NEW ASYMPTOTIC FORMULAS FOR THE RIEMANN ZETA FUNCTION

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I certify that this thesis is the true and accurate version of the thesis

approved by the examiners.

Signed: [redacted] (Director of Studies) Date: 21/03/97

Declaration:

I, Shuang Cang, hereby certify that this thesis has been composed by myself, that it is a record of my own work and that it has not been presented in partial or complete fulfillment for any other degree or professional qualification.

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Certificate

I hereby certify that the candidate, Shuang Cang has fulfilled the conditions of the Resolution and Regulations appropriate to the degree of Doctor of Philosophy.

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SUMMARY

This thesis proposes two new asymptotic formulas for the Riemann zeta function $\zeta(s)$. The first is a Riemann-Siegel type formula. This representation for $\zeta(s)$ is given as an absolutely convergent expansion involving incomplete gamma functions which is valid for all finite complex values of s ($\neq 1$). It is then shown how use of the uniform asymptotics of the incomplete gamma function lead to an asymptotic representation for $\zeta(s)$ on the critical line $s = \frac{1}{2} + it$ when $t \to \infty$. This result involves an error function smoothing of the infinite sum. The terms in the smoothed Dirichlet series effectively "switch off" when n attains roughly the Riemann-Siegel cut-off value $N_t = \text{int}\,[(t/2\pi)^{\frac{1}{2}}]$. The second is an exponentially-smoothed Gram-type formula for the Riemann zeta function in the critical strip $0 < \sigma < 1$ when $t \to \infty$. This is a special case of the expansion in terms of special functions such as the confluent hypergeometric function and the incomplete gamma function with a free parameter. It is found that, in the critical strip $0 < \sigma < 1$ when t is large, the "cut-off" in this smoothed sum occurs after $O(|t|/2\pi)$ terms. Extensive numerical results demonstrate the validity of these proposed asymptotic formulas.

The asymptotic behaviour of the coefficients $c_r(\eta)$, which appear in the uniform asymptotic expansion of the incomplete gamma function, is developed for large r. It is found that the behaviour of the coefficients is a 'factorial divided by a power' multiplied by a slowly varying function of the form $\{\Gamma(r + \frac{1}{2})/(2\pi)^{r+1}\}f_r(\eta)$. When η is real, $f_r(\eta)$ is slowly decaying for real $\eta > 0$ and has an oscillatory domain for $\eta < 0$. When η is in the domains D ($D : |1 - e^{\mp\frac{\pi}{2}i}\eta^2/4\pi| < 1$) in the complex η plane, the values of the functions $f_r(\eta)$ become large. Thus, there are two lobes ($\eta \in D$) in the complex η-plane situated symmetrically either side of the negative real η axis for the coefficients $c_r(\eta)$. Numerical results and analysis confirm the above conclusions.

The asymptotic behaviour of the late terms in the Riemann-Siegel type asymptotic formula which is given in Chapter 2 is studied. It is found that the expansion of the late terms in this asymptotic formula is divergent and also possesses the 'facto-

rial divided by a power' dependence characteristic of an asymptotic series multiplied by a slowly varying function.

CHAPTER 1

INTRODUCTION

The theory of the Riemann zeta function and its generalisations represent one of the most beautiful developments in mathematics. The Riemann zeta function is a meromophic function whose properties can be investigated by the techniques of complex analysis. The Riemann zeta function occupies a central place in analytic number theory. Many problems from multiplicative number theory directly depend on properties of the zeta function. Therefore a better understanding of the theory of this function helps number theorists to obtain various arithmetical results. Another important aspect of the theory of the Riemann zeta function is that the zeta function is the simplest of a large class of Dirichlet series known as zeta functions. General zeta functions occur in many branches of mathematics, including algebraic and analytic number theory and quantum chaology, the results from the theory of the zeta function in many instances generalize to other zeta functions.

The Riemann Hypothesis plays a central role in the theory of the Riemann zeta function in that it indicates what one expects to be true. In the theory of the Riemann zeta function the classical approach is based on the approximate functional equation which allows one to approximate the zeta function well in the critical strip and especially on the critical line.

This chapter includes four Sections: some basic definitions of the Riemann zeta function are introduced in the first section. Then the properties which are required in this thesis are given in Section 1.2. The previous main asymptotic expansions of the Riemann zeta function are reviewed in Section 1.3. The last section is the structure of the thesis which also highlights the main contribution of the work.

1.1 Definitions of the Riemann Zeta Function

The celebrated Riemann zeta function is defined by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad \operatorname{Re} s > 1. \tag{1.1.1}$$

This function seems to have been studied first by L. Euler (1707-1783), who considered only real values of s. The present notation and the notation of $\zeta(s)$ as a function of the complex variable s are due to B. Riemann (1826-1866), who made a number of startling discoveries about $\zeta(s)$. Riemann wrote $s = \sigma + it$ for the complex variable s, and this notation has be used in this thesis. From Cauchy's integral test one knows that the series (1.1.1) converges in the region described; moreover if $\sigma > 1$ then the series is dominated term by term by the absolutely convergent series $\sum_{n=1}^{\infty} n^{-\sigma}$, so that, by Weierstrass's criterion, it converges uniformly in $\sigma > 1$.

There is a second representation of $\zeta(s)$ due to Euler in 1749 which is perhaps more fundamental and which is the reason for the significance of the Riemann zeta function. This is

$$\zeta(s) = \prod (1 - p^{-s})^{-1}, \quad \operatorname{Re} s > 1, \tag{1.1.2}$$

where the product is taken over all prime numbers p. This is called the Euler product representation of the Riemann zeta function and gives analytic expression to the fundamental theorem of arithmetic. The infinite product is absolutely convergent for $\sigma > 1$.

The analytic continuation of $\zeta(s)$ is obtainable by means of a loop integral of Hankel's type

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \int_{\infty}^{(0+)} \frac{(-z)^s}{e^z - 1} \frac{dz}{z}, \tag{1.1.3}$$

where the contour does not encircle any of the points $\pm 2\pi n i$, $n = 1, 2, \cdots$ and the phase of z at the beginning of the contour is zero. The symbol $\int_{\infty}^{(0+)}$ indicates a path of integration which begins at $+\infty$, encircles the origin once in the positive counter-clockwise direction and returns to $+\infty$. Now this integral is uniformly convergent in any finite region, and so represents an integral function of s. This enables us to continue $\zeta(s)$ over the whole plane. Hence $\zeta(s)$ is analytic for all values of s except for a simple pole at $s = 1$, with residue 1.

1.2 Properties and Hypotheses of $\zeta(s)$

(1). The functional relation satisfied by $\zeta(s)$

By using the residue theorem, (1.1.3) embodies the well-known functional equation

$$\zeta(s) = \chi(s)\zeta(1-s), \quad \chi(s) = 2^s \pi^{s-1} \sin\frac{1}{2}\pi s\, \Gamma(1-s) = \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{2}-\frac{s}{2})}{\Gamma(\frac{s}{2})}. \quad (1.2.1)$$

This relation combined with the Dirichlet series in (1.1.1) enables (in principle) the computation of $\zeta(s)$ throughout the s plane except in and on the boundaries of the critical strip $0 < \sigma < 1$.

(2). Riemann's hypothesis concerning the zeros of $\zeta(s)$

From (1.2.1), it may be deduced that $\zeta(-2m) = 0$ for $m = 1, 2, \cdots$. These zeros of $\zeta(s)$ are known as the "trivial" zeros. If s is an integer, the values of $\zeta(s)$ can be obtained from the definition (1.1.3). The function $z/(e^z - 1)$ is analytic near $z = 0$; therefore it can be expanded as a power series $z/(e^z - 1) = \sum_{n=0}^{\infty} B_n z^n/n!$ valid in the disk $|z| < 2\pi$, where the coefficients B_n of this expansion are by definition the Bernoulli numbers. This expansion can be used in (1.1.3) to obtain

$$\zeta(0) = -\frac{1}{2}, \quad \zeta(-2m) = 0, \quad \zeta(2m) = \frac{(-)^{m+1}(2\pi)^{2m} B_{2m}}{2(2m)!}, \quad m = 1, 2, \cdots. \quad (1.2.2)$$

It can be shown that $\zeta(s)$ has no zeros at which $\sigma > 1$ in (1.1.2). From the functional relation (1.2.1), it is now apparent that the only zeros of $\zeta(s)$ for which $\sigma < 0$ are the "trivial" zeros and all the complex zeros (the "non trivial" zeros) must be distributed symmetrically about the line $\sigma = \frac{1}{2}$. For $\sigma = 1$, it has been proved that $\zeta(1 + it) \neq 0$ and $\zeta(s)$ has an infinity of zeros [Titchmarsh 1930, p. 3]. Hence all the zeros of $\zeta(s)$ except the "trivial" zeros lie in that strip of the domain of the complex variable s which is defined by $0 < \sigma < 1$.

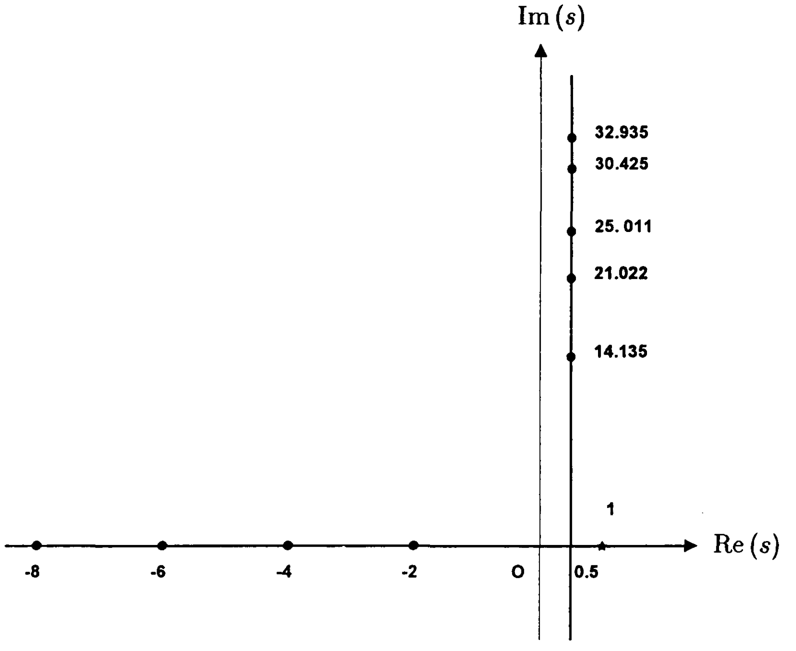


Figure 1.1: The zeros and the simple pole of $\zeta(s)$

It was conjectured by Riemann (**the Riemann Hypothesis**) that: **all the complex zeros of $\zeta(s)$ are situated on the line $\sigma = \frac{1}{2}$, which is called the critical line.** However, the proof of this claim is still open. It has been proved by Hardy that an infinity of zeros of $\zeta(s)$ actually lie on the critical line. The numerical evidence is very convincing; it is known through very extensive computations that the first 1 500 000 001 zeros of the Riemann zeta function lie on the critical line. Nevertheless, there have been a number of examples of assertions based on very extensive computation in analytic number theory which have later been disproved by theoretical considerations without a counter example being found. The zeros and the simple pole of $\zeta(s)$ situated at $s = 1$ are shown in Fig. 1.1.

(3). The Lindelöf Hypothesis

The Lindelöf Hypothesis is a further conjecture which would be a consequence of the Riemann Hypothesis but which is not sufficient to imply the Riemann Hypothesis in the present state of knowledge. It represents the limit of what one might expect to be able to prove with classical techniques. The Lindelöf Hypothesis can be phrased as expressing that for each positive ϵ as $t \to \infty$

$$\zeta(\frac{1}{2} + it) = O(|t|^{\epsilon}). \tag{1.2.3}$$

The formulation of the Lindelöf Hypothesis just given involves measuring the 'size' of the zeta function by $\mathbf{Sup}_{\{0<t<T\}}|\zeta(\frac{1}{2}+it)|$. It is possible to replace the 'sup norm' here with various integral means, and this has the further advantage that it fractionates the Lindelöf Hypothesis into a family of assertions which can be investigated by special methods.

1.3 Background of Asymptotic Formulas for $\zeta(s)$

Of fundamental importance is the behaviour of $\zeta(s)$ in the critical strip and, in particular, on the critical line $s = \frac{1}{2} + it$. It is often necessary to be able to represent $\zeta(\frac{1}{2} + it)$ in a simple way high in the critical strip and calculate it there with great accuracy. It is usually more convenient to work with the real even function $Z(t)$ defined by

$$Z(t) = e^{i\vartheta(t)}\zeta(\frac{1}{2} + it), \tag{1.3.1}$$

where the phase-angle $\vartheta(t)$ has the representation

$$\vartheta(t) = \arg\{\pi^{-\frac{1}{2}it}\Gamma(\frac{1}{4} + \frac{1}{2}it)\} = \operatorname{Im}\log\Gamma(\frac{1}{4} + \frac{1}{2}it) - \frac{1}{2}t\log\pi, \tag{1.3.2}$$

and possesses the asymptotic expansion for $t \to +\infty$ [Haselgrove, 1963]

$$\vartheta(t) \sim \frac{1}{2}t\,(\log\frac{t}{2\pi} - 1) - \frac{\pi}{8} + \frac{1}{48t} + \frac{7}{5760t^3} + \frac{31}{80640t^5} + \cdots, \tag{1.3.3}$$

which follows from Stirling's formula for the logarithm of the gamma function.

The two well-established methods of computing $\zeta(s)$ in the critical strip are the Gram and the Riemann-Siegel formulas which have been known for the better part of a century. In 1903, Gram gave an asymptotic expansion for $\zeta(s)$ based on the Euler-Maclaurin summation formula in the form

$$\zeta(s) \sim \sum_{n=1}^{M-1} n^{-s} + \frac{M^{1-s}}{s-1} + \frac{1}{2}M^{-s} + M^{-s}\sum_{r=1}^{\infty}\frac{B_{2r}\,(s)_{2r-1}}{(2r)!M^{2r-1}}, \tag{1.3.4}$$

where B_{2r} are the Bernoulli numbers and $(s)_r = \Gamma(s+r)/\Gamma(s)$. For the terms in this expansion to initially decrease, M must be chosen such that $M > |s|/2\pi$, for large t, however, this means that the number of terms in the finite "main sum" then becomes $O(t/2\pi)$ and the time of computation for very large t becomes prohibitively long.

An alternative, more powerful expansion given by Siegel in 1932 from Riemann's manuscripts of the 1850's is the so-called Riemann-Siegel formula, which can be obtained by appropriate manipulation of the contour and expansion of the integrand in (1.1.3) about its saddle point. The Riemann-Siegel formula is [Haselgrove, 1963; Edwards, 1974; Titchmarsh, 1986]

$$Z(t) = 2\sum_{n=1}^{N_t} \frac{\cos(\vartheta(t) - t\log n)}{n^{\frac{1}{2}}} + (-)^{N_t-1}(\frac{t}{2\pi})^{-\frac{1}{4}} \sum_{r=0}^{m} (-)^r (\frac{t}{2\pi})^{-\frac{1}{2}r} \psi_r(p) + R_m(t), \tag{1.3.5}$$

where N_t and $p(t)$ are defined by

$$N_t = \text{int}\,[(t/2\pi)^{\frac{1}{2}}], \quad p = p(t) = (t/2\pi)^{\frac{1}{2}} - N_t, \tag{1.3.6}$$

and the int [ ] denotes the integer part. The function $\vartheta(t)$ is defined in (1.3.2) and R_m denotes the remainder. The functions $\psi_r(p)$ are combinations of derivatives of the function

$$\psi_0(p) = \frac{\cos 2\pi(p^2 - p - \frac{1}{16})}{\cos 2\pi p},$$

and the first few of the functions $\psi_r(p)$ are given by

$$\begin{aligned}
\psi_1(p) &= \frac{1}{12\pi^2}\psi_0^{(3)}(p), \\
\psi_2(p) &= \frac{1}{288\pi^4}\psi_0^{(6)}(p) + \frac{1}{16\pi^2}\psi_0^{(2)}(p), \\
\psi_3(p) &= \frac{1}{10368\pi^6}\psi_0^{(9)}(p) + \frac{1}{120\pi^4}\psi_0^{(5)}(p) + \frac{1}{32\pi^2}\psi_0^{(1)}(p), \\
\psi_4(p) &= \frac{1}{497664\pi^8}\psi_0^{(12)}(p) + \frac{11}{23040\pi^6}\psi_0^{(8)}(p) + \frac{19}{1536\pi^4}\psi_0^{(4)}(p) + \frac{1}{128\pi^2}\psi_0(p).
\end{aligned}$$

The asymptotic formula (1.3.5) requires $N_t \simeq (t/2\pi)^{\frac{1}{2}}$ terms in the main sum and thus represents a considerable saving in computational time for large t compared with the Gram formula. The main sum in (1.3.5), which usually dominates $Z(t)$, is however a discontinuous function of t because of the discrete upper limit N_t. The correction terms in (1.3.5) 'smooth out' the discontinuity in successive derivatives of this approximation to $Z(t)$ at the truncation point.

In 1992, a new asymptotic representation for $\zeta(\frac{1}{2}+it)$ suitable for computational purposes when t is very large was given by Berry and Keating. [Berry and Keating, 1992] as an example of a technique for dealing with zeta function-like series encountered in quantum chaology. By means of Cauchy's integral formula applied to $Z(t)$, they derived an asymptotic expansion for $Z(t)$ for which the leading term is given by the convergent sum

$$Z_0(t,\kappa) = 2\mathrm{Re}\sum_{n=1}^{\infty} n^{-\frac{1}{2}} e^{i(\vartheta(t)-t\log n)} \times \frac{1}{2}\mathrm{erfc}\,\{\sqrt{\frac{t}{2}\frac{\xi(n,t)}{Q(\kappa,t)}}\}, \tag{1.3.7}$$

where $\xi(n,t) = \log n - \vartheta'(t)$, $Q^2(\kappa,t) = \kappa^2 - it\,\vartheta''(t)$ and κ is a real parameter chosen freely. Other terms of the asymptotic expansion for $Z(t)$ (other than the leading term) can be evaluated in terms of Hermite polynomials. The dominant contribution $Z_0(t,\kappa)$ is a convergent sum over the integers n of the Dirichlet series, resembling the finite 'main sum' of the Riemann-Siegel formula, but with the sharp cut-off smoothed by a complementary error function. The main sum and first correction term of the Riemann-Siegel formula are included in the leading term $Z_0(t,\kappa)$. It is possible that more, perhaps all, terms of the Riemann-Siegel formula are contained in $Z_0(t,\kappa)$. As a consequence, it is found numerically that $Z_0(t,\kappa)$ is always a better approximation that the main sum of the Riemann-Siegel formula, with the addition of higher correction terms in the expansion yielding further improvement comparable with the Riemann-Siegel formula.

In 1994, an alternative asymptotic representation for $\zeta(s)$ on the critical line was derived by Paris [Paris, 1994]. This asymptotic formula resulted from application of the Poisson summation formula to the tail of the Dirichlet series representation of $\zeta(s)$ followed by analytic continuation. Suitable approximation using the uniform asymptotic expansion of the incomplete gamma function $Q(a,z) = \Gamma(a,z)/\Gamma(a)$, then yields an approximation for $Z(t)$ in the form

$$Z(t) \simeq 2\sum_{k=0}^{N} \frac{\cos(\vartheta(t) - t\log k)}{\sqrt{k}} + \mathrm{Re}\,\{e^{-i\vartheta(t)} E_m(t) - \frac{ie^{i\vartheta(t)}\Gamma^*(a)}{2R^{1-a}\sin w} \sum_{r=0}^{m-1} \frac{B_r(w)(a/4R^2)^r}{\Gamma^*_{m-r}(a)}\}, \tag{1.3.8}$$

where $m = 1, 2, \cdots$ and N is arbitrary. $\eta_\pm$ and w are defined by

$$\begin{aligned} \frac{1}{2}\eta_\pm^2 &= \pm i\frac{\mu}{a} - 1 - \log(\pm i\frac{\mu}{a}), \quad \mu = 2\pi kR = (2N+1)\pi k, \\ w &\equiv i\frac{a}{2R} = \frac{(t+\frac{1}{2}i)}{(2N+1)}. \end{aligned}$$

$\Gamma_n^*(a)$ denotes the expansion of the 'scaled' gamma function truncated after n terms. The convergent sum $E_m(t)$ is given by

$$E_m(t) = \frac{1}{2}\sum_{k=1}^{\infty} \delta_N k^{it-\frac{1}{2}} \text{erfc}\,((a/2)^{\frac{1}{2}}\delta_N \eta_-; m), \quad \delta_N = \begin{cases} -1 & k \le N, \\ +1 & k > N, \end{cases} \tag{1.3.9}$$

where erfc $(z; m)$ is the modified complementary error function defined by

$$\text{erfc}\,(z; m) = \text{erfc}\, z - \frac{e^{-z^2}}{z\sqrt{\pi}} \sum_{r=0}^{m-1} D_r (2z^2)^{-r}, \quad m = 1, 2, \cdots. \ (z \neq 0),$$

and the first few coefficients $B_r(w)$ are defined in the form

$$\begin{aligned} B_0(w) &= 1, \\ B_1(w) &= \frac{1}{2} + \cot^2 w - w^{-1}\cot w, \\ B_2(w) &= \frac{1}{8}(1 - 20\text{cosec}^2 w + 24\text{cosec}^4 w) + \frac{5}{6}w^{-1}(1 - 6\text{cosec}^2 w)\cot w \\ &\quad + w^{-2}(1 + 2\cot^2 w). \end{aligned}$$

The asymptotic formula (1.3.8) consists of the finite main sum of $N+1$ terms which is analogous to the main sum in the Riemann-Siegel formula, together with a series of correction terms involving an infinite sum of modified complementary error functions. Numerical results suggest that term for term (1.3.8) is at least as accurate as the Riemann-Siegel formula and yields a comparable accuracy to the Berry-Keating formula.

1.4 Structure of the Thesis

This thesis is basically organised in two parts. The first part (Chapters 2-3) proposes a new Riemann-Siegel type and Gram-type asymptotic formulas for $\zeta(s)$ when t is large. The second part (Chapters 4-5) is concerned with the asymptotic behaviour of the coefficients which appear in the uniform asymptotic expansion of $Q(a, z)$ and the late term appearing in the Riemann-Siegel type asymptotic formula for $\zeta(s)$.

Chapter 2 proposes a new Riemann-Siegel type asymptotic formula for $\zeta(s)$ when t is large. This chapter mainly consists of two parts. The first part (Sections 2.2-2.6) develops a new Riemann-Siegel type asymptotic formula for $\zeta(s)$ on the critical line. The special case for the proposed asymptotic formula of $Z(t)$ is considered and the method of calculation of the coefficients is given. Discussion and numerical results demonstrate the validity of the asymptotic formula for $Z(t)$. The second part (Sections 2.7-2.10) studies the case when t is near a critical value which is defined by $2\pi \times N_t^2$, where N_t is positive integer. A modified Riemann-Siegel type asymptotic formula is proposed in the neighbourhood of a critical value and the calculation of the coefficients is also discussed. Discussion and numerical results are given.

Chapter 3 proposes new Gram-type asymptotic formulas for $\zeta(s)$ which are the incomplete gamma function-smoothed Gram-type and the confluent hypergeometric-smoothed Gram-type asymptotic formulas for $\zeta(s)$ in Sections 3.2 and 3.3. Extensive numerical results demonstrate the validity of these asymptotic formulas. A bound on the remainder term appearing in the Gram-type asymptotic formulas is studied in Section 3.6. Estimation of the order of the main sum in the asymptotic formulas and the order of $\zeta(\frac{1}{2} + it)$ is given using the new method in Sections 3.8 and 3.9.

Chapter 4 studies the asymptotic behaviour of the coefficients $c_r(\eta)$ in the uniform asymptotic expansion of $Q(a, z)$ for large r. Derivation of the asymptotic form of $c_r(\eta)$ for large r is given in Section 4.2. The representation and asymptotic expansion of the function $F_r(\eta)$ appearing in the asymptotic form of $c_r(\eta)$ for large r is developed in Section 4.3. Extensive numerical results are presented in Section 4.4. The asymptotic behaviour of the function $G_r(z)$ appearing in the function $F_r(\eta)$ is studied in Section 4.5. Discussion and numerical results are included in Section 4.6. The asymptotic behaviour of the function $f_r(\eta)$, which is defined by $f_r(\eta) = \{(2\pi)^{r+1}/\Gamma(r + \frac{1}{2})\}c_r(\eta)$, is developed in Section 4.7.

Chapter 5 deals with the asymptotic behaviour of the late terms in the Riemann-Siegel type asymptotic formula for $\zeta(s)$ given in Chapter 2. For this purpose, the asymptotic behaviour of the coefficients $\alpha_k^{(r)}$ for large r is studied in Section 5.2.

Then the behaviour of the late terms in the asymptotic formula for $\zeta(s)$ when t is large is given in Section 5.3. The behaviour of the late terms in the asymptotic formula for $\zeta(s)$ when t ($\gg 1$) is in the neighbourhood of a critical value is considered in Section 5.4.

Chapter 6 presents the conclusions of this thesis and gives possible future research topics.

Some useful formulas are included in Appendices A-C.

CHAPTER 2

A RIEMANN-SIEGEL TYPE ASYMPTOTIC FORMULA FOR $Z(t)$

2.1 Introduction

A general representation for the Riemann zeta function on the critical line $s = \frac{1}{2} + it$ when $t \to +\infty$ is given in this chapter. This result involves an error function smoothing of an infinite sum. We also propose a modification of this asymptotic formula in the neighbourhood of a critical value, where the critical value is defined by $t = 2\pi \times N_t^2$, N_t being a positive integer. The cut-off value, which is the transition point of the incomplete gamma function in the main sum of the asymptotic formula, is the same as the cut-off value of the Riemann-Siegel asymptotic formula. Therefore this asymptotic formula is called a Riemann-Siegel type formula.

The outline of this chapter is as follows: The derivation of the basic formula for $Z(t)$ is given in Section 2.2. A general representation for the Riemann zeta function on the critical line $s = \frac{1}{2} + it$, when $t \to +\infty$, is proposed in Section 2.3. The method of obtaining the general coefficients in the asymptotic formula is developed in Section 2.4. A discussion and extensive numerical results of the general asymptotic formula are given in Sections 2.5-2.6. An asymptotic formula in the neighbourhood of critical values t is proposed in Section 2.7. The calculation of the coefficients $B_r^*(w)$, which are defined by (2.8.1), in the asymptotic formula given in Section 2.7, is studied in Section 2.8. A discussion and numerical results in the neighbourhood of critical values of t are given in Sections 2.9-2.10. Conclusions are given in the last section. The details of the proof of some formulas are included in Appendix A.

2.2 A Basic Formula for $Z(t)$

2.2.1 The Derivation of the Expansion for $\zeta(s)$

We derive an asymptotic representation for $Z(t)$ based on the expansion for $\zeta(s)$ given by

$$\zeta(s) = \frac{(\pi\xi)^{\frac{s}{2}}}{\Gamma(\frac{s}{2})}(\frac{\xi^{-\frac{1}{2}}}{s-1} - \frac{1}{s}) + \sum_{n=1}^{\infty} n^{-s} Q(\frac{s}{2}, \pi n^2 \xi) + \chi(s) \sum_{n=1}^{\infty} n^{s-1} Q(\frac{1-s}{2}, \pi n^2/\xi), \tag{2.2.1}$$

where

$$\chi(s) = 2^s \pi^{s-1} \sin\frac{\pi s}{2} \Gamma(1-s) = \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{2} - \frac{s}{2})}{\Gamma(\frac{s}{2})}, \tag{2.2.2}$$

and ξ is a complex parameter satisfying $|\arg\xi| \le \frac{\pi}{2}$, this result involving the normalised incomplete gamma function $Q(a, z)$, which is defined by

$$Q(a, z) = \frac{\Gamma(a, z)}{\Gamma(a)} = \frac{1}{\Gamma(a)} \int_z^{\infty} u^{a-1} e^{-u} du. \tag{2.2.3}$$

The result (2.2.1) is the expansion given in [Lavrik, 1968] for the Dirichlet L-function specialized to $\zeta(s)$. We remark that the case $\xi = 1$ is effectively embodied in Riemann's 1859 paper, though he did not explicitly identity the incomplete gamma functions.

