Technische Universität Ilmenau
Institut für Mathematik

Preprint No. M 19/07

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Juni 2019

URN: urn:nbn:de:gbv:ilm1-2019200307

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About Mathematical Models in Mechanics

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June 12, 2019

Abstract

This paper must not be seen as a report on actual investigations nor as a stroll along stringent analytical mechanics. It is just a revision of a lecture recently given by the author during a seminary of students, pre- and postdocs in the mechanics departments at Technical University Ilmenau. Focussing on current research fields it is to enlighten the general connections of any real object and the hierarchy of models which form the base for theoretical investigations. Therefore the paper should primarily be understood as a pragmatic support of actual scientific meditations and activities by theoretically engaged young people at TU Ilmenau.

The readers are supposed to have knowledge in elementary analysis and applied mechanics.

MSC[2010]: 70G,74K

1 Introduction

In order to set a frame for the following discussions we start with the statement that, actually and locally, primary research interests lie in the fields *construction elements, mechatronics*, and *bionics*. Then the real objects to

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be investigated are (parts of) mechanical or biological devices. Aim of these (both practical and theoretical) investigations is to learn how these parts behave during rest or motion under the influence of neighbored bodies. Afterwards, the gained knowledge might be used for an improvement of the device the object was taken from, or - quite obvious in bionics - one recognizes the utilizability of certain inner structures and processes for to create a mechanical device that mimics the biological paradigm.

As a rule, every research needs interdisciplinary work. It is unquestionable that scientists from different branches come along with different concrete interests for natural objects, their structure, interconnections and interactions, and they show preferences for different investigation methods. Although there is one common research object, several scientific branches are present: material sciences, biology, measuring techniques, mechanics, mathematics. (Think of a mechanist interested in linking bodies together - why shouldn't he cooperate with somebody from zoology?)

Let us take the actual research group '*Tactile Sensors'* as a striking example. There is one common general research object - *bio-inspired sensors* - and all scientific branches mentioned above are present.

Certainly, it is the business of biologists and zoologists to investigate natural objects (e.g., animal sensory organs like *vibrissae* - these "hairs" in an animal's face region, which are not for beauty alone!) in a maximum of details: geometry, material, behavior under external influences, structure and mode of operation of the sensing nervous elements in the base domain. This task blows up due to the multiple forms of vibrissae, distinguished by their place on the surface of a living organism (mystacial : snout region; carpal : foot region; as examples) in connection with various purposes (object perception quite near or afar). General and special questions arise: which structure (created through evolution) makes the vibrissa optimally fit for its task? - and which internal processes are the very reason for this fitness? It may be true that the answers to these questions are primarily sought by means of both *in vivo* and *in vitro* observations and measuring. Their results then lead to hypotheses and, further on, to theories which are to describe structure and functioning of a group of natural objects.

On the other hand, co-operating engineering scientists are primarily interested in the *principles* of structure and functioning with the final aim to design an artificial object that (under choice aspects) comes close to the live paragon. For this end they take up the results gained by the life scientists as the basis of their own work that specifically uses techniques from technology, physics, and mathematics.

And this is just the **Principle of Bionics**:

See the nature and adopt evolutionary achievements to technology.

The overwhelming complexity of natural objects excludes, from the very beginning, the investigation of such objects as a whole, i.e., as they show up with all their details. Any investigation has to focus on a *model* of the object, and this means, take the (possibly incomplete) image of the object presented by the observing scientists (biologists, zoologists), dissect this image and take away all pieces of (actual or guessed) non-interest. The rest then forms a *virtual object*, which all considerations to come have to be concentrated on.

Next, this virtual object must be described by means of physical terms, this description represents a *physical model* of the natural object. Finally, applying corresponding physical theories, and turning physical terms into adequate mathematical ones, a *mathematical model* of the natural object has appeared. As a rule, this represents itself as a system of (fixed or adjustable) constants (called system parameters) and variables (maybe, time-dependent), combined by equations of any kind. Possibly, based on such a model and on results of its mathematical analysis, engineers could design a *hardware model* to be used for demonstration or measuring or even application.

All these steps should strictly follow this

General Guide in Modeling:

Make the model as simple as possible (to enable a thorough analysis) *and as comprehensive and complicated as necessary* (to capture all important items).

Obviously, the extent of performing these claims depends on both objective necessities and subjective abilities to master the coming steps. (To quote Albert Einstein: *"Everything should be made as simple as possible, but no simpler"*.)

It is not a must to surrender if the results of the model analysis are not satisfactory though every mathematics in system structure and evaluation is correct. One should just throw a critical glance at the working hypotheses which coin the steps in modeling, try some improvements and a new analysis. Worst case, of course, is to end up with definitely wrong results. The *positive* result then is: *never this* model again!

2 Modeling Slender Bodies

In the following we shall focus our considerations to objects which have to do with tactile sensors, compliant connecting pieces, straight or curved parts of frameworks or tensegrity structures, and similar ones. All these things are members of the general object class *'slender bodies'*. In daily view the set of such objects is huge indeed: walking-stick, fishing-rod, hose, power-cable, rope, human vein, hair, elephant's trunk, jet of water, vapor-trail, etc., (just not to forget: anybody's sweetheart after a successful diet-period). All these objects are different in origin, dimension, structure, behavior, function, ..., but they share at least one geometric feature: to occupy in space a close neighborhood of a curve; this neighborhood is filled with matter of any kind, which then obviously determines the behavior (being rigid or deformable or fluid).

Later on, our interests will be concentrated on *solid slender bodies*. Slenderness then means "transverse dimensions are small in comparison to length", and "small" remains during deformation. The acceptable rate of slenderness may depend on the context. It will be seen that the above mentioned curves play the leading role in investigations. In order to keep these investigations mathematically on a modest level we confine our considerations to the plane $\mathbb{R}^2$.

2.1 Sketch: Curves in $\mathbb{R}^2$

Fixed Cartesian coordinate system $\{O, x, y\}$ with coordinate frame $\{\mathbf{e}_x, \mathbf{e}_y\}$ (positive sense of orientation $x \to y$), position vector of point $P(x, y) : \overrightarrow{OP} = \mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y$.

A parameterized curve is a map from an interval (a, b) into $\mathbb{R}^2$, k-fold continuously differentiable on the interval,

$$\mathbf{r}(\cdot) \in \mathbf{C}^k(a, b),\ k \geq 2,\ \ s \mapsto \mathbf{r}(s) = x(s)\mathbf{e}_x + y(s)\mathbf{e}_y.$$

Tangent vector at s: $\mathbf{e}_1(s) := \mathbf{r}'(s)$, iff scalar product $(\mathbf{e}_1 \mid \mathbf{e}_1) \equiv 1$, then s is the *arc-length* parameter of the curve. Then orthonormal (in positive sense) to $\mathbf{e}_1(s)$ is the *normal vector* $\mathbf{e}_2(s)$; the orthonormal pair $\{\mathbf{e}_1, \mathbf{e}_2\}$ is called *moving frame* (begleitendes Zweibein). It satisfies the *Frenet equations*

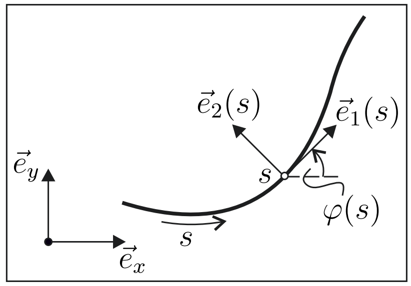

Figure 1: Curve in plane with moving frame.

