^\infty\big((0,\ell_0(X))\big)$, iterated integration by parts yields

$$|\widehat{\varphi}_T(z)| = \left|\widehat{\varphi}_1\left(\frac{z}{T}\right)\right| \leq c\left(1+\left|\frac{z}{T}\right|\right)^{-3} \times \begin{cases} \exp\left(\frac{\ell_0(X)}{T}\mathrm{Im}(z)\right) & \text{if } \mathrm{Im}(z) \geq 0 \\ 1 & \text{if } \mathrm{Im}(z) \leq 0, \end{cases} \tag{3.29}$$

for all $z \in \mathbb{C}$ and $T > 0$, where $c > 0$ is a constant depending only on $\ell_0(X)$ and the choice of φ_1.

Recall that for each resonance $s \in \mathcal{R}(\widetilde{X})$ we have

$$\mathrm{Im}\left(i\left(s-\frac{1}{2}\right)\right) = \mathrm{Re}(s) - \frac{1}{2} \leq \delta - \frac{1}{2}.$$

From (3.29) it follows that

$$\left|\widehat{\varphi}_T\left(i\left(s-\frac{1}{2}\right)\right)\right| \leq c\left(1+\left|\frac{s-\frac{1}{2}}{T}\right|\right)^{-3} \leq c_2.$$

Thus,

$$\sum_{\substack{s\in\mathcal{R}(\widetilde{X})\\|s|\le r}} \left|\widehat{\varphi}_T\left(i\left(s-\frac{1}{2}\right)\right)\right| \le c_2 N_{\widetilde{X}}(r). \tag{3.30}$$

Using (3.29) again, we find

$$\sum_{\substack{s\in\mathcal{R}(\widetilde{X})\\|s|>r}} \left|\widehat{\varphi}_T\left(i\left(s-\frac{1}{2}\right)\right)\right| \le c \sum_{\substack{s\in\mathcal{R}(\widetilde{X})\\|s|>r}} \left(1+\left|\frac{s-1/2}{T}\right|\right)^{-3}. \tag{3.31}$$

The sum on right hand side of (3.31) can be bounded by a Stieltjes integral as follows:

$$\begin{aligned}\sum_{\substack{s\in\mathcal{R}(\widetilde{X})\\|s|>r}} \left(1+\left|\frac{s-1/2}{T}\right|\right)^{-3} &\le \sum_{\substack{s\in\mathcal{R}(\widetilde{X})\\|s|>r}} \left(1-\frac{1}{2T}+\left|\frac{s}{T}\right|\right)^{-3}\\ &\le T^3 \sum_{\substack{s\in\mathcal{R}(\widetilde{X})\\|s|>r}} |s|^{-3}\\ &\le T^3 \int_r^\infty \frac{1}{t^3}\,dN_{\widetilde{X}}(t).\end{aligned}$$

Note that the integral converges, since $N_{\widetilde{X}}(t) = O(t^2)$ as $t\to\infty$.

By Theorem 3.1(i) (which is already proven above) there exists $C>0$ (independent of $\widetilde{X}$) such that $N_{\widetilde{X}}(r) \le C\,0\text{-vol}(\widetilde{X})r^2$ for all $r\ge 1$. (Here we use the relation $0\text{-vol}(\widetilde{X}) = \mathrm{dcov}(\widetilde{X},X)\,0\text{-vol}(X)$.) It follows that

$$\begin{aligned}\int_r^\infty \frac{1}{t^3}\,dN_{\widetilde{X}}(t) &= \lim_{R\to\infty} R^{-3}N_{\widetilde{X}}(R) - r^{-3}N_{\widetilde{X}}(r) + 3\int_r^\infty \frac{N_{\widetilde{X}}(t)}{t^4}\,dt\\ &\le r^{-3}N_{\widetilde{X}}(r) + 3C\,0\text{-vol}(\widetilde{X})\int_r^\infty \frac{dt}{t^2}\\ &\le 4C\,0\text{-vol}(\widetilde{X})r^{-1}.\end{aligned}$$

Thus, we have established

$$\sum_{\substack{s\in\mathcal{R}(\widetilde{X})\\|s|>r}} \left|\widehat{\varphi}_T\left(i\left(s-\frac{1}{2}\right)\right)\right| \le c_3\,0\text{-vol}(\widetilde{X})T^3r^{-1} \tag{3.32}$$

for all $r\ge 1$ and $T\ge 1$, where $c_3 := 4C\cdot c$.

Gathering (3.27), (3.30) and (3.32) leads to the inequality

$$c_1\,0\text{-vol}(\widetilde{X})T^2 \le c_2 N_{\widetilde{X}}(r) + c_3\,0\text{-vol}(\widetilde{X})T^3r^{-1}, \tag{3.33}$$

which is valid for all $r\ge 1$ and $T\ge 1$.

Finally set $a := (2c_3)^{-1}c_1 > 0$ and $r_0 := \max\{1, a^{-1}\}$, and notice that these constants only depend on X. For all $r\ge r_0$ we apply (3.33) with $T := ar \ge 1$ to obtain

$$N_{\widetilde{X}}(r) \ge c_4\,0\text{-vol}(\widetilde{X})r^2,$$

where
$$c_4 := \frac{c_1 a^2 - c_3 a^3}{c_2} = \frac{c_1^3}{8 c_2 c_3^2} > 0.$$

This completes the proof of Theorem 3.1(ii). □

3.4 Proof of Theorem 3.2

In this section we provide a proof of Theorem 3.2. This proof follows a route similar to the one taken by Jakobson and Naud for the proof of [38, Theorem 1.3]. The main novelties here are the use of twisted transfer operators and a new separation lemma (Lemma 3.8).

Throughout this section let
$$X = \Gamma \backslash \mathbb{H} \tag{3.34}$$
be a fixed Schottky surface, and $\delta = \delta(X) = \dim \Lambda(\Gamma)$ the Hausdorff dimension of the limit set of Γ.

For any finite cover $\widetilde{X} = \widetilde{\Gamma} \backslash \mathbb{H}$ we can estimate the number $M_{\widetilde{X}}(\sigma, T)$ of resonances of $\widetilde{X}$ in the box
$$R(\sigma, T) := [\sigma, \delta] + i[T-1, T+1]$$
by counting the number of zeros of the Selberg zeta function $Z_{\widetilde{\Gamma}}$ in R, and we can use the identities
$$Z_{\widetilde{\Gamma}}(s) = L_\Gamma(s, \lambda) = \det\left(1 - \mathcal{L}_{s,\lambda}\right), \tag{3.35}$$
where $\lambda = \mathrm{Ind}_{\widetilde{\Gamma}}^{\Gamma} \mathbf{1}_{\widetilde{\Gamma}}$ is the induction of the trivial character of $\widetilde{\Gamma}$ to Γ, and $\mathcal{L}_{s,\lambda}$ is the transfer operator associated to Γ twisted with λ.

However, instead of using $\mathcal{L}_{s,\lambda}$ we will use a suitable power $\mathcal{L}_{s,\lambda}^N$ of this transfer operator. Since we have the identity
$$1 - \mathcal{L}_{s,\lambda}^N = (1 - \mathcal{L}_{s,\lambda})(1 + \mathcal{L}_{s,\lambda} + \cdots + \mathcal{L}_{s,\lambda}^{N-1}), \quad N \in \mathbb{N}$$
and since $\mathcal{L}_{s,\lambda}$ (and all of its iterates) are trace class operators, we obtain
$$\begin{aligned} \det(1 - \mathcal{L}_{s,\lambda}^N) &= \det(1 - \mathcal{L}_{s,\lambda}) \cdot \det(1 + \mathcal{L}_{s,\lambda} + \cdots + \mathcal{L}_{s,\lambda}^{N-1}) \\ &= Z_{\widetilde{\Gamma}}(s) \cdot \det(1 + \mathcal{L}_{s,\lambda} + \cdots + \mathcal{L}_{s,\lambda}^{N-1}). \end{aligned}$$

It follows that the Fredholm determinant of any iterate of $\mathcal{L}_{s,\lambda}$ is just some multiple of the Selberg zeta function of $\widetilde{\Gamma}$. In particular, for any $N \in \mathbb{N}$,
$$M_{\widetilde{X}}(\sigma, T) \le \#\{s \in R(\sigma, T) \colon \det(1 - \mathcal{L}_{s,\lambda}^N) = 0\}.$$

The goal of this section is to derive an improved growth estimate for $\det(1 - \mathcal{L}_{s,\lambda}^N)$, for a suitable choice of N. The bound on the number of resonances is then deduced from Titchmarsh's Number of Zeros Theorem.

The domain of definition on which $\mathcal{L}_{s,\lambda}^N$ acts is a 'refined' Hilbert space parametrized by $h > 0$, and will be introduced in Section 3.4.1 below. During the proof, the variables h and N will be chosen, so as to optimize the estimate.

Throughout let Γ be chosen such that the disks $\mathcal{D}_1, \ldots, \mathcal{D}_{2m}$ (see (2.5)) used in the geometric construction of Γ are all contained in $\mathbb{C}$, let $\gamma_1, \ldots, \gamma_m$ be the associated generators of Γ (see Section 2.2), and set

$$\mathcal{D} := \bigcup_{j=1}^{2m} \mathcal{D}_j.$$

3.4.1 Refined Hilbert spaces and iterates of the transfer operator

We recall from [33] the definition of a family of Hilbert spaces, depending on a parameter $h > 0$, which we use as domain of definition for appropriate powers of the transfer operators.

Throughout let $\Lambda := \Lambda(\Gamma)$ denote the limit set of Γ. For $h > 0$ we let

$$\Lambda(h) := (-h, h) + \Lambda.$$

By [33] we find $h_0 > 0$ such that for all $h \in (0, h_0)$, the set $\Lambda(h)$ is bounded, has finitely many connected components, say $N(h)$ many, its connected components

$$I_p(h), \quad p = 1, \ldots, N(h),$$

are intervals of lengths at most Ch for some $C > 0$ independent of h, each connected component is contained in some connected component of $\mathcal{D}$, and

$$N(h) = O(h^{-\delta}) \quad \text{as } h \searrow 0,$$

where $\delta = \delta(X) = \dim \Lambda$ is the Hausdorff dimension of Λ.

For each $h \in (0, h_0)$ and $p \in \{1, \ldots, N(h)\}$ let $\mathcal{E}_p(h)$ be the open Euclidean disk in $\mathbb{C}$ with center in $\mathbb{R}$ such that

$$\mathcal{E}_p(h) \cap \mathbb{R} = I_p(h),$$

and let

$$\mathcal{E}(h) := \bigcup \mathcal{E}_p(h).$$

For each finite-dimensional unitary space V let $H^2(\mathcal{E}_p(h); V)$ denote the Hilbert Bergman space of V-valued functions on $\mathcal{E}_p(h)$, and let

$$H^2(\mathcal{E}(h); V) := \bigoplus_{p=1}^{N(h)} H^2(\mathcal{E}_p(h); V).$$

A slight adaptation of [33] shows that there exists $N_1 \in \mathbb{N}$ (independent of $h \in (0, h_0)$) such that for all finite-dimensional unitary spaces V, all unitary representations $\rho\colon \Gamma \to U(V)$ and all $N \geq N_1$, the N-th power of $\mathcal{L}_{s,\rho}$ defines on operator on $H^2(\mathcal{E}(h); V)$:

$$\mathcal{L}_{s,\rho}^N\colon H^2(\mathcal{E}(h); V) \to H^2(\mathcal{E}(h); V).$$

Considered as an operator on $H^2(\mathcal{E}(h); V)$, the transfer operator $\mathcal{L}_{s,\rho}^N$ remains to be of trace class, and its Fredholm determinant is identical to the one of $\mathcal{L}_{s,\rho}^N$ as an operator on the Hilbert space from Section 2.6.

We need an explicit formula for the N-th iterate of $\mathcal{L}_{s,\rho}$. To that end, let us introduce some notations. First, for any $n \in \mathbb{N}$ let $[n] := \{1, \ldots, n\}$. Now, for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_N) \in [2m]^N$, set

$$\gamma_\alpha := \gamma_{\alpha_1} \cdots \gamma_{\alpha_N}.$$

Further, let

$$\mathcal{W}_N := \left\{(\alpha_1, \ldots, \alpha_N) \in [2m]^N : \forall\, j \in [N-1]\colon \alpha_{j+1} \neq \alpha_j + m \mod 2m\right\}$$

denote the set of multi-indices in $[2m]^N$ corresponding to the elements in Γ of *reduced* word length N over the alphabet $\{\gamma_1, \ldots, \gamma_{2m}\}$. Finally, for $j \in [2m]$ we set

$$\mathcal{W}_N^j := \{\alpha \in \mathcal{W}_N : \alpha_1 \neq j + m \mod 2m\}.$$

With these notations in place, one can inductively show that we have

$$\mathcal{L}_{s,\rho}^N = \sum_{j=1}^{2m} 1_{\mathcal{D}_j} \sum_{\alpha \in \mathcal{W}_N^j} \nu_{s,\rho}(\gamma_\alpha). \tag{3.36}$$

3.4.2 Separation lemmas

The results of this subsection are crucial in the proof of growth bounds of the Fredholm determinant of $\mathcal{L}_{s,\rho}^N$, see Proposition 3.10 below.

What we prove is roughly speaking the following statement: given two distinct words $\alpha \neq \beta$ of the same length N and $z \in \mathcal{E}(h)$, the images $\gamma_\alpha^{-1}.z$ and $\gamma_\beta^{-1}.z$ lie in different components (that is, they are separated), provided N is of moderate size.

Throughout, X refers to the fixed Schottky surface (3.34). All the implied constants are allowed to depend on the base surface X.

Lemma 3.6. *Let $C > 0$. Then there exists $h_1 \in (0,1)$ (depending on X and C) and $C_1 > 0$ (depending on X, C, h_1) such that for all $j \in \{1, \ldots, 2m\}$, for all $z \in \mathcal{D}_j$, for all $h \in (0, h_1)$, for all $N \in \mathbb{N}$ with $N < C_1 \log h^{-1}$ and for all $\alpha, \beta \in \mathcal{W}_N^j$ the bound*

$$\left|\gamma_\alpha^{-1}.z - \gamma_\beta^{-1}.z\right| < Ch$$

implies $\alpha = \beta$.

Proof. By [38, Lemma 4.4] we find $c > 0$ and $\rho \in (0,1)$ such that for all $j \in \{1, \ldots, 2m\}$, for all $z \in \mathcal{D}_j$, for all $N \in \mathbb{N}$, for all $\alpha, \beta \in \mathcal{W}_N^j$ with $\alpha \neq \beta$ we have

$$\left|\gamma_\alpha^{-1}.z - \gamma_\beta^{-1}.z\right| \geq c\rho^N. \tag{3.37}$$

Let $C > 0$. Suppose that we have $h \in (0,1)$, $j \in \{1,\ldots,2m\}$, $z \in \mathcal{D}_j$, $N \in \mathbb{N}$, $\alpha, \beta \in \mathcal{W}_N^j$ such that $\alpha \neq \beta$ and

$$\left|\gamma_\alpha^{-1}.z - \gamma_\beta^{-1}.z\right| < Ch. \tag{3.38}$$

Combining (3.37) and (3.38) yields $c\rho^N < Ch$, which implies (note that $\log\rho < 0$)

$$N > c_1 + c_2 \log h^{-1}.$$

where

$$c_1 := \frac{\log C - \log c}{\log\rho}, \qquad c_2 := \frac{1}{\log\rho^{-1}} > 0.$$

Now pick $h_1 \in (0,1)$ so small that

$$c_2 > \frac{c_1}{\log h_1^{-1}}$$

and pick

$$C_1 \in \left(0, c_2 - \frac{c_1}{\log h_1^{-1}}\right).$$

Then, for all $h \in (0, h_1)$ we have

$$N > C_1 \log h^{-1}.$$

This completes the proof of the lemma. $\square$

Lemma 3.7. *There exists $C_2 > 0$ (depending on X) such that for all finite covers $\widetilde{X} = \widetilde{\Gamma}\backslash\mathbb{H}$ of X, for all $N \in \mathbb{N}$ with $N < C_2\ell_0(\widetilde{X})$ and for all $\alpha, \beta \in \mathcal{W}_N$ with $\operatorname{Tr}\operatorname{Ind}_{\widetilde{\Gamma}}^{\Gamma}\mathbf{1}_{\widetilde{\Gamma}}(\gamma_\alpha\gamma_\beta^{-1}) \neq 0$ we have $\alpha = \beta$.*

Proof. Let $\|\cdot\|_F$ denote the Frobenius norm on $\mathrm{SL}_2(\mathbb{R})$. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ be hyperbolic. Then we have

$$\begin{aligned} \|g\|_F^2 &= \operatorname{Tr}(g^\top g) = a^2 + b^2 + c^2 + d^2 = (a+d)^2 + (b-c)^2 - 2 \\ &\geq (\operatorname{Tr} g)^2 - 2 = \left(e^{\ell(g)/2} + e^{-\ell(g)/2}\right)^2 - 2 \\ &\geq e^{\ell(g)}, \end{aligned}$$

or equivalently,

$$\|g\|_F \geq e^{\ell(g)/2}. \tag{3.39}$$

Set

$$K := \max\left\{\|\gamma_j\|_F : j \in \{1,\ldots,2m\}\right\}$$

and

$$C_2 := \frac{1}{4\log K}.$$

Notice that $K > 1$, which implies $C_2 > 0$. Let $\widetilde{\Gamma}$ be any subgroup of Γ of finite index. We argue by contradiction. Let $N \in \mathbb{N}$ with $N < C_2 \ell_0(\widetilde{X})$ and suppose that there exist $\alpha, \beta \in \mathcal{W}_N$ such that $\alpha \neq \beta$ and $\operatorname{Tr} \operatorname{Ind}_{\widetilde{\Gamma}}^{\Gamma} \mathbf{1}_{\widetilde{\Gamma}}(\gamma_\alpha \gamma_\beta^{-1}) \neq 0$. Let

$$g := \gamma_\alpha \gamma_\beta^{-1}.$$

Since $\alpha, \beta \in \mathcal{W}_N$, the element g is the product of at most $2N$ matrices, all of which are in the set $\{\gamma_1, \dots, \gamma_{2m}\}$. Using the sub-multiplicativity of the Frobenius norm, we obtain

$$\|g\|_F \leq K^{2N}. \tag{3.40}$$

Since $\operatorname{Tr} \operatorname{Ind}_{\widetilde{\Gamma}}^{\Gamma} \mathbf{1}_{\widetilde{\Gamma}}(g) \neq 0$, it follows from the Frobenius formula (2.7) that g is conjugated to some element in $\widetilde{\Gamma}$. In other words, there exists $p \in \Gamma$ is such that

$$pgp^{-1} \in \widetilde{\Gamma}.$$

Furthermore we know that g is hyperbolic, since $\alpha \neq \beta$. Hence,

$$\ell(g) = \ell\big(pgp^{-1}\big) \geq \ell_0\big(\widetilde{X}\big).$$

Invoking (3.39) gives

$$\|g\|_F \geq e^{\ell(g)/2} \geq e^{\ell_0(\widetilde{X})/2}. \tag{3.41}$$

Combining (3.40) and (3.41) yields

$$N \geq \frac{1}{4 \log K} \ell_0(\widetilde{X}) = C_2 \ell_0(\widetilde{X}),$$

a contradiction. This completes the proof. □

The combination of Lemmas 3.6 and 3.7 yields the following result.

Lemma 3.8. *Let $C > 0$. Then there exists $h_1 \in (0,1)$ and $\varepsilon_0 > 0$ such that for all $h \in (0, h_1)$, for all finite covers $\widetilde{X} = \widetilde{\Gamma}\backslash\mathbb{H}$ of X, for all $N \in \mathbb{N}$ with $N \leq \varepsilon_0\big(\ell_0(\widetilde{X}) + \log h^{-1}\big)$, for all $j \in \{1, \dots, 2m\}$, for all $z \in \mathcal{D}_j$, for all $\alpha, \beta \in \mathcal{W}_N^j$ the following is satisfied: if*

$$\operatorname{Tr} \operatorname{Ind}_{\widetilde{\Gamma}}^{\Gamma} \mathbf{1}_{\widetilde{\Gamma}}\big(\gamma_\alpha \gamma_\beta^{-1}\big) \neq 0 \quad \textit{and} \quad \left| \gamma_\alpha^{-1}.z - \gamma_\beta^{-1}.z \right| < Ch$$

then $\alpha = \beta$.

3.4.3 Bounds on Fredholm determinants

In this subsection, we provide growth bounds on the Fredholm determinants of iterates of the transfer operator $\mathcal{L}_{s,\lambda}$, where $\lambda = \operatorname{Ind}_{\widetilde{\Gamma}}^{\Gamma} \mathbf{1}_{\widetilde{\Gamma}}$ for finite covers $\widetilde{X} = \widetilde{\Gamma}\backslash\mathbb{H}$ of $X = \Gamma\backslash\mathbb{H}$. These estimates together with an application of Titchmarsh's Number of Zeros Theorem allow us to prove Theorem 3.2, see Section 3.4.4 below.

Throughout, $X = \Gamma\backslash\mathbb{H}$ is the fixed Schottky surface (3.34), $\delta = \delta(X)$ denotes the Hausdorff dimension of the limit set of Γ, and all powers of transfer operators are defined on the Hilbert spaces from Section 3.4.1 for some $h \in (0, h_0)$. All constants may depend on X.

Proposition 3.9. *There exists a constant $C > 0$ such that for all finite covers $\widetilde{X}$ of X, for all $N \in \mathbb{N}$, for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \delta$ we have*

$$-\log \left|\det\left(1 - \mathcal{L}_{s,\lambda}^N\right)\right| \leq CN \operatorname{dcov}(\widetilde{X}, X) \frac{\mathrm{Re}(s)}{\mathrm{Re}(s) - \delta} e^{-(\mathrm{Re}(s)-\delta)\ell_0(\widetilde{X})},$$

where $\lambda := \mathrm{Ind}_{\widetilde{\Gamma}}^{\Gamma} \mathbf{1}_{\widetilde{\Gamma}}$ denotes the representation of Γ that is induced by the trivial character $\mathbf{1}_{\widetilde{\Gamma}}$ of $\widetilde{\Gamma}$.

Proof. Since $\mathrm{Re}(s) > \delta$, we can expand the Fredholm determinant using (A.5). We obtain

$$\det\left(1 - \mathcal{L}_{s,\lambda}^N\right) = \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{Tr}\left(\mathcal{L}_{s,\lambda}^{nN}\right)\right).$$

This leads to

$$\left|\det\left(1 - \mathcal{L}_{s,\lambda}^N\right)\right| = \exp\left(-\mathrm{Re} \sum_{n=1}^{\infty} \frac{1}{n} \mathrm{Tr}\left(\mathcal{L}_{s,\lambda}^{nN}\right)\right),$$

and therefore

$$-\log\left|\det\left(1 - \mathcal{L}_{s,\lambda}^N\right)\right| = \mathrm{Re} \sum_{n=1}^{\infty} \frac{1}{n} \mathrm{Tr}\left(\mathcal{L}_{s,\lambda}^{nN}\right) \leq \sum_{n=1}^{\infty} \frac{1}{n} \left|\mathrm{Tr}\left(\mathcal{L}_{s,\lambda}^{nN}\right)\right|. \tag{3.42}$$

By adding extra non-negative terms to the infinite sum, we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n} \left|\mathrm{Tr}\left(\mathcal{L}_{s,\lambda}^{nN}\right)\right| = N \sum_{n=1}^{\infty} \frac{1}{Nn} \left|\mathrm{Tr}\left(\mathcal{L}_{s,\lambda}^{nN}\right)\right| \leq N \sum_{m=1}^{\infty} \frac{1}{m} \left|\mathrm{Tr}\left(\mathcal{L}_{s,\lambda}^{m}\right)\right|. \tag{3.43}$$

Let $L_S(\gamma)$ denote the word length of γ with respect to the generating set $S = \{\gamma_1, \ldots, \gamma_{2m}\}$ of Γ. We denote by $\mathrm{WL}(\gamma) = \min\{L_S(g) : g \in [\gamma]\}$ the minimal word length of any element in the conjugacy class of γ. By (A.8), we have the following formula for the traces of iterates of $\mathcal{L}_{s,\lambda}$:

$$\mathrm{Tr}(\mathcal{L}_{s,\lambda}^m) = \sum_{d|m} \sum_{\substack{[\gamma]\in[\Gamma]_p \\ \mathrm{WL}(\gamma)=d}} d\chi_\lambda(\gamma^{m/d}) \frac{e^{-s\ell(\gamma)\frac{m}{d}}}{1 - e^{-\ell(\gamma)\frac{m}{d}}}, \tag{3.44}$$

where

$$\chi_\lambda(\gamma) := \mathrm{Tr}\,\mathrm{Ind}_{\widetilde{\Gamma}}^{\Gamma} \mathbf{1}_{\widetilde{\Gamma}}(\gamma).$$

Combining (3.42)–(3.44) yields

$$-\log\left|\det\left(1 - \mathcal{L}_{s,\lambda}^N\right)\right| \leq N \sum_{m=1}^{\infty} \frac{1}{m} \sum_{d|m} \sum_{\substack{[\gamma]\in[\Gamma]_p \\ \mathrm{WL}(\gamma)=d}} d\chi_\lambda(\gamma^{m/d}) \frac{e^{-\mathrm{Re}(s)\ell(\gamma)\frac{m}{d}}}{1 - e^{-\ell(\gamma)\frac{m}{d}}}.$$

Introducing the new variable $k = m/d$ and rearranging the above sum accordingly, leads to

$$\begin{aligned} -\log\left|\det\left(1 - \mathcal{L}_{s,\lambda}^N\right)\right| &\leq N \sum_{k=1}^{\infty} \sum_{[\gamma]\in[\Gamma]_p} \frac{1}{k} \chi_\lambda(\gamma^k) \frac{e^{-\mathrm{Re}(s)\ell(\gamma)k}}{1 - e^{-\ell(\gamma)k}} \\ &\leq N \sum_{k=1}^{\infty} \sum_{[\gamma]\in[\Gamma]_p} \chi_\lambda(\gamma^k) \frac{e^{-\mathrm{Re}(s)\ell(\gamma)k}}{1 - e^{-\ell(\gamma)k}}, \end{aligned}$$

where we have dropped the $1/k$-terms in the last estimate. Further,

$$
\begin{aligned}
-\log\left|\det\left(1-\mathcal{L}_{s,\lambda}^{N}\right)\right| &\leq N\sum_{[\gamma]\in[\Gamma]}\chi_\lambda(\gamma)\frac{e^{-\mathrm{Re}(s)\ell(\gamma)}}{1-e^{-\ell(\gamma)}}\\
&\leq \frac{N}{1-e^{-\ell_0(X)}}\sum_{[\gamma]\in[\Gamma]}\chi_\lambda(\gamma)e^{-\mathrm{Re}(s)\ell(\gamma)}.
\end{aligned}
$$

Recall that $\chi_\lambda(\gamma) > 0$ implies that γ is conjugate to an element in $\widetilde{\Gamma}$, and hence $\ell(\gamma) \geq \ell_0(\widetilde{X})$. Moreover, $\chi_\lambda \leq [\Gamma : \widetilde{\Gamma}]$. Thus,

$$
-\log\left|\det\left(1-\mathcal{L}_{s,\lambda}^{N}\right)\right| \leq \frac{N[\Gamma:\widetilde{\Gamma}]}{1-e^{-\ell_0(X)}}\sum_{\substack{[\gamma]\in[\Gamma]\\ \ell(\gamma)\geq\ell_0(\widetilde{X})}} e^{-\ell(\gamma)\mathrm{Re}(s)}. \tag{3.45}
$$

By the prime geodesic theorem [46, Corollary 11.2] we find a constant $C > 0$ such that

$$
\Pi_\Gamma(t) := \#\{[\gamma]\in[\Gamma] : \ell(\gamma)\leq t\} \leq Ce^{\delta t}. \tag{3.46}
$$

Interpreting the right hand side of (3.45) as a Stieltjes integral and using (3.46), we obtain

$$
\sum_{\substack{[\gamma]\in[\Gamma]\\ \ell(\gamma)\geq\ell_0(\widetilde{X})}} e^{-\ell(\gamma)\mathrm{Re}(s)} = \mathrm{Re}(s)\int_{\ell_0(\widetilde{X})}^{\infty} e^{-\mathrm{Re}(s)x}\Pi_\Gamma(x)dx \leq C\frac{\mathrm{Re}(s)e^{-(\mathrm{Re}(s)-\delta)\ell_0(\widetilde{X})}}{\mathrm{Re}(s)-\delta}.
$$

This completes the proof of Proposition 3.9. $\square$

Proposition 3.10. *There exists $\varepsilon_0 > 0$, $N_0 \in \mathbb{N}$, and a map $\eta\colon \mathbb{R}\to\mathbb{R}$ that is strictly concave, strictly increasing and has a unique zero at $\delta/2$ such that for each pair $\sigma_1 > \sigma_0 \geq 0$ and each $T_0 \in \mathbb{R}$ there exists a constant $c > 0$ (depending continuously on T_0) such that for all $T\in\mathbb{R}$ and all $s \in (\sigma_0,\sigma_1)+i(T-T_0,T+T_0)$ and each finite cover $\widetilde{X}$ of X we have*

$$
\log\left|\det\left(1-\mathcal{L}_{s,\lambda}^{2N(T,\widetilde{X})}\right)\right| \leq c\,\mathrm{dcov}(\widetilde{X},X)e^{-\eta(\sigma_0)\ell_0(\widetilde{X})}\langle T\rangle^{\delta-\eta(\sigma_0)}
$$

with

$$
N(T,\widetilde{X}) = \left\lfloor \varepsilon_0\left(\ell_0(\widetilde{X})+\log\langle T\rangle\right)+N_0+1\right\rfloor,
$$

where $\lambda := \mathrm{Ind}_{\widetilde{\Gamma}}^{\Gamma}\mathbf{1}_{\widetilde{\Gamma}}$ denotes the representation of Γ that is induced by the trivial character $\mathbf{1}_{\widetilde{\Gamma}}$ of $\widetilde{\Gamma}$.

Proof. Throughout let

$$
d := \dim\lambda = \mathrm{dcov}(\widetilde{X},X) = [\Gamma:\widetilde{\Gamma}].
$$

and let V be the d-dimensional unitary vector space on which λ represents Γ.

In this proof we consider the iterates of the transfer operator $\mathcal{L}_{s,\lambda}$ as an operator on the Hilbert space $H^2(\mathcal{E}(h);V)$ for a specific h, approximately of size $\langle T\rangle^{-1}$. To be more precise, we fix some parameters:

- Recall the parameter $h_0 > 0$ from Section 3.4.1. We fix $C > 0$ such that for all $h \in (0, h_0)$ and all $p \in \{1, \dots, N(h)\}$ we have $\operatorname{diam} \mathcal{E}_p(h) < Ch$.
- Depending on the choice of C, we fix $h_1 = h_1(C) \in (0, h_0)$ and $\varepsilon_0 = \varepsilon_0(C) > 0$ such that the conclusions of Lemma 3.8 are valid.
- For all $h \in (0, h_1)$ and $j \in \{1, \dots, 2m\}$ let
$$\mathcal{P}_j(h) := \{p \in \{1, \dots, N(h)\} \colon \mathcal{E}_p(h) \subseteq \mathcal{D}_j\}.$$
By [59, Lemma 3.2] we find $N_0 \in \mathbb{N}$ such that for all $N > N_0$, for all $h \in (0, h_1)$, all $j \in \{1, \dots, 2m\}$, all $\alpha \in \mathcal{W}_N^j$, all $p \in \mathcal{P}_j(h)$ there exists a unique $q \in \{1, \dots, N(h)\}$ such that
$$\gamma_\alpha^{-1}\big(\mathcal{E}_p(h)\big) \subseteq \mathcal{E}_q(h) \quad \text{and} \quad d\big(\gamma_\alpha^{-1}(\mathcal{E}_p(h)), \partial\mathcal{E}_q(h)\big) \geq \frac{h}{2}. \tag{3.47}$$
Recall the number $N_1 \in \mathbb{N}$ from Section 3.4.1. We fix N_0 such that
$$e^{-(N_0+1)/\varepsilon_0} < h_1$$
and $N_0 > N_1$.
- Let $T \in \mathbb{R}$. We set $h := e^{-(N_0+1)/\varepsilon_0} \langle T \rangle^{-1}$.
- We set $N := N_{(\widetilde{X},T)} := \left\lfloor \varepsilon_0 \left(\ell_0(\widetilde{X}) + \log h^{-1} \right) \right\rfloor$. Note that
$$N \geq \varepsilon_0 \log h^{-1} - 1 \geq N_0.$$

By [83, Lemma 3.3] and the relation between the trace norm and the Hilbert-Schmidt norm (denoted by $\|\cdot\|_{\mathrm{HS}}$), we get
$$\log \left| \det \left(1 - \mathcal{L}_{s,\lambda}^{2N} \right) \right| \leq \left\| \mathcal{L}_{s,\lambda}^{2N} \right\|_1 \leq \left\| \mathcal{L}_{s,\lambda}^{N} \right\|_{\mathrm{HS}}^2 \tag{3.48}$$
for all $s \in \mathbb{C}$. Thanks to (3.48), proving Proposition 3.10 amounts to estimating the Hilbert-Schmidt norm of $\mathcal{L}_{s,\lambda}^N$, for which there is a nice explicit expression.

We take advantage of an explicit Hilbert basis for the Hilbert Bergman space $H^2(\mathcal{E}(h); \mathbb{C})$. First we fix an orthonormal basis $\mathbf{e}_1, \dots, \mathbf{e}_d$ of V.

For $p \in \{1, \dots, N(h)\}$ let $r_p := r_p(h)$ and $c_p := c_p(h)$ denote the radius and center of $\mathcal{E}_p = \mathcal{E}_p(h)$, respectively. For $q \in \mathbb{N}_0$ set
$$\kappa_{p,q} := \kappa_{p,q}^{(h)} \colon \mathcal{E}_p \to \mathbb{C}, \quad \kappa_{p,q}(z) := \sqrt{\frac{q+1}{\pi r_p^2}} \left(\frac{z - c_p}{r_p} \right)^q.$$

Then $\{\kappa_{p,q}\}_{q \in \mathbb{N}_0}$ is a Hilbert basis for $H^2(\mathcal{E}_p; \mathbb{C})$. We extend each function $\kappa_{p,q}$ to a function $\varphi_{p,q} \colon \mathcal{E} \to \mathbb{C}$ by setting
$$\varphi_{p,q}(z) := \begin{cases} \kappa_{p,q}(z) & \text{for } z \in \mathcal{E}_p \\ 0 & \text{otherwise.} \end{cases}$$

For $k \in \{1,\ldots,d\}$, $p \in \{1,\ldots,N(h)\}$, $q \in \mathbb{N}_0$ define $\psi_{k,p,q} := \psi^{(h)}_{k,p,q} \colon \mathcal{E} \to V$ by

$$\psi_{k,p,q}(z) := \varphi_{p,q}(z)\mathbf{e}_k$$

Then the family of functions

$$\{\psi_{k,p,q} \colon 1 \le k \le d,\ 1 \le p \le N(h),\ q \in \mathbb{N}_0\}$$

is a Hilbert basis for $H^2(\mathcal{E}; V)$, which in turn allows us to write

$$\left\|\mathcal{L}^N_{s,\lambda}\right\|^2_{\mathrm{HS}} = \sum_{q\in\mathbb{N}_0}\sum_{p=1}^{N(h)}\sum_{k=1}^{d}\left\|\mathcal{L}^N_{s,\lambda}\psi_{k,p,q}\right\|^2. \tag{3.49}$$

In what follows, we evaluate step by step the right hand side of (3.49), proceeding from the most inner norm to the final outer series.

Let $k \in \{1,\ldots,d\}$, $p \in \{1,\ldots,N(h)\}$, $q \in \mathbb{N}_0$. Then using the expression for the iterates of the transfer operator (3.36) we have

$$\begin{aligned}
\left\|\mathcal{L}^N_{s,\lambda}\psi_{k,p,q}\right\|^2 &= \int_{\mathcal{E}}\left|\mathcal{L}^N_{s,\lambda}\psi_{k,p,q}(z)\right|^2 \mathrm{dvol}(z)\\
&= \sum_{j=1}^{2m}\int_{\mathcal{E}\cap\mathcal{D}_j}\sum_{\alpha,\beta\in\mathcal{W}^j_N}\left\langle \nu_{s,\rho}(\gamma_\alpha)\psi_{k,p,q}(z), \nu_{s,\rho}(\gamma_\beta)\psi_{k,p,q}(z)\right\rangle \mathrm{dvol}(z)\\
&= \sum_{j=1}^{2m}\sum_{\alpha,\beta\in\mathcal{W}^j_N}\langle\lambda(\gamma_\alpha)\mathbf{e}_k, \lambda(\gamma_\beta)\mathbf{e}_k\rangle\\
&\quad\times\int_{\mathcal{E}\cap\mathcal{D}_j}\left((\gamma_\alpha^{-1})'(z)\right)^s\overline{\left((\gamma_\beta^{-1})'(z)\right)^s}\varphi_{p,q}(\gamma_\alpha^{-1}.z)\overline{\varphi_{p,q}(\gamma_\beta^{-1}.z)}\ \mathrm{dvol}(z).
\end{aligned}$$

Let $\chi_\lambda := \operatorname{Tr}\lambda(\gamma)$ be the character associated with the representation λ. Using

$$\sum_{k=1}^{d}\left\langle\lambda(\gamma_\alpha)\mathbf{e}_k, \lambda(\gamma_\beta)\mathbf{e}_k\right\rangle = \operatorname{Tr}\lambda\left(\gamma_\alpha\gamma_\beta^{-1}\right) = \chi_\lambda\left(\gamma_\alpha\gamma_\beta^{-1}\right),$$

we can evaluate the sum over k in (3.49):

$$\begin{aligned}
\sum_{k=1}^{d}\left\|\mathcal{L}^N_{s,\lambda}\psi_{k,p,q}\right\|^2 &= \sum_{j=1}^{2m}\sum_{\alpha,\beta\in\mathcal{W}^j_N}\chi_\lambda\left(\gamma_\alpha\gamma_\beta^{-1}\right)\\
&\quad\times\int_{\mathcal{E}\cap\mathcal{D}_j}\left((\gamma_\alpha^{-1})'(z)\right)^s\overline{\left((\gamma_\beta^{-1})'(z)\right)^s}\varphi_{p,q}(\gamma_\alpha^{-1}.z)\overline{\varphi_{p,q}(\gamma_\beta^{-1}.z)}\ \mathrm{dvol}(z).
\end{aligned} \tag{3.50}$$

Lemma 3.8 implies that in (3.50) only the summands with $\alpha = \beta$ contribute. Hence

$$\sum_{k=1}^{d}\left\|\mathcal{L}^N_{s,\lambda}\psi_{k,p,q}\right\|^2 = d\sum_{j=1}^{2m}\sum_{\alpha\in\mathcal{W}^j_N}\int_{\mathcal{E}\cap\mathcal{D}_j}\left|\left((\gamma_\alpha^{-1})'(z)\right)^s\right|^2\left|\varphi_{p,q}(\gamma_\alpha^{-1}.z)\right|^2 \mathrm{dvol}(z).$$

For $j \in \{1, \ldots, 2m\}$, $\alpha \in \mathcal{W}_N^j$, $u \in \mathcal{P}_j$ let $v = v(u, \alpha) \in \{1, \ldots, N(h)\}$ be the unique element such that $\gamma_\alpha^{-1}(\mathcal{E}_u) \subseteq \mathcal{E}_v$. Then

$$\begin{aligned}\sum_{p=1}^{N(h)} \sum_{k=1}^{d} \left\| \mathcal{L}_{s,\lambda}^N \psi_{k,p,q} \right\|^2 &= d \sum_{j=1}^{2m} \sum_{\alpha \in \mathcal{W}_N^j} \sum_{u \in \mathcal{P}_j} \sum_{p=1}^{N(h)} \int_{\mathcal{E}_u} \left| \left((\gamma_\alpha^{-1})'(z) \right)^s \right|^2 \left| \varphi_{p,q}(\gamma_\alpha^{-1}.z) \right|^2 \mathrm{dvol}(z) \\ &= d \sum_{j=1}^{2m} \sum_{\alpha \in \mathcal{W}_N^j} \sum_{u \in \mathcal{P}_j} \int_{\mathcal{E}_u} \left| \left((\gamma_\alpha^{-1})'(z) \right)^s \right|^2 \left| \varphi_{v(u,\alpha),q}(\gamma_\alpha^{-1}.z) \right|^2 \mathrm{dvol}(z).\end{aligned}$$

To evaluate the final outer series in (3.49) we use that, for each $v \in \{1, \ldots, N(h)\}$, the series

$$\sum_{q \in \mathbb{N}_0} \varphi_{v,q} \overline{\varphi}_{v,q}$$

converges compactly to the Bergman kernel $B_{\mathcal{E}_v}$ of $\mathcal{E}_v$, and that due to the specific shape of $\mathcal{E}_v$ (a complex ball), the Bergman kernel $B_{\mathcal{E}_v}$ is given by a rather easy explicit formula (which in this case also follows from a straightforward calculation). More precisely, for all $(z, w) \in \mathcal{E} \times \mathcal{E}$ we have

$$\sum_{q \in \mathbb{N}_0} \varphi_{v,q}(z) \overline{\varphi_{v,q}(w)} = B_{\mathcal{E}_v}(z, w) = \frac{r_v^2}{\pi \left(r_v^2 - (w - c_v)(\overline{z} - c_v) \right)^2}. \tag{3.51}$$

Hence, we obtain the final expression for the Hilbert–Schmidt norm of $\mathcal{L}_{s,\lambda}^N$:

$$\begin{aligned}\left\| \mathcal{L}_{s,\lambda}^N \right\|_{\mathrm{HS}}^2 &= \sum_{q \in \mathbb{N}_0} \sum_{p=1}^{N(h)} \sum_{k=1}^{d} \left\| \mathcal{L}_{s,\lambda}^N \psi_{k,p,q} \right\|^2 \\ &= d \sum_{j=1}^{2m} \sum_{\alpha \in \mathcal{W}_N^j} \sum_{u \in \mathcal{P}_j} \int_{\mathcal{E}_u} \left| \left((\gamma_\alpha^{-1})'(z) \right)^s \right|^2 B_{\mathcal{E}_{v(u,\alpha)}} \left(\gamma_\alpha^{-1}.z, \gamma_\alpha^{-1}.z \right) \mathrm{dvol}(z).\end{aligned} \tag{3.52}$$

For all $j \in \{1, \ldots, 2m\}$, all $\alpha \in \mathcal{W}_N^j$, $u \in \mathcal{P}_j$, $z \in \mathcal{E}_u$, the combination of (3.51) with (3.47) yields that

$$\left| B_{\mathcal{E}_{v(u,\alpha)}} \left(\gamma_\alpha^{-1}.z, \gamma_\alpha^{-1}.z \right) \right| \leq c_1 h^{-2} \tag{3.53}$$

where

$$c_1 := \frac{16}{\pi} C$$

depends on X only.

From now on, let $\sigma_1 > \sigma_0 \geq 0$ and $T_0 \in \mathbb{R}$ be fixed, and set

$$D := (\sigma_0, \sigma_1) + i(T - T_0, T + T_0).$$

By [59, below (11)] there exists $c_2 = c_2(\sigma_0, \sigma_1, T_0) > 0$ such that for all $j \in \{1, \ldots, 2m\}$ and all $s \in D$ we have

$$\sup \left\{ \left| \left(\left(\gamma_\alpha^{-1} \right)' (z) \right)^s \right| : \alpha \in \mathcal{W}_N^j,\ z \in \mathcal{E}_u,\ \mathcal{E}_u \subseteq \mathcal{D}_j \right\} \leq c_2 \sup_{x \in I_j} \left(\left(\gamma_\alpha^{-1} \right)' (x) \right)^{\mathrm{Re}(s)}. \tag{3.54}$$

In [59] this estimate is shown for the case that $|\mathrm{Im}(s)| = h^{-1}$. In this case, the constant c_2 is independent of h (and hence of T). Any continuous perturbation of T then results in a continuous perturbation of c_2. Thus, applied to all $s \in D$, the constant c_2 remains independent of T but depends (continuously) on T_0.

Using (3.53) and (3.54) in (3.52) we get for all $s \in D$,

$$\left\| \mathcal{L}_{s,\lambda}^N \right\|_{\mathrm{HS}}^2 \leq c_3 d h^{-2} \sum_{j=1}^{2m} \sum_{\alpha \in \mathcal{W}_N^j} \sup_{x \in I_j} \left(\left(\gamma_\alpha^{-1} \right)' (x) \right)^{2\mathrm{Re}(s)} \sum_{u \in \mathcal{P}_j} \int_{\mathcal{E}_u} 1 \; \mathrm{dvol}(z) \tag{3.55}$$

for a constant $c_3 > 0$ with the same dependencies as c_2. From [88] (see also [33, Section 5] or [14, Proof of Theorem 15.12]) it follows that

$$\#\mathcal{P}_j \leq N(h) \leq c_4 h^{-\delta}$$

for some constant $c_4 > 0$ depending on X only. Further, for any $u \in \{1, \ldots, N(h)\}$,

$$\int_{\mathcal{E}_u} 1 \; \mathrm{dvol}(z) \leq \pi \left(\frac{Ch}{2} \right)^2 .$$

By [59, Lemma 3.1] there exists a map $p \colon \mathbb{R} \to \mathbb{R}$ that is strictly convex, strictly decreasing, and has a unique zero which is precisely δ and a constant $c_5 = c_5(\sigma_0, \sigma_1)$ such that for all $s \in \mathbb{R}$ with $\mathrm{Re}(s) \in (\sigma_0, \sigma_1)$ we have

$$\sum_{j=1}^{2m} \sum_{\alpha \in \mathcal{W}_N^j} \sup_{x \in I_j} \left(\left(\gamma_\alpha^{-1} \right)' (x) \right)^{\mathrm{Re}(s)} \leq c_5 e^{N p(\sigma_0)}. \tag{3.56}$$

The function p is a rescaled variant of the topological pressure of the discrete dynamical system that gives rise to the transfer operator $\mathcal{L}_s$. We refer to [59] for more details.