We will prove (2.2.1) by using a slight modification of Riemann's original analysis. He obtained the result

$$\pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s) = \int_0^{\infty} x^{\frac{s}{2}-1} \psi(x) dx, \tag{2.2.4}$$

where $\mathrm{Re}\,(s) > 1$, and $\psi(x)$ denotes the sum

$$\psi(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}.$$

The function $\psi(x)$ satisfies the well-known Poisson summation formula

$$2\psi(x)+1=x^{-\frac{1}{2}}\{2\psi(1/x)+1\}, \quad |\arg x|<\frac{\pi}{2}. \tag{2.2.5}$$

Instead of dividing the path of integration in (2.2.4) into $[0,1]$ and $[1,\infty)$, we divide the path into $[0,\xi]$ and $[\xi,\infty)$, where, for the moment, we restrict ξ to lie in $|\arg\xi|<\frac{\pi}{2}$. Then, from (2.2.4) and (2.2.5) we find

$$\begin{aligned}\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) &= \int_0^{\xi} x^{\frac{s}{2}-1}\psi(x)dx+\int_{\xi}^{\infty} x^{\frac{s}{2}-1}\psi(x)dx\\ &= \frac{\xi^{\frac{s}{2}-\frac{1}{2}}}{s-1}-\frac{\xi^{\frac{s}{2}}}{s}+\int_0^{\xi} x^{\frac{s}{2}-\frac{3}{2}}\psi(1/x)dx+\int_{\xi}^{\infty} x^{\frac{s}{2}-1}\psi(x)dx\\ &= \frac{\xi^{\frac{s}{2}-\frac{1}{2}}}{s-1}-\frac{\xi^{\frac{s}{2}}}{s}+\int_{1/\xi}^{\infty} x^{-\frac{s}{2}-\frac{1}{2}}\psi(x)dx+\int_{\xi}^{\infty} x^{\frac{s}{2}-1}\psi(x)dx. \end{aligned}\tag{2.2.6}$$

Reversal of the order of summation and integration, which is justified by absolute convergence, followed by introduction of the new variable $u=\pi n^2x$, enables $\zeta(s)$ to be written in the form

$$\begin{aligned}\zeta(s) &= \frac{(\pi\xi)^{\frac{s}{2}}}{\Gamma(\frac{s}{2})}(\frac{\xi^{-\frac{1}{2}}}{s-1}-\frac{1}{s})+\frac{1}{\Gamma(\frac{s}{2})}\sum_{n=1}^{\infty}n^{-s}\int_{\pi n^2\xi}^{\infty}x^{\frac{s}{2}-1}e^{-u}du\\ &\quad+\frac{\chi(s)}{\Gamma(\frac{1-s}{2})}\sum_{n=1}^{\infty}n^{s-1}\int_{\pi n^2/\xi}^{\infty}u^{-\frac{s}{2}-\frac{1}{2}}e^{-u}du. \end{aligned}\tag{2.2.7}$$

The resulting integrals can be expressed in terms of the normalised incomplete gamma function $Q(a,z)$. From (2.2.7) and (2.2.3), we obtain the expansion (2.2.1).

Since $\Gamma(a,\,z)\sim z^{a-1}e^{-z}$ as $|z|\to\infty$ in $|\arg z|<\frac{3\pi}{2}$, then both sums in (2.2.1) converge absolutely for all values of s, with late terms behave like $n^{-2}\exp(-\pi n^2\xi^{\pm 1})$, respectively. The result (2.2.1), which was derived for $\mathrm{Re}(s)>1$ and $|\arg\,\xi|<\frac{\pi}{2}$, holds for all values of s (except $s=1$) and ξ satisfying $|\arg\,\xi|\leq\frac{\pi}{2}$ by analytic continuation.

We will use the Mellin-Barnes representation of the incomplete gamma function to give an alternative proof of the expansion (2.2.1) in Section 3.2.

2.2.2 The Derivation of the Basic Formula for $Z(t)$

We study the special case of the result (2.2.1) when s is on the critical line , that is $s=\frac{1}{2}+it$. By the symmetry of the zeta function, it is sufficient to take $t\geq 0$. In this case it is more convenient to consider the real even function $Z(t)$ given by

$$Z(t) = e^{i\vartheta(t)}\zeta(\frac{1}{2} + it), \tag{2.2.8}$$

where $\vartheta(t)$ is defined in (1.3.2). Moreover, $\vartheta(t)$ is a real and odd function of t. It is noted that for large t, we may approximate $\vartheta(t)$ by Stirling's asymptotic expansion for the gamma function given in (1.3.3).

We now specialise the result (2.2.1) to the critical line. When $s = \frac{1}{2}+it$, (2.2.2) can be written as the following form

$$\chi(\frac{1}{2} + it) = \pi^{it}\frac{\overline{\Gamma(\frac{1}{4} + \frac{it}{2})}}{\Gamma(\frac{1}{4} + \frac{it}{2})} = e^{-2i\vartheta(t)}. \tag{2.2.9}$$

In order to obtain a symmetrical form, we require $|\xi| = 1$ and write $\xi = e^{i\phi}$, where ϕ is real. According to $Q(\bar{a}, \bar{z}) = \overline{Q(a, z)}$ (where the bar denotes the complex conjugate), substitution of (2.2.1) into (2.2.8) and using of (2.2.9), considering the factor $\pi^{\frac{s}{2}}e^{i\vartheta(t)}/\Gamma(\frac{s}{2})$ is real, $(s = \frac{1}{2} + it)$ then yields the elegant result [Paris, 1993]

$$\begin{aligned} Z(t) &= e^{i\vartheta(t)}\frac{(\pi e^{i\phi})^{\frac{s}{2}}}{\Gamma(\frac{s}{2})}(\frac{e^{-\frac{1}{2}i\phi}}{s-1} - \frac{1}{s}) + e^{i\vartheta(t)}\sum_{n=1}^{\infty} n^{-s}Q(\frac{s}{2}, \pi n^2 e^{i\phi}) \\ &\quad + e^{-i\vartheta(t)}\sum_{n=1}^{\infty} n^{s-1}Q(\frac{1-s}{2}, \pi n^2/e^{i\phi}) \\ &= -e^{i\vartheta(t)}\frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2})}(\frac{e^{\frac{1}{2}i\phi(s-1)}}{1-s} + \frac{e^{\frac{1}{2}i\phi s}}{s}) + e^{i\vartheta(t)}\sum_{n=1}^{\infty} n^{-s}Q(\frac{s}{2}, \pi n^2 e^{i\phi}) \\ &\quad + e^{-i\vartheta(t)}\sum_{n=1}^{\infty} n^{s-1}Q(\frac{1-s}{2}, \pi n^2 e^{-i\phi}) \\ &= 2\mathrm{Re}\, e^{i\vartheta(t)}\{\sum_{n=1}^{\infty} n^{-s}Q(\frac{s}{2}, \pi n^2 e^{i\phi}) - \frac{\pi^{\frac{1}{2}s}\, e^{\frac{1}{2}i\phi s}}{s\Gamma(\frac{s}{2})}\}, \end{aligned} \tag{2.2.10}$$

where $s = \frac{1}{2} + it$ and $|\phi| \le \frac{\pi}{2}$.

2.3 A New Asymptotic Formula for $Z(t)$

To obtain an approximation for $Z(t)$ as $t \to +\infty$, we employ the asymptotic expansion of the incomplete gamma function $Q(a, z)$ which holds uniformly as $a \to \infty$ for $|z| \in [0, \infty)$ in the domains $|\arg a| \le \pi - \epsilon_1$ and $|\arg(z/a)| \le 2\pi - \epsilon_2$, where ϵ_1, ϵ_2 are positive numbers satisfying $0 < \epsilon_1 < \pi$, $0 < \epsilon_2 < 2\pi$. The uniform asymptotic expansion of $Q(a,z)$ given in terms of the complementary error function, can be written as [Temme, 1979]

$$Q(a,z) = \frac{1}{2}\mathrm{erfc}\,((a/2)^{\frac{1}{2}}\eta) + \frac{e^{-\frac{1}{2}a\eta^2}}{\sqrt{2\pi a}}\{\sum_{r=0}^{m-1} a^{-r}c_r(\eta) + a^{-m}G_m(a,\eta)\}, \qquad (2.3.1)$$

with

$$c_r(\eta) = (-)^r\frac{Q_r(\mu)}{\mu^{2r+1}} - \frac{D_r}{\eta^{2r+1}}, \qquad (2.3.2)$$

$$\lambda = \frac{z}{a}, \qquad \mu = \lambda - 1, \qquad \frac{1}{2}\eta^2 = \lambda - 1 - \log\lambda, \qquad (2.3.3)$$

where $m = 1, 2, \cdots$ and $G_m(a,\eta)$ is the remainder in the expansion truncated after m terms. The choice of the square root branch for $\eta(\lambda)$ is made such that $\eta(\lambda)$ and $\lambda - 1$ have the same sign when $\lambda > 0$. We note that $\eta \simeq 0$ when $\lambda \simeq 1$. The coefficients $c_r(\eta)$ are specified in terms of the polynomials $Q_r(\mu)$ in μ of degree $2r$, and $D_r = (-2)^r\Gamma(r+\frac{1}{2})/\Gamma(\frac{1}{2})$. The polynomials $Q_r(\mu)$ can be written as

$$Q_r(\mu) = (1+\mu)P_r(\mu) + (-)^r\gamma_r\mu^{2r}, \qquad (2.3.4)$$

where $P_r(\mu)$ is a polynomial of degree $2r-2\,(r \geq 1)$. Writing

$$P_r(\mu) = p_0^{(r)} + p_1^{(r)}\mu + \cdots + p_{2r-2}^{(r)}\mu^{2r-2}, \qquad (2.3.5)$$

the coefficients $p_k^{(r)}$ satisfy the relations

$$\begin{aligned} p_0^{(r)} &= (2r-1)p_0^{(r-1)}, \\ p_k^{(r)} &= (2r-1-k)(p_k^{(r-1)} + p_{k-1}^{(r-1)}), \quad k = 1, 2, \cdots, 2r-4, \\ p_{2r-3}^{(r)} &= 2p_{2r-4}^{(r-1)}, \\ p_{2r-2}^{(r)} &= (-)^{r-1}\gamma_{r-1}, \end{aligned} \qquad (2.3.6)$$

with as starting polynomial $P_1(\mu) = 1$, or $p_0^{(1)} = 1$.

We now define the modified complementary error function [Paris, 1994] by

$$\mathrm{erfc}\,(z;m) = \mathrm{erfc}\,z - \frac{e^{-z^2}}{z\sqrt{\pi}}\sum_{r=0}^{m-1} D_r(2z^2)^{-r}, \qquad m = 1, 2, \cdots. \; (z \neq 0), \qquad (2.3.7)$$

which corresponds to the removal from the complementary error function $\mathrm{erfc}\,z$ of the first m terms of its asymptotic expansion for $|z| \to \infty$ in $|\arg z| < \frac{3}{4}\pi$. Using the definition (2.3.7), the uniform asymptotic expansion (2.3.1) can then be written in the modified form more suitable for the present application as

$$Q(a,z) = \frac{1}{2}\text{erfc}\,((a/2)^{\frac{1}{2}}\eta; m) + \frac{e^{-\frac{1}{2}a\eta^2}}{\sqrt{2\pi a}}\{\sum_{r=0}^{m-1}(-)^r a^{-r}\frac{Q_r(\mu)}{\mu^{2r+1}} + a^{-m}G_m(a,\eta)\}. \quad (2.3.8)$$

In view of (2.2.10), which involves the incomplete gamma function with $a \equiv \frac{s}{2}$, $z = \pi n^2 e^{i\phi}$, then

$$a \equiv \frac{1}{4} + \frac{1}{2}it, \qquad \lambda = \frac{2\pi n^2 e^{i\phi}}{s}.$$

It follows from (2.3.3) that $\eta \equiv \eta_n(t)$ and $\mu \equiv \mu_n(t)$; for simplicity in presentation we shall omit the dependence on t, except where it is essential. We can also show that

$$\exp\,(-\frac{1}{2}a\eta_n^2) = \exp(-\pi n^2 e^{i\phi})n^s(\frac{2\pi e}{s})^{\frac{s}{2}}e^{\frac{1}{2}i\phi s}. \quad (2.3.9)$$

The term $\pi^{\frac{s}{2}}\,e^{\frac{1}{2}i\phi s}/s\Gamma(\frac{s}{2})$ in (2.2.10) can be expressed as

$$\frac{\pi^{\frac{1}{2}s}\,e^{\frac{1}{2}i\phi s}}{s\Gamma(\frac{s}{2})} = \frac{1}{2}(\frac{2\pi e}{s})^{\frac{s}{2}}\frac{e^{\frac{1}{2}i\phi s}}{\sqrt{\pi s}\Gamma^*(\frac{s}{2})}. \quad (2.3.10)$$

We have employed the well-known expansion of the inverse of the "scaled" gamma function

$$\Gamma^*(z) = (2\pi)^{-\frac{1}{2}}z^{\frac{1}{2}-z}e^z\Gamma(z), \quad (2.3.11)$$

for large z in the form

$$\frac{1}{\Gamma^*(z)} = \sum_{r=0}^{m-1}\gamma_r z^{-r} + z^{-m}H_m(z), \qquad m = 1,\,2,\,\cdots, \quad (2.3.12)$$

where $H_m(z)$ is a remainder and the first few Stirling coefficients are

$$\gamma_0 = 1,\ \gamma_1 = -\frac{1}{12},\ \gamma_2 = \frac{1}{288},\ \gamma_3 = \frac{139}{51840},\ \gamma_4 = -\frac{571}{2488320},\ \gamma_5 = -\frac{163879}{209018880}.$$

Substituting (2.3.8) into (2.2.10) and using (2.3.9), (2.3.10) and (2.3.12), we obtain

$$Z(t) \;=\; \text{Re}\,e^{i\vartheta}\{\sum_{n=1}^{\infty}n^{-s}\text{erfc}\,(\frac{1}{2}\eta_n\sqrt{s}; m) + \frac{e^{\frac{1}{2}i\phi s}}{\sqrt{\pi s}}(\frac{2\pi e}{s})^{\frac{1}{2}s}[\sum_{r=0}^{m-1}(-)^{r-1}(\frac{s}{2})^{-r}A_r(s) + (\frac{s}{2})^{-m}\bar{R}_m]\}. \quad (2.3.13)$$

The coefficients $A_r(s)$ and the remainder $\bar{R}_m$ are defined by

$$A_r(s) = (-)^r\gamma_r - 2\sum_{n=1}^{\infty}\exp\,(-\pi n^2 e^{i\phi})\frac{Q_r(\mu_n)}{\mu_n^{2r+1}}, \quad (2.3.14)$$

$$\bar{R}_m = 2\sum_{n=1}^{\infty} \exp(-\pi n^2 e^{i\phi}) G_m(\frac{s}{2}, \eta_n) - H_m(\frac{s}{2}). \tag{2.3.15}$$

We can use the reflection formula for the modified complementary error function

$$\operatorname{erfc}(z; m) + \operatorname{erfc}(-z; m) = 2,$$

to make the sum of modified complementary error functions in (2.3.13) contain the finite main sum over N terms,

$$2\sum_{n=1}^{N} n^{-\frac{1}{2}} \cos(\vartheta(t) - t\log n),$$

where N is an arbitrary positive integer. Define the absolutely convergent sum, when $s = \frac{1}{2} + it$,

$$E_m(t; N) = \sum_{n=1}^{\infty} \delta_N n^{-s} \operatorname{erfc}(\frac{1}{2}\delta_N \eta_n \sqrt{s}; m), \tag{2.3.16}$$

where

$$\delta_N = \begin{cases} -1 & n \leq N \\ 1 & n > N. \end{cases} \tag{2.3.17}$$

Then the following identity can be derived

$$\sum_{n=1}^{\infty} n^{-s} \operatorname{erfc}(\frac{1}{2}\eta\sqrt{s}; m) = 2\sum_{n=1}^{N} n^{-s} + E_m(t; N).$$

This enables us to write $Z(t)$ from (2.3.13) in the form [Paris and Cang, 1996]

$$\begin{aligned} Z(t) &= 2\sum_{n=1}^{N} \frac{\cos(\vartheta(t) - t\log n)}{\sqrt{n}} + \operatorname{Re} e^{i\vartheta(t)}\{E_m(t; N) \\ &+ \frac{e^{\frac{1}{2}i\phi s}}{\sqrt{\pi s}} (\frac{2\pi e}{s})^{\frac{s}{2}} [\sum_{r=0}^{m-1} (-)^{r-1} (\frac{s}{2})^{-r} A_r(s) + (\frac{s}{2})^{-m} \bar{R}_m]\}, \end{aligned} \tag{2.3.18}$$

where $m = 1, 2, \cdots,\ |\phi| \leq \frac{\pi}{2}$ and we recall that in this formula $s = \frac{1}{2} + it$.

2.4 A Special Case of the Formula (2.3.18)

2.4.1 The Case of (2.3.18) for $\phi = \frac{\pi}{2}$

We observe that the factor $e^{\frac{1}{2}i\phi s}(2\pi e/s)^{\frac{s}{2}}/\sqrt{\pi s}$ appearing in (2.3.18) contains the exponential term $\exp\{(\frac{1}{4}\pi - \frac{1}{2}\phi)t\}$ for large t. There is a compensating factor which also appears in the terms of $E_m(t;N)$ when $\eta_n\sqrt{s} > 1$; this results from the asymptotic behaviour of the modified complementary error function [Paris, 1994]. Thus if $\phi < \frac{\pi}{2}$, the factor $e^{\frac{1}{2}i\phi s}(2\pi e/s)^{\frac{s}{2}}/\sqrt{\pi s}$ becomes large as $t \to +\infty$. In order to avoid this numerically large term, we are effectively forced to set $\phi = \frac{\pi}{2}$. We also can choose $\phi = \frac{\pi}{2} - \phi_\epsilon$, where ϕ_ϵ is a very small argument. However, the compensating coefficients (2.3.14) have not been computed.

Throughout the rest of this chapter, we accordingly let $\phi = \frac{\pi}{2}$ and define the variable w by

$$w^2 \equiv -\frac{1}{2}\pi i s = \frac{1}{2}\pi(t - \frac{1}{2}i). \tag{2.4.1}$$

Then, the coefficients $A_r(s)$ in (2.3.14) are given by the simpler form

$$A_r(s) = (-)^r\gamma_r - 2\sum_{n=1}^{\infty}(-)^n\frac{Q_r(\mu_n)}{\mu_n^{2r+1}}, \qquad \mu_n = (\frac{\pi n}{w})^2 - 1. \tag{2.4.2}$$

For convenience, define

$$A_r(s) = 2^{-2r}w^{2r+1}B_r(w). \tag{2.4.3}$$

The expansion for $Z(t)$ in (2.3.18) on the critical line then takes the final form

$$\begin{aligned} Z(t) &= 2\sum_{n=1}^{N}\frac{\cos(\vartheta(t) - t\log n)}{\sqrt{n}} + \mathrm{Re}\, e^{i\vartheta(t)}\{E_m(t;N) \\ &\quad -(2i)^{-\frac{1}{2}}(\frac{\pi}{w})^s e^{\frac{s}{2}}[\sum_{r=0}^{m-1}(\pi i/4)^r B_r(w) + (w/2)^{-2m-1}R_m]\}, \end{aligned} \tag{2.4.4}$$

where the remainder R_m is defined by

$$R_m = (-)^{m+1}\frac{1}{2}(\pi i/4)^m\bar{R}_m, \tag{2.4.5}$$

with $\bar{R}_m$ given in (2.3.15), $m = 1, 2 \cdots$, N is an arbitrary positive integer and $B_r(w)$ are coefficients which are evaluated in the next section.

2.4.2 Calculation of the Coefficients $B_r(w)$ for $\phi = \frac{\pi}{2}$

If $\phi = \frac{\pi}{2}$, the coefficients $A_r(s)$ given in (2.4.2) include $Q_r(\mu)$ defined in (2.3.4). The polynomial $Q_r(\mu)$ is of degree $2r$ in μ and can be written as

$$Q_r(\mu) = \sum_{k=0}^{2r}\alpha_k^{(r)}\mu^k, \tag{2.4.6}$$

where the coefficients $\alpha_k^{(r)}$ satisfy the recurrence relation which is given in (5.2.5); the values of $\alpha_k^{(r)}$ are also presented in Table 5.1 for $0 \leq r \leq 10$, $0 \leq k \leq 2r$. Substitution of (2.3.6) into (2.3.4) then leads to the following polynomials

$$\begin{aligned}
Q_0(\mu) &= 1, \\
Q_1(\mu) &= 1+\mu+\frac{1}{12}\mu^2, \\
Q_2(\mu) &= 3+5\mu+\frac{25}{12}\mu^2+\frac{1}{12}\mu^3+\frac{1}{288}\mu^4, \\
Q_3(\mu) &= 15+35\mu+\frac{105}{4}\mu^2+\frac{77}{12}\mu^3+\frac{49}{288}\mu^4+\frac{1}{288}\mu^5-\frac{139}{51840}\mu^6, \\
Q_4(\mu) &= 105+315\mu+\frac{1365}{4}\mu^2+\frac{1883}{12}\mu^3+\frac{2513}{96}\mu^4+\frac{149}{288}\mu^5+\frac{221}{51840}\mu^6 \\
&\quad -\frac{139}{51840}\mu^7-\frac{571}{2488320}\mu^8, \\
&\vdots
\end{aligned} \tag{2.4.7}$$

It is noted that $\alpha_k^{(r)}$ satisfy the condition

$$\sum_{k=0}^{2r-1}(-)^k\alpha_k^{(r)}=0, \quad (r\geq 1). \tag{2.4.8}$$

The details of the proof of (2.4.8) will be given in Section 5.2.

Substitution of (2.4.6) into (2.4.2) and use of (2.4.8) shows that

$$\begin{aligned}
A_r(s) &= (-)^r\gamma_r+2\sum_{k=0}^{2r}(-)^k\alpha_k^{(r)}\sum_{n=1}^{\infty}(-)^n\left(\frac{w^2}{w^2-\pi^2n^2}\right)^{2r+1-k} \\
&= \sum_{k=0}^{2r}(-)^k\alpha_k^{(r)}\sum_{n=-\infty}^{\infty}(-)^n\left(\frac{w^2}{w^2-\pi^2n^2}\right)^{2r+1-k}.
\end{aligned} \tag{2.4.9}$$

The inner sum can be expressed in terms of the functions $S_k(w)$ defined by

$$S_k(w)=2^{k-1}\sum_{n=-\infty}^{\infty}(-)^n\frac{w^k}{(w^2-\pi^2n^2)^k}, \quad k=1,\,2,\,\cdots. \tag{2.4.10}$$

$S_k(w)$ can be written in terms of $\operatorname{cosec} w$ and its derivatives. The recursive relation for $S_k(w)$ is

$$\begin{aligned}
S_{k+1}(w) &= \frac{1}{w}S_k(w)-\frac{1}{k}\frac{dS_k(w)}{dw}, \quad k=1,\,2,\,\cdots, \\
S_1(w) &= \operatorname{cosec} w.
\end{aligned} \tag{2.4.11}$$

The proof of (2.4.11) is given in A.1.

By the definition of (2.4.3), the coefficients $B_r(s)$ can then be finally written in the form

$$B_r(w) = \sum_{k=0}^{2r} (-)^k \alpha_k^{(r)} (w/2)^{-k} S_{2r+1-k}(w). \tag{2.4.12}$$

The explicit representation of the first few coefficients $B_r(w)$ is therefore given by

$$\begin{aligned}
B_0(w) &= S_1(w), \\
B_1(w) &= S_3(w) - \frac{S_2(w)}{w/2} + \frac{1}{12}\frac{S_1(w)}{(w/2)^2}, \\
B_2(w) &= 3S_5(w) - 5\frac{S_4(w)}{w/2} + \frac{25}{12}\frac{S_3(w)}{(w/2)^2} - \frac{1}{12}\frac{S_2(w)}{(w/2)^3} + \frac{1}{288}\frac{S_1(w)}{(w/2)^4}, \\
B_3(w) &= 15S_7(w) - 35\frac{S_6(w)}{w/2} + \frac{105}{4}\frac{S_5(w)}{(w/2)^2} - \frac{77}{12}\frac{S_4(w)}{(w/2)^3} + \frac{49}{288}\frac{S_3(w)}{(w/2)^4} \\
&\quad - \frac{1}{288}\frac{S_2(w)}{(w/2)^5} - \frac{139}{51840}\frac{S_1(w)}{(w/2)^6}, \\
B_4(w) &= 105S_9(w) - 315\frac{S_8(w)}{w/2} + \frac{1365}{4}\frac{S_7(w)}{(w/2)^2} - \frac{1883}{12}\frac{S_6(w)}{(w/2)^3} + \frac{2513}{96}\frac{S_5(w)}{(w/2)^4} \\
&\quad - \frac{149}{288}\frac{S_4(w)}{(w/2)^5} + \frac{221}{51840}\frac{S_3(w)}{(w/2)^6} + \frac{139}{51840}\frac{S_2(w)}{(w/2)^7} \\
&\quad - \frac{571}{2488320}\frac{S_1(w)}{(w/2)^8}.
\end{aligned} \tag{2.4.13}$$

2.5 Discussion of $Q(s/2, \pi n^2 i)$ and $E_m(t; N)$

We notice that the sum in (2.2.1) represents the original Dirichlet series "smoothed" by the incomplete gamma function $Q(\frac{s}{2}, \pi n^2 i)$. For large t, the behaviour of this incomplete gamma function changes abruptly in the neighbourhood of its transition point given by $\frac{s}{2} = \pi n^2 i$; that is, when n attains roughly the Riemann-Siegel cut-off value N_t defined in (1.3.6). Then for values of n near N_t, we obtain from (2.3.1) the approximation

$$\begin{aligned}
Q(\frac{s}{2}, \pi n^2 i) &\sim \frac{1}{2}\operatorname{erfc}\,[\pi t^{-\frac{1}{2}}(n^2 - \frac{t}{2\pi})e^{\frac{\pi}{4}i}] \\
&\approx \frac{1}{2}\operatorname{erfc}\,[\sqrt{2\pi i}(n - N_t - p(t))], \quad t \to +\infty.
\end{aligned} \tag{2.5.1}$$

Thus, for large t, we have $Q(\frac{s}{2}, \pi n^2 i) \sim 1$ when $n \leq N_t$; while $Q(\frac{s}{2}, \pi n^2 i)$ decays to zero when $n \geq N_t$. This means that the terms in the smoothed Dirichlet series

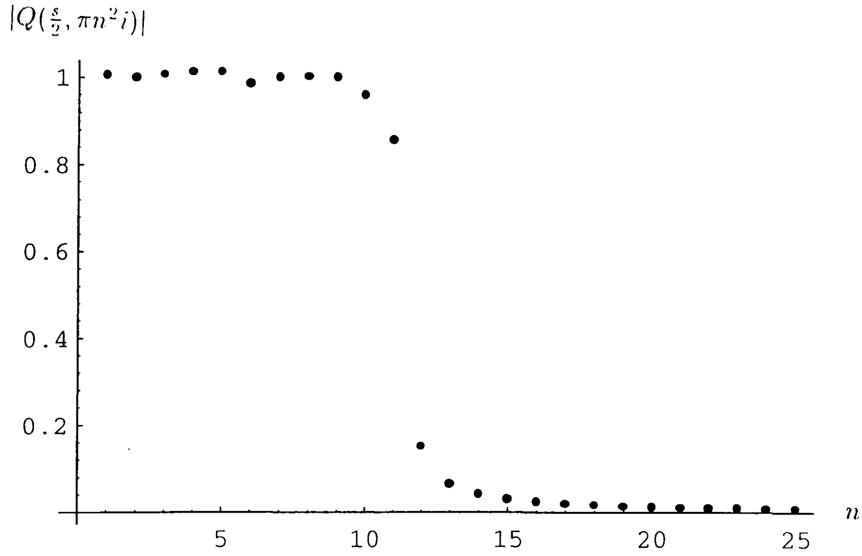


Figure 2.1: The behaviour of $|Q(\frac{s}{2}, \pi n^2 i)|$ for the case $s = \frac{1}{2} + 800i$ when $n = 1, 2, \cdots, 25$

effectively "switch off" when n is near N_t. This behaviour of $Q(\frac{s}{2}, \pi n^2 i)$ is illustrated in Fig. 2.1 for the particular case $t = 800$.