(Ableitungsgleichungen, $'$ means $\frac{d}{ds}$ here and further on)

$$\boxed{\begin{array}{rl} \mathbf{e}_1' = & \kappa\mathbf{e}_2 \\ \mathbf{e}_2' = & -\kappa\mathbf{e}_1 \end{array}} \tag{1}$$

where $\kappa(s) := (\mathbf{e}_1'(s) \mid \mathbf{e}_2(s))$ is called *curvature* at s. Let $\varphi(s)$ be the slope of the curve at s, then

$$\left\{ \begin{array}{rl} \mathbf{e}_1(s) = & \cos\varphi(s)\mathbf{e}_x + \sin\varphi(s)\mathbf{e}_y \ , \\ \mathbf{e}_2(s) = & -\sin\varphi(s)\mathbf{e}_x + \cos\varphi(s)\mathbf{e}_y \ , \end{array} \right. \tag{2}$$

and there follows by differentiation

$$\boxed{\kappa(s) = \varphi'(s)\ .} \tag{3}$$

If $\kappa(s) > 0,\ < 0,\ = 0$ then the curve is *turning left*, *turning right*, or *flat* at s.

Theorem 1 *For any given continuous real-valued function $k(s)$ there exists a curve in $\mathbb{R}^2$ which is unique up to translation and rotation and has $\kappa(s) = k(s)$ as curvature.*

When dealing with curves in $\mathbb{R}^2$ we meet as the central item the *differential initial value problem for a curve:*

$$\left\{ \begin{array}{rcl} x'(s) = \cos\varphi(s) & , & x(s_0) = x_0 \ , \\ y'(s) = \sin\varphi(s) & , & y(s_0) = y_0 \ , \\ \varphi'(s) = \kappa(s) & , & \varphi(s_0) = \varphi_0 \ . \end{array} \right. \tag{4}$$

In the differential equation of a concrete problem, $\kappa(s)$ can be given in the form $k(x(s), y(s), \varphi(s))$. The common uniqueness theorem for initial value problems entails the proof of the above theorem.

2.2 A First Model for General Slender Bodies

For short, a slender body has been visualized above as a curve (normally in $\mathbb{R}^3$) closely surrounded by a cloud of material particles. The cloud exhibits various local material properties like density of mass, diverse elasticities, viscosities and conductivities, flow velocities, density of electrical charge, external forces (e.g., weight), and so on. Now, tempted by the smallness of cross-sections, one may apply certain summation- and averaging-processes on each cross-section s, thereby coming to a density $\mu(s)$ of mass per unit of length, internal forces and moments resulting from mechanical stresses, an external force $\mathbf{q}(s)$ per unit of length, and further quantities, everyone associated to the curve point at s. So, e.g., we have a function $s \mapsto \mu(s)$ with $s \in (a, b)$, and this is nothing else but a scalar field on the curve. Same interpretaton is valid for any quantity of interest, in particular for purely geometric ones like area $A(s)$ and inertia moments $I(s)$ of the cross-section or curvature properties of the curve itself.

In this way we have come to a very comprehensive class of models:

Any slender body is modeled as a $1-$dimensional continuum   *(i.e. curve in space) that is the support*   *of geometrical, internal and external physical fields.*

If the continuum shows bifurcations then it should be dissected into separate branches which then represent the body as a set of (geometrically and physically) coupled slender bodies.

Everything in this model (curve, field) is allowed to depend on time t if motion problems are to be investigated. Naturally, the fields may be coupled and vary in time following respective physical laws. For short, one may speak about this model as *1-dimensional body embedded in space.*

Remark 1 *In doing analysis we use the following notations:*
total derivatives: $\frac{d}{ds}f(s) =: f'(s)$, $\frac{d}{dt}f(t) =: \dot{f}(t)$;
partial derivatives: $\frac{\partial}{\partial s}f(s,t) =: f_{,s}(s,t)$, $\frac{\partial}{\partial t}f(s,t) =: f_{,t}(s,t)$;
scalar product: $(\mathbf{a}|\mathbf{b})$, *vector product:* $\mathbf{a} \times \mathbf{b}$.
We do not use special measuring units as done elsewhere.

3 1-dimensional Solid Bodies in $\mathbb{R}^2$

In the following we focus our interest on slender bodies made from solid material (excluding fluids an gases). For any configuration we suppose the model curve to be plane and smooth with arc-length $s \in [0, l]$ and to be the geometric locus of the centroids of the cross-sections. The cross-sections are supposed to be symmetric w.r.t. the plane background $\mathbb{R}^2$.

Frequently, one particular configuration is used as a *reference configuration* (preferably, that one with zero load - the 'original configuration'). Later on we denote the respective arc-length ξ (used as a body-fixed parameter). Note that in general $s \neq \xi$!

Firstly, we have the geometric fields $s \mapsto \{\kappa(s), \mathbf{e}_1(s), \mathbf{e}_2(s)\}$. Assume that there are mechanical stresses on the cross-sections which reduce to the *cut-resultants*[1]

$$force: \ \mathbf{F}(s) = F_1(s)\mathbf{e}_1(s) + F_2(s)\mathbf{e}_2(s), \quad moment: \ \mathbf{M}(s) = M_z(s)\mathbf{e}_z.$$

External loads shall be described by *force and moment per unit of length* [2]:

$$\mathbf{q}(s) = q_1(s)\mathbf{e}_1(s) + q_2(s)\mathbf{e}_2(s), \qquad \mathbf{m}(s) = m_z(s)\mathbf{e}_z \ .$$

Single external forces and moments could be seen as sharp extrema of $\mathbf{q}$ and $\mathbf{m}$ (possibly represented by Dirac's δ-functional), or could be handled by considering separate open subintervals of the curve which are free of singular loads (and, after re-connecting the intervals, living with discontinuities of the cut-resultants).

Now a coupling of the fields comes into play by the following

Working hypothesis: (stiffness principle, Erstarrungsprinzip [1])
In every state the model satisfies all rigid-body-equilibrium) conditions.*

For both rigid or deformable bodies *) means: a) during rest every dissected part under action of the impressed forces and the cut-reactions is governed by the rigid-body equilibrium conditions; b) the same holds during motion for every dissected part in any *fixed* (*frozen*) state if regarding the inertia forces.

[1] Common German notation: $F_1 = N$ (Längskraft), $F_2 = Q$ (Querkraft); z: vertical to $\mathbb{R}^2$.

[2] Common notation also $q_1 = q_t$ (tangent), $q_2 = q_n$ (normal).

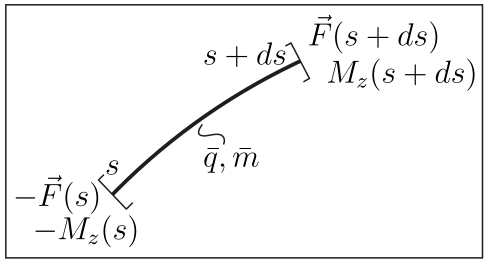

Figure 2: Load and reactions at differential part of a curve in plane.