Using these estimates in (3.55) we get for all $s \in \mathbb{C}$, $\mathrm{Re}(s) \in (\sigma_0, \sigma_1)$,

$$\left\| \mathcal{L}_{s,\lambda}^N \right\|_{\mathrm{HS}}^2 \leq c_6 d h^{-\delta} e^{N p(2\sigma_0)}$$

where c_6 depends on σ_0, σ_1, T_0 and X only, and the dependence on T_0 is continuous.

Inserting the values for h and N as defined in the beginning of this proof, using $d = \mathrm{dcov}(\widetilde{X}, X)$, and combining with (3.48) completes the proof. □

3.4.4 Proof of Theorem 3.2

If X is an elementary Schottky surface (that is to say, a hyperbolic cylinder) and $\widetilde{X}$ is a finite cover of X (hence also a hyperbolic cylinder), then the statement of Theorem 3.2 is vacuously true, since both $M_{\widetilde{X}}(\sigma, T)$ and 0-vol$(\widetilde{X})$ vanish.

Therefore, we assume for the rest of this section that X is non-elementary. Fix $\sigma > \delta/2$, let $\widetilde{X}$ be a finite cover of X, and let $T \in \mathbb{R}$. Recall that the quantity $M_{\widetilde{X}}(\sigma, T)$ which we seek to estimate is the number of resonances in the rectangle

$$R(\sigma, T) := [\sigma, \delta] + i[T-1, T+1].$$

We will use Titchmarsh's Number of Zeros Theorem (Lemma 3.5) to obtain such an estimate. However, Lemma 3.5 is formulated to count zeros in disks, not rectangles. To bypass this issue, we use suitable disks containing $R(\sigma, T)$. Let

$$z_0 := 2 + iT$$

and fix $r_2 > r_1 > 0$ such that

$$R(\sigma, T) \subseteq \overline{D(z_0; r_1)}$$

and $2 - r_2 > \delta/2$, thus

$$\overline{D(z_0; r_2)} \subseteq \{z \in \mathbb{C} : \mathrm{Re}(z) > \delta/2\}.$$

Note that the choice of r_1, r_2 may depend on σ but it is independent of T.

Further let $N(T, \widetilde{X})$ be as in Proposition 3.10 and set $f\colon \mathbb{C} \to \mathbb{C}$,

$$f(s) := \det\left(1 - \mathcal{L}_{s,\lambda}^{2N(T,\widetilde{X})}\right).$$

Then

$$M_{\widetilde{X}}(\sigma, T) \leq \#\{s \in \mathcal{R}(\widetilde{X}) : s \in \overline{D(z_0; r_1)},\ f(s) = 0\}.$$

Titchmarsh's Number of Zeros Theorem yields

$$M_{\widetilde{X}}(\sigma, T) \leq \frac{1}{\log(r_2/r_1)}\left(\log \max_{|s-z_0|=r_2} |f(s)| - \log|f(z_0)|\right). \tag{3.57}$$

Since $\mathrm{Re}(z_0) = 2 > \delta$ we can use Proposition 3.9 to estimate the second summand in (3.57). To estimate the first summand, we use Proposition 3.10 with $\sigma_0 := 2 - r_2$, $\sigma_1 := 2 + r_2$, $T_0 = r_2$. Thus, there is a function $\eta\colon \mathbb{R} \to \mathbb{R}$ with the properties as stated in Proposition 3.10 and constants $c_1, c_2 > 0$ depending on σ (and X) only such that

$$M_{\widetilde{X}}(\sigma, T) \leq c_1\, 0\text{-vol}(\widetilde{X}) e^{-\eta(\sigma_0)\ell_0(\widetilde{X})} \langle T\rangle^{\delta-\eta(\sigma_0)} + c_2\, 0\text{-vol}(\widetilde{X}) N(T, \widetilde{X}) e^{-(2-\delta)\ell_0(\widetilde{X})}.$$

Recall from Proposition 3.10 that

$$N(T, \widetilde{X}) \approx c_3 \ell_0(\widetilde{X}) + c_4 \log\langle T\rangle + c_5$$

for certain constants $c_3, c_4, c_5 > 0$ depending on X only. Since

$$0 \leq \log\langle T\rangle \leq c_\varepsilon \langle T\rangle^\varepsilon$$

for all $\varepsilon > 0$, and

$$\ell_0(\widetilde{X}) e^{-(2-\delta)\ell_0(\widetilde{X})}$$

is bounded as $\ell_0(\widetilde{X}) \to \infty$, we find $\tau_1(\sigma) > 0$, $\tau_2(\sigma) \in (0, \delta)$, $c > 0$ depending on σ and X only such that

$$M_{\widetilde{X}}(\sigma, T) \leq c\, 0\text{-vol}(\widetilde{X}) e^{-\tau_1(\sigma)\ell_0(\widetilde{X})} \langle T\rangle^{\delta-\tau_2(\sigma)}.$$

Due to the properties of η, the functions $\tau_j\colon \sigma \mapsto \tau_j(\sigma)$ $(j = 1, 2)$ can be chosen as stated in Theorem 3.2. This completes the proof of Theorem 3.2.

3.5 Examples for covers with long shortest geodesics

The goal of this section is to give examples of sequences $(X_j)_j$ of finite covers of Schottky surfaces, for which $\ell_0(X_j)$ tends to infinity as $j \to \infty$. For such sequences (3.20) holds true. If we assume the stronger condition that $\ell_0(X_j)$ grows logarithmically with the zero-volume of X_j, i.e. if we have $\ell_0(X_j) \gg \log\big(0\text{-vol}(X_j)\big)$, then one obtains a power-savings estimate for the number of L^2-eigenvalues analogous to (3.17). That is, there exists some $\varepsilon > 0$ only depending on Γ such that

$$\#\Omega(X_j) = \#\{\lambda \text{ Laplace } L^2\text{-eigenvalue of } X_j\} = O(0\text{-vol}(X_j)^{1-\varepsilon}) \quad \text{as } j \to \infty \tag{3.58}$$

Recall that a cover $\widetilde{X} = \widetilde{\Gamma}\backslash\mathbb{H}$ of $X = \Gamma\backslash\mathbb{H}$ is called *abelian* if $\widetilde{\Gamma}$ is a finite normal subgroup of Γ, and the quotient group $\Gamma/\widetilde{\Gamma}$ is abelian. We point out that for any abelian sequence $(X_j)_j$, the minimal length $\ell_0(X_j)$ remains bounded as $j \to \infty$. (To see this, see for instance Figure 4.2 for a general intuition, or combine Theorem 3.2 with Theorem 4.2 in Chapter 4.) Hence, along sequences of abelian covers such a growth behavior is not possible.

This suggests that we should be looking for regular covers $\widetilde{\Gamma}\backslash\mathbb{H} \to \Gamma\backslash\mathbb{H}$ with non-abelian quotient group $\mathbf{G} = \Gamma/\widetilde{\Gamma}$. 'Congruence' surfaces are good candidates for covers having long shortest geodesics, since the corresponding quotient groups $\mathbf{G}$ (also called Galois groups) are subgroups of the highly non-abelian group $\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})$. The term 'highly non-abelian' is justified by the fact that the dimension of non-trivial representations of $\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})$ is typically very large, as opposed to abelian groups, whose representations are all one-dimensional.

Let $X = \Gamma\backslash\mathbb{H}$ be an integral Schottky surface, thus, $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$. We consider X and Γ to be fixed throughout this section. For $q \in \mathbb{N}$ we consider (allowing a slight abuse of notation for notational convenience) the following three families of congruence subgroups:

$$\begin{aligned}
\Gamma_0(q) &:= \{g \in \Gamma : g \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \mod q\}\,, \\
\Gamma_1(q) &:= \{g \in \Gamma : g \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \mod q\}\,, \\
\Gamma_2(q) &:= \{g \in \Gamma : g \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \mod q\} = \Gamma(q).
\end{aligned}$$

For $j \in \{0, 1, 2\}$, we set

$$X_j(q) := \Gamma_j(q)\backslash\mathbb{H}.$$

As shown in Proposition 3.11 below, along any sequence $(X_q)_{q\in\mathbb{N}}$ of covers of X sandwiched between $(X_0(q))_{q\in\mathbb{N}}$ and $(X_2(q))_{q\in\mathbb{N}}$, an analog of (3.17) holds. The sequence $(X_1(q))_{q\in\mathbb{N}}$ is one such example. For the sequence $(X_2(q))_q$, Proposition 3.11 recovers the result by Jakobson–Naud.

Proposition 3.11. *For each $q \in \mathbb{N}$ let Γ_q be a Schottky group such that $\Gamma_2(q) \subseteq \Gamma_q \subseteq \Gamma_0(q)$ and set $X_q := \Gamma_q\backslash\mathbb{H}$. Then there exists functions $\alpha, \beta\colon \mathbb{R} \to \mathbb{R}$ that are strictly concave, increasing, and positive on $(\delta/2, \delta]$ such that for each $\sigma > \delta/2$ there exists $C > 0$ such that for all $T \geq 1$ and all $q \in \mathbb{N}$ (not necessarily prime), we have*

$$M_{X_q}(\sigma, T) \leq C[\Gamma : \Gamma_q]^{1-\alpha(\sigma)}\langle T\rangle^{\delta-\beta(\sigma)}. \tag{3.59}$$

In particular, there exists $\alpha > 0$ *such that the number of* L^2*-eigenvalues satisfies*

$$\#\Omega(X_q) = O\big([\Gamma : \Gamma_q]^{1-\alpha}\big) \qquad \text{as } q \to \infty.$$

Proposition 3.11 follows immediately from Proposition 3.13 below in combination with Theorem 3.2. Before we discuss Proposition 3.13, we present the following lemma on the growth of the covers.

Lemma 3.12. *For* $j \in \{0, 1, 2\}$ *we have*

$$[\Gamma : \Gamma_j(q)] \asymp q^{j+1} \quad \text{as } q \to \infty, q \text{ prime.}$$

Proof. Let $q \in \mathbb{N}$ be prime. Let

$$\pi_q : \Gamma \to \mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z}), \quad g \mapsto g \mod q.$$

For $j \in \{0, 1, 2\}$ we let $H_j(q)$ denote the subgroup of $\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})$ given by

$$H_0 := \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}, \quad H_1 := \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}, \quad H_2 := \{\mathrm{id}\}.$$

Then

$$\Gamma_j(q) := \pi_q^{-1}\big(H_j(q)\big), \quad (j \in \{0, 1, 2\}).$$

By [30, Section 2], the map π_q is surjective if q is sufficiently large. Thus, the isomorphism theorems for groups show that for all such sufficiently large q and each $j \in \{0, 1, 2\}$ we have

$$[\Gamma : \Gamma_j(q)] = [\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z}) : H_j(q)] = \frac{|\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})|}{|H_j(q)|}. \tag{3.60}$$

As is well-known, $|\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})| = q(q^2-1)$. Obviously, $|H_2(q)| = 1$ and $|H_1(q)| = q$. Since q is prime, $\mathbb{Z}/q\mathbb{Z}$ is a field and hence contains $q-1$ multiplicatively invertible elements. Thus, there are $q-1$ possibilities for the pair of diagonal entries of an element of H_0. Hence, $|H_0| = q(q-1)$. Using these element counts in (3.60) completes the proof. □

Proposition 3.13. *Under the hypotheses of Proposition 3.11 there exists* $c_0 > 0$ *such that for all* $q \in \mathbb{N}$ *we have*

$$\ell_0(X_q) \geq c_0 \log[\Gamma : \Gamma_q].$$

Proof. For any $q \in \mathbb{N}$ we have

$$[\Gamma : \Gamma_0(q)] \leq [\Gamma : \Gamma_q] \leq [\Gamma : \Gamma_2(q)] \leq |\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})| = q^3 \prod_{\substack{p \text{ prime} \\ \text{divisor of } q}} \left(1 - \frac{1}{p^2}\right) < q^3.$$

Thus, it suffices to establish the existence of $c_0 > 0$ such that

$$\ell_0(X_q) \geq c_0 \log q \tag{3.61}$$

for all $q \in \mathbb{N}$. Since for each $q \in \mathbb{N}$ the group Γ_q is contained in $\Gamma_0(q)$, the shortest geodesic on X_q is at least as long as the shortest geodesic on $X_0(q)$. Hence, to establish (3.61) it suffices to prove the existence of $c_0 > 0$ such that for all $q \in \mathbb{N}$ we have

$$\ell_0\big(X_0(q)\big) \geq c_0 \log q. \tag{3.62}$$

To that end let $q \in \mathbb{N}$ and let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q)$$

be hyperbolic. Since, necessarily, $|b| \geq 1$ and $|c| \geq q$, it follows that

$$|\operatorname{Tr} g| = |a+d| \geq \frac{1}{2}\sqrt{|ad|} = \frac{1}{2}\sqrt{|1+bc|} \geq \frac{1}{2}\sqrt{q-1} \geq \sqrt{\frac{q}{8}}.$$

Thus,

$$\ell(g) \geq 2\log\frac{|\operatorname{Tr} g|}{2} \geq \log\frac{q}{32}.$$

Hence, we get

$$\ell_0\big(X_0(q)\big) \geq c_0 \log q$$

for some suitably small constant $c_0 > 0$, completing the proof. □

3.6 Regular covers and Cayley graphs

Throughout this section let $X = \Gamma\backslash\mathbb{H}$ be a fixed non-elementary Schottky surface, let S be a fixed set of generators for Γ and suppose that S is symmetric (i. e., $S^{-1} = S$). For convenience, we suppose that $S = \{\gamma_1, \ldots, \gamma_{2m}\}$ is the set of generators arising from a geometric construction of Γ, see Section 2.2.

Let $\widetilde{X} = \widetilde{\Gamma}\backslash\mathbb{H}$ be a finite *regular* cover of X, that is, $\widetilde{\Gamma}$ is normal in Γ. Let $\mathbf{G} := \Gamma/\widetilde{\Gamma}$ be the quotient group. The group $\mathbf{G}$ is also called the Galois group of the covering $\widetilde{X} \to X$. Let $\pi\colon \Gamma \to \mathbf{G}$ be the natural projection. We associate to the pair $(X, \widetilde{X})$ the Cayley graph

$$\mathcal{G} := \operatorname{Cay}\left(\mathbf{G}, \pi(S)\right)$$

of $\mathbf{G}$ with respect to $\pi(S)$. Let us recall the construction of Cayley graphs. Note that $\pi(S)$ is a symmetric generating set for $\mathbf{G}$. The vertices of $\mathcal{G}$ are given by the elements of $\mathbf{G}$. Two vertices $x, y \in \mathbf{G}$ are connected if and only if $xy^{-1} \in \pi(S)$. Since $\pi(S)$ is a symmetric generating set for $\mathbf{G}$, one can easily verify that $\mathcal{G}$ is a simple, connected graph. Recall that the girth of $\mathcal{G}$, denoted by $\operatorname{girth}(\mathcal{G})$, is the length of the shortest cycle in $\mathcal{G}$ or, equivalently, the length of the shortest non-trivial relation in the group $\mathbf{G}$ with respect to the generating set $\pi(S)$.

In this section we show the following bound of the resonance counting function $M_{\widetilde{X}}$, which is essentially a corollary of Theorem 3.2, and can be seen as an algebraic reformulation of it.

Corollary 3.14. *Let $\widetilde{X} \to X$ be a finite regular cover and let $\mathcal{G}$ be the associated Cayley graph. Then for all $\sigma > \delta/2$ and $T \in \mathbb{R}$ we have*

$$M_{\widetilde{X}}(\sigma, T) \leq C|\mathbf{G}|e^{-\alpha_1 \operatorname{girth}(\mathcal{G})}\langle T\rangle^{\delta-\alpha_2},$$

for constants $C, \alpha_1, \alpha_2 > 0$ depending solely on σ and X.

Remark 3.15. (i) *If* $\mathbf{G}$ *is abelian (i.e. the covering* $\widetilde{X} \to X$ *is abelian), then* $\mathrm{girth}(\mathcal{G}) \leq 4$*, since* $\mathcal{G}$ *must contain a 4-cycle* $(a + b - a - b = 0)$*. By contrast, Cayley graphs* $\mathcal{G}$ *of the group* $\mathbf{G} = \mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})$ *with* q *prime have logarithmic girth, i.e.* $\mathrm{girth}(\mathcal{G}) \gg \log |\mathbf{G}|$*, see [30, Section 2].*

(ii) *Let* $(X_j)_j$ *be a sequence of finite covers of a Schottky surface* X *and let* $(\mathcal{G}_j)_j$ *be the associated sequence of Cayley graphs. Corollary 3.14 shows that if* $\mathrm{girth}(\mathcal{G}_j) \to \infty$*, then the number of resonances satisfy*

$$\frac{M_{X_j}(\sigma, T)}{|\mathbf{G}_j|} \to 0 \quad as \quad j \to \infty,$$

for fixed $\sigma > \delta/2$ *and* $T \in \mathbb{R}$*.*

For $\gamma \in \Gamma$ let $L_S(\gamma)$ denote the minimal word length of γ over the alphabet S (i. e., the representing word of γ does not contain neighboring pairs of mutually inverses). Further, let

$$\mathrm{WL}(\gamma) := \min_{g \in [\gamma]} L_S(g)$$

be the shortest word length of a representative in the conjugacy class of γ. Note that $\mathrm{WL}(\gamma) = \mathrm{WL}_S(\gamma)$ depends on the set S of generators of the group Γ. While different choices of the generating sets typically lead to different word-lengths, all the word-lengths are in some sense equivalent. Indeed, given another generating set S', there is a constant $C > 0$ such that

$$C^{-1} \cdot \mathrm{WL}_S(\gamma) \leq \mathrm{WL}_{S'}(\gamma) \leq C \cdot \mathrm{WL}_S(\gamma),$$

since every element in S' can be written as a finite word in the alphabet S (and vice-versa). We will drop the set S from the notation, assuming it to be fixed. The crucial (and less obvious) observation for the proof of Corollary 3.14 is that $\mathrm{WL}(\gamma)$ is controlled by the hyperbolic displacement length $\ell(\gamma)$.

Lemma 3.16. *There exists a constant* $C > 0$ *(depending only on* Γ *and* S*) such that for all* $\gamma \in \Gamma \setminus \{\mathrm{id}\}$ *we have*

$$\mathrm{WL}(\gamma) \leq C \cdot \ell(\gamma).$$

Proof. Let $\overline{\mathcal{N}} \subset \mathbb{H}$ be the *Nielsen region* of $X = \Gamma \setminus \mathbb{H}$, that is, the union of all geodesic arcs connecting two points in the limit set $\Lambda(\Gamma)$ of Γ. Let $\mathcal{N} := \Gamma \setminus \overline{\mathcal{N}}$ denote the *convex core* of X. Since X is a Schottky surface and hence convex co-compact, $\mathcal{N}$ is compact. Let $\widetilde{\mathcal{N}}$ be a compact subset of $\overline{\mathcal{N}}$ that contains at least one representative for each point in $\mathcal{N}$. Let $d_{\mathbb{H}}$ denote the hyperbolic metric on $\mathbb{H}$.

By Knopp–Sheingorn [44] we find constants $c_1, c_2 > 0$ such that for every $\gamma \in \Gamma \setminus \{\mathrm{id}\}$ we have

$$L_S(\gamma) \leq c_1 d_{\mathbb{H}}(\gamma i, i) + c_2.$$

Since $\widetilde{\mathcal{N}}$ is compact we find $c_3 > 0$ such that for all $z' \in \widetilde{\mathcal{N}}$ we have $d_{\mathbb{H}}(z', i) \leq c_3$. Now let $z \in \overline{\mathcal{N}}$ be arbitrary. Clearly, there exists $h \in \Gamma$ such that $z' := hz \in \widetilde{\mathcal{N}}$. Note that

$$\mathrm{WL}(\gamma) \leq L_S(h\gamma h^{-1}) \leq c_1 \cdot d_{\mathbb{H}}(h\gamma h^{-1} i, i) + c_2$$

Using the triangle inequality and exploiting left-invariance of $d_{\mathbb{H}}$ leads to

$$\begin{aligned} d_{\mathbb{H}}(h\gamma h^{-1}i, i) &\leq d_{\mathbb{H}}(h\gamma h^{-1}i, h\gamma z) + d_{\mathbb{H}}(h\gamma z, hz) + d_{\mathbb{H}}(hz, i) \\ &= d_{\mathbb{H}}(h^{-1}i, z) + d_{\mathbb{H}}(\gamma z, z) + d_{\mathbb{H}}(z', i) \\ &= d_{\mathbb{H}}(i, z') + d_{\mathbb{H}}(\gamma z, z) + d_{\mathbb{H}}(z', i) \\ &= d_{\mathbb{H}}(\gamma z, z) + 2d_{\mathbb{H}}(z', i) \\ &\leq d_{\mathbb{H}}(\gamma z, z) + 2c_3. \end{aligned}$$

Thus, there exists $c_4 > 0$ such that for every $z \in \overline{\mathcal{N}}$ and every $\gamma \in \Gamma \setminus \{\mathrm{id}\}$ we have

$$\mathrm{WL}(\gamma) \leq c_1 d_{\mathbb{H}}(\gamma z, z) + c_4.$$

For each $\gamma \in \Gamma \setminus \{\mathrm{id}\}$, there is an element, say η, in the conjugacy class $[\gamma]$ such that the geodesic on $\mathbb{H}$ connecting the two fixed points of η passes through $\widetilde{\mathcal{N}}$. Let $\alpha(\gamma)$ be the geodesic arc connecting the two fixed points of γ. Thus, there is $z \in \alpha(\gamma) \subset \overline{\mathcal{N}}$ such that $\ell(\gamma) = \ell(\eta) = d_{\mathbb{H}}(\eta z, z)$. Since $\ell(\gamma) = \ell(\eta)$ is bounded from below by $\ell_0(X) > 0$, we obtain

$$\mathrm{WL}(\gamma) \leq c_1 \ell(\gamma) + c_4 \leq c_5 \ell(\gamma)$$

for a constant $c_5 > 0$ depending on Γ only. $\square$

Proof of Corollary 3.14. In view of Theorem 3.2 it suffices to show that

$$\ell_0(\widetilde{X}) \geq c \cdot \mathrm{girth}(\mathcal{G})$$

for some constant $c > 0$ only depending on Γ.

Clearly, every element $\gamma \in \Gamma \setminus \{\mathrm{id}\}$ can be written as a reduced word $\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_L}$ with $L = L_S(\gamma) > 0$ and indices $i_1, \ldots, i_L \in \{1, \ldots, 2m\}$.

Pick an element $\gamma \in \widetilde{\Gamma} \setminus \{\mathrm{id}\}$ with *minimal* word length. By the assumption of minimality, we can write γ as a reduced word $\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_L}$ with $L = \mathrm{WL}(\gamma)$. Set $g_i := \pi(\gamma_i) \in \mathbf{G}$ for each $i \in \{1, \ldots, 2m\}$. Clearly, since $\gamma \in \widetilde{\Gamma} \setminus \{\mathrm{id}\}$ we have

$$\mathrm{id}_{\mathbf{G}} = \pi(\gamma) = g_{i_1} g_{i_2} \cdots g_{i_L}.$$

For $j = 1, \ldots, L$ set $x_j := g_{i_1} g_{i_2} \cdots g_{i_j} \in \mathbf{G}$. Using again the assumption of minimality of L, it is easy to see that the elements $x_1, \ldots, x_L = \mathrm{id}_{\mathbf{G}}$ are all distinct. This yields the cycle

$$\mathrm{id}_{\mathbf{G}} \to x_1 \to \cdots \to x_L = \mathrm{id}_{\mathbf{G}}$$

in $\mathcal{G}$ of length L. Since the girth of $\mathcal{G}$ is by definition the length of its shortest cycle, it follows that

$$\min_{\gamma \in \widetilde{\Gamma} \setminus \{\mathrm{id}\}} \mathrm{WL}(\gamma) \geq \mathrm{girth}(\mathcal{G}). \tag{3.63}$$

By Lemma 3.16, we have $\ell(\gamma) \geq C^{-1}\mathrm{WL}(\gamma)$, which combined with (3.63) yields

$$\ell_0(\widetilde{X}) = \min_{\gamma \in \widetilde{\Gamma} \setminus \{\mathrm{id}\}} \ell(\gamma) \geq C^{-1}\mathrm{girth}(\mathcal{G}).$$

The proof of Corollary 3.14 is complete. $\square$

Chapter 4

Abelian covers and spectral gap

4.1 Introduction and motivation

Throughout this chapter let $X = \Gamma\backslash\mathbb{H}$ be a geometrically finite, non-elementary hyperbolic surface. Let $\delta = \delta(\Gamma) = \dim_H \Lambda(\Gamma)$ be the Hausdorff dimension of the limit set of Γ. By the work of Patterson [66, 67] we know that there exists one resonance at the value $s = \delta$ of multiplicity one, and that there are no other resonances in the half-plane $\mathrm{Re}(s) \geq \delta$. This result can be translated in terms of the Selberg zeta function as follows. The function $Z_\Gamma(s)$ has a zero at $s = \delta$ of order one, and no other zeros in the half-plane $\mathrm{Re}(s) \geq \delta$.

One consequence of this fact is an asymptotic formula for the prime geodesic counting function. To state this result, let $\mathcal{L}(X)$ be the *primitive length spectrum* of X, that is, the set of lengths of the primitive periodic geodesics on X (counted with multiplicities). Then, as $t \to \infty$, we have

$$\pi_X(t) := \#\{\ell \in \mathcal{L}(X) : \ell \leq t\} \sim \mathrm{Li}(e^{\delta t}), \tag{4.1}$$

where Li is the (Eulerian) logarithmic integral

$$\mathrm{Li}(x) := \int_2^x \frac{dt}{\log t} \sim \frac{x}{\log x}.$$

The fact that the resonance at $s = \delta$ is always the 'first' resonance of X (the resonance with largest real part) leads us to the following definition of the *spectral gap* of X:

$$\mathrm{Gap}(X) := \inf_{s \in \mathcal{R}(X) \smallsetminus \{\delta\}} (\delta - \mathrm{Re}(s)).$$

Suppose that $\delta > \frac{1}{2}$. In this case, $s = \delta$ corresponds to the lowest L^2-eigenvalue $\lambda_0(X) = \delta(1-\delta)$ of the Laplacian Δ_X, which is a simple eigenvalue. All the remaining resonances $s \in \mathcal{R}(X)$ with $\mathrm{Re}(s) > \frac{1}{2}$ correspond to L^2-eigenvalues, all of which are contained in the interval $(0, \frac{1}{4})$. Since there only finitely many of them, we immediately conclude that X has a positive spectral gap if $\delta > \frac{1}{2}$.

Now suppose that X has $\delta \leq \frac{1}{2}$. In this case, it is not at all clear whether $\mathrm{Gap}(X) > 0$, since we can no longer argue using the basic spectral theory of X. Nevertheless, Naud [57] gave an ingeneous proof for $\mathrm{Gap}(X) > 0$, by adapting the techniques introduced by Dolgopyat [22]. More recently, Bourgain–Dyatlov [15] proved a refined version of Naud's result, using different techniques.

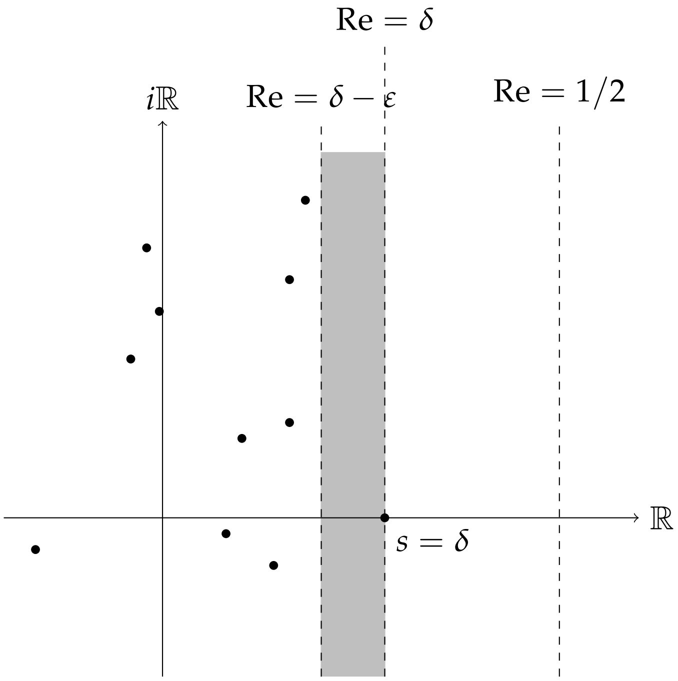

Figure 4.1: Schematic representation of spectral gap for $\delta \leq \frac{1}{2}$: the gray strip is resonance-free, except for $s = \delta$

Therefore, for every non-elementary, geometrically finite hyperbolic surface X (in which case $0 < \delta \leq 1$) there exists an $\varepsilon > 0$ such that

$$\mathcal{R}(X) \cap \{\mathrm{Re}(s) > \delta - \varepsilon\} = \{\delta\}. \tag{4.2}$$

Having a positive spectral gap allows us to estimate the error term in some asymptotic formulas for counting functions associated to the geometry and dynamics of hyperbolic surfaces. For instance, if $\varepsilon > 0$ is chosen as in (4.2), we obtain the following refined asymptotics for the length spectrum:

$$\pi_X(t) := \#\{\ell \in \mathcal{L}(X) : \ell \leq t\} = \mathrm{Li}(e^{\delta t}) + O\left(e^{(\delta - \varepsilon/2)t}\right), \tag{4.3}$$

see Naud [58, 57]. In view of applications such as (4.3), it would certainly be interesting to have an estimate for the spectral gap ε. However, the methods used in [57] and [15] to prove positivity of the spectral gap are not effective, in the sense that they do not provide *explicit* bounds on the spectral gap.

Thus the following natural question arises: Can the spectral gap of a geometrically finite, non-elementary hyperbolic surface X be arbitrarily small? The primary aim of this chapter is to show that this question can be answered affirmatively. More precisely, we will show that for every $\varepsilon > 0$, there exists an *abelian* cover X' of X such that $\mathrm{Gap}(X') < \varepsilon$. This is essentially the content of Theorems 4.1 and 4.2 (note that Theorem 4.2 gives a much more precise statement).

Recall that $X' \to X$ is said to be a (finite) abelian cover if and only if $\pi_1(X')$ is a normal subgroup of $\pi_1(X)$ and the Galois group $\mathbf{G} := \pi(X)/\pi(X')$ is a (finite) abelian group. The existence of abelian covers is a priori not obvious. However, it turns out that hyperbolic surfaces have an abundance of abelian covers.

4.1.1 Abelian covers

An efficient way to manufacture abelian covers is to use the first homology group with integral coefficients,

$$H^1(X, \mathbb{Z}) \simeq \Gamma/[\Gamma, \Gamma],$$

where $[\Gamma, \Gamma]$ is the commutator subgroup of Γ. If we assume that $X = \Gamma\backslash\mathbb{H}$ is non-compact, then Γ is actually a free group[3]. In particular, this implies that there is some integer $m \geq 1$ such that

$$H^1(X, \mathbb{Z}) \simeq \mathbb{Z}^m. \tag{4.4}$$

We can therefore choose a surjective homomorphism $P : \Gamma \to \mathbb{Z}^m$. Let $N := (N_1, N_2, \dots, N_m)$ be an m-tuple of positive integers and consider the surjective map π_N given by

$$\pi_N : \begin{cases} \mathbb{Z}^r \to \mathbb{Z}/N_1\mathbb{Z} \times \mathbb{Z}/N_2\mathbb{Z} \times \dots \times \mathbb{Z}/N_m\mathbb{Z} \\ x = (x_1, \dots, x_m) \mapsto (x_1 \bmod N_1, \dots, x_m \bmod N_m) \end{cases}$$

One can then check that

$$\widetilde{\Gamma} := \ker(\pi_N \circ P)$$

is a normal subgroup of Γ with Galois group

$$\mathbf{G} := \widetilde{\Gamma}/\Gamma \simeq \mathbb{Z}/N_1\mathbb{Z} \times \mathbb{Z}/N_2\mathbb{Z} \times \dots \times \mathbb{Z}/N_m\mathbb{Z}.$$

In particular, $\mathbf{G}$ is a finite abelian group, which means that $\widetilde{\Gamma}\backslash\mathbb{H} \to \Gamma\backslash\mathbb{H}$ is a finite abelian cover.

[3]It's a pure fact of algebraic topology that the fundamental group of a non-compact surface with finite geometry is free, see for example [86].

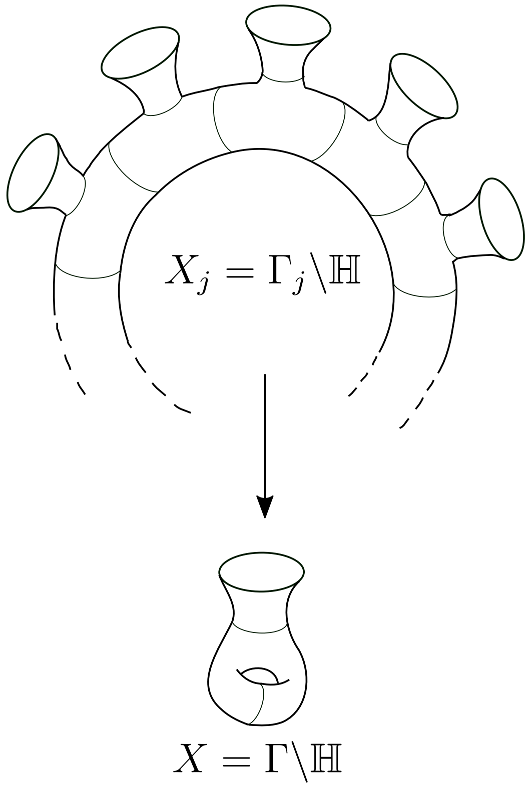

Figure 4.2: A cyclic cover with Galois group $\mathbf{G}_j = \mathbb{Z}/j\mathbb{Z}$. This is a special case of an abelian cover

It should be noted that whenever $\widetilde{\Gamma}$ is a finite index subgroup of Γ, the limit sets $\Lambda(\Gamma)$ and $\Lambda(\widetilde{\Gamma})$ have the same Hausdorff dimensions: $\delta = \delta(\Gamma) = \delta(\widetilde{\Gamma})$. That is to say, the first resonance at $s = \delta$ of X remains unchanged after moving to a finite cover $\widetilde{X}$.

For the remainder of this chapter we assume that X is a non-elementary, geometrically finite, and *non-compact* hyperbolic surface. Additional assumptions will be stated explicitly. We consider infinite sequences $(X_j)_j$ of finite abelian covers (indexed by $j \in \mathbb{N}$) of the fixed base surface X. For any such sequence there exists a sequence of m-tuples of positive integers $N^{(j)} = (N_1^{(j)}, \ldots, N_m^{(j)})$ such that the Galois group of the covering $X_j \to X$ is isomorphic to

$$\mathbf{G}_j = \mathbb{Z}/N_1^{(j)}\mathbb{Z} \times \mathbb{Z}/N_2^{(j)}\mathbb{Z} \times \ldots \times \mathbb{Z}/N_m^{(j)}\mathbb{Z},$$

where m is defined through (4.4). The first result of this chapter is the following.

Theorem 4.1. *Assume that $X = \Gamma\backslash\mathbb{H}$ has at least one cusp, and let $(X_j)_j$ be a sequence of abelian covers with Galois group $\mathbf{G}_j$ as above with $|\mathbf{G}_j| \to \infty$ as $j \to \infty$. Then for all $\varepsilon > 0$, one can find an index j such that $X_j = \Gamma_j\backslash\mathbb{H}$ has at least one non-trivial resonance s with $|s - \delta| \leq \varepsilon$.*

We call a resonance $s \in \mathcal{R}(X)$ 'non-trivial' if $s \neq \delta$.

In the case of *compact* hyperbolic surfaces, this is a known result proved by Randol[4] [75] in 1974. Note that in the compact case, it follows also from min-max

[4] although there is no interpretation in terms of abelian covers in this early work.

techniques and the Buser inequality, see for example in the book of Bergeron [11, Chapter 3]. In the case of abelian covers of the modular surface $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$, this fact was first observed by Selberg, see [82]. For more general compact manifolds, we mention the work of Brooks [19] (based on the Cheeger constant) which gives sufficient conditions on the fundamental group that guarantee existence of coverings with arbitrarily small spectral gaps.

The outline of the proof is as follows. Since X has a cusp, we have $\delta = \delta(X) > \frac{1}{2}$ and therefore, resonances close to δ are actually L^2-eigenvalues. One can then use the fact that Cayley graphs of abelian groups are never expanders, combined with some L^2 techniques and Fell's continuity of induction to prove the result. We follow earlier ideas of Gamburd [30]. The proof of Theorem 4.1 is rather different than the rest of this work, since we use mainly representation theoretic techniques, rather than thermodynamic formalism. It can be found in the last section of this chapter.

If we assume in addition that X is convex cocompact (no cusps), we can actually prove a much more precise result which goes as follows. Recall from Section 2.2 that in this case, X is a Schottky surface.

Theorem 4.2. *Assume that Γ is a non-elementary Schottky group. Let $X_j := \Gamma_j\backslash\mathbb{H}$ be a sequence of abelian covers with Galois group $\mathbf{G}_j$ as above with $|\mathbf{G}_j| \to \infty$ as $j \to \infty$. Then, up to a sequence extraction, there exists a small open set $\mathcal{U}$ with $\delta \in \mathcal{U} \subset \mathbb{C}$ such that for all j large we have $\mathcal{R}(X_j) \cap \mathcal{U} \subset \mathbb{R}$. Moreover, for all test functions $\varphi \in C_c^\infty(\mathcal{U})$, we have*

$$\lim_{j\to\infty} \frac{1}{|\mathbf{G}_j|} \sum_{\lambda\in\mathcal{R}(X_j)\cap\mathcal{U}} \varphi(\lambda) = \int_I \varphi d\mu,$$

where μ is a finite positive measure which is absolutely continuous with respect to the Lebesgue measure on an interval $I = [a, \delta]$ for some $a < \delta$.

- The absolutely continuous measure μ depends dramatically on the sequence of covers: a more detailed description of this density is provided at the end of Section 4.2.

- Since δ belongs to the support of μ, a simple approximation argument shows that for all $\varepsilon > 0$ small enough, we have as $j \to \infty$,

  $$\#\{\lambda \in \mathcal{R}(X_j) \ : \ |\lambda - \delta| < \varepsilon\} \sim C_\varepsilon |\mathbf{G}_j|,$$

  for some constant $C_\varepsilon > 0$ only depending on ε.

- Another obvious corollary is that for all $\varepsilon > 0$ one can find a finite abelian cover X_j of X such that X_j has a non-trivial resonance ε-close to δ. Both Theorems 4.1 and 4.2 fully cover the case of all geometrically finite surfaces. We have existence of surfaces with arbitrarily small spectral gap, which was *not known so far*.

- Note that the non-trivial resonances obtained here are real: for $\delta > 1/2$, this is clear because when close enough to δ they are actually L^2-eigenvalues. However, when $\delta \leq 1/2$, this is not an obvious fact.

- In the general context of scattering theory on spaces with negative curvature, it is to the knowledge of the author the first *exact* asymptotic result on the distribution of resonances, apart from the 'trivial' cases of elementary groups or cylindrical manifolds where resonances can be explicitly computed. For a review of the current knowledge on counting results for resonances, we refer to the recent exhaustive survey of Zworski [92].

The proof of Theorem 4.2 mostly uses thermodynamic formalism and L-functions to carefully analyze the contribution of L-factors related to characters which are close to the identity. In particular we use in a fundamental way dynamical L-functions related to characters of $\mathbb{Z}^m$ and their representation as Fredholm determinants of suitable transfer operators, see Section 2.6. We point out that using the coding available for compact hyperbolic surfaces [73], the proof of the above equidistribution result carries through without modification in the compact case, which to the knowledge of the author is also new (though less surprising). In [75], Randol showed that the number of small eigenvalues in $(0, 1/4)$ can be as large as wanted, by passing to a finite abelian cover. However Randol's technique, which is based on the 'twisted' trace formula, prevented him from further investigating the distribution of these small eigenvalues.

4.2 Equidistribution of resonances and abelian covers

In this section we prove Theorem 4.2. Recall the geometric definition of a Schottky group Γ from Section 2.2. As in Section 2.2, we let $\mathcal{D}_1, \ldots, \mathcal{D}_{2m}$ and $\gamma_1, \ldots, \gamma_{2m}$ denote the corresponding configuration of disks and isometries used in the construction of Γ. The Schottky group $\Gamma = \langle \gamma_1^{\pm}, \ldots, \gamma_m^{\pm} \rangle$ is isomorphic to a free group on m symbols. Moreover, we set

$$\mathcal{D} := \bigcup_{i=1}^{2m} \mathcal{D}_i.$$

Recall that we are considering a family of abelian covers of the surface $X = \Gamma \backslash \mathbb{H}$ given by normal subgroups $\Gamma_j \lhd \Gamma$ with Galois group

$$\mathbf{G}_j = \mathbb{Z}/N_1^{(j)}\mathbb{Z} \times \mathbb{Z}/N_2^{(j)}\mathbb{Z} \times \ldots \times \mathbb{Z}/N_m^{(j)}\mathbb{Z}.$$

Since we assume that $|\mathbf{G}_j| \to \infty$ as $j \to \infty$, we can extract a sequence (and reindex) such that

$$\mathbf{G}_j = \mathbb{Z}/N_1^{(j)}\mathbb{Z} \times \ldots \times \mathbb{Z}/N_r^{(j)}\mathbb{Z} \times \mathbb{Z}/N_{r+1}\mathbb{Z} \times \ldots \times \mathbb{Z}/N_m\mathbb{Z},$$

with $\min\{N_1^{(j)}, \ldots, N_r^{(j)}\} \to \infty$ as $j \to \infty$ and $N_{r+1}, \ldots N_m$ are fixed (and could be 1). Since the groups $\mathbf{G}_j$ are abelian, all their irreducible representations are one-dimensional (i.e. characters). The characters of $\mathbf{G}_j$ are given by

$$\chi_\alpha(g) := \exp\left(2i\pi \sum_{\ell=1}^{m} \frac{\alpha_\ell}{N_\ell} g_\ell\right),$$

where $g = (g_1, \dots, g_m)$ and $\alpha = (\alpha_1, \dots, \alpha_m)$ with $\alpha_\ell \in \{0, \dots, N_\ell - 1\}$. Thanks to the Venkov–Zograf factorization formula, Theorem A.3, we can factorize the Selberg zeta function of Γ_j as a product of L-functions,

$$Z_{\Gamma_j}(s) = \prod_{\alpha} L_\Gamma(s, \chi_\alpha), \tag{4.5}$$

where α belongs to the above specified set product. The case $\alpha = 0$ corresponds to the trivial representation $\chi_0 = \mathbf{1}_\mathbb{C}$, in which case we have $L_\Gamma(s, \mathbf{1}_\mathbb{C}) = Z_\Gamma(s)$. Recall that Z_Γ has a simple zero at $s = \delta$. Roughly speaking, we need to split this product into two separate factors: the one corresponding to 'small α's' which will produce a zero close to $s = \delta$ via an implicit function theorem, and the other one for which we have to show that they do not vanish in a small neighbourhood of δ. To that effect, we will introduce an auxiliary L-function that is related to characters of the homology group $H^1(X, \mathbb{Z}) \simeq \mathbb{Z}^m$.

Given $\theta \in \mathbb{C}^m$, we define the 'twisted' transfer operator $\mathcal{L}_{s,\theta} : H^2(\mathcal{D}) \to H^2(\mathcal{D})$ by

$$\mathcal{L}_{s,\theta} f(z) = \sum_{\substack{i=1 \\ i \neq j+m}}^{2m} e^{2i\pi\theta \bullet P(\gamma_i)} \left[(\gamma_i^{-1})'(z) \right]^s f(\gamma_i^{-1}.z) \quad \text{if } z \in \mathcal{D}_j \tag{4.6}$$

where $s \in \mathbb{C}$ is the spectral parameter, $P : \Gamma \to \mathbb{Z}^m$ is the projection in the first homology group. In addition we have denoted by $\theta \bullet a$ the pairing

$$\theta \bullet a := \sum_{k=1}^{m} \theta_k a_k.$$

Notice that $\mathcal{L}_{s,\theta}$ is nothing else but the transfer operator associated to the one-dimensional representation $\chi_\theta : \Gamma \to \mathbb{S}^1$, as defined by equation (2.13).