The rate of convergence of the sum $E_m(t; N)$ is a crucial factor in (2.4.4) when computing $Z(t)$ for large t. From the asymptotic behaviour $\text{erfc}\,(z; m) \sim (2/\pi)^{\frac{1}{2}} D_m (2z^2)^{-m-\frac{1}{2}} e^{-z^2}$ valid as $z \to \infty$ in $|\arg z| < \frac{3}{4}\pi$ together with the result, from (2.3.3), $\frac{1}{2}\eta_n^2 \sim \lambda = 2\pi n^2 i/s$ as $n \to \infty$, we find that the terms in $E_m(t; N)$ ultimately lose their n^{-s} dependence to behave like

$$\begin{aligned} |n^{-s} \text{erfc}\,(\frac{1}{2}\eta_n \sqrt{s}; m)| &\sim (2/\pi)^{\frac{1}{2}} |D_m|\, |(\frac{1}{2}\eta_n^2 s)^{-m-\frac{1}{2}}| (t/2\pi)^{-\frac{1}{4}} \\ &= (2\pi n^2)^{-m-\frac{1}{2}} O((t/2\pi)^{-\frac{1}{4}}). \end{aligned} \quad (2.5.2)$$

Thus, although the decay of the terms in $E_m(t; N)$ is algebraic rather than exponential (since the phase of the modified complementary error functions is $\simeq \pi/4$ for large t), this algebraic decay is controlled by n^{-2m-1} together with a scaling factor depending weakly on t like $t^{-\frac{1}{4}}$. This nonetheless represents a considerable improvement of the convergence for moderate values of m.

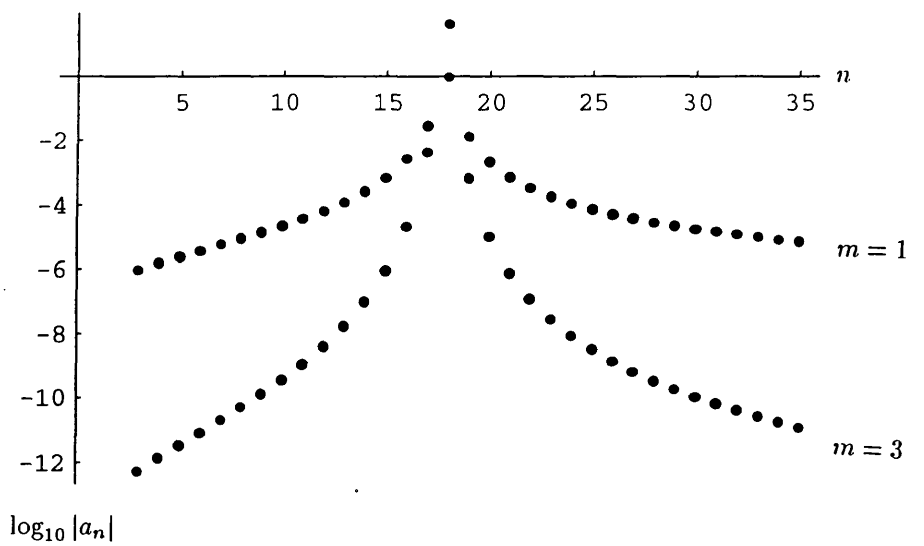
We now choose the number of terms N in the finite main sum in (2.4.4) equal to the Riemann-Siegel cut-off value N_t given in (1.3.6). This choice has the consequence of making the terms in the sum $E_m(t; N)$ decay rapidly either side of $n = N_t$. In Fig.

2.2, we illustrate this decay for different values of m in the particular case $t = 2000$ when $N = N_t$; we also show the consequence of choosing $N < N_t$. An important feature in the calculations is that this rapid fall-off of the terms in $E_m(t; N_t)$ away from $n = N_t$ is essentially independent of t. This can be seen from (2.5.1) which shows that the argument of the modified complementary error functions in $E_m(t; N)$ has the same form when $n \simeq N_t$, thus revealing that the decay on both sides is controlled only by the difference $|n - N_t|$.

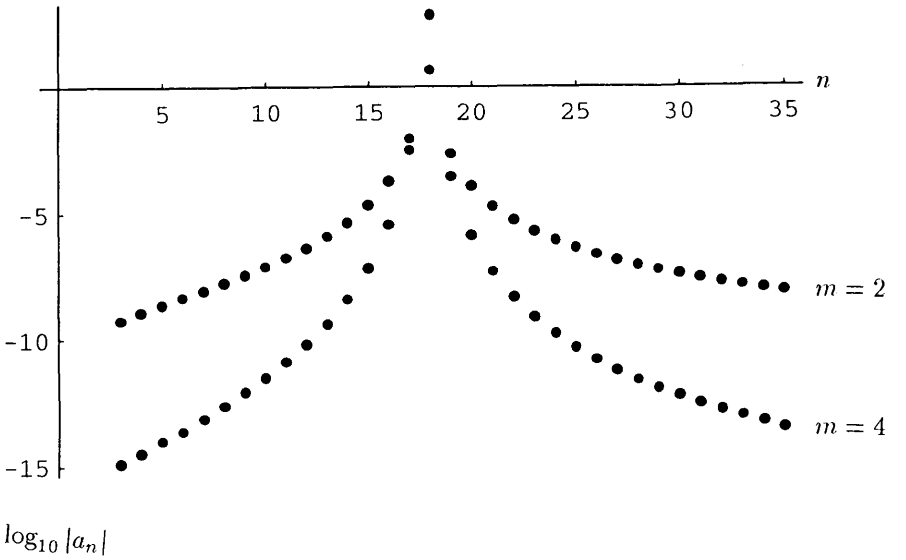
We truncate the sum $E_m(t; N_t)$ at n_1 and n_2, where $n_1 < N_t < n_2$ are determined by when the modulus of the argument of the modified complementary error function attains a prescribed value, that is $|\frac{1}{2}\eta_n\sqrt{s}| \simeq K\sqrt{2\pi}$, where K is an integer. From (2.5.1), this occurs roughly when $|n - N_t| = K$; we therefore take $n_{1,2} = N_t \pm K$. From (2.5.2), we have that the magnitude of the terms in $E_m(t; N_t)$ at these truncation values is then given approximately by

$$(2/\pi)^{\frac{1}{2}}|D_m|(t/2\pi)^{-\frac{1}{4}}(4\pi K^2)^{-m-\frac{1}{2}}.$$

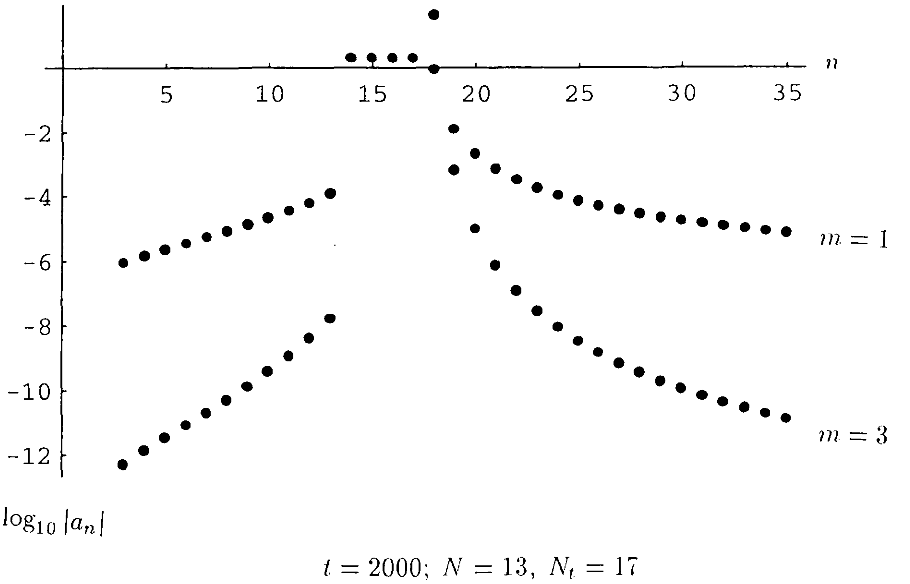
For example, if we choose $K = 10$ this corresponds to neglecting terms in $E_m(t; N)$ when $t \gg 1$ of magnitude smaller than roughly $2.63 \times 10^{-10}t^{-\frac{1}{4}}$ when $m = 3$ and $1.07 \times 10^{-14}t^{-\frac{1}{4}}$ when $m = 5$.



$t = 2000$; $N = N_t = 17$



$t = 2000;\ N = N_t = 17$



$t = 2000;\ N = 13,\ N_t = 17$

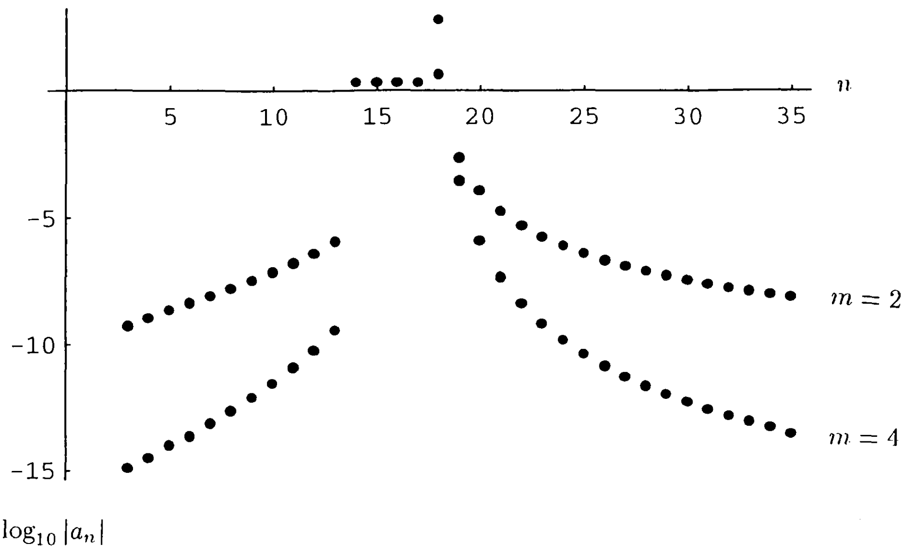


Figure 2.2: The behaviour of $a_n = \text{erfc}\,(\frac{1}{2}\delta_N \eta_n \sqrt{s}; m)$ appearing in $E_m(t; N_t)$ for (a) $m = 1$, $m = 3$; $m = 2$, $m = 4$ when $t = 2000$, $N = N_t$, (b) $m = 1$, $m = 3$; $m = 2$, $m = 4$ when $t = 2000$, $N < N_t$

2.6 Numerical Results of (2.4.4)

To illustrate the accuracy of (2.4.4) truncated after m terms we present the results for the three cases

$$t = 18, \quad t = 7005.08186, \quad t = 250000.$$

The first value of t is very low and is situated approximately midway between the first two non-trivial zeros of $\zeta(s)$. In this case $N_t = 1$, so that summation in $E_m(t; N_t)$ is carried out over $1 \leq n \leq n_2$. The second value corresponds to the first pair of close zeros in between which $Z(t)$ is very small. The third value chosen corresponds to a point high up on the critical line in the truly asymptotic range.

The results are presented in Tables 2.1-2.3 for different values of m and different truncation indices n_1, n_2. For the lowest value of t we computed $\vartheta(t)$ by means of (1.3.2), whereas for large values of t it was sufficient to use the well-known asymptotic expansion for $\vartheta(t)$ given in (1.3.3). For comparison, we give the values of $Z(t)$ computed using **Mathematica** and the values of $Z(t)$ computed using the Riemann-Siegel (R-S) formula with five correction terms.

Table 2.1: Computations of $Z(t)$ from (2.4.4) for $t = 18$

t=18 $\vartheta(t) = 0.0809$	$Z(t)$=2.33679 96899 16 $p(t) = 0.693$	$N_t = 1$
	m=2	
n_2	Z_{approx}	$\|Z(t) - Z_{approx}\|$
2	2.33612 94955	6.7×10^{-4}
3	2.33682 83433	2.8×10^{-5}
4	2.33678 65806	1.3×10^{-5}
5	2.33679 40559	5.6×10^{-6}
	m=3	
n_2	Z_{approx}	$\|Z(t) - Z_{approx}\|$
2	2.33652 96514	2.7×10^{-4}
3	2.33680 73737	7.7×10^{-6}
4	2.33679 91542	5.4×10^{-7}
5	2.33679 99857	3.0×10^{-7}
	m=4	
n_2	Z_{approx}	$\|Z(t) - Z_{approx}\|$
2	2.33688 188644	8.2×10^{-5}
4	2.33679 91036	5.9×10^{-7}
5	2.33679 96660	2.4×10^{-8}
	m=5	
n_2	Z_{approx}	$\|Z(t) - Z_{approx}\|$
2	2.33684 54911	4.6×10^{-5}
3	2.33679 95068	1.8×10^{-7}
4	2.33679 96947	4.8×10^{-9}
5	2.33679 96896	3.7×10^{-10}
R-S value	2.33679 617	3.5×10^{-6}

Table 2.2: Computations of $Z(t)$ from (2.4.4) for $t = 7005.08186$

t=7005.08186 $\vartheta(t) = 21072.6941$	$Z(t) = 0.00396\ 73572\ 77190\ 50701\ 38402$ $p(t) = 0.390$	$N_t = 33$
	m=2	
$n_1,\ n_2$	Z_{approx}	$\lvert Z(t) - Z_{approx}\rvert$
25, 40	0.00396 73466 99530	1.1×10^{-8}
20, 45	0.00396 73581 32790	8.6×10^{-10}
15, 50	0.00396 73571 15795	1.6×10^{-10}
10, 55	0.00396 73573 23010	4.6×10^{-11}
5, 60	0.00396 73572 60638	1.7×10^{-11}
1, 80	0.00396 73572 75878	1.3×10^{-12}
	m=3	
$n_1,\ n_2$	Z_{approx}	$\lvert Z(t) - Z_{approx}\rvert$
25, 40	0.00396 73570 94439 13302	1.8×10^{-10}
20, 45	0.00396 73572 82710 06236	5.5×10^{-12}
15, 50	0.00396 73572 76642 23029	5.5×10^{-13}
10, 55	0.00396 73572 77288 14461	9.8×10^{-14}
5, 60	0.00396 73572 77165 78273	2.4×10^{-14}
1, 80	0.00396 73572 77189 81290	6.9×10^{-16}
	m=4	
$n_1,\ n_2$	Z_{approx}	$\lvert Z(t) - Z_{approx}\rvert$
25, 40	0.00396 73572 78008 75339 81159	2.9×10^{-12}
20, 45	0.00396 73572 77181 53319 69707	9.0×10^{-15}
15, 50	0.00396 73572 77190 97396 11026	4.7×10^{-16}
10, 55	0.00396 73572 77190 45497 58332	5.2×10^{-17}
5, 60	0.00396 73572 77190 51617 73206	9.2×10^{-18}
1, 80	0.00396 73572 77190 50710 54550	9.2×10^{-20}
1, 120	0.00396 73572 77190 50701 56007	1.8×10^{-21}
	m=5	
$n_1,\ n_2$	Z_{approx}	$\lvert Z(t) - Z_{approx}\rvert$
25, 40	0.00396 73572 77216 38928 99974 24396	2.6×10^{-14}
20, 45	0.00396 73572 77190 39853 44751 62935	1.1×10^{-16}
15, 50	0.00396 73572 77190 51001 29593 39808	3.0×10^{-18}
10, 55	0.00396 73572 77190 50680 40140 69187	2.1×10^{-19}
5, 60	0.00396 73572 77190 50703 94288 18603	2.6×10^{-20}
1, 80	0.00396 73572 77190 50701 39322 22910	9.2×10^{-23}
1, 110	0.00396 73572 77190 50701 38412 28027	1.1×10^{-24}
R-S value	0.00396 73572 77296 1	1.2×10^{-13}

Table 2.3: Computations of $Z(t)$ from (2.4.4) for $t = 250000$

t=250000 $\vartheta(t) = 1198916.9986$	$Z(t) = -0.78556\ 62503\ 91741\ 40097$ $52314\ 33303$ $p(t) = 0.471$	$N_t = 199$
	m=2	
n_1, n_2	Z_{approx}	$\lvert Z(t) - Z_{approx}\rvert$
195, 205	−0.78556 62603 41323 49713	9.9×10^{-9}
185, 215	−0.78556 62504 05226 17559	1.3×10^{-11}
175, 225	−0.78556 62503 91703 94678	3.7×10^{-14}
160, 240	−0.78556 62503 91852 44540	1.1×10^{-13}
150, 250	−0.78556 62503 91797 57472	5.6×10^{-14}
100, 300	−0.78556 62503 91745 87341	4.5×10^{-15}
50, 350	−0.78556 62503 91742 32636	9.2×10^{-16}
1, 400	−0.78556 62503 91741 70624	3.1×10^{-16}
1, 600	−0.78556 62503 91741 41618	1.5×10^{-17}
1, 800	−0.78556 62503 91741 40564	5.0×10^{-18}
	m=3	
n_1, n_2	Z_{approx}	$\lvert Z(t) - Z_{approx}\rvert$
195, 205	−0.78556 62507 86823 34222	4.0×10^{-10}
185, 215	−0.78556 62503 91815 13885	7.4×10^{-14}
175, 225	−0.78556 62503 91741 31837	8.3×10^{-17}
160, 240	−0.78556 62503 91741 49139	9.0×10^{-17}
150, 250	−0.78556 62503 91741 43034	2.9×10^{-17}
100, 300	−0.78556 62503 91741 40156	5.8×10^{-19}
50, 350	−0.78556 62503 91741 40103	5.0×10^{-20}
1, 400	−0.78556 62503 91741 40098 41650	8.9×10^{-21}
1, 450	−0.78556 62503 91741 40097 73547	2.1×10^{-21}
	m=4	
n_1, n_2	Z_{approx}	$\lvert Z(t) - Z_{approx}\rvert$
195, 205	−0.78556 62503 87855 03469 95247 02796	3.9×10^{-12}
185, 215	−0.78556 62503 91741 31051 37297 02262	9.0×10^{-17}
175, 225	−0.78556 62503 91741 40101 32104 92281	3.8×10^{-20}
160, 240	−0.78556 62503 91741 40095 89730 77506	1.6×10^{-20}
150, 250	−0.78556 62503 91741 40097 18392 52601	3.4×10^{-21}
100, 300	−0.78556 62503 91741 40097 52146 40716	1.7×10^{-23}
50, 350	−0.78556 62503 91741 40097 52307 72376	6.6×10^{-25}
1, 400	−0.78556 62503 91741 40097 52313 70093	6.3×10^{-26}
1, 450	−0.78556 62503 91741 40097 72314 23382	9.9×10^{-27}

m=5		
n_1, n_2	Z_{approx}	$\|Z(t) - Z_{approx}\|$
195, 205	−0.78556 62503 91508 46382 69843 63431	2.3×10^{-13}
185, 215	−0.78556 62503 91741 40020 54094 38755	7.7×10^{-19}
175, 225	−0.78556 62503 91741 40097 53615 52319	1.3×10^{-22}
160, 240	−0.78556 62503 91741 40097 52106 42307	2.1×10^{-23}
150, 250	−0.78556 62503 91741 40097 52286 42267	2.8×10^{-24}
100, 300	−0.78556 62503 91741 40097 52314 29814	3.5×10^{-27}
50, 350	−0.78556 62503 91741 40097 52314 33242	6.1×10^{-29}
1, 400	−0.78556 62503 91741 40097 52314 33300	3.0×10^{-30}
1, 450	−0.78556 62503 91741 40097 52314 33303	0.1×10^{-31}
R-S value	−0.78556 62503 91741 39954	1.4×10^{-18}

From Tables 2.1-2.3, it can be seen that for fixed m, the smaller is n_1 or the larger is n_2, the smaller is the error. On the other hand, when n_1 and n_2 are fixed, the error decreases as m increases. When t is large, the asymptotic formula (2.4.4) gives better results than the Riemann-Siegel asymptotic formula does for the same number of correction terms.

2.7 Derivation of an Asymptotic Formula in the Neighbourhood of a Critical Value

The asymptotic formula (2.4.4) is an attractive alternative to the Riemann-Siegel formula and its coefficients $B_r(w)$ can be calculated to as high an order as required by a simple recursive relation (2.4.11), which involves the parameter w defined in (2.4.1). The expansion (2.4.4) is found to yield very accurate results comparable with the Riemann-Siegel formula and Berry-Keating formulas. However, a disadvantage with this formula appears when we attempt to compute $Z(t)$ for values of t which makes w lie close to an integer multiple of π; this corresponds to the transition point of the incomplete gamma function $Q(\frac{s}{2}, \pi n^2 i)$. Thus, when t is large, this arises when $p(t) \simeq 0$ or $p(t) \simeq 1$, that means at a discontinuity in N_t. Since w always has a small imaginary part, although the coefficients $B_r(w)$ (and the sum $E_m(t;N)$) are not singular for such critical t values, there will be loss of accuracy due to round-

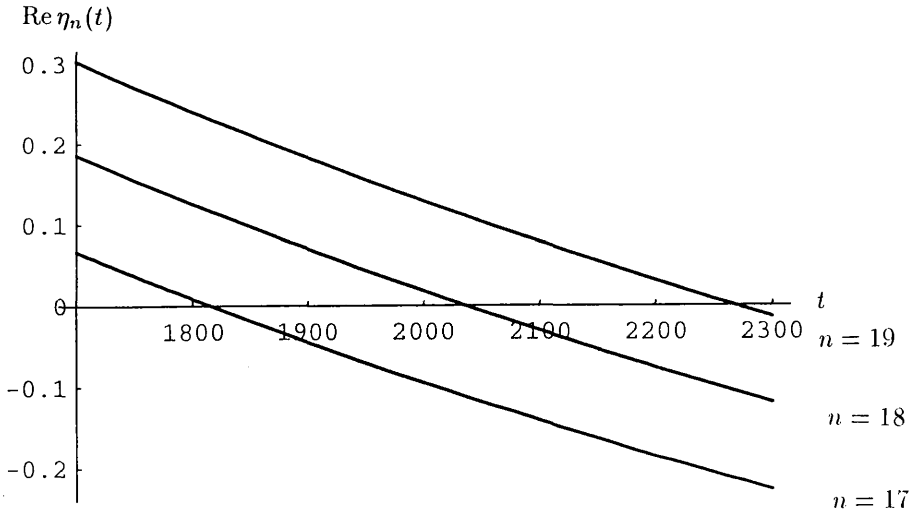


Figure 2.3: The behaviour of the real part of $\eta_n(t)$, when $n = 17, 18, 19$

off error when computing with fixed-decimal arithmetic which represents a major inconvenience.

In order to deal with this difficulty, we propose a modification of the asymptotic formula for $Z(t)$. We do this by subtracting off from $E_m(t; N)$ the terms responsible for this behaviour and combining them with corresponding terms in the coefficients $B_r(w)$. In Fig. 2.3, we show the behaviour of the real part of $\eta_n(t)$, for different n as a function of t in the neighbourhood containing the discontinuous change from $N_t - 1$ to N_t. The imaginary part of $\eta_n(t)$ is very small and slowly varying [see Appendix A.2]. It is seen that $\mathrm{Re}\,\eta_n(t)$ corresponding to $n = 17, 18, 19$ becomes very small near this change and inclusion of these terms in the modified complementary error functions in (2.3.16) will result in $E_m(t; N)$ becoming large.

We modify $E_m(t; N)$ by deleting from the sum the term involving D_r in the modified complementary error function which corresponds to $n = n_*$ and define

$$E_m^*(t; N) = \Big(\sum_{n=0}^{n_*-1} + \sum_{n=n_*+1}^{\infty}\Big)\delta_N n^{-s} \mathrm{erfc}\,(\frac{1}{2}\delta_N \eta_n \sqrt{s}; m) + \delta_N n_*^{-s} \mathrm{erfc}\,(\frac{1}{2}\delta_N \eta_* \sqrt{s}). \tag{2.7.1}$$

For simplicity in presentation, we have written η_* for the value of η_n when $n = n_*$. Comparing (2.3.16) with (2.7.1) and using (2.4.1), we obtain

$$\begin{aligned} E_m(t;N) &= E_m^*(t;N) - 2n_*^{-s}\frac{e^{-\frac{1}{2}a\eta_*^2}}{\sqrt{\pi s}}\sum_{r=0}^{m-1}\frac{D_r(\frac{1}{2}s)^{-r}}{\eta_*^{2r+1}} \\ &= E_m^*(t;N) + (-)^{n_*-1}(2i)^{-\frac{1}{2}}(\frac{\pi}{w})^s e^{\frac{s}{2}}\sum_{r=0}^{m-1}\frac{(-)^r(\pi i/4)^r D_r}{(\frac{1}{2}w\eta_*)^{2r+1}}. \quad (2.7.2) \end{aligned}$$

Substitution of (2.7.2) into (2.4.4), then gives

$$\begin{aligned} Z(t) &= 2\sum_{n=1}^{N}\frac{\cos(\vartheta(t) - t\log n)}{\sqrt{n}} + \mathrm{Re}\, e^{i\vartheta(t)}\{E_m^*(t;N) - (2i)^{-\frac{1}{2}}(\frac{\pi}{w})^s e^{\frac{s}{2}} \\ &\quad [\sum_{r=0}^{m-1}(\pi i/4)^r(B_r(w) + (-)^{n_*+r}\frac{D_r}{(\frac{1}{2}w\eta_*)^{2r+1}}) + (w/2)^{-2m-1}R_m]\} \quad (2.7.3) \end{aligned}$$

Let

$$B_r^*(w) = B_r(w) + (-)^{n_*+r}\frac{D_r}{(\frac{1}{2}w\eta_*)^{2r+1}}. \quad (2.7.4)$$

The final asymptotic formula for $Z(t)$ can be written as the following when t is in the neighbourhood of the critical value

$$\begin{aligned} Z(t) &= 2\sum_{n=1}^{N}\frac{\cos(\vartheta - t\log n)}{\sqrt{n}} + \mathrm{Re}\, e^{i\vartheta}\{E_m^*(t;N) \\ &\quad -(2i)^{-\frac{1}{2}}(\frac{\pi}{w})^s e^{\frac{s}{2}}[\sum_{r=0}^{m-1}(\pi i/4)^r B_r^*(w) + (w/2)^{-2m-1}R_m]\}, \quad (2.7.5) \end{aligned}$$

where $B_r^*(w)$ are coefficients evaluated in the next section, R_m is the remainder given in (2.4.5) and N is an arbitrary positive integer.