Figure 2 shows the cut out differential part $(s,\ s+ds)$ with all mechanical actions and (cut-) reactions (where $\overline{\mathbf{q}}$, $\overline{\mathbf{m}}$ mean any intermediate values). Then equilibrium holds if

$$\tfrac{d}{ds}\mathbf{F}(s) + \mathbf{q}(s) = 0, \qquad \tfrac{d}{ds}M_z(s) + Q(s) + m_z(s) = 0,$$

more explicitly, in components, these are called
Equations by Kirchhoff and Clebsch[3]:

$$\boxed{\begin{array}{l} \tfrac{d}{ds}N - \kappa Q + q_t = 0, \\ \tfrac{d}{ds}Q + \kappa N + q_n = 0, \\ \tfrac{d}{ds}M_z + Q + m_z = 0. \end{array}} \tag{5}$$

These differential equations for (N, Q, M_z), together with boundary conditions at $s = 0$ and $s = l$ are the equilibrium conditions for the model of solid bodies with any material properties. Note that κ means the *actual* curvature; if using, in case of deformability, the original curvature, then the equations represent a *first approximation* of the equilibrium conditions. Featuring an observed particular behavior or regarding a particular original inner structure or material property can be realized with the help of supplementing *working hypotheses* (*constitutive laws,* e.g., Hooke's law). This shall be shown by some simple examples next and later in some more extension.

Remark 2 *Intuitively connecting N with stretching and $\{Q,\ M_z\}$ with bending we observe in the Kirchhoff-Clebsch equations a coupling of both (trends of) deformation via the curvature κ (being original or actual). This becomes*

[3]Kirchhoff, Clebsch, about 1860.

more impressive after elimination of Q:

$$\begin{cases} N' + \kappa M_z' + \kappa m_z + q_t = 0, \\ M_z'' - \kappa N - q_n + m_z' = 0. \end{cases}$$

If there is an interest, Q afterwards comes up as $Q = -M' - m_z$.
On the other hand, if the loads q_t and q_n are handable functions, the first two lines in (5) together with boundary conditions yield N and Q, the rest remains $\frac{d}{ds}M_z - Q = 0$. Thereby the Kirchhoff-Clebsch equations are exhausted - no result concerning κ and the curve!

3.1 The Thread

By daily observation of a thread we state about its behavior under external influences: there is no resistance under transverse forces or moments or under pressing longitudinal force, but there is some under pulling logitudinal force ('no thread fits as a walking stick, at most it is good for hangmen's use'). This leads (for the model of threads) to the working hypothesis**:**
A thread is characterized by the **constitutive law**

$$\boxed{M_z = 0, \quad Q = 0, \quad N \geq 0.}$$

The Kirchhoff-Clebsch equations for the thread then are

$$\begin{cases} N' + q_t = 0, \\ \kappa N + q_n = 0, \\ m_z = 0. \end{cases}$$

Example 3.1.1: Let a thread of length l be totally wound around a circular cylinder of radius a (everyday scenario). Then take $\kappa = 1/a$ (mind: only approximative if $l > 2\pi a$ because forming a coil!). The equations yield

$$a q_n' = q_t \ .$$

Case 1: all surfaces (thread and cylinder) are smooth, i.e., $q_t = 0$. This entails $q_n' = 0, \ N' = 0$, hence

$$N(s) = N(0) := N_0 \ \ (> 0 \ !), \quad q_n = -N_0/a \ .$$

Case 2: contacting surfaces with Coulomb stiction (coefficient μ_0), i.e., $q_t = -\mu_0 q_n$. This entails $N' = \mu_0 q_n = -\frac{\mu_0}{a}N$, and we get

$$N(s) = N(0)\exp(-\frac{\mu_0}{a}s).$$

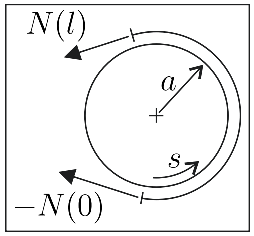

Figure 3: Thread contacting a wheel.

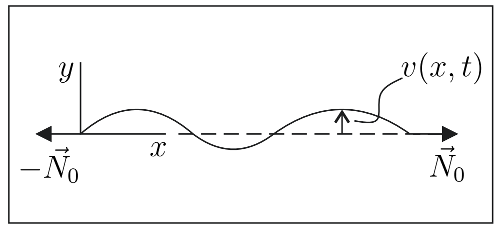

Figure 4: Oscillating string.

Finally, there holds $N(l) = N(0)\exp(-\frac{\mu_0}{a}l) < N(0)$ - a pleasant result for fasten boats on to the quay. (*Criticism*: result still correct if taking a real rope instead of a real thread?).

Example 3.1.2: String under tension, oscillating (*crude simplification!*):

$$M_z = 0,\ Q = 0,\ N = N_0 > 0,\ N_0 \text{ prescribed.}$$

Resting without external loads: straight along the x-axis; under external excitement: *small deformation*, state at time t: $x \approx s,\ y = v(x,t)$, $\kappa \approx v_{,xx}$, inertial load (μ =mass per unit of length) $q_n = -\mu v_{,tt}$. Then we get the motion equation (wave equation)

$$v_{,xx} - \frac{\mu}{N_0} v_{,tt} = 0 \ .$$

Example 3.1.3: The catenary curve (thread, end points fixed, under dead weight):

Let the mass density per unit of length be μ, then we have $\mathbf{q} = -\ \mu g\, \mathbf{e}_y$, and we get

$$q_t = (\mathbf{q} \mid \mathbf{e}_1) = -\mu g \sin\varphi, \quad q_n = (\mathbf{q} \mid \mathbf{e}_2) = -\mu g \cos\varphi,$$

and the equilibrium equations now are

$$\begin{cases} N' - \mu g \sin\varphi = 0, \\ \varphi' N - \mu g \cos\varphi = 0, \end{cases}$$
$$\begin{cases} x' = \cos\varphi, \\ y' = \sin\varphi. \end{cases}$$

The first two equations yield in turn

$$N'/\varphi' = \frac{dN}{d\varphi} = N\tan\varphi, \quad N(\varphi) = N_0/\cos\varphi,$$

$$\varphi' = \frac{\mu g}{N_0}\cos^2\varphi, \quad \varphi(s) = \arctan(\frac{\mu g}{N_0}s).$$

Finally, the last two equations describe the well-known curve shape by

$$y(x) = \frac{N_0}{\mu g}\cosh(\frac{\mu g}{N_0}x).$$

Criticism: Practical usefulness? - In practice one has to do with power-cables of a more complex internal structure than any real thread has; thereby also the interconnection of $N_0,\ l,\ y(0)$, and the coordinates of the suspensions (where l and $N(l)$ take their values during setting-up) need to be considered. External influences like ice or warm-up may then qualitatively change everything.

4 The Bending Rod in $\mathbb{R}^2$

The elastic bending rod plays a central role in the research domains focussed here. This is the reason for giving it a separate section. Our aim is to point out several steps of modeling with increasing precision for different applications - certainly a way not usual in common teaching.

Preliminary note. In all that follows we use the slopes φ (original) and Φ (actual) of the axis as central configuration variable. This is because we want to capture, e.g., sensor hairs as objects with pre-curvature and undergoing large deformations. The final relations, in case of small deformations, can then be transformed into an approximate linear theory (for straight rods or beams the common $x-v-$scheme).

At the present level of our considerations the state of the corresponding model is given by the Kirchhoff-Clebsch equations together with the differential equations of the axis geometry. Two simple examples may give a hint to necessary enhancements of the model. Consider a horizontal cantilever under external load at the tip:
a) a positive moment (couple of forces),
b) a vertical upward force.
Under either load the axis is *expected* (or observed) to become curved upward in some way. The shape of this curve certainly depends on both the load and the rod's 'flexibility' which may depend on the geometry and material structure of the rod, and may vary along the axis (imagine a cantilever originally either cylindrical or conically tapered).