This family (of trace class operators) depends holomorphically on (s, θ). Therefore the Fredholm determinant

$$L_\Gamma(s, \theta) := \det(I - \mathcal{L}_{s,\theta})$$

is holomorphic on $\mathbb{C} \times \mathbb{C}^m$. By equation (2.14), $L_\Gamma(s, \theta) = L_\Gamma(s, \chi_\theta)$ is the L-function associated to (Γ, χ_θ). Using this auxiliary function, we can rewrite (4.5) as

$$Z_{\Gamma_j}(s) = \prod_{k=(k_1,\dots,k_m) \in \mathcal{S}_j} L_\Gamma(s, k_1/N_1^{(j)}, \dots, k_r/N_r^{(j)}, \dots, k_m/N_m), \tag{4.7}$$

where

$$\mathcal{S}_j = \{0, \dots, N_1^{(j)} - 1\} \times \dots \times \{0, \dots, N_r^{(j)} - 1\} \times \dots \times \{0, \dots, N_m - 1\}.$$

4.2.1 A non-vanishing result for $L_\Gamma(s, \theta)$

The goal of this subsection is to establish the following fact which is crucial in the analysis of resonances close to $s = \delta$. Recall that $\Gamma = \langle \gamma_1^\pm, \dots, \gamma_m^\pm \rangle$ is a Schottky group.

Proposition 4.3. *Using the above notations, we have for $\theta \in \mathbb{R}^m$,*

$$L_\Gamma(\delta,\theta) = 0 \Leftrightarrow \theta \in \mathbb{Z}^m.$$

Proof. Obviously if $\theta \in \mathbb{Z}^m$, then $L_\Gamma(s,\theta) = Z_\Gamma(s)$ vanishes at $s = \delta$. The converse will follow from a convexity argument that is similar to what has been used by Parry and Pollicott [65, Chapter 5] to analyze dynamical Ruelle zeta functions on the line $\{\mathrm{Re}(s) = 1\}$. First we need to recall the usual 'normalizing trick' which is essential in the latter part of the argument. By the Ruelle–Perron–Frobenius Theorem (see [65, Theorem 2.2]), the operator

$$\mathcal{L}_{\delta,0} : H^2(\mathcal{D}) \to H^2(\mathcal{D})$$

has 1 as a simple eigenvalue and the associated eigenspace is spanned by a real-analytic function H which satisfies $H(x) > 0$ for all $x \in \Lambda(\Gamma)$. For $j \in \{1, \ldots, 2m\}$ let $I_j := \mathcal{D}_j \cap \mathbb{R}$. By setting (we work on $\Lambda(\Gamma)$)

$$\mathcal{M}_\delta(F)(x) := \sum_{\substack{i=1\\ i\neq j+m}}^{2m} e^{g_i(x)} f(\gamma_i^{-1}.x), \quad x \in I_j \cap \Lambda(\Gamma),$$

where

$$g_i(x) = \delta \log\left[(\gamma_i^{-1})'(x)\right] - \log H(x) + \log H(\gamma_i^{-1}.x),$$

we obtain an operator

$$\mathcal{M}_\delta : C^0(\Lambda(\Gamma)) \to C^0(\Lambda(\Gamma))$$

which satisfies $\mathcal{M}_\delta(\mathbf{1}) = \mathbf{1}$. Assume now that $L_\Gamma(\delta,\theta) = 0$ for some $\theta \in \mathbb{R}^m$. Then $\mathcal{L}_{\delta,\theta}$ has 1 as an eigenvalue. Pick an associated non-trivial 1-eigenfunction W, obviously continuous on $\Lambda(\Gamma)$. By writing

$$H^{-1}\mathcal{L}_{\delta,\theta}(H.(H^{-1}W)) = H^{-1}W,$$

we deduce that

$$\sum_{\substack{i=1\\ i\neq j+m}}^{2m} e^{g_i(x)} e^{2i\pi\theta\bullet P(\gamma_i)} \widetilde{W}(\gamma_i^{-1}.x) = \widetilde{W}(x), \quad x \in I_j \cap \Lambda(\Gamma) \tag{4.8}$$

where we have set

$$\widetilde{W}(x) = H^{-1}(x)W(x).$$

Choosing $x_0 \in \Lambda(\Gamma)$ (say in $I_j \cap \Lambda(\Gamma)$) such that

$$|\widetilde{W}(x_0)| = \sup_{\xi\in\Lambda(\Gamma)} |\widetilde{W}(\xi)|,$$

we get by the triangle inequality

$$\sup_{\xi\in\Lambda(\Gamma)} |\widetilde{W}(\xi)| \leq \mathcal{M}_\delta(|\widetilde{W}|)(x_0) \leq \sup_{\xi\in\Lambda(\Gamma)} |\widetilde{W}(\xi)|.$$

The same conclusion holds when iterating $\mathcal{M}_\delta$ so that for all $N \geq 0$, we have

$$\sup_{\xi \in \Lambda(\Gamma)} |\widetilde{W}(\xi)| = \mathcal{M}_\delta^N(|\widetilde{W}|)(x_0).$$

Because $\mathcal{M}_\delta^N$ are normalized, this forces

$$\sup_{\xi \in \Lambda(\Gamma)} |\widetilde{W}(\xi)| = |\widetilde{W}(\gamma_\alpha^{-1}.x_0)|$$

for all words $\alpha \in \mathcal{W}_N^j$. Since the set $\{\gamma_\alpha^{-1}.x_0\}_{\alpha \in \mathcal{W}_N^j}$ is dense in $\Lambda(\Gamma)$ as $N \to \infty$, we deduce that $|\widetilde{W}|$ is constant on $\Lambda(\Gamma)$. Without loss of generality we may assume that

$$|\widetilde{W}| = 1.$$

By strict convexity of the unit Euclidean ball in $\mathbb{C}$, we deduce from (4.8) that for all $i \neq j + m$, we have

$$e^{2i\pi\theta \bullet P(\gamma_i)} \widetilde{W}(\gamma_i^{-1}.x) = \widetilde{W}(x), \quad x \in I_j \cap \Lambda(\Gamma)$$

Writing

$$\widetilde{W}(x) = e^{2i\pi V(x)},$$

where $V : \Lambda(\Gamma) \to \mathbb{R}$ is a continuous lift, we end up with the identity (for all $j \neq i + m$ and all $x \in I_j \cap \Lambda(\Gamma)$)

$$\theta \bullet P(\gamma_i) = V(x) - V(\gamma_i^{-1}.x) + M_{x,i}, \tag{4.9}$$

where $M_{x,i}$ is an integer. Now let $x_i \in I_i$ be the unique attracting fixed point of the map

$$\gamma_i^{-1} : I_i \to I_i.$$

By inserting $x = x_i$ into equation (4.9), we obtain

$$\theta \bullet P(\gamma_i) = M_{x_i,i} \in \mathbb{Z}. \tag{4.10}$$

for every $i \in \{1, \dots, m\}$. Recall that Γ is a free group on m elements generated by the elements $\gamma_1, \dots, \gamma_m$ (and its inverses). Therefore

$$(P(\gamma_1), \dots, P(\gamma_m))$$

is a $\mathbb{Z}$-basis of $H^1(X, \mathbb{Z}) \simeq \mathbb{Z}^m$. As a consequence, the $m \times m$ matrix whose rows are given by the vectors $P(\gamma_1), \dots, P(\gamma_m)$ has determinant ± 1 and is thus invertible with integer coefficients: this implies by (4.10) that $\theta \in \mathbb{Z}^m$. The proof is complete. □

A direct corollary, which is what we will actually use in the proof of Theorem 4.2, is the following.

Corollary 4.4. *Using the above notations, for all $\varepsilon > 0$ small enough, one can find a complex neighbourhood $\mathcal{V}$ of δ such that for all $s \in \mathcal{V}$ and $\theta \in \mathbb{R}^m$,*

$$L_\Gamma(s, \theta) = 0 \Rightarrow \text{dist}(\theta, \mathbb{Z}^m) < \varepsilon.$$

Proof. We argue by contradiction. Fix some $\varepsilon > 0$. If the above statement is not true, then one can find a sequence

$$(s_j, \theta_j) \in \mathbb{C} \times \mathbb{R}^m$$

such that for all j we have $L_\Gamma(s_j, \theta_j) = 0$ and $\mathrm{dist}(\theta_j, \mathbb{Z}^m) \geq \varepsilon$ and $\lim_j s_j = \delta$. Using the $\mathbb{Z}^m$-periodicity of $L_\Gamma(s, \theta)$ with respect to θ, we can assume that θ_j remains in a bounded subset of $\mathbb{R}^m$ and use compactness to extract a subsequence such that $\theta_j \to \widetilde{\theta}$ with $\mathrm{dist}(\widetilde{\theta}, \mathbb{Z}^m) \geq \varepsilon$. We then have $L_\Gamma(\delta, \widetilde{\theta}) = 0$, which contradicts Proposition 4.3. $\square$

4.2.2 Proof of Theorem 4.2

We are now ready to prove Theorem 4.2. Consider the holomorphic map

$$\mathbb{C} \times \mathbb{C}^m \to \mathbb{C}, \quad (s, \theta) \mapsto L_\Gamma(s, \theta).$$

Since $L_\Gamma(\delta, 0) = 0$, we have (recall that $s = 0$ is a simple zero of $Z_\Gamma(s)$)

$$\partial_s L_\Gamma(\delta, 0) = Z'_\Gamma(\delta) \neq 0.$$

Hence, we can apply the Holomorphic Implicit Function Theorem, which states that there exists an open set $\mathcal{O} \subset \mathbb{C}$ with $\delta \in \mathcal{O}$ and some $\varepsilon > 0$ such that for all $(s, \theta) \in \mathcal{O} \times B_\infty(0, \varepsilon)$,

$$L_\Gamma(s, \theta) = 0 \Longleftrightarrow s = \phi(\theta),$$

where $\phi : B_\infty(0, \varepsilon) \to \mathcal{O}$ is a real-analytic map and

$$B_\infty(0, \varepsilon) := \{x = (x_1, \dots, x_m) \in \mathbb{R}^m \ : \ \max_{1 \leq l \leq m} |x_l| < \varepsilon\}$$

is the ε-ball in $\mathbb{R}^m$ centered at 0 with respect to the infinity norm.

Using Corollary 4.4 with the above ε, we deduce that if $s \in \mathcal{U} := \mathcal{O} \cap \mathcal{V}$ is a such that

$$L_\Gamma(s, \theta) = 0,$$

for some $\theta \in \mathbb{R}^m$, then $\mathrm{dist}(\theta, \mathbb{Z}^m) < \varepsilon$, and $s = \phi(\widetilde{\theta})$ where $\widetilde{\theta} = \theta \bmod \mathbb{Z}^m$ and $\widetilde{\theta} \in B_\infty(0, \varepsilon)$.

Now pick $\varphi \in C_c^\infty(\mathcal{U})$, using the factorization formula (4.7), we observe that provided ε is taken small enough we have

$$\sum_{\lambda \in \mathcal{R}(X_j) \cap \mathcal{U}} \varphi(\lambda) = \sum_{\substack{\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r \\ |k_1| < \varepsilon N_1^{(j)}, \dots, |k_r| < \varepsilon N_r^{(j)}}} \varphi \circ \phi \left(\frac{k_1}{N_1^{(j)}}, \dots, \frac{k_r}{N_r^{(j)}}, 0, \dots, 0 \right).$$

Next we will apply the following Lemma.

Lemma 4.5. *Fix $\varepsilon > 0$ and assume that $\psi \in C^\infty(\mathbb{R}^r)$ is a compactly supported function on $B_\infty(0,\varepsilon) \subset \mathbb{R}^r$. Then we have*

$$\lim_{j\to\infty} \frac{1}{N_1^{(j)} \dots N_r^{(j)}} \sum_{\substack{\mathbf{k}=(k_1,\dots,k_r)\in\mathbb{Z}^r \\ |k_1|<\varepsilon N_1^{(j)},\dots,|k_r|<\varepsilon N_r^{(j)}}} \psi\left(\frac{k_1}{N_1^{(j)}},\dots,\frac{k_r}{N_r^{(j)}}\right) = \int_{B_\infty(0,\varepsilon)} \psi(x)\,dx.$$

Proof. Using the Poisson summation formula, we can write

$$\frac{1}{N_1^{(j)} \dots N_r^{(j)}} \sum_{\substack{\mathbf{k}=(k_1,\dots,k_r)\in\mathbb{Z}^r \\ |k_1|<\varepsilon N_1^{(j)},\dots,|k_r|<\varepsilon N_r^{(j)}}} \psi\left(\frac{k_1}{N_1^{(j)}},\dots,\frac{k_r}{N_r^{(j)}}\right)$$

$$= \frac{1}{N_1^{(j)} \dots N_r^{(j)}} \sum_{\mathbf{k}\in\mathbb{Z}^r} \psi\left(\frac{k_1}{N_1^{(j)}},\dots,\frac{k_r}{N_r^{(j)}}\right)$$
$$= \int_{\mathbb{R}^r} \psi(x)dx + \sum_{\mathbf{k}\in\mathbb{Z}^r,k\neq\mathbf{0}} \widehat{\psi}(2\pi N_1k_1,\dots,2\pi N_rk_r),$$

where $\widehat{\psi}$ is as usual the Fourier transform defined by

$$\widehat{\psi}(\xi) = \int_{\mathbb{R}^r} \psi(x)e^{-i\xi.x}dx.$$

Since $\widehat{\psi}$ has rapid decay (Schwartz class), a simple summation argument gives

$$\frac{1}{N_1^{(j)} \dots N_r^{(j)}} \sum_{\substack{\mathbf{k}=(k_1,\dots,k_r)\in\mathbb{Z}^r \\ |k_1|<\varepsilon N_1^{(j)},\dots,|k_r|<\varepsilon N_r^{(j)}}} \psi\left(\frac{k_1}{N_1^{(j)}},\dots,\frac{k_r}{N_r^{(j)}}\right)$$

$$= \int \psi(x)\,dx + O_\alpha\left(\frac{1}{(\min\{N_1^{(j)},\dots N_r^{(j)}\})^\alpha}\right),$$

for all integers α. The proof is complete. □

Applying the above lemma with $\psi(x) = \varphi \circ \phi(x,0)$ we get as $j \to \infty$,

$$\lim_{j\to\infty} \frac{1}{|\mathbf{G}_j|} \sum_{\lambda\in\mathcal{R}(X_j)\cap\mathcal{U}} \varphi(\lambda) = N_{r+1}\dots N_m \int_{\mathbb{R}^r} \varphi\circ\phi(x,0)\,dx := \int \varphi\,d\mu,$$

where $\phi(x,0) = \phi(x_1,\dots,x_r,0,\dots,0)$.

The measure μ is nothing else but the *push-forward of Lebesgue measure* on the ball $B_\infty(0,\varepsilon)$ via the map ϕ. It is clear from the above formula that δ belongs to the support of μ since $\phi(0) = \delta$. What remains to show is:

- The maps $x \mapsto \phi(x,0)$ are real valued. This implies that all the resonances in the vicinity of $s = \delta$ are actually real.

- The maps $x \mapsto \phi(x, 0)$ are non-constant.
- The corresponding push-forward measure μ is absolutely continuous.

Since all the resonances in $\mathcal{R}(X_j)$ (also all zeros of $s \mapsto L_\Gamma(s,\theta)$ for $\theta \in \mathbb{R}^m$) are in the half plane $\{\mathrm{Re}(s) \leq \delta\}$, we must have $\nabla \mathrm{Re}(\phi)(0) = 0$. However, one can actually show that $\mathrm{Im}(\phi) = 0$ identically. Indeed, recall that by using the same ideas leading to the Fredholm determinant identity (2.14), one can show that for all $\mathrm{Re}(s) > \delta$ and all $\theta \in \mathbb{Z}^m$, we have

$$L_\Gamma(s,\theta) = \exp\left(- \sum_{[\gamma]\in[\Gamma]_p} \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} \frac{\chi_\theta(\gamma^m)}{m} e^{-(s+k)m\ell(\gamma)} \right),$$

where the first sum runs over prime conjugacy classes. By complex conjugation and uniqueness of analytic continuation, we obtain the identity valid for all $s \in \mathbb{C}$ and all $\theta \in \mathbb{Z}^m$:

$$\overline{L_\Gamma(s,\theta)} = L_\Gamma(\bar{s}, -\theta). \tag{4.11}$$

This implies that for all $\theta \in B_\infty(0,\varepsilon)$, we have

$$\phi(-\theta) = \overline{\phi(\theta)}.$$

On the other hand, if $[\gamma] \in [\Gamma]_p$, then $[\gamma^{-1}] \in [\Gamma]_p$ and $\ell(\gamma) = \ell(\gamma^{-1})$, while $\chi_\theta(\gamma^{-1}) = \chi_{-\theta}(\gamma)$. Therefore 'time reversal' invariance of $[\Gamma]_p$ yields another identity (again using unique continuation) valid for all $s \in \mathbb{C}$ and all $\theta \in \mathbb{Z}^m$:

$$L_\Gamma(s,\theta) = L_\Gamma(s,-\theta) \tag{4.12}$$

Combining (4.11) and (4.12) shows that for all $\theta \in B_\infty(0,\varepsilon)$, we must have $\overline{\phi}(\theta) = \phi(-\theta) = \phi(\theta)$. In particular, ϕ is real-valued. This fact was observed in previous works related to prime orbit counting (in homology classes) for geodesics flows, see for example [65, Chapter 12]. By the same arguments as above, we know that the Hessian matrix $\nabla^2 \mathrm{Re}(\phi)(0)$ must be negative. Because the zeta functions $Z_{\Gamma_j}(s)$ all have a simple zero at $s = \delta$, the maps $x \mapsto \phi(x,0)$ have to be non-constant.

One can actually show, using that the length spectrum of X is not a lattice, that (see for example the arguments in [65, page 199]) we have

$$\det\left(\nabla^2 \mathrm{Re}(\phi)(0)\right) \neq 0,$$

i.e. that the associated quadratic form is negative-definite. Historically, the non-degeneracy of this critical point has played an important role in works related to prime orbit counting in homology classes, see [2, 42, 45, 69, 72]. Since each map $(x_1, \ldots, x_r) \mapsto \phi(x_1, \ldots, x_r, 0) \in \mathbb{R}$ is non-constant, the (closure of the) image is a closed interval $I = [a, \delta]$ for some $a < \delta$. Moreover, because

$$(x_1, \ldots, x_r) \mapsto F(x) := \phi(x_1, \ldots, x_r, 0)$$

is real-analytic (and non-constant), the set of points $x = (x_1, \ldots, x_r) \in B_\infty(0,\varepsilon)$ such that $\nabla F(x) = 0$, has zero Lebesgue measure. It follows from standard arguments (see for example in [74]) that F has the '0-set' property: the pre-image of each set of zero Lebesgue measure has zero Lebesgue measure. We can apply the Radon–Nikodym theorem and conclude that μ is absolutely continuous with respect to Lebesgue on I. The proof of Theorem 4.2 is now complete.

Let us now give a more precise description of the measure μ. It is possible to describe the Radon-Nikodym derivative $\frac{d\mu}{dm}(u)$ in the vicinity of δ, where m is Lebesgue measure on I. Indeed, we know from the above that locally,

$$\phi(x) = \delta - Q(x) + O(\|x\|^3),$$

where $Q(x)$ is a positive definite quadratic form.

The Morse lemma implies that for all $\varepsilon > 0$ small enough there an open neighbourhood $\tilde{U} \subset \mathbb{R}^r$ of 0 and a diffeomorphism

$$\Psi : B_\infty(0, \varepsilon) \to \tilde{U}, \quad (x_1, \dots, x_r) \mapsto (y_1, \dots, y_r)$$

such that $\Psi(0) = 0$ and $\phi \circ \Psi^{-1}(y) = \delta - y_1^2 - \cdots - y_r^2$. Therefore, for any $\varphi \in C_0^\infty(\mathcal{U})$ we have

$$\begin{aligned}
\int \varphi \, d\mu &= \int_{\mathbb{R}^r} \varphi \circ \phi(x, 0) dx \\
&= \int_{\tilde{U}} \varphi(\delta - y_1^2 - \cdots - y_r^2) \cdot |D\Psi^{-1}(y)| \, dy \\
&\asymp \int_{\tilde{U}} \varphi(\delta - y_1^2 - \cdots - y_r^2) \, dy,
\end{aligned}$$

where $|D\Psi^{-1}(y)|$ is the Jacobian determinant. Choosing polar coordinates yields

$$\int \varphi d\mu \asymp \int_{\mathbb{R}^+} \varphi(\delta - R^2) R^{r-1} dR.$$

With one last change of variables $R \mapsto \xi = R^2$ we obtain

$$\int \varphi \, d\mu \asymp \int_{\mathbb{R}^+} \varphi(\delta - \xi) \xi^{\frac{r-2}{2}} \, d\xi.$$

We conclude that there exists a constant $C > 0$ such that for all u close enough to delta ($u < \delta$)

$$C^{-1}(\delta - u)^{\frac{r-2}{2}} \leq \frac{d\mu}{dm}(u) \leq C(\delta - u)^{\frac{r-2}{2}},$$

where r is defined above as the number of unbounded cyclic factors in the sequence of abelian groups $\mathbf{G}_j$. In particular we observe a drastic difference in the density shape when $r = 1, 2$ and $r > 2$.

We conclude this section by a remark on the case of elementary groups (which we have excluded so far). Given a non-trivial hyperbolic isometry γ in $\mathrm{PSL}_2(\mathbb{R})$, we set $\Gamma = \langle \gamma \rangle$ and $X = \Gamma \backslash \mathbb{H}$ the corresponding hyperbolic cylinder. It is easy to check that all finite covers of X are (obviously) abelian and given by

$$X_N = \Gamma_N \backslash \mathbb{H}, \ \Gamma_N = \langle \gamma^N \rangle,$$

with $N \geq 1$. In this case, we can explicitly compute the Selberg zeta function of X_N (see Borthwick [14, Chapter 10]):

$$Z_{X_N}(s) = \prod_{k \geq 0} \left(1 - e^{-(s+k)N\ell(\gamma)}\right)^2,$$

where $\ell(\gamma)$ is the displacement length of γ. The zero-set of $Z_{X_N}(s)$ is therefore the half-lattice

$$\frac{2i\pi}{N\ell(\gamma)}\mathbb{Z} - \mathbb{N}_0,$$

from which we can see that resonances accumulate as $N \to \infty$ on the axis

$$\{\mathrm{Re}(s) = \delta = 0\}.$$

Notice that for the hyperbolic cylinder, each resonance has multiplicity *two*, which explains why the perturbative argument does not work here.

4.3 Fell's continuity and Cayley graphs of abelian groups

In this section we prove Theorem 4.1. The arguments follow closely those of Gamburd in [30]. Roughly speaking, since Cayley graphs of finite abelian groups can never form a family of expanders, one should also expect that there is no uniform spectral gap in the family of covers $X_j = \Gamma_j \backslash \mathbb{H}$. We give a rigorous proof of this fact using Fell's continuity.

Let $\mathcal{G}$ be a finite graph with set of vertices $\mathcal{V}$ and of degree k. That is, for every vertex $x \in \mathcal{V}$ there are k edges adjacent to x. For a subset of vertices $A \subset \mathcal{V}$ we define its boundary ∂A as the set of edges with one extremity in A and the other in $\mathcal{G} - A$. The *Cheeger isoperimetric constant* $h(\mathcal{G})$ is defined as

$$h(\mathcal{G}) := \min\left\{\frac{|\partial A|}{|A|} : A \subset \mathcal{V} \text{ and } 1 \leq |A| \leq \frac{|\mathcal{V}|}{2}\right\}. \tag{4.13}$$

Let $L^2(\mathcal{V})$ be the Hilbert space of complex-valued functions on $\mathcal{V}$, endowed with the inner product

$$\langle F, G\rangle_{L^2(\mathcal{V})} = \sum_{x\in\mathcal{V}} F(x)\overline{G(x)}.$$

Let Δ be the discrete Laplace operator acting on $L^2(\mathcal{V})$ by

$$\Delta F(x) = F(x) - \frac{1}{k}\sum_{y\sim x} F(y),$$

where $F \in L^2(\mathcal{V})$, $x \in \mathcal{V}$ is a vertex of $\mathcal{G}$, and $y \sim x$ means that y and x are connected by an edge. The operator Δ is self-adjoint and positive. Let $\lambda_1(\mathcal{G})$ denote the first non-zero eigenvalue of Δ.

The following result due to Alon and Milman [1] relates the spectral gap $\lambda_1(\mathcal{G})$ and Cheeger's isoperimetric constant.

Proposition 4.6. *For finite graphs $\mathcal{G}$ of degree k we have*

$$\frac{1}{2}k \cdot \lambda_1(\mathcal{G}) \leq h(\mathcal{G}) \leq k\sqrt{\lambda_1(\mathcal{G})(2 - \lambda_1(\mathcal{G}))}.$$

We note that large first non-zero eigenvalue $\lambda_1(\mathcal{G})$ implies fast convergence of random walks on $\mathcal{G}$, that is, high connectivity (see Lubotzky [49]).

Definition 4.7. *A family of finite graphs $\{\mathcal{G}_j\}$ of bounded degree is called a family of expanders if there exists a constant $c > 0$ such that $h(\mathcal{G}_j) \geq c$.*

The family of graphs we are interested in is built as follows. Let $\Gamma = \langle S \rangle$ be a Fuchsian group generated by a finite set $S \subset \mathrm{PSL}_2(\mathbb{R})$. We will assume that S is symmetric, i.e. $S^{-1} = S$. Given a sequence Γ_j of finite index normal subgroups of Γ, let S_j be the image of S under the natural projection $r_{\mathbf{G}_j} : \Gamma \to \mathbf{G}_j = \Gamma/\Gamma_j$. Notice that S_j is a symmetric generating set for the group $\mathbf{G}_j$. Let $\mathcal{G}_j = \mathrm{Cay}(\mathbf{G}_j, S_j)$ denote the Cayley graph of $\mathbf{G}_j$ with respect to the generating set S_j. That is, the vertices of $\mathbf{G}_j$ are the elements of $\mathbf{G}_j$ and two vertices x and y are connected by an edge if and only if $xy^{-1} \in S_j$.

The connection of uniform spectral gap with the graphs constructed above comes from the following result.

Proposition 4.8. *Assume that $\delta = \delta(\Gamma) > \frac{1}{2}$ and assume that there exists $\varepsilon > 0$ such that for all j all non-trivial resonances s of $X_j = \Gamma_j\backslash\mathbb{H}$ satisfy $|s - \delta| > \varepsilon$. Then the Cayley graphs $\mathcal{G}_j$ form a family of expanders.*

Let us see how Proposition 4.8 implies Theorem 4.1.

Proof of Theorem 4.1. Since $X = \Gamma\backslash\mathbb{H}$ has at least one cusp by assumption, we have $\delta > \frac{1}{2}$ so that we can apply Proposition 4.8. Suppose by contradiction that there exists $\varepsilon > 0$ such that for all j we have $|s - \delta| > \varepsilon$ for all non-trivial resonances s of X_j. Then Proposition 4.8 implies that the Cayley graphs $\mathcal{G}_j = \mathrm{Cay}(\mathbf{G}_j, S_j)$ form a family of expanders. We will show that this is never true for the sequence of abelian groups $\mathbf{G}_j$ defined in Section 4.1.1, thus showing Theorem 4.1. Write

$$\mathbf{G}_j = \mathbb{Z}/N_1^{(j)}\mathbb{Z} \times \mathbb{Z}/N_2^{(j)}\mathbb{Z} \times \cdots \times \mathbb{Z}/N_m^{(j)}\mathbb{Z}.$$

The space $L^2(\mathbf{G}_j)$ is spanned by the characters χ_α given by

$$\chi_\alpha(x) = \exp\left(2\pi i \sum_{\ell=1}^{m} \frac{\alpha_\ell}{N_\ell^{(j)}} x_\ell\right)$$

where $x = (x_1, \ldots, x_m)$ and $\alpha = (\alpha_1, \ldots, \alpha_m)$ with $\alpha_\ell \in \{0, \ldots, N_\ell^{(j)} - 1\}$. Note that the trivial character $\chi_\alpha \equiv 1$ corresponds to $\alpha = 0$. Applying the discrete

Laplace operator to χ_α yields

$$\begin{aligned}
\Delta\chi_\alpha(x) &= \chi_\alpha(x) - \frac{1}{|S_j|}\sum_{s\in S_j}\chi_\alpha(x+s) \\
&= \chi_\alpha(x) - \frac{1}{|S_j|}\sum_{s\in S_j}\exp\left(2\pi i\sum_{\ell=1}^{m}\frac{\alpha_\ell}{N_\ell^{(j)}}s_\ell\right)\chi_\alpha(x) \\
&= \chi_\alpha(x) - \frac{1}{|S_j|}\sum_{s\in S_j}\cos\left(2\pi i\sum_{\ell=1}^{m}\frac{\alpha_\ell}{N_\ell^{(j)}}s_\ell\right)\chi_\alpha(x) \\
&= \left(1 - \frac{1}{|S_j|}\sum_{s\in S_j}\cos\left(2\pi i\sum_{\ell=1}^{m}\frac{\alpha_\ell}{N_\ell^{(j)}}s_\ell\right)\right)\chi_\alpha(x),
\end{aligned}$$

where we exploited the symmetry of the set S_j in the third line. Thus every character χ_α is an eigenfunction of Δ with eigenvalue

$$\lambda_\alpha^{(j)} := \frac{1}{|S_j|}\sum_{s\in S_j}\left(1-\cos\left(2\pi i\sum_{\ell=1}^{m}\frac{\alpha_\ell}{N_\ell^{(j)}}s_\ell\right)\right).$$

Note that we can view S_j as a subset of

$$\{0,\dots,N_1^{(j)}-1\}\times\cdots\times\{0,\dots,N_m^{(j)}-1\}\subset\mathbb{Z}^m.$$

Since the generating set S is a finite subset of $\mathrm{PSL}_2(\mathbb{R})$, there exists a constant $M>0$ independent of j such that $\max_{s\in S_j}\|s\|_\infty \le M$, where $\|s\|_\infty = \max_{1\le\ell\le m}|s_\ell|$ is the supremum norm. Since we assume that $|\mathbf{G}_j|\to\infty$, we may assume (after extracting a sequence and re-indexing) that $N_1^{(j)}\to\infty$. Set $\alpha=(1,0,\dots,0)$. Then we have

$$0\le\eta^{(j)} := \max_{s\in S_j}\sum_{\ell=1}^{m}\frac{\alpha_\ell}{N_\ell^{(j)}}s_\ell = \max_{s\in S_j}\frac{1}{N_1^{(j)}}s_1 \le \frac{M}{N_1^{(j)}}\to 0$$

as $j\to\infty$. Using the bound $1-\cos x \ll x^2$, we obtain

$$\lambda_\alpha^{(j)} \ll (\eta^{(j)})^2 \to 0$$

as $j\to\infty$. We need to exclude the possibility that $\lambda_\alpha^{(j)}$ is zero. Note that $\mathcal{G}_j$ is a connected graph because S_j is a generating set for $\mathbf{G}_j$. Hence the zero eigenvalue of the discrete Laplacian is simple and therefore

$$\lambda_\alpha^{(j)} = 0 \Leftrightarrow \alpha = 0.$$

In particular, for $\alpha=(1,0,\dots,0)$ we have $\lambda_\alpha^{(j)}>0$. We have thus shown that the spectral gap $\lambda_1(\mathcal{G}_j)$ of $\mathcal{G}_j$ tends to zero as $j\to\infty$, up to a sequence extraction. By Proposition 4.6 this implies that the $\mathcal{G}_j$ do not form a family of expanders. The proof of Theorem 4.1 is therefore complete. □

4.3.1 Proof of Proposition 4.8

A very similar statement to that of Proposition 4.8 was given by Gamburd [30, Section 7]. The key ingredient in Gamburd's proof is Fell's continuity of induction and we will follow this line of thought.

For the remainder of this section set $G = \mathrm{SL}_2(\mathbb{R})$ and let $\hat{G}$ be its unitary dual, that is, the set of equivalence classes of (continuous) irreducible unitary representations of G. We endow the set $\hat{G}$ with the *Fell topology*. The Fell topology can actually be defined on more general sets of unitary representations of G, not only irreducible ones. We refer the reader to [27] and [10, Chapter F] for more background on the Fell topology.

A representation of G is called *spherical* if it has a non-zero K-invariant vector, where $K = \mathrm{SO}(2)$. Let us consider the subset $\hat{G}^1 \subset \hat{G}$ of irreducible spherical unitary representations. According to Lubotzky [50, Chapter 5], the set $\hat{G}^1$ can be parametrized as

$$\hat{G}^1 = i\mathbb{R}^+ \cup \left[0, \frac{1}{2}\right],$$

where $s \in i\mathbb{R}^+$ corresponds to the spherical unitary principal series representations, $s \in (0, \frac{1}{2})$ corresponds to the complementary series representation, and $s = \frac{1}{2}$ corresponds to the trivial representation. See also Gelfand–Graev–Pyatetskii-Shapiro [31, Chapter 1 §3] for a classification of the irreducible (spherical and non-spherical) unitary representations with a different parametrization. Moreover the Fell topology on $\hat{G}^1$ is the same as that induced by viewing the set of parameters s as a subset of $\mathbb{C}$, see [50, Chapter 5]. In particular, the spherical unitary principal series representations are bounded away from the identity.

Let us now recall the connection between the exceptional eigenvalues $\lambda \in (0, \frac{1}{4})$ and the complementary series representation. Consider the (left) quasiregular representation $(\lambda_{G/\Gamma}, L^2(G/\Gamma))$ of G defined by

$$\lambda_{G/\Gamma}(g) f(h\Gamma) = f(hg^{-1}\Gamma).$$

(We will denote this representation simply by $L^2(G/\Gamma)$.) Define the function $s(\lambda) = \sqrt{1/4 - \lambda}$ for $\lambda \in (0, \frac{1}{4})$. Then, $\lambda \in (0, \frac{1}{4})$ is an exceptional eigenvalue of $\Delta_{\Gamma\backslash\mathbb{H}}$ if and only if the complementary series $\pi_{s(\lambda)}$ occurs as a subrepresentation of $L^2(G/\Gamma)$. This is the so-called Duality Theorem [31, Chapter 1§4].

Let us return to the proof of Proposition 4.8. Let Γ and Γ_j be as in Proposition 4.8. Let $\Omega(\Gamma)$ denote eigenvalues of the Laplacian Δ_X on $X = \Gamma\backslash\mathbb{H}$. Let $\lambda_0(\Gamma) = \delta(1-\delta) = \inf \Omega(\Gamma)$ denote the bottom of the spectrum. Since Γ_j is by assumption a finite-index subgroup of Γ, we have $\delta(\Gamma_j) = \delta$ and consequently

$$\lambda_0(\Gamma_j) = \lambda_0(\Gamma) =: \lambda_0$$

for all j. Let V_{s_0} be the invariant subspace corresponding to the representation π_{s_0} and let $L^2_0(G/\Gamma_j)$ be its orthogonal complement in $L^2(G/\Gamma_j)$. For each j we can decompose the quasiregular representation of G into direct sum of subrepresentations

$$L^2(G/\Gamma_j) = L^2_0(G/\Gamma_j) \oplus V_{s_0}.$$

Recall that λ_0 is a simple eigenvalue by the result of Patterson [66]. By the Duality Theorem it follows that V_{s_0} is one-dimensional. The following lemma provides us with a link between uniform spectral gap and representation theory.

Lemma 4.9. *Let $\mathcal{R} \subset \hat{G}^1$ be the following set:*

$$\mathcal{R} = \bigcup_j \{(\pi, \mathcal{H}) : \pi \textit{ is spherical irreducible unitary subrep. of } L_0^2(G/\Gamma_j)\}/\sim,$$

where $\sim$ denotes the equivalence of representations. Let us further define the set

$$\mathcal{R}' = \bigcup_j \{(\pi, \mathcal{H}) : \pi \textit{ is spherical unitary subrep. of } L_0^2(G/\Gamma_j)\}/\sim$$

Then the following statements are equivalent.

(i) There exists $\varepsilon_0 > 0$ such that $|s - \delta| > \varepsilon_0$ for all j and all non-trivial resonances s of X_j.

(ii) The representation π_{s_0} is isolated in the set $\mathcal{R} \cup \{\pi_{s_0}\}$ with respect to the Fell topology.

(iii) The representation π_{s_0} is isolated in the set $\mathcal{R}' \cup \{\pi_{s_0}\}$ with respect to the Fell topology.

Proof. Equivalence of (ii) and (iii) is clear, since one can decompose every representation into irreducible ones. It suffices to prove the equivalence of (i) and (ii).

Since the resonances s of $X_j = \Gamma_j\backslash\mathbb{H}$ with $\mathrm{Re}(s) > \frac{1}{2}$ correspond to the eigenvalues $\lambda = s(1-s) \in [\lambda_0, \frac{1}{4})$, the uniform spectral gap condition (i) can be stated as follows. There exists $\varepsilon_1 > 0$ such that for all j we have

$$\Omega(\Gamma_j) \cap [0, \lambda_0 + \varepsilon_1) = \{\lambda_0\}. \tag{4.14}$$

By the discussion preceding the lemma, eigenvalues correspond to subrepresentations of $L_0^2(G/\Gamma)$, which allows us to reformulate (4.14) in representation-theoretic language. Set $s_0 = s(\lambda_0)$. Then by the Duality Theorem, there exists $\varepsilon > 0$ such that for all j and all $s \in (s_0 - \varepsilon, \frac{1}{2}]$, the complementary series representation π_s does not occur as a subrepresentation of $L^2(G/\Gamma_j)$. Since V_{s_0} is one-dimensional (and each representation π_s with $s \neq \frac{1}{2}$ is infinite-dimensional), (i) is equivalent to

$$\mathcal{R} \cap \left(s_0 - \varepsilon, \frac{1}{2}\right] = \{s_0\}. \tag{4.15}$$

Since the Fell topology on $\hat{G}^1$ is equivalent to the one induced by viewing $\hat{G}^1$ as the subset $i\mathbb{R}^+ \cup \left[0, \frac{1}{2}\right]$ of the the complex plane, the equivalence of (i) and (ii) is now evident. $\square$

Let 1_{Γ_j} denote the trivial representation of Γ_j on $\mathbb{C}$. Then the induced representation $\mathrm{Ind}_{\Gamma_j}^{\Gamma} 1_{\Gamma_j}$ is equivalent to the (left) quasiregular representation $(\lambda_{\mathbf{G}_j}, L^2(\mathbf{G}_j))$ of Γ defined by

$$(\lambda_{\mathbf{G}_j}(\gamma)F)(h\Gamma_j) = (\gamma.F)(h\Gamma_j) = F(h\gamma^{-1}\Gamma_j).$$

The action of Γ on $L^2(\mathbf{G}_j)$ given by $\gamma.F = \lambda_{\mathbf{G}_j}(\gamma)F$ is transitive. Hence the only Γ-fixed vectors are the constants. Thus we can decompose the representation of Γ on $L^2(\mathbf{G}_j)$ into a direct sum of subrepresentations

$$L^2(\mathbf{G}_j) = L^2_0(\mathbf{G}_j) \oplus \mathbb{C},$$

where $L^2_0(\mathbf{G}_j)$ is the subspace of functions orthogonal to the constant function, and $(1_\Gamma, \mathbb{C})$ does not occur as a subrepresentation of $L^2_0(\mathbf{G}_j)$.

Consider

$$\mathcal{T} = \bigcup_{j\in\mathbb{N}} \{(\rho, V) : \rho \text{ is unitary subrepresentation of } L^2_0(\mathbf{G}_j)\}/\sim .$$

We claim the following.

Lemma 4.10. *Assume that one of the equivalent statements in Lemma 4.9 holds true. Then the trivial representation 1_Γ is isolated in $\mathcal{T} \cup \{1_\Gamma\}$ with respect to the Fell topology.*

Proof. Let us start with some general definitions. Let K be a closed subgroup of a locally compact group H. Given a unitary representation (π, V) of K, the induced representation $\operatorname{Ind}_K^H \pi$ of H is defined as follows. Let μ be a quasi-invariant regular Borel measure on H/K and set

$$\operatorname{Ind}_K^H \pi := \{f : H \to V : f(hk) = \pi(k^{-1})f(h) \text{ for all } k \in K \text{ and } f \in L^2_\mu(H/K)\}. \tag{4.16}$$

Note that the requirement $f \in L^2_\mu(H/K)$ makes sense, since the norm of $f(g)$ is constant on each left coset of H. The action of G on $\operatorname{Ind}_H^G \pi$ is defined by

$$g.f(x) = f(g^{-1}x)$$

for all $x, g \in G$, $f \in \operatorname{Ind}_H^G \pi$. We also note that the equivalence class of the induced representation $\operatorname{Ind}_K^H \pi$ is independent of the choice of μ. We refer the reader to [10, Chapter E] for a more thorough discussion on properties of induced representations.

If two representations $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ are equivalent, we write $\mathcal{H}_1 = \mathcal{H}_2$ by abuse of notation. Using induction by stages (see [28] or [29] for a proof) we have

$$\begin{aligned}
V_{s_0} \oplus L^2_0(G/\Gamma_j) &= L(G/\Gamma_j)\\
&= \operatorname{Ind}_{\Gamma_j}^G 1_{\Gamma_j}\\
&= \operatorname{Ind}_{\Gamma}^G \operatorname{Ind}_{\Gamma_j}^{\Gamma} 1_{\Gamma_j}\\
&= \operatorname{Ind}_{\Gamma}^G L^2(\mathbf{G}_j)\\
&= \operatorname{Ind}_{\Gamma}^G 1_\Gamma \oplus \operatorname{Ind}_{\Gamma}^G L^2_0(\mathbf{G}_j)\\
&= V_{s_0} \oplus L^2_0(G/\Gamma) \oplus \operatorname{Ind}_{\Gamma}^G L^2_0(\mathbf{G}_j).
\end{aligned}$$

Choose an index j and a unitary subrepresentation (τ, V) of $L^2_0(\mathbf{G}_j)$. The above calculation implies that $\operatorname{Ind}_\Gamma^G \tau$ is a subrepresentation of $L^2_0(G/\Gamma_j)$. Since τ is unitary, so is $\operatorname{Ind}_\Gamma^G \tau$. Moreover $\operatorname{Ind}_\Gamma^G \tau$ is a spherical representation of G, since any non-zero function $f \in L^2(\mathbb{H}/\Gamma)$ and non-zero vector $v \in V$ gives rise to a non-zero K-invariant function $F \in \operatorname{Ind}_\Gamma^G \tau$. Indeed, we have $\mathbb{H} \cong K\backslash G$, so that we may view f as function $f : G \to \mathbb{C}$ satisfying $f(kg\gamma) = f(g)$ for all $g \in G, k \in K, \gamma \in \Gamma$. Now one easily verifies that $F = fv : G \to V$ belongs to $\operatorname{Ind}_\Gamma^G \tau$ and is invariant under K. We have thus shown that $\operatorname{Ind}_\Gamma^G \tau$ is a spherical unitary subrepresentation of $L_0^2(G/\Gamma_j)$. In other words, $\operatorname{Ind}_\Gamma^G \tau$ belongs to the set $\mathcal{R}'$, which we defined in the statement of Lemma 4.9.

Now suppose the lemma is false. Then there exists a sequence $(\tau_n)_{n\in\mathbb{N}} \subset \mathcal{T}$ that converges to 1_Γ as $n \to \infty$. On the other hand, π_{s_0} is weakly contained in $\operatorname{Ind}_\Gamma^G 1_\Gamma$. By Fell's continuity of induction [27] we have

$$\pi_{s_0} \prec \operatorname{Ind}_\Gamma^G 1_\Gamma = \lim_{n\to\infty} \operatorname{Ind}_\Gamma^G \tau_n \in \overline{\mathcal{R}'},$$

which contradicts Lemma 4.9. □

We can now prove Proposition 4.8.

Proof of Proposition 4.8. Assume that every non-trivial resonance s of $X_j = \Gamma_j\backslash\mathbb{H}$ satisfies $|s - \delta| > \varepsilon$ for some $\varepsilon > 0$ uniform in j. By the preceding lemma, this statement implies that every subrepresentation of $L_0^2(\mathbf{G}_j)$ is bounded away (with respect to the Fell topology) from the trivial representation, uniformly in j. The goal now is to show that the latter statement implies that the graphs $\mathcal{G}_j = \operatorname{Cay}(\mathbf{G}_j, S_j)$ yield a family of expanders. This implication seems to well-known in the literature and a proof of this fact appears in Gamburd [30, Section 7]. For the sake of completeness, we provide a detailed proof here.