2.8 Calculation of the Coefficients $B_r^*(w)$

In order to use the asymptotic formula (2.7.5), the coefficients $B_r^*(w)$ should be calculated first. Using the definition of the coefficients $c_r(\eta)$ in (2.3.2), (2.7.4) can be written as

$$B_r^*(w) = B_r(w) + (-)^{n_*}\frac{Q_r(\mu_*)}{(\frac{1}{2}w\mu_*)^{2r+1}} - (-)^{n_*+r}(w/2)^{-2r-1}c_r(\eta_*), \quad (2.8.1)$$

where η_* and μ_* denote the values of η_n and μ_n when $n = n_*$.

For convenience, define

$$\bar{B}_r^*(w) = B_r(w) + (-)^{n_*}\frac{Q_r(\mu_*)}{(\frac{1}{2}w\mu_*)^{2r+1}}, \quad (2.8.2)$$

substituting (2.8.2) into (2.8.1) gives

$$B_r^*(w) = \bar{B}_r^*(w) + (-)^{n_*+r-1}(w/2)^{-2r-1}c_r(\eta_*). \tag{2.8.3}$$

From (2.4.6), (2.4.10) and (2.4.12), the first part of (2.8.3) can be written as

$$\bar{B}_r^*(w) = \sum_{k=0}^{2r}(-)^k\alpha_k^{(r)}(w/2)^{-k}S_{2r+1-k}^*(w), \tag{2.8.4}$$

where

$$S_k^*(w) = 2^{k-1}\sum_{-\infty}^{\infty}{}'(-)^n\frac{w^k}{(w^2-\pi^2n^2)^k}, \quad k=1,\,2,\,\cdots. \tag{2.8.5}$$

with the prime on the summation sign denoting the deletion of the terms corresponding to $n=\pm n_*$. Thus the coefficients $\bar{B}_r^*(w)$ are obtained from the same expressions for $B_r(w)$ in (2.4.12), but with the sums $S_k(w)$ replaced by the deleted sums $S_k^*(w)$. Clearly, the sums $S_k^*(w)$ satisfy the same recursion as $S_k(w)$ in (2.4.11), and

$$S_1^*(w) = \operatorname{cosec} w - \frac{2(-)^{n_*}w}{w^2-\pi^2n_*^2}. \tag{2.8.6}$$

To calculate the sums $S_k^*(w)$, we choose n_* to be the value of N_t immediately to the right of the critical t value, where N_t changes discontinuously by unity. Since t is close to the critical value, this is equivalent to w being close to πn_*. Let $w = \pi n_* + \epsilon(t)$, where $\epsilon(t)$ is a function of t and is very small complex variable in the neighbourhood of the transition. Then it is easily established that

$$S_1^*(w) = (-)^{n_*}(\operatorname{cosec}\epsilon - \frac{1}{\epsilon} - \frac{1}{2\pi n_* + \epsilon}), \tag{2.8.7}$$

with the sums $S_k^*(w)$ for $k \geq 2$ satisfying the same recursion as $S_k(w)$ in (2.4.11). All the sums $S_k^*(w)$ are thus seen to be finite as $\epsilon(t) \to 0$ and can be computed either by using higher-precision arithmetic or by employing a straightforward expansion in powers of $\epsilon(t)$. As $\epsilon \to 0$, the expansion is

$$\operatorname{cosec}\epsilon - \frac{1}{\epsilon} \sim \frac{1}{6}\epsilon + \frac{7}{360}\epsilon^3 + \cdots + \frac{(-)^{n-1}2(2^{2n-1}-1)B_{2n}}{(2n)!}\epsilon^{2n-1} + O(\epsilon^{2n}),$$

where B_{2n} denotes the Bernoulli numbers.

Next, the calculation of the second part of (2.8.3) is given. The coefficients $c_r(\eta_*)$ have a removable singularity at the origin $\eta = 0$ and are consequently analytic functions of η. Since η_* is close to zero, $c_r(\eta_*)$ can be expanded in powers of η_* in the form [Temme, 1979]

$$c_r(\eta_*) = \sum_{k=0}^{\infty} \beta_{r\,k}\eta_*^k, \quad |\eta_*| < 2\sqrt{\pi}. \tag{2.8.8}$$

For $r = 0$, we have

$$\beta_{0\,0} = -\frac{1}{3}, \quad \beta_{0\,k} = (k+2)\alpha_{k+2}, \quad k \geq 1, \tag{2.8.9}$$

where α_k are the coefficients of the expansion of $\mu(\eta_*)$, that is

$$\mu(\eta_*) = \alpha_1\eta_* + \alpha_2\eta_*^2 + \cdots. \tag{2.8.10}$$

The first few α_k are

$$\alpha_1 = 1,\ \alpha_2 = \frac{1}{3},\ \alpha_3 = \frac{1}{36},\ \alpha_4 = -\frac{1}{270},\ \alpha_5 = \frac{1}{4320}. \tag{2.8.11}$$

For general $r \geq 1$, the recursion is

$$\beta_{r\,k} = \gamma_r\beta_{0\,k} + (k+2)\beta_{r-1\,k+2}, \quad k \geq 0,$$

or, in terms of $\beta_{0\,k}$,

$$\beta_{r\,k} = \gamma_r\beta_{0\,k} + \gamma_{r-1}(k+2)\beta_{0\,k+2} + \cdots + \gamma_0(k+2)\cdots(k+2r)\beta_{0\,k+2r}. \tag{2.8.12}$$

The values of $\beta_{r\,k}$ are shown in Table 2.4 for $0 \leq r \leq 5$ and $0 \leq k \leq 9$.

2.9 Discussion of a Choice of N in (2.7.5)

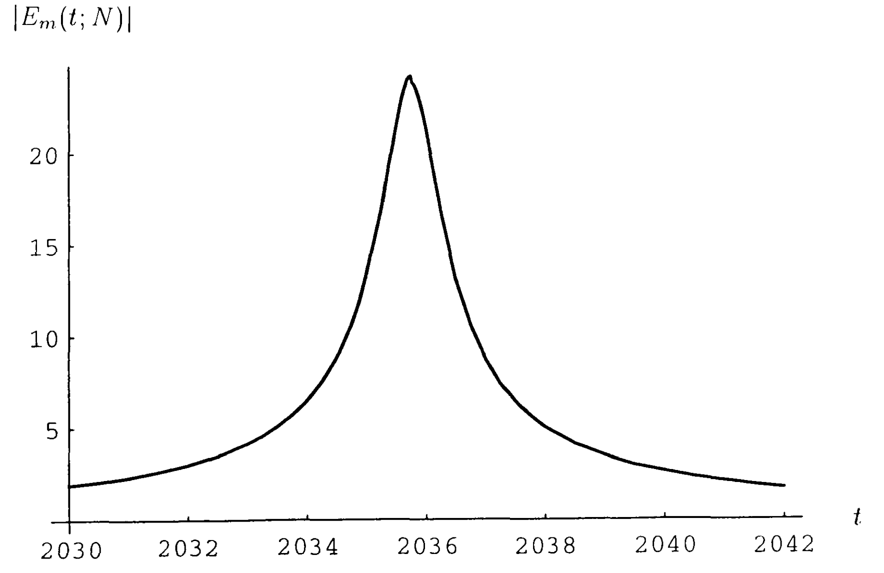
When t is in the neighbourhood of the critical value $2\pi N_t^2$, $Z(t)$ can be computed using the asymptotic formula (2.7.5). For a fixed t, the fixed positive integer N_t can be computed by $\text{int}\,[(t/2\pi)^{\frac{1}{2}}]$, which is the Riemann-Siegal cut off value. We also known $(t/2\pi)^{\frac{1}{2}} = N_t + p(t)$ given in (1.3.6). If $p(t) < 0.5$, we set $n_* = N_t$. If $p(t) \geq 0.5$, then t is in the neighbourhood of $2\pi(N_t+1)^2$. Thus, we can set $n_* = N_t + 1$. When t is chosen on the right side of $2\pi N_t^2$, Fig. 2.3 shows that $\text{Re}\,(\eta_{N_t})$ is very small, which means η_{N_t} is very small, since the imaginary part of η_* is very small. It can be seen that the term which corresponds to N_t in the sum

Table 2.4: The coefficients $\beta_{r\,k}$ for $0 \le r \le 5$ and $0 \le k \le 9$

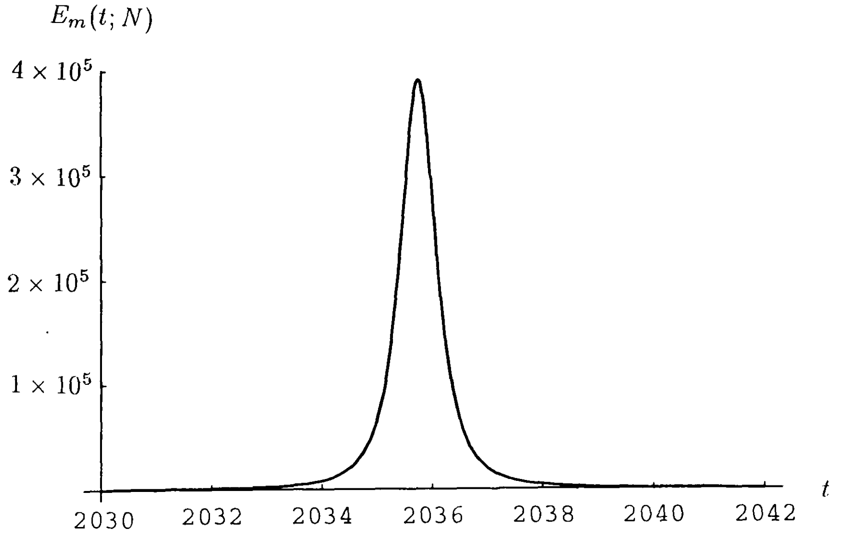
$r \setminus k$					
0	$k=0$ $-\frac{1}{3}$ $k=5$ $-\frac{139}{777600}$	$k=1$ $\frac{1}{12}$ $=6$ $\frac{1}{25515}$	$k=2$ $-\frac{2}{135}$ $k=7$ $-\frac{571}{261273600}$	$k=3$ $\frac{1}{864}$ $k=8$ $-\frac{281}{151559100}$	$k=4$ $\frac{1}{2835}$ $k=9$ $\frac{163879}{197522841600}$
1	$k=0$ $-\frac{1}{540}$ $k=5$ $-\frac{1}{2488320}$	$k=1$ $-\frac{1}{288}$ $k=6$ $-\frac{2743}{151559100}$	$k=2$ $\frac{1}{378}$ $k=7$ $\frac{41969}{5486745600}$	$k=3$ $-\frac{77}{77760}$ $k=8$ $-\frac{11}{6823440}$	$k=4$ $\frac{1}{4860}$ $k=9$ $\frac{47207}{10158317568000}$
2	$k=0$ $\frac{25}{6048}$ $k=5$ $\frac{5531}{104509440}$	$k=1$ $-\frac{139}{51840}$ $k=6$ $-\frac{1219}{95528160}$	$k=2$ $\frac{1}{1296}$ $k=7$ $\frac{19321}{564350976000}$	$k=3$ $\frac{1}{497664}$ $k=8$ $\frac{121}{88179840}$	$k=4$ $-\frac{6199}{57736800}$ $k=9$ $-\frac{5118973}{8126654054400}$
3	$k=0$ $\frac{101}{155520}$ $k=5$ $-\frac{19321}{80621568000}$	$k=1$ $\frac{571}{2488320}$ $k=6$ $\frac{14659}{1322697600}$	$k=2$ $-\frac{54179}{115473600}$ $k=7$ $-\frac{19215991}{3386105856000}$	$k=3$ $\frac{41969}{156764160}$ $k=8$ $\frac{201596239}{141660912960000}$	$k=4$ $-\frac{20639}{272937600}$ $k=9$ $-\frac{326041}{117023818338336000}$
4	$k=0$ $-\frac{3184811}{3695155200}$ $k=5$ $-\frac{5118973}{128994508800}$	$k=1$ $\frac{163879}{209018880}$ $k=6$ $\frac{6445983473}{566643651840000}$	$k=2$ $-\frac{8707}{29113344}$ $k=7$ $-\frac{326041}{1300264648704000}$	$k=3$ $-\frac{47207}{32248627200}$ $k=8$ $-\frac{24542153}{14475602534400}$	$k=4$ $\frac{66931}{1007769600}$ $k=9$ $\frac{30958999147807}{3475607405985792000}$
5	$k=0$ $-\frac{2745493}{8151736320}$ $k=5$ $\frac{12301049}{86684309913600}$	$k=1$ $-\frac{5246819}{75246796800}$ $k=6$ $-\frac{1377474269}{101329217740800}$	$k=2$ $\frac{260801}{940584960}$ $k=7$ $\frac{54189828403651}{67581255116390400000}$	$k=3$ $-\frac{3599669}{18059231232}$ $k=8$ $-\frac{146648839}{63997400678400}$	$k=4$ $\frac{51358922059}{755524869120000}$ $k=9$ $-\frac{26856326641}{825720062512988160000}$

$E_m(t;N)$ becomes very large. On the other hand, when t is chosen on the left side of $2\pi N_t^2$, we also found Re (η_{N_t}) is very small from Fig. 2.3, which means η_{N_t} is very small. It can be seen that the term which corresponds to N_t in the sum $E_m(t;N)$ become very large. Thus, we can fix N to be the value of N_t on either side, since N can be chosen to be an arbitrary positive integer. **It is not necessary to change N_t discontinuously as we pass through a critical t value. When t is on the left side of $2\pi N_t^2$, the choice of η_* in $E_m^*(t;N_t)$ is the principal root of $\eta_* = \{2\mu_* - 2\log(1+\mu_*)\}^{\frac{1}{2}}$ which given in (2.3.3).**

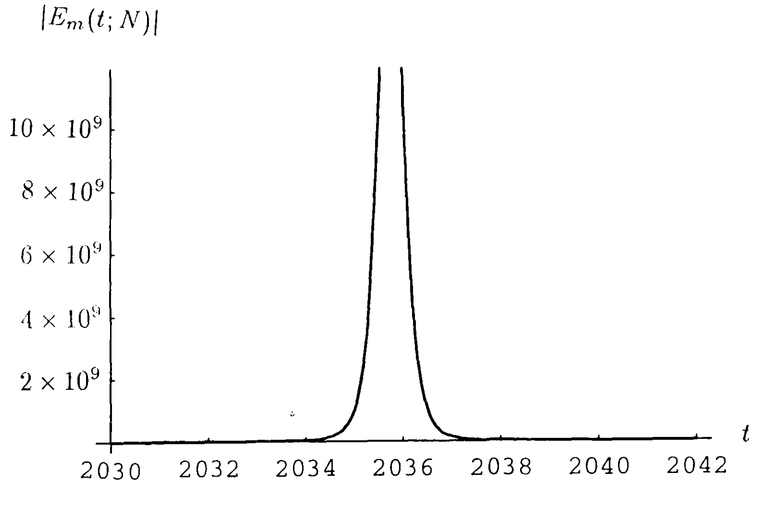
To illustrate the accuracy of the formula (2.7.5), we first show in Fig. 2.4 the behaviour of the modules of $E_m(t;N)$ and $E_m^*(t;N)$ when $m = 1, 2, 3$ for the values of t in the range containing the critical value $t = 2\pi \times N_t^2 \simeq 2035.75$, where $n_* = N_t = 18$. It is seen that without the deletion of the terms corresponding to $n_* = 18$, the modified sum $E_m(t;N)$ increases dramatically in the neighbourhood of the critical value $t \simeq 2035.75$, and the values of modified sum $E_m(t;N)$ become larger as m is larger, while the modified sum $E_m^*(t;N)$ remains $O(1)$ through a critical t value.



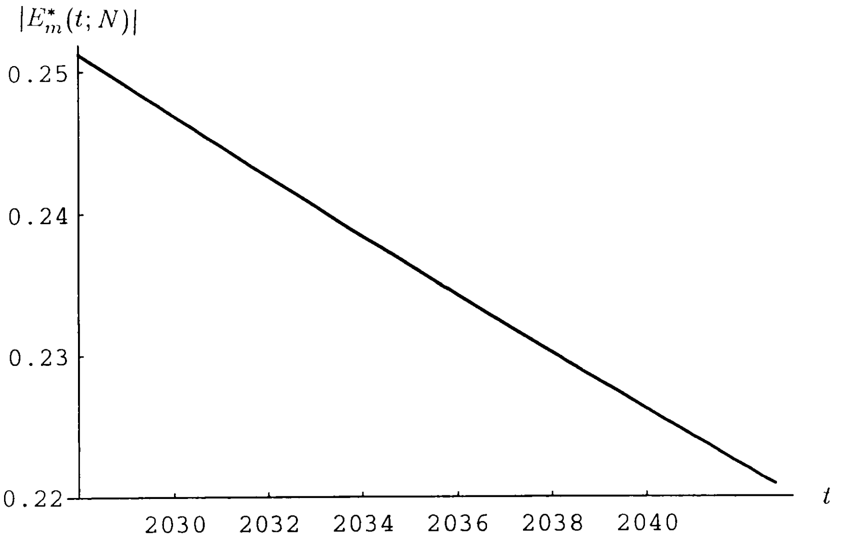
$m = 1$


$E_m(t; N)$
4×10^5
3×10^5
2×10^5
1×10^5
2030
2032
2034
2036
2038
2040
2042
t

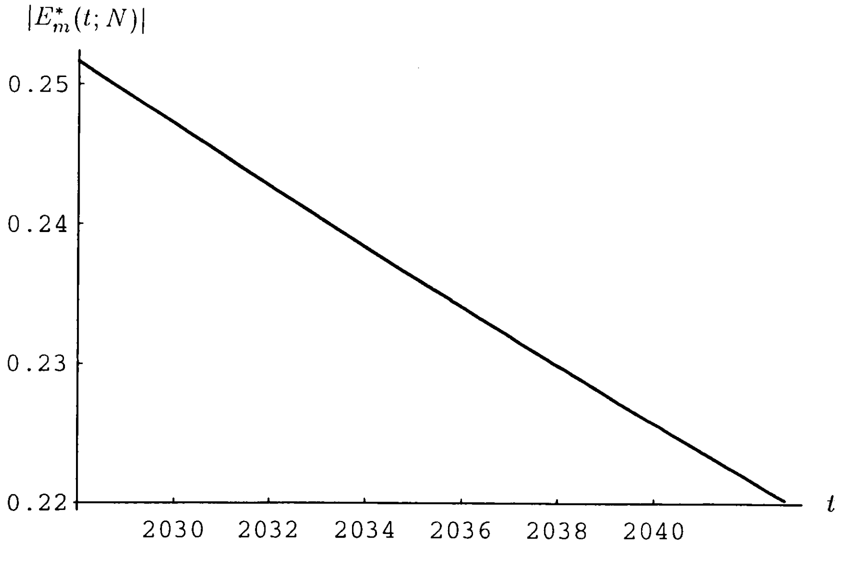
$m = 2$


$|E_m(t; N)|$
10×10^9
8×10^9
6×10^9
4×10^9
2×10^9
2030
2032
2034
2036
2038
2040
2042
t

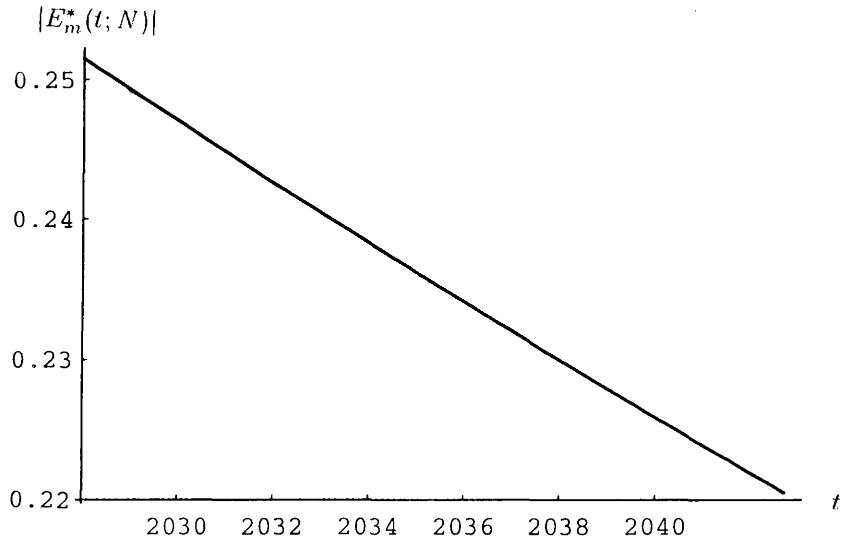
$m = 3$


$|E_m^*(t;N)|$
0.25
0.24
0.23
0.22
2030
2032
2034
2036
2038
2040
t

$m = 1$


$|E_m^*(t;N)|$
0.25
0.24
0.23
0.22
2030
2032
2034
2036
2038
2040
t

$m = 2$



$m = 3$

Figure 2.4: The behaviour of $|E_m(t; N)|$, $|E_m^*(t; N)|$ for $m = 1, 2, 3$ when t is near $2\pi \times N_t^2 \simeq 2035.75$, where $N = N_t = 18$

2.10 Numerical Results of (2.7.5)

To compare the results from (2.7.5) and (2.4.4) when t is in the neighbourhood of a critical value, the numerical results have been illustrated in Table 2.5. The critical t value is chosen as $2\pi \times 18^2 \simeq 2035.75$ and 10 decimal-arithmetic is used for different m. If we choose higher decimal-arithmetic, we can get the same results from (2.7.5) and (2.4.4).

Table 2.5 shows that if the number of decimals is not enough, the error of (2.4.4) increases dramatically from both sides when t is close to the critical value. The formula (2.7.5), however, still maintains the same accuracy even when t tends to the critical value. Table 2.6 illustrates the results of computing $Z(t)$ from (2.7.5) when t is near the critical value $t = 2\pi \times 18^2 \simeq 2035.75$.

2.11 Conclusion

New Riemann-Siegel type asymptotic formulas have been given in this chapter. The asymptotic formula (2.4.4) can be used when t is far away from a critical value

Table 2.5: Comparison of the accuracy in the neighbourhood of the critical value $2\pi \times 18^2$. $Z_{(2.4.4)}(t)$ denotes the values from (2.4.4), $Z_{(2.7.5)}(t)$ denotes the values from (2.7.5), $Z(t)$ denotes the exact value.

m=2				
t	$Z_{(2.4.4)}(t)$	$\lvert Z(t) - Z_{(2.4.4)}(t)\rvert$	$Z_{(2.7.5)}(t)$	$\lvert Z(t) - Z_{(2.7.5)}(t)\rvert$
2030	7.746063334	9.0×10^{-9}	7.746063331	$< 10^{-10}$
2032	−0.695748494	4.9×10^{-8}	−0.695748543	$< 10^{-10}$
2034	−1.976210722	1.1×10^{-6}	−1.976211813	$< 10^{-10}$
2035	0.200325323	1.6×10^{-4}	0.200163787	$< 10^{-10}$
2036	−2.175744936	6.5×10^{-4}	−2.176394634	$< 10^{-10}$
2037	0.776504766	2.5×10^{-6}	0.776507285	$< 10^{-10}$
2038	−3.956261424	1.4×10^{-8}	−3.956261410	$< 10^{-10}$
2040	−3.486183065	3.0×10^{-9}	−3.486183062	$< 10^{-10}$
m=3				
t	$Z_{(2.4.4)}(t)$	$\lvert Z(t) - Z_{(2.4.4)}(t)\rvert$	$Z_{(2.7.5)}(t)$	$\lvert Z(t) - Z_{(2.7.5)}(t)\rvert$
2030	7.746064569	1.2×10^{-6}	7.746063331	$< 10^{-10}$
2032	−0.695711646	3.7×10^{-5}	−0.695748543	$< 10^{-10}$
2034	−1.963834703	1.7×10^{-4}	−1.976211813	$< 10^{-10}$
2035	2.849245814	2.65	0.200163787	$< 10^{-10}$
2036	−45.01366979	42.84	−2.176394634	$< 10^{-10}$
2037	0.773071018	3.4×10^{-3}	0.776507283	$< 10^{-10}$
2038	−3.955925024	3.4×10^{-4}	−3.956261410	$< 10^{-10}$
2040	−3.486177293	5.8×10^{-6}	−3.486183062	$< 10^{-10}$
m=4				
t	$Z_{(2.4.4)}(t)$	$\lvert Z(t) - Z_{(2.4.4)}(t)\rvert$	$Z_{(2.7.5)}(t)$	$\lvert Z(t) - Z_{(2.7.5)}(t)\rvert$
2030	7.741459534	4.6×10^{-3}	7.746063331	$< 10^{-10}$
2032	−0.796235362	0.10	−0.695748543	$< 10^{-10}$
2034	52.55834574	54.53	−1.976211813	$< 10^{-10}$
2035	29854.13126	29853.93	0.200163787	$< 10^{-10}$
2036	2.36×10^6	2.36×10^6	−2.176394634	$< 10^{-10}$
2037	522.98	522.20	0.776507283	$< 10^{-10}$
2038	−5.275710942	1.32	−3.956261410	$< 10^{-10}$
2040	−3.485155385	1.0×10^{-3}	−3.486183062	$< 10^{-10}$

Table 2.6: Computations of $Z(t)$ from (2.7.5) for $t = 2036$

t=2036 $\vartheta(t) = 4866.5282$	$Z(t)$=-2.17639 46337 82407 $p(t) = 0.001$	$N_t = 18$
	m=1	
n_1, n_2	Z_{approx}	$\|Z(t) - Z_{approx}\|$
15, 20	−2.17624 88971	1.4×10^{-4}
10, 25	−2.17640 21868	7.6×10^{-6}
5, 30	−2.17639 29879	1.7×10^{-6}
1, 35	−2.17639 50679	4.3×10^{-7}
	m=2	
n_1, n_2	Z_{approx}	$\|Z(t) - Z_{approx}\|$
15, 20	−2.17639 65132 1174	1.9×10^{-6}
10, 25	−2.17639 46210 9124	1.3×10^{-8}
5, 30	−2.17639 46348 4889	1.1×10^{-9}
1, 35	−2.17639 46335 9074	1.9×10^{-10}
	m=3	
n_1, n_2	Z_{approx}	$\|Z(t) - Z_{approx}\|$
15, 20	−2.17639 48212 4772	1.9×10^{-7}
10, 25	−2.17639 46335 7425	2.1×10^{-10}
5, 30	−2.17639 46337 8953	7.1×10^{-12}
1, 35	−2.17639 46337 8171	7.0×10^{-13}
	m=4	
n_1, n_2	Z_{approx}	$\|Z(t) - Z_{approx}\|$
15, 20	−2.17639 46275 4872	6.2×10^{-9}
10, 25	−2.17639 46337 8334	9.3×10^{-13}
5, 30	−2.17639 46337 82396	1.1×10^{-14}
1, 35	−2.17639 46337 82404	3.1×10^{-15}

$2\pi \times N_t^2$. If t is near a critical value, more decimals are necessary to maintain the accuracy of (2.4.4). However, the asymptotic formula (2.7.5) gives very good results when t is near a critical value. Extensive numerical results demonstrate the accuracy of the asymptotic formula (2.4.4) and (2.7.5).

To proceed further with the analysis of (2.4.4) and (2.7.5) would require a bound on the remainder term R_m defined in (2.4.5). This requirement will be investigated and the general coefficients $A_r(s)$ given in (2.3.14) when ϕ is close to $\frac{\pi}{2}$ will be considered in a future study.

Based on the formula (2.2.10), a version of the asymptotic formula (A.3.1) had been given in [Paris, 1994] in which the coefficients were more complicated to evaluate [see Appendix A.3]. This asymptotic formula is at least as accurate as the Berry-Keating formula. A general and efficient method of calculating the coefficients A_r ($\phi = \frac{\pi}{2}$) will be given in Appendix A.3.