Now we look for the outcome of the Kirchhoff-Clebsch model. In both cases we are given $(q_t,\ q_n,\ m_z) = 0$; after the cut resultants (N, Q, M_z) are determined via equilibrium of the cut-off right part of the (deformed!) axis we check the Kirchhoff-Clebsch equations to be satisfied. There is no information about κ and the *actual* arc-length, i.e., the actual shape of the rod is kept unknown, the model shows up to be unsufficient for a comprehensive description of the body's behavior, (no wonder, since the Kirchhoff-Clebsch equations describe the equilibrium in *any* 1-dimensional body).

In order to find suitable supplementary working hypotheses which then allow to capture also the shape of our model, we must decide about the subclass of objects of proper interest. As a rule, bending rods are made from mostly homogeneous (sometimes layered) material like metal, plastics, rubber or something organic, their cross-sections are mainly circular or rectangular, and they may have axes which are originally pre-curved and could be stretched under load. The following working hypotheses are commonly named *Bernoulli-Euler hypothesis*, they describe a well-defined internal behavior under deformation and, together with our up-to-now model, match our goals.[4]

	Working hypothesis (Bernoulli-Euler)
(B1)	The axis deforms into a smooth $\mathbb{R}^2$-curve.
(B2)	Cross-sections accomplish a rigid motion.
(B3)	Cross-sections remain orthogonal to the axis.
(B4)	Cross-sections do not rotate about the axis-tangent.
(B5)	Homogeneous isotropical Hooke material.

The next figure shall visualize the contents of (B1),..,(B4). Note that we allow for change of arc-length during deformation! Therefore we introduce once for all the following

Agreement: *the arc-length in the* original *(undeformed) state is named* ξ *(serves throughout as body-fixed coordinate); s then means the arc-length in the considered* actual *state.*

[4]Naturally, the theory of bending rods has been established by Johann Bernoulli (about 1690) and Daniel Bernoulli and Leonhard Euler (about 1740) in a totally different way than one is going today [3]. Also our composition of the hypothesis may differ from others. This is because we move in our frame of modeling and use notions (like stress and strain) from modern times.

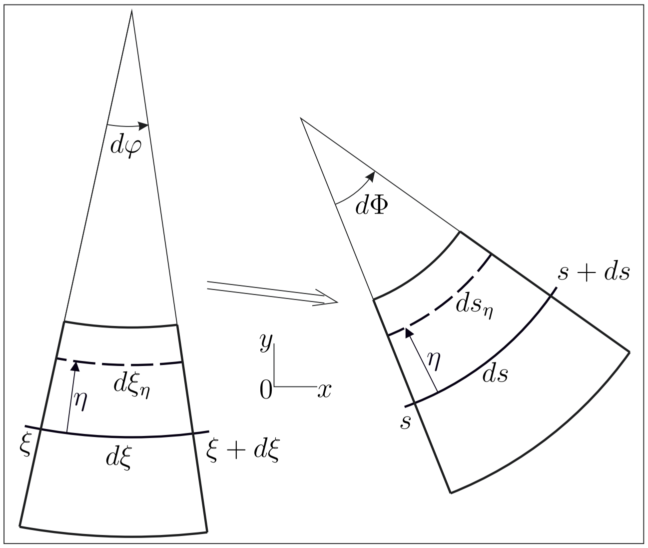

Figure 5: Differential part of bending rod, original and actual.

The *local axis strain* at point ξ then is called $\varepsilon_0(\xi)$. Mind that ξ keeps its character as body-fixed coordinate though it is not the arc-length anymore if $\varepsilon_0 \neq 0$!

If originally the axis is not a straight line, we denote the *pre-curvature* by $\kappa_0(\xi)$. The cross-section at ξ will be equipped with the coordinate system $\{C; \eta, \zeta\}$ where C is the centroid, η in normal direction, ζ normal to $\mathbb{R}^2$.

We put together a list of events during deformation ($\longrightarrow$):

$$\begin{array}{lcl} (x(\xi), y(\xi), \varphi(\xi)) & \longrightarrow & (X(\xi), Y(\xi), \Phi(\xi)) \\ d\xi = \frac{1}{\kappa_0(\xi)} d\varphi & \longrightarrow & ds = (1 + \varepsilon_0(\xi)) d\xi = \frac{1}{\kappa(\xi)} d\Phi \\ \kappa_0(\xi) = \frac{d}{d\xi}\varphi(\xi) & \longrightarrow & \kappa(\xi) = \frac{d}{ds}\Phi = \frac{1}{1+\varepsilon_0(\xi)} \frac{d}{d\xi}\Phi(\xi) \\ d\xi_\eta = (1 - \eta\kappa_0(\xi)) d\xi & \longrightarrow & ds_\eta = (1 - \eta\kappa(\xi)) ds \; . \end{array}$$

From the list we get the strain of the fiber η at ξ, $\varepsilon_\xi(\xi, \eta) := (ds_\eta - d\xi_\eta)/d\xi_\eta$:

$$\boxed{\varepsilon_\xi(\xi, \eta) = \frac{1}{1-\eta\kappa_0(\xi)} \varepsilon_0(\xi) - \frac{\eta}{1-\eta\kappa_0(\xi)} \frac{d}{d\xi}(\Phi(\xi) - \varphi(\xi)) \; .} \tag{6}$$

We emphasize that this is a purely geometric relation resulting from the *rigid* relative motion of the cross-sections at ξ and $\xi + d\xi$. No strains in further directions appear.

Working hypothesis (B5) now adjoins stresses in axial direction,

$$\sigma_\xi(\xi,\eta) = E\varepsilon_\xi(\xi,\eta), \quad E = const,$$

distributed over the cross-section ξ; the distribution is non-linear iff $\kappa_0(\xi) \neq 0$. By means of integrations we get the corresponding resultants in dependence of the kinematical quantities *axis-strain* and *change of curvature* (mind that, due to (B2), neither area $A(\xi)$ nor η have changed),

$$N(\xi) = \int_{A(\xi)} \sigma_\xi(\xi,\eta)da, \quad M_z(\xi) = \int_{A(\xi)} -\eta\sigma_\xi(\xi,\eta)da, \tag{7}$$

and finally, we have the
constitutive law of the Bernoulli-Euler bending rod,

$$\boxed{\begin{array}{rcl} N(\xi) & = & E\mu_0(\xi)\varepsilon_0(\xi) - E\mu_1(\xi)\frac{d}{d\xi}(\Phi(\xi) - \varphi(\xi)), \\ M_z(\xi) & = & -E\mu_1(\xi)\varepsilon_0(\xi) + E\mu_2(\xi)\frac{d}{d\xi}(\Phi(\xi) - \varphi(\xi)), \end{array}} \tag{8}$$

where

$$\mu_i(\xi) := \int_{A(\xi)} \frac{\eta^i}{1-\eta\kappa_0(\xi)}da, \quad i = 0,1,2\ .$$

These three quantities have to be seen as geometric fields $\xi \mapsto \mu_i(\xi)$ on the axis, determined by the shape of the cross-sections and the pre-curvature of the axis, see [2]. An originally *straight* rod carries the fields

$$\mu_0(\xi) = A(\xi), \quad \mu_1(\xi) = 0, \quad \mu_2(\xi) = I_z(\xi)\,,$$

so, μ_0, μ_2 modify the classical stiffnesses $EA(\xi)$ (stretching) and $EI_z(\xi)$ (bending) if the rod is pre-curved, whereas μ_1 causes a (*kinematic!*) coupling of N and M_z in that case.[5]

It is easy to verify the recursions $\mu_1 = \kappa_0\mu_2$, $\mu_0 = A + \kappa_0^2\mu_2$, and $\det(\mu) = A\mu_2 \neq 0$: the constitutive equations can be solved for ε_0 and $\frac{d}{d\xi}(\Phi(\xi) - \varphi(\xi))$.