Let us recall the definition of the Fell topology on $\hat{\Gamma}$ (for further reading consult [10, Chapter F]). For an irreducible unitary representation (π, V) of Γ, for a unit vector $\xi \in V$, for a finite set $Q \subset \Gamma$, and for $\varepsilon > 0$ let us define the set $W(\pi, \xi, Q, \varepsilon)$ that consists of all irreducible unitary representations (π', V') of Γ with the following property. There exists a unit vector $\xi' \in V'$ such that

$$\sup_{\gamma\in Q} |\langle \pi(\gamma)\xi, \xi\rangle_V - \langle \pi'(\gamma)\xi', \xi'\rangle_{V'}| < \varepsilon.$$

The Fell topology is generated by the sets $W(\pi, \xi, Q, \varepsilon)$. By Lemma 4.10 and the definition of the Fell topology, there exists $c_0 = c_0(\Gamma, S) > 0$ only depending on Γ and the generating set S of Γ, but not on j, such that for all functions $f \in L_0^2(\mathbf{G}_j)$ with $\|f\|_{L^2(\mathbf{G}_j)} = 1$ we have

$$\sup_{\gamma\in S} |\langle \gamma.f, f\rangle_{L^2(\mathbf{G}_j)} - 1| \geq c_0. \tag{4.17}$$

Let us drop the index $L_0^2(\mathbf{G}_j)$ from the notation and write $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ instead. Let $F \in L_0^2(\mathbf{G}_j)$ be an arbitrary non-zero function. Applying (4.17) to the unit vector

$$f = \frac{F}{\|F\|} \in L_0^2(\mathbf{G}_j)$$

yields the more practical inequality

$$\sup_{\gamma\in S} |\langle \gamma.F - F, F\rangle| \geq c_0\|F\|^2. \tag{4.18}$$

By the Cauchy-Schwarz inequality, (4.18) implies

$$\sup_{\gamma\in S}\|\gamma.F-F\|\geq c_0\|F\|. \tag{4.19}$$

Now fix a non-empty subset A of $\mathbf{G}_j$ with $|A|\leq\frac{1}{2}|\mathbf{G}_j|$ and define the function

$$F(x):=-|A|+|\mathbf{G}_j|\cdot\mathbf{1}_A(x),$$

where $\mathbf{1}_A$ denotes the indicator function of A. By construction, $F\in L^2_0(\mathbf{G}_j)$. A computation shows that

$$\|F\|^2=|A||\mathbf{G}_j|(|\mathbf{G}_j|-|A|).$$

On the other hand (recall that $(\gamma.F)(x)=F(\gamma^{-1}x)$),

$$\begin{aligned}\|\gamma.F-F\|^2&=\sum_{x\in\mathbf{G}_j}\Big(F(\gamma^{-1}x)-F(x)\Big)^2\\&=|\mathbf{G}_j|^2\sum_{x\in\mathbf{G}_j}\big(\mathbf{1}_{\gamma A}(x)-\mathbf{1}_A(x)\big)^2\\&=|\mathbf{G}_j|^2\cdot|\gamma A\bigtriangleup A|,\end{aligned}$$

where $A\bigtriangleup B$ denotes the symmetric difference of A and B. Thus, invoking (4.19) leads to

$$|\gamma A\bigtriangleup A|=\frac{\|\gamma.F-F\|^2}{|\mathbf{G}_j|^2}\geq c_0^2\frac{\|F\|^2}{|\mathbf{G}_j|^2}=c_0^2\left(1-\frac{|A|}{|\mathbf{G}_j|}\right)|A|\geq\frac{c_0^2}{2}|A|$$

for some element $\gamma\in S$. Hence, we obtain a lower bound for the size of the boundary of A in the graph $\mathcal{G}_j=\mathrm{Cay}(\mathbf{G}_j,S_j)$:

$$|\partial A|\geq\frac{1}{2}\sup_{\gamma\in S}|\gamma A\bigtriangleup A|\geq\frac{c_0^2}{4}|A|.$$

From the definition of the Cheeger isoperimetric constant in (4.13), it follows that

$$h(\mathcal{G}_j)\geq\frac{c_0^2}{4}$$

for all j. Consequently, the graphs $\mathcal{G}_j$ form a family of expanders, completing the proof of Proposition 4.8. □

Chapter 5

Fractal Weyl bounds and Hecke triangle groups

5.1 Introduction and statement of results

Hecke triangle groups are, in some sense, natural generalizations of the more prominent modular group

$$\Gamma^{\mathbb{Z}} = \mathrm{PSL}_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{R}) : a, b, c, d \in \mathbb{Z} \right\},$$

which is generated by the two elements

$$T := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad S := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

On the hyperbolic plane $\mathbb{H}$, these elements act by $S(z) = -1/z$ and $T(z) = z + 1$. In [37], Hecke introduced the groups Γ_w generated by $S(z) := -1/z$ and $T_w(z) := z + w$ and showed that Γ_w is discrete if and only if $w = 2\cos(\pi/q)$ for $q \in \mathbb{N}_{\geq 3}$, or $w \geq 2$. In both cases, these groups came to be known as *Hecke triangle groups with cusp width w*. He found that

$$\mathcal{F}(w) = \left\{ z \in \mathbb{H} : |\mathrm{Re}(z)| < \frac{w}{2},\ |z| > 1 \right\}$$

provides a fundamental domain for the action of Γ_w on $\mathbb{H}$.

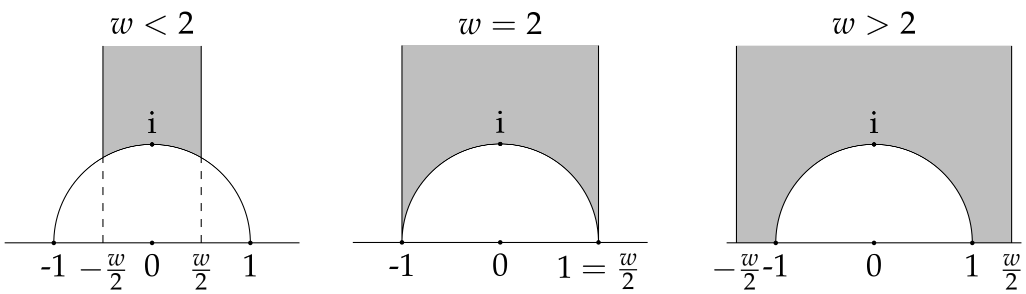

Figure 5.1: Fundamental domain for three Hecke triangle groups

The fundamental domain $\mathcal{F}(w)$ meets the real line in an open interval (which leads to a funnel in the quotient $\Gamma_w\backslash\mathbb{H}$) if and only if $w > 2$. Consequently, the Hecke triangle groups with $w \leq 2$ (that is $w = 2$ or $w = 2\cos(\pi/q)$ for $q \in \mathbb{N}_{\geq 3}$) are Fuchsian groups of the first kind, meaning that the quotient $\Gamma_w\backslash\mathbb{H}$ is of finite area. On the other hand, for $w > 2$, Γ_w is a Fuchsian group of the second kind, i.e. $\Gamma_w\backslash\mathbb{H}$ has infinite area.

In the present chapter we focus on Hecke groups of the second kind ($w > 2$) and their finite-index subgroups. In particular, the limit set $\Lambda(\Gamma_w)$ for $w > 2$ is a Cantor-like fractal, whose Hausdorff dimension we denote by δ_w. For certain values of w, the Hausdorff dimension δ_w has been estimated numerically by Phillips–Sarnak [77]. For instance, we have $\delta_3 = 0.753 \pm 0.003$. Note that for any finite-index subgroup $\widetilde{\Gamma} \leqslant \Gamma_w$, we have $\delta(\widetilde{\Gamma}) = \delta(\Gamma_w) = \delta_w$. In the Appendix, Subsection A.4, we investigate the bahaviour of δ_w as $w \to \infty$. We write $\delta = \delta_w$ if the cusp width w is fixed.

Our principal aim is to establish a fractal growth estimate on the Selberg zeta function in strips parallel to the imaginary axis and bounded away from the real axis. In other words, we are looking for an estimate of the type

$$\log|Z_\Gamma(s)| \ll_{\varepsilon,\sigma} |\mathrm{Im}(s)|^{\delta+\varepsilon}, \tag{5.1}$$

for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) \geq \sigma$ and $|\mathrm{Im}(s)| \geq 10$ (here, the number 10 may be replaced by any other positive number). We refer to estimates of this type as *fractal* upper bounds, since they involve the Hausdorff dimension of the limit set $\Lambda(\Gamma)$. For cofinite Fuchsian groups (in which case $\delta = 1$) the estimate (5.1) is true without the ε-loss in the exponent and can be proven within the framework of Selberg theory, see for instance [90, Lemma 5.2.3]. Guillopé–Lin–Zworski [33] proved a fractal upper bound without the ε-loss for convex co-compact Schottky groups acting on the $(n+1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}$. Their estimate leads to new estimates on the number of resonances and scattering poles of the Laplacian on $X = \Gamma\backslash\mathbb{H}^{n+1}$, often referred to as 'fractal Weyl upper bounds'. Their result provides a rigorous statement in the direction of the fractal Weyl conjecture, proposed by Sjöstrand [85] and Lu–Sridhar–Zworski [48]. In the case of hyperbolic surfaces this conjecture reads as follows. If $\mathcal{R}(X)$ denotes the set of resonances of the Laplacian on a non-elementary hyperbolic surface X, then for all σ negative enough,

$$N_X(\sigma, T) := \#\{s \in \mathcal{R}(X) : \mathrm{Re}(s) \geq \sigma,\ |\mathrm{Im}(s)| \leq T\} \asymp T^{1+\delta}, \tag{5.2}$$

as $T \to \infty$. Using the fractal upper bound for the Selberg zeta function, Guillopé–Lin–Zworski proved that for convex co-compact Schottky manifolds X one has

$$M_X(\sigma, T) := \#\big\{s \in \mathcal{R}(X) : \mathrm{Re}(s) \geq \sigma,\ |\mathrm{Im}(s) - T| \leq 1\big\} \ll_\sigma T^\delta. \tag{5.3}$$

Clearly, the upper bound in (5.2) follows by integrating (5.3) along T. This settles one half of the fractal Weyl law conjecture for all geometrically finite hyperbolic surfaces *without* cusps.

As of today no analogues of (5.1), (5.3), or (5.3) have been proven for Fuchsian groups of the second kind containing parabolic elements. In the present chapter, we widen the validity of (5.1) to Hecke triangle groups of the second kind and their finite-index subgroups. We actually prove a more general result for L-functions (twisted Selberg zeta functions). To state it, let $\rho\colon \Gamma_w \to U(V)$ be a finite-dimensional unitary representation with representation space V, and recall that for $\mathrm{Re}(s) > \delta$ the L-function associated to (Γ_w, ρ) is defined by the product

$$L_{\Gamma_w}(s,\rho) = \prod_{[\gamma]\in[\Gamma_w]_p} \prod_{k=0}^{\infty} \det\left(1-\rho(\gamma)e^{-(s+k)\ell(\gamma)}\right).$$

Our first main result is the following.

Theorem 5.1. *Let Γ_w be the Hecke triangle group with cusp width $w > 2$, let $\rho\colon \Gamma_w \to U(V)$ be a finite-dimensional unitary representation of Γ_w, and let $\delta = \delta_w$ be the Hausdorff dimension of $\Lambda(\Gamma_w)$. Then $L_{\Gamma_w}(s,\rho)$ extends to a meromorphic function on $\mathbb{C}$ and all its poles are contained in $\frac{1}{2}(1-\mathbb{N}_0)$. For all $s = \sigma + iT$ with $\sigma, T \in \mathbb{R}$, and $|T| \geq 2$ there exists $C = C(\sigma, w, \rho)$ such that*

$$\log|L_{\Gamma_w}(s,\rho)| \leq C|T|^{\delta}(\log|T|)^{2-\delta}.$$

Theorem 5.1 has various corollaries. By the Venkov–Zograf factorization formula (see Theorem A.3), the Selberg zeta function $Z_{\widetilde{\Gamma}}$ of a finite-index subgroup $\widetilde{\Gamma} \leqslant \Gamma_w$ is equal to the L-function associated to (Γ_w, λ), where λ is the representation of $\widetilde{\Gamma}$ that is induced from the trivial one-dimensional representation of Γ_w. We immediately deduce the following growth estimate on the Selberg zeta function of arbitrary finite-index subgroups of Hecke triangle groups.

Corollary 5.2. *Let $w > 2$ and let $\widetilde{\Gamma}$ be a finite-index subgroup of Γ_w. Then for all $s = \sigma + iT$ with $\sigma, T \in \mathbb{R}$, and $|T| \geq 2$ there exists $C = C(\sigma, w, \widetilde{\Gamma})$ such that*

$$\log|Z_{\widetilde{\Gamma}}(s)| \leq C|T|^{\delta}(\log|T|)^{2-\delta}.$$

Up to the logarthmic loss, Corollary 5.2 is analogous to the result of Guillopé–Lin–Zworski [33] for Schottky groups. It is likely that Corollary 5.2 holds without this logarithmic term, though our methods do not allow it to be removed.

From Borthwick–Judge–Perry [13] we know that if Γ is a finitely generated, *torsion-free*[5] Fuchsian group, then the resonances for $X = \Gamma\backslash\mathbb{H}$ correspond one-to-one to the zeros of the Selberg zeta function Z_Γ, with the exception of a set of well-understood real zeros. Therefore, using a standard argument of complex analysis, we can convert the growth estimate of the preceding corollary to upper bounds on the resonance counting functions $N_X(\sigma, T)$ and $M_X(\sigma, T)$ defined in (5.2) and (5.3).

Corollary 5.3. *Let $\widetilde{\Gamma}$ be torsion-free, finite-index subgroup of some Hecke triangle group Γ_w with $w > 2$. Set $\widetilde{X} = \widetilde{\Gamma}\backslash\mathbb{H}$. Then for all $\sigma \in \mathbb{R}$ there exists $C = C(\sigma, w, \widetilde{\Gamma})$ such that*

$$M_{\widetilde{X}}(\sigma, T) \leq C|T|^{\delta}(\log|T|)^{2-\delta}, \quad |T| > 2$$

and

$$N_{\widetilde{X}}(\sigma, T) \leq CT^{1+\delta}(\log T)^{2-\delta}, \quad T > 2.$$

[5] this means that Γ has no elliptic elements. Consequently, the quotient $\Gamma\backslash\mathbb{H}$ has no conical singularities and therefore it qualifies as a surface in the sense of Riemannian geometry

Corollary 5.3 gives, for the first time, a fractal Weyl upper bound for a class of hyperbolic surfaces *with cusps*, at the expense of the logarithmic factor. Notice that Corollary 5.3 does not apply to the Hecke triangle group Γ_w itself, since it contains the elliptic element S. The quotient $X_w := \Gamma_w \backslash \mathbb{H}$ has one cusp, one funnel, and one conical singularity. Because of the latter, X_w is an orbifold (rather than a manifold).

There exist plenty of torsion-free subgroups of Γ_w. By passing to finite-index, torsion-free subgroups $\widetilde{\Gamma} \leqslant \Gamma_w$, one can obtain plenty of geometrically finite groups without elliptic elements, producing examples of Riemannian surfaces with several cusps and funnels. In Subsection 5.1.2 below we give some examples of such subgroups of Γ_w.

Another consequence of Theorem 5.1 (although far less direct) is an explicit strip in the complex plane containing infinitely many resonances for hyperbolic surfaces arising from torsion-free, finite-index subgroups of Γ_w.

Theorem 5.4. *Let $w > 2$ and let $\widetilde{\Gamma}$ be a torsion-free, finite-index subgroup of Γ_w. Then for every $\varepsilon > 0$, the hyperbolic surface $\widetilde{X} = \widetilde{\Gamma} \backslash \mathbb{H}$ has infinitely many resonances in the half-plane*

$$\mathrm{Re}(s) \geq \frac{\delta}{2} - \delta^2 - \varepsilon.$$

If we assume further that the parameter w is the square-root of some integer ≥ 5, then for every $\varepsilon > 0$, $\widetilde{X}$ has infinitely many resonances in the half-plane

$$\mathrm{Re}(s) \geq \frac{\delta}{2} - \frac{1}{4} - \varepsilon.$$

Theorem 5.4 is proved by closely following the analysis of Jakobson–Naud [40]. It ultimately follows from a lower bound on the *essential spectral gap*. We refer to [40] for a stronger conjecture on the size of the essential spectral gap and its heuristic justification, and for an application concerning the error term of the hyperbolic lattice counting problem.

The present chapter is roughly organized as follows. In Section 5.2 we develop a general framework for transfer operators acting on spaces of vector-valued holomorphic functions. In Section 5.3 we prove Theorem 5.1 and Corollary 5.3. In Section 5.4 we prove Theorem 5.4. We continue to use the notational conventions that are already in place, see Subsection 2.7.

5.1.1 Overview of main ideas

Let us give an overview of the main ideas leading to Theorem 5.1. For the sake of exposition, we will only outline the argument for the group Γ_w. The generalization to arbitrary finite-index subgroups $\widetilde{\Gamma} \leqslant \Gamma_w$ is similar and relies on a vector-valued extension of this argument.

We will adopt a similar a view as in Guillopé–Lin–Zworski [33], where thermodynamic formalism and transfer operators techniques play a crucial role. Recall from Subsection 2.6 that for Fuchsian Schottky groups Γ, the Selberg zeta function can be expressed as the Fredholm determinant

$$Z_\Gamma(s) = \det(1 - \mathcal{L}_s)\,, \tag{5.4}$$

where $\mathcal{L}_s$ is the standard tranfer operator arising from the geometric contruction of Γ. By a clever exploitation of (5.4), Guillopé–Lin–Zworski [33] managed to derive fractal estimates for the Selberg zeta function in strips parallel to the imaginary axis.

For Fuchsian groups Γ with parabolic elements (that is, when $X = \Gamma\backslash\mathbb{H}$ has cusps) it is less obvious to establish Fredholm determinant representations such as (5.4). Mayer [51] discovered that the Selberg zeta function of the modular group $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ satisfies (5.4) in the half-plane $\mathrm{Re}(s) > \frac{1}{2}$, where the transfer operator $\mathcal{L}_s$ is given by

$$\mathcal{L}_s f(z) = \sum_{n=1}^{\infty} \left(\frac{1}{z+n}\right)^{2s} f\left(\frac{1}{z+n}\right). \tag{5.5}$$

We point out that $\mathcal{L}_1$ is precisely the Perron–Ruelle operator for the Gauss map on continued fractions. Here, the operator $\mathcal{L}_s$ acts on the Banach space of functions f which are holomorphic on the open disk $D(1, \frac{3}{2})$ and continuous on the closure $\overline{D(1, \frac{3}{2})}$.

Morita [54] proved analogues of (5.4) (up to certain correction factors) for general cofinite Fuchsian groups. More recently Pohl [71] and Fedosova–Pohl [26] considered geometrically finite (cofinite and non-cofinite) Fuchsian groups and proved existence of transfer operators satisfying the Fredholm determinant identity (5.4), under an additional geometric condition.

A particularly nice class of examples are the Hecke triangle groups Γ_w with $w > 2$, for which there is a relatively simple transfer operator satisfying (5.4). Not surprisingly, since $\Gamma_1 = \mathrm{PSL}_2(\mathbb{Z})$, the transfer operator for Γ_w is very similar to Mayer's operator (5.5), see Subsection 5.3.1 for a description.

This new point of view on the Selberg zeta function enables us to prove fractal upper bounds in the same spirit of [33]. The main idea is to let the transfer operator act on 'refined' function spaces. This means that we work on a small neighbourhood $\Omega(h)$ in $\mathbb{C}$ of (a portion of) the limit set $\Lambda(\Gamma_w)$ of Γ_w. Here, the parameter $h > 0$ determines how close we are to $\Lambda(\Gamma_w)$. The Hilbert space on which $\mathcal{L}_s$ acts is then given by the Bergman space $H^2(\Omega(h))$ on $\Omega(h)$.

Identity (5.4) remains valid independently of the parameter h (provided it is small enough to ensure that $\mathcal{L}_s : H^2(\Omega(h)) \to H^2(\Omega(h))$ is of trace class). Thus there is some freedom in the choice of h. On the other hand, the singular values of $\mathcal{L}_s$ on which our estimates are based, heavily depend on h.

As we let $h \searrow 0$, that is, as we come closer to the limit set, its fractal nature becomes more visible and $\Omega(h)$ has more and more connected components. Whereas the analogue of $\Omega(h)$ in [33] is a union of Euclidean disks all of which have diameter $\asymp h$, in our proof the connected components of $\Omega(h)$ are 'stretched balls' having diameters ranging from h to $\asymp \sqrt{h}$. The lack of structure in our setting poses new technical difficulties that were not present in the Schottky-scenario. Nevertheless, for $\mathrm{Re}(s) > \frac{1}{2}$ the singular values of $\mathcal{L}_s : H^2(\Omega(h)) \to H^2(\Omega(h))$

can be shown to satisfy an upper bound of the type

$$\mu_k(\mathcal{L}_s) \ll_\sigma h^{-A} e^{B|\mathrm{Im}(s)|h} \exp(-Ch^\delta k), \quad k \in \mathbb{N} \tag{5.6}$$

for some constants $A, B, C > 0$ possibly depending on $\sigma = \mathrm{Re}(s)$, but not on h, see Proposition 5.18. Hence the sequence of singular values decays exponentially fast, and the decay rate is governed by h^δ (it is only here where the Hausdorff dimension makes its appearance). Using (5.6) in conjunction with Weyl's inequality on determinants, we get

$$\log |Z_{\Gamma_w}(s)| \leq \sum_{k=1}^{\infty} \log(1 + \mu_k(\mathcal{L}_s)) \ll_\sigma h^{-\delta} \log\left(1 + h^{-A} e^{B|\mathrm{Im}(s)|h}\right)^2 \tag{5.7}$$

for all $\mathrm{Re}(s) > \frac{1}{2}$ and for all h small enough. A computation reveals that we can optimize this bound by choosing

$$h = \left(\frac{|\mathrm{Im}(s)|}{\log|\mathrm{Im}(s)|}\right)^{-1}.$$

Note that this is a valid choice for large $|\mathrm{Im}(s)|$. With this choice, we obtain

$$\log |Z_{\Gamma_w}(s)| \ll_\sigma |\mathrm{Im}(s)|^\delta \cdot (\log |\mathrm{Im}(s)|)^{2-\delta}, \tag{5.8}$$

which establishes Theorem 5.1 in the half-plane $\mathrm{Re}(s) > \frac{1}{2}$. Notice that the log-loss in (5.8) is caused solely by the h^{-A}-term in (5.6).

To deal with the case $\mathrm{Re}(s) \leq \frac{1}{2}$, we are forced to work with a meromorphic continuation of the operator $\mathcal{L}_s$, making the proof of Theorem 5.1 much more subtle in this range. It should be noted that the Selberg zeta function of a Fuchsian group Γ has poles in $\frac{1}{2}(1 - \mathbb{N}_0)$ if and only if Γ contains parabolic elements. This is an additional technical difficulty that only occurs in the presence of cusps. Luckily, the meromorphic continuation for $\mathcal{L}_s$ in the case of Hecke triangle groups is constructive, see Proposition 5.24. It turns out that we can write $\mathcal{L}_s$ as a sum of a finite-rank operator (which extends meromorphically to $\mathbb{C}$) and an operator which is holomorphic on the entire complex plane. Understanding the meromorphic continuation of $\mathcal{L}_s$ amounts to understanding the analytic properties of the naturally appearing *Lerch zeta function*, see Subsection 5.3.4

The bulk of the proof of Theorem 5.1 is concerned with establishing (5.6). Fortunately, we can draw upon the work of Bandtow–Jenkinson [7], for which we give an independent treatment in Section 5.2.

5.1.2 Examples of torsion-free subgroups of Γ_w and one consequence

As promised in the introduction of this chapter, we now give some examples of finite-index subgroups $\widetilde{\Gamma} \leqslant \Gamma_w$ with the property that $\widetilde{\Gamma}$ does not contain elliptic elements. These are precisely the groups that come under the purview of Corollary 5.3 and Theorem 5.4.

If the parameter $w > 2$ is an integer, then clearly the group Γ_w only consists of matrices having integer coefficients. In this case we can fabricate families of torsion-free subgroups using congruence subgroups. More precisely, let $m \geq 3$ and $q \geq 2$ be integers and let $\Gamma^{\mathbb{Z}}(q) := \ker(\pi_q)$ be the principal congruence subgroup of level q, where

$$\pi_q \colon \mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z}), \quad g \mapsto g \mod q.$$

is the reduction map modulo q. Consider the group $\Gamma_{m,q} := \Gamma_m \cap \Gamma^{\mathbb{Z}}(q)$. Since $\Gamma^{\mathbb{Z}}(q)$ is known to be torsion-free, so is $\Gamma_{m,q}$. Moreover, its index as a subgroup of Γ_m is finite, since $[\Gamma_m : \Gamma_{m,q}] \leq |\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})| < \infty$. Hence, the groups $\Gamma_{m,q}$ with $m \geq 3$ and $q \geq 2$ are all torsion-free, finite-index subgroups of the Hecke triangle group Γ_m.

For arbitrary $w > 2$ we can produce a torsion-free subgroup $\Gamma_w^0 \leqslant \Gamma_w$ of index 2 as follows. Let $\rho : \Gamma_w \to \mathbb{C}^\times$ be the one-dimensional representation defined by $\rho(T_w) = 1$ and $\rho(S) = -1$ and set $\Gamma_w^0 := \ker(\rho)$. The group Γ_w^0 is a normal subgroup of Γ_w (being the kernel of a homomorphism), is freely generated by the elements T_w and ST_wS, and contains no elliptic elements. Moreover, we have

$$\Gamma_w/\Gamma_w^0 \simeq \{\mathrm{id}, S\} \simeq \mathbb{Z}/2\mathbb{Z}.$$

The action of Γ_w^0 on $\mathbb{H}$ has the fundamental domain

$$\mathcal{F}^0(w) = \mathcal{F}(w) \cup S.\mathcal{F}(w),$$

see Figure 5.2.

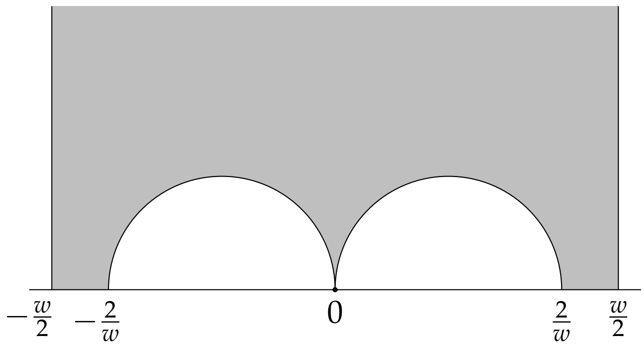

Figure 5.2: Fundamental domain for Γ_w^0 with $w > 2$

The quotient $X_w^0 := \Gamma_w^0\backslash\mathbb{H}$ is a hyperbolic surface (no conical singularities!) with one funnel ($n_f = 1$), two cusps ($n_c = 2$), and genus zero ($g = 0$). In particular, X_w^0 has Euler characteristic $\chi(X_w^0) = 2 - 2g - n_c - n_f = -1$. A consequence of this example is the following side result.

Corollary 5.5. *Fix $w > 2$. The Selberg zeta functions of the groups Γ_w and Γ_w^0 have no zeros s with $\mathrm{Re}(s) > \frac{1}{2}$, except at $s = \delta$.*

Proof. All the zeros of $Z_{\Gamma_w^0}(s)$ with $\mathrm{Re}(s) > \frac{1}{2}$ correspond to the L^2-eigenvalues of the Laplacian on X_w^0 in the interval $(0, \frac{1}{4})$ (so-called 'small' eigenvalues of X_w^0). By the result of Ballmann–Mathiesen–Mondal [5], the number of small eigenvalues is bounded by $-\chi(X^0_w) = 1$. Hence, the eigenvalue $\delta(1-\delta)$ corresponding to the zero $s = \delta$ is the only eigenvalue, which proves Corollary 5.5 for Γ^0_w.

To prove the result for Γ_w we invoke the Venkov–Zograf factorization formula, Theorem A.3, from which we deduce that

$$Z_{\Gamma^0_w}(s) = Z_{\Gamma_w}(s) L_{\Gamma_w}(s,\rho),$$

where ρ is the representation given by $\rho(T_w) = 1$ and $\rho(S) = -1$. From Theorem 5.1 we know that $L_{\Gamma_w}(s,\rho)$ is holomorphic on the half-plane $\{\mathrm{Re}(s) > \frac{1}{2}\}$. Therefore, in $\{\mathrm{Re}(s) > \frac{1}{2}\}$ zeros of $Z_{\Gamma_w}(s)$ must be zeros of $Z_{\Gamma^0_w}(s)$. The result for Γ_w now follows from the result for Γ^0_w. $\square$

5.2 Vector-valued transfer operators and singular value estimates

5.2.1 Setup and Notation

In this section we consider a rather general type of transfer operator and prove an estimate for their singular values. The main result of this section is Theorem 5.10 below. We use some methods in the paper of Bandtlow–Jenkinson [7], in which they prove precise estimates for the eigenvalues for transfer operators acting on spaces of holomorphic functions. The statements that we prove are tailored specifically to our situation.

One of the main novelties in our approach is that we consider *vector-valued* transfer operators. Moreover we do not have to introduce the notion of *exponential classes* developed by Bandtlow in [6] (prior to [7]), which was crucial in deriving the results in [7]. Instead, our approach relies solely on some rather well-known properties of singular values, which we have deferred to the Appendix, Subsection A.1.1.

For the remainder of this section let V be a finite-dimensional complex vector space, endowed with the hermitian inner product $\langle\cdot,\cdot\rangle_V$. Moreover, given a non-empty open subset $\Omega \subset \mathbb{C}$, we consider the vector-valued Bergman space on Ω,

$$H^2(\Omega;V) := \left\{ f\colon \Omega \to V \text{ holomorphic} \;\middle|\; \|f\|^2_{L^2(\Omega)} := \int_\Omega \|f(z)\|^2_V \,\mathrm{dvol}(z) < \infty \right\}, \tag{5.9}$$

where $\|v\|_V := \sqrt{\langle v,v\rangle_V}$ and vol is the Lebesgue measure. Endowed with the inner product

$$\langle f,g\rangle := \int_\Omega \langle f(z), g(z)\rangle_V \,\mathrm{dvol}(z),$$

the space $H^2(\Omega;V)$ is a Hilbert space. Note that $H^2(\Omega;\mathbb{C}) = H^2(\Omega)$ is the usual Bergman space on Ω. The norm of an endomorphism $M \in \mathrm{End}(V)$ is defined as

$$\|M\|_{\mathrm{End}(V)} := \sup_{v\in V\setminus\{0\}} \frac{\|Mv\|_V}{\|v\|_V}.$$

Furthermore, given sets A, B, we write $A \Subset B$ to mean that the closure of A is a compact subset of B. The following definition generalizes the terminology used in [7].

Definition 5.6. *Let $\Omega, \Omega' \subset \mathbb{C}$ be two non-empty, open, bounded subsets and let $\mathcal{I}$ be a countable index set. The quintuple $(\Omega, \Omega', V, \phi_n, W_n)_{n\in\mathcal{I}}$ is called a vector-valued holomorphic map-weight system if*

(i) $(\phi_n)_{n\in\mathcal{I}}$ is a family of holomorphic maps $\phi_n : \Omega \to \Omega'$ such that

$$\bigcup_{n\in\mathcal{I}} \phi_n(\Omega) \Subset \Omega',$$

(ii) the weights W_n are functions in $H^2(\Omega; \mathrm{End}(V))$ satisfying

$$\sum_{n\in\mathcal{I}} \|W_n(\cdot)\|_{\mathrm{End}(V)} \in L^2(\Omega, \mathrm{dvol}).$$

If $\Omega' = \Omega$ we write $(\Omega, V, \phi_n, W_n)_{n\in\mathcal{I}}$ instead of $(\Omega, \Omega, V, \phi_n, W_n)_{n\in\mathcal{I}}$.

5.2.2 Generic transfer operators

Given a vector-valued holomorphic map-weight system

$$(\Omega, \Omega', V, \phi_n, W_n)_{n\in\mathcal{I}}$$

we can associate to it the (initially only formal) *transfer operator*

$$\mathcal{L}f(z) = \sum_{n\in\mathcal{I}} W_n(z) f(\phi_n(z)) \tag{5.10}$$

acting on functions $f \in H^2(\Omega'; V)$. The following result shows that (5.10) defines a bounded operator

$$\mathcal{L} : H^2(\Omega'; V) \to H^2(\Omega; V),$$

under the conditions given by Definition 5.6.

Proposition 5.7. *Let $(\Omega, \Omega', V, \phi_n, W_n)_{n\in\mathcal{I}}$ be a vector-valued holomorphic map-weight system. Define $\rho_n := \mathrm{dist}\big(\phi_n(\Omega), \partial\Omega'\big)$ and set $\rho := \inf_{n\in\mathcal{I}} \rho_n > 0$.*

Then the operator $\mathcal{L} : H^2(\Omega'; V) \to H^2(\Omega; V)$ is well-defined and bounded with norm

$$\|\mathcal{L}\|_{H^2(\Omega';V)\to H^2(\Omega;V)} \le \rho^{-1} \left\| \sum_{n\in\mathcal{I}} \|W_n(\cdot)\|_{\mathrm{End}(V)} \right\|_{L^2(\Omega)}.$$

For the proof of Proposition 5.7 we need the following result.

Lemma 5.8. *Let $\Omega \subset \mathbb{C}$ be an open subset of the complex plane. Given $f \in H^2(\Omega; V)$ and $z_0 \in \Omega$ we have*

$$\|f(z_0)\|_V \le \mathrm{dist}(z_0, \partial\Omega)^{-1} \|f\|_{L^2(\Omega)}.$$

In particular, for a compact subset $K \subset \Omega$ we have

$$\sup_{z\in K} \|f(z)\|_V \le \mathrm{dist}(K, \partial\Omega)^{-1} \|f\|_{L^2(\Omega)}.$$

Proof. Fix $0 < r < \operatorname{dist}(z_0, \partial\Omega)$, so that $\overline{D(z_0, r)} \subset \Omega$. Let $d = \dim V$ and fix an orthonormal basis $\mathbf{e}_1, \ldots, \mathbf{e}_d$ for V. Then for each $l \in \{1, \ldots, d\}$ the function

$$f_l := \langle f, \mathbf{e}_l \rangle_V$$

is holomorphic on Ω. In particular, we can Taylor-expand f_l as

$$f_l(z) = \sum_{k=0}^{\infty} a_{k,l}(z - z_0)^k, \quad \forall z \in D(z_0, r)$$

for some suitable $a_{k,l} \in \mathbb{C}$. A simple calculation shows

$$\frac{1}{\operatorname{vol}(D(z_0, r))} \int_{D(z_0,r)} f_l(z) \operatorname{dvol}(z) = a_{0,l} = f_l(z_0).$$

By Cauchy–Schwarz we have

$$|f_l(z_0)|^2 \leq \frac{1}{\operatorname{vol}(D(z_0, r))} \int_{D(z_0,r)} |f_l(z)|^2 \operatorname{dvol}(z).$$

Summing over l yields

$$\begin{aligned}
\|f(z_0)\|_V^2 &= \sum_{l=1}^{d} |f_l(z_0)|^2 \\
&\leq \frac{1}{\operatorname{vol}(D(z_0, r))} \int_{D(z_0,r)} \sum_{l=1}^{d} |f_l(z)|^2 \operatorname{dvol}(z) \\
&= \frac{1}{\operatorname{vol}(D(z_0, r))} \int_{D(z_0,r)} \|f(z)\|_V^2 \operatorname{dvol}(z) \\
&\leq r^{-2} \|f\|_{L^2(\Omega)}^2.
\end{aligned}$$

Sending $r \nearrow \operatorname{dist}(z_0, \partial\Omega)$ finishes the proof. $\square$

Proof of Proposition 5.7. Without loss of generality we may assume that $\mathcal{I} = \mathbb{N}$. Consider the truncated transfer operator

$$\mathcal{L}_N f(z) := \sum_{n=1}^{N} W_n(z) f(\phi_n(z)), \quad f \in H^2(\Omega'; V),\ z \in \Omega.$$

Clearly, for each $N \in \mathbb{N}$ the operator $\mathcal{L}_N : H^2(\Omega'; V) \to H^2(\Omega; V)$ is well-defined. By Lemma 5.8 we have

$$\sup_{z \in \Omega} \|f(\phi_n(z))\|_V \leq \rho_n^{-1} \|f\|_{L^2(\Omega')} \leq \rho^{-1} \|f\|_{L^2(\Omega')},$$

for all $f \in H^2(\Omega'; V)$ and all $n \in \mathbb{N}$. Hence, by the triangle-inequality and the definition of the norm on $\operatorname{End}(V)$, we have

$$\|\mathcal{L}_N f(z)\|_V \leq \rho^{-1} \|f\|_{L^2(\Omega')} \sum_{n=1}^{N} \|W_n(z)\|_{\operatorname{End}(V)}, \tag{5.11}$$

for all $z \in \Omega$. Integrating (the square) of (5.11) over Ω leads to

$$\begin{aligned}\|\mathcal{L}_N f\|_{L^2(\Omega)} &\leq \rho^{-1}\|f\|_{L^2(\Omega')} \left\| \sum_{n=1}^{N} \|W_n(\cdot)\|_{\mathrm{End}(V)} \right\|_{L^2(\Omega)} \\ &\leq \rho^{-1}\|f\|_{L^2(\Omega')} \left\| \sum_{n=1}^{\infty} \|W_n(\cdot)\|_{\mathrm{End}(V)} \right\|_{L^2(\Omega)},\end{aligned}$$

where in the last line we used the assumption

$$\sum_{n=1}^{\infty} \|W_n(\cdot)\|_{\mathrm{End}(V)} \in L^2(\Omega). \tag{5.12}$$

Since $f \in H^2(\Omega'; V)$ was arbitrary, we obtain

$$\|\mathcal{L}_N\|_{H^2(\Omega';V)\to H^2(\Omega;V)} \leq \rho^{-1} \left\| \sum_{n=1}^{\infty} \|W_n(\cdot)\|_{\mathrm{End}(V)} \right\|_{L^2(\Omega)}.$$

We now claim that for every fixed $f \in H^2(\Omega'; V)$, we have $\mathcal{L}_N f \to \mathcal{L} f$ in $H^2(\Omega; V)$ as $N \to \infty$. Fix an arbitrary $\varepsilon > 0$. Without loss of generality, we may assume that $f \neq 0$. By (5.12), there exists $N_0 \in \mathbb{N}$ such that

$$\left\| \sum_{n=N_0}^{\infty} \|W_n(\cdot)\|_{\mathrm{End}(V)} \right\|_{L^2(\Omega)} < \frac{\varepsilon \cdot \rho}{\|f\|_{L^2(\Omega')}}.$$

Using the same estimates as above, we obtain for all $N \geq N_0$ the bound

$$\|\mathcal{L}_N f - \mathcal{L} f\|_{L^2(\Omega)} \leq \rho^{-1}\|f\|_{L^2(\Omega')} \left\| \sum_{n=N}^{\infty} \|W_n(\cdot)\|_{\mathrm{End}(V)} \right\|_{L^2(\Omega)} < \varepsilon,$$

which proves the claim. By the uniform boundedness principle (also called the Banach–Steinhaus theorem) we deduce that $\mathcal{L} : H^2(\Omega'; V) \to H^2(\Omega; V)$ is a well-defined operator, whose norm is bounded by

$$\|\mathcal{L}\|_{H^2(\Omega';V)\to H^2(\Omega;V)} \leq \sup_{N\in\mathbb{N}} \|\mathcal{L}_N\|_{H^2(\Omega';V)\to H^2(\Omega;V)} \leq \rho^{-1} \left\| \sum_{n=1}^{\infty} \|W_n(\cdot)\|_{\mathrm{End}(V)} \right\|_{L^2(\Omega)}.$$

The proof is complete. $\square$

5.2.3 Main estimate on singular values

In order to state the main estimate, let us introduce some new language. Given a Euclidean disk in the complex plane

$$D = D(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\},$$

we will denote its η-dilate $D(z_0, \eta r)$ by $D(\eta)$, for ease of notation. In other words, the disk $D(\eta)$ has the same center as D but its radius is rescaled by the factor η.

Definition 5.9. *Let Ω_1 and Ω_2 be open sets of the complex plane and assume that $\Omega_1 \Subset \Omega_2$. A* relative (N,η)-cover *of the pair (Ω_1,Ω_2) is a family of open Euclidean disks $\{D_j\}_{j=1}^N$ in the complex plane such that the following two conditions hold:*

$$\Omega_1 \subseteq \bigcup_{j=1}^{N} D_j \tag{5.13}$$

and

$$\bigcup_{j=1}^{N} D_j(\eta) \subseteq \Omega_2. \tag{5.14}$$

The main theorem in this section is the following estimate.

Theorem 5.10. *Let $(\Omega, V, \phi_n, W_n)_{n\in\mathcal{I}}$ be a vector-valued holomorphic map-weight system and let $\mathcal{L} : H^2(\Omega;V) \to H^2(\Omega;V)$ be the associated transfer operator as defined in* (5.10). *Let*

$$\widetilde{\Omega} := \bigcup_{n\in\mathcal{I}} \phi_n(\Omega).$$

Let Ω' be an intermediate set satisfying

$$\widetilde{\Omega} \Subset \Omega' \Subset \Omega,$$

such that the pair (Ω',Ω) has a relative (N,η)-cover with $\eta > 1$. Assume further that

$$\rho := \inf_{n\in\mathcal{I}} \operatorname{dist}\big(\phi_n(\Omega'), \partial\Omega\big) > 0.$$

Then the k-th singular value of $\mathcal{L}$ satisfies

$$\mu_k(\mathcal{L}) \le \rho^{-1} N \eta^{-k/(N\dim V)+1} \left\| \sum_{n\in\mathcal{I}} \|W_n(\cdot)\|_{\mathrm{End}(V)} \right\|_{L^2(\Omega)}.$$

We immediately obtain the following corollary. If the operator

$$\mathcal{L} : H^2(\Omega;V) \to H^2(\Omega;V)$$

satisfies the hypotheses of Theorem 5.10, then the sequence $(\mu_k(\mathcal{L}))_k$ is summable in k. This implies that $\mathcal{L}$ is of trace class, i.e.

$$\|\mathcal{L}\|_1 := \sum_{k=1}^{\infty} \mu_k(\mathcal{L}) < \infty.$$

5.2.4 Canonical embeddings

Given two open sets of the complex plane Ω_1 and Ω_2 with $\Omega_1 \Subset \Omega_2$, we define the *canonical embedding*

$$J : H^2(\Omega_2;V) \to H^2(\Omega_1;V), \quad Jf = f\restriction_{\Omega_1}.$$

That is, Jf is the restriction of f to Ω_1. The following lemma is a vector-valued version of [7, Proposition 3.4 (ii)].

Lemma 5.11. *Let $D \subset \mathbb{C}$ be a non-empty open Euclidean disk. If $\eta > 1$ then for all $k \in \mathbb{N}$ the k-th singular value of the canonical embedding $J : H^2(D(\eta); V) \to H^2(D; V)$ satisfies*

$$\mu_k(J) \leq \eta^{-k/\dim V}.$$

Proof. By translation invariance of the Lebesgue measure we may assume that the disk D is centered at the origin, i.e. $D = D(0, r)$ and $D(\eta) = D(0, \eta r)$ for some radius $r > 0$. Let $d = \dim V$ and fix an orthonormal basis $\mathbf{e}_1, \ldots, \mathbf{e}_d$ for V. Then the family of functions

$$\varphi_{n,l}(z) = \sqrt{\frac{n+1}{(\eta r)^{2n+2}\pi}}\, z^n \mathbf{e}_l, \quad n \in \mathbb{N}_0, l \in \{1, \ldots, d\}.$$

provides an orthonormal basis for $H^2(D(\eta); V)$. In particular,

$$\langle \varphi_{n,l}, \varphi_{m,l'} \rangle_{H^2(D(\eta);V)} = \int_{D(\eta)} \langle \varphi_{n,l}(z), \varphi_{m,l'}(z) \rangle_V \,\mathrm{dvol}(z) = \delta_{nm}\delta_{ll'}$$

for all $n, m \in \mathbb{N}_0$ and all $l, l' \in \{1, \ldots, d\}$. By the definition of J we have

$$\begin{aligned}
\langle J^* J \varphi_{n,l}, \varphi_{m,l'} \rangle_{H^2(D(\eta);V)} &= \langle J\varphi_{n,l}, J\varphi_{m,l'} \rangle_{H^2(D(\eta);V)} \\
&= \langle \varphi_{n,l}, \varphi_{m,l'} \rangle_{H^2(D;V)} \\
&= \int_D \langle \varphi_{n,l}(z), \varphi_{m,l'}(z) \rangle_V \,\mathrm{dvol}(z) \\
&= \eta^{-2(n+1)} \delta_{nm}\delta_{ll'}.
\end{aligned}$$

We deduce that the operator $J^*J : H^2(D(\eta); V) \to H^2(D(\eta); V)$ is diagonal with respect to the basis $(\varphi_{n,l})$ with set of eigenvalues being equal to

$$\{\eta^{-2(n+1)} : n \geq 0\},$$

and each eigenvalue having multiplicity exactly d. Hence, the set of singular values of J is equal to $\{\eta^{-1}, \eta^{-2}, \eta^{-3}, \ldots\}$, where each element is repeated d times. We conclude that $\mu_k(J) \leq \eta^{-k/d}$ for all $k \geq 1$, as claimed. □

5.2.5 Proof of Theorem 5.10

Before we can prove the main result of this section, Theorem 5.10, we need an intermediate result. For the proof of the latter we use the following notion which can also be found in [7]: $\{\widetilde{D}_j\}_{j=1}^N$ is called a *disjointification* of $\{D_j\}_{j=1}^N$ if $\{\widetilde{D}_j\}_{j=1}^N$ is a partition of $\bigcup_{j=1}^N D_j$ (up to sets of zero Lebesgue measure) and if $\widetilde{D}_j \subseteq D_j$ for all $j \in \{1, \ldots, N\}$.