CHAPTER 3

GRAM-TYPE ASYMPTOTIC FORMULAS FOR THE RIEMANN ZETA FUNCTION

3.1 Introduction

Important and effective asymptotic formulas for the Riemann zeta function $\zeta(s)$ as a function of complex variable $s = \sigma + it$ are given in this chapter. The terms of the original Dirichlet series (valid in $\sigma > 1$) in these formulas are smoothed by some special functions. It is found that the switch off in the smoothed sum occurs after $O(|t|/2\pi)$ terms, when $t \to +\infty$ and s is in the critical strip $0 < \sigma < 1$. Therefore these asymptotic formulas are called Gram-type. The special case when s is on the critical line $s = \frac{1}{2} + it$ is studied in this chapter.

The outline of this chapter is as follows: The Gram-type asymptotic formulas for the Riemann zeta function are proposed. Two different methods of the incomplete gamma function-smoothed Gram-type asymptotic formula for $\zeta(s)$ are given in Section 3.2. A more general asymptotic formula which is smoothed by the confluent hypergeometric function is developed in Section 3.3. Section 3.4 shows that the asymptotic formula given in Section 3.2 is a special case of the asymptotic formula given in Section 3.3. Extensive numerical results shown in Section 3.5 demonstrate the performance of the asymptotic formula. A bound for the remainder term of the asymptotic formula for general positive a is studied in Section 3.6. In Section 3.7, we demonstrate the main contribution of the remainder term R_M. The order of the main sum in the asymptotic formula is given in Section 3.8. Further study of the order of the Riemann zeta function using the new method is carried out in Sections 3.9. Concluding remarks are given in Section 3.10. Some useful formulas and properties used in the main Sections are included in Appendix B.

3.2 The Incomplete Gamma Function-Smoothed Gram-Type Asymptotic Formula for $\zeta(s)$

3.2.1 Derivation of the Incomplete Gamma Function-Smoothed Gram-Type Asymptotic Formula for $\zeta(s)$

The incomplete gamma function-smoothed Gram-type asymptotic formula for $\zeta(s)$ will be derived using the Mellin integral representation of the incomplete gamma function. Using w instead of s in Appendix (B.1.6), the Mellin integral representation of the incomplete gamma function becomes

$$\Gamma(a,z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(a+w)}{w} z^{-w} dw, \tag{3.2.1}$$

where

$$c > 0, \quad |\arg z| < \frac{\pi}{2}.$$

Let $z = (n/K)^{2p}$, where p is a positive number, K denotes arbitrary complex parameter which will be restricted to satisfy $|\arg K| < \pi/4p$ and a be a positive value. Putting $z = (n/K)^{2p}$ into (3.2.1), we have

$$\begin{aligned}\Gamma(a,(n/K)^{2p}) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(a+w)(n/K)^{-2pw}\frac{dw}{w} \\ &= \frac{1}{2\pi i}\int_{2pc-i\infty}^{2pc+i\infty} \Gamma(a+w/2p)(n/K)^{-w}\frac{dw}{w}. \end{aligned}\tag{3.2.2}$$

Multiply both sides of (3.2.2) by n^{-s} then sum them for n from 1 to infinity and, using the definition $Q(a,z) = \Gamma(a,z)/\Gamma(a)$ given in (2.2.3), we establish the Perrontype formula

$$\sum_{n=1}^{\infty} n^{-s} Q(a, (n/K)^{2p}) = \frac{1}{2\pi i \Gamma(a)} \int_{c-i\infty}^{c+i\infty} \Gamma(a + \frac{w}{2p}) K^w \zeta(s+w) \frac{dw}{w}. \qquad (3.2.3)$$

The integrand in (3.2.3) has simple poles at $w = 0$, $w = 1 - s$ and $w = -2pa - 2pm$, where $m = 0, 1, \cdots$. As we are primarily interested in s values situated in the critical strip $0 < \sigma < 1$, we only consider the case when $0 < \sigma < 1$ in the following presentation. For simplicity in presentation, we define

$$f(w) = \Gamma(a + \frac{w}{2p}) K^w \zeta(s+w)/w.$$

Choose $c > 1 - \sigma$, K a positive number and $0 < c_0 < 2p\mathrm{Re}\,(a)$, $0 < c_1 < 2p$. Let the domain D_0 be denoted by $\{c - i\infty, c + i\infty, -c_0 + i\infty, -c_0 - i\infty\}$ and the domain D_1 be denoted by $\{c - i\infty, c + i\infty, -c_1 - 2pa - 2p(M-1) + i\infty, -c_1 - 2pa - 2p(M-1) - i\infty\}$. Then we apply the residue theorem to the function $f(w)$ in the domains D_0 and D_1 respectively, to find the following two formulas

$$\frac{1}{2\pi i} \{ \int_{c-i\infty}^{c+i\infty} f(w) dw - \int_{-c_0-i\infty}^{-c_0+i\infty} f(w) dw \} = \mathrm{Res}\,(f, 0) + \mathrm{Res}\,(f, 1-s), \qquad (3.2.4)$$

$$\frac{1}{2\pi i} \{ \int_{c-i\infty}^{c+i\infty} f(w) dw - \int_{-c_1-2p(M-1+a)-i\infty}^{-c_1-2p(M-1+a)+i\infty} f(w) dw \}$$
$$= \mathrm{Res}\,(f, 0) + \mathrm{Res}\,(f, 1-s) + \sum_{m=0}^{M-1} \mathrm{Res}\,(f, -2pa - 2pm), \qquad (3.2.5)$$

where $\mathrm{Res}\,(f, w_0)$ indicates the residue of the function $f(w)$ at w_0. Substitution of (3.2.3) into (3.2.4) and evaluation of the residues at the simple poles $w = 0$ and $w = 1 - s$ in (3.2.4), then shows that

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} Q(a, (n/K)^{2p}) + \frac{\Gamma(a + \frac{1-s}{2p})}{\Gamma(a)} \frac{K^{1-s}}{s-1} - J, \qquad (3.2.6)$$

where

$$J = \frac{1}{2\pi i \Gamma(a)} \int_{-c_0-i\infty}^{-c_0+i\infty} \Gamma(a + \frac{w}{2p}) K^w \zeta(s+w) \frac{dw}{w}. \qquad (3.2.7)$$

Substitution of (3.2.3) into (3.2.5) and calculation of the residues at the simple poles $w = 0$, $w = 1 - s$ and $w = -2pa - 2pm$, $m = 0, 1, 2, \cdots, M-1$ in (3.2.5), then yields the following asymptotic formula

$$\begin{aligned}\zeta(s) &= \sum_{n=1}^{\infty} n^{-s} Q(a,(n/K)^{2p}) + \frac{\Gamma(a+\frac{1-s}{2p})}{\Gamma(a)} \frac{K^{1-s}}{s-1} \\ &+ \frac{1}{\Gamma(a)} \sum_{m=0}^{M-1} \frac{(-)^m}{m!} \frac{K^{-2p(m+a)}}{m+a} \zeta(s-2p(m+a)) - R_M, \end{aligned} \tag{3.2.8}$$

where the remainder R_M is given by

$$R_M = \frac{1}{2\pi i \Gamma(a)} \int_{-c_1-2p(M-1+a)-i\infty}^{-c_1-2p(M-1+a)+i\infty} \Gamma(a+\frac{w}{2p}) K^w \zeta(s+w) \frac{dw}{w}. \tag{3.2.9}$$

Using the Riemann zeta functional relation in (3.2.8), which is

$$\zeta(s) = \chi(s)\zeta(1-s), \tag{3.2.10}$$

where $\chi(s)$ is defined in (2.2.2), we finally have

$$\begin{aligned}\zeta(s) &= \sum_{n=1}^{\infty} n^{-s} Q(a,(n/K)^{2p}) + \frac{\Gamma(a+\frac{1-s}{2p})}{\Gamma(a)} \frac{K^{1-s}}{s-1} \\ &+ \frac{\chi(s)}{\Gamma(a)} \sum_{m=0}^{M-1} \frac{(-)^m}{m!} \frac{(2\pi K)^{-2p(m+a)}}{m+a} A_m - R_M, \end{aligned} \tag{3.2.11}$$

where $M = 1, 2, \cdots$ and the coefficients A_m are defined by

$$A_m = \frac{\sin\frac{\pi}{2}(s-2p(m+a))}{\sin\frac{\pi s}{2}} \frac{\Gamma(1-s+2p(m+a))}{\Gamma(1-s)} \zeta(1-s+2p(m+a)). \tag{3.2.12}$$

This representation of $\zeta(s)$ is seen to involve the original Dirichlet series smoothed by the incomplete gamma function. We remark that the coefficients A_m involve the zeta function itself. The expansion (3.2.11) will therefore be of computational use only if $\mathrm{Re}\,(a) > \sigma/2p$, then the zeta function can be computed simply from the convergent Dirichlet series. For large m, $\zeta(1+2p(m+a)-s)$ tends to 1.

The expansion formula (3.2.11) was given in [Paris, 1994]. The incomplete gamma function contained in this asymptotic formula has a free parameter a, which can be selected. A suitable choice of a parameter a then enables us to present an expansion in which the main sum can be smoothed by a special function. For example, if $a = 1$, the main sum is smoothed by the real exponential factor; if $a = \frac{1}{2}$, the main sum is then smoothed by the complementary error function. These special cases will be discussed in Subsection 3.2.3.

3.2.2 An Alternative Proof of (3.2.11)

Next we will give an alternative method of proof of (3.2.11) based on the identity

$$P(a,z)+Q(a,z)=1, \tag{3.2.13}$$

the function $P(a,z)$ is the complementary normalised incomplete gamma function defined by

$$P(a,z)=\frac{1}{\Gamma(a)}\int_0^z u^{a-1}e^{-u}du.$$

If we choose $z=(n/K)^{2p}$, where K, p are the same conditions as being introduced in Subsection 3.2.1, a is a positive number in (3.2.13) and considering the region $\mathrm{Re}\,(s)>1$ in the s complex plane, then the Dirichlet series for $\zeta(s)$ can be expressed as

$$\begin{aligned}\zeta(s) &= \sum_{n=1}^{\infty} n^{-s}\{P(a,(n/K)^{2p})+Q(a,(n/K)^{2p})\} \\ &= \sum_{n=1}^{\infty} n^{-s}Q(a,(n/K)^{2p})+\sum_{n=1}^{\infty} n^{-s}P(a,(n/K)^{2p}).\end{aligned} \tag{3.2.14}$$

The complementary normalised incomplete gamma function and the confluent hypergeometric function have the following relation [Luke, vol I, 1968, p. 220]

$$P(a,z)=\frac{z^a}{a\Gamma(a)}M(a,a+1,-z), \quad \mathrm{Re}\,(a)>0. \tag{3.2.15}$$

The definition of the confluent hypergeometric function is given by the integral representation [Luke, vol I, 1968, p. 116]

$$M(a,b,z)=\frac{\Gamma(b)}{2\pi i\Gamma(a)}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(a+w)\Gamma(-w)}{\Gamma(b+w)}(-z)^w dw, \tag{3.2.16}$$

where

$$0>\gamma>-\mathrm{Re}\,(a), \quad |\arg(-z)|<\frac{\pi}{2}.$$

Let $z=-(n/K)^{2p}$, $b=a+1$ in (3.2.16) and choose $\gamma<0$, $0<\gamma+\mathrm{Re}\,(a)<\mathrm{Re}\,(s-1)/2p$, to find

$$M(a,a+1,-(n/K)^{2p})=\frac{a}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(-w)(n/K)^{2pw}}{a+w}dw. \tag{3.2.17}$$

Define

$$g(w)=\Gamma(-w)K^{-2pa-2pw}\zeta(s-2pa-2pw)/(a+w).$$

Putting (3.2.17) into (3.2.15), then the second sum on the right-hand side of (3.2.14) is

$$\frac{1}{2\pi i\Gamma(a)}\int_{\gamma-i\infty}^{\gamma+i\infty} g(w)dw. \tag{3.2.18}$$

The integrand in (3.2.18) has the simple poles at $w=-a+(s-1)/2p$, $w=m$, $m=0,\,1,\,2,\,\cdots$ in the domain $D_2:\{c_1/2p+M-1-i\infty, c_1/2p+M-1+i\infty, \gamma+i\infty, \gamma-i\infty\}$. Here $\zeta(s-2pa-2pw)$ is regular for all values of w in the w plane, except for a simple pole at $w=-a+(s-1)/2p$, with residue $1/2p$.

Using the residue theorem in the domain D_2 and evaluation of the residues in (3.2.18), we then find

$$\frac{1}{2\pi i\Gamma(a)}\{\int_{\gamma-i\infty}^{\gamma+i\infty} g(w)dw-\int_{c_1/2p+(M-1)-i\infty}^{c_1/2p+(M-1)+i\infty} g(w)dw\}=$$
$$\frac{\Gamma(a+\frac{1-s}{2p})K^{1-s}}{\Gamma(a)(s-1)}+\frac{1}{\Gamma(a)}\sum_{m=0}^{M-1}\frac{(-)^m}{m!}\frac{K^{-2p(m+a)}}{m+a}\zeta(s-2p(m+a)). \tag{3.2.19}$$

Putting (3.2.19) into (3.2.14), we then have the following formula

$$\begin{aligned}\zeta(s) &= \sum_{n=1}^{\infty} n^{-s}Q(a,(n/K)^{2p})+\frac{\Gamma(a+\frac{1-s}{2p})}{\Gamma(a)}\frac{K^{1-s}}{s-1}\\ &+\frac{1}{\Gamma(a)}\sum_{m=0}^{M-1}\frac{(-)^m}{m!}\frac{K^{-2p(m+a)}}{m+a}\zeta(s-2p(m+a))+\bar{R}_M,\end{aligned} \tag{3.2.20}$$

where

$$\bar{R}_M=\frac{1}{2\pi i\Gamma(a)}\int_{c_1/2p+(M-1)-i\infty}^{c_1/2p+(M-1)+i\infty}\frac{\Gamma(-w)}{a+w}K^{-2pa-2pw}\zeta(s-2pa-2pw)dw. \tag{3.2.21}$$

Introduction of the new variable $u=-2pa-2pw$ in (3.2.21), then shows that $\bar{R}_M=-R_M$, where R_M is given in (3.2.9). Use of the functional relation (3.2.10) in (3.2.20) then leads to the same formula (3.2.11).

We have only dealt with the case $\mathrm{Re}\,(s)>1$, but the formula (3.2.20) is quite general and valid for all s, except $s=1$, by analytic continuation.

3.2.3 Discussion of the Asymptotic Formula (3.2.11)

Here we consider the asymptotic formula (3.2.11), in which the main term contains the free parameter a. For different choices of a, we can obtain different asymptotic formulas. The dominant term in (3.2.11) is $\sum_{n=1}^{\infty} n^{-s}Q(a,(n/K)^{2p})$, which is an

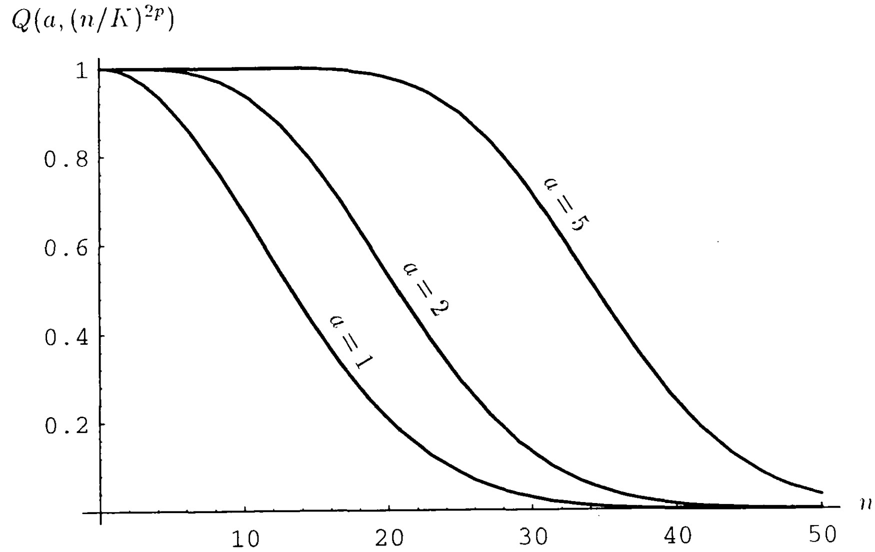


Figure 3.1: The Behaviour of $Q(a,(n/K)^{2p})$ for $a = 1, 2, 5$ when $p = 1$, $K = t/2\pi$ and $t = 100$

absolutely convergent series, since $\Gamma(a,z) \sim z^{a-1}e^{-z}$ as $|z| \to +\infty$ in $|\arg z| < \frac{3}{2}\pi$. Thus, the late terms in the main sum behave like $n^{2pa-2p-\sigma}\exp\left(-n^{2p}/K^{2p}\right)/K^{2pa-2p}$, so that this main sum in (3.2.11) converges absolutely for all values of s by analytic continuation. Here, we suppose that p is a positive integer.

The behaviour of the incomplete gamma function $Q(a,(n/K)^{2p})$ can be expected to change suddenly when n satisfies the relation $a \simeq (n/K)^{2p}$, since the transition point of $Q(a,z)$, is situated at $z = a$; that is, when n attains the critical value $n^* = \text{int}\,[K\,a^{1/2p}]$. Thus, we have $Q(a,(n/K)^{2p}) \sim 1$, when $n \leq n^*$, while $Q(a,(n/K)^{2p})$ decays to zero, when $n \geq n^*$. The behaviour of the incomplete gamma function $Q(a,(n/K)^{2p})$ is shown in the Fig. 3.1.

Next, we give the representations of (3.2.11) for different a. If we let $a = 1$ in (3.2.11), the main sum is smoothed by the real exponential factor $\exp\left(-(n/K)^{2p}\right)$, since $Q(1,z) = e^{-z}$. If p is a positive integer, the formula (3.2.11) then becomes the following simple asymptotic formula

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}\exp(-(n/K)^{2p}) - \frac{K^{1-s}}{2p}\Gamma(\frac{1-s}{2p}) - \chi(s)\sum_{m=1}^{M}\frac{(-)^m}{m!}(2\pi K)^{-2pm}A_{m-1} - R_M, \tag{3.2.22}$$

where

$$A_{m-1} = (-)^{pm}(1-s)_{2pm}\,\zeta(1-s+2pm), \quad m = 1,\, 2,\, \cdots,\, M \tag{3.2.23}$$

and

$$R_M = \frac{1}{4\pi i p}\int_{-c_1-2pM-i\infty}^{-c_1-2pM+i\infty}\Gamma(\frac{w}{2p})K^w\zeta(s+w)dw. \tag{3.2.24}$$

The asymptotic formula (3.2.22) exhibits an exponential smoothing of the terms in the infinite main sum which effectively "switch off" for values of n given by $n^* \sim K$. Also, the asymptotic formula (3.2.22) contains the sum $\sum_{m=1}^{M} a_m$, where

$$a_m = (-)^{pm+m}(2\pi K)^{-2pm}(1-s)_{2pm}\zeta(1-s+2pm)/m!, \quad m = 1,\, 2,\, \cdots,\, M. \tag{3.2.25}$$

This results from the large m-behaviour of the terms in the finite sum in (3.2.22) which, since $\zeta(1-s+2pm) \to 1$ as $m \to \infty$, is controlled essentially by the behaviour of

$$(\frac{t}{2\pi K})^{2pm}\frac{(1-s)_{2pm}}{m!\,t^{2pm}}, \quad m = 1,\, 2,\, \cdots.$$

It is easily shown that, for the terms in this sequence to possess an asymptotic character, it is necessary to choose $K \geq t/2\pi$. Since the main sum is smoothed after $n^* \simeq t/2\pi$ terms, the formula (3.2.22) consequently has the character of a Gram-type formula, rather than that of the Riemann-Siegel-type formulas discussed in Chapter 2. We also can choose $K = 2t/3\pi$, $K = t/\pi$, etc. If $K = t/2\pi$, the terms will first decrease to a minimum value at $m = M_0$, which is the optimal truncation point before finally diverging in asymptotic fashion.

Using $|a_m| \sim |a_{m+1}|$ at $m = M_0$, we have

$$\frac{(2pm)^{2p}|1+\frac{ti}{2pm}|^{2p}}{mt^{2p}} \sim 1.$$

Assuming $m \gg t$, the optimal truncation point M_0 is given approximately by

$$M_0 \simeq (t/2p)^{2p/(2p-1)}. \tag{3.2.26}$$

For the special case when $p = 1$, the optimal truncation point is $M_0 \simeq t^2/4$.

If let $a = \frac{1}{2}$, the main sum of the asymptotic formula (3.2.11) is then smoothed by the complementary error function erfc $(n/K)^p$, since $Q(\frac{1}{2}, z^2) = \text{erfc}\,(z)$. Then the asymptotic formula (3.2.11) becomes

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}\text{erfc}(n/K)^p + \frac{\Gamma(\frac{1}{2} + \frac{1-s}{2p})}{\Gamma(\frac{1}{2})}\frac{K^{1-s}}{s-1} + \frac{\chi(s)}{\Gamma(\frac{1}{2})}\sum_{m=0}^{M-1}\frac{(-)^m}{m!}\frac{(2\pi K)^{-2pm-p}}{m+a}A_m - R_M, \tag{3.2.27}$$

where

$$A_m = \frac{\sin\frac{\pi}{2}(s-2pm-p)}{\sin\frac{\pi s}{2}}\frac{\Gamma(1-s+2pm+p)}{\Gamma(1-s)}\zeta(1-s+2pm+p), \tag{3.2.28}$$

and

$$R_M = \frac{1}{2\pi i\Gamma(a)}\int_{-c_1-2p(M-1/2)-i\infty}^{-c_1-2p(M-1/2)+i\infty}\Gamma(a+\frac{w}{2p})K^w\zeta(s+w)\frac{dw}{w}. \tag{3.2.29}$$

If p is a even integer, then A_m can be simplified to

$$A_m = (-)^{pm+\frac{p}{2}}(1-s)_{2pm+p}\,\zeta(1-s+2pm+p).$$

If p is a odd integer, then

$$A_m = (-)^{pm+\frac{p}{2}+\frac{1}{2}}\cot\frac{\pi s}{2}\,(1-s)_{2pm+p}\,\zeta(1-s+2pm+p).$$

For the finite sum in (3.2.27) to possess an asymptotic character, it is again required that $K \geq t/2\pi$.

Other choices of the parameter a can also be made. However the mathematical description of the main sum will not be in terms of such a simple function. We only can express it using the incomplete gamma function. The numerical results of the asymptotic formula (3.2.11) for different choices of a are given in Section 3.6.

To conclude this section, we demonstrate how the result in (3.2.6) contains the expansion (2.2.1) as a special case. For convenience, we rewrite (2.2.1) as follows

$$\zeta(s) = \frac{(\pi\xi)^{\frac{s}{2}}}{\Gamma(\frac{s}{2})}(\frac{\xi^{-\frac{1}{2}}}{s-1} - \frac{1}{s}) + \sum_{n=1}^{\infty} n^{-s}Q(\frac{s}{2}, \pi n^2\xi) + \chi(s)\sum_{n=1}^{\infty} n^{s-1}Q(\frac{1-s}{2}, \pi n^2/\xi), \tag{3.2.30}$$

valid for all $s\,(\neq 1)$, where $\chi(s)$ is defined by (2.2.2), and ξ denotes an arbitrary complex constant such that $|\arg\xi| \leq \frac{\pi}{2}$.

Let $a = s/2$, $K = (\pi\xi)^{-1/2}$ and $p = 1$. Then the integrand in (3.2.7) has only a single simple pole at $w = -s$ on the left of the path of integration, the remaining poles of the gamma function being cancelled by the trivial zeros of $\zeta(s+w)$ situated at $w = -s-2k$, $k = 1, 2, \cdots$ and $\zeta(0) = -\frac{1}{2}$. We choose $c' > \mathrm{Re}\,(s)$, $0 < c_0 < \mathrm{Re}\,(s)$ and $|\arg K| < \frac{\pi}{4}$, thus, $|\arg \xi| \leq \frac{\pi}{2}$. By the residue theorem, we have

$$J - \frac{1}{2\pi i\Gamma(\frac{s}{2})}\int_{-c'-i\infty}^{-c'+i\infty}\Gamma(\frac{s+w}{2})(\pi\xi)^{-\frac{w}{2}}\zeta(s+w)\frac{dw}{w} = \frac{(\pi\xi)^{\frac{s}{2}}}{s\Gamma(\frac{s}{2})}. \tag{3.2.31}$$

We then obtain

$$\begin{aligned} J &= \frac{(\pi\xi)^{\frac{s}{2}}}{s\Gamma(\frac{s}{2})} + \frac{\chi(s)}{2\pi i\Gamma(\frac{s}{2})}\int_{-c'-i\infty}^{-c'+i\infty}\Gamma(\frac{s+w}{2})(\pi\xi)^{-\frac{w}{2}}\frac{\chi(s+w)}{\chi(s)}\zeta(1-s-w)\frac{dw}{w} \\ &= \frac{(\pi\xi)^{\frac{s}{2}}}{s\Gamma(\frac{s}{2})} - \frac{\chi(s)}{2\pi i\Gamma(\frac{1-s}{2})}\int_{c'-i\infty}^{c'+i\infty}\Gamma(\frac{1-s}{2}+\frac{w}{2})(\frac{\xi}{\pi})^{\frac{w}{2}}\zeta(1-s+w)\frac{dw}{w}. \end{aligned} \tag{3.2.32}$$

In the above derivation, we have used the functional relation (3.2.10) and replaced the variable w by $-w$. The second part of (3.2.32) is seen to be of the same form as that in (3.2.3) when $p = 1$, with s replaced by $1-s$. Hence, we obtain

$$J = \frac{(\pi\xi)^{\frac{s}{2}}}{s\Gamma(\frac{s}{2})} - \chi(s)\sum_{n=1}^{\infty} n^{s-1}Q(\frac{1-s}{2}, \pi n^2/\xi). \tag{3.2.33}$$

Substituting (3.2.33) into (3.2.6) leads to the expansion (3.2.30) valid for all values of $s\,(\neq 1)$ by analytic continuation.

3.3 The Confluent Hypergeometric-Smoothed Gram-Type Asymptotic Formula for $\zeta(s)$

3.3.1 Derivation of the Confluent Hypergeometric-Smoothed Gram-Type Asymptotic Formula for $\zeta(s)$

We employ the Mellin-Barnes integral representation of the confluent hypergeometric function $M(a, b, z)$ to derive another asymptotic formula for $\zeta(s)$. Recall the definition of the integral representation of the confluent hypergeometric function given in (3.2.16).