The constitutive law (8) says the following: the kinematics and the material property postulated by (B1),..,(B4), and (B5) require that the cut

[5]Obviously, the deviation from the classical stiffnesses becomes remarkable if the integrands come close to zero, i.e., if η comes close to the curvature radius. This could happen in fat strongly curved rods like crane hooks, whereas in common bending problems effects were shown to be negligible. But: whether relevant effects could appear in vibration problems - seemingly, nobody knows yet.

resultants ($N,\ M_z$) which are ruling the equilibrium have to match with the resultants given by (8). Equivalently: to ensure the postulated local kinematics (B1),..,(B4) during deformation the constitutive law is a necessary condition. Thereby the axis geometry (up to now as s and κ implicit to the Kirchhoff-Clebsch equations) enters these equations (and so does the original rod geometry, e.g., $A(\xi),\ I_z(\xi)$, too) via the unknowns N and M_z , and feeds the hope to find the actual axis shape through further calculations.

We note the constitutive law for two important particular cases:
Originally straight rod:

$$\boxed{N(\xi) = EA(\xi)\,\varepsilon_0(\xi), \quad M_z(\xi) = EI_z(\xi)\tfrac{d}{d\xi}\Phi(\xi)\ .} \tag{9}$$

Originally straight, and unstretchable rod:

$$\boxed{M_z(\xi) = EI_z(\xi)\tfrac{d}{d\xi}\Phi(\xi)\ .} \tag{10}$$

In all cases with $\varepsilon_0 = 0$, i.e. $\xi = s$, N is simply the reaction force to this constraint. The shear force Q does not appear at all in the constitutive laws, it simply acts as reaction force to the constraints (B2), (B3).

The most beloved - and indeed important - bending rod in teaching comes along as straight ($\kappa_0 = 0\,,\ \varphi(\xi) = \varphi_0 = const$), as, say, cylindrical (constant $A(\xi) = A_0,\ I_z(\xi) = I_0$), and with $\varepsilon_0 = 0$ (pragmatic assumption, looks reasonable for, e.g., stiff beams or levers of very small deformations; but what about very sensible joints or sensors?). The latter assumption brings in some comfort, $s = \xi,\ \kappa = \frac{d}{d\xi}\Phi$, and yields the well-known classical bending equation $\kappa = \frac{1}{EI_0}M_z$, where M_z commonly follows from cut methods.[6] Introducing this κ to the curve equations (4) marks the origin of the curve as *mechanics*.

A standard example is given in the next subsection.

Before, let us state some

Criticisms:

a)The postulated rigidity of the cross-sections entails zero η- and ζ-strains, this implies that the rod's elastic matter is with Poisson number $\nu = 0$ (else one must have also transversal strains $\varepsilon_\eta = \varepsilon_\zeta = -\nu\varepsilon_\xi(\xi,\eta) \neq 0$).

b) The common assumption $\varepsilon_0 = 0$ contradicts (B5): one had to accept $E = \infty$ along the $\eta = 0$ fiber whereas else E is finite (no homogeneity!).

[6]Things become comparatively trivial in case of small deformation: $\mid \Phi - \varphi_0 \mid \ll 1,\ x' = \cos\Phi \approx 1 : x \approx s,\ \kappa \approx y''$, finally $M_z(x) = EI_0\frac{d^2}{dx^2}y(x)$.

4.1 Bending Rod With Tip-Load

Let us consider a Bernoulli rod that is originally straight ($\kappa_0 = 0$) and with fixed shape and area of cross-sections ($A(\xi)$, $I_z(\xi)$ constant for $\xi \in [0, l]$). Let the end $\xi = 0$ be supported (clamp or pivoting) at the fixed point (x_0, y_0), and let a constant force $\mathbf{f} = f\,(\sin\alpha\,\mathbf{e}_x - \cos\alpha\,\mathbf{e}_y)$ act upon the rod's tip $\xi = l$. Suppose $\varepsilon_0 = 0$, thus $\xi = s$ in every configuration. An actual configuration is sketched in figure 6, *just for comfort we prefer here and in the following the symbols x, y, φ, s instead of the capitals used above.*

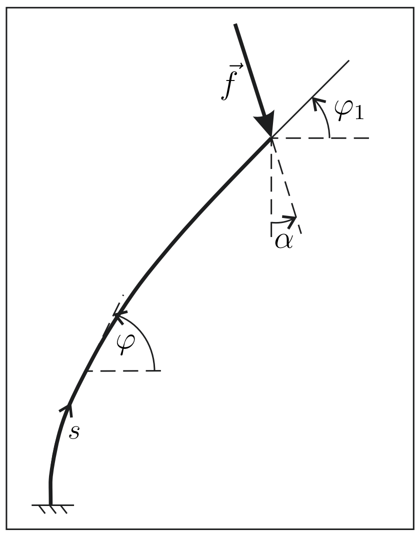

Figure 6: Bending rod, deformed under tip-load.

4.1.1 Problem formulation 1 (common use)

The geometry of the elastic line is governed by the differential equations

$$x' = \cos\varphi, \quad y' = \sin\varphi, \quad \varphi' = \kappa$$

of *any* curve, but now, to capture that special curve *'elastic line'*, physics comes in through the corresponding constitutive law, which we steal from our room-mate: $M_z = EI_z\kappa$, and M_z has to be learned by cut method as

$$M_z(s) = f\,[(x(s) - x_1)\cos\alpha + (y(s) - y_1)\sin\alpha].$$

Finally we end up with a problem like this (we abbreviate $f/E\,I_z =: g$):

$$\left.\begin{array}{l}\text{Find } x(s),\ y(s) \text{ such that, with } s \in (0,l)\\ \ x'(s) = \cos\varphi(s), \quad x(0) = x_0,\ x(l) = x_1,\\ \ y'(s) = \sin\varphi(s), \quad y(0) = y_0,\ y(l) = y_1,\\ \ \varphi'(s) = g\,[(x(s) - x_1)\cos\alpha + (y(s) - y_1)\sin\alpha].\end{array}\right\} \tag{11}$$

Look at the structure of this system: Three differential equations each of first order for the unknown functions x, y, φ with given values x_0, y_0, φ_0 (the latter in case of clamp) at $s = 0$ - so far a harmless initial value problem if - alas! - there were not the unknown tip coordinates x_1, y_1 within. So, hopefully, some shooting procedure on the computer could be tried.

A reasonable trick helps: enlarge the order of the system by introducing the curvature κ as a supplementing unknown, and see the problem separated into two subproblems:

$$(\mu)\ \left\{\begin{array}{lll}\varphi' = \kappa, & \varphi(0) = \varphi_0, & \\ \kappa' = g\,\cos(\varphi - \alpha), & & \kappa(l) = 0;\end{array}\right. \tag{12}$$

$$(\gamma)\ \left\{\begin{array}{ll}x' = \cos\varphi, & x(0) = x_0,\\ y' = \sin\varphi, & y(0) = y_0.\end{array}\right. \tag{13}$$

The first part (μ) represents all the mechanics of the full problem and it is a two-point boundary value problem, whereas the second part (γ) is a simple initial value problem that solves through integration after (μ) has been solved. Finally, also the tip coordinates appear as $x_1 = x(l),\ y_1 = y(l)$.