Observe that we can always obtain a disjointification by setting

$$\widetilde{D}_1 := D_1 \quad \text{and} \quad \widetilde{D}_j := \operatorname{int}\left(D_j \setminus \bigcup_{i=1}^{j-1} D_i \right) \quad \text{for} \quad j = 2, \ldots, N.$$

Proposition 5.12. *Let Ω_1 and Ω_2 be non-empty open sets of the complex plane with $\Omega_1 \Subset \Omega_2$ and assume that (Ω_1, Ω_2) has a relative (N, η)-cover with $\eta > 1$. Then the k-th singular value of the canonical embedding $J : H^2(\Omega_2; V) \to H^2(\Omega_1; V)$ satisfies*

$$\mu_k(J) \leq N\eta^{-k/(N \dim V)+1}.$$

Proof. By assumption there exists a family $\{D_j\}_{j=1}^N$ of Euclidean disks satisfying conditions (5.13) and (5.14). Let $\{\widetilde{D}_j\}_{j=1}^N$ be a disjointification of $\{D_j\}_{j=1}^N$. For notational convenience set

$$D := \bigcup_{j=1}^N D_j = \bigcup_{j=1}^N \widetilde{D}_j,$$

where the second equality is to be understood up to sets of zero Lebesgue measure.

For each $n = 1, \dots, N$ define the operator $T_n : H^2(D_n; V) \to H^2(D; V)$ implicitly by

$$\langle T_n f, g \rangle_{H^2(D;V)} := \int_{\widetilde{D}_n} \langle f, g \rangle_V \,\mathrm{dvol}\,.$$

We claim that $T_n : H^2(D_n; V) \to H^2(D; V)$ is a bounded operator with norm at most 1. Indeed, for any $f \in H^2(D_n; V)$ and $g \in H^2(D; V)$ we have by Cauchy-Schwarz

$$\left|\langle T_n f, g \rangle_{H^2(D;V)}\right|^2 \leq \left(\int_{\widetilde{D}_n} \|f\|_V^2 \,\mathrm{dvol}\right)\left(\int_{\widetilde{D}_n} \|g\|_V^2 \,\mathrm{dvol}\right) \leq \|f\|^2_{H^2(D_n;V)} \|g\|^2_{H^2(D;V)}. \tag{5.15}$$

Applying (5.15) to $g = T_n f$ shows that

$$\|T_n f\|_{H^2(D;V)} \leq \|f\|_{H^2(D_n;V)}.$$

Since $f \in H^2(D_n; V)$ was arbitrary, this gives

$$\|T_n\|_{H^2(D_n;V) \to H^2(D;V)} \leq 1, \tag{5.16}$$

as claimed. Now consider the canonical identifications

$$\widetilde{J}_n : H^2(\Omega_2; V) \to H^2(D_n(\eta); V),$$

$$J_n : H^2(D_n(\eta); V) \to H^2(D_n; V),$$

and

$$\widetilde{J} : H^2(D; V) \to H^2(\Omega_1; V).$$

We claim that $J : H^2(\Omega_2; V) \to H^2(\Omega_1; V)$ can be written as

$$J = \sum_{n=1}^N \widetilde{J} T_n J_n \widetilde{J}_n. \tag{5.17}$$

To see this pick arbitrary functions $f \in H^2(\Omega_2; V)$ and $g \in H^2(\Omega_1; V)$. A step by step computation gives

$$\begin{aligned}
\left\langle \sum_{n=1}^{N} \widetilde{J} T_n J_n \widetilde{J}_n f, g \right\rangle_{H^2(\Omega_1;V)} &= \sum_{n=1}^{N} \left\langle \widetilde{J} T_n J_n \widetilde{J}_n f, g \right\rangle_{H^2(\Omega_1;V)} \\
&= \sum_{n=1}^{N} \left\langle T_n J_n \widetilde{J}_n f, \widetilde{J}^* g \right\rangle_{H^2(D_n;V)} \\
&= \sum_{n=1}^{N} \int_{\widetilde{D}_n} \left\langle J_n \widetilde{J}_n f, \widetilde{J}^* g \right\rangle_V \mathrm{dvol} \quad \text{(definition of } T_n\text{)} \\
&= \sum_{n=1}^{N} \int_{\widetilde{D}_n} \left\langle f, \widetilde{J}^* g \right\rangle_V \mathrm{dvol} \qquad \text{(since } J_n \widetilde{J}_n f = f \text{ on } \widetilde{D}_n\text{)} \\
&= \int_{D} \left\langle f, \widetilde{J}^* g \right\rangle_V \mathrm{dvol} \qquad \text{(disjointification)} \\
&= \left\langle f, \widetilde{J}^* g \right\rangle_{H^2(D)} \\
&= \left\langle \widetilde{J} f, g \right\rangle_{H^2(\Omega_1)} \\
&= \langle J f, g \rangle_{H^2(\Omega_1)} \qquad (\Omega_1 \subset D \subset \Omega_2).
\end{aligned}$$

This proves (5.17). Notice that we have the trivial estimates

$$\|\widetilde{J}_n\|_{H^2(\Omega_2;V) \to H^2(D_n(\eta);V)} \leq 1 \quad \text{and} \quad \|\widetilde{J}\|_{H^2(D;V) \to H^2(\Omega_1;V)} \leq 1,$$

as well as (5.16). Hence, we can apply (A.2) and Lemma A.1 (1) (in that order) to obtain

$$\mu_k(J) \leq \sum_{n=1}^{N} \mu_{\lfloor \frac{k+N-1}{N} \rfloor} \left(\widetilde{J} T_n J_n \widetilde{J}_n \right) \leq \sum_{n=1}^{N} \mu_{\lfloor \frac{k+N-1}{N} \rfloor} (J_n).$$

Using Lemma 5.11 and noticing that

$$\left\lfloor \frac{k+N-1}{N} \right\rfloor \geq \frac{k}{N} - 1,$$

we obtain (recall that $\eta > 1$ by assumption)

$$\mu_k(J) \leq \sum_{n=1}^{N} \eta^{-\lfloor \frac{k+N-1}{N} \rfloor / \dim V} = N \eta^{-\lfloor \frac{k+N-1}{N} \rfloor / \dim V} \leq N \eta^{-k/(N \dim V)+1},$$

which completes the proof of Proposition 5.12. □

We can now prove Theorem 5.10:

Proof of Theorem 5.10. The operator $\mathcal{L} : H^2(\Omega; V) \to H^2(\Omega; V)$ lifts to an operator $\mathcal{L}' : H^2(\Omega'; V) \to H^2(\Omega; V)$. Let $J : H^2(\Omega; V) \to H^2(\Omega'; V)$ be the canonical embeding associated to the pair (Ω', Ω). Notice that the operator $\mathcal{L}$ factorizes as $\mathcal{L} = \mathcal{L}' J$. Therefore, by Proposition 5.12, we arrive at

$$\mu_k(\mathcal{L}) \leq \|\mathcal{L}'\|_{H^2(\Omega';V) \to H^2(\Omega;V)} \cdot \mu_k(J) \leq \|\mathcal{L}'\|_{H^2(\Omega';V) \to H^2(\Omega;V)} \cdot N \eta^{-k/(N \dim V)+1}. \tag{5.18}$$

By Lemma 5.7, the norm of $\mathcal{L}' : H^2(\Omega'; V) \to H^2(\Omega; V)$ we can be estimated as

$$\|\mathcal{L}'\|_{H^2(\Omega';V)\to H^2(\Omega;V)} \leq \rho^{-1} \left\| \sum_{n\in\mathfrak{I}} \|W_n(\cdot)\|_{\mathrm{End}(V)} \right\|_{L^2(\Omega)}. \tag{5.19}$$

Combining (5.18) and (5.19) gives

$$\mu_k(\mathcal{L}) \leq \rho^{-1} N \eta^{-k/(N \dim V)+1} \left\| \sum_{n\in\mathfrak{I}} \|W_n(\cdot)\|_{\mathrm{End}(V)} \right\|_{L^2(\Omega)},$$

which finishes the proof of Theorem 5.10. □

5.3 Growth of L-functions for Hecke triangle groups

In this section we prove Theorem 5.1 and Corollary 5.3.

5.3.1 Hecke triangle groups and transfer operators

Recall that

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{R})$$

acts on $\mathbb{H}$ by $z \mapsto (az+b)/(cz+d)$.

The Hecke triangle group Γ_w with cusp width $w > 2$ is the subgroup of $\mathrm{PSL}_2(\mathbb{R})$ generated by the two elements

$$T := T_w := \begin{bmatrix} 1 & w \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad S := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Throughout this section the parameter $w > 2$ is fixed and all constants are allowed to depend on w.

Let Λ be the limit set of the group Γ_w (viewed as a subset of $\partial\mathbb{H} = \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$). We will only consider a small portion of the limit set, namely $\Lambda_0 := \Lambda \cap (-1, 1)$. For $h > 0$ let

$$\Omega(h) := \Lambda_0 + D(0, h)$$

be the *complex* h-neighbourhood of Λ_0. Throughout we will assume that $h > 0$ is small enough. The parameter h will be decreased whenever necessary.

Throughout this section, let $\rho\colon \Gamma_w \to U(V)$ be a finite-dimensional representation of Γ_w. The representation space V is endowed with the inner-product $\langle \cdot, \cdot \rangle_V$ with respect to which ρ is unitary. Let $H^2(\Omega(h); V)$ be the V-valued Bergman space of $\Omega(h)$, as defined in (5.9).

We will work with the (initially only formal) operator

$$\mathcal{L}_{s,\rho} = \sum_{n \in \mathbb{Z} \smallsetminus \{0\}} \nu_{s,\rho}(T^{-n} S) \tag{5.20}$$

acting on $H^2(\Omega(h); V)$, where $\nu_{s,\rho}(\gamma)$ is defined as the action of the element $\gamma \in \Gamma$ on functions $f\colon U \to V$, given by

$$\big(\nu_{s,\rho}(\gamma)f\big)(z) := \Big[(\gamma^{-1})'(z)\Big]^{s} \rho(\gamma)f(\gamma^{-1}.z), \quad z \in U,$$

whenever this makes sense. More concretely, the transfer operator is defined for all $f \in H^2(\Omega(h); V)$ by the equation

$$\mathcal{L}_{s,\rho}f(z) = \sum_{n\in\mathbb{Z}\smallsetminus\{0\}} \gamma_n'(z)^s \rho(\gamma_n)^{-1} f(\gamma_n(z)), \quad z \in \Omega(h),$$

where we have set $\gamma_n := ST^n$ and

$$\gamma_n(z) := \frac{-1}{z+nw}$$

for notational convenience. Note that γ_n' is a positive on the real line. Therefore, the complex power $\gamma_n'(z)^s$ is well-defined for all $z \in \Omega(h)$ with h small enough, by setting

$$\gamma_n'(z)^s = e^{s\mathrm{Log}(\gamma_n'(z))},$$

where Log is a complex logarithm defined on $\mathbb{C} \smallsetminus (-\infty, 0]$.

The reason we are interested in the operator $\mathcal{L}_{s,\rho}$ will become apparent in Subsection 5.3.6 below, where we prove that its Fredholm determinant is identical to the L-function $L_{\Gamma_w}(s,\rho)$. However, the connection with L-functions will not be used until we actually finish the proof of Theorem 5.1 in Subsection 5.3.8.

Lemma 5.13. *Notations being as above, there exist $h_0 > 0$ and $0 < \alpha < 1$ such that for all $h \in (0, h_0)$ we have*

$$\widetilde{\Omega}(h) := \bigcup_{n\in\mathbb{Z}\smallsetminus\{0\}} \gamma_n(\Omega(h)) \subseteq \Omega(\alpha h).$$

Proof. First note that we can write

$$\Omega(h) = \{z \in \mathbb{C} : \exists p \in \Lambda_0 \ : |z-p| < h\}.$$

Let $n \in \mathbb{Z} \smallsetminus \{0\}$ be arbitrary. Then

$$\gamma_n(\Omega(h)) = \{z' \in \mathbb{C} : \exists p \in \Lambda_0 \ : |\gamma_n^{-1}(z') - p| < h\}.$$

Suppose $z' \in \mathbb{C}$ and $p \in \Lambda_0$ are such that $|\gamma_n^{-1}(z') - p| < h$. Then by setting $q := \gamma_n(p) \in \Lambda_0$ we obtain the bound

$$|z'-q| \le \sup_{y\in\Omega(h)} |\gamma_n'(y)| \cdot |\gamma_n^{-1}(z') - p| \le \sup_{y\in\Omega(h)} |\gamma_n'(y)| \cdot h.$$

Choose $h_0 > 0$ such that $\Omega(h_0) \subset D(0,1)$. Then for all $h \in (0, h_0)$ we have

$$\sup_{y\in\Omega(h)} |\gamma_n'(y)| \le \sup_{y\in D(0,1)} |\gamma_n'(y)| \le \frac{1}{(w-1)^2}.$$

Thus we have shown that

$$\gamma_n(\Omega(h)) \subseteq \Omega(\alpha h),$$

where $\alpha := \frac{1}{(w-1)^2}$. Since $w > 2$, we have $\alpha < 1$ and the lemma follows. □

Proposition 5.14. *There exists $h_0 > 0$ such that for all $h \in (0, h_0)$, the operator $\mathcal{L}_{s,\rho} : H^2(\Omega(h); V) \to H^2(\Omega(h); V)$ is well-defined and bounded on the half-plane $\{\mathrm{Re}(s) > \frac{1}{2}\}$.*

Proof. Let h_0 be as in Lemma 5.13 and let $h \in (0, h_0)$. Notice that there is some constant $C > 0$ such that for all $z \in \Omega(h), n \in \mathbb{N}$ we have

$$\left|\gamma_n'(z)^s\right| \le \frac{C}{|n|^{2\mathrm{Re}(s)}} e^{C|\mathrm{Im}(s)|h}.$$

Moreover, since ρ is unitary, we have $\|\rho(\gamma)\|_{\mathrm{End}(V)} = 1$ for all $\gamma \in \Gamma_w$. It follows that

$$\left\|(\gamma_n')^s \rho(\gamma_n)^{-1}\right\|_{\mathrm{End}(V)} \le \frac{C}{|n|^{2\mathrm{Re}(s)}} e^{C|\mathrm{Im}(s)|h}$$

and consequently

$$\sum_{n \in \mathbb{Z} \smallsetminus \{0\}} \left\|(\gamma_n')^s \rho(\gamma_n)^{-1}\right\|_{\mathrm{End}(V)} \in L^2(\Omega(h)),$$

provided $\mathrm{Re}(s) > \frac{1}{2}$. Therefore for all $\mathrm{Re}(s) > \frac{1}{2}$, the quintuple

$$\left(\Omega, \Omega, V, \gamma_n, (\gamma_n')^s \rho(\gamma_n)^{-1}\right)_{n \in \mathbb{Z} \smallsetminus \{0\}} \tag{5.21}$$

is a vector-valued holomorphic map-weight system in the sense of Definition 5.6. Notice that $\mathcal{L}_{s,\rho}$ is precisely the transfer operator associated to (5.21). Therefore, the statement follows from Lemma 5.13 and Lemma 5.7. □

5.3.2 Structure of the limit set

In this subsection we prove a crucial upper bound on the volume of the set $\Omega(h)$. We use the finite version of the Basic Covering Lemma, sometimes also referred to as 'Vitali's Covering Lemma', although the latter stands for different statements in the literature. To avoid confusion we state what we need here:

Lemma 5.15. *Let $B_1, \ldots, B_n$ be a finite collection of balls in an arbitrary metric space. Then there exists a subcollection $B_{j_1}, \ldots, B_{j_m}$ of these balls which are mutually disjoint and satisfy*

$$\bigcup_{i=1}^{n} B_i \subset \bigcup_{k=1}^{m} 3B_{j_k},$$

where $3B$ denotes the ball with the same center as B but three times its radius.

The result which we seek to prove in this subsection is the following:

Proposition 5.16. *There exists $h_0 > 0$ and $C > 0$ such that for all $h \in (0, h_0)$ we have*

$$\mathrm{vol}(\Omega(h)) \le C h^{2-\delta}.$$

Remark 5.17. *Proposition 5.16 shows that the Hausdorff dimension and the Minkowski dimension of Λ_0 are identical.*

Proof. Consider the *real* h-neighbourhood of Λ_0, that is, the set $\Lambda_0(h) := \Lambda_0 + [-h, h]$. Notice that $\Omega(h) \subseteq \Lambda_0(h) + i[-h, h]$, which implies that

$$\mathrm{vol}(\Omega(h)) \leq 2h \cdot |\Lambda_0(h)|,$$

where $|\cdot|$ denotes the Lebesgue measure. Thus it suffices to prove that

$$|\Lambda_0(h)| \leq Ch^{1-\delta}. \tag{5.22}$$

Observe that

$$\overline{\Lambda_0(h)} \subset \Lambda_0(2h) = \Lambda_0 + [-2h, 2h] = \bigcup_{p \in \Lambda_0} I(p, 2h),$$

where $I(p, r) := (p - r, p + r)$. Since $\overline{\Lambda_0(h)}$ is compact, there exists a finite set of points $\{p_1, \ldots, p_M\} \subset \Lambda_0$ such that

$$\Lambda_0(h) \subset \bigcup_{i=1}^{M} I(\xi_i, 2h).$$

By Lemma 5.15, there exists a subset $\{p'_1, \ldots, p'_N\} \subseteq \{p_1, \ldots, p_M\}$ such that the intervals $I(p'_j, 2h)$, $j = 1, \ldots, N$ are mutually disjoint and such that

$$\Lambda_0(h) \subset \bigcup_{j=1}^{N} I(p'_j, 6h). \tag{5.23}$$

For convenience we now switch to the unit disk model $\mathbb{D}$ of hyperbolic geometry, in which the boundary at infinity $\mathbb{S}^1 = \partial\mathbb{D}$ is treated in a uniform way. Let $\Lambda_{\mathbb{S}^1}$ denote the limit set of Γ_w viewed as a subset of $\mathbb{S}^1$. Fix a Cayley transform $\phi : \overline{\mathbb{H}} \to \overline{\mathbb{D}}$ so that $\Lambda_{\mathbb{S}^1} = \phi(\Lambda)$.

Recall the family of Patterson-Sullivan measures $\mu_{z,z'}$ on $\mathbb{S}^1$ associated with Γ_w with vantage point $z \in \mathbb{D}$ and base point $z' \in \mathbb{D}$. For a construction of these measures we refer to [66] or [14, Chapter 14]. It is well-known that $\mu_{z,z'}$ is a probability measure supported on the limit set $\Lambda_{\mathbb{S}^1}$, for any choice $z, z' \in \mathbb{D}$. We will work with $\mu := \mu_{0,0}$, where 0 is the origin of $\mathbb{D}$.

For notational convenience set $\xi_i := \phi(p_i)$ for $i = 1, \ldots, M$ and $\xi'_j := \phi(p'_j)$ for $j = 1, \ldots, N$. Furthermore let $I_{\mathbb{S}^1}(\xi_i, r) := \phi(I(p_i, r))$ and note that $I_{\mathbb{S}^1}(\xi_i, r)$ is an interval on $\mathbb{S}^1$ with center ξ_i. For $r > 0$ small enough (smaller than some r_1, say) we have $I(p, r) \subseteq [-1, 1]$ for all $p \in \Lambda_0$. The map ϕ restricted to the interval $[-1, 1]$ has bounded length distortion. This implies that there exist constants c_1, c_2 such that the length of the intervals $I_{\mathbb{S}^1}(\xi_i, r)$, are bounded from below by $c_1 r$ and from above by $c_2 r$, provided that $r \in (0, r_1)$. In other words, for every $i = 1, \ldots, M$ and every h small enough, $I_{\mathbb{S}^1}(\xi_i, 2h)$ is an interval on $\mathbb{S}^1$ centered at the point $\xi_i \in \Lambda_{\mathbb{S}^1}$ and of size comparable to h.

Given a point $\xi \in \mathbb{S}^1$, let s_ξ be the ray from the origin 0 in $\mathbb{D}$ to ξ. If A is a subset of $\mathbb{D}$, then the *shadow at infinity* of A is defined by

$$\{\xi \in \mathbb{S}^1 : s_\xi \cap A \neq \emptyset\}.$$

For $t > 0$, let $\xi(t)$ be the point on s_ξ which lies at hyperbolic distance t from the origin. Let $b(\xi(t))$ denote the shadow at infinity of the geodesic arc which is orthogonal to s_ξ and which intersects s_ξ at the point $\xi(t)$. It can be shown, using elementary hyperbolic geometry, that $b(\xi(t))$ is an interval on $\mathbb{S}^1$ centered at ξ with length comparable to e^{-t}.

We deduce that all the intervals $I_{\mathbb{S}^1}(\xi_i, 2h)$ contain some shadow $b(\xi_i(t))$ where h is comparable to e^{-t}. More precisely, there exist $h_0, c_3, c_4 > 0$ such that for all $i = 1, \dots, M$ and all $h \in (0, h_0)$ we have

$$I_{\mathbb{S}^1}(\xi_i, 2h) \supseteq b(\xi_i(t))$$

where $c_4 e^{-t} \le h \le c_3 e^{-t}$. By the result of Stratmann–Urbański [87, Theorem 2], we obtain

$$\mu(I_{\mathbb{S}^1}(\xi_i, 2h)) \ge \mu(b(\xi_i(t))) \ge c_5 e^{-\delta t} \ge c_6 h^\delta.$$

for some constants $c_5, c_6 > 0$ not depending on t or h. (The additional factor appearing in the lower bound of [87, Theorem 2] is larger or equal than 1 and thus we can ignore it.)

Since the sets $I(p'_j, 2h)$, $j = 1, \dots, N$ are mutually disjoint, so are the sets $I_{\mathbb{S}^1}(\xi'_j, 2h)$. This gives

$$c_6 h^\delta N \le \sum_{j=1}^{N} \mu(I_{\mathbb{S}^1}(\xi'_j, 2h)) = \mu\left(\bigcup_{j=1}^{N} I_{\mathbb{S}^1}(\xi'_j, 2h)\right) \le 1,$$

which implies

$$N \le c_6^{-1} h^{-\delta}.$$

It follows from (5.23) that

$$|\Lambda_0(h)| \le \sum_{j=1}^{N} |I(\xi'_j, 6h)| = 6hN \le 6c_6^{-1} h^{1-\delta},$$

proving (5.22) and thus concluding the proof of Proposition 5.16. □

5.3.3 Singular value estimate

In this section we prove the crucial upper bound for the *singular values* of the transfer operator $\mathcal{L}_{s,\rho}$ defined in (5.20) for $\mathrm{Re}(s) > \frac{1}{2}$. As we shall see in the subsequent subsections, having estimates for the singular values in the half-plane $\{\mathrm{Re}(s) > \frac{1}{2}\}$ is sufficient to control the growth of the L-function $L_{\Gamma_w}(s, \rho)$ on the *entire* complex plane (and bounded away from the real line, where some poles may live.)

The goal here is to specialize the main theorem of the previous section, Theorem 5.10, to the Hecke triangle groups setting. We obtain the following estimate.

Proposition 5.18. *There exists $h_0 > 0$ such that for every finite-dimensional representation $\rho\colon \Gamma_w \to U(V)$, for all $\sigma := \mathrm{Re}(s) > \frac{1}{2}$, and all $h \in (0, h_0)$ the singular values of $\mathcal{L}_{s,\rho} : H^2(\Omega(h); V) \to H^2(\Omega(h); V)$ satisfy*

$$\mu_k(\mathcal{L}_{s,\rho}) \le C_1 \cdot h^{-3\delta/2} \cdot e^{C_2 |\mathrm{Im}(s)| h} \cdot \exp\left(-C_3 h^\delta k\right),$$

where $C_j = C_j(\sigma, w, \rho)$, $j = 1, 2, 3$ are positive constants depending only on σ, w, and the representation ρ.

Remark 5.19. *In a nutshell, Proposition 5.18 asserts that the sequence of singular values of $\mathcal{L}_{s,\rho}$ decays exponentially where the exponent of the decay rate is proportional to h^{δ}. It is precisely here where the fractal bound for $L_{\Gamma_w}(s, \rho)$ comes from.*

To make the proof of Proposition 5.18 more readable, let us start with the following general result, which is a consequence of the Basic Covering Lemma.

Lemma 5.20. *Let $S \subset \mathbb{C}$ be a bounded subset and let $0 < \beta < 1$ and $h > 0$. Consider the two complex neighbourhoods $S_1 = S + D(0, \beta h)$ and $S_2 = S + D(0, h)$. Then the pair (S_1, S_2) possesses a relative $(N, 2)$-cover with*

$$N \leq \frac{36}{\pi(1-\beta)^2} h^{-2} \operatorname{vol}(S_2).$$

Proof. Set $h_1 := \beta h > 0$ and $h_2 := \frac{1-\beta}{6} h > 0$ so that $h_1 + 6h_2 = h$. Clearly we can cover $\overline{S_1}$ by open disks of radius h_2, all of which are centered in S_1. Since $\overline{S_1}$ is compact, we can do so with only a finite number of disks, meaning that there exists a finite set of points $\{p_1, \ldots, p_M\} \subseteq S_1$ such that

$$S_1 \subset \bigcup_{i=1}^{M} D(p_i, h_2).$$

By Lemma 5.15, there exists a subset $\{p'_1, \ldots, p'_N\} \subseteq \{p_1, \ldots, p_M\}$ such that the disks $D(p'_j, h_2)$, $j = 1, \ldots, N$ are mutually disjoint, and such that

$$S_1 \subset \bigcup_{j=1}^{N} D(p'_j, 3h_2). \tag{5.24}$$

We claim that $\{D(p'_j, 3h_2)\}_{j=1}^{N}$ provides a relative $(N, 2)$-cover for (S_1, S_2) where N is as in the statement. By exploiting the disjointness of the disks $D(p'_j, h_2)$ we obtain

$$\begin{aligned} N \cdot \pi h_2^2 &= \sum_{j=1}^{N} \operatorname{vol}(D(p'_j, h_2)) \\ &= \operatorname{vol}\left(\bigcup_{j=1}^{N} D(p'_j, h_2)\right) \\ &\leq \operatorname{vol}(S_1 + D(0, h_2)) \\ &= \operatorname{vol}(S + D(0, h_1 + h_2)) \\ &\leq \operatorname{vol}(S_2). \end{aligned}$$

Hence, the number N of disks used in the cover (5.24) is bounded by

$$N \leq (\pi h_2^2)^{-1} \operatorname{vol}(S_2) = \frac{36}{\pi(1-\beta)^2} h^{-2} \operatorname{vol}(S_2).$$

To finish the proof notice that the 2-dilate of each disk in (5.24) satisfies

$$D(p'_j, 6h_2) = p'_j + D(0, 6h_2) \subseteq S_1 + D(0, 6h_2) = S + D(0, h_1 + 6h_2) = S_2,$$

so $\{D(p'_j, 3h_2)\}_{j=1}^N$ is indeed a relative $(N, 2)$-cover for (S_1, S_2). $\square$

Proof of Proposition 5.18. Fix $h_0 > 0$ such that the conclusions of Lemma 5.13 and Proposition 5.16 hold true for all $h \in (0, h_0)$ and fix such an h for the rest of the proof. By Lemma 5.13, we can find $\alpha < 1$ such that

$$\widetilde{\Omega}(h) = \bigcup_{n \in \mathbb{Z} \smallsetminus \{0\}} \gamma_n(\Omega(h)) \subseteq \Omega(\alpha h).$$

Set $\beta := \frac{1+\alpha}{2}$ so that $\alpha < \beta < 1$. Then $\Omega(\beta h)$ is an intermediate set in the sense that

$$\widetilde{\Omega}(h) \Subset \Omega(\beta h) \Subset \Omega(h).$$

When applied to the set $S = \Lambda_0$, Lemma 5.20 shows that the pair $\big(\Omega(\beta h), \Omega(h)\big)$ has a relative $(N, 2)$-cover with

$$N \leq c_1 h^{-2} \operatorname{vol}(\Omega(h)),$$

Using Proposition 5.16 we can further estimate $\operatorname{vol}(\Omega(h))$, leading to

$$N \leq c_2 h^{-\delta}.$$

Using Lemma 5.13 again, we obtain furthermore

$$\rho := \inf_n \operatorname{dist}\big(\gamma_n(\Omega), \partial\Omega(\beta h)\big) \geq \operatorname{dist}\big(\Omega(\alpha h), \partial\Omega(\beta h)\big) \geq (\beta - \alpha)h = \frac{1-\alpha}{2} h.$$

Thus for all $h \in (0, h_0)$ and all $\operatorname{Re}(s) > \frac{1}{2}$ the conditions in Theorem 5.10 for the transfer operator

$$\mathcal{L}_{s,\rho} : H^2(\Omega(h); V) \to H^2(\Omega(h); V)$$

are satisfied with $\Omega = \Omega(h)$, $\widetilde{\Omega} = \widetilde{\Omega}(h)$, $\Omega' = \Omega(\beta h)$, $N \leq c_2 h^{-\delta}$, $\eta = 2$, and $\rho \geq \frac{1-\alpha}{2} h$. By setting $c_3 := \log 2 > 0$, we arrive at

$$\begin{aligned}
\mu_k(\mathcal{L}_{s,\rho}) &\leq c_4 \rho^{-1} N \exp\left(-c_3 k / (N \dim \rho)\right) \left\| \sum_{n \in \mathbb{Z} \smallsetminus \{0\}} \left\| (\gamma'_n)^s \rho(\gamma_n)^{-1} \right\|_{\operatorname{End}(V)} \right\|_{L^2(\Omega(h))} \\
&\leq c_5 h^{-1-\delta} \exp\left(-c_6 h^{\delta} (\dim \rho)^{-1} k\right) \left\| \sum_{n \in \mathbb{Z} \smallsetminus \{0\}} \left\| (\gamma'_n)^s \rho(\gamma_n)^{-1} \right\|_{\operatorname{End}(V)} \right\|_{L^2(\Omega(h))}.
\end{aligned}$$

It remains to estimate the norm in the last line. Using $\|\rho(\gamma)\|_{\operatorname{End}(V)} = 1$ and Proposition 5.16, we obtain

$$\begin{aligned}
\left\| \sum_{n \in \mathbb{Z} \smallsetminus \{0\}} \left\| (\gamma'_n)^s \rho(\gamma_n)^{-1} \right\|_{\operatorname{End}(V)} \right\|^2_{L^2(\Omega(h))} &\leq c_7 \operatorname{vol}(\Omega(h)) \sum_{n \in \mathbb{Z} \smallsetminus \{0\}} \frac{e^{c_8 |\operatorname{Im}(s)| h}}{|n|^{2\sigma}} \\
&\leq c_9 h^{2-\delta} e^{c_8 |\operatorname{Im}(s)| h} \sum_{n \in \mathbb{Z} \smallsetminus \{0\}} \frac{1}{|n|^{2\sigma}}.
\end{aligned}$$

Combining this with the elementary estimate

$$\sum_{n\in\mathbb{Z}\smallsetminus\{0\}} \frac{1}{|n|^{2\sigma}} = 2\sum_{n=1}^{\infty}\frac{1}{n^{2\sigma}} \leq 2\left(1+\int_1^\infty \frac{dx}{x^{2\sigma}}\right) = \frac{4\sigma}{2\sigma-1},$$

we obtain

$$\mu_k(\mathcal{L}_{s,\rho}) \leq c_{10}\cdot h^{-3\delta/2}\cdot e^{c_8|\mathrm{Im}(s)|h}\cdot\sqrt{\frac{\sigma}{2\sigma-1}}\cdot\exp\left(-c_6h^\delta(\dim\rho)^{-1}k\right).$$

Relabelling the constants finishes the proof of Proposition 5.18. □

5.3.4 Lerch zeta function

In this subsection we digress briefly into questions related to the Lerch zeta function. As will become evident in the next subsections, understanding its analytic properties is mandatory in our proof of Theorem 5.1 in the range $\mathrm{Re}(s)\leq\frac{1}{2}$. The Lerch zeta function is defined for $\mathrm{Re}(s)>1$, $z\in\mathbb{C}\smallsetminus(-\infty,0]$, and $\lambda\in(0,1]$ by the absolutely convergent series

$$\phi(z,s,\lambda) := \sum_{n=0}^{\infty} e^{2\pi i\lambda}(n+z)^{-s}.$$

The Lerch zeta function may be regarded as a far-reaching generalization of the Riemann zeta function. It is important to notice that the complex power $(n+z)^s$ makes sense by setting

$$(n+z)^s = e^{s\,\mathrm{Log}(n+z)},$$

where Log is defined on $\mathbb{C}\smallsetminus(-\infty,0]$ by

$$\mathrm{Log}(1+z) := z\int_0^1 \frac{dt}{1+tz}.$$

By uniform convergence, $(z,s)\mapsto\phi(z,s,\lambda)$ is holomorphic on

$$\mathbb{C}\smallsetminus(-\infty,0]\times\{\mathrm{Re}(s)>1\}.$$

For our purposes it is slightly more convenient to work with

$$H(z,s,\lambda) := \sum_{n=1}^{\infty} e^{2\pi i\lambda}(n+z)^{-s}, \tag{5.25}$$

which defines, for a fixed number $\lambda\in(0,1]$, a holomorphic function $(z,s)\mapsto H(z,s,\lambda)$ on

$$\mathbb{C}\smallsetminus(-\infty,-1]\times\{\mathrm{Re}(s)>1\}.$$

This subsection has two purposes. First, we prove that $s\mapsto H(z,s,\lambda)$ (for fixed $|z|<1$ and $\lambda\in(0,1]$) extends to a meromorphic function with poles contained in the set $\{1,0,-1,-2,\dots\}$. Second, we establish a growth estimate on $H(s,z,\lambda)$ in vertical lines of the s-plane. We borrow some ideas of Murty–Sinha [56].

The first result is probably well-known but we give an independent proof here for the sake of completeness.

Proposition 5.21. *If $|z| < 1$ and $\lambda \in (0,1]$ are fixed, then $s \mapsto H(z,s,\lambda)$ extends to a meromorphic function on the complex plane. Its poles are all contained in the set*

$$1 - \mathbb{N}_0 = \{1, 0, -1, -2, \dots\}.$$

Moreover, for fixed $s \notin 1 - \mathbb{N}_0$ and $0 < r < 1$, the function $z \mapsto H(z,s,\lambda)$ is holomorphic on $\overline{D(0,r)}$.

Proof. For $\mathrm{Re}(s) > 1$ consider the series

$$\sum_{n=1}^{\infty} \frac{e^{2\pi i \lambda n}}{n^s} =: H(0,s,\lambda). \tag{5.26}$$

It is well-known that $s \mapsto H(0,s,\lambda)$ extends to a meromorphic function on the complex plane $\mathbb{C}$, whose poles are contained in $1 - \mathbb{N}_0$, see for instance [3, Equation 1.3]. Clearly, for some fixed $s \in \mathbb{C} \setminus \{1, 0, -1, \dots\}$, the sequence $(|H(s+k, 0, \lambda)|)_{k \in \mathbb{N}}$ is bounded by a constant $M(s,\lambda)$ (depending on s and λ).

Now recall that if $|z| < 1$, we have the absolutely convergent series expansion

$$(1+z)^{-s} = \sum_{k=0}^{\infty} \frac{(-s)_k}{k!} z^k,$$

where $(u)_0 := 1$ and $(u)_k := u \cdot (u-1) \cdot \ldots \cdot (u-k+1)$ for $k \geq 1$. Assuming $\mathrm{Re}(s) > 1$, we can write

$$H(z,s,\lambda) = \sum_{n=1}^{\infty} \frac{e^{2\pi i \lambda n}}{n^s} \left(1 + \frac{z}{n}\right)^{-s} = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{e^{2\pi i \lambda n}}{n^{s+k}} \frac{(-s)_k}{k!} z^k$$

By an easy application of the theorems of Lebesgue and Fubini (in that order), we obtain the series

$$H(z,s,\lambda) = \sum_{k=0}^{\infty} H(s+k, 0, \lambda) \frac{(-s)_k}{k!} z^k, \tag{5.27}$$

Formula (5.27) is a priori only valid for $\mathrm{Re}(s) > 1$ but it can be used to obtain analytic continuation of $s \mapsto H(z,s,\lambda)$ to the complex plane, except for the points $s \in \{1, 0, -1, \dots\}$. Indeed, for all $s \in \mathbb{C} \setminus \{1, 0, -1, \dots\}$, we can bound the summands in (5.27) individually as

$$\left| H(s+k, 0, \lambda) \frac{(-s)_k}{k!} z^k \right| \leq M(s,\lambda) \cdot A_k(|s|) \cdot |z|^k$$

where

$$A_k(u) := \begin{cases} A_0(u) = 1, \\ A_1(u) = u, \\ u \prod_{j=1}^{k-1} \left(1 + \frac{u}{j}\right), & k \geq 2 \end{cases} \tag{5.28}$$

Now notice that $A_k(u)$ grows only polynomially as $k \to \infty$. Indeed, for $u > 0$ and $k \geq 2$ we have

$$A_k(u) \leq u \prod_{j=1}^{k-1} \exp\left(\frac{u}{j}\right) = u \exp\left(u \sum_{j=1}^{k-1} \frac{1}{j}\right) \leq u \exp\left(u(1 + \log k)\right) = e^u k^{u+1}.$$

Since $|z| < 1$, it is now clear that the series (5.27) converges absolutely for all $s \in \mathbb{C} \setminus \{1, 0, -1, \dots\}$. Hence, for all $0 < r < 1$, (5.27) defines a holomorphic map

$$D(0,r) \times (\mathbb{C} \setminus \{1, 0, -1, \dots\}) \to \mathbb{C}, \quad (z,s) \mapsto H(z,s,\lambda),$$

thus proving Proposition 5.21. □

Let us now turn to our second goal of estimating $H(z,s,\lambda)$. To this end we recall the result of Katsurada [43, Lemma 1], which goes as follows. For $\sigma \in \mathbb{R}$ define the quantity

$$\mu(x,\sigma,\lambda) := \limsup_{t \to \pm\infty} \frac{\log|\phi(x,\sigma+it,\lambda)|}{\log|t|},$$

which measures the polynomial growth rate of $\phi(x,s,\lambda)$ in vertical strips of the s-plane. Then we have for any $x \in (0,1]$ and $\lambda \in (0,1]$

$$\mu(x,\sigma,\lambda) \le \begin{cases} 1/2 - \sigma & \text{if } \sigma \le 0 \\ (1-\sigma)/2 & \text{if } 0 \le \sigma \le 1 \\ 0 & \text{if } \sigma \ge 1 \end{cases}.$$

In particular, this result gives us a polynomial upper bound for the Lerch zeta function in strips parallel to the imaginary axis. This bound carries over to the modified function $H(x,s,\lambda)$. Indeed, observe that $H(x,s,\lambda) = \phi(x,s,\lambda) - x^{-s}$ and that $|x^{-s}| \le x^{-\mathrm{Re}(s)}$ is bounded in strips parallel to the imaginary axis. It follows that

$$H(x,\sigma+it,\lambda) \le C_0'(x,\sigma,\lambda) \cdot |t|^{\alpha_0(\sigma)} \tag{5.29}$$

for all $|t| \ge 1$ (say), $x \in (0,1)$, and $\lambda \in (0,1]$. Now suppose that $x \in (-1,0)$. In this case we use the relation

$$H(x,\sigma+it,\lambda) = x^{-s} + e^{2\pi i\lambda} H(1+x,\sigma+it,\lambda),$$

which gives (because $0 < 1+x < 1$)

$$|H(x,\sigma+it,\lambda)| \le x^{-\mathrm{Re}(s)} + |H(1+x,\sigma+it)| \le x^{-\mathrm{Re}(s)} + C_0'(1+x,\sigma,\lambda) \cdot |t|^{\alpha_0(\sigma)}.$$

Hence, up to a change of the multiplicative factor, we obtain

$$H(x,\sigma+it,\lambda) \le C_0(x,\sigma,\lambda) \cdot |t|^{\alpha_0(\sigma)} \tag{5.30}$$

for all $|t| \ge 1$, $x \in (-1,1)$, and $\lambda \in (0,1]$. What is important to notice here is that we can choose $C_0(x,\sigma,\lambda)$ and $\alpha_0(\sigma)$ that only depend on x,σ,λ and σ, respectively. The next result is an extension of estimate (5.30) when the first argument is complex.

Proposition 5.22. *Let $\lambda \in (0,1]$ and $r \in (0,1)$. Write s and z in cartesian coordinates as $s = \sigma + it$, $z = x + iy$, and assume that $|z| \le r$ and $|t| \ge 1$. Then we have*

$$|H(z,s,\lambda)| \le C(r,\sigma,\lambda) \cdot |t|^{\alpha(\sigma)} \cdot e^{\frac{2}{1-r}|y||s|\log(1+|s|)}.$$

Remark 5.23. *Proposition 5.22 shows that if $|\mathrm{Im}(z)| \ll |s|^{-1}$, then $H(z,s,\lambda)$ is at most polynomial in $|\mathrm{Im}(s)|$.*

Proof. Let s and z be as in the statement. Since we assume that $|z| \leq r$, we have $x \in [-r, r]$.

First note that for k large enough, $|H(x, s+k, \lambda)|$ is bounded by some constant depending only on the variables σ, x, and λ. Hence we can define the quantities

$$C_1(x, \sigma, \lambda) := \max_{k \in \mathbb{N}} \{C_0(x, \sigma + k, \lambda)\} < \infty$$

and

$$\alpha_1(\sigma) := \max_{k \in \mathbb{N}} \{\alpha_0(\sigma + k)\} < \infty,$$

where $C_0(x, \sigma, \lambda)$ and $\alpha_0(\sigma, \lambda)$ are constants satisfying (5.30). With these definitions in place, we can clearly write

$$|H(x, s+k, \lambda)| \leq C_1(x, \sigma, \lambda)|t|^{\alpha_1(\sigma)} \tag{5.31}$$

for all $k \in \mathbb{N}$. We can write

$$\begin{aligned} H(z, s, \lambda) &= \sum_{n=1}^{\infty} e^{2\pi i \lambda n}(n+z)^{-s} \\ &= \sum_{n=1}^{\infty} e^{2\pi i \lambda n}(n+x)^{-s}\left(1 + \frac{iy}{n+x}\right)^{-s} \\ &= \sum_{n=1}^{\infty} e^{2\pi i \lambda n}(n+x)^{-s} \sum_{k=0}^{\infty} \frac{(-s)_k}{k!}\left(\frac{iy}{n+x}\right)^{k}. \end{aligned}$$

Interchanging sums (as we did in (5.27)) yields the expression

$$H(z, s, \lambda) = \sum_{k=0}^{\infty} H(x, s+k, \lambda)\frac{(-s)_k}{k!}(iy)^k. \tag{5.32}$$

Using (5.31), we can estimate the summands individually as

$$\left| H(x, s+k, \lambda)\frac{(-s)_k}{k!}(iy)^k \right| \leq C_0(x, \sigma, \lambda)|t|^{\beta_0(\sigma)} A_k(|s|)|y|^k,$$

where $A_k(\cdot)$ is defined by (5.28). Using similar arguments as in the proof of Proposition 5.21, it is now easy to prove that the series (5.32) converges.

To actually obtain an estimate from (5.32) we set $r_1 := \frac{1+r}{2}$, so that $r < r_1 < 1$, and

$$p := \left\lceil \frac{|s||y|}{r_1 - |y|} \right\rceil + 1,$$

so that

$$|y|\left(1 + \frac{|s|}{p}\right) \leq r_1.$$

Then for all $k \geq p$ we have

$$A_k(|s|)|y|^k \leq (1+|s|)^{p+1}|y|^p\left(1 + \frac{|s|}{p}\right)^{k-p}|y|^{k-p} \leq (1+|s|)^{p+1} r_1^{k-p}.$$

To finish the proof, we simply write (notice that $|y| \leq |z| \leq r$)

$$\begin{aligned}
|H(z,s,\lambda)| &\leq \sum_{k=0}^{p-1} \left| H(x,s+k)\frac{(-s)_k}{k!}(iy)^k \right| + \sum_{k=p}^{\infty} \left| H(x,s+k)\frac{(-s)_k}{k!}(iy)^k \right| \\
&\leq C_1(x,\sigma,\lambda) \cdot |t|^{\alpha_1(\sigma)} \cdot \left(\sum_{k=0}^{p-1} A_k(|s|)r^k + \sum_{k=p}^{\infty} (1+|s|)^{p+1} r_1^{k-p} \right) \\
&\leq C_1(x,\sigma,\lambda) \cdot |t|^{\alpha_1(\sigma)} \cdot (1+|s|)^{p+1} \left(\frac{1}{1-r} + \frac{1}{1-r_1} \right) \\
&\leq C_2(x,\sigma,\lambda,r) \cdot |t|^{\alpha_1(\sigma)} \cdot \exp\left(\left(\frac{|y||s|}{r_1-|y|} + 3 \right) \log(1+|s|) \right) \\
&\leq C_3(x,\sigma,\lambda,r) \cdot |t|^{\alpha_1(\sigma)+3} \cdot \exp\left(\frac{2}{1-r}|y||s| \log(1+|s|) \right)
\end{aligned}$$

Setting $\alpha(\sigma) := \alpha_1(\sigma) + 3$ and

$$C(r,\sigma,\lambda) := \sup_{x \in [-r,r]} C_3(x,\sigma,\lambda,r)$$

finally yields

$$|H(z,s,\lambda)| \leq C(\sigma,\lambda,r) \cdot |t|^{\alpha(\sigma)} \cdot e^{\frac{2}{1-r}|s||y|\log(1+|s|)},$$

concluding the proof. $\square$

5.3.5 Meromorphic continuation

The transfer operator family $\mathcal{L}_{s,\rho}$ defined in (5.20) is given by an infinite sum which only converges in the half-plane $\{\mathrm{Re}(s) > \frac{1}{2}\}$. To pass beyond the line $\{\mathrm{Re}(s) = \frac{1}{2}\}$, we show the meromorphic continuability of $s \mapsto \mathcal{L}_{s,\rho}$ similarly to the proof of Mayer [51] for his transfer operator family for the modular group $\mathrm{PSL}_2(\mathbb{Z})$. To deal with the twist ρ, we diagonalize the unitary map $\rho(T)$, as was done in Pohl [71]. Moreover, we have to take into account the fact that $\Omega(h)$ may consist of more than one (but only finitely many) connected components. As we will see below, the properties of the meromorphic continuation of $\mathcal{L}_{s,\rho}$ rely on the properties of the Lerch zeta function, which we studied in the previous subsection.