Supposing $\gamma = -c$ and satisfy $\mathrm{Re}(a) > c > 0$ and $|\arg(-z)| < \frac{\pi}{2}$, then the formula (3.2.16) can be written as

$$\frac{\Gamma(a)}{\Gamma(b)}M(a,b,z) = \frac{1}{2\pi i}\int_{-c-i\infty}^{-c+i\infty}\frac{\Gamma(a+w)\Gamma(-w)}{\Gamma(b+w)}(-z)^w dw. \tag{3.3.1}$$

Letting $z = -(n/K)^{2p}$ in (3.3.1), we have

$$\frac{\Gamma(a)}{\Gamma(b)}M(a,b,-(n/K)^{2p}) = \frac{1}{2\pi i}\int_{-c-i\infty}^{-c+i\infty}\frac{\Gamma(a+w)\Gamma(-w)}{\Gamma(b+w)}n^{2pw}K^{-2pw}dw$$
$$= \frac{1}{4\pi p i}\int_{2pc-i\infty}^{2pc+i\infty}\frac{\Gamma(a-w/2p)\Gamma(w/2p)}{\Gamma(b-w/2p)}n^{-w}K^w dw. \tag{3.3.2}$$

Summing both sides of (3.3.2) multiplied by n^{-s} over n from 1 to infinity, (3.3.2) becomes

$$\frac{\Gamma(a)}{\Gamma(b)}\sum_{n=1}^{\infty}n^{-s}M(a,b,-(n/K)^{2p}) = \frac{1}{4\pi p i}\int_{2pc-i\infty}^{2pc+i\infty}\frac{\Gamma(a-w/2p)\Gamma(w/2p)}{\Gamma(b-w/2p)}K^w\zeta(s+w)dw. \tag{3.3.3}$$

Letting $c > (1-\sigma)/2p$ and according to the residue theorem, we then obtain

$$\frac{1}{2\pi i}\{\int_{2pc-i\infty}^{2pc+i\infty}f(w)dw - \int_{-c_1-2p(M-1)-i\infty}^{-c_1-2p(M-1)+i\infty}f(w)dw\}$$
$$= \mathrm{Res}\,(f,1-s) + \mathrm{Res}\,(f,0) + \sum_{m=1}^{M-1}\mathrm{Res}\,(f,-2pm), \tag{3.3.4}$$

where $0 < c_1 < 2p$ and $f(w) = \frac{\Gamma(a-w/2p)\Gamma(w/2p)}{\Gamma(b-w/2p)}K^w\zeta(s+w)$. Substitution of (3.3.3) into (3.3.4) and evaluation of the residues, we then find

$$\zeta(s) = \sum_{n=1}^{\infty}n^{-s}M(a,b,-(n/K)^{2p}) - \frac{\Gamma(b)\Gamma(\frac{1-s}{2p})\Gamma(a-\frac{1-s}{2p})}{2p\Gamma(a)\Gamma(b-\frac{1-s}{2p})}K^{1-s}$$
$$-\frac{\Gamma(b)}{\Gamma(a)}\sum_{m=1}^{M-1}\frac{(-)^m}{m!}\frac{\Gamma(a+m)}{\Gamma(b+m)}K^{-2pm}\zeta(s-2pm) - R_M, \tag{3.3.5}$$

where

$$R_M = \frac{\Gamma(b)}{4\pi p i\Gamma(a)}\int_{-c_1-2p(M-1)-i\infty}^{-c_1-2p(M-1)+i\infty}\frac{\Gamma(a-w/2p)\Gamma(w/2p)}{\Gamma(b-w/2p)}K^w\zeta(s+w)dw. \tag{3.3.6}$$

Using the functional relation (3.2.10), we find

$$\zeta(s-2pm) = \chi(s-2pm)\zeta(1-s+2pm) = \chi(s)\frac{\chi(s-2pm)}{\chi(s)}\zeta(1-s+2pm)$$
$$= \chi(s)(2\pi)^{-2pm}\frac{\sin\frac{\pi}{2}(s-2pm)}{\sin\frac{\pi}{2}s}(1-s)_{2pm}\zeta(1-s+2pm). \tag{3.3.7}$$

Putting (3.3.7) into (3.3.5), we get the final asymptotic formula

$$\begin{aligned}\zeta(s) &= \sum_{n=1}^{\infty} n^{-s} M(a,b,-(n/K)^{2p}) - \frac{\Gamma(b)\Gamma(\frac{1-s}{2p})\Gamma(a-\frac{1-s}{2p})}{2p\Gamma(a)\Gamma(b-\frac{1-s}{2p})} K^{1-s} \\ &\quad -\frac{\Gamma(b)}{\Gamma(a)}\chi(s)\sum_{m=1}^{M-1}\frac{(-)^m}{m!}\frac{\Gamma(a+m)}{\Gamma(b+m)}(2\pi K)^{-2pm}A_{m-1} - R_M, \qquad (3.3.8)\end{aligned}$$

where

$$A_{m-1} = \frac{\sin\frac{\pi}{2}(s-2pm)}{\sin\frac{\pi s}{2}}(1-s)_{2pm}\zeta(1-s+2pm). \qquad (3.3.9)$$

When p is a positive integer, we have the simple representation of the coefficients

$$A_{m-1} = (-)^{pm}(1-s)_{2pm}\zeta(1-s+2pm), \quad m = 1, 2, \cdots, M-1.$$

3.3.2 Discussion of (3.3.8)

The formula (3.3.8) is a general form and includes the formula (3.2.11) as a special case. It contains two free parameters a and b. By choosing different values for a and b, we can obtain different asymptotic formulas. For example, if $b = a + 1$, we obtain (3.2.11). The coefficients A_m in (3.2.11) contain the parameter a, but A_{m-1} in (3.3.8) is independent of the parameters a and b.

The main sum in (3.3.8) contains the sum involving the confluent hypergeometric function. Since $M(a,b,z) \sim \Gamma(b)(-z)^{-a}/\Gamma(b-a)$ as $z \to -\infty$ in its principal branch domain $-\pi < \arg z \leq \pi$, the behaviour of the late terms of the main sum in (3.3.8) is given by $K^{2pa}/n^{2pa+\sigma}$. Thus, the main sum in (3.3.8) converges absolutely for all s when $a > (1-\sigma)/2p$.

The main sum of (3.3.8) is smoothed by a confluent hypergeometric function. The choice of a and b are free, but here we assume a is a positive value. We only consider three particular cases which can be expressed in terms of some special functions:

Case 1: a = b

In this case, if we take $a = b$, the confluent hypergeometric function $M(a,b,-(n/K)^{2p})$ equals the exponential factor $\exp(-(n/K)^{2p})$. Thus the formula (3.3.8) becomes

$$\zeta(s)=\sum_{n=1}^{\infty}n^{-s}\exp(-(n/K)^{2p})-\frac{K^{1-s}}{2p}\Gamma(\frac{1-s}{2p})-\chi(s)\sum_{m=1}^{M-1}\frac{(-)^m}{m!}(2\pi K)^{-2pm}A_{m-1}-R_M, \tag{3.3.10}$$

where

$$A_{m-1}=\frac{\sin\frac{\pi}{2}(s-2pm)}{\sin\frac{\pi s}{2}}(1-s)_{2pm}\zeta(1-s+2pm), \tag{3.3.11}$$

$$R_M=\frac{1}{4\pi pi}\int_{-c_1-2p(M-1)-i\infty}^{-c_1-2p(M-1)+i\infty}\Gamma(w/2p)K^w\zeta(s+w)dw. \tag{3.3.12}$$

This is the result obtained in (3.2.22). The main sum smoothed by an exponential function is an absolutely convergent series.

Case 2: $\mathbf{b=a+1}$

The confluent hypergeometric function $M(a,b,-(n/K)^{2p})$ becomes

$$\begin{aligned} M(a,b,-(n/K)^{2p}) &= a(n/K)^{-2pa}\gamma(a,(n/K)^{2p}) \\ &= a\Gamma(a)(n/K)^{-2pa}\{1-Q(a,(n/K)^{2p})\}, \end{aligned} \tag{3.3.13}$$

where $\gamma(a,z)$ is the complementary incomplete gamma function defined by

$$\gamma(a,z)=\int_0^z t^{a-1}e^{-t}dt.$$

The main sum is consequently

$$\begin{aligned}\sum_{n=1}^{\infty}n^{-s}M(a,b,-(n/K)^{2p}) &= aK^{2pa}\Gamma(a)\zeta(s+2pa) \\ &\quad -a\Gamma(a)K^{2pa}\sum_{n=1}^{\infty}n^{-s-2pa}Q(a,(n/K)^{2p}).\end{aligned}$$

The complete asymptotic formula for this case is

$$\begin{aligned}\zeta(s) &= -a\Gamma(a)K^{2pa}\sum_{n=1}^{\infty}n^{-s-2pa}Q(a,(n/K)^{2p})-\frac{a\Gamma(\frac{1-s}{2p})}{2p(a-\frac{1-s}{2p})}K^{1-s}+ \\ & a\Gamma(a)K^{2pa}\zeta(s+2pa)-a\chi(s)\sum_{m=1}^{M-1}\frac{(-)^m}{m!}\frac{(2\pi K)^{-2pm}}{m+a}A_{m-1}-R_M, \end{aligned}\tag{3.3.14}$$

where

$$R_M=\frac{a}{4\pi pi}\int_{-c_1-2p(M-1)-i\infty}^{-c_1-2p(M-1)+i\infty}\frac{\Gamma(w/2p)}{a-w/2p}K^w\zeta(s+w)dw, \tag{3.3.15}$$

the coefficients A_{m-1} are the same as in (3.3.11). In (3.3.14), we require $\mathrm{Re}\,(a)>(1-\sigma)/2p$.

Case 3: $\mathbf{a = 1/2,\ b = 3/2}$

This is a special case of $b = a + 1$. From (3.3.14), we then derive the following asymptotic formula

$$\zeta(s) = -\frac{1}{2}\pi^{\frac{1}{2}}K^p \sum_{n=1}^{\infty} n^{-s-p}\text{erfc}\,(n/K)^p - \frac{\Gamma(\frac{1-s}{2p})}{2p(1-\frac{1-s}{p})}K^{1-s}$$
$$+\frac{1}{2}\pi^{\frac{1}{2}}K^p\zeta(s+p) - \frac{1}{2}\chi(s)\sum_{m=1}^{M-1}\frac{(-)^m}{m!}\frac{(2\pi K)^{-2pm}}{m+\frac{1}{2}}A_{m-1} - R_M, \quad (3.3.16)$$

where

$$R_M = \frac{1}{8\pi pi}\int_{-c_1-2p(M-1)-i\infty}^{-c_1-2p(M-1)+i\infty}\frac{\Gamma(w/2p)}{(p-w)/2p}K^w\zeta(s+w)dw. \quad (3.3.17)$$

The coefficients A_{m-1} are the same as in (3.3.11). In (3.3.16), we require $p > 1-\sigma$. Clearly, this condition is valid since normally we assume that s is in the critical strip and p is a positive integer.

3.4 Equivalence of the Asymptotic Formulas (3.2.11) and (3.3.14)

In Section 3.3, we derived the asymptotic formula (3.3.14) for the case $b = a+1$ of (3.3.8). In this section, we will prove that the formula (3.3.14) is equivalent to the formula (3.2.11). To see this, we replace the variable s by $s' - 2pa$ in (3.3.14) to find

$$\zeta(s'-2pa) = -a\Gamma(a)K^{2pa}\sum_{n=1}^{\infty}n^{-s'}Q(a,(n/K)^{2p}) - \frac{a\Gamma(\frac{1-s'+2pa}{2p})}{2p(a-\frac{1-s'+2pa}{2p})}K^{1-s'+2pa}$$
$$+a\Gamma(a)K^{2pa}\zeta(s') - a\chi(s'-2pa)\sum_{m=1}^{M-1}\frac{(-)^m}{m!}\frac{(2\pi K)^{-2pm}}{m+a}A'_{m-1} - R'_M, \quad (3.4.1)$$

where

$$A'_{m-1} = \frac{\sin\frac{\pi}{2}(s'-2pa-2pm)}{\sin\frac{\pi}{2}(s'-2pa)}\frac{\Gamma(1-s'+2pa+2pm)}{\Gamma(1-s'+2pa)}\zeta(1-s'+2pa+2pm), \quad (3.4.2)$$

$$R'_M = \frac{a}{4\pi pi}\int_{-c_1-2p(M-1)-i\infty}^{-c_1-2p(M-1)+i\infty}\frac{\Gamma(w/2p)}{a-w/2p}K^w\zeta(s'-2pa+w)dw. \quad (3.4.3)$$

Using the functional equation $\zeta(s'-2pa) = \chi(s'-2pa)\zeta(1-s'+2pa)$ in (3.4.1), we obtain

$$\zeta(s') = \sum_{n=1}^{\infty} n^{-s'} Q(a,(n/K)^{2p}) + \frac{\Gamma(a+\frac{1-s'}{2p})}{\Gamma(a)(s'-1)} K^{1-s'} + \frac{\chi(s'-2pa)}{\Gamma(a)K^{2pa}}$$
$$\left\{\frac{\zeta(1-s'+2pa)}{a} + \sum_{m=1}^{M-1} \frac{(-)^m}{m!} \frac{(2\pi K)^{-2pm}}{m+a} A'_{m-1}\right\} + R''_M, \tag{3.4.4}$$

where

$$R''_M = \frac{1}{4\pi p i \Gamma(a) K^{2pa}} \int_{-c_1-2p(M-1)-i\infty}^{-c_1-2p(M-1)+i\infty} \frac{\Gamma(w/2p)}{a-w/2p} K^w \zeta(s'-2pa+w)dw. \tag{3.4.5}$$

Using the simple rearrangement and the definition (2.2.2), we have

$$\begin{aligned}\frac{\chi(s'-2pa)}{\Gamma(a)K^{2pa}} &= \frac{\chi(s')}{\Gamma(a)} \times \frac{\chi(s'-2pa)}{\chi(s')K^{2pa}} \\ &= \frac{\chi(s')}{\Gamma(a)}(2\pi K)^{-2pa} \frac{\sin\frac{\pi}{2}(s'-2pa)}{\sin\frac{\pi}{2}s'}(1-s')_{2pa}\end{aligned}$$

and, for convenience, replacing the variable s' by s in (3.4.4), we finally obtain the following result

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} Q(a,(n/K)^{2p}) + \frac{\Gamma(a+\frac{1-s}{2p})}{\Gamma(a)} \frac{K^{1-s}}{s-1}$$
$$+\frac{\chi(s)}{\Gamma(a)} \sum_{m=0}^{M-1} \frac{(-)^m}{m!} \frac{(2\pi K)^{-2p(m+a)}}{m+a} A_m + R''_M \tag{3.4.6}$$

where the coefficients A_m are the same as (3.2.12).

In fact, the remainder R''_M, which is defined in (3.4.5), equals the remainder $-R_M$ defined in (3.2.9), as can be shown by introducing the new variable $w = u+2pa$.

3.5 Numerical Results of (3.2.11)

The numerical illustration of the asymptotic formula (3.2.11) is presented to demonstrate the above analysis. On the critical line $s = \frac{1}{2} + it$, the function $\chi(\frac{1}{2}+it)$ can be expressed as $e^{-2i\vartheta(t)}$, which $\vartheta(t)$ is defined in (1.3.2). Table 3.1 shows the values of $|a_m|$ in the finite sum of (3.2.11) near the optimal truncation point M_0 for fixed t and different a values, where a_m is defined in (3.2.25). Using (3.2.26) for $t = 50$, it is found that the optimal truncation point is $M_0 \simeq 625$.

Table 3.1: Values of $|a_m|$ near optimal truncation for $t = 50$; $p = 1$, $K = t/2\pi$.

m	$\lvert a_m\rvert$ $a = \frac{1}{2}$	$\lvert a_m\rvert$ $a = 1$	$\lvert a_m\rvert$ $a = 2$	$\lvert a_m\rvert$ $a = 5$	$\lvert a_m\rvert$ $a = 10$
450	5.63×10^{-230}	1.02×10^{-228}	3.31×10^{-226}	1.18×10^{-218}	5.01×10^{-206}
500	6.58×10^{-236}	1.31×10^{-234}	5.31×10^{-232}	3.55×10^{-224}	4.22×10^{-211}
550	1.13×10^{-239}	2.49×10^{-238}	1.21×10^{-235}	1.42×10^{-227}	4.34×10^{-214}
600	1.81×10^{-241}	4.35×10^{-240}	2.52×10^{-237}	4.99×10^{-229}	3.57×10^{-215}
625	1.11×10^{-241}	2.78×10^{-240}	1.75×10^{-237}	4.41×10^{-229}	4.74×10^{-215}
650	1.86×10^{-241}	4.83×10^{-240}	3.28×10^{-237}	1.05×10^{-228}	1.66×10^{-214}
700	8.86×10^{-240}	2.49×10^{-238}	1.96×10^{-235}	9.73×10^{-227}	3.20×10^{-212}
750	1.51×10^{-236}	4.53×10^{-235}	4.09×10^{-232}	3.06×10^{-223}	2.00×10^{-208}
800	7.14×10^{-232}	2.29×10^{-230}	2.35×10^{-227}	2.59×10^{-218}	3.20×10^{-203}

The computation of the asymptotic formula (3.2.11) for different values of t with the value $a = 1$ is shown in Table 3.2. When $t = 10$, optimal truncation yields a value accurate to 5 decimals only. For the other t values M was chosen to yield an accuracy of 25 decimals. K was chosen as $t/2\pi$. The exact value $Z(t)$ can be computed from $Z(t) = e^{i\vartheta(t)}\zeta(\frac{1}{2} + it)$.

Table 3.2: Computations of $Z(t)$ from (3.2.11) for different values of t with $p = 1$, $a = 1$, $K = t/2\pi$.

t	M	$Z_{approx}(t)$	$Z(t)$	$\lvert Z_{approx}(t) - Z(t)\rvert$
10	24	−1.54918 98594	−1.54919 45461	4.7×10^{-6}
20	50	1.14784 24121 85197	1.14784 24121 85197	
		27763 50340 5	27763 50340 87	4.0×10^{-26}
30	33	0.59602 85192 39884	0.59602 85192 39884	
		95531 85143 14725	95531 85143 09520	5.2×10^{-27}
40	29	−1.30888 23934 56599	−1.30888 23934 56599	
		15901 61454 38381	15901 61454 47440	9.1×10^{-27}
50	27	−0.34073 50059 55024	−0.34073 50059 55024	
		98275 33166 17482	98275 33166 39750	2.2×10^{-26}

The computation of the asymptotic formula (3.2.11) for different values of p, K and fixed a when $t = 100$ and $M = 30$ is shown in Tables 3.3-3.4.

Tables 3.3-3.4 show that when p is increased, the accuracy decreases. Increasing K clearly increases the attainable accuracy; this is at the cost of increasing the

Table 3.3: Computations of $Z(t)$ from (3.2.11) for different values of p, K and fixed $a = 1$ when $t = 100$, $M = 30$

$a = 1$	$K = t/2\pi$	$K = 3t/4\pi$	$K = t/\pi$
p	$\lvert Z_{approx}(t) - Z(t)\rvert$	$\lvert Z_{approx}(t) - Z(t)\rvert$	$\lvert Z_{approx}(t) - Z(t)\rvert$
1	3.1×10^{-33}	3.9×10^{-44}	7.1×10^{-52}
2	1.5×10^{-24}	5.5×10^{-47}	1.8×10^{-62}
3	3.1×10^{-5}	4.3×10^{-21}	4.3×10^{-32}

Table 3.4: Computations of $Z(t)$ from (3.2.11) for different values of p, K and fixed $a = \frac{1}{2}$ when $t = 100$ and $M = 30$

$a = \frac{1}{2}$	$K = t/2\pi$	$K = 3t/4\pi$	$K = t/\pi$
p	$\lvert Z_{approx}(t) - Z(t)\rvert$	$\lvert Z_{approx}(t) - Z(t)\rvert$	$\lvert Z_{approx}(t) - Z(t)\rvert$
1	5.6×10^{-34}	1.1×10^{-44}	2.7×10^{-52}
2	7.2×10^{-25}	7.5×10^{-47}	4.2×10^{-62}
3	2.8×10^{-6}	1.4×10^{-19}	8.5×10^{-35}

effective cut-off point $n^* \sim Ka^{1/2p}$ of the smoothed infinite sum.

We will show the results of the asymptotic formula (3.2.11) for $t = 50$ and $t = 100$ in Tables 3.6-3.9 together with the bound of the remainder R_M in the next section.

3.6 A Bound on the Remainder R_M for $a > 0$

We will obtain a bound on the remainder term R_M in (3.2.9) for the particular case when p is a positive integer and a, K are positive numbers. Substituting the functional relation (3.2.10) into (3.2.9) and choosing $c_1 = p$, we can write R_M as

$$R_M = \frac{1}{2\pi i\Gamma(a)} \int_{-2p\delta-i\infty}^{-2p\delta+i\infty} \Gamma(a + \frac{w}{2p}) K^w \chi(s+w)\zeta(1-s-w)\frac{dw}{w}, \tag{3.6.1}$$

where $\delta = M + a - \frac{1}{2}$. Introduction of the new variable $w = -2p\delta + 2iu$, $u = y - \frac{t}{2}$ with $s = \sigma + it$ $(t > 0)$, R_M becomes

$$R_M = \frac{\pi^{s-\frac{3}{2}}}{2\Gamma(a)} \int_{-\infty}^{\infty} \frac{\Gamma(-M + \frac{1}{2} + \frac{iu}{p})}{iu - p\delta} (\pi K)^{-2p\delta+2iu} \zeta(1 - s + 2p\delta - 2iu)$$

$$\times \frac{\Gamma(\frac{1-\sigma}{2} + p\delta - iy)}{\Gamma(\frac{\sigma}{2} - p\delta + iy)} dy. \quad (3.6.2)$$

In the following study, we suppose $0 < \sigma < 1$ and define

$$F(y) = \left| \frac{\Gamma(\alpha + iy)}{\Gamma(\frac{1}{2} - \alpha + iy)} \right|, \quad \alpha = \frac{1-\sigma}{2} + p\delta. \quad (3.6.3)$$

In Appendix B.3.1, we prove that $F(y)$ is an even, monotonically increasing function of y for $y > 0$, since $\alpha = \frac{1-\sigma}{2} + p\delta > \frac{1}{4}$. The gamma function in (3.6.2) can be replaced by

$$\Gamma(-M + \frac{1}{2} + \frac{iu}{p}) = \frac{(-)^M \Gamma(\frac{1}{2} + \frac{iu}{p})}{(\frac{1}{2} - \frac{iu}{p})(\frac{1}{2} + 1 - \frac{iu}{p}) \cdots .(\frac{1}{2} + M - 1 - \frac{iu}{p})}. \quad (3.6.4)$$

From the inequality

$$|\Gamma(\frac{1}{2} + \frac{iu}{p})| = (\frac{\pi}{\cosh \pi u/p})^{\frac{1}{2}} < (2\pi)^{\frac{1}{2}} e^{-\pi|u|/2p}, \quad (3.6.5)$$

it follows that

$$|\Gamma(-M + \frac{1}{2} + \frac{iu}{p})| < \frac{(2\pi)^{\frac{1}{2}} e^{-\pi|u|/2p} \Gamma(\frac{1}{2})}{\Gamma(M + \frac{1}{2})}. \quad (3.6.6)$$

Using the inequalities $|\zeta(x + iy)| \leq \zeta(x)$ for real $x > 1$; real y and substitution of (3.6.6) into (3.6.2), we then find

$$|R_M| < \frac{\pi^{\sigma - \frac{1}{2}} 2^{-\frac{1}{2}} (\pi K)^{-2p\delta} \zeta(1 - \sigma + 2p\delta)}{\Gamma(a)\Gamma(M + \frac{1}{2}) p\delta} \times I, \quad (3.6.7)$$

where

$$I = \int_{-\infty}^{\infty} F(u + t/2) e^{-\pi|u|/2p} du.$$

From Stirling's formula, we have

$$F(y) \sim y^{2\alpha - \frac{1}{2}} \{1 + O(p^3\delta^3/y^2)\} \quad y \to +\infty.$$

Letting the scaled gamma function ratio be

$$G_M(y) = y^{\frac{1}{2} - 2\alpha} F(y), \quad (3.6.8)$$

then $G_M(y) \to 1$, as $y \to +\infty$, and its numerical value is $O(1)$ provided $p^{\frac{3}{2}} M^{\frac{3}{2}} \leq y$ for fixed M. The function $G_M(y)$ is also discussed in Appendix B.3.2, where it is established that $G_M(y)$ is a monotonically decreasing function of y for $y > 0$ when

$\alpha \geq 1$. In fact, the numerical results show that $G_M(y)$ has the same property when $\alpha \geq c_0$, where $c_0 \simeq 0.63605$.

The integral I can also be written as

$$\begin{aligned} I &= \{\int_{-\infty}^{-t} + \int_{-t}^{0} + \int_{0}^{\infty}\} F(u+t/2)e^{-\pi|u|/2p}du \\ &= (1+e^{-\pi t/2p})\int_{0}^{\infty} F(u+t/2)e^{-\pi u/2p}du \\ &\quad + \int_{0}^{t} F(-u+t/2)e^{-\pi u/2p}du. \end{aligned} \tag{3.6.9}$$

Using the properties of $F(y)$, we can write (3.6.9) as

$$\begin{aligned} I &= (1+e^{-\pi t/2p})G_M(t/2)\int_{0}^{\infty}(u+t/2)^{\frac{1}{2}-\sigma+2p\delta}e^{-\pi u/2p}du + F(t/2)\int_{0}^{t} e^{-\pi u/2p}du \\ &= (1+e^{-\pi t/2p})G_M(t/2)(\pi/2p)^{-N-1}e^{X}\Gamma(N+1,X) \\ &\quad + F(t/2)(2p/\pi)(1-e^{-\pi t/2p}), \end{aligned} \tag{3.6.10}$$

where $X = \pi t/4p$, $N = \frac{1}{2} - \sigma + 2p\delta$. In the above derivation, we have used the formula [Gradshteyn & Ryzhik, 1965, p. 318]

$$\int_{0}^{\infty}(x+\beta)^{\nu}e^{-\mu x}dx = \mu^{-\nu-1}e^{\beta\mu}\Gamma(\nu+1,\beta\mu), \tag{3.6.11}$$

where $\mathrm{Re}\,\mu > 0$, $|\arg\beta| < \pi$. Substitution of (3.6.10) into (3.6.7), shows that the bound on the remainder term R_M is expressed as

$$|R_M| < 2^{\frac{1}{2}}\pi^{\sigma-3/2}(\frac{t}{2})^{\frac{1}{2}-\sigma}(\frac{t}{2\pi K})^{2p\delta} \times \frac{\zeta(1-\sigma+2p\delta)}{\Gamma(a)\Gamma(M+\frac{1}{2})(M-\frac{1}{2}+a)} \times C_M(t), \tag{3.6.12}$$

where

$$C_M(t) = \{1 - e^{-\pi t/2p} + (1+e^{-\pi t/2p}) \times \frac{e^X}{X^N}\Gamma(N+1,X)\} \times G_M(t/2). \tag{3.6.13}$$

The function $G_M(t/2)$ is determined by direct computation from (3.6.3) and (3.6.8). In Table 3.5 we show values of $G_M(t/2)$ for different M when $p = 1$ and $a = \frac{1}{2}$, $a = 1$ for $t = 50$ and $t = 100$. For low values of M, $G_M(t/2) \simeq 1$, while for larger values of M, it is found that $G_M(t/2)$ ceases to be $O(1)$ once $M \gtrsim (t/2)^{\frac{2}{3}}/p$. Consequently, we would expect the bound (3.6.12) to be quite realistic only for values of M satisfying $M \lesssim (t/2)^{\frac{2}{3}}/p$.

In order to illustrate the accuracy of the asymptotic formula (3.2.11) and the sharpness of the bound (3.6.12), we give numerical results for $Z_{approx}(t)$ using the

Table 3.5: Values of the coefficient $G_M(t/2)$ when $p = 1$ and $a = \frac{1}{2}$, $a = 1$ for $t = 50$ and $t = 100$

$a = \frac{1}{2}$			$a = 1$		
M	$G_M(25)$	$G_M(50)$	M	$G_M(25)$	$G_M(50)$
5	1.06793	1.01671	5	1.09122	1.02230
10	1.66401	1.14075	10	1.79844	1.16447
15	5.11858	1.54996	15	5.99574	1.62045
20	37.7341	2.77013	20	48.5599	2.98869
25	7337.75	6.98094	25	1042.88	7.82060

Table 3.6: Computations of $Z_{approx}(t)$ and R_M for $t = 50$, $a = \frac{1}{2}$, $K = t/2\pi$, $p = 1$, $M_0 = 625$.

Z(t)	−0.340735005955024982753316639750814878139663427		
M	$Z_{approx}(t)$	$\lvert Z_{approx}(t) - Z(t)\rvert$	$\lvert R_M\rvert$
5	−0.33992 97514	8.053×10^{-4}	2.284×10^{-3}
10	−0.34073 49818 43000	2.411×10^{-8}	1.100×10^{-7}
15	−0.34073 50059 54995 87282	2.911×10^{-14}	1.156×10^{-12}
20	−0.34073 50059 55024 98288 06415	1.273×10^{-19}	9.219×10^{-18}
25	−0.34073 50059 55024 98275 33178 10318	1.171×10^{-24}	1.608×10^{-22}
30	−0.34073 50059 55024 98275 33166 39750 35059	4.643×10^{-31}	1.455×10^{-26}
35	−0.34073 50059 55024 98275 33166 39750 81487 35331	4.607×10^{-36}	8.022×10^{-30}

Table 3.7: Computations of $Z_{approx}(t)$ and R_M for $t = 50$, $a = 1$, $K = t/2\pi$, $p = 1$, $M_0 = 625$.