Remark 3 *It is obvious that the representation of bending problems used in this paper - main state function is the axis slope $\varphi(s)$ - diverges considerably from that one commonly used in teaching and practice. The reason for this fact: many applications need a theory which allows for large deformations (e.g., biologically inspired tactile sensors, see later), whereas small deformations (important, e.g., for sensible connecting pieces) can be captured by the supposition $\mid \varphi(s) - \varphi(0) \mid \ll 1$. Then $x' \approx 1$, i.e. $s \approx x,\ y' \approx \varphi,\ \kappa \approx y''(x) \approx \frac{1}{EI_z} M_z(x)$, and equilibrium considerations can approximately be done in the original state. Therefore everything needed in the approximate $x - y-$theory has a predecessor in the above $s - \varphi-$theory.*

Now we might feel satisfied that the preceding considerations ended up in differential equations and boundary conditions which then yield a solution of the problem *bending rod with tip load* by means of pertinent analytical or computational methods. Indeed, we do so because this is the common familiar successful way. But in fact this is a short-cut of the way our cautiously developed theory has opened and which is always in the background. The following re-consideration of the bending problem is to enlighten that fact.

4.1.2 Problem formulation 2 (commonly not in use)

Remind that the Kirchhoff-Clebsch equations match *any* solid 1-dimensional body (what kind of material and original geometry ever). At the tip let $s = l$ (actual length), slope $\varphi(l) = \varphi_1$. Then we are given the following facts:
- there are no distributed loads,

$$q_t = q_n = 0,\ m_z = 0.$$

- at the tip $s = l$ (actual length) there acts the force $\mathbf{f} = f\,(\sin\alpha\,\mathbf{e}_x - \cos\alpha\,\mathbf{e}_y)$ with components

$$N(l) = -f\ \sin(\varphi_1 - \alpha), \quad Q(l) = -f\ \cos(\varphi_1 - \alpha),$$

and the moment

$$M_z(l) = 0.$$

- the foot is clamped,

$$\varphi(0) = \tfrac{\pi}{2}.$$

The Kirchhoff-Clebsch equations,

$$\begin{aligned} &N' - \kappa Q = 0, \\ &Q' + \kappa N = 0, \\ &M_z' + Q = 0, \\ &\varphi' = \kappa, \end{aligned} \tag{14}$$

appear as differential equations for the unknowns $N(s),\ Q(s),\ M_z(s), \varphi(s),\ s \in (0, l)$, and corresponding boundary conditions are the preceding relations.

Now the first two equations yield $NN' + QQ' = 0$, hence by integration and obeying the boundary values at $s = l$ there follows

$$\frac{1}{2}(N^2 + Q^2) = const = \frac{1}{2}f^2.$$

So N and Q are structured

$$N(s) = f\ \sin(\beta(s)),\ \ Q(s) = f\ \cos(\beta(s))$$

with some still unknown function β. But the first and the fourth equation entail

$$f\,\cos\beta\,\beta' - \varphi' f\,\cos\beta = 0, \quad \text{i.e.,} \quad \beta' = \varphi'.$$

Therefore $\beta(s) = \varphi(s) + \gamma$, $N(s) = f\,\sin(\varphi(s) + \gamma)$, where γ is any constant. At $s = l$ we get $-f\,\sin(\varphi_1 - \alpha) = f\,\sin(\varphi_1 + \gamma)$ which gives $\gamma = -\alpha \pm \pi$, and

$$\begin{aligned} N(s) &= -f\,\sin(\varphi(s) - \alpha), \\ Q(s) &= -f\,\cos(\varphi(s) - \alpha). \end{aligned} \tag{15}$$

(Of course, the same follows by considering the equilibrium of the cut-off part $[s, l]$.)
Finally, the last differential equation writes:

$$M_z'(s) = f\,\cos(\varphi(s) - \alpha). \tag{16}$$

This looks a bit tempting, but $\varphi(s)$, which is the base for finding the elastic line, remains unknown! A fact that was already mentioned before the Bernoulli hypotheses: all the preceding results hold for *any* geometry (straight or pre-curved, cylindrical or tapered), and *any* material featuring the concrete rod that undergoes modeling. At this stage a constitutive law is unavoidable. That means, a kind of ingenious stunt is to establish a function $(geometry, kinematics) \to M_z$ which matches all relevant features of the concrete rod under investigation and turns (16) to a differential equation for $\varphi(s)$. One could think about

$$M_z(s) = \mathbb{M}(s, \varphi(s), \kappa(s), \kappa'(s), ...)$$

as a general structure.

Remind the foregoing section. A careful analysis of the Bernoulli hypotheses had led us to the relation $M_z = EI_z\kappa$ (via (10) for the simplest case: straight and unstretchable). So, if we introduce (with some constant 'stiffness' B)

$$M_z(s) := B\,\kappa(s), \tag{17}$$

the last step to the elastic line is the well-known boundary value problem

$$\varphi'(s) = \kappa(s), \quad \kappa'(s) = \frac{f}{B}\cos(\varphi(s) - \alpha), \quad \varphi(0) = \frac{\pi}{2}, \ \kappa(l) = 0.$$

In this simple classical case both formulations of the problem come out with the same end. That means, the common law (17), also in use with non-constant bending stiffness B, does the job. And, in section 4.1.1, the jump from (11) to (12), (13) by differentiation is indeed not so tricky, but immanent to the problem itself.

4.1.3 Analytical solution of the problem (sketch)

The core of the problem is (μ) in (12),(13). The differential equation is autonomous if the system parameter g does not depend on s (enhancement in next section). This implies the existence of a *first integral*. First, let (φ, κ) be any solution of the differential equation then there holds

$$\kappa\kappa' = g\,\cos(\varphi - \alpha)\,\varphi' \equiv (g\,\sin(\varphi - \alpha)\,)'\,,$$

and by integration we get

$$\frac{1}{2}\kappa^2(s) - g\,\sin(\varphi(s) - \alpha) = const\,.$$

So the function $\mathcal{F} : (\varphi, \kappa) \mapsto \kappa^2 - 2g\,\sin(\varphi - \alpha)$ takes a constant value along every solution $(\varphi(s), \kappa(s))$. To be more specific we go to $s = l$, there we know $\kappa(l) = 0$; if we denote $\varphi(l) =: \varphi_1$, then we get $const = -g\,\sin(\varphi_1 - \alpha)$. (*Equivalently*, we could take $s = 0$, $\varphi(0) = \varphi_0$, $\kappa(0) =: \kappa_0$, then $const = \frac{1}{2}\kappa_0^2 - g\,\sin(\varphi_0 - \alpha)$).

Having an eye to a configuration sketched in figure 7, where $\kappa \le 0$, then we have

$$\kappa(s) = \boxed{\varphi'(s) = -\sqrt{2g}\sqrt{\sin(\varphi(s) - \alpha) - \sin(\varphi_1 - \alpha)}}. \tag{18}$$

This was the most important step in proceeding towards a formal analytic solution. Firstly, (μ) has been reduced to one differential equation for only one unknown, $\varphi(s)$. (Indeed, under punishment by appearing a new unknown constant, φ_1.) But mind that φ' is of fixed (negative) sign, so $s \mapsto \varphi(s)$ is monotonic, and φ can serve as an axis parameter. Then (γ) in (12),(13) can be rewritten, using $\frac{dx}{ds} = \frac{dx}{d\varphi}\frac{d\varphi}{ds}$, ..., as

$$\begin{aligned} \tfrac{dx}{d\varphi} &= -\tfrac{1}{\sqrt{2g}}\cos\varphi\,[\sin(\varphi - \alpha) - \sin(\varphi_1 - \alpha)]^{-1/2}, \quad x(\varphi_0) = x_0, \\ \tfrac{dy}{d\varphi} &= -\tfrac{1}{\sqrt{2g}}\sin\varphi\,[\sin(\varphi - \alpha) - \sin(\varphi_1 - \alpha)]^{-1/2}, \quad y(\varphi_0) = y_0\,. \end{aligned}$$