Let us be more precise. We will show that the map $s \mapsto \mathcal{L}_{s,\rho}$ (viewed as a map from $\{\mathrm{Re}(s) > \frac{1}{2}\}$ to the Banach space of trace class operators $H^2(\Omega(h);V) \to H^2(\Omega(h);V)$) extends to a meromorphic function on $\mathbb{C}$. In this context, meromorphic continuation is to be understood as follows. There exists a discrete set $P \subset \mathbb{C}$ of poles such that for every $s \in \mathbb{C} \setminus P$ there exists a trace class operator

$$\widetilde{\mathcal{L}}_{s,\rho} : H^2(\Omega(h);V) \to H^2(\Omega(h);V),$$

which agrees with $\mathcal{L}_{s,\rho}$ whenever $\mathrm{Re}(s) > \frac{1}{2}$. Moreover, given $f \in H^2(\Omega(h);V)$ and $z \in \Omega(h)$ the function $s \mapsto \widetilde{\mathcal{L}}_{s,\rho}f(z)$ is meromorphic with poles in P.

Once this fact is established, we immediately obtain meromorphic continuation for the Fredholm determinants $s \mapsto \det(1 - \mathcal{L}_{s,\rho})$ with the same set of (potential) poles P.

We will actually prove a refined version of the above statement, which is better suited for our purposes:

Proposition 5.24. *For each $k \in \mathbb{N}$ there exists an operator*

$$\Psi_k : H^2(\Omega(h); V) \to H^2(\Omega(h); V)$$

and for each $k \in \mathbb{N}$ and $\mathrm{Re}(s) > \frac{1}{2}$ there exists a finite-rank operator

$$\mathcal{F}_{s,\rho,k} : H^2(\Omega(h); V) \to H^2(\Omega(h); V)$$

such that the following holds true:

1. *For all $\mathrm{Re}(s) > \frac{1}{2}$ we have the formula*

$$\mathcal{L}_{s,\rho} = \mathcal{F}_{s,\rho,k} + \mathcal{L}_{s+\frac{k}{2},\rho}\Psi_k.$$

2. *For each $k \in \mathbb{N}_0$, the map $s \mapsto \mathcal{F}_{s,\rho,k}$ extends to a meromorphic function with poles contained in $\frac{1}{2}(1 - \mathbb{N}_0)$.*

3. *We have the identities*

$$\Psi_k := \Psi_1^k, \quad \mathcal{F}_{s,\rho,k} := \sum_{j=0}^{k-1} \mathcal{F}_{s+\frac{j}{2},\rho,1}\Psi_1^j.$$

4. *The rank of $\mathcal{F}_{s,\rho,k}$ is at most $\dim \rho \cdot k$.*

Remark 5.25. *Parts* (1) *and* (2) *together provide an meromorphic continuation for $\mathcal{L}_{s,\rho}$ on the half-plane $\{\mathrm{Re}(s) > \frac{1-k}{2}\}$ for every $k \in \mathbb{N}$. Although Proposition 5.24 is formulated in a rather abstract way, its proof is constructive in the sense that the 'auxiliary operators' Ψ_k and $\mathcal{F}_{s,\rho,k}$ are given by explicit formulas. These explicit formulas will enable us to control their operator norms in Subsection 5.3.7.*

Proof. Let us first consider the case $k = 1$.

Let $M = M(h)$ be the number of connected components of $\Omega(h)$ and denote them by $\Omega_1(h), \ldots, \Omega_M(h)$. Let $\Omega_1(h)$ be the connected component that contains the point $z = 0$.

We have the direct sum of Hilbert spaces

$$H^2(\Omega(h); V) = \bigoplus_{j=1}^{M} H^2(\Omega_j(h); V).$$

In other words, we can (and will) view every function $f \in H^2(\Omega(h); V)$ as a vector

$$(f^1, \ldots, f^M),$$

where the j-th component is the restriction of f to the j-th component, that is, $f^j = f\mathbf{1}_{\Omega_j(h)}$.

It is not hard to see that for every $f \in H^2(\Omega(h); V)$ there exists a function $\widetilde{f} = (\widetilde{f}^1, \dots, \widetilde{f}^M) \in H^2(\Omega(h))$ satisfying

$$f^1(z) = f^1(0) + z\widetilde{f}^1(z), \quad z \in \Omega_1(h) \tag{5.33}$$

and

$$f^j(z) = z\widetilde{f}^j(z), \quad z \in \Omega_j(h) \quad \text{for} \quad j \in \{2, \dots, M\} \tag{5.34}$$

Indeed, we can set

$$\widetilde{f}^1(z) := \begin{cases} \frac{1}{z}(f^1(z) - f^1(0)) & \text{if } z \neq 0 \\ (f^1)'(0) & \text{if } z = 0 \end{cases}, \quad \widetilde{f}^2(z) := \frac{1}{z}f^2(z), \dots, \widetilde{f}^M(z) := \frac{1}{z}f^M(z).$$

Given a pair $(i,j) \in [M] \times [M]$ we define the index set

$$\mathcal{S}(i,j) := \{n \in \mathbb{Z} \setminus \{0\} : \gamma_n(\Omega_i(h)) \subset \Omega_j(h)\}. \tag{5.35}$$

From general topology we know that the continuous image of a connected set is connected. Hence, for every $n \in \mathbb{N}$ and every index $i \in \{1, ..., M\} =: [M]$ there exists an index $j \in [M]$ such that $\gamma_n(\Omega_i(h)) \subset \Omega_j(h)$. Consequently, for fixed i the family $\{\mathcal{S}(i,j)\}_{j=1}^M$ is a partition of $\mathbb{Z} \setminus \{0\}$ and similarly, for fixed j the family $\{\mathcal{S}(i,j)\}_{i=1}^M$ is a partition of $\mathbb{Z} \setminus \{0\}$.

Moreover, for all $z \in \mathbb{C}$, we have

$$\lim_{|n| \to \infty} \gamma_n(z) = 0 \in \Omega_1(h).$$

Hence for all $|n|$ sufficiently large, we have $\gamma_n(\Omega_i(h)) \subset \Omega_1(h)$. This shows that $\mathcal{S}(i,j)$ with $j \neq 1$ is a finite set, while $\mathcal{S}(i,1)$ contains every integer with sufficiently large absolute value. (See Lemma (5.26) below for a quantitative version of this fact.)

Keeping the above properties in mind and isolating the indices $n \in \mathcal{S}(i,1)$, we can write

$$\begin{aligned}
\mathcal{L}_s f(z) &= \sum_{n \in \mathbb{Z} \setminus \{0\}} \gamma_n'(z)^s \rho(\gamma_n)^{-1} f(\gamma_n(z)) \\
&= \sum_{i=1}^M \sum_{n \in \mathbb{Z} \setminus \{0\}} \gamma_n'(z)^s \rho(\gamma_n)^{-1} f(\gamma_n(z)) \mathbf{1}_{\Omega_i(h)}(z) \\
&= \sum_{i=1}^M \sum_{j=1}^M \sum_{n \in \mathcal{S}(i,j)} \gamma_n'(z)^s \rho(\gamma_n)^{-1} f^j(\gamma_n(z)) \mathbf{1}_{\Omega_i(h)}(z) \\
&= \sum_{i=1}^M \sum_{n \in \mathcal{S}(i,1)} \gamma_n'(z)^s \rho(\gamma_n)^{-1} f^1(\gamma_n(z)) \mathbf{1}_{\Omega_i(h)}(z) \\
&\qquad + \sum_{i=1}^M \sum_{j=2}^M \sum_{n \in \mathcal{S}(i,j)} \gamma_n'(z)^s \rho(\gamma_n)^{-1} f^j(\gamma_n(z)) \mathbf{1}_{\Omega_i(h)}(z).
\end{aligned}$$

Inserting equations (5.33) and (5.34) into the previous line and rearranging, leads to

$$\mathcal{L}_{s,\rho} f(z) = \underbrace{\left[\sum_{i=1}^{M} \sum_{n \in \mathcal{S}(i,1)} \gamma_n'(z)^s \rho(\gamma_n)^{-1} \mathbf{1}_{\Omega_i(h)}(z) \right]}_{=:G(z,s,\rho)} f^1(0)$$
$$- \underbrace{\sum_{i=1}^{M} \sum_{j=1}^{M} \sum_{n \in \mathcal{S}(i,j)} \gamma_n'(z)^{s+\frac{1}{2}} \rho(\gamma_n)^{-1} \widetilde{f}^j\left(\gamma_n(z)\right) \mathbf{1}_{\Omega_i(h)}(z)}_{=\mathcal{L}_{s+\frac{1}{2},\rho} \widetilde{f}(z)} .$$

In the last line we used the identity

$$\gamma_n'(z)^s \gamma_n(z) = -\gamma_n'(z)^{s+1/2}. \tag{5.36}$$

(It is enough to check (5.36) for real z. The result follows for all $z \in \Omega(h)$ by analytic continuation.) Hence, we have established

$$\mathcal{L}_{s,\rho} f(z) = G(z,s,\rho) f_1(0) - \mathcal{L}_{s+\frac{1}{2},\rho} \widetilde{f}(z).$$

We can put this in the more abstract form

$$\mathcal{L}_{s,\rho} = \mathcal{F}_{s,\rho,1} + \mathcal{L}_{s+\frac{1}{2},\rho} \Psi_1, \tag{5.37}$$

where $\Psi_1 : H^2(\Omega(h); V) \to H^2(\Omega(h); V)$ is the operator sending $f = (f^1, \dots, f^M)$ to

$$-\widetilde{f} = (-\widetilde{f}^1, \dots, -\widetilde{f}^M)$$

and $\mathcal{F}_{s,\rho,1}$ is the operator defined by

$$\mathcal{F}_{s,\rho,1} f(z) := G(z,s,\rho) f^1(0). \tag{5.38}$$

Notice that for $\mathrm{Re}(s) > \frac{1}{2}$ we have $G(\cdot,s,\rho) \in H^2(\Omega(h); \mathrm{End}(V))$. Thus,

$$\mathcal{F}_{s,\rho,1} : H^2(\Omega(h); V) \to H^2(\Omega(h); V)$$

is well-defined for $\mathrm{Re}(s) > \frac{1}{2}$.

Note that the second term on right hand side of (5.37) is well-defined in the half-plane $\mathrm{Re}(s) > 0$, while the first term is a priori only defined for $\mathrm{Re}(s) > \frac{1}{2}$. To extend the domain of existence of the first term further to the left, we will show in the next few lines that $s \mapsto G(z,s,\rho)$ extends to a meromorphic function (with possible poles in $\frac{1}{2}(1 - \mathbb{N}_0)$ and that $z \mapsto G(z,s,\rho)$ is holomorphic on $\overline{\Omega(h)}$ for all points s not being a pole.

Since the sets $\mathcal{S}(i,1)$ contain every integer with sufficiently large absolute value, we can write

$$\mathcal{S}(i,1) = (\mathbb{Z} \setminus \{0\}) \setminus \mathcal{S}_i,$$

where $\mathcal{S}_i \subset \mathbb{Z} \setminus \{0\}$ is a finite set. Therefore,

$$G(z,s,\rho) = \sum_{i=1}^{M} \sum_{n\in\mathcal{S}(i,1)} \gamma_n'(z)^s \rho(\gamma_n)^{-1} \mathbf{1}_{\Omega_i(h)}(z)$$

$$= \sum_{n\in\mathbb{Z}\setminus\{0\}} \gamma_n'(z)^s \rho(\gamma_n)^{-1} - \sum_{i=1}^{M} \sum_{n\in\mathcal{S}_i} \gamma_n'(z)^s \rho(\gamma_n)^{-1} \mathbf{1}_{\Omega_i(h)}(z) \tag{5.39}$$

Observe that the functions

$$\Omega(h) \to \mathrm{End}(V), \quad z \mapsto \sum_{i=1}^{M} \sum_{n\in\mathcal{S}_i} \gamma_n'(z)^s \rho(\gamma_n)^{-1} \mathbf{1}_{\Omega_i(h)}(z)$$

are holomorphic for all $s \in \mathbb{C}$, since $\mathcal{S}_i$ are finite sets. Now let us have a closer look at the infinite sum appearing in (5.39). Since $\rho(T) \in \mathrm{End}(V)$ is a unitary map, there exists a basis $\mathbf{e}_1, \ldots, \mathbf{e}_d$ of V (where $d = \dim V = \dim \rho$) with respect to which $\rho(T)$ acts diagonally. That is, we can find numbers $\lambda_1, \ldots, \lambda_d \in (0,1]$, such that under this basis, we can write

$$\rho(T) = \mathrm{diag}\big(e^{-2\pi i\lambda_1}, \ldots, e^{-2\pi i\lambda_d}\big). \tag{5.40}$$

Then (we continue to work with the basis $\mathbf{e}_1, \ldots, \mathbf{e}_d$) we can write

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} \gamma_n'(z)^s \rho(\gamma_n)^{-1} = \sum_{n\in\mathbb{Z}\setminus\{0\}} \gamma_n'(z)^s \rho(T^{-n}S)$$

$$= \left(\sum_{n\in\mathbb{Z}\setminus\{0\}} \gamma_n'(z)^s \rho(T^{-n}) \right) \rho(S)$$

$$= \mathrm{diag}\left(\sum_{n\in\mathbb{Z}\setminus\{0\}} \gamma_n'(z)^s e^{2\pi i n\lambda_1}, \ldots, \sum_{n\in\mathbb{Z}\setminus\{0\}} \gamma_n'(z)^s e^{2\pi i n\lambda_d} \right) \rho(S)$$

Let us consider the diagonal elements in the last line individually. Fix some index $l \in \{1, \ldots, d\}$. Using the definition of the complex powers $\gamma_n'(z)^s$ we can write

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} \gamma_n'(z)^s e^{2\pi i n\lambda_l} = \sum_{n=1}^{\infty} e^{2\pi i n\lambda_l}(nw+z)^{-2s} + \sum_{n=1}^{\infty} e^{-2\pi i n\lambda_l}(nw-z)^{-2s}$$

$$= w^{-2s} H\left(\frac{z}{w}, 2s, \lambda_l\right) + w^{-2s} H\left(-\frac{z}{w}, 2s, -\lambda_l\right),$$

where $H(z,s,\lambda)$ is the (modified) Lerch zeta function defined by (5.25). Putting everything together, we obtain the following final expression for $G(z,s,\rho)$:

$$G(z,s,\rho) = \mathrm{diag}\left(w^{-2s} H\left(\frac{z}{w}, 2s, \lambda_l\right) + w^{-2s} H\left(-\frac{z}{w}, 2s, -\lambda_l\right) \right)_{l=1}^{d} \cdot \rho(S) \tag{5.41}$$
$$- \sum_{i=1}^{M} \sum_{n\in\mathcal{S}_i} \gamma_n'(z)^s \rho(\gamma_n)^{-1} \mathbf{1}_{\Omega_i(h)}(z).$$

Invoking the analytic properties of $H(z,s,\lambda)$ from Proposition 5.21, we deduce from the above expression that $s \mapsto G(z,s,\rho)$ extends to a meromorphic function and that $z \mapsto G(z,s,\rho)$ is holomorphic on $\overline{\Omega(h)}$ provided $s \notin \frac{1}{2}(1-\mathbb{N}_0)$, as claimed. It follows that

$$s \mapsto \left(\mathcal{F}_{s,\rho,1} : H^2(\Omega(h);V) \to H^2(\Omega(h);V)\right)$$

extends to a meromorphic map, whose poles are contained in the set $\frac{1}{2}(1-\mathbb{N}_0)$. Thus we have established (1) and (2) for $k=1$.

To obtain (1) and (2) for general $k \in \mathbb{N}$ we simply iterate the recursion equation (5.37) k times, where in each iteration step the 'current' s gets replaced by $s+\frac{1}{2}$. This leads to

$$\mathcal{L}_{s,\rho} = \mathcal{F}_{s,\rho,k} + \mathcal{L}_{s+\frac{k}{2},\rho}\Psi_k, \tag{5.42}$$

where

$$\Psi_k := \Psi_1^k \tag{5.43}$$

and

$$\mathcal{F}_{s,\rho,k} := \sum_{j=0}^{k-1} \mathcal{F}_{s+\frac{j}{2},\rho,1}\Psi_1^j. \tag{5.44}$$

This completes the proof of (1) (for all $k \in \mathbb{N}$) and (3).

Moreover, by the analytic properties of $\mathcal{F}_{s,\rho,1}$ already established above, the right hand side of equation (5.44) immediately reveals that

$$s \mapsto \left(\mathcal{F}_{s,\rho,k} : H^2(\Omega(h);V) \to H^2(\Omega(h);V)\right)$$

extends to a meromorphic map, whose poles are contained in the set $\frac{1}{2}\left(1-\mathbb{N}_0\right)$, thus proving (2) for all $k \in \mathbb{N}$.

It remains to estimate the rank of the operator $\mathcal{F}_{s,\rho,k}$. From the definition of Ψ_1, one easily verifies that

$$(\Psi_1^j f)(0) = \frac{(-1)^j}{j!} f^{(j)}(0).$$

Here, $f^{(j)}(z) = \partial_z^j f(z)$ denotes the usual j-th derivative (applied to each component of $f = (f^1, \ldots, f^M)$). In particular,

$$\mathcal{F}_{s+\frac{j}{2},\rho,1}\Psi_1^j f(z) = \frac{(-1)^j}{j!} G(z, s+j/2, \rho) f^{(j)}(0).$$

Notice that linear map $f \mapsto f^{(j)}(0)$ has rank equal to 1 and for fixed s, z, j, we have $G(z, s+j/2, \rho) \in \mathrm{End}(V)$. Thus, we have

$$\operatorname{rank}\left(\mathcal{F}_{s+\frac{j}{2},\rho,1}\Psi_1^j\right) \leq \dim V = \dim \rho.$$

Using (5.44) once again, we find

$$\operatorname{rank}(\mathcal{F}_{s,\rho,k}) \leq \sum_{j=0}^{k-1} \operatorname{rank}\left(\mathcal{F}_{s+\frac{j}{2},\rho,1}\Psi_1^j\right) \leq \dim \rho \cdot k,$$

proving (4). The proof of Proposition 5.24 is now complete. □

Let us finish this subsection with the following result, which we used in the above proof of Proposition 5.24. Moreover, we give a quantitative version of it, which will be useful later in Subsection 5.3.7.

Lemma 5.26. *Let $\mathcal{S}(i,j)$ be the sets defined by* (5.35)*. For each $i \in \{1,\dots,M\}$ there exists a finite set $\mathcal{S}_i \subset \mathbb{Z} \smallsetminus \{0\}$ such that*

$$\mathcal{S}(i,1) = (\mathbb{Z} \smallsetminus \{0\}) \smallsetminus \mathcal{S}_i.$$

Moreover, the sets $\mathcal{S}_i$ are not too large in the sense that there exists a constant $C > 0$ independent of h and i such that $\max_{n\in\mathcal{S}_i} |n| \leq Ch^{-1}$.

Proof. Pick an arbitrary $i \in \{1,\dots,M\}$ and let $z \in \Omega_i(h)$. The statement follows essentially from the estimate

$$|\gamma_n(z)| = \left|\frac{1}{z+nw}\right| \leq \frac{1}{||n|w - |z||},$$

which implies that for all integers n with $|n| > Ch^{-1}$ for some large enough $C > 0$ (independent of h and i), we must have

$$\operatorname{dist}(\gamma_n(z), 0) = |\gamma_n(z)| < h.$$

This means that for all $|n| > Ch^{-1}$ we have $\gamma_n(z) \in \Omega_1(h)$. In other words,

$$\{n \in \mathbb{Z} : |n| > Ch^{-1}\} \subseteq \mathcal{S}(i,1).$$

This concludes the proof. □

5.3.6 Fredholm determinant representation

The next theorem states that we can realize the L-function associated to (Γ_w, ρ) as the Fredholm determinant of $\mathcal{L}_{s,\rho}$.

Theorem 5.27. *For all $h > 0$ small enough and $\operatorname{Re}(s) > \frac{1}{2}$ the operator*

$$\mathcal{L}_{s,\rho} : H^2(\Omega(h); V) \to H^2(\Omega(h); V)$$

is trace class and we have the identity

$$L_{\Gamma_w}(s,\rho) = \det(1 - \mathcal{L}_{s,\rho}).$$

Moreover, $L_{\Gamma_w}(s,\rho)$ extends to a meromorphic function on $\mathbb{C}$. All the poles of $L_{\Gamma_w}(s,\rho)$ are contained in $\frac{1}{2}(1-\mathbb{N}_0)$.

Proof. Fix $h_0 > 0$ small enough such that the conclusion of Proposition 5.18 holds true for all $h \in (0,h_0)$. Fix $s \in \mathbb{C}$ with $\operatorname{Re}(s) > \frac{1}{2}$ and $h \in (0,h_0)$. Then

$$\mathcal{L}_{s,\rho} : H^2(\Omega(h); V) \to H^2(\Omega(h); V)$$

is a trace class operator, since the sequence of singular values $(\mu_k(\mathcal{L}_s))_{k\in\mathbb{N}}$ decays exponentially as $k \to \infty$ and is therefore summable in k.

Consider the entire function $u \mapsto \det(1 - u\mathcal{L}_{s,\rho})$. For all $|u|$ small enough we can use (A.5) to expand the determinant as

$$\det(1 - u\mathcal{L}_{s,\rho}) = \exp\left(-\sum_{N=1}^{\infty} \frac{u^N}{N} \operatorname{tr}\left(\mathcal{L}_{s,\rho}^N\right)\right). \tag{5.45}$$

In light of this formula, proving Theorem 5.27 amounts to finding a suitable expression for the traces for the iterates of $\mathcal{L}_{s,\rho}$. We have

$$\mathcal{L}_{s,\rho}^N = \sum_{\gamma \in P_N} \nu_{s,\rho}(\gamma), \tag{5.46}$$

where

$$P_N := \{T^{n_1} S T^{n_2} S \cdots T^{n_N} S : n_1, \dots, n_N \in \mathbb{Z} \setminus \{0\}\}.$$

Set

$$P := \bigcup_{N \in \mathbb{N}} P_N.$$

Let $[\gamma] \in [\Gamma_w]_h$ be a conjugacy class represented by a hyperbolic element $\gamma \in \Gamma_w$. Let $m(\gamma)$ denote the unique positive integer m satisfying $\gamma = \gamma_0^m$ with γ_0 primitive (i.e. $[\gamma_0] \in [\Gamma_w]_p$).

The following properties can be checked easily:

- Every element in P is hyperbolic.
- Every conjugacy class $[\gamma] \in [\Gamma_w]_h$ has a representative in P, say in P_N (and $N = N(\gamma)$ is unique with this property).
- Every conjugacy class $[\gamma] \in [\Gamma_w]_h$ has precisely $N(\gamma)/m(\gamma)$ distinct representatives in $P_{N(\gamma)}$.

Moreover, for every hyperbolic element $\gamma \in \Gamma_w$ with $\overline{\gamma^{-1}(\Omega(h))} \subset \Omega(h)$ we have

$$\operatorname{tr}(\nu_{s,\rho}(\gamma)) = \frac{e^{-s\ell(\gamma)}}{1 - e^{-\ell(\gamma)}} \chi(\gamma), \tag{5.47}$$

where $\chi = \operatorname{Tr} \rho$ is the character associated to ρ. Equation (5.47) is widely known in the literature, at least in the case $\rho = \mathbf{1}_{\mathbb{C}}$. For a proof of (5.47) for arbitrary twists, we refer to Pohl [71, Lemma 5.2] and the references therein.

Taking traces on both sides of (5.46), and using (5.47) and a geometric series expansion, we obtain

$$\operatorname{tr}\left(\mathcal{L}_{s,\rho}^N\right) = \sum_{\gamma \in P_N} \frac{e^{-s\ell(\gamma)}}{1 - e^{-\ell(\gamma)}} \chi(\gamma) = \sum_{k=0}^{\infty} \sum_{\gamma \in P_N} \chi(\gamma) e^{-(s+k)\ell(\gamma)}.$$

Using the above properties, we can rewrite the inner sum in the previous line as

$$\begin{aligned}
\sum_{\gamma \in P_N} e^{-(s+k)\ell(\gamma)} \chi(\gamma) &= \sum_{m=1}^{\infty} \sum_{\substack{\gamma \in P_N \\ \gamma = \gamma_0^m,\, [\gamma_0] \in [\Gamma_w]_p}} \chi(\gamma_0^m) e^{-m(s+k)\ell(\gamma_0)} \\
&= \sum_{m=1}^{\infty} \sum_{\substack{[\gamma_0] \in [\Gamma_w]_p \\ N(\gamma_0) \cdot m = N}} \frac{N}{m} \chi(\gamma_0^m) e^{-m(s+k)\ell(\gamma_0)}
\end{aligned}$$

Hence, going back to (5.45), we obtain

$$\begin{aligned}
-\log\det(1-u\mathcal{L}_{s,\rho}) &= \sum_{N=1}^{\infty}\frac{u^N}{N}\operatorname{tr}\left(\mathcal{L}_s^N\right)\\
&= \sum_{N=1}^{\infty}\frac{u^N}{N}\sum_{k=0}^{\infty}\sum_{m=1}^{\infty}\sum_{\substack{[\gamma_0]\in[\Gamma_w]_p\\ N(\gamma_0)\cdot m=N}}\frac{N}{m}\chi(\gamma_0^m)e^{-(s+k)\ell(\gamma)}
\end{aligned}$$

Rearranging the order of summation (which is justified for $\mathrm{Re}(s)$ large enough by absolute convergence) leads to

$$\begin{aligned}
-\log\det(1-u\mathcal{L}_s) &= \sum_{k=0}^{\infty}\sum_{[\gamma_0]\in[\Gamma_w]_p}\sum_{m=1}^{\infty}\frac{u^{N(\gamma_0)\cdot m}}{m}\chi(\gamma_0^m)e^{-m(s+k)\ell(\gamma_0)}\\
&= -\sum_{k=0}^{\infty}\sum_{[\gamma_0]\in[\Gamma_w]_p}\log\det\left(1-u^{N(\gamma_0)}\rho(\gamma_0)e^{-(s+k)\ell(\gamma_0)}\right)\\
&= -\log\prod_{k=0}^{\infty}\prod_{[\gamma_0]\in[\Gamma_w]_p}\det\left(1-u^{N(\gamma_0)}\rho(\gamma_0)e^{-(s+k)\ell(\gamma_0)}\right).
\end{aligned}$$

Notice that since the expression in the last line converges at $u=1$, provided $\mathrm{Re}(s)$ is large enough, we obtain the Fredholm determinant identity

$$L_\Gamma(s,\rho)=\det(1-\mathcal{L}_{s,\rho}),\quad \mathrm{Re}(s)\gg 0. \tag{5.48}$$

The validity of (5.48) immediately extends to $\mathrm{Re}(s)>\frac{1}{2}$, since the right hand side of (5.48) defines a holomorphic function in the range $\mathrm{Re}(s)>\frac{1}{2}$. Finally, meromorphic continuation to the entire complex plane with poles contained in $\frac{1}{2}(1-\mathbb{N}_0)$ follows from Proposition 5.24, completing the proof of Theorem 5.27. $\square$

5.3.7 Controlling the norms of Ψ_k and $\mathcal{F}_{s,\rho,k}$

The aim of this subsetion is to develop estimates for the operator-norm of the operators Ψ_k and $\mathcal{F}_{s,\rho,k}$ arising from the meromorphic continuation in Proposition 5.24. These estimates are crucial for the proof of Theorem 5.1 when dealing with the case $\mathrm{Re}(s)\le\frac{1}{2}$.

Lemma 5.28. *There exists $h_0>0$ such that for all $h\in(0,h_0)$ we have*

$$\|\Psi_1\|_{H^2(\Omega(h);V)\to H^2(\Omega(h);V)}\le 4h^{-3/2}.$$

Proof. As in the proof of Proposition 5.24, let $\Omega_1(h),\dots,\Omega_M(h)$ denote the connected components of $\Omega(h)$, and let $\Omega_1(h)$ be the connected component containing $z=0$. Given a function $f\in H^2(\Omega(h))$, let $f^j=f\mathbf{1}_{\Omega_j(h)}$ be its restriction to the component $\Omega_j(h)$. Recall the definition of the operator Ψ_1 from equation (5.37):

$$\Psi_1 f := -\widetilde{f} = (-\widetilde{f}^1,\widetilde{f}^2,\dots,-\widetilde{f}^M),$$

where $\widetilde{f}$ is defined by (5.33) and (5.34). The goal of this proof is to find an upper bound for $\|\widetilde{f}\|_{L^2(\Omega(h))}$ in terms of $\|f\|_{L^2(\Omega(h))}$.

It turns out that we can reduce all considerations to the one-dimensional case ($d = 1$). To this end, fix an orthonormal basis $\mathbf{e}_1, \dots, \mathbf{e}_d$ for V. For each $j \in \{1, \dots, M\}$ and $l \in \{1, \dots, d\}$ set

$$f_l^j(z) := \langle f^j(z), \mathbf{e}_l \rangle_V, \quad \widetilde{f}_l^j(z) := \langle \widetilde{f}^j(z), \mathbf{e}_l \rangle_V$$

and

$$f_l(z) := \langle f(z), \mathbf{e}_l \rangle_V, \quad \widetilde{f}_l(z) := \langle \widetilde{f}(z), \mathbf{e}_l \rangle_V.$$

Let us choose an index $l \in \{1, \dots, d\}$ that shall remain fixed until the end of the proof. Notice that the functions $\widetilde{f}_l^j$ inherit the relations (5.33) and (5.34):

$$f_l^1(z) = f_l^1(0) + z\widetilde{f}^1(z), \quad z \in \Omega_1(h) \tag{5.49}$$

and

$$f_l^j(z) = z\widetilde{f}_l^j(z), \quad z \in \Omega_j(h) \quad \text{for} \quad j \in \{2, \dots, M\}. \tag{5.50}$$

Let us fix an Euclidean disk $D(0,r)$ centered around $z = 0$ with radius $r = h/2$, say, so that $\overline{D(0,r)} \subset \Omega_1(h)$. Since f_l^1 is a holomorphic function, we can write

$$f_l^1(z) = \sum_{k=0}^{\infty} a_{k,l} z^k, \quad \forall z \in D(0,r),$$

and therefore

$$\widetilde{f}_l^1(z) = \sum_{k=0}^{\infty} a_{k+1,l} z^k, \quad \forall z \in D(0,r).$$

Using the orthogonality relation

$$\int_{D(0,r)} z^n \overline{z^m} \, \mathrm{dvol}(z) = \pi \cdot \frac{r^{2n+2}}{n+1} \delta_{nm}$$

for all $n, m \in \mathbb{N}_0$, we calculate

$$\int_{D(0,r)} |f_l^1|^2 \, \mathrm{dvol} = \pi \sum_{k=0}^{\infty} \frac{|a_{k,l}|^2 r^{2k+2}}{k+1} \tag{5.51}$$

and

$$\int_{D(0,r)} |\widetilde{f}_l^1|^2 \, \mathrm{dvol} = \pi \sum_{k=0}^{\infty} \frac{|a_{k+1,l}|^2 r^{2k+2}}{k+1}. \tag{5.52}$$

Comparing the left hand sides of (5.51) and (5.52) shows that

$$\begin{aligned}\int_{D(0,r)} |\widetilde{f}_l^1|^2 \, \mathrm{dvol} &= \pi \sum_{k=0}^{\infty} \frac{|a_{k+1,l}|^2 r^{2k+2}}{k+1} = \pi \sum_{k=1}^{\infty} \frac{|a_{k,l}|^2 r^{2k}}{k} \\ &\leq 2\pi r^{-2} \sum_{k=1}^{\infty} \frac{|a_{k,l}|^2 r^{2k+2}}{k+1} \leq 8\pi h^{-2} \int_{D(0,r)} |f_l^1|^2 \, \mathrm{dvol}.\end{aligned}$$

Since we can trivially bound the integral in the last line by $\|f_l^1\|^2_{L^2(\Omega(h))}$, the previous estimate yields

$$\int_{D(0,r)} |\widetilde{f}_l^1|^2 \,\mathrm{dvol} \leq 8\pi h^{-2} \|f_l^1\|^2_{L^2(\Omega(h))}. \tag{5.53}$$

Using the elementary estimate

$$|f_l^1(z) - f_l^1(0)|^2 \leq 2|f_l^1(z)|^2 + 2|f_l^1(0)|^2,$$

we obtain

$$\begin{aligned}
\int_{\Omega_1(h)\smallsetminus D(0,r)} |\widetilde{f}_l^1|^2 \,\mathrm{dvol} &= \int_{\Omega_1(h)\smallsetminus D(0,r)} \left| \frac{f_l^1(z) - f_l^1(0)}{z} \right|^2 \mathrm{dvol} \\
&\leq r^{-2} \int_{\Omega_1(h)\smallsetminus D(0,r)} \left(2|f_l^1(z)|^2 + 2|f_l^1(0)|^2\right) \mathrm{dvol}(z) \\
&\leq 8h^{-2} \|f_l^1\|^2_{L^2(\Omega(h))} + 8\,\mathrm{vol}(\Omega_1(h)) h^{-2} |f_l^1(0)|^2.
\end{aligned}$$

By Lemma 5.8 we have

$$|f_l^1(0)| \leq h^{-1} \|f_l^1\|_{L^2(\Omega(h))}. \tag{5.54}$$

Moreover, we have $\mathrm{vol}(\Omega_1(h)) \leq h$ for h sufficiently small. Hence, we obtain (by further decreasing h if necessary)

$$\begin{aligned}
\int_{\Omega_1(h)\smallsetminus D(0,r)} |\widetilde{f}_l^1|^2 \,\mathrm{dvol} &\leq 8h^{-2} \|f_l^1\|^2_{L^2(\Omega(h))} + 8h^{-3} \|f_l^1\|^2_{L^2(\Omega(h))} \\
&\leq 9h^{-3} \|f_l^1\|^2_{L^2(\Omega(h))}.
\end{aligned} \tag{5.55}$$

Adding up the estimates (5.53) and (5.55) yields (again, after decreasing h if necessary) the estimate

$$\begin{aligned}
\|\widetilde{f}_l^1\|^2_{L^2(\Omega(h))} = \int_{\Omega_1(h)} |\widetilde{f}_l^1|^2 \,\mathrm{dvol} &\leq 8\pi h^{-2} \|f_l^1\|^2_{L^2(\Omega(h))} + 9h^{-3} \|f_l^1\|^2_{L^2(\Omega(h))} \\
&\leq 10h^{-3} \|f_l^1\|^2_{L^2(\Omega(h))}.
\end{aligned} \tag{5.56}$$

For the remaining connected components, indexed by $j = 2, \ldots, M$, we have

$$\|\widetilde{f}_l^j\|^2_{L^2(\Omega(h))} = \int_{\Omega_j(h)} |\widetilde{f}_l^j|^2 \,\mathrm{dvol} = \int_{\Omega_j(h)} \frac{|f_l^j(z)|^2}{|z|^2} \,\mathrm{dvol}(z) \leq h^{-2} \|f_l^j\|^2_{L^2(\Omega(h))}, \tag{5.57}$$

where we simply recorded the fact that $|z| \geq \mathrm{dist}(0, \Omega_j(h)) \geq h$ for all $z \in \Omega_j(h)$. Summing up the estimates (5.56) and (5.57) leads to

$$\|\widetilde{f}_l\|^2_{L^2(\Omega(h))} = \sum_{j=1}^{M} \|\widetilde{f}_l^j\|^2_{L^2(\Omega(h))} \leq 10h^{-3} \sum_{j=1}^{M} \|f_l^j\|^2_{L^2(\Omega(h))} = 10h^{-3} \|f_l\|^2_{L^2(\Omega(h))}. \tag{5.58}$$

We can finally sum inequality (5.58) over all indices $l \in \{1, \ldots, d\}$ to obtain

$$\|\Psi f\|^2_{L^2(\Omega(h))} = \|\widetilde{f}\|^2_{L^2(\Omega(h))} = \sum_{l=1}^{d} \|\widetilde{f}_l\|^2_{L^2(\Omega(h))}$$

$$\leq 10h^{-3} \sum_{l=1}^{d} \|f_l\|^2_{L^2(\Omega(h))} = 10h^{-3}\|f\|_{L^2(\Omega(h))}.$$

Since $f \in H^2(\Omega(h); V)$ was arbitrary, we conclude that

$$\|\Psi_1\|_{H^2(\Omega(h);V)\to H^2(\Omega(h);V)} \leq \sqrt{10h^{-3}} < 4h^{-3/2}.$$

The proof is complete. □

Lemma 5.29. *There exists $h_0 > 0$ such that for all $h \in (0, h_0)$ and all $s \in \mathbb{C}$ with $|\mathrm{Im}(s)| \geq 1$ we have*

$$\|\mathcal{F}_{s,\rho,1}\|_{H^2(\Omega(h);V)\to H^2(\Omega(h);V)} \leq C(\sigma, \rho)h^{-A(\sigma)}|\mathrm{Im}(s)|^{\alpha(\sigma)}e^{Ch|s|\log(1+|s|)}.$$

Proof. Choose $h_0 > 0$ sufficiently small such that $\Omega(h) \subset D(0, 1)$ for all $h \in (0, h_0)$. From (5.38) we know that the operator $\mathcal{F}_{s,\rho,1}$ is given by

$$\mathcal{F}_{s,\rho,1}f(z) := G(z, s, \rho)f_1(0), \quad f \in H^2(\Omega(h); V).$$

Recall from (5.41) that we have the expression

$$G(z, s, \rho) = \mathrm{diag}\left(w^{-2s}H\left(\frac{z}{w}, 2s, \lambda_l\right) + w^{-2s}H\left(-\frac{z}{w}, 2s, -\lambda_l\right)\right)_{l=1}^{d} \cdot \rho(S)$$
$$- \sum_{i=1}^{M} \sum_{n\in\mathcal{S}_i} \gamma_n'(z)^s \rho(\gamma_n)^{-1}\mathbf{1}_{\Omega_i(h)}(z),$$

where $\mathcal{S}_i$ are the finite sets given by Lemma 5.26.

The first goal in this proof is to estimate $G(z, s, \rho)$ pointwise. To this end, fix some $z \in \Omega(h)$ (say $z \in \Omega_i(h)$). Using the estimate

$$\|\,\mathrm{diag}(\xi_1, \ldots, \xi_d)\|_{\mathrm{End}(V)} \leq \max_{1\leq l\leq d} |\xi_l|,$$

and the triangle inequality, we obtain

$$\|G(z, s, \rho)\|_{\mathrm{End}(V)} \leq \max_{1\leq l\leq d} \left\{w^{-2\sigma}\left|H\left(\frac{z}{w}, 2s, \lambda_l\right)\right| + w^{-2\sigma}\left|H\left(-\frac{z}{w}, 2s, -\lambda_l\right)\right|\right\}$$
$$+ \sum_{n\in\mathcal{S}_i} |\gamma_n'(z)^s|,$$

Recall that we have $|\gamma_n'(z)^s| \leq \frac{C}{|n|^{2\sigma}}e^{C|\mathrm{Im}(s)|h}$ for some $C > 0$ independent of h. Using Lemma 5.26 and increasing the constant C if necessary, we can estimate the sum in the last line as

$$\sum_{n\in\mathcal{S}_i} |\gamma_n'(z)^s| \leq 2 \sum_{1\leq n\leq Ch^{-1}} \frac{C}{n^{2\sigma}}e^{C|\mathrm{Im}(s)|h} \leq c_1(\sigma)e^{C|\mathrm{Im}(s)|h}h^{2\sigma-1}. \tag{5.59}$$

Now we invoke the estimate on the Lerch zeta function, Proposition 5.22. Note that for all $z \in D(0,1)$ we have $z/w \in D(0, \frac{1}{2})$. Hence, applying Proposition 5.22 with $r = 1/2$, we get for all $z \in D(0,1)$ the estimate

$$\left| H\left(\pm\frac{z}{w}, 2s, \pm\lambda\right) \right| \leq c_2(\sigma,\lambda)|\mathrm{Im}(s)|^{\alpha(\sigma)} e^{4|\mathrm{Im}(z)||s|\log(1+|s|)}. \tag{5.60}$$

Recall that the numbers $\lambda_1, \ldots, \lambda_d$ are defined through equation 5.40. As such, they only depend on the representation ρ. Hence, combining (5.59) and (5.60) we obtain

$$\|G(z,s,\rho)\|_{\mathrm{End}(V)} \leq c_3(\sigma,\rho)|\mathrm{Im}(s)|^{\alpha(\sigma)} e^{4|\mathrm{Im}(z)||s|\log(1+|s|)} + c_1(\sigma) e^{C|\mathrm{Im}(s)|h} h^{2\sigma-1}, \tag{5.61}$$

where $c_3(\sigma,\rho)$ depends solely on σ and the representation ρ. By merging the two summands on the left of (5.61), we get a pointwise estimate of the type

$$\|G(z,s,\rho)\|_{\mathrm{End}(V)} \leq c_4(\sigma,\rho) h^{2\sigma-1} |\mathrm{Im}(s)|^{\alpha(\sigma)} e^{C|\mathrm{Im}(z)||s|\log(1+|s|)}. \tag{5.62}$$

To finish the proof, pick an arbitrary function $f \in H^2(\Omega(h); V)$. Using Lemma 5.8, we find

$$\begin{aligned}
\|\mathcal{F}_{s,\rho,1} f\|_{L^2(\Omega(h))} &= \left(\int_{\Omega(h)} \|G(z,s,\rho) f^1(0)\|_V^2 \,\mathrm{dvol}(z) \right)^{1/2} \\
&\leq \left(\int_{\Omega(h)} \|G(z,s,\rho)\|_{\mathrm{End}(V)}^2 \,\mathrm{dvol}(z) \right)^{1/2} \|f^1(0)\|_V \\
&\leq \left(\int_{\Omega(h)} \|G(z,s,\rho)\|_{\mathrm{End}(V)}^2 \,\mathrm{dvol}(z) \right)^{1/2} h^{-1} \|f\|_{L^2(\Omega(h))}.
\end{aligned}$$

Using the pointwise estimate in (5.62) and noticing that for all $z \in \Omega(h)$ we have $|\mathrm{Im}(z)| \leq h$, we can crudely bound the integral in the previous line, leading to

$$\begin{aligned}
&\leq \mathrm{vol}(\Omega(h)) \cdot c_4(\sigma,\rho) h^{-A_1(\sigma)} |\mathrm{Im}(s)|^{\alpha(\sigma)} e^{Ch|s|\log(1+|s|)} \|f\|_{L^2(\Omega(h))} \\
&= c_5(\sigma,\rho) h^{-A_2(\sigma)} |\mathrm{Im}(s)|^{\alpha(\sigma)} e^{Ch|s|\log(1+|s|)} \|f\|_{L^2(\Omega(h))}.
\end{aligned}$$

Since f was arbitrary, we obtain (after relabelling the constants) the claimed estimate for the norm of $\mathcal{F}_{s,\rho,1}$. $\square$

We can finally prove the last result of this subsection, which is what we actually need in the proof of Theorem 5.1.