Z(t)	−0.34073 50059 55024 98275 33166 39750 81487 81396 63426 67267 96489		
M	$Z_{approx}(t)$	$\lvert Z_{approx}(t) - Z(t)\rvert$	$\lvert R_M\rvert$
5	−0.33936 96628	1.365×10^{-3}	4.048×10^{-3}
10	−0.34073 49606	4.540×10^{-8}	2.070×10^{-7}
15	−0.34073 50059 55145 49923	1.205×10^{-13}	2.464×10^{-12}
20	−0.34073 50059 55024 98222 07131	5.326×10^{-19}	2.330×10^{-17}
25	−0.34073 50059 55024 98275 33203 24412	3.685×10^{-24}	5.198×10^{-22}
30	−0.34073 50059 55024 98275 33166 39759 15453	8.340×10^{-30}	6.184×10^{-26}
40	−0.34073 50059 55024 98275 33166 39750 81487 81398 66857	2.034×10^{-40}	1.612×10^{-31}
50	−0.34073 50059 55024 98275 33166 39750 81487 81396 63426 67267 24770	7.172×10^{-51}	1.812×10^{-34}

Table 3.8: Computations of $Z_{approx}(t)$ and R_M for $t = 100$, $a = \frac{1}{2}$, $K = t/2\pi$, $p = 1$.

Z(t)	2.69269 70566 64463 47499 53798 28685 03242 06190 21637 67271 34374 02731 19279		
M	$Z_{approx}(t)$	$\lvert Z_{approx}(t) - Z(t)\rvert$	$\lvert R_M\rvert$
5	2.69334 31671	6.461×10^{-4}	2.116×10^{-3}
10	2.69269 70653 68957	8.704×10^{-9}	5.967×10^{-8}
15	2.69269 70566 64462 41105	1.064×10^{-15}	2.037×10^{-13}
20	2.69269 70566 64463 47500 45590	9.179×10^{-21}	1.945×10^{-19}
25	2.69269 70566 64463 47499 53798 39462	1.078×10^{-26}	8.304×10^{-26}
30	2.69269 70566 64463 47499 53798 28685 03186 03162	5.603×10^{-34}	2.184×10^{-32}
40	2.69269 70566 64463 47499 53798 28685 03242 06190 21637 67175 50959	9.583×10^{-49}	1.022×10^{-45}
50	2.69269 70566 64463 47499 53798 28685 03242 06190 21637 67271 34374 02730 96799	2.248×10^{-61}	2.359×10^{-58}

Table 3.9: Computations of $Z_{approx}(t)$ and R_M for $t = 100$, $a = 1$, $K = t/2\pi$, $p = 1$.

Z(t)	2.69269 70566 64463 47499 53798 28685 03242 06190 21637 67271 34374 02731 19279

M	$Z_{approx}(t)$	$\lvert Z_{approx}(t) - Z(t)\rvert$	$\lvert R_M\rvert$
5	2.6936 70837	9.738×10^{-4}	3.454×10^{-3}
10	2.69269 70762 25807	1.956×10^{-8}	1.038×10^{-7}
15	2.69269 70566 64484 47128	2.124×10^{-14}	3.698×10^{-13}
20	2.69269 70566 64463 47503 36646	3.828×10^{-20}	3.688×10^{-19}
25	2.69269 70566 64463 47499 53798		
	39462	1.990×10^{-26}	1.654×10^{-25}
30	2.69269 70566 64463 47499 53798		
	28685 02927 95836	3.141×10^{-33}	4.604×10^{-32}
40	2.69269 70566 64463 47499 53798		
	28685 03242 06190 21637 62688		
	46531	4.583×10^{-47}	2.521×10^{-45}
50	2.69269 70566 64463 47499 53798		
	28685 03242 06190 21637 67271		
	34374 02731 09698	9.581×10^{-62}	7.557×10^{-58}

asymptotic formula (3.2.11) and the bound R_M using (3.6.12) for different values of t and a.

In Tables 3.6-3.9, we show the accuracy of the asymptotic formula (3.2.11) truncated after M terms for $t = 50$ or $t = 100$, when $a = \frac{1}{2}$ and 1. It is seen that when $M \leq 25$ or $M \leq 50$ the accuracy of the bound is acceptable. However, when $M > 25$ or $M > 50$ the bound is not very accurate; thus the bound is realistic when $M \leq 25$ or $M \leq 50$.

From Tables 3.6-3.9, we find that the bound (3.6.12) is not sharp enough to establish the asymptotic nature of the expansion in (3.2.11). When M is not too large, this bound turns out to be very realistic, and remains so until $M \simeq \text{int}[t/2]$ when $p = 1$. For large values of M, the bound (3.6.12) is too crude for values of M near optimal truncation. In fact, the main contribution to the bound R_M in the integral in (3.2.9) arises when y is near $\frac{t}{2}$ for the special case $p = 1$. We shall show this in Section 3.7.

3.7 Discussion of the Main Contribution of the Remainder R_M

We will demonstrate that the integral for R_M given in (3.2.9) can be divided into several parts over its domain of integration. We will show that only one of them yields a dominant contribution and rest of them are exponentially small and can be neglected.

For convenience, R_M is rewritten as follows

$$R_M = \frac{1}{2\pi i\Gamma(a)}\int_{-c_1-2p(M-1+a)-i\infty}^{-c_1-2p(M-1+a)+i\infty}\Gamma(a+\frac{w}{2p})K^w\zeta(s+w)\frac{dw}{w}. \tag{3.7.1}$$

Here we consider the case $0 < c_1 < 2, p = 1$ and a is a positive number. Putting (3.2.10) into (3.7.1), we have

$$R_M = \frac{1}{2\pi i\Gamma(a)}\int_{-c_1-2(M-1+a)-i\infty}^{-c_1-2(M-1+a)+i\infty}\Gamma(a+\frac{w}{2p})K^w\pi^{w+s-\frac{1}{2}}\frac{\Gamma(\frac{1}{2}-\frac{s+w}{2})}{\Gamma(\frac{s+w}{2})}\zeta(1-s-w)\frac{dw}{w}. \tag{3.7.2}$$

With $w = 2z+1-s$ and $\nu = a - s/2$, this becomes

$$R_M = \frac{\pi^{s-\frac{1}{2}}}{2\pi i\Gamma(a)}\int_{-c_2-(M-1+a)-i\infty}^{-c_2-(M-1+a)+i\infty}\frac{\Gamma(\frac{1}{2}+z+\nu)\Gamma(-z)}{\Gamma(z+\frac{1}{2})(z-\frac{s}{2}+\frac{1}{2})}(\pi^2K^2)^{z-\frac{s}{2}+\frac{1}{2}}\zeta(-2z)dz, \tag{3.7.3}$$

where $c_2 = \frac{c_1}{2}+\frac{1}{4}$ and $\frac{1}{4} < c_2 < \frac{5}{4}$ $(\sigma = \frac{1}{2})$. Letting $z = -c_2-(M-1+a)+iy$, we can write (3.7.3) as

$$\begin{aligned} R_M &= \frac{\pi^{s-\frac{1}{2}}}{2\pi i\Gamma(a)}\int_{-\infty}^{\infty}\frac{\Gamma(\delta_1-M+iu)\Gamma(c_2+(M-1+a)-iy)}{\Gamma(\frac{1}{2}-c_2-(M-1+a)+iy)(-M-a+\delta_1+iu)} \\ &\quad\times(\pi^2K^2)^{-M+\delta_1-a+iu}\zeta(2c_2+2(M-1+a)-2iy)dy, \end{aligned} \tag{3.7.4}$$

where $u = y-\frac{t}{2}$, $\delta_1 = \frac{5}{4}-c_2$ and δ_1 satisfies $0 < \delta_1 < 1$.

If we take $\alpha = c_2+(M-1+a)$ and $\beta = \frac{1}{2}-\alpha$ in (B.3.1), $H(y)$ changes to $F_1(y)$ which is defined as

$$F_1(y) = \left|\frac{\Gamma(c_2+(M-1+a)-iy)}{\Gamma(\frac{1}{2}-c_2-(M-1+a)+iy)}\right|. \tag{3.7.5}$$

Then $F_1(y)$ has the following properties (see Appendix B.3):

(i) $F_1(y)$ is an even increasing function of y for $y > 0$ since $c_2+(M-1+a) > \frac{1}{4}$;

(ii) $F_1(y) \sim y^{2(M+a)-2\delta_1}$ when $y \to +\infty$;

(iii) $F_1(0) = \left|\frac{\Gamma(c_2+(M-1+a))}{\Gamma(\frac{1}{2}-c_2-(M-1+a))}\right|$.

Define

$$G_M^1(y) = y^{2\delta_1-2(M+a)}F_1(y), \quad y > 0. \tag{3.7.6}$$

Thus, $G_M^1(y)$ is a decreasing function of y for $y > 0$ when (at least) $c_2+(M-1+a) \geq \frac{3}{4}$ and $G_M^1(y) \sim 1$ when $y \to +\infty$. The proof of these statements is given in B.3.

For convenience, we define the integral $\bar{I}$

$$\bar{I} = \int_{-\infty}^{\infty} F_2(y)dy, \tag{3.7.7}$$

where the function $F_2(y)$ denotes

$$\begin{aligned} F_2(y) &= \frac{\Gamma(\delta_1 - M + iu)\Gamma(c_2 + (M-1+a) - iy)}{\Gamma(\frac{1}{2} - c_2 - (M-1+a) + iy)(-M-a+\delta_1+iu)} \\ &\quad \times(\pi^2K^2)^{-M+\delta_1-a+iu}\zeta(2c_2+2(M-1+a)-2iy). \end{aligned}$$

We divide $\bar{I}$ into the following four parts:

$$\begin{aligned} \bar{I} &= \{\int_{-\infty}^{0} + \int_{0}^{\frac{t}{2}-\Delta} + \int_{\frac{t}{2}-\Delta}^{\frac{t}{2}+\Delta} + \int_{\frac{t}{2}+\Delta}^{+\infty}\}F_2(y)dy \\ &= \bar{I}_1 + \bar{I}_2 + \bar{I}_3 + \bar{I}_4, \end{aligned} \tag{3.7.8}$$

where Δ is a function of t which satisfies $\Delta = \Delta(t) \to +\infty$ and $\Delta \ll \frac{t}{2}$ as $t \to +\infty$. For example, we can choose $\Delta(t) \sim t^{\nu}$, where $0 < \nu < 1$.

Since $2c_2 + 2(M-1+a) > 1$, the first part of $\bar{I}$ can be written as

$$\begin{aligned} |\bar{I}_1| &\leq (\pi^2K^2)^{-(M+a-\delta_1)}\zeta(2c_2+2(M-1+a)) \\ &\quad \times \int_{-\infty}^{0} F_1(y)\frac{|\Gamma(\delta_1 - M + i(y-\frac{t}{2}))|}{\{(M+a-\delta_1)^2+(y-\frac{t}{2})^2\}^{\frac{1}{2}}}dy \\ &\leq 2(\pi^2K^2)^{-(M+a-\delta_1)}t^{-1}\zeta(2c_2+2(M-1+a)) \\ &\quad \times \int_{0}^{+\infty} F_1(y)|\Gamma(\delta_1 - M + i(y+t/2))|dy. \end{aligned} \tag{3.7.9}$$

The gamma function satisfies the inequality

$$|\Gamma(\chi+iu)| < K_1|u|^{\chi-\frac{1}{2}}e^{-\frac{\pi}{2}|u|}, \quad |\chi+iu| > K_2, \tag{3.7.10}$$

where K_1 is a computable coefficient ($K_1 \sim \sqrt{2\pi}$ when $u \to +\infty$) and K_2 is a positive constant. Replacing χ by $\delta_1 - M$ in (3.7.10) and applying this result to the integral (3.7.9), we have

$$\begin{aligned} |\bar{I}_1| &\leq 2K_1(\pi^2K^2)^{-(M+a-\delta_1)}t^{-1}\zeta(2c_2+2(M-1+a)) \\ &\quad\times \int_0^{+\infty} F_1(y)(y+t/2)^{\delta_1-M-\frac{1}{2}}e^{-\frac{\pi}{2}(y+\frac{t}{2})}dy \\ &< 2K_1(\pi^2K^2)^{-(M+a-\delta_1)}t^{-1}\zeta(2c_2+2(M-1+a))e^{-\frac{\pi t}{4}} \\ &\quad\times \int_0^{+\infty} F_1(y)(y+1)^{\delta_1-M-\frac{1}{2}}e^{-\frac{\pi}{2}y}dy, \end{aligned} \tag{3.7.11}$$

where $t > 2$ (t is large). The last integral is independent of t and is finite, since $F_1(y) \sim y^{2(M+a)-2\delta_1}$ when $y \to +\infty$. We suppose K is a function of t such that $K = K(t) \to +\infty$ as $t \to +\infty$. From the above inequality, we therefore obtain

$$|\bar{I}_1| \sim O(K^{-2(M+a-\delta_1)}e^{-\frac{\pi t}{4}}t^{-1}), \quad t \to +\infty,$$

so that, $|\bar{I}_1|$ is exponentially small when t is large.

Analysing the contribution from the other parts of $|\bar{I}|$, we have

$$\begin{aligned} |\bar{I}_2| &\leq (\pi^2K^2)^{-(M+a-\delta_1)}\zeta(2c_2+2(M-1+a)) \\ &\quad\times \int_0^{\frac{t}{2}-\Delta} F_1(y)\frac{|\Gamma(\delta_1-M+i(y-\frac{t}{2}))|}{\{(M+a-\delta_1)^2+(y-\frac{t}{2})^2\}^{\frac{1}{2}}}dy \\ &< K_1F_1(\frac{t}{2}-\Delta)(\pi^2K^2)^{-(M+a-\delta_1)}\zeta(2c_2+2(M+a-1)) \\ &\quad e^{-\frac{\pi\Delta}{2}}\Delta^{\delta_1-M-\frac{3}{2}}(t/2-\Delta), \end{aligned} \tag{3.7.12}$$

so that we obtain

$$|\bar{I}_2| \sim O(K^{-2(M+a-\delta_1)}\Delta^{\delta_1-M-\frac{3}{2}}(t/2-\Delta)^{2(M+a-\delta_1)+1}e^{-\pi\Delta/2}), \quad t \to +\infty.$$

This part is also exponentially small when t is large.

The part of $|\bar{I}_4|$ will be analysed next.

$$\begin{aligned} |\bar{I}_4| &\leq (\pi^2K^2)^{-(M+a-\delta_1)}\zeta(2c_2+2(M-1+a)) \\ &\quad \int_{\frac{t}{2}+\Delta}^{+\infty} F_1(y)\frac{|\Gamma(\delta_1-M+i(y-\frac{t}{2}))|}{\{(M+a-\delta_1)^2+(y-\frac{t}{2})^2\}^{\frac{1}{2}}}dy \\ &< K_1(\pi^2K^2)^{-(M+a-\delta_1)}G_M^1(t/2+\Delta)\zeta(2c_2+2(M+a-1))\Delta^{-1} \end{aligned}$$

$$\times \int_{\frac{t}{2}+\Delta}^{+\infty} y^{2(M+a-\delta_1)}(y-t/2)^{\delta_1-M-\frac{1}{2}}e^{-\frac{\pi}{2}(y-\frac{t}{2})}dy$$
$$< K_1(\pi^2K^2)^{-(M+a-\delta_1)}G_M^1(t/2+\Delta)\zeta(2c_2+2(M+a-1))\Delta^{\delta_1-M-\frac{3}{2}}$$
$$\times(2/\pi)^{2(M+a)-2\delta_1+1}e^{\frac{\pi t}{4}}\Gamma(2(M+a-\delta_1)+1,\pi(t/2+\Delta)/2). \quad (3.7.13)$$

Thus, we have the estimate of the order of $|\bar{I}_4|$ as

$$|\bar{I}_4| \sim O(K^{-2(M+a-\delta_1)}\Delta^{\delta_1-M-\frac{3}{2}}(t/2+\Delta)^{2(M+a-\delta_1)}e^{-\pi\Delta/2}), \quad t\to+\infty.$$

The part $|\bar{I}_4|$ is exponentially small too when t is large.

The only part left is $|\bar{I}_3|$, the integral whose interval is the neighbourhood of $t/2$, which is $(t/2-\Delta, t/2+\Delta)$. Then

$$|\bar{I}_3| \le (\pi^2K^2)^{-(M+a-\delta_1)}\zeta(2c_2+2(M-1+a))(M+a-\delta_1)^{-1}$$
$$\times \int_{\frac{t}{2}-\Delta}^{\frac{t}{2}+\Delta} F_1(y)|\Gamma(\delta_1-M+i(y-t/2))|dy. \quad (3.7.14)$$

Clearly, the parts $\bar{I}_1$, $\bar{I}_2$, $\bar{I}_4$ are very small terms compared with the term $\bar{I}_3$, when t is large. Therefore, the main contribution to the integral I results from the part $\bar{I}_3$. To demonstrate the accuracy of the above analysis, some numerical results are given in Tables 3.10 and 3.11. The computable coefficient K_1 in Tables 3.10-3.11 is given in (3.7.10), $K_1 \sim \sqrt{2\pi}$ when t is large. It can be seen that $R_{m3} \sim |Z_{approx}(t) - Z(t)|$ and $R_{mi} \ll R_{m3}$ (i = 1, 2, 4). For example, when M = 25 in Table 3.11, the smallest term is R_{m1} which is 1.2×10^{-75}, the second smallest term is R_{m2} which is 7.3×10^{-66} and the value of $|Z_{approx}(t) - Z(t)|$ is 2.0×10^{-26} which is the same order as R_{m3} given by 5.6×10^{-26}. These results agree with the theoretical analysis.

Table 3.10: Computations of R_{mi} (i = 1, 2, 3, 4) for $t = 50$, $a = 1$, $K = t/2\pi$, $c_1 = p = 1$, $\Delta = t/4$, where $R_{mi} = \frac{1}{2\pi\Gamma(a)K_1}|\bar{I}_i|$ when i = 1, 2, 4 and $R_{m3} = \frac{1}{2\pi\Gamma(a)}|\bar{I}_3|$

M	R_{m1}	R_{m2}	R_{m3}	R_{m4}	$\|Z_{approx}(t) - Z(t)\|$
5	2.9×10^{-32}	1.1×10^{-18}	2.0×10^{-3}	8.7×10^{-15}	1.4×10^{-3}
10	8.4×10^{-38}	1.9×10^{-26}	8.3×10^{-8}	2.6×10^{-18}	4.5×10^{-8}
15	7.0×10^{-41}	2.3×10^{-33}	6.8×10^{-13}	1.2×10^{-21}	1.2×10^{-13}
20	1.8×10^{-42}	2.1×10^{-39}	2.8×10^{-18}	9.0×10^{-25}	5.3×10^{-19}
25	6.2×10^{-43}	1.1×10^{-44}	9.4×10^{-24}	1.5×10^{-27}	3.7×10^{-24}

Table 3.11: Computations of R_{mi} $i = (1, 2, 3, 4)$ for $t = 100$, $a = 1$, $K = t/2\pi$, $c_1 = p = 1$, $\Delta = t/4$, where $R_{mi} = \frac{1}{2\pi\Gamma(a)K_1}|\bar{I}_i|$ when $i = 1, 2, 4$ and $R_{m3} = \frac{1}{2\pi\Gamma(a)}|\bar{I}_3|$

M	R_{m1}	R_{m2}	R_{m3}	R_{m4}	$\lvert Z_{approx}(t) - Z(t)\rvert$
5	6.2×10^{-53}	7.7×10^{-29}	1.8×10^{-3}	3.5×10^{-25}	9.7×10^{-4}
10	1.8×10^{-61}	1.3×10^{-38}	5.0×10^{-8}	2.4×10^{-30}	1.9×10^{-8}
15	1.4×10^{-67}	4.2×10^{-48}	1.6×10^{-13}	1.9×10^{-35}	2.2×10^{-14}
20	3.7×10^{-72}	3.4×10^{-57}	1.5×10^{-19}	1.6×10^{-40}	3.8×10^{-20}
25	1.2×10^{-75}	7.3×10^{-66}	5.6×10^{-26}	1.7×10^{-45}	2.0×10^{-26}
30	3.0×10^{-78}	4.3×10^{-74}	1.2×10^{-32}	2.3×10^{-50}	3.1×10^{-33}
40	2.2×10^{-81}	2.6×10^{-89}	2.3×10^{-46}	9.1×10^{-60}	4.6×10^{-47}
50	2.6×10^{-82}	5.7×10^{-103}	2.5×10^{-60}	1.4×10^{-68}	9.6×10^{-62}

3.8 The Order of the Main Sum in (3.2.6)

We notice that the asymptotic expansion (3.2.6) for $\zeta(s)$ has three parts, the first part is the main sum over n smoothed by the incomplete gamma function, the second part is exponentially small for fixed a when t is large, and the last part J can be represented by the integral given in (3.2.7). We will use the new formula to estimate $\zeta(\frac{1}{2} + it)$. For this purpose, the estimation of the order of the main sum in (3.2.6) will be developed in this section and the approximation of J will be given in the next section. These estimates will be used to determine the growth of $\zeta(\frac{1}{2} + it)$.

We will estimate the order of the sum $|\sum_{n=1}^{\infty} n^{-s}Q(a, n^2/K^2)|$ when s is on the critical line $\sigma = \frac{1}{2}$. First we consider the incomplete gamma function $Q(a, x)$ defined in (2.2.3), which we rewrite here as

$$Q(a, x) = \frac{1}{\Gamma(a)} \int_x^{\infty} t^{a-1}e^{-t}dt, \tag{3.8.1}$$

where we suppose that $x \geq 0$ and $a > 0$. Using the following change of variables in (3.8.1),

$$\frac{t}{a} = \tau, \quad \tau - 1 - \log\tau = \frac{1}{2}\xi^2, \tag{3.8.2}$$

we obtain

$$\begin{aligned} Q(a, x) &= \frac{e^{-a}a^a}{\Gamma(a)} \int_{x/a}^{\infty} e^{-a(\tau-1-\log\tau)}\frac{d\tau}{\tau} \\ &= \frac{1}{\Gamma^*(a)}\sqrt{\frac{a}{2\pi}} \int_{\eta}^{\infty} e^{-\frac{1}{2}a\xi^2} f(\xi)d\xi, \end{aligned} \tag{3.8.3}$$

where $\lambda = x/a$ and η satisfies the equation $\lambda - 1 - \log\lambda = \frac{1}{2}\eta^2$, with the choice of the square root branch for η being made such that η and $\lambda - 1$ have the same sign and $\Gamma^*(a) = (a/2\pi)^{\frac{1}{2}}e^a a^{-a}\Gamma(a)$. The function $f(\xi)$ involved in (3.8.3) is defined by

$$f(\xi) = \frac{\xi}{\tau(\xi) - 1}, \quad \tau(\xi) \neq 1 \tag{3.8.4}$$

and has the following properties:

(i) $f(\xi)$ is a continuous decreasing function of ξ;

(ii) $f(\xi) \sim 1$, when $\xi \sim 0$.

The proof of these properties is given in Appendix B.4.

Using the bound $f(\xi) \leq 1$ when $\xi > 0$ in (3.8.3), we can then show that the incomplete gamma function satisfies

$$\begin{aligned} Q(a,x) &< \frac{1}{\Gamma^*(a)\pi^{1/2}}\int_{\eta\sqrt{a/2}}^{\infty} e^{-w^2}dw \\ &= \frac{1}{2\Gamma^*(a)}\text{erfc}\,(\eta\sqrt{\frac{a}{2}}). \end{aligned} \tag{3.8.5}$$

Also, $\Gamma(a)$ can be expressed as [Abramowitz & Stegun, 1968, p. 257]:

$$\Gamma(a) = \sqrt{2\pi}a^{a-\frac{1}{2}}e^{-a}e^{\theta/12a}, \quad a > 0,\ 0 < \theta < 1.$$

From the above equation and the definition of $\Gamma^*(a)$, we have

$$\Gamma^*(a) = e^{\theta/12a}.$$

Thus, $\Gamma^*(a) > 1$ for $a > 0$.

The complementary error function satisfies the following inequality for $x \geq 0$,

$$\begin{aligned} \text{erfc}\,x &= \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-u^2}du = \frac{2}{\sqrt{\pi}}e^{-x^2}\int_0^{\infty} e^{-2xw-w^2}dw \\ &\leq \frac{2}{\sqrt{\pi}}e^{-x^2}\int_0^{\infty} e^{-w^2}dw = e^{-x^2}. \end{aligned} \tag{3.8.6}$$

Substituting (3.8.6) into (3.8.5), we therefore have the following inequality for $\eta > 0$,

$$Q(a,x) \leq \frac{1}{2}\exp\,(-\frac{a}{2}\eta^2), \quad a > 0,\ x \geq 0. \tag{3.8.7}$$

We also have $0 < Q(a,x) \leq 1$ for $a > 0$ and $x \geq 0$ by the definition of $Q(a,x)$.

Next, we investigate the sum $|\sum_{n=1}^{\infty} n^{-s}Q(a, n^2/K^2)|$, where $K = K(t) \to +\infty$ as $t \to +\infty$ as mentioned in the last section. Letting

$$n^* = \text{int}\,[K\sqrt{a}], \quad x_0 = K\sqrt{a} = n^* + p, \quad 0 \leq p < 1,$$

we have

$$\lambda = \frac{n^2}{aK^2}.$$

Thus, for $n \leq n^*$, $\lambda \leq 1$ which means that $\eta \leq 0$, and for $n > n^*$, $\lambda > 1$ which means that $\eta > 0$.

We divide the sum $\sum_{n=1}^{\infty} n^{-s}Q(a, n^2/K^2)$ into two parts as follows ($\sigma = \frac{1}{2}$):

$$\begin{aligned} |\sum_{n=1}^{\infty} n^{-s}Q(a, n^2/K^2)| &\leq \sum_{n=1}^{\infty} n^{-\frac{1}{2}}Q(a, n^2/K^2) \\ &= \sum_{n=1}^{n^*} n^{-\frac{1}{2}}Q(a, n^2/K^2) + \sum_{n=n^*+1}^{\infty} n^{-\frac{1}{2}}Q(a, n^2/K^2). \end{aligned} \tag{3.8.8}$$

The first part of (3.8.8) which is $\eta \leq 0$ can be estimated by

$$\sum_{n=1}^{n^*} n^{-\frac{1}{2}}Q(a, n^2/K^2) \leq \sum_{n=1}^{n^*} n^{-\frac{1}{2}} < \int_0^{n^*} x^{-\frac{1}{2}}dx = 2n^{*\frac{1}{2}}. \tag{3.8.9}$$

Thus, the first part is $O(n^{*\frac{1}{2}})$. Using the inequality (3.8.7) and considering that the function $e^{-a((n/x_0)^2-1)}(n/x_0)^{2a-1/2}$ is a decreasing function of n, the second part, which is the smoothed tail when $\eta > 0$, is

$$\begin{aligned} \sum_{n=n^*+1}^{\infty} n^{-\frac{1}{2}}Q(a, n^2/K^2) &\leq \frac{1}{2}\sum_{n=n^*+1}^{\infty} n^{-\frac{1}{2}}e^{-\frac{1}{2}a\eta^2} \\ &< \frac{1}{2}n^{*-\frac{1}{2}} + \frac{1}{2x_0^{\frac{1}{2}}}\sum_{n=n^*+2}^{\infty} e^{-a((\frac{n}{x_0})^2-1)}(\frac{n}{x_0})^{2a-\frac{1}{2}} \\ &< \frac{1}{2}n^{*-\frac{1}{2}} + \frac{1}{2x_0^{\frac{1}{2}}}\int_{n=n^*+1}^{\infty} e^{-a((\frac{x}{x_0})^2-1)}(\frac{x}{x_0})^{2a-\frac{1}{2}}dx \\ &< \frac{1}{2}n^{*-\frac{1}{2}} + \frac{1}{2x_0^{\frac{1}{2}}}\int_{x_0}^{\infty} e^{-a((\frac{x}{x_0})^2-1)}(\frac{x}{x_0})^{2a-\frac{1}{2}}dx \\ &< \frac{1}{2}n^{*-\frac{1}{2}} + x_0^{\frac{1}{2}}\frac{\Gamma(a+\frac{1}{4}, a)e^a}{4a^{a+\frac{1}{4}}}. \end{aligned} \tag{3.8.10}$$

When a is large, we find $\Gamma(a+\frac{1}{4}, a)e^a/4a^{a+\frac{1}{4}} \sim O(a^{-\frac{1}{2}})$. Since $x_0 \sim n^*$, by the definition of x_0, the second part is consequently $O(n^{*\frac{1}{2}})$.