Proceeding from here, it is trivial matter to write the functions x and y in form of integrals, e.g., $x(\varphi) - x_0 = -\frac{1}{\sqrt{2g}}\int_{\varphi_0}^{\varphi}\cos\tau\,[...]^{-1/2}d\tau$. After some transformations in the integrands, integrals like this one can be expressed by means of the *elliptic integrals of first and second kind, with module* $k < 1$

$$\begin{aligned} &\text{elliptic integral of 1st kind:} \quad \mathbb{F}(z, k) := \int_0^z \tfrac{1}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}dt, \\ &\text{elliptic integral of 2nd kind:} \quad \mathbb{E}(z, k) := \int_0^z \tfrac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}dt\,. \end{aligned} \tag{19}$$

As a rule, these integrals are standard functions in computer software (to be activated like sin and cos), accordingly it is possible to define frequently called functions like the above $x(\varphi)$ as 'custom standards' in a current software.

At first place we have to find the still unknown parameter φ_1: from above we have $\varphi' = -\sqrt{2g}[...]^{1/2}$, then $\frac{ds}{d\varphi} = -\frac{1}{\sqrt{2g}}[...]^{-1/2}$, and by integration we get the equation

$$\sqrt{2g}l = -\int_{\varphi_0}^{\varphi_1} [\sin(\tau - \alpha) - \sin(\varphi_1 - \alpha)]^{-1/2} d\tau$$

to be solved (numerically) for φ_1. Further evaluations are done in turn on the computer by solving finite equations (see [6]).

4.2 This and That Towards the End

In the following we sketch some applications and enlargements of the items given in the foregoing sections.

4.2.1 Scanning object contours

In sections 4.1.1 and 4.1.3 we considered a vertically clamped straight bending rod acted on by a given tip force $\mathbf{f} = f\,(\sin\alpha\,\mathbf{e}_x - \cos\alpha\,\mathbf{e}_y)$. Investigating the boundary value resp. initial value problems (12) and (13) we found the elastic line $(x(s), y(s), \varphi(s))$ which then in particular gave us the coordinates $(x_1, y_1, \varphi_1) = (x(l), y(l), \varphi(l))$ of the tip, that point where the force $\mathbf{f}$ acts - no matter 'who' this force applies. Furthermore we got the clamp reactions $(r_x, r_y, r_z) = -f \cdot\ (\sin\alpha,\ -\cos\alpha,\ (x_0 - x_1)\cos\alpha + (y_0 - y_1)\sin\alpha\ )$.

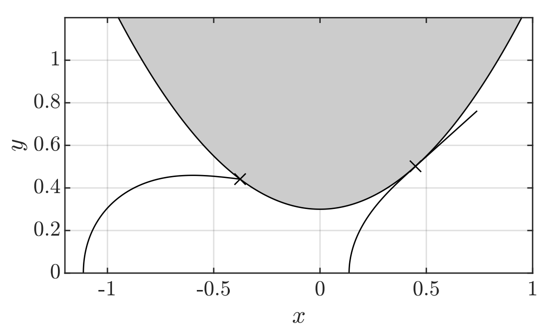

Figure 7: Scanning an object by means of a tactile sensor.

Now think of the following practical scenario in the $x-y-$plane: Given a strictly convex smooth curve (contour of an object) which is swept by an originally straight flexible tactile sensor, see figure.

We model the sensor as a Bernoulli bending rod with vertically clamped foot at (x_0, y_0) where y_0 is fixed below the minimum of the contour, and x_0 runs (slow or step by step) to the left side. During a certain interval the rod touches the contour, gets bent by a contact force which causes corresponding clamp reactions, the latter can be measured. If we assume ideal contact (no friction) then the force is orthogonal to the contour, it appears exactly in the form used above, where α equals the slope of the contour at the contact point.

Scanning the contour means: Exploiting the above scenario by means of measurements in order to find sufficiently many contour points. This task will be done by solving the *inverse problem* to that one in section 4.1:

- measured: $x_0,\ y_0,\ r_x,\ r_y,\ r_z$; implies: $f^2 = r_x^2 + r_y^2$, $\tan\alpha = -r_x/r_y$;
- wanted: $x_1,\ y_1,\ \varphi_1$ (contact point and slope of rod there).

Now solve the bending equations from (11), but use $M_z = r_z + x r_y - y r_x$ and the initial conditions $x(0) = x_0,\ y(0) = y_0,\ \varphi(0) = \frac{\pi}{2}$. Contact then is at $(x_1, y_1) = (x(s_1), y(s_1))$, where s_1 is the least axis point with $\kappa(s_1) = 0$. If $s_1 < l$ then there is tangential contact ($\varphi_1 = \alpha$), and on (s_1, l) the elastica remains a straight line (see [6]).

4.2.2 Surface texture detection

In a way, the following problem is a counterpart to the one above. In order to explore the shape of an object we used a tactile sensor which tenderly slides along the object contour, whereas for to detect the contour roughness we let the sensor tip scratch along the surface (as we do in daily life, too). We consider a plane surface.

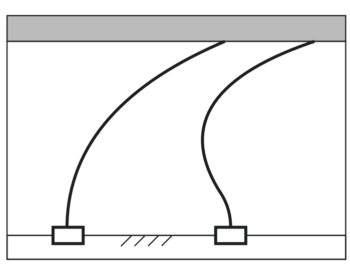

Figure 8: Detection of surface texture by means of a tactile sensor.

Figure 8 shows the scenario: A straight rod of length l, vertically clamped at $(x_0, 0)$ is bent to the right under a horizontal plane of altitude $h < l$. At the contact point we assume a reaction force $\mathbf{f} = -f\,\mathbf{e}_y, \quad f > 0$. Now let the clamp be in *slow motion to the right side*. If the contact was ideal then for every x_0 we have the same vertical reaction force at a displaced contact point. Now assume that the rough surface can be described by Coulomb stiction, then at each position x_0 of the clamp there acts also a horizontal reaction of the form $-\mu\, f\,\mathbf{e}_x$, whith $0 < \mu \leq \mu_0$, where the *friction coefficient* μ_0 characterizes the grade of roughness. Indeed, it is just this μ_0 the investigations aim at. It is obvious that the model is the same as in the section before, a bending rod under tip load

$$\mathbf{f} = f\,(\sin\alpha\,\mathbf{e}_x - \cos\alpha\,\mathbf{e}_y) \quad \text{where} \quad \tan\alpha = -\mu\ .$$

So, measuring the clamp reactions while x_0 is increasing, we get for every x_0 the coefficient $\mu = r_x/r_y$. If μ reaches μ_0 then the horizontal reaction breaks down, some 'bang' is observed and the last measured value μ is the wanted μ_0 characterizing the roughness of the surface at the tip coordinate x_1 which can be observed or calculated. After a pause the process can be continued. If x_0 increases continuously then a characteristic stick-slip sound is observed.

If the theory is needed for certain details (e.g., to find x_1 corresponding to x_0) one has to accept that possibly the curvature κ changes its sign - in contrast to the fixed sign in (18). For theory see [4].

4.2.3 About small bending vibrations (sketch)

At last we take up the bending rod from the last sections again, but now we want to describe a model for its vibrations. In order not to overload the formalism we restrict our considerations to some simple case.

Suppose first that the deformations are without stretching, $\varepsilon_0 = 0$, and, focussing on a linear theory, the deformations are of small amplitude.