Proposition 5.30. *There exist $C > 0$ and $h_0 > 0$ such that for all $h \in (0, h_0)$, $s \in \mathbb{C}$ with $|\mathrm{Im}(s)| \geq 1$, and all $k \in \mathbb{N}$ there exist constants $\alpha_1 = \alpha_1(\sigma,k)$, $A_1 = A_1(\sigma,k)$ and $C_1 = C_1(\sigma,k,\rho)$ such that*

$$\|\mathcal{F}_{s,\rho,k}\|_{H^2(\Omega(h);V) \to H^2(\Omega(h);V)} \leq C_1 h^{-A_1} |\mathrm{Im}(s)|^{\alpha_1} e^{Ch(|s|+\frac{k}{2})\log(1+|s|+\frac{k}{2})} \tag{5.63}$$

and

$$\|\Psi_k\|_{H^2(\Omega(h);V) \to H^2(\Omega(h);V)} \leq 4^k h^{-3k/2}. \tag{5.64}$$

Proof. For ease of notation we write $\|\cdot\|$ instead of $\|\cdot\|_{H^2(\Omega(h);V)\to H^2(\Omega(h);V)}$. By Proposition 5.24 (3) and Lemma 5.28, we have

$$\|\Psi_k\| \leq \|\Psi_1\|^k \leq 4^k h^{-3k/2},$$

proving (5.64). Applying the triangle inequality to the recursion equation for $\mathcal{F}_{s,\rho,k}$ in Proposition 5.24 (3), and using the already established norm estimate in Lemma 5.29 for $\mathcal{F}_{s,\rho,1}$, we get

$$\|\mathcal{F}_{s,\rho,k}\| \leq \sum_{j=0}^{k-1} \|\mathcal{F}_{s+\frac{j}{2},\rho,1}\| \cdot \|\Psi_1\|^j$$

$$\leq \sum_{j=0}^{k-1} C\left(\sigma + \frac{j}{2}, \rho\right) h^{-A\left(\sigma+\frac{j}{2}\right)} |\mathrm{Im}(s)|^{\alpha\left(\sigma+\frac{j}{2}\right)} e^{Ch|s+\frac{j}{2}|\log\left(1+|s+\frac{j}{2}|\right)} \cdot 4^j h^{-3j/2}$$

$$\leq \sum_{j=0}^{k-1} 4^j C\left(\sigma + \frac{j}{2}, \rho\right) h^{-\left(A\left(\sigma+\frac{j}{2}\right)+\frac{3j}{2}\right)} |\mathrm{Im}(s)|^{\alpha\left(\sigma+\frac{j}{2}\right)} e^{Ch\left(|s|+\frac{j}{2}\right)\log\left(1+|s|+\frac{j}{2}\right)}.$$

Choosing (without trying to optimize the constants)

$$\alpha_1 := \max_{0\leq j\leq k-1}\left\{\alpha\left(\sigma+\frac{j}{2}\right)\right\}, \quad A_1 := \max_{0\leq j\leq k-1}\left\{A\left(\sigma+\frac{j}{2}\right)+\frac{3j}{2}\right\},$$

and

$$C_1 := k \cdot \max_{0\leq j\leq k-1}\left\{4^j C\left(\sigma+\frac{j}{2},\rho\right)\right\},$$

we obtain

$$\|\mathcal{F}_{s,\rho,k}\| \leq C_1 h^{-A_1} |\mathrm{Im}(s)|^{\alpha_1} e^{Ch(|s|+\frac{k}{2})\log(1+|s|+\frac{k}{2})},$$

thus proving (5.63). □

5.3.8 Proof of Theorem 5.1

We can now prove Theorem 5.1 by gathering all the results established in previous subsections. The following inequality on Fredholm determinants allows us to separate the finite-rank part ($\mathcal{F}_{s,\rho,k}$) arising in Proposition 5.24 from the holomorphic part ($\mathcal{L}_{s+\frac{k}{2},\rho}\Psi_k$) of the transfer operator. For the latter we have a rather precise singular value estimate, thanks to Proposition 5.18.

Lemma 5.31. *Let $\mathcal{H}$ be a separable Hilbert space, $\mathcal{F}: \mathcal{H} \to \mathcal{H}$ a finite-rank operator, and $\mathcal{T}: \mathcal{H} \to \mathcal{H}$ an arbitrary trace class operator. Then*

$$\log|\det(1+\mathcal{F}+\mathcal{T})| \leq \mathrm{rank}(\mathcal{F})\log(1+\|\mathcal{F}\|) + \sum_{m=1}^{\infty}\log(1+\mu_m(\mathcal{T})).$$

Proof. By the well-known Weyl inequality (see (A.4)), we have

$$|\det(1+\mathcal{F}+\mathcal{T})| \leq \det(1+|\mathcal{F}+\mathcal{T}|).$$

By a result of Seiler–Simon [78], we have

$$\det(1+|\mathcal{F}+\mathcal{T}|) \leq \det(1+|\mathcal{F}|)\det(1+|\mathcal{T}|).$$

Since $\mathcal{F}$ is a finite-rank operator, we have $\mu_m(\mathcal{F}) = 0$ for all $m > \operatorname{rank}(\mathcal{F})$. Thus, taking logarithms leads to

$$\begin{aligned}\log|\det(1+\mathcal{F}+\mathcal{T})| &\leq \log\det(1+|\mathcal{F}|) + \log\det(1+|\mathcal{T}|)\\ &= \sum_{m=1}^{\infty}\log(1+\mu_m(\mathcal{F})) + \sum_{m=1}^{\infty}\log(1+\mu_m(\mathcal{T}))\\ &\leq \operatorname{rank}(\mathcal{F})\log(1+\|\mathcal{F}\|) + \sum_{m=1}^{\infty}\log(1+\mu_m(\mathcal{T})).\end{aligned}$$

The proof is complete. □

Also helpful is the following elementary estimate.

Lemma 5.32. *For all $M \geq 0$ we have, as $\eta \searrow 0$,*

$$\sum_{m=0}^{\infty}\log(1+Me^{-\eta m}) = O\left(\eta^{-1}\log(2+M)^2\right).$$

Proof. Set $m_0 := \lfloor \eta^{-1}\log(2+M)\rfloor$. Then for all integers $m > m_0$ we have $Me^{-\eta m} < 1$, which implies

$$\begin{aligned}\sum_{m=0}^{\infty}\log(1+Me^{-\eta m}) &\leq m_0\log(1+M) + \sum_{j=1}^{\infty}\log(1+e^{-\eta j})\\ &\leq \eta^{-1}\log(2+M)^2 + \sum_{j=0}^{\infty}e^{-\eta j}\\ &= \eta^{-1}\log(2+M)^2 + \frac{1}{1-e^{-\eta}}.\end{aligned}$$

Since $e^{-\eta} = 1-\eta+o(\eta)$ as $\eta \searrow 0$, we conclude that

$$\sum_{m=0}^{\infty}\log(1+Me^{-\eta m}) \leq \eta^{-1}\log(2+M)^2 + O(\eta^{-1}) = O\left(\eta^{-1}\log(2+M)^2\right),$$

as claimed. □

Proof of Theorem 5.1. From Theorem 5.27 we already know that $L_{\Gamma_w}(s,\rho)$ extends to a meromorphic function of $s \in \mathbb{C}$, and that its poles are contained in the set $\frac{1}{2}(1-\mathbb{N}_0)$.

So it remains to prove the growth estimate. To this end, fix some $h_0 > 0$ small enough such that the conclusions of Proposition 5.24, Proposition 5.18, and Proposition 5.30 hold true for all $h \in (0, h_0)$. Fix $s = \sigma + iT$ where $\sigma = \mathrm{Re}(s)$ and $T = \mathrm{Im}(s)$, and assume that $|T| \geq 2$. Set $h := h_0 \left(\frac{|T|}{1+\log|T|}\right)^{-1} \in (0, h_0)$ and

$$k := \begin{cases} \lceil -2\sigma + 1 \rceil + 1 & \text{if } \sigma \leq \frac{1}{2} \\ 0 & \text{otherwise,} \end{cases}$$

so that $\mathrm{Re}(s) + \frac{k}{2} > \frac{1}{2}$. By Theorem 5.27 and Proposition 5.24 we can express the L-function as

$$L_{\Gamma_w}(s, \rho) = \det(1 - \mathcal{F}_{s,\rho,k} - \mathcal{L}_{s+\frac{k}{2},\rho} \Psi_k). \tag{5.65}$$

In the case $k = 0$ (that is, when $\sigma > \frac{1}{2}$) we simply set $\mathcal{F}_{s,\rho,0} = 0$ and $\Psi_0 = \mathrm{id}$, to avoid an artificial case distinction. Again, we simplify the notation by writing $\|\cdot\|$ instead of $\|\cdot\|_{H^2(\Omega(h);V)\to H^2(\Omega(h);V)}$.

Applying Lemma 5.31 to the right hand side of (5.65) leads to

$$\log |L_{\Gamma_w}(s, \rho)| \leq S_1 + S_2,$$

with the two terms

$$S_1 := \mathrm{rank}(\mathcal{F}_{s,\rho,k}) \cdot \log(1 + \|\mathcal{F}_{s,\rho,k}\|)$$

and

$$S_2 := \sum_{m=1}^{\infty} \log(1 + \mu_m(\mathcal{L}_{s+\frac{k}{2},\rho} \Psi_k)).$$

The goal is to estimate these two terms individually, starting with S_1. By Proposition 5.24 and the definition of k, the rank of $\mathcal{F}_{s,\rho,k}$ can be bounded solely in terms of σ, ρ:

$$\mathrm{rank}(\mathcal{F}_{s,\rho,k}) \leq \dim \rho \cdot k \ll_{\sigma,\rho} 1.$$

Combining this with the norm estimate for $\mathcal{F}_{s,\rho,k}$ in (5.63) and noticing that the constants C_1, A_1, α_1 appearing in (5.63) are now bounded solely in terms of σ, ρ, we obtain

$$\begin{aligned} S_1 &\ll_{\sigma,\rho} \log \left(1 + C_1 h^{-A_1} |T|^{\alpha_1} e^{Ch(|s|+\frac{k}{2})\log(1+|s|+\frac{k}{2})}\right) \\ &\ll_{\sigma,\rho} \log(h^{-1}) + \log|T| + h(|s| + k/2)\log(1 + |s| + k/2). \end{aligned}$$

Observe that we trivially have $|s| + k/2 \ll_\sigma |T|$, from which we deduce the final bound on S_1:

$$S_1 \ll_{\sigma,\rho} \log(h^{-1}) + \log|T| + h|T|\log|T| \ll_{\sigma,\rho} (\log|T|)^2. \tag{5.66}$$

We proceed with estimating S_2. First notice that

$$\mu_m(\mathcal{L}_{s+\frac{k}{2}} \Psi_k) \leq \mu_m(\mathcal{L}_{s+\frac{k}{2}}) \cdot \|\Psi_k\|.$$

We can now invoke the singular value estimate from Proposition 5.18 and the norm estimate for Ψ_k in (5.64). Combining both of these estimates leads to an estimate of the type

$$\mu_m(\mathcal{L}_{s+\frac{k}{2}} \Psi_k) \leq A h^{-A} e^{B|T|h} \exp(-Ch^\delta m),$$

where $A, B, C > 0$ may depend on σ and ρ, but not on h. We can now invoke Lemma 5.32, which gives

$$\begin{aligned}
S_2 &\le \sum_{m=1}^{\infty} \log\left(1 + Ah^{-A}e^{B|T|h}\exp(-Ch^{\delta}m)\right) \\
&\ll_{\sigma,\rho} h^{-\delta}\log\left(2 + Ah^{-A}e^{B|T|h}\right)^2 \\
&\ll_{\sigma,\rho} \left(\frac{|T|}{\log|T|}\right)^{\delta}\log\left(2 + Ah_0^{-A}\left(\frac{|T|}{1+\log|T|}\right)^{A} e^{Bh_0^{-1}\log|T|}\right)^2 \\
&\le \left(\frac{|T|}{\log|T|}\right)^{\delta}\log\left(2 + Ah_0^{-A}e^{(A+Bh_0^{-1})\log|T|}\right)^2
\end{aligned}$$

Noticing that $\log\left(2 + Ah_0^{-A}e^{(A+Bh_0^{-1})\log|T|}\right) \ll_{\sigma} \log|T|$ finally leads to the bound

$$S_2 \ll_{\sigma,\rho} \left(\frac{|T|}{\log|T|}\right)^{\delta}(\log|T|)^2 = |T|^{\delta}(\log|T|)^{2-\delta}. \tag{5.67}$$

Adding together (5.66) and (5.67) yields

$$\log|L_{\Gamma_w}(s,\rho)| \ll_{\sigma,\rho} (\log|T|)^2 + |T|^{\delta}(\log|T|)^{2-\delta}. \tag{5.68}$$

Clearly, the first term in (5.68) gets absorbed by the second. The proof of Theorem 5.1 is complete. □

5.3.9 Proof of Corollary 5.3

We can now prove Corollary 5.3. Let $\widetilde{\Gamma}$ be a torsion-free, finite-index subgroup of Γ_w, let $\widetilde{X} = \widetilde{\Gamma}\backslash\mathbb{H}$ be its corresponding hyperbolic surface, and let $N_X(\sigma, T)$ and $M_X(\sigma, T)$ be the resonance counting functions defined in (5.2) and (5.3), which we seek to estimate. Notice that it suffices to prove the estimate for $M_{\widetilde{X}}(\sigma, T)$, since

$$N_{\widetilde{X}}(\sigma, T) \le \int_{-T}^{T} M_{\widetilde{X}}(\sigma, t)dt.$$

By the result of Borthwick–Judge–Perry [13], the zeros of $Z_{\widetilde{\Gamma}}(s)$ correspond with multiplicities to the resonances for $\widetilde{X}$, except for some zeros on the real line. More formally, we have the equality of sets (including multiplicities)

$$\#\left\{s \in \mathbb{C} \smallsetminus \mathbb{R} : Z_{\widetilde{\Gamma}}(s) = 0\right\} = \mathcal{R}(\widetilde{X}) \smallsetminus \mathbb{R}. \tag{5.69}$$

Hence, for fixed $\sigma \in \mathbb{R}$ we have

$$M_{\widetilde{X}}(\sigma, T) = M(\sigma, T) - O(1),$$

where

$$M(\sigma, T) := \#\left\{s \in \mathbb{C} : Z_{\widetilde{\Gamma}}(s) = 0,\ \mathrm{Re}(s) \ge \sigma,\ |\mathrm{Im}(s) - T| \le 1\right\}.$$

Therefore, it is enough to estimate the zero-counting function $M(\sigma, T)$. Recall that if $\sigma > \delta$, then the Selberg zeta function $Z_{\widetilde{\Gamma}}(s)$ has no zeros in the half-plane

$\{\mathrm{Re}(s) \geq \sigma\}$. We can therefore assume that $\sigma \leq \delta$ in which case $M(\sigma, T)$ is equal to the number of zeros of $Z_{\widetilde{\Gamma}}(s)$ within the box

$$R(\sigma, T) := [\sigma, \delta] + i[T-1, T+1].$$

Set $s_0 = 1 + iT$ and observe that

$$R(\sigma, T) \subset D(s_0, r_1) \subset D(s_0, r_2),$$

with radii $r_1 = \sqrt{1 + (1-\sigma)^2} + 1$ and $r_2 = 2r_1$. Notice that r_1, r_2 depend solely on σ. By increasing the multiplicative constant in the statement of Corollary 5.3, we may assume that $|T| > r_2 + 2$, so that for each $s \in D(s_0, r_2)$ we have $|\mathrm{Im}(s)| \geq 2$. In particular, the disk $D(s_0, r_2)$ does not meet the real line. Since all the poles of $Z_{\widetilde{\Gamma}}(s)$ lie on the real line, the function $Z_{\widetilde{\Gamma}}(s)$ is holomorphic on $\overline{D(s_0, r_2)}$, allowing us to apply Titchmarsh's Number of Zeros Theorem (see Lemma 3.5). We obtain

$$M(\sigma, T) \leq \frac{1}{\log(r_2/r_1)} \left(\max_{|s-s_0|=r_2} \log |Z_{\widetilde{\Gamma}}(s)| - \log |Z_{\widetilde{\Gamma}}(s_0)| \right). \tag{5.70}$$

Using the Euler product representation for the Selberg zeta function $Z_{\widetilde{\Gamma}}(s)$ (which is valid for all $\mathrm{Re}(s) > \delta$), we find

$$|Z_{\widetilde{\Gamma}}(s_0)| = |Z_{\widetilde{\Gamma}}(1 + iT)| \geq Z_{\widetilde{\Gamma}}(1). \tag{5.71}$$

Hence, inserting (5.71) and Corollary 5.2 into (5.70) gives

$$M(\sigma, T) \leq \frac{1}{\log(2)} \left(\max_{|s-s_0|=r_2} \log |Z_{\widetilde{\Gamma}}(s)| - \log |Z_{\widetilde{\Gamma}}(1)| \right) \ll_\sigma |T|^\delta (\log |T|)^{2-\delta},$$

as claimed.

5.4 Essential spectral gap

The goal of this section is to prove the last result stated in the introduction of this chapter, Theorem 5.4. It will follow from Theorem 5.33 below, which is a more abstract result that applies to every non-elementary, finitely generated, torsion-free Fuchsian group.

Let Γ be a finitely generated Fuchsian group and let Z_Γ be its Selberg zeta function. We define the *essential spectral gap* of Γ as

$$G(\Gamma) := \inf \{\sigma : Z_\Gamma(s) \text{ has finitely many zeros in } \{\mathrm{Re}(s) \geq \sigma\}\}.$$

Notice that the statement $G(\Gamma) \geq \sigma$ is equivalent with saying that for every $\varepsilon > 0$ the Selberg zeta function $Z_\Gamma(s)$ has *infinitely* many zeros with $\mathrm{Re}(s) \geq \sigma - \varepsilon$. Thus, an explicit lower bound on the essential gap leads to an explicit vertical strip containing infinitely many zeros. If we assume furthermore that Γ is torsion-free, then the zeros of $Z_\Gamma(s)$ in $\{\mathrm{Re}(s) \geq \sigma - \varepsilon\}$ are (up to a finite number of exceptions) resonances of $X = \Gamma \backslash \mathbb{H}$.

In this section, we follow the ideas of Jakobson–Naud [40] to show how one can utilize vertical growth estimates for the Selberg zeta function to derive lower bounds for $G(\Gamma)$.

To adequately formulate our main result, we define for each $\sigma \in \mathbb{R}$ the quantity $\kappa(\sigma)$, which measures the growth of the Selberg zeta function $Z_\Gamma(s)$ in the half-plane $\{\mathrm{Re}(s) \geq \sigma\}$ and bounded away from the real axis. More precisely, we let $\kappa(\sigma)$ be the infimum over all $\kappa > 0$ such that for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) \geq \sigma$ and $|\mathrm{Im}(s)| \geq 1$ there exists $C = C(\sigma, \kappa)$ such that

$$\log |Z_\Gamma(s)| \leq C |\mathrm{Im}(s)|^\kappa.$$

Notice that the condition $|\mathrm{Im}(s)| \geq 1$, which ensures that s is bounded away from the real axis, may be replaced by $|\mathrm{Im}(s)| \geq c$ for any other fixed positive value c without changing the value $\kappa(\sigma)$.

The fractal upper bound for Schottky groups Γ due to Guillopé–Lin–Zworski [33] immediately implies that $\kappa_\Gamma(\sigma) \leq \delta$ for all $\sigma \in \mathbb{R}$. Similarly, Corollary 5.2 shows that for any finite-index subgroup $\widetilde{\Gamma}$ of a Hecke triangle group Γ_w we have $\kappa_{\widetilde{\Gamma}}(\sigma) \leq \delta$. On the other hand, using the product definition for the Selberg zeta function, it is not hard to show that for every finitely generated Fuchsian group Γ and all $\sigma > \delta$, we have $\kappa_\Gamma(\sigma) = 0$.

The main result of the present section relates the essential spectral gap $G(\Gamma)$ with the growth $\kappa_\Gamma(\sigma)$ of the Selberg zeta function Z_Γ.

Theorem 5.33. *Let Γ be a non-elementary, finitely generated, torsion-free Fuchsian group and let $\delta = \delta(\Gamma)$ be the Hausdorff dimension of the limit set $\Lambda(\Gamma)$. Then*

$$G(\Gamma) \geq \frac{\delta}{2} - \kappa_\Gamma(G(\Gamma))\delta.$$

Moreover, if Γ has the bounded cluster property and $\delta(\Gamma) > \frac{1}{2}$, we have

$$G(\Gamma) \geq \delta - \frac{\kappa_\Gamma(G(\Gamma))}{2} - \frac{1}{4}.$$

The term 'bounded cluster property' refers to a property of the trace spectrum of Γ and will be recalled in Subsection 5.4.2.

This section is mainly devoted to the proof of Theorem 5.33. It is only in the last subsection, Subsection 5.4.3, where we will show how Theorem 5.4 can be deduced from Theorem 5.33.

5.4.1 An approximate trace formula

The next result can be seen as an approximate trace formula with error term. It is essentially an extension of [40, Proposition 3.1] to the case when $X = \Gamma\backslash\mathbb{H}$ has cusps. Recall that for any given function $\varphi \in C_c^\infty\big((0,\infty)\big)$ we let

$$\widehat{\varphi}(z) := \int_{-\infty}^{\infty} e^{-ixz} \varphi(x)\, dx.$$

be its Fourier transform. Moreover, we denote by $\mathcal{Z}_\Gamma$ the set of zeros of Z_Γ.

Proposition 5.34. *Let Γ and δ be as in Theorem 5.33. Let $\sigma < \delta$ and $\kappa > 0$. Assume that $\kappa > \kappa_\Gamma(\sigma)$ and assume that $Z_\Gamma(s)$ has only finitely many zeros in the half-plane $\mathrm{Re}(s) \geq \sigma$. Then for all test functions $\varphi \in C_c^\infty\big((0,\infty)\big)$ and all $\varepsilon > 0$ sufficiently small we have*

$$\sum_{[\gamma]\in[\Gamma]_p} \sum_{m=1}^{\infty} \frac{\ell(\gamma)\varphi(m\ell(\gamma))}{1-e^{-m\ell(\gamma)}} \tag{5.72}$$

$$= \sum_{\substack{\zeta\in\mathcal{Z}_\Gamma \\ \mathrm{Re}(\zeta)\geq\sigma}} \widehat{\varphi}(i\zeta) - S_\Gamma(\sigma) + O_{\varepsilon,\sigma,\kappa}\left(\int_{-\infty}^{\infty} \langle x\rangle^{\kappa} |\widehat{\varphi}(x+i(\sigma+\varepsilon))| dx\right), \tag{5.73}$$

where

$$S_\Gamma(\sigma) = \begin{cases} 0 & \text{if } \sigma > \frac{1}{2} \\ n_c \sum_{k=0}^{\lceil 1-2\sigma\rceil} \widehat{\varphi}\left(i\frac{1-k}{2}\right) & \text{if } \sigma \leq \frac{1}{2} \end{cases}$$

is the contribution coming from the possible poles of Z_Γ and $n_c \geq 0$ denotes the number of cusps of $X = \Gamma\backslash\mathbb{H}$.

For the proof of Proposition 5.34 we need the Borel-Carathéodory inequality (for a proof see Titchmarsh [89, Chapter V]):

Lemma 5.35. *Let f be a holomorphic function on a neighbourhood of $\{|z| \leq R\}$. Then for any $r < R$ we have*

$$\max_{|z|=r} |f'(z)| \leq \frac{8R}{(R-r)^2}\left(\max_{|z|=R} |\mathrm{Re} f(z)| + |f(0)|\right).$$

Proof of Proposition 5.34. For all $\mathrm{Re}(s) > \delta$, the Euler product definition of the Selberg zeta function leads to the following formula for its logarithmic derivative:

$$\frac{Z_\Gamma'(s)}{Z_\Gamma(s)} = \sum_{[\gamma]\in[\Gamma]_p} \sum_{m=1}^{\infty} \frac{\ell(\gamma)e^{-sm\ell(\gamma)}}{1-e^{-m\ell(\gamma)}}. \tag{5.74}$$

Fix some real number $a > \delta$ and consider the contour integral

$$I(a) := \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=a} \frac{Z_\Gamma'(s)}{Z_\Gamma(s)} \widehat{\varphi}(is) ds. \tag{5.75}$$

By (5.74), we have

$$\left|\frac{Z_\Gamma'(s)}{Z_\Gamma(s)}\right| \leq \frac{Z_\Gamma'(a)}{Z_\Gamma(a)}$$

for all $\mathrm{Re}(s) \geq a$. Moreover, $\widehat{\varphi}(is)$ decreases rapidly as $|\mathrm{Im}(s)| \to \infty$, since $\varphi \in C_c^\infty(\mathbb{R})$. Therefore (5.75) converges absolutely. Using formula (5.74) and interchanging summation and integration (justified by the absolute convergence of the contour integral), we obtain

$$I(a) = \sum_{[\gamma]\in[\Gamma]_p} \sum_{m=1}^{\infty} \frac{\ell(\gamma)}{1-e^{-m\ell(\gamma)}} \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=a} \widehat{\varphi}(is) e^{-sm\ell(\gamma)} ds.$$

Deforming the contour to the imaginary axis and using the Fourier inversion formula, we obtain

$$\begin{aligned}\frac{1}{2\pi i}\int_{\mathrm{Re}(s)=a}\widehat{\varphi}(is)e^{-sm\ell(\gamma)}ds &= \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=0}\widehat{\varphi}(is)e^{-sm\ell(\gamma)}ds\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\xi m\ell(\gamma)}\widehat{\varphi}(\xi)d\xi\\ &= \varphi(m\ell(\gamma)).\end{aligned}$$

Thus,

$$I(a) = \sum_{[\gamma]\in[\Gamma]_p}\sum_{m=1}^{\infty}\frac{\ell(\gamma)}{1-e^{-m\ell(\gamma)}}\varphi(m\ell(\gamma)).$$

Let us return to (5.75). By assumption, $Z_\Gamma(s)$ has only finitely many zeros in the half-plane $\mathrm{Re}(s)\geq\sigma$. By moving the contour in (5.75) to the left, we have (at least formally)

$$I(a) = \sum_{\substack{\zeta\in\mathcal{Z}_\Gamma\\ \mathrm{Re}(\zeta)\geq\sigma}}\widehat{\varphi}(i\zeta) - S_\Gamma(\sigma) + \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=\sigma+\varepsilon}\frac{Z_\Gamma'(s)}{Z_\Gamma(s)}\widehat{\varphi}(is)ds, \tag{5.76}$$

for all $\varepsilon > 0$ small enough, where $S_\Gamma(\sigma)$ is the contribution coming from the possible poles of Z_Γ in the half-plane $\{\mathrm{Re}(s)\geq\sigma\}$. From the result of Bortwick–Judge–Perry [13] (see also Section 2.5), we know that the possible poles of Z_Γ are located at the points $\frac{1}{2}(1-\mathbb{N}_0)$, and each pole has multiplicity n_c. (Note that if Γ has no parabolic elements ($n_c = 0$), then Z_Γ has no poles.) It follows that

$$S_\Gamma(\sigma) = \begin{cases}0 & \text{if } \sigma > \frac{1}{2}\\ n_c\sum_{k=0}^{\lceil 1-2\sigma\rceil}\widehat{\varphi}\left(i\frac{1-k}{2}\right) & \text{if } \sigma\leq\frac{1}{2}\end{cases}.$$

We now claim that there exists a constant $M\geq 1$ such that

$$\left|\frac{Z_\Gamma'(s)}{Z_\Gamma(s)}\right| \ll_{a,\varepsilon,\sigma,\kappa} |\mathrm{Im}(s)|^\kappa \tag{5.77}$$

for all $s\in\mathbb{C}$ with $\sigma+\varepsilon\leq\mathrm{Re}(s)\leq a$ and $|\mathrm{Im}(s)| > M$. Before we prove this claim, let us see how it can be used to finish the proof. First, notice that the estimate (5.77) shows that the integral appearing in (5.76) converges absolutely, and therefore justifies the contour deformation used in (5.76). Using (5.77), we obtain

$$I(a) = \sum_{\substack{\zeta\in\mathcal{Z}_\Gamma\\ \mathrm{Re}(\zeta)\geq\sigma}}\widehat{\varphi}(i\zeta) - S_\Gamma(\sigma) + O_{\varepsilon,\sigma,\kappa}\left(\int_{-\infty}^{\infty}\langle x\rangle^\kappa|\widehat{\varphi}(x+i(\sigma+\varepsilon))|dx\right),$$

and Proposition 5.34 follows.

It remains to establish estimate (5.77). For $t\in\mathbb{R}$, consider the meromorphic function

$$z\mapsto\frac{Z_\Gamma(z+a+it)}{Z_\Gamma(a+it)}. \tag{5.78}$$

Using the assumption that $Z_\Gamma(s)$ has only finitely many zeros in $\mathrm{Re}(s) \geq \sigma$, we deduce that there exists $M \geq 1$ such that for all $|t| > M$, (5.78) defines a non-vanishing holomorphic function on the closed disk $\overline{D}(0, a - \sigma - \varepsilon/2)$. This allows us to define the complex holomorphic logarithm

$$f_t(z) := \log\left(\frac{Z_\Gamma(z + a + it)}{Z_\Gamma(a + it)}\right),$$

satisfying $f_t(0) = 0$ and

$$\mathrm{Re}(f_t(z)) = \log\left|\frac{Z_\Gamma(z + a + it)}{Z_\Gamma(a + it)}\right|.$$

Applying the Lemma 5.35 to the function f_t with radii $R = a - \sigma - \varepsilon/2$ and $r = a - \sigma - \varepsilon$, we obtain

$$\max_{|z| = a - \sigma - \varepsilon}\left|\frac{Z'_\Gamma(z + a + it)}{Z_\Gamma(z + a + it)}\right| \leq 32R\varepsilon^{-2} \max_{|z| = a - \sigma} \log\left|\frac{Z_\Gamma(z + a + it)}{Z_\Gamma(a + it)}\right| \tag{5.79}$$

$$= 32R\varepsilon^{-2}\left(\max_{|z| = a - \sigma - \varepsilon/2} \log|Z_\Gamma(z + a + it)| - \log|Z_\Gamma(a + it)|\right) \tag{5.80}$$

Since $a > \delta$, we can use the product definition of Z_Γ to obtain the lower bound

$$|Z_\Gamma(a + it)| \geq Z_\Gamma(a) \tag{5.81}$$

for all $t \in \mathbb{R}$. On the other hand, by the definition of $\kappa(\sigma)$ and by the assumption that $\kappa > \kappa(\sigma)$, we know that for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) \geq \sigma$ and $|\mathrm{Im}(s)| \geq M$ we have the growth estimate

$$\log|Z_\Gamma(s)| \ll_{\sigma,\kappa} |\mathrm{Im}(s)|^\kappa.$$

In particular, we have

$$|Z_\Gamma(z + a + it)| \ll_{a,\sigma,\kappa} |t|^\kappa. \tag{5.82}$$

for all complex z with $|z| = a - \sigma - \varepsilon/2$. The claimed estimate (5.77) now follows from inserting (5.81) and (5.82) into (5.80). $\square$

5.4.2 Bounded cluster property

A Fuchsian group Γ is said to have the *bounded cluster property* if there exists a constant $C = C(\Gamma)$ such that for every $N \in \mathbb{N}_0$ we have

$$|\mathrm{Tr}(\Gamma) \cap [N, N+1]| \leq C,$$

where

$$\mathrm{Tr}(\Gamma) := \{|\mathrm{tr}(\gamma)| : \gamma \in \Gamma\}$$

is the *trace spectrum* of Γ (without multiplicities!). The bounded cluster property imposes strong implications on the length spectrum of $X = \Gamma\backslash\mathbb{H}$.

Recall that the (primitive) length spectrum of $X = \Gamma\backslash\mathbb{H}$ is the countable set

$$\mathcal{L}(X) = \{\ell(\gamma) : [\gamma] \in [\Gamma]_p\}.$$

Each element $\ell \in \mathcal{L}(X)$ in this set appears with a certain multiplicity, which we denote by $M_\Gamma(\ell)$. The following lemma is a quantitative version of the fact that, on average, the lengths in the length spectrum of groups having bounded cluster property occur with high multiplicity.

Lemma 5.36. *Let Γ be a finitely generated Fuchsian group with $\delta = \delta(\Gamma) > \frac{1}{2}$. Assume that Γ has the bounded cluster property. Then for any $c > 0$ we have*

$$\sum_{\substack{\ell \\ \tau - c \le \ell \le \tau + c}} M_\Gamma(\ell)^2 \gg \frac{e^{(2\delta - 1/2)\tau}}{\tau^2}, \quad \text{as } \tau \to \infty, \tag{5.83}$$

where the summation is taken over each length ℓ appearing in $\mathcal{L}(X) \cap [\tau - c, \tau + c]$. Moreover, the implied constant in (5.83) *depends on Γ and c only.*

Proof. The bounded cluster property implies that there exists $C' = C'(\Gamma)$ such that for all $x \ge 1$ we have

$$\#\{t \in \mathrm{Tr}(\Gamma) : t \le x\} \le C' x. \tag{5.84}$$

Combining (5.84) with the identity

$$|\mathrm{tr}(\gamma)| = 2\cosh\left(\frac{\ell(\gamma)}{2}\right),$$

shows that there exists some $C'' = C''(\Gamma)$ such that

$$\#\{\ell \in \mathcal{L}(X) : \ell \le \tau\} \le C'' \cosh(\tau/2) \le C'' e^{\tau/2},$$

where the set on the left is to be understood *without* multiplicities. Hence, using the Cauchy-Schwarz Inequality leads to

$$\left(\sum_{\substack{\ell \\ \tau - c \le \ell \le \tau + c}} M_\Gamma(\ell) \right)^2 \le C'' e^{\tau/2} \sum_{\substack{\ell \in \mathcal{L}(X) \\ \tau - c \le \ell \le \tau + c}} M_\Gamma(\ell)^2, \tag{5.85}$$

On the other hand, by the prime geodesic theorem (see Borthwick [14, Theorem 14.20] and the references therein), we have

$$\sum_{\substack{\ell \\ \tau - c \le \ell \le \tau + c}} M_\Gamma(\ell) = \pi_\Gamma(\tau + c) - \pi_\Gamma(\tau - c) \gg \frac{e^{\delta\tau}}{\tau}, \tag{5.86}$$

with implied constant only depending on c and Γ. Combining (5.85) and (5.86) finishes the proof. $\square$

The next result, which is needed only in the proof of the second part of Theorem 5.4, asserts that Hecke triangle groups Γ_w with cusp width w have the bounded cluster property if w^2 is an integer.

Lemma 5.37. *Let Γ_w be the Hecke group with parameter $w = \sqrt{n}$ for some $n \in \mathbb{N}$. Then every element of Γ_w is either of the form*

$$\begin{bmatrix} a & b\sqrt{n} \\ c\sqrt{n} & d \end{bmatrix}, \quad a, b, c, d \in \mathbb{Z}$$

or

$$\begin{bmatrix} a\sqrt{n} & b \\ c & d\sqrt{n} \end{bmatrix}, \quad a, b, c, d \in \mathbb{Z}.$$

Moreover, Γ_w has the bounded cluster property.

Proof. Set $\Gamma := \Gamma_w$ and $T := T_w$. Every element $\gamma \in \Gamma$ is of the form

$$\gamma = T^{\alpha_k} S T^{\alpha_{k-1}} \cdots S T^{\alpha_0} \tag{5.87}$$

for some $k \in \mathbb{N}$ and $\alpha_0, \alpha_1, \ldots, \alpha_k \in \mathbb{Z}$ (note that $\alpha_j = 0$ is allowed and we do not require this representation to be unique). For fixed $k \in \mathbb{N}$ let $\Gamma^{(k)}$ denote the set of all the elements in Γ for which (5.87) is satisfied for some $\alpha_0, \alpha_1, \ldots, \alpha_k \in \mathbb{Z}$. Let $\mathcal{M} \subset \mathrm{PSL}_2(\mathbb{R})$ be the following subset:

$$\mathcal{M} := \left\{ \begin{bmatrix} a & b\sqrt{n} \\ c\sqrt{n} & d \end{bmatrix} : a, b, c, d \in \mathbb{Z} \right\} \cup \left\{ \begin{bmatrix} a\sqrt{n} & b \\ c & d\sqrt{n} \end{bmatrix} : a, b, c, d \in \mathbb{Z} \right\}.$$

We will show that $\Gamma^{(k)} \subseteq \mathcal{M}$ for every $k \in \mathbb{N}$. For $k = 1$ the statement is clear, since we have the expression

$$T^{\alpha_0} S T^{\alpha_1} = \begin{bmatrix} \alpha_0\sqrt{n} & -1 + \alpha_0\alpha_1 n \\ 1 & \alpha_1\sqrt{n} \end{bmatrix}$$

An elementary calculation shows the inclusions

$$T \cdot \mathcal{M} \subseteq \mathcal{M} \quad \text{and} \quad S \cdot \mathcal{M} \subseteq \mathcal{M},$$

and hence the statements for all $k \in \mathbb{N}$ follow by induction. Thus $\Gamma \subseteq \mathcal{M}$.

Consequently, the trace spectrum of Γ is contained in the set $\mathbb{Z} \cup \sqrt{n}\mathbb{Z}$, which implies that Γ satisfies the bounded cluster property. The proof is complete. □

5.4.3 Proof of Theorem 5.33

This subsection is entirely devoted to the proof of Theorem 5.33. Throughout let Γ be a non-elementary, finitely generated, torsion-free Fuchsian group with $\delta = \delta(\Gamma)$. All the implied constants during the proof may depend on Γ.

Fix $\sigma > G(\Gamma)$ and then $\kappa > \kappa_\Gamma(\sigma)$. We will show that any such choice of σ and κ must satisfy the inequality

$$\sigma \geq \frac{\delta}{2} - \kappa\delta. \tag{5.88}$$

Taking the limit $\kappa \searrow \kappa_\Gamma(\sigma)$ in (5.88) leads to

$$\sigma \geq \frac{\delta}{2} - \kappa_\Gamma(\sigma)\delta.$$

Notice that by the definition of κ_Γ we have $\limsup_{\sigma\searrow\sigma_0}\kappa(\sigma)=\kappa(\sigma_0)$ for all $\sigma_0\in\mathbb{R}$. Hence, taking the limit $\sigma\searrow G(\Gamma)$ in the previous inequality yields the desired conclusion

$$G(\Gamma)\geq\frac{\delta}{2}-\kappa_\Gamma(G(\Gamma))\delta.$$

If we assume furthermore that Γ has the bounded cluster property, we will show that for any choice of σ and κ as above, we must have

$$\sigma\geq\delta-\frac{\kappa}{2}-\frac{1}{4}. \tag{5.89}$$

After sending first $\kappa\searrow\kappa_\Gamma(\sigma)$ and then $\sigma\searrow G(\Gamma)$, we obtain

$$G(\Gamma)\geq\delta-\frac{\kappa_\Gamma(G(\Gamma))}{2}-\frac{1}{4}. \tag{5.90}$$

Thus, our goal is to prove (5.88) (respectively (5.89) in the case that Γ has the bounded cluster property) for any fixed pair σ and κ satisfying $\sigma>G(\Gamma)$ and $\kappa>\kappa(\sigma)$.

By the definition of $G(\Gamma)$, we know that $Z_\Gamma(s)$ has only a finite number of zeros in the half-plane $\{\mathrm{Re}(s)\geq\sigma\}$. In particular, σ and κ satisfy the conditions of Proposition 5.34, which we want to a apply to a suitable family $\varphi_{\xi,\tau}$ of test functions.

Fix a positive function $\varphi\in C_c^\infty(\mathbb{R})$, with support in the interval $[-1,1]$ and such that $\varphi=1$ on $\left[-\frac{1}{2},\frac{1}{2}\right]$. For real parameters ξ and τ, set

$$\varphi_{\xi,\tau}(x):=e^{-i\xi t}\varphi(x-\tau)$$

Define the quantity

$$S(\xi,\tau):=\sum_{[\gamma]\in[\Gamma]_p}\sum_{m=1}^\infty\frac{\ell(\gamma)\varphi_{\xi,\tau}(m\ell(\gamma))}{1-e^{-m\ell(\gamma)}}=\sum_{[\gamma]\in[\Gamma]_p}\sum_{m=1}^\infty\frac{\ell(\gamma)e^{-i\xi m\ell(\gamma)}}{1-e^{-m\ell(\gamma)}}\varphi(m\ell(\gamma)-\tau).$$

Note that for all $\tau>2$, we have $\mathrm{supp}(\varphi_{\xi,\tau})\subset(0,\infty)$. Hence, applying Proposition 5.34 to $\varphi_{\xi,\tau}$, we obtain

$$S(\xi,\tau)=\sum_{\substack{\zeta\in\mathcal{Z}_\Gamma\\ \mathrm{Re}(\zeta)\geq\sigma}}\widehat{\varphi_{\xi,\tau}}(i\zeta)-S_\Gamma(\sigma)+E(\xi,\tau), \tag{5.91}$$

where the error term $E(\xi,\tau)$ satisfies for all $\varepsilon>0$ small enough

$$E(\xi,\tau)=O_{\varepsilon,\sigma,\kappa}\left(\int_{-\infty}^\infty\langle x\rangle^\kappa|\widehat{\varphi_{\xi,\tau}}(x+i(\sigma+\varepsilon))|dx\right)$$

To estimate the integral in the error term, note that

$$\widehat{\varphi_{\xi,\tau}}(x)=e^{-i(x+\xi)}\widehat{\varphi}(x+\xi),$$

which follows from basic properties of the Fourier transform. Moreover, since φ is compactly supported, its Fourier transform decays rapidly, which implies that

$$|\widehat{\varphi_{\xi,\tau}}(x+i(\sigma+\varepsilon))|\ll_m e^{(\sigma+\varepsilon)\tau}\left(1+|x+\xi+i(\sigma+\varepsilon)|\right)^{-m}$$

for any positive integer m. Taking $m = 3$, say, we obtain the estimate

$$\int_{-\infty}^{\infty} \langle x\rangle^{\kappa} |\widehat{\varphi_{\xi,\tau}}(x + i(\sigma+\varepsilon))| dx = O_{\varepsilon}(e^{(\sigma+\varepsilon)\tau}\langle\xi\rangle^{\kappa}). \tag{5.92}$$

Hence, we obtain the following estimate on the error term:

$$E(\xi,\tau) = O_{\varepsilon,\sigma,\kappa}(e^{(\sigma+\varepsilon)\tau}\langle\xi\rangle^{\kappa}) \tag{5.93}$$

The *Gaussian average* of $S(\xi,\tau)$ with parameter $\lambda > 0$ is defined as

$$G(\lambda,\tau) := \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} |S(\xi,\tau)|^2 e^{-\lambda\xi^2} d\xi. \tag{5.94}$$

The strategy of the proof is to give a lower and an upper bound bound for $G(\lambda,\tau)$ (the upper bound will be optimized with respect to the parameter λ), and then to compare lower and upper bounds as $\tau \to \infty$ and $\varepsilon \searrow 0$.

Let us start with the upper bound. By (5.91) we can estimate the Gaussian average from above as follows:

$$G(\lambda,\tau) \le 2\sqrt{\frac{\lambda}{\pi}} \left(\int_{-\infty}^{\infty} \left| \sum_{\substack{\zeta\in\mathcal{Z}_\Gamma \\ \mathrm{Re}(\zeta)\ge\sigma}} \widehat{\varphi_{\xi,\tau}}(i\zeta) - S_\Gamma(\sigma) \right|^2 e^{-\lambda\xi^2} d\xi + \int_{-\infty}^{\infty} |E(\xi,\tau)|^2 e^{-\lambda\xi^2} d\xi \right). \tag{5.95}$$

Recall that for any integer $m \ge 2$ we have

$$|\widehat{\varphi_{\xi,\tau}}(i\zeta)| \ll_m e^{\mathrm{Re}(\zeta)\tau}\langle\xi\rangle^{-m}.$$

Recall that $S_\Gamma(\sigma)$ is the contribution coming from the poles of the Selberg zeta function $Z_\Gamma(s)$ in the half-plane $\mathrm{Re}(s) \ge \sigma$. Since all the zeros and poles of the Selberg zeta function $Z_\Gamma(s)$ lie in the half-plane $\mathrm{Re}(s) \le \delta$, and since there are only finitely many zeros and poles in the half-plane $\mathrm{Re}(s) \ge \sigma$, we obtain (by taking $m = 1$)

$$\int_{-\infty}^{\infty} \left| \sum_{\substack{\zeta\in\mathcal{Z}_\Gamma \\ \mathrm{Re}(\zeta)\ge\sigma}} \widehat{\varphi_{\xi,\tau}}(i\zeta) - S_\Gamma(\sigma) \right|^2 e^{-\lambda\xi^2} d\xi \ll_\sigma e^{2\delta\tau} \int_{-\infty}^{\infty} \langle\xi\rangle^{-2} e^{-\lambda\xi^2} d\xi \ll e^{2\delta\tau}. \tag{5.96}$$

Using the bound for the error term $E(\xi,\tau)$ in (5.93), we have

$$\int_{-\infty}^{\infty} |E(\xi,\tau)|^2 e^{-\lambda\xi^2} d\xi \ll_{\varepsilon,\sigma,\kappa} e^{2(\sigma+\varepsilon)\tau} \int_{-\infty}^{\infty} \langle\xi\rangle^{2\kappa} e^{-\lambda\xi^2} d\xi \ll_{\varepsilon,\sigma,\kappa} e^{2(\sigma+\varepsilon)\tau}\lambda^{-\kappa-1/2}. \tag{5.97}$$

Going back to (5.95) and summing (5.96) and (5.97), we obtain the upper bound

$$G(\lambda,\tau) \ll_{\varepsilon,\sigma,\kappa} e^{2\delta\tau}\lambda^{1/2} + e^{2(\sigma+\varepsilon)\tau}\lambda^{-\kappa} \tag{5.98}$$

We now proceed to give a lower bound for the Gaussian average $G(\lambda,\tau)$. Using its definition in (5.94) we compute

$$G(\lambda,\tau) = \tag{5.99}$$

$$\sum_{[\gamma]_1,[\gamma]_2\in[\Gamma]_p}\sum_{m_1,m_2\geq 1}\frac{\ell(\gamma_1)\ell(\gamma_2)\varphi(m_1\ell(\gamma_1)-\tau)\varphi(m_2\ell(\gamma_2)-\tau)}{(1-e^{-m_1\ell(\gamma_1)})(1-e^{-m_2\ell(\gamma_2)})}I(m_1,\gamma_1;m_2,\gamma_2), \tag{5.100}$$

where

$$I(m_1,\gamma_1;m_2,\gamma_2) = \sqrt{\frac{\lambda}{\pi}}\int_{-\infty}^{\infty} e^{-\lambda\xi^2}e^{-i\xi(m_1\ell(\gamma_1)-m_2\ell(\gamma_2))}d\xi \tag{5.101}$$

Using the formula for the Fourier transform of the Gaussian, we find

$$I(m_1,\gamma_1;m_2,\gamma_2) = \exp\left(-\frac{(m_1\ell(\gamma_1)-m_2\ell(\gamma_2))^2}{4\lambda}\right)$$

Thus, the Gaussian average can be written as the sum

$$G(\lambda,\tau) = \sum_{[\gamma]_1,[\gamma]_2\in[\Gamma]_p}\sum_{m_1,m_2\geq 1}\frac{\ell(\gamma_1)\ell(\gamma_2)\varphi(m_1\ell(\gamma_1)-\tau)\varphi(m_2\ell(\gamma_2)-\tau)}{(1-e^{-m_1\ell(\gamma_1)})(1-e^{-m_2\ell(\gamma_2)})} \times \exp\left(-\frac{(m_1\ell(\gamma_1)-m_2\ell(\gamma_2))^2}{4\lambda}\right).$$

All the terms of the above sum are positive. By restricting this sum to the terms $m_1 = m_2 = 1$ and all $[\gamma_2],[\gamma_1] \in [\Gamma]_p$ with $\ell(\gamma_1) = \ell(\gamma_2)$, we obtain the lower bound

$$G(\lambda,\tau) \geq \sum_{\ell}\frac{\ell^2\varphi(\ell-\tau)^2}{(1-e^{-\ell})^2}M_\Gamma(\ell)^2 \geq \sum_{\ell}\ell^2\varphi(\ell-\tau)^2M_\Gamma(\ell)^2, \tag{5.102}$$

where the summation is over each length appearing in the primitive length spectrum $\mathcal{L}(X)$ of $X = \Gamma\backslash\mathbb{H}$, and $M_\Gamma(\ell)$ denotes the multiplicity with which ℓ appears in $\mathcal{L}(X)$.