Collecting the above results together, we have

$$|\sum_{n=1}^{\infty} n^{-s}Q(a, n^2/K^2)| = O(n^{*\frac{1}{2}}), \tag{3.8.11}$$

where we recall that $n^* = \text{int}\,[K\sqrt{a}]$.

3.9 Estimation of the Order of $\zeta(\frac{1}{2}+it)$

The famous Lindelöf Hypothesis is that $\zeta(\frac{1}{2}+it) = O(t^{\varepsilon})$ as $t \to \infty$, where ε is an arbitrarily small positive number. It also has been shown by Walfsz that van der Corput's method gives $\zeta(\frac{1}{2}+it) = O(t^{163/988})$, where $163/988 \simeq 0.16498$ [Titchmarsh, 1930, p. 26]. At the present, the best-known result obtained by classical methods is that due to Kolesnik which is $\zeta(\frac{1}{2}+it) = O(|t|^{35/216+\varepsilon})$ for all $\varepsilon > 0$ as $|t| \to \infty$, where $35/216 \simeq 0.16204$. A new method in which estimates of Kloosterman sums obtained from the theory of modular forms are used has been applied by Bombieri and Iwaniec to yield the estimate $\zeta(\frac{1}{2}+it) = O(|t|^{9/56+\varepsilon})$ for all $\varepsilon > 0$ as $|t| \to \infty$. In this case $9/56 \simeq 0.16071$ [Patterson, 1988, p. 100]. In this section, using a similar method to Section 3.7, we explore the consequences of the new formula for $\zeta(s)$ in (3.2.6). We first study the remainder term J and then the order of the main sum in (3.2.6), which is estimated in Section 3.8. Use of these results then yields an estimate for the order of $\zeta(\frac{1}{2}+it)$ as $t \to \infty$: this is found to be $\zeta(\frac{1}{2}+it) = O(t^{\frac{1}{4}+\varepsilon})$ for $\varepsilon > 0$. This result is, of course, not very sharp and can be found in the standard text by Titchmarsh [Titchmarsh, 1986, p. 95]. Further improvement in this estimate would require sharper bounds in the estimation of J and the main sum.

For convenience, we write J $(p=1)$ as

$$J = \frac{1}{2\pi i \Gamma(a)} \int_{-c_0-i\infty}^{-c_0+i\infty} \Gamma(a+\frac{w}{2}) K^w \zeta(s+w) \frac{dw}{w}, \tag{3.9.1}$$

where a is a positive number and $0 < c_0 < 2a$. Letting $w = 2z+1-s$, $\nu = a - s/2$, and using (3.2.10), J can be written as

$$J = \frac{\pi^{s-\frac{1}{2}}}{2\pi i \Gamma(a)} \int_{-c_3-i\infty}^{-c_3+i\infty} \frac{\Gamma(\frac{1}{2}+z+\nu)\Gamma(-z)}{\Gamma(z+\frac{1}{2})(z-\frac{s}{2}+\frac{1}{2})} (\pi^2 K^2)^{z-\frac{s}{2}+\frac{1}{2}} \zeta(-2z) dz, \tag{3.9.2}$$

where $c_3 = \frac{c_0}{2} + \frac{1}{4}$ and $\frac{1}{4} < c_3 < a + \frac{1}{4}$.

We deal with the case that s is on the critical line $(\sigma = \frac{1}{2})$. To simplify expression (3.9.2), let $\delta_2 = c_3 - \frac{1}{4}$, $\rho = a - \delta_2$. Clearly $\delta_2 > 0$, $\rho > 0$. We can also choose c_3 to satisfy $\rho > \frac{1}{2}$ which requires $a > \frac{1}{2}$. It is noted that this choice of c_3 is not necessary but just for convenience in presentation. Letting $z = -c_3 + iy$

in (3.9.2), then J becomes

$$J = -\frac{\pi^{s-\frac{1}{2}}}{2\pi\Gamma(a)}\int_{-\infty}^{\infty}\frac{\Gamma(\rho+iu)\Gamma(c_3-iy)}{\Gamma(\frac{1}{2}-c_3+iy)(\delta_2-iu)}(\pi^2K^2)^{iu-\delta_2}\zeta(2c_3-2iy)dy, \tag{3.9.3}$$

where $u = y - \frac{t}{2}$. We divide J into the following four parts

$$\begin{aligned} J &= -\frac{\pi^{s-\frac{1}{2}}}{2\pi\Gamma(a)}\{\int_{-\infty}^{0} + \int_{0}^{\frac{t}{2}-\Delta} + \int_{\frac{t}{2}-\Delta}^{\frac{t}{2}+\Delta} + \int_{\frac{t}{2}+\Delta}^{+\infty}\}F_3(y)dy \\ &= J_1 + J_2 + J_3 + J_4, \end{aligned} \tag{3.9.4}$$

where

$$F_3(y) = \frac{\Gamma(\rho+iu)\Gamma(c_3-iy)}{\Gamma(\frac{1}{2}-c_3+iy)(\delta_2-iu)}(\pi^2K^2)^{iu-\delta_2}\zeta(2c_3-2iy),$$

and Δ is a function of t which satisfies

$$\Delta = \Delta(t) \to +\infty, \quad \Delta^2 << t, \quad t \to +\infty. \tag{3.9.5}$$

Define

$$\bar{F}(y) = \left|\frac{\Gamma(c_3-iy)}{\Gamma(\frac{1}{2}-c_3+iy)}\right|, \quad y > 0. \tag{3.9.6}$$

and

$$\bar{G}(y) = y^{-2\delta_2}\bar{F}(y). \tag{3.9.7}$$

In Appendix B.3, it is shown that $\bar{F}(y)$ is an even monotonically increasing function and $\bar{G}(y)$ is a decreasing function of $y > 0$ when $c_3 \geq 1$. We can choose $2c_3 > 1$ and use a similar method in Section 3.7. We have

$$\begin{aligned} |J_1| &\leq \frac{1}{2\pi\Gamma(a)}(\pi^2K^2)^{-\delta_2}\zeta(2c_3)\int_{-\infty}^{0}\bar{F}(y)\frac{|\Gamma(\rho+i(y-\frac{t}{2}))|}{\{\delta_2^2+(y-\frac{t}{2})^2\}^{\frac{1}{2}}}dy \\ &\leq \frac{1}{2\pi\Gamma(a)}(\pi^2K^2)^{-\delta_2}\zeta(2c_3)\int_{0}^{+\infty}\bar{F}(y)\frac{|\Gamma(\rho+i(y+\frac{t}{2}))|}{\{\delta_2^2+(y+\frac{t}{2})^2\}^{\frac{1}{2}}}dy. \end{aligned} \tag{3.9.8}$$

Using $\chi = \rho$ and $u = y + \frac{t}{2}$ in (3.7.10), (3.9.8) becomes

$$\begin{aligned} |J_1| &\leq \frac{K_1}{\pi t\Gamma(a)}(\pi^2K^2)^{-\delta_2}\zeta(2c_3)e^{-\pi t/4}\int_{0}^{+\infty}\bar{F}(y)(y+t/2)^{\rho-\frac{1}{2}}e^{-\frac{\pi}{2}y}dy \\ &< \frac{K_1}{\pi t\Gamma(a)}(\pi^2K^2)^{-\delta_2}\zeta(2c_3)e^{-\pi t/4}(t/2)^{\rho-\frac{1}{2}} \\ &\quad \times \int_{0}^{+\infty}\bar{F}(y)(y+1)^{\rho-\frac{1}{2}}e^{-\frac{\pi}{2}y}dy, \end{aligned} \tag{3.9.9}$$

where we recall that $\rho > \frac{1}{2}$. The last integral is independent of t and is finite, since $\bar{F}(y) \sim y^{2\delta_2}$ as $t \to +\infty$. Hence

$$|J_1| \sim O(K^{-2\delta_2} e^{-\pi t/4} t^{\rho-\frac{3}{2}}).$$

Using the same procedure as in the analysis of $|J_1|$, we also have

$$\begin{aligned} |J_2| &\leq \frac{1}{2\pi\Gamma(a)}(\pi^2K^2)^{-\delta_2}\zeta(2c_3)\int_0^{\frac{t}{2}-\Delta}\bar{F}(y)\frac{|\Gamma(\rho+i(y-\frac{t}{2}))|}{\{\delta_2^2+(y-\frac{t}{2})^2\}^{\frac{1}{2}}}dy \\ &< \frac{K_1}{2\pi\Gamma(a)\Delta}(\pi^2K^2)^{-\delta_2}\zeta(2c_3)\bar{F}(t/2-\Delta)\int_0^{\frac{t}{2}-\Delta}(t/2-y)^{\rho-\frac{1}{2}}e^{-\frac{\pi}{2}(\frac{t}{2}-y)}dy \\ &< \frac{K_1}{2\pi\Gamma(a)\Delta}(\pi^2K^2)^{-\delta_2}\zeta(2c_3)\bar{F}(t/2-\Delta)(t/2)^{\rho-\frac{1}{2}}(t/2-\Delta)e^{-\pi\Delta/2}, \quad (3.9.10) \end{aligned}$$

to yield

$$|J_2| \sim O(K^{-2\delta_2}\Delta^{-1}(t/2-\Delta)^{2\delta_2+1}e^{-\pi\Delta/2}t^{\rho-\frac{1}{2}}),$$

while the part J_4 is given by

$$\begin{aligned} |J_4| &\leq \frac{1}{2\pi\Gamma(a)}(\pi^2K^2)^{-\delta_2}\zeta(2c_3)\int_{\frac{t}{2}+\Delta}^{+\infty}\bar{F}(y)\frac{|\Gamma(\rho+i(y-\frac{t}{2}))|}{\{\delta_2^2+(y-\frac{t}{2})^2\}^{\frac{1}{2}}}dy \\ &< \frac{K_1}{2\pi\Gamma(a)\Delta}(\pi^2K^2)^{-\delta_2}\zeta(2c_3)\bar{G}(t/2+\Delta)e^{\pi t/4}\int_{\frac{t}{2}+\Delta}^{+\infty}y^{2\delta_2}(y-t/2)^{\rho-\frac{1}{2}}e^{-\frac{\pi}{2}y}dy \\ &< \frac{K_1}{2\pi\Gamma(a)\Delta}(\pi^2K^2)^{-\delta_2}\zeta(2c_3)\bar{G}(t/2+\Delta)e^{\pi t/4}(t/2)^{\rho-\frac{1}{2}} \\ &\quad \times\int_{\frac{t}{2}+\Delta}^{+\infty}y^{2\delta_2}(y-1)^{\rho-\frac{1}{2}}e^{-\frac{\pi}{2}y}dy. \quad (3.9.11) \end{aligned}$$

The integral in (3.9.11) satisfies the inequality

$$\int_{\frac{t}{2}+\Delta}^{+\infty}y^{2\delta_2}(y-1)^{\rho-\frac{1}{2}}e^{-\frac{\pi}{2}y}dy < \int_{\frac{t}{2}+\Delta}^{+\infty}y^{2\delta_2+\rho-\frac{1}{2}}e^{-\frac{\pi}{2}y}dy;$$

the right side of the above inequality can be expressed in terms of an incomplete gamma function by

$$(2/\pi)^{2\delta_2+\rho+\frac{1}{2}}\Gamma(2\delta_2+\rho+1/2, \pi(t/2+\Delta)/2),$$

so that

$$|J_4| \sim O(K^{-2\delta_2}\Delta^{-1}\,t^{\rho-\frac{1}{2}}(t/2+\Delta)^{2\delta_2+\rho-\frac{1}{2}}e^{-\pi\Delta/2}).$$

The parts J_1, J_2 and J_4 are thus exponentially small when t is large. The asymptotic form of the main part J_3 will be studied next. J_3 can be written as

$$J_3 = -\frac{\pi^{s-\frac{1}{2}}}{2\pi\Gamma(a)}\int_{\frac{t}{2}-\Delta}^{\frac{t}{2}+\Delta}\bar{F}(y)\frac{\Gamma(\rho+i(y-\frac{t}{2}))}{(\delta_2-i(y-\frac{t}{2}))}(\pi^2K^2)^{i(y-\frac{t}{2})-\delta_2}\zeta(2c_3-2iy)dy. \quad (3.9.12)$$

Letting $w = i(y - \frac{t}{2}) - \delta_2$, we write (3.9.12) as

$$J_3 = \frac{\pi^{s-\frac{1}{2}}}{2\pi i\Gamma(a)} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}-it}} \int_{-\delta_2-i\Delta}^{-\delta_2+i\Delta} \frac{\Gamma(\frac{1}{4} - w - \frac{t}{2}i)}{\Gamma(\frac{1}{4} + w + \frac{t}{2}i)} \frac{\Gamma(\rho + \delta_2 + w)}{w} (\pi^2 K^2 n^2)^w dw, \tag{3.9.13}$$

and recall that $2c_3 > 1$ with $\delta_2 = c_3 - \frac{1}{4}$.

The asymptotic expansion of $\log \Gamma(z)$ valid for $|z| \to +\infty$ in $|\arg z| < \pi$ is

$$\log \Gamma(z) = (z - \frac{1}{2}) \log z - z + \frac{1}{2} \log (2\pi) + O(\frac{1}{|z|}). \tag{3.9.14}$$

Since the range of the variable w is $(-\delta_2 - i\Delta,\ -\delta_2 + i\Delta)$ in (3.9.13), then $w + \frac{t}{2}i \to \infty$ as $t \to +\infty$. Using (3.9.14), the asymptotic behaviour of $\Gamma(\frac{1}{4} - w - \frac{t}{2}i)/\Gamma(\frac{1}{4} + w + \frac{t}{2}i)$ can be expressed as

$$\begin{aligned}
\frac{\Gamma(\frac{1}{4} - w - \frac{it}{2})}{\Gamma(\frac{1}{4} + w + \frac{it}{2})} &= \exp\{\log \Gamma(\frac{1}{4} - w - \frac{it}{2}) - \log \Gamma(\frac{1}{4} + w + \frac{it}{2})\} \\
&= \exp\{2w + it + (-\frac{1}{4} - w - \frac{it}{2})[\log(-\frac{it}{2}) + \log(1 - \frac{\frac{1}{4} - w}{\frac{it}{2}})] \\
&\quad -(-\frac{1}{4} + w + \frac{it}{2})[\log \frac{it}{2} + \log(1 + \frac{\frac{1}{4} + w}{\frac{it}{2}})] + O(\frac{1}{t})\} \\
&= \exp\{2w + it + (-\frac{1}{4} - w - \frac{it}{2}) \log(-\frac{it}{2}) \\
&\quad +(-\frac{1}{4} - w - \frac{it}{2})[\frac{w - \frac{1}{4}}{\frac{it}{2}} - O(\frac{\Delta^2}{(\frac{t}{2})^2})] \\
&\quad -(-\frac{1}{4} + w + \frac{it}{2}) \log(\frac{it}{2}) - (-\frac{1}{4} + w + \frac{it}{2})[\frac{\frac{1}{4} + w}{\frac{it}{2}} - O(\frac{\Delta^2}{(\frac{t}{2})^2})]\}.
\end{aligned} \tag{3.9.15}$$

Using the condition (3.9.5) in (3.9.15), we have

$$\begin{aligned}
\frac{\Gamma(\frac{1}{4} - w - \frac{it}{2})}{\Gamma(\frac{1}{4} + w + \frac{it}{2})} &\sim \exp\{it + (-\frac{1}{4} - \frac{it}{2}) \log(-\frac{it}{2}) - (-\frac{1}{4} + \frac{it}{2}) \log \frac{it}{2} \\
&\qquad -w \log(-\frac{it}{2}) - w \log \frac{it}{2}\} \\
&\sim \exp\{it + \frac{i\pi}{4} - it \log \frac{t}{2}\} (\frac{t^2}{4})^{-w}.
\end{aligned} \tag{3.9.16}$$

Putting (3.9.16) into (3.9.13), we find

$$J_3 \simeq e^{it - it \log \frac{t}{2} + \frac{\pi}{4} i} \frac{\pi^{it}}{2\pi i\Gamma(a)} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}-it}} \int_{-\delta_2-i\Delta}^{-\delta_2+i\Delta} \frac{\Gamma(\rho + \delta_2 + w)}{w} (\frac{t^2}{4\pi^2 K^2 n^2})^{-w} dw. \tag{3.9.17}$$

Since $\Delta = \Delta(t)$ is large as $t \to +\infty$, applying the residue theorem to the integral in (3.9.17), we find

$$\begin{aligned}
&\int_{-\delta_2-i\Delta}^{-\delta_2+i\Delta} \frac{\Gamma(\rho+\delta_2+w)}{w}(\frac{t^2}{4\pi^2K^2n^2})^{-w}dw \\
= \quad & -2\pi i\Gamma(\rho+\delta_2) + \int_{\delta_2-i\Delta}^{\delta_2+i\Delta} \frac{\Gamma(\rho+\delta_2+w)}{w}(\frac{t^2}{4\pi^2K^2n^2})^{-w}dw. \qquad (3.9.18)
\end{aligned}$$

From (B.1.6), we have

$$\lim_{\Delta\to+\infty}\frac{1}{2\pi i}\int_{\delta_2-i\Delta}^{\delta_2+i\Delta} \frac{\Gamma(\rho+\delta_2+w)}{w}(\frac{t^2}{4\pi^2K^2n^2})^{-w}dw = \Gamma(\rho+\delta_2, \frac{t^2}{4\pi^2K^2n^2}).$$

By the definition of lim in the above formula and (3.9.18), we obtain

$$\frac{1}{2\pi i}\int_{-\delta_2-i\Delta}^{-\delta_2+i\Delta} \frac{\Gamma(\rho+\delta_2+w)}{w}(\frac{t^2}{4\pi^2K^2n^2})^{-w}dw = -\Gamma(\rho+\delta_2)+\Gamma(\rho+\delta_2, \frac{t^2}{4\pi^2K^2n^2})+\mathbf{E.S.T.}, \qquad (3.9.19)$$

where **E.S.T.** denotes the exponentially small term. Substitution of (3.9.19) into (3.9.17), then yields

$$\begin{aligned}
J_3 \quad &\sim \quad e^{it-it\log\frac{t}{2}+\frac{\pi}{4}i}\frac{\pi^{it}}{\Gamma(a)}\sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{2}-it}}\{-\Gamma(\rho+\delta_2)+\Gamma(\rho+\delta_2, \frac{t^2}{4\pi^2K^2n^2})\} \\
&\sim \quad -e^{it-it\log\frac{t}{2}+\frac{\pi}{4}i}\pi^{it}\sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{2}-it}}P(a, \frac{t^2}{4\pi^2K^2n^2}), \qquad (3.9.20)
\end{aligned}$$

where we recall that $a = \rho + \delta_2$. The modulus of J_3 is then given approximately by the sum

$$|J_3| \sim \sum_{n=1}^{\infty} n^{-\frac{1}{2}}P(a, \frac{t^2}{4\pi^2K^2n^2}). \qquad (3.9.21)$$

The parameter K is an increasing function of t, and we can choose $K = t^{\nu}$ with $0 < \nu \leq 1$. We have proved that $\sum_{n=1}^{\infty} n^{-\frac{1}{2}}Q(a, n^2/K^2) = O(n_1^{*\frac{1}{2}})$ in Section 3.8 for $n_1^* = \text{int}\,[K\sqrt{a}]$. Thus, for finite a, we have

$$\sum_{n=1}^{\infty} n^{-s}Q(a, n^2/K^2) = O(K^{\frac{1}{2}}) = O(t^{\frac{\nu}{2}}).$$

The function $P(a, \frac{t^2}{4\pi^2K^2n^2}) \simeq 1$ for $n \leq n_2^*$ and decays to zero when $n > n_2^*$, where $n_2^* = \text{int}\,[t/(2\pi K\sqrt{a})]$. Thus, the series $\sum_{n=1}^{\infty} n^{-\frac{1}{2}}P(a, \frac{t^2}{4\pi^2K^2n^2})$ is convergent. We also have

$$\sum_{n=1}^{\infty} n^{-\frac{1}{2}}P(a, \frac{t^2}{4\pi^2K^2n^2}) = O(n_2^{*\frac{1}{2}}) = O(\sqrt{t/K}) = O(t^{\frac{1}{2}-\frac{\nu}{2}}).$$

Since the term $\{\Gamma(a+\frac{1-s}{2})K^{1-s}\}/\{(1-s)\Gamma(a)\}$ in (3.2.6) is exponentially small for large t, we therefore find that

$$\zeta(\frac{1}{2}+it) = O(t^{\frac{\nu}{2}}) + O(t^{\frac{1}{2}-\frac{\nu}{2}}).$$

The best choice of ν should satisfy $\frac{\nu}{2} = \frac{1}{2} - \frac{\nu}{2}$, that is $\nu = \frac{1}{2}$, which leads to the estimate

$$\zeta(\frac{1}{2}+it) = O(t^{\frac{1}{4}}). \tag{3.9.22}$$

3.10 Conclusions

The principal result of this chapter is that we gave some asymptotic formulas for the Riemann zeta function $\zeta(s)$, which hold in the strip $0 < \sigma < 1$ in the s plane. The general asymptotic formula (3.3.8) consists of a main sum smoothed by a confluent hypergeometric function together with a series of truncated terms and a remainder term. We can obtain different asymptotic formulas for different choices of a and b in (3.3.8). Some special cases of (3.3.8) were given, such as the asymptotic formula (3.2.11), which is smoothed by the incomplete gamma function, and the asymptotic formula (3.2.22) which is smoothed by the exponential factor. We also derived (3.2.11) using a different method.

Extensive numerical results of the asymptotic formula (3.2.11) were given for different a and t. It is shown that formula (3.2.11) is more accurate when the truncation index M large.

To establish the asymptotic nature of the asymptotic formulas would require a detailed treatment of these remainder terms. We have obtained a bound on the remainder term R_M for general positive a, and that the bound is quite realistic until $M \simeq \text{int}\,[t/2p]$. We also demonstrated that the main contribution of the remainder arises from the part $\bar{I}_3$ of the integral which is given in (3.7.8) and explored the consequences of the new formula (3.2.6) by estimating the order of the Riemann zeta function $\zeta(\frac{1}{2}+it)$ as $t \to \infty$.

CHAPTER 4

ASYMPTOTIC BEHAVIOUR OF THE COEFFICIENTS $c_r(\eta)$ FOR LARGE r

4.1 Introduction

The asymptotic behaviour of the coefficients $c_r(\eta)$ appearing in the uniformly asymptotic expansion of the incomplete gamma function (2.3.1) for large r is studied in this chapter.

The outline of this chapter is as follows: The asymptotic form of the coefficients $c_r(\eta)$ defined in (2.3.2) for large r is developed in Section 4.2. The function $F_r(z)$ which is defined in (4.2.23) and involved in the asymptotic form of $c_r(\eta)$ is studied in Section 4.3. Extensive numerical results and discussion of the asymptotic form of $c_r(\eta)$ for large r are presented in Section 4.4. Since the function $G_r(z)$ defined in (4.3.7) is contained in the representation of the first and second leading term of $F_r(z)$, the asymptotic formula of $G_r(z)$ involving the complementary error function is proposed in Section 4.5. Numerical results which compare the values of the asymptotic formula and the exact value of $G_r(z)$ are given in Section 4.6. Application of the asymptotic formula of $G_r(z)$ to the main asymptotic form of $c_r(\eta)$, an additional asymptotic form of $c_r(\eta)$ for large r is considered in Section 4.7. Some useful formulas are included in Appendix C.

4.2 Asymptotic Form of the Coefficients $c_r(\eta)$ for Large r

In this section, the asymptotic form of the coefficients $c_r(\eta)$ defined in (2.3.2), for large r is proposed. For clarification, (2.3.2) is rewritten here

$$c_r(\eta) = (-)^r \frac{Q_r(\mu)}{\mu^{2r+1}} - \frac{D_r}{\eta^{2r+1}}, \tag{4.2.1}$$

where

$$\lambda = z/a, \ \mu = \lambda - 1, \ \frac{1}{2}\eta^2 = \mu - \log(1+\mu), \tag{4.2.2}$$

$Q_r(\mu)$ is a polynomial in μ of degree $2r$ and $D_r = (-2)^r \Gamma(r+\frac{1}{2})/\Gamma(\frac{1}{2})$. The coefficients $c_r(\eta)$ satisfy the following recurrence relation [Temme, 1979]

$$\begin{aligned} c_0(\eta) &= \frac{1}{\mu} - \frac{1}{\eta}, \\ c_r(\eta) &= \frac{1}{\eta}\frac{d}{d\eta}c_{r-1}(\eta) + \frac{\gamma_r}{\mu}, \quad r \geq 1, \end{aligned} \tag{4.2.3}$$

where γ_r are the Stirling coefficients [see (2.3.12)]. The above recurrence relation can be written as

$$\begin{aligned} c_0(\eta) &= \frac{1}{\mu} - \frac{1}{\eta}, \\ \frac{dc_{r-1}(\eta)}{d\eta^2} &= \frac{1}{2}(c_r(\eta) - \frac{1}{\mu}\gamma_r), \quad r \geq 1. \end{aligned} \tag{4.2.4}$$

The coefficients $c_r(\eta)$ have a removable singularity at $\eta = 0$, so that $c_r(\eta)$ are analytic at $\eta = 0$.

We study the leading asymptotic behaviour of the coefficients $c_r(\eta)$ for large r by means of the Maclaurin series representation

$$c_r(\eta) = \sum_{k=0}^{\infty} \beta_{r\,k}\eta^k, \quad |\eta| < 2\sqrt{\pi}.$$

The coefficients $\beta_{r\,k}$ satisfy the relation [Temme, 1979]

$$\beta_{r\,k} = \gamma_r\beta_{0\,k} + \gamma_{r-1}(k+2)\beta_{0\,k+2} + \cdots + \gamma_0(k+2)\cdots(k+2r)\beta_{0\,k+2r}, \tag{4.2.5}$$

where

$$\beta_{00} = -\frac{1}{3}, \quad \beta_{0k} = (k+2)\alpha_{k+2}, \quad k \geq 1, \tag{4.2.6}$$

and the first few α_r are given in (2.8.11). The relation between α_r and γ_r is

$$\gamma_r = (-)^r 1 \cdot 3 \cdot 5 \cdots (2r+1)\alpha_{2r+1}, \quad r = 0, 1, 2, \cdots. \tag{4.2.7}$$

Substitution of (4.2.6) into (4.2.5) leads to the result

$$\beta_{rk} = \frac{2^{r+1}}{(\frac{k}{2})!} \sum_{j=0}^{r} \frac{\gamma_j}{2^j} (\frac{k}{2} + r + 1 - j)