This means in detail

- **original**: $(x, y, \varphi, \kappa_0)(\xi),\ \xi \in (0, l);\ \ (x, y, \varphi)(0) = (0, 0, \frac{\pi}{2})$,
  moving frame: $\mathbf{e}_1 = c(\xi)\mathbf{e}_x + s(\xi)\mathbf{e}_y,\ \ \mathbf{e}_2 = -s(\xi)\mathbf{e}_x + c(\xi)\mathbf{e}_y$,
  abbreviations: $c(\xi) := \cos(\varphi(\xi)),\ c(\xi, \rho) := \cos(\varphi(\xi) - \varphi(\rho))$, etc.
- **actual**: $(X, Y, \Phi, \kappa)(\xi, t),\ \Phi(\xi, t) = \varphi(\xi) + \chi(\xi, t),\ |\chi|, |\chi_{,\xi}\,| \ll 1;$ [7]

[7] This smallness shall be handled by neglecting all terms at least quadratic in χ and χ'.

because of $\varepsilon_0 = 0$ we may use the body-fixed coordinate ξ and we have

$$\begin{aligned} X'(\xi,t) &= \cos\Phi(\xi) \approx \cos\varphi(\xi) - \chi(\xi,t)\,\sin\varphi(\xi), \\ Y'(\xi,t) &= \sin\Phi(\xi) \approx \sin\varphi(\xi) + \chi(\xi,t)\,\cos\varphi(\xi)\;. \end{aligned} \tag{20}$$

displacement: $\mathbf{u} = u_x\mathbf{e}_x + u_y\mathbf{e}_y$, $u_x = X - x$, ..;

$$u_x(\xi,t) = -\int_0^\xi \chi(\rho,t)\,s(\rho)\,d\rho, \quad u_y(\xi,t) = \int_0^\xi \chi(\rho,t)\,c(\rho)\,d\rho,$$

$$\mathbf{u}(\xi,t) = \int_0^\xi \chi(\rho,t)\,\mathbf{e}_2(\rho)\,d\rho\,.$$

To come up with some differential equation for the vibrations we use the Kirchhoff-Clebsch equations under consideration of all inertia forces; doing so, we accept the following *approximations*:
- using the equations along the original axis (due to small amplitudes),
- drop the moment load m_z (rotational inertia of small cross-sections).
So we get

$$\tfrac{\partial}{\partial\xi}\mathbf{F}(\xi,t) + \mathbf{q}(\xi,t) = 0, \qquad \tfrac{\partial}{\partial\xi}M_z(\xi,t) + Q(\xi,t) = 0,$$

with inertia force per unity of length (μ: mass per unit of length)

$$\mathbf{q}(\xi,t) = -\mu\,\ddot{\mathbf{u}}(\xi,\mathbf{t}).$$

Letting the external tip force depend on the time t via $(f(t),\alpha(t))$, integration yields

$$\mathbf{F}(\xi,t) - \mathbf{F}(l,t) - \mu\int_l^\xi \ddot{\mathbf{u}}(\rho,t)\,d\rho = \mathbf{0}\;.$$

Keeping an eye on M_z we need the shear force $Q(\xi,t) = (\mathbf{F}(\xi,t) \mathbin{\rceil} \mathbf{e}_2(\xi))$, and we get

$$Q(\xi,t) = -f(t)\,\cos(\varphi(\xi) - \alpha(t)) - \mu\int_l^\xi\int_0^\rho \ddot{\chi}(\sigma,t)\,c(\xi,\sigma)\,d\sigma\,d\rho\;.$$

From the constitutive law (8) we have $M_z(\xi,t) = E\,\mu_2(\xi)\,\frac{\partial}{\partial\xi}\chi(\xi,t)$. Therefore the last Kirchhoff-Clebsch equation yields the vibration equation in the form

$$\boxed{E\,[\mu_2(\xi)\,\chi'(\xi,t)]' = f(t)\cos(\varphi(\xi) - \alpha(t)) - \mu\int_l^\xi\int_0^\rho \ddot{\chi}(\sigma,t)\,c(\xi,\sigma)\,d\sigma\,d\rho} \tag{21}$$

of a linear partial integro-differential equation (the exploitation of which is still waiting for activities).

Particular case: $\mu_2(\xi) = I_z = const$, $\kappa_0 = const$. Let $\frac{EI_z}{\mu} =: B_0$. Four times differentiating w.r.t. ξ we get a linear partial differential equation of order 6 in ξ and order 2 in t :

$$B_0[\chi^{v\prime}(\xi,t) + 2\kappa_0^2\,\chi^{\prime v}(\xi,t) + \kappa_0^4\,\chi''(\xi,t)] = -\ddot{\chi}''(\xi,t) + \kappa_0^2\ddot{\chi}(\xi,t) \qquad (22)$$

(mind that a mixed derivative has occured; **f** has gone to the backstage, but comes in again via boundary condition).

Particular case: $\mu_2(\xi) = I_z = const$, $\kappa_0 = 0$, $\varphi(\xi) = \frac{\pi}{2}$, $\alpha = \frac{\pi}{2}$. Differentiating twice w.r.t. ξ, there remains the well-known equation

$$B_0\frac{\partial^4}{\partial\xi^4}\chi(\xi,t) + \frac{\partial^2}{\partial t^2}\chi(\xi,t) = \frac{1}{\mu}f(t)\ , \qquad (23)$$

which is commonly written for a displacement u instead for our preferred slope deviation χ.

In each case the necessary boundary conditions must be found from the respective equations after the different steps of differentiation.

Excitation of vibrations could be achieved, e.g., through $f(t)$ (modeling some contact as in foregoing sections) or $\chi_0(t) := \chi(0,t)$. Both scenarios match the activity of an animal vibrissa, whose foot is supported in the so called *follicle sinus complex* (FSC) which serves for both driving oscillations and sensing external perturbations as nervous impulses to be transmitted to the central nervous system (which in our model are the measured clamp reactions).

5 Conclusion

Well, so far. Paper fini. Cui bono?

The aim of the paper, as told at the very beginning, was not to serve as a little textbook or as a research report. It was to remember a young theoretical researcher what he is actually doing while investigating any object. In everyday work the modus operandi (often under pressure) is a bit crude: for instance, knowing that the object under consideration deforms by bending then one usually takes the bending equations from memory or from a book or (better?) calls it on the PC by pushing a button - and then : away with it to the computer! The latter then replies with numbers or curves which are taken as a description of the object's behavior.

If a check by comparison with measurement results shows some quantitative deviations then this might be overcome by refined measuring. If, however, the deviations are of *qualitative* nature, then, suddenly, we again become aware that these computer results in fact describe the behavior of a model of the object (which we do not keep in mind with all its details). Last hope: back to the roots, look at the evolution of the model, find an improved model, and start again!

To bring this scenario back to the minds of those people who slightly forgot it, or giving them an engaging evening - this was the aim of the paper.

Of course, there are (practically important) slender bodies of more complex structure which are not captured by the modeling scheme used here. Such objects are, e.g., a hose with internally streaming fluid, a piezo-electrical strip, a tube-like elastic rod filled with some fluid of controllable pressure. Already the aim of investigation and, then, the way of modeling might be different from ours. This becomes obvious by the last cited example, which could be related to a medical vein dilation; here, the primary interest does not lie on bending but on the pressure-caused extension of the tube radius (see [5]).

If there are criticisms of any kind with respect to this all, or somebody finds an error, then, please, give a hint.

Thanks are to Lukas Merker for his kind help in finishing the paper.

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