By the assumption that $\varphi = 1$ on the interval $\left[-\frac{1}{2},\frac{1}{2}\right]$, we have

$$G(\lambda,\tau) \geq \sum_{\substack{\ell \\ \tau-1/2\leq\ell\leq\tau+1/2}}\ell^2M_\Gamma(\ell)^2 \gg \tau^2\sum_{\substack{\ell \\ \tau-1/2\leq\ell\leq\tau+1/2}}M_\Gamma(\ell)^2. \tag{5.103}$$

Hence, using the prime geodesic theorem the prime geodesic theorem (see Borthwick [14, Theorem 14.20] and the references therein), we obtain the final lower bound for the Gaussian average:

$$G(\lambda,\tau) \gg \tau^2\left(\pi_\Gamma(\tau+1/2)-\pi_\Gamma(\tau-1/2)\right) \gg e^{\delta\tau}. \tag{5.104}$$

We can finally compare the upper bound (5.98) with the lower bound (5.104). This comparison yields

$$e^{\delta\tau} \ll_{\varepsilon,\sigma,\kappa} e^{2\delta\tau}\lambda^{1/2} + e^{2(\sigma+\varepsilon)\tau}\lambda^{-\kappa}. \tag{5.105}$$

Now we set $\lambda = e^{-\frac{2(\delta-\sigma)}{\kappa+1/2}\tau}$, so as to optimize the left hand side of (5.105). This choice gives, for every τ large enough and for every $\varepsilon > 0$ small enough, the bound

$$e^{\delta\tau} \ll_{\varepsilon,\sigma,\kappa} e^{\left(\frac{2\delta\kappa+\sigma}{\kappa+1/2}+\varepsilon\right)\tau} \tag{5.106}$$

Hence, taking the logarithm on both sides of (5.106), dividing by τ, and finally sending $\tau \to \infty$ and $\varepsilon \searrow 0$, yields

$$\delta \leq \frac{2\delta\kappa + \sigma}{\kappa + 1/2} \iff \sigma \geq \frac{\delta}{2} - \kappa\delta,$$

which proves (5.88). Thus, the proof of the first part of Theorem 5.33 is complete.

Let us now prove the second part and assume that Γ has the bounded cluster property. In this case, the proof requires only one modification in the above argument. By invoking Lemma 5.36, we obtain a sharper lower bound on the Gaussian average, when $\delta > \frac{1}{2}$. Indeed, combining (5.103) and Lemma 5.36 leads to

$$G(\lambda,\tau) \gg \tau^2 \sum_{\substack{\ell\in\mathcal{L}(X)\\ \tau-1/2\leq\ell\leq\tau+1/2}} M_\Gamma(\ell)^2 \gg e^{(2\delta-1/2)\tau} \tag{5.107}$$

for $\tau \to \infty$. Therefore, under the assumption that Γ has the bounded cluster property and $\delta > \frac{1}{2}$, Inequality (5.106) gets replaced by

$$e^{(2\delta-1/2)\tau} \ll_{\varepsilon,\sigma,\kappa} e^{\left(\frac{2\delta\kappa+\sigma}{\kappa+1/2}+\varepsilon\right)\tau} \tag{5.108}$$

for all τ large enough and for all $\varepsilon > 0$ sufficiently small. Arguing as above, this leads to

$$2\delta - \frac{1}{2} \leq \frac{2\delta\kappa + \sigma}{\kappa + 1/2} \iff \sigma \geq \delta - \frac{\kappa}{2} - \frac{1}{4},$$

proving (5.89). The proof of Theorem 5.33 is now complete.

5.4.4 Proof of Theorem 5.4

Using the results of this section, we are now in place to prove Theorem 5.4.

Proof of Theorem 5.4. Let Γ_w be the Hecke triangle group with cusp width $w > 2$ and let $\widetilde{\Gamma} \leqslant \Gamma_w$ be a torsion-free, finite-index subgroup. From the fractal growth bound on $Z_{\widetilde{\Gamma}}$ in Corollary 5.2, we find $\kappa_{\widetilde{\Gamma}}(\sigma) \leq \delta$ for all $\sigma \in \mathbb{R}$. Applying the first part of Theorem 5.33 leads to

$$G(\widetilde{\Gamma}) \geq \frac{\delta}{2} - \kappa_{\widetilde{\Gamma}}(G(\widetilde{\Gamma}))\delta \geq \frac{\delta}{2} - \delta^2. \tag{5.109}$$

Let us now assume in addition that $w = \sqrt{n}$ is the square-root of some integer $n \geq 5$. Then, by Lemma 5.37, the group Γ_w has the bounded cluster property. Therefore $\widetilde{\Gamma}$ inherits this property, being a subgroup of Γ_w. By the second part of Theorem 5.33, we obtain

$$G(\widetilde{\Gamma}) \geq \delta - \frac{\kappa_{\widetilde{\Gamma}}(G(\widetilde{\Gamma}))}{2} - \frac{1}{4} \geq \frac{\delta}{2} - \frac{1}{4}. \tag{5.110}$$

This completes the proof. $\square$

We remark that in Theorem 5.4 we assumed $\widetilde{\Gamma}$ to be *torsion-free*, since we were mainly interested in resonances for the corresponding *surface* $\widetilde{X} = \widetilde{\Gamma}\backslash\mathbb{H}$. However, we may drop this assumption on $\widetilde{\Gamma}$. Following the steps that led to Theorem 5.33 and using Corollary 5.2 (which is valid for *all* finite-index subgroups $\widetilde{\Gamma} \leqslant \Gamma_w$), we obtain the same lower bound on the essential spetral gap of $\widetilde{\Gamma}$ as in (5.109) and (5.110). This in turn leads to an explicit strip in the complex plane containing infinitely many zeros for the Selberg zeta function $Z_{\widetilde{\Gamma}}$ of *all* finite-index subgroups $\widetilde{\Gamma} \leqslant \Gamma_w$.

Chapter 6

Conclusion and outlook

Although we gained further insight into the spectral theory of infinite-area hyperbolic surfaces, this subject is by no means closed and many interesting questions remain open. In this last chapter, we briefly outline some of these questions. The goal here is not an exhaustive list of all the problems that might be interesting. Rather, we provide a list of some problems that we feel could be attacked using methods similar to those used and developed in this thesis.

In Chapter 3, we proved global estimates for resonances for covers of Schottky surfaces, in terms of their 0-volume. We believe the same estimate to hold true for *all* geometrically finite hyperbolic surfaces. The main reason we restricted to *Schottky* surfaces was the relative simplicity of the associated transfer operators, which allowed us to perform the decoupling trick needed to establish Proposition 3.4.

Problem 6.1. *Let X be an arbitrary geometrically finite, infinite-area hyperbolic surface and let (X_j) be a family of finite covers of X with* $0\text{-vol}(X_j) \to \infty$ *as* $j \to \infty$*. Can one prove that* $N_{X_j}(r) \asymp 0\text{-vol}(X_j) r^2$ *as* $r \to \infty$*, or even the stronger asymptotics* $N_{X_j}(r) \sim C_X\, 0\text{-vol}(X_j) r^2$ *as* $r, j \to \infty$*?*

In Chapter 4 we considered the notion of spectral gap in the case of abelian covers X_j. By exploiting the 'abelianness' of these covers, we frabicated hyperbolic surfaces with arbitrarily small spectral gap. Abelianness was absolutely crucial for doing so, and there exist examples for families of covers for which precisely the opposite is true. For instance, from the work of Bourgain–Gamburd–Sarnak [17] and Oh–Winter [63] we know that *congruence covers* $X(q)$ lead to *uniform spectral gap*. We say that a family of covers (X_j) has 'uniform spectral gap' if there exists $\varepsilon_0 = \varepsilon_0(X) > 0$ such that $\mathrm{Gap}(X_j) \geq \varepsilon_0$ for all j. We point out that uniform spectral gap for congruence subgroups of the modular group $\mathrm{PSL}_2(\mathbb{Z})$ is given by the famous 'Selberg $\frac{3}{16}$-theorem' [81], which Selberg proved using Weil's bounds on Kloosterman sums.

We believe that the example of congruence covers is merely a 'distraction'. From the deep work of Bourgain–Gamburd [16] we know that certain Cayley graphs of $\mathrm{SL}_2(\mathbb{F}_p)$, p prime, give rise to families of *expanders*. This key fact was used in [17] and [63] to prove uniform spectral gap of congruence covers. In view of [17] and [63], it seems that uniform spectral gap is caused solely by this expanding property of the sequence of Galois groups $\mathbf{G}_j$ of the covering $X_j \to X$. Our problem can thus be formulated in terms of the Cayley graphs $\mathcal{G}_j = \mathrm{Cay}(\mathbf{G}_j, \pi_j(S))$ from Proposition 4.8.

Problem 6.2. *Let X be a geometrically finite hyperbolic surface with $\delta(X) \leq \frac{1}{2}$. Let $(X_j)_j$ be a sequence of finite covers of X. Assume that the associated Cayley graphs $(\mathcal{G}_j)_j$ form a family of expanders. Is it then true that $(X_j)_j$ has uniform spectral gap?*

In the case $\delta(X) > \frac{1}{2}$, Problem 6.2 is solved in [17] by purely spectral methods. The same techniques are no longer accessible when $\delta(X) \leq \frac{1}{2}$. Note that every geometrically finite hyperbolic surface with $\delta(X) \leq \frac{1}{2}$ must be a Schottky surface, for reasons explained in Subsection 2.1.2. Hence Problem 6.2 could (at least in principle) be attacked using the transfer operators $\mathcal{L}_{s,\rho}$ defined in Section 2.6.

Another interesting problem to consider is a possible generalization of the equidistribution result gained in Chapter 4:

Problem 6.3. *Can we generalize Theorem 4.2 to arbitrary finitely generated Fuchsian groups Γ?*

We believe that this question can be answered affirmatively, using more transfer operators for more general Fuchsian groups. Again, using transfer operators for non-Schottky groups will certainly give rise to new non-trivial obstacles.

In Chapter 5 we went on to prove fractal Weyl bounds for Hecke triangle groups. Transfer operator techniques were absolutely crucial. In order to use them effectively, we developed estimates for singular values of rather general vector-valued transfer operators. Inspired by the work of Bandtlow–Jenkinson [7], we proved Theorem 5.10, which we subsequently specialized to the Hecke triangle group setting.

It would be interesting to prove 'sharp' estimates for generic transfer operators acting on subsets of $\mathbb{C}^d$ with $d \geq 1$. So far we have only considered the case $d = 1$, which was sufficient for the goals of this thesis. Even better, we could try to derive some estimates for their *eigenvalues*.

Problem 6.4. *Can we prove estimates for singular values and eigenvalues of general vector-valued transfer operators acting on subsets of $\mathbb{C}^d$?*

Once established, such estimates could be applied to prove fractal Weyl bounds for discrete subgroups of $\mathrm{PSL}_2(\mathbb{C})$ or even more general Kleinian groups acting on the $(d+1)$-dimensional hyperbolic space $\mathbb{H}^{d+1}$ for arbitrary $d \geq 1$. In the case of three dimensions, i.e. $d = 2$, we may consider for instance the analogues of the Hecke triangle groups Γ_w generated by the three elements

$$\begin{bmatrix} 1 & w \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & iw \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Estimates analogous to Theorem 5.10 are also likely to lead to growth estimates on *Ruelle zeta functions*. These are zeta-type functions associated to more general dynamical systems, which were introduced in Ruelle's famous work [76].

However, Problem 6.4 is interesting for its own sake, regardless of the possible applications it may have in the future.

The main theorem in Chapter 5 and its direct corollaries have a logarithmic loss in it. The methods employed in this work do not allow the logarithm to be removed and we do not know if our upper bounds are optimal.

Problem 6.5. *Can we remove the logarithmic loss in Theorem 5.1, or in Corollaries 5.2 and 5.3? Can we improve the exponent δ in Theorem 5.1 for values $\sigma < \delta$ close to δ ?*

It is likely that we can improve Corollary 5.3. We strongly believe that working with iterates of the transfer operator (as is done towards our proof of Theorem 3.2) will yield at least logarithmic improvements, without the need for new key insights.

The last problem we would like to mention is work in progress, which is being pursued jointly with Pohl and Naud.

Problem 6.6. *Can we prove fractal Weyl upper bounds analogous to Theorem 5.1 for arbitrary finitely generated Fuchsian groups?*

The work of Pohl [71] and Fedosova–Pohl [26] allows us to represent the Selberg zeta function (and more generally, L-functions) of very general Fuchsian groups in terms of transfer operators. We believe that the methods that led to Theorem 5.1 are robust enough, so as to generalize it to the general geometrically finite case. In particular, meromorphic continuation of transfer operators and the delicate covering arguments of the limit set seem to be applicable in a more general setting. Conceptually, no big changes seem to be required, but the level of generality we are aiming at will certainly add more technical and notational complexity to the proof.

Appendix A

Background material

A.1 Some elements of functional analysis

A.1.1 Sigular values

We recall a few elements of functional analysis that are used throughout this work. For proofs and more details we refer to [83] or any other standard reference.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be separable Hilbert spaces, and let $A\colon \mathcal{H}_1 \to \mathcal{H}_2$ be a trace class operator. We note that some parts of this subsection apply to operators more general than trace class. However, such generalizations are not needed for our purposes.

The (operator) norm of $A\colon \mathcal{H}_1 \to \mathcal{H}_2$ is defined by

$$\|A\|_{\mathcal{H}_1\to\mathcal{H}_2} := \sup_{v\in\mathcal{H}_1\setminus\{0\}} \|Av\|_{\mathcal{H}_2}.$$

When it is clear from the context which Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ are being used, we simply write $\|A\|$ instead of $\|A\|_{\mathcal{H}_1\to\mathcal{H}_2}$.

Let $A^*\colon \mathcal{H}_2 \to \mathcal{H}_1$ denote the adjoint of A. Then $A^*A : \mathcal{H}_1 \to \mathcal{H}_1$ is positive semi-definite, and hence the *absolute value*

$$|A| := \left(A^*A\right)^{\frac{1}{2}}.$$

of A exists. The *singular values* of A are the non-zero eigenvalues of $|A|$. Let $(\mu_k(A))_{k=1}^{S(A)}$ be the sequence of singular values (with multiplicities) of A, arranged by decreasing order:

$$\mu_1(A) \geq \mu_2(A) \geq \mu_3(A) \geq \cdots$$

If necessary, we turn this sequence into an infinite one by filling it up with zeros at the end. The trace norm of A is

$$\|A\|_1 := \sum_{j=1}^{\infty} \mu_j(A).$$

Singular values enjoy some nice properties, which we use frequently in this work. Recall the following characterization of the k-th singular value as the following infimum over operators $L\colon \mathcal{H}_1 \to \mathcal{H}_2$ of rank less than k:

$$\mu_k(A) = \inf\{\|A - L\| : L\colon \mathcal{H}_1 \to \mathcal{H}_2,\ \mathrm{rank}(L) < k\}. \tag{A.1}$$

In particular, $\mu_1(A) = \|A\|$. A proof of (A.1) can be found in [32].

Lemma A.1. *Let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4$ be separable Hilbert spaces, and let $A\colon \mathcal{H}_3 \to \mathcal{H}_4$, $B\colon \mathcal{H}_2 \to \mathcal{H}_3$, $C\colon \mathcal{H}_1 \to \mathcal{H}_2$ be operators:*

$$\mathcal{H}_1 \xrightarrow{C} \mathcal{H}_2 \xrightarrow{B} \mathcal{H}_3 \xrightarrow{A} \mathcal{H}_4.$$

Furthermore, let $A_1, A_2\colon \mathcal{H}_1 \to \mathcal{H}_2$ be operators. Then, notation being as above, we have

1. $\mu_k(ABC) \le \|A\|\mu_k(B)\|C\|$.
2. *For all $i, j \in \mathbb{N}$ we have $\mu_{i+j-1}(A_1 + A_2) \le \mu_i(A_1) + \mu_j(A_2)$.*

Proof. We will the characterization (A.1) to prove both (1) and (2).

Let us start with (1). Observe that if $K\colon \mathcal{H}_2 \to \mathcal{H}_3$ is an operator of rank $< k$, then

$$AKC\colon \mathcal{H}_1 \to \mathcal{H}_4$$

is also an operator of rank $< k$. Thus, for any $k \in \mathbb{N}$ we have

$$\begin{aligned}
\mu_k(ABC) &= \inf\{\|ABC - L\| : L\colon \mathcal{H}_1 \to \mathcal{H}_4,\ \mathrm{rank}(L) < k\} \\
&\le \inf\{\|ABC - AKC\| : K\colon \mathcal{H}_2 \to \mathcal{H}_3,\ \mathrm{rank}(K) < k\} \\
&\le \|A\| \cdot \inf\{\|B - K\| : K\colon \mathcal{H}_2 \to \mathcal{H}_3,\ \mathrm{rank}(K) < k\} \cdot \|C\| \\
&= \|A\|\mu_k(B)\|C\|.
\end{aligned}$$

Let us now prove (2): Suppose $L_1\colon \mathcal{H}_1 \to \mathcal{H}_2$ is an operator of rank $< i$, and $L_2\colon \mathcal{H}_1 \to \mathcal{H}_2$ is an operator of rank $< j$. Then $L_1 + L_2$ is of rank $< i + j - 1$. Thus

$$\begin{aligned}
\mu_{i+j-1}(A_1 + A_2) &= \inf\{\|A_1 + A_2 - L\| : L\colon \mathcal{H}_1 \to \mathcal{H}_2\ \mathrm{rank}(L) < i + j - 1\} \\
&\le \inf\{\|A_1 + A_2 - (L_1 + L_2)\| : L_1, L_2\colon \mathcal{H}_1 \to \mathcal{H}_2,\ \mathrm{rank}(L_1) < i,\ \mathrm{rank}(L_2) < j\} \\
&\le \inf\{\|A_1 - L_1\| + \|A_2 - L_2\| : L_1, L_2\colon \mathcal{H}_1 \to \mathcal{H}_2,\ \mathrm{rank}(L_1) < i,\ \mathrm{rank}(L_2) < j\} \\
&= \inf\{\|A_1 - L_1\| : L_1\colon \mathcal{H}_1 \to \mathcal{H}_2,\ \mathrm{rank}(L_1) < i\} \\
&\quad + \inf\{\|A_2 - L_2\| : L_2\colon \mathcal{H}_1 \to \mathcal{H}_2,\ \mathrm{rank}(L_2) < j\} \\
&= \mu_i(A_1) + \mu_j(A_2).
\end{aligned}$$

The proof of Lemma A.1 is finished. □

A direct corollary of Part (2) of Lemma A.1 is the following iterated version, which is sometimes more useful. Given $N \ge 1$ operators $A_1, \dots, A_N\colon \mathcal{H}_1 \to \mathcal{H}_2$ and $k \in \mathbb{N}$ we have

$$\mu_k\left(\sum_{j=1}^{N} A_j\right) \le \mu_{N\lfloor \frac{k+N-1}{N} \rfloor - (N-1)}\left(\sum_{j=1}^{N} A_j\right) \le \sum_{j=1}^{N} \mu_{\lfloor \frac{k+N-1}{N} \rfloor}(A_j). \tag{A.2}$$

A.1.2 Fredholm determinants

For the purposes of this subsection, assume that $A : \mathcal{H}_1 \to \mathcal{H}_1$ is a trace class operator from a separable Hilbert space $\mathcal{H}_1$ to itself. Let $(\lambda_j(A))_{j=1}^{E(A)}$ be the sequence of eigenvalues (with multiplicities) of A, arranged by decreasing absolute value:

$$|\lambda_1(A)| \geq |\lambda_2(A)| \geq \cdots ,$$

As in the case of sigular values, we turn this sequence into an infinite one by filling it up with zeros at the end, if $E(A) < \infty$. Then the Fredholm determinant of A is given by

$$\det(1 + A) = \prod_{j=1}^{\infty}(1 + \lambda_j(A)). \tag{A.3}$$

By the Weyl inequality we have for each $N \in \mathbb{N}$,

$$\prod_{j=1}^{N} \left(1 + |\lambda_j(A)|\right) \leq \prod_{j=1}^{N} \left(1 + \mu_j(A)\right).$$

In particular, letting $N \to \infty$, we get

$$\left|\det(1 + A)\right| = \prod_{j=1}^{\infty} \left(1 + \lambda_j(A)\right) \leq \prod_{j=1}^{\infty} \left(1 + \mu_j(A)\right) = \det\left(1 + |A|\right). \tag{A.4}$$

If we assume further $\|A\|_1 < 1$, then we can express the Fredholm determinant in terms of the traces of iterates of A:

$$\det(1 + A) = \exp\left(\sum_{N=1}^{\infty} \frac{(-1)^{N+1}}{n} \operatorname{Tr}\left(A^N\right)\right). \tag{A.5}$$

A.2 Fredholm determinant identity for Schottky groups

The following result, which we announced in Section 2.6, was crucial in Chapters 3 and 4.

Proposition A.2 (Determinant identity for Schottky groups). *Let Γ be a non-elementary Schottky group and $\rho\colon \Gamma \to U(V)$ a finite-dimensional unitary representation of Γ. Let $\mathcal{H} = H^2(\mathcal{D}; V)$, and let*

$$\mathcal{L}_{s,\rho} : \mathcal{H} \to \mathcal{H}$$

be the transfer-operator corresponding to (Γ, ρ), as defined in Section 2.6. Then for all $\mathrm{Re}(s) \gg 0$, *we have the identity*

$$L_\Gamma(s, \rho) = \det(1 - \mathcal{L}_{s,\rho}).$$

In particular, $L_\Gamma(s, \rho)$ extends to an entire function.

We provide a proof here for the benefit of the reader. We point out that Proposition A.2 for the trivial representation appears in [33]. For arbitary unitary representations it can also be found in [41]. A proof of this identity in a more general setting can be found in the work of Fedosova–Pohl [26].

Proof. We will use the transfer operator $\mathcal{L}_{s,\rho}$ defined in Section 2.6. The main step in the proof is to derive a formula for the trace of the iterates of the transfer operator, $\mathcal{L}_{s,\rho}^N$.

Set $d = \dim V$ and let $\mathbf{e}_1, \dots, \mathbf{e}_d$ be an orthonormal basis for V. For each Schottky disk $\mathcal{D}_j$ let $(\varphi_\ell^j)_{\ell\in\mathbb{N}_0}$ be a Hilbert basis for the Bergman space $H^2(\mathcal{D}_j)$. One possible choice would be

$$\varphi_\ell^j(z) = \sqrt{\frac{l+1}{\pi}}\frac{1}{r_j}\left(\frac{z-c_j}{r_j}\right)^\ell, \quad \ell \in \mathbb{N}_0,$$

where r_j is the radius and c_j is the center of $\mathcal{D}_j$. Then the family defined by

$$\Psi_{j,\ell,k}(z) := \begin{cases} \varphi_\ell^j(z)\mathbf{e}_k \text{ if } z \in \mathcal{D}_j \\ 0 \text{ otherwise,} \end{cases}$$

is a Hilbert basis of $\mathcal{H} = H^2(\mathcal{D}; V)$. Using the formula for the N-th power of $\mathcal{L}_{s,\rho}$ (see for instance equation (3.36)) we can write

$$\langle \mathcal{L}_{\rho,s}^N(\Psi_{j,\ell,k}), \Psi_{j,\ell,k}\rangle_{\mathcal{H}} = \sum_{\alpha\in\mathcal{W}_N^j}\int_{\mathcal{D}_j}\left[(\gamma_\alpha^{-1})'(z)\right]^s \varphi_\ell^j(\gamma_\alpha^{-1}z)\overline{\varphi_\ell^j(z)}\langle\rho(\gamma_\alpha)\mathbf{e}_k, \mathbf{e}_k\rangle_V \,\mathrm{dvol}(z).$$

Therefore, summing over all indices, we obtain

$$\mathrm{Tr}(\mathcal{L}_{\rho,s}^N) = \sum_{j=1}^{2m}\sum_{\ell\in\mathbb{N}_0}\sum_{k=1}^{d}\langle \mathcal{L}_{\rho,s}^N(\Psi_{j,\ell,k}), \Psi_{j,\ell,k}\rangle_{\mathcal{H}}$$

$$= \sum_{j=1}^{2m}\sum_{\ell\in\mathbb{N}_0}\sum_{k=1}^{d}\sum_{\alpha\in\mathcal{W}_N^j}\int_{\mathcal{D}_j}\left[(\gamma_\alpha^{-1})'(z)\right]^s \varphi_\ell^j(\gamma_\alpha^{-1}z)\overline{\varphi_\ell^j(z)}\langle\rho(\gamma_\alpha)\mathbf{e}_k, \mathbf{e}_k\rangle_{V_\rho} \,\mathrm{dvol}(z).$$

Since

$$\sum_{\ell,j}\varphi_\ell^j(z)\overline{\varphi_\ell^j(w)}$$

converges uniformly on compact sets of $\mathcal{D}\times\mathcal{D}$ to the Bergman kernel $B_{\mathcal{D}}(z,w)$, we can rearrange sums to write

$$\mathrm{Tr}(\mathcal{L}_{\rho,s}^N) = \sum_j\sum_{\alpha\in\mathcal{W}_N^j}\chi(\gamma_\alpha)\int_{\mathcal{D}_j}\left[(\gamma_\alpha^{-1})'(z)\right]^s B_{\mathcal{D}}(\gamma_\alpha^{-1}z, z)\,\mathrm{dvol},$$

where $\chi = \mathrm{tr}\,\rho$ is the character of ρ.

Now observe that since $\mathcal{D}$ is a disjoint union of disks, the Bergman kernel satisfies $B_{\mathcal{D}}(z,w) = 0$ whenever z and w lie in different disks. On the other hand, if z, w both lie in $\mathcal{D}_j$ we have $B_{\mathcal{D}}(z,w) = B_{\mathcal{D}_j}(z,w)$, where $B_{\mathcal{D}_j}$ is the Bergman kernel of

$\mathcal{D}_j$. Given a word $\alpha \in \mathcal{W}_N^j$ and a point $z \in \mathcal{D}_j$, we have $\gamma_\alpha^{-1} z \in \mathcal{D}_j$ if and only if $\alpha_N = j$. Hence, the above expression reduces to

$$\operatorname{Tr}(\mathcal{L}_{\rho,s}^N) = \sum_j \sum_{\substack{\alpha \in \mathcal{W}_N^j \\ \alpha_N = j}} \chi(\gamma_\alpha) \int_{\mathcal{D}_j} \left[(\gamma_\alpha^{-1})'(z) \right]^s B_{\mathcal{D}_j}(\gamma_\alpha^{-1} z, z)\, \mathrm{dvol}(z). \tag{A.6}$$

We are left with the computation of the integral appearing in (A.6). To this end, recall that there is an explicit formula for the Bergmann kernel of $\mathcal{D}_j = D(c_j, r_j)$:

$$B_{\mathcal{D}_j}(w, z) = \frac{r_j^2}{\pi \left[r_j^2 - (w - c_j)(\overline{z} - c_j) \right]^2}.$$

It is now an exercise involving Stoke's and Cauchy formula (for details we refer to Borthwick [14, Lemma 15.9]) to obtain the Lefschetz identity

$$\int_{\mathcal{D}_j} \left[(\gamma_\alpha^{-1})'(z) \right]^s B_{\mathcal{D}_j}(\gamma_\alpha^{-1} z, z)\, \mathrm{dvol}(z) = \frac{\left[(\gamma_\alpha^{-1})'(x_\alpha) \right]^s}{1 - (\gamma_\alpha^{-1})'(x_\alpha)},$$

where x_α is the unique fixed point of $\gamma_\alpha^{-1} : \mathcal{D}_j \to \mathcal{D}_j$. Moreover,

$$(\gamma_\alpha^{-1})'(x_\alpha) = e^{-\ell(\gamma_\alpha)},$$

where $\ell(\gamma_\alpha)$ is the displacement length of γ_α. We have therefore achieved

$$\operatorname{Tr}(\mathcal{L}_{\rho,s}^N) = \sum_j \sum_{\substack{\alpha \in \mathcal{W}_N^j \\ \alpha_N = j}} \chi(\gamma_\alpha) \frac{e^{-s\ell(\gamma_\alpha)}}{1 - e^{-\ell(\gamma_\alpha)}}. \tag{A.7}$$

Our next goal is to rewrite the right hand side of this equation as a sum over primitive conjugacy classes in Γ. This requires a combinatorial argument.

Let $L_S(\gamma)$ denote the word length of γ with respect to the generating set $S = \{\gamma_1, \ldots, \gamma_{2m}\}$ of Γ. We denote by $\mathrm{WL}(\gamma) = \min\{L_S(g) : g \in [\gamma]\}$ the minimal word length of any element in the conjugacy class of γ. There is a one-to-one correspondence between prime reduced words (up to circular permutations) in

$$\bigcup_{N \geq 1} \bigcup_{j=1}^{2m} \{\alpha \in \mathcal{W}_N^j \text{ such that } \alpha_N = j\},$$

and prime conjugacy classes in Γ (see Borthwick [14, Proposition 15.6]). Consequently, there is a one-to-one correspondence between prime reduced words (again, up to circular permutations) in

$$\bigcup_{j=1}^{2m} \{\alpha \in \mathcal{W}_N^j \text{ such that } \alpha_N = j\},$$

and the primitive conjugacy classes of word length m dividing N. Hence, (A.7) can be rewritten in terms of the minimal word length as

$$\operatorname{Tr}(\mathcal{L}_{\rho,s}^N) = \sum_{m | N} \sum_{\substack{[\gamma] \in [\Gamma]_P \\ \mathrm{WL}(\gamma) = m}} m \chi(\gamma^{N/m}) \frac{e^{-s\ell(\gamma)\frac{N}{m}}}{1 - e^{-\ell(\gamma)\frac{N}{m}}}. \tag{A.8}$$

Recall from Subsection 2.6 that $\mathcal{L}_{s,\rho}$ is a trace class operator. Therefore, for all $z \in \mathbb{C}$ with $|z|$ small enough, we can use (A.5) to obtain the following expansion for the Fredholm determinant:

$$\det(I - z\mathcal{L}_{s,\rho}) = \exp\left(-\sum_{N\geq 1} \frac{z^N}{N} \mathrm{Tr}(\mathcal{L}_{s,\rho}^N)\right)$$

for $|z|$ small enough. Inserting formula (A.8) into the above expansion and using a geometric series expansion, we obtain

$$\det\left(1 - z\mathcal{L}_{s,\rho}\right) = \exp\left(-\sum_{N\geq 1} \frac{z^N}{N} \sum_{k\geq 0} \sum_{m|N} \sum_{\substack{[\gamma]\in[\Gamma]_p \\ \mathrm{WL}(\gamma)=m}} m e^{-\ell(\gamma)(s+k)\frac{N}{m}} \chi\left(\gamma^{N/m}\right)\right).$$

Introducing the new variable $d = N/m$ and rearranging the sums accordingly leads to

$$\begin{aligned}
\det\left(1 - z\mathcal{L}_{s,\rho}\right) &= \exp\left(-\sum_{k\geq 0}\sum_{d\geq 1}\sum_{[\gamma]\in[\Gamma]_p} \frac{z^{\mathrm{WL}(\gamma)d}}{d} e^{-\ell(\gamma)(s+k)j} \chi(\gamma^d)\right) \\
&= \exp\left(\sum_{k\geq 0}\sum_{[\gamma]\in[\Gamma]_p} \log\det\left(1 - \rho(\gamma) z^{\mathrm{WL}(\gamma)} e^{-(s+k)\ell(\gamma)}\right)\right) \\
&= \prod_{[\gamma]\in[\Gamma]_p}\prod_{k\geq 0} \det\left(1 - \rho(\gamma) z^{\mathrm{WL}(\gamma)} e^{-(s+k)\ell(\gamma)}\right).
\end{aligned}$$

For $\mathrm{Re}(s)$ large enough the result converges at $z = 1$ and we obtain the result. □

A.3 Venkov–Zograf factorization formula

In this short subsection we state (without proof) a result of Venkov–Zograf [91, 90], which was frequently used in the main body of this thesis.

It was first proved for cofinite Fuchsian groups, using a version of the Selberg trace formula with unitary representations. A simpler proof using only properties of finite-dimensional representations and which applies to *all* finitely generated Fuchsian groups can be found in the paper of Fedosova–Pohl [26, Theorem 6.1].

Theorem A.3. *Let Γ be a finitely generated Fuchsian group and let $\widetilde{\Gamma}$ be a subgroup of finite index. Let $\rho\colon \Gamma \to U(V)$ be a unitary representation.*

(i) If $\rho = \bigoplus_{j=1}^m \rho_j$ decomposes into a finite direct sum representations ρ_j of Γ then

$$L_\Gamma(s,\rho) = \prod_{j=1}^m L_\Gamma(s,\rho_j).$$

(ii) Let $\widetilde{\rho}$ be a finite-dimensional representation of $\widetilde{\Gamma}$ such that $\rho = \mathrm{Ind}_{\widetilde{\Gamma}}^{\Gamma}(\widetilde{\rho})$ is its induced representation on Γ. Then

$$L_{\widetilde{\Gamma}}(s, \widetilde{\rho}) = L_{\Gamma}(s, \rho).$$

(iii) Assume $\widetilde{\Gamma}$ is a normal subgroup of Γ and let $\mathbf{G} := \Gamma/\widetilde{\Gamma}$ be the quotient group. Then

$$Z_{\widetilde{\Gamma}}(s) = \prod_{\rho \in \widehat{\mathbf{G}}} L_{\Gamma}(s, \rho)^{\dim(\rho)},$$

where $\widehat{\mathbf{G}}$ denotes the unitary dual of the group $\mathbf{G}$.

Recall that the *unitary dual* $\widehat{\mathbf{G}}$ of a finite group $\mathbf{G}$ is the set of irreducible representations of $\mathbf{G}$, up to equivalence of representations.

A.4 A remark on the Hausdorff dimension of Hecke triangle groups

The calculation and estimation of the Hausdorff dimension of limit sets of Fuchsian groups is a hard problem. In this section we analyze the Hausdorff dimension of the limit set of Hecke triangle groups. We will use the notation from Chapter 5.

Let $T_w := T$ and let $\Gamma := \Gamma_w = \langle T_w, S \rangle$ be the Hecke triangle group with cusp width $w > 2$, as we defined it in Chapter 5. We want to consider the Hausdorff dimension $\delta = \delta_w$ of (the limit set of) Γ_w as a function of the parameter w.

Phillips–Sarnak [77] estimated the numerical value of δ_w for certain values of w. For instance, they found $\delta_3 = 0.753 \pm 0.003$. Moreover, they showed that the lowest eigenvalue $\lambda_0(w) = \delta_w(1 - \delta_w)$ is convex, as a function of $w \in (2, \infty)$. It is also known that $w \mapsto \delta_w$ is decreasing, see [77] and the references therein. On the other hand, since Γ_w contains a parabolic element, we must have $\delta_w > \frac{1}{2}$ for every w. The next result shows how δ_w and $\lambda_0(w)$ behave as $w \to \infty$.

Proposition A.4. *For each $w > 2$, let δ_w be the Hausdorff dimension of the limit set of Γ_w and let $\lambda_0(w)$ be the lowest eigenvalue of the Laplacian on $X_w = \Gamma_w \backslash \mathbb{H}$. Then we have $\delta_w \leq 1/2 + 4/w$ and $\lambda_0(w) \geq 1/4 - 16/w^2$.*

We need two Lemmas to prove Proposition A.4. Recall that $\gamma_n(x) = \frac{-1}{x+nw}$ for $n \in \mathbb{Z}$. Let us introduce some notation. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{Z}^k$ set $\gamma_\alpha := \gamma_{\alpha_1} \circ \cdots \circ \gamma_{\alpha_k}$ and $I_\alpha := \gamma_\alpha([-1, 1])$. Note that $I_\alpha \subset \mathbb{R}$ is an interval.

Let us further define

$$\mathbb{Z}_*^k := \left\{(\alpha_1, \ldots, \alpha_k) \in \mathbb{Z}^k : \alpha_j \neq 0 \text{ for all } j \in \{1, \ldots, k\}\right\}.$$

Observe that for all $k \in \mathbb{N}$, for all $\alpha \in \mathbb{Z}_*^k$, and for every interval $A \subset [-1, 1]$, we have $\gamma_\alpha(A) \subset [-1, 1]$.

Lemma A.5. *For every interval $A \subset [-1,1]$ and multi-index $\alpha \in \mathbb{Z}_*^k$, we have*

$$|\gamma_\alpha(A)| \le \frac{|A|}{(w-1)^{2k}(\alpha_1 \cdots \alpha_k)^2},$$

where $|\cdot|$ denotes the Lebesgue measure. In particular, for every $\alpha \in \mathbb{Z}_*^k$ we have the estimate

$$|I_\alpha| \le \frac{2}{(w-1)^{2k}(\alpha_1 \cdots \alpha_k)^2}.$$

Proof. For all $\alpha \in \mathbb{N}$ and $x \in [-1,1]$ we have

$$|\gamma'_\alpha(x)| = \frac{1}{(x+\alpha w)^2} \le \frac{1}{(w-1)^2\alpha^2}.$$

Therefore, writing $A = [a,b]$ yields

$$\frac{|\gamma_\alpha(A)|}{|A|} = \left|\frac{\gamma_\alpha(a)-\gamma_\alpha(b)}{a-b}\right| \le \frac{1}{(w-1)^2\alpha^2} \tag{A.9}$$

and the statement for $k=1$ follows. To obtain the the statement for general k one argues by induction. Assume that the statement of the lemma is true for $k \in \mathbb{N}$. Now pick $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{k+1}) \in \mathbb{Z}_*^{k+1}$ and write $\tilde{\alpha} := (\alpha_2, \dots, \alpha_{k+1}) \in \mathbb{Z}_*^k$. Using (A.9) and the induction hypothesis gives

$$|\gamma_\alpha(A)| = |\gamma_{\alpha_1}(\gamma_{\tilde{\alpha}}(A))| \le \frac{|\gamma_{\tilde{\alpha}}(A)|}{(w-1)^2\alpha_1^2} \le \frac{|A|}{(w-1)^{2(k+1)}\alpha_1^2\alpha_2^2\cdots\alpha_{k+1}^2}.$$

The proof is complete. □

Lemma A.6. *Assume that $r > \frac{1}{2}$ satisfies $\frac{2\zeta(2r)}{(w-1)^{2r}} < 1$, where ζ is the Riemann zeta function. Then $\delta_w \le r$.*

Proof. Let r be as in the statement.

Recall that Λ is the limit set of $\Gamma = \langle T, S \rangle$ viewed as subset of $\overline{\mathbb{R}}$ and $\Lambda_0 = \Lambda \cap (-1,1)$. Using T-invariance of Λ, we can write Λ as a countable disjoint union of translates of Λ_0, that is,

$$\Lambda = \bigcup_{n\in\mathbb{Z}} T^n \Lambda_0. \tag{A.10}$$

From (A.10) it follows that both Λ_0 and Λ have the same Hausdorff dimension. Using S-invariance of Λ, we find furthermore

$$\Lambda = S.\Lambda = \bigcup_{n\in\mathbb{Z}} ST^n.\Lambda_0.$$

Notice that $ST^n.x = \gamma_n(x)$. Hence, we have the inclusion

$$\Lambda = \bigcup_{n\in\mathbb{Z}} \gamma_n(\Lambda_0) \subset \bigcup_{n\in\mathbb{Z}} I_n,$$

where the intervals I_n are defined as above.

Observe that γ_n maps the interval $[-1,1]$ into itself if and only if $n \neq 0$. We deduce that

$$\Lambda_0 \subset \{0\} \cup \bigcup_{n \in \mathbb{Z} \setminus \{0\}} I_n. \tag{A.11}$$

Since $\gamma_n(\Lambda) \subset \Lambda_0$ for all $n \neq 0$, the k-fold iterate of (A.11) gives

$$\Lambda_0 \subset \{0\} \cup \bigcup_{\alpha \in \mathbb{Z}_*^k} I_\alpha. \tag{A.12}$$

Now let us briefly recall the definition of the Hausdorff dimension. Let $\varepsilon > 0$. Given a set $E \subset \mathbb{R}$, let

$$\mathcal{H}_\varepsilon^r(E) = \inf \left\{ \sum_{i \in I} \operatorname{diam}(E)^r : E \subset \bigcup_{i \in I} E_i, \quad \operatorname{diam}(E_i) < \varepsilon \right\}.$$

Then the r-dimensional Hausdorff measure is defined as

$$\mathcal{H}^r(E) = \lim_{\varepsilon \searrow 0} \mathcal{H}_\varepsilon^r(E),$$

and the Hausdorff dimension of E is given by

$$\dim_{\mathrm{H}}(E) = \inf\{r : \mathcal{H}^r(E) = 0\}.$$

By Lemma A.5 there exists $k_0(\varepsilon)$ such that for all $k > k_0(\varepsilon)$ and for all $\alpha \in \mathbb{Z}_*^k$ we have $|I_\alpha| < \varepsilon$. Therefore we can use (A.12) with some $k > k_0(\varepsilon)$ and the definition of $\mathcal{H}_\varepsilon^r$, to obtain

$$\mathcal{H}_\varepsilon^r(\Lambda_0) \leq \sum_{\alpha \in \mathbb{Z}_*^k} |I_\alpha|^r.$$

Using Lemma A.5 we can further estimate

$$\begin{aligned}
\mathcal{H}_\varepsilon^r(\Lambda_0) &\leq \sum_{\alpha \in \mathbb{Z}_*^k} \left(\frac{2}{(w-1)^{2k}(\alpha_1 \cdots \alpha_k)^2} \right)^r \\
&= \left(\frac{2}{(w-1)^{2k}} \right)^r \left(\sum_{\alpha \in \mathbb{Z} \setminus \{0\}} \left(\frac{1}{\alpha^2} \right)^r \right)^k \\
&= 2^r \left(\frac{2\zeta(2r)}{(w-1)^{2r}} \right)^k.
\end{aligned}$$

By assumption, we have $\frac{2\zeta(2r)}{(w-1)^{2r}} < 1$. Therefore we can send $k \to \infty$ to obtain $\mathcal{H}_\varepsilon^r(\Lambda_0) = 0$. Consequently, $\mathcal{H}^r(\Lambda_0) = 0$ and therefore, by the definition of the Hausdorff dimension, we have $\delta_w = \dim_H(\Lambda_0) \leq r$. This completes the proof of Lemma A.6. □

We can now prove Proposition A.4.

Proof of Propositon A.4. Observe that for $w \leq 8$ we have $\delta_w < 1 \leq 1/2 + 4/w$. Hence the estimates for δ_w and $\lambda_0(w)$ in Proposition A.4 are trivially satisfied in the range $w \leq 8$. Let $w > 8$ and set $r := 1/2 + 4/w$. Notice that for all $0 < \xi < 1$ we have the elementary estimate

$$\zeta(1+\xi) = \sum_{n=1}^{\infty} \frac{1}{n^{1+\xi}} \leq 1 + \int_1^{\infty} \frac{dx}{x^{1+\xi}} = \frac{1+\xi}{\xi} < \frac{2}{\xi}.$$

Conseqently, since $w > 8$, we obtain

$$\frac{2\zeta(2r)}{(w-1)^{2r}} = \frac{2\zeta(1+8/w)}{(w-1)^{2r}} < \frac{w}{2(w-1)} < 1. \tag{A.13}$$

By Lemma A.6, the latter implies that $\delta_w \leq r = 1/2 + 4/w$ and consequently $\lambda_0(w) = \delta_w(1-\delta_w) \geq 1/4 - 16/w^2$, thus completing the proof of Proposition A.4. $\square$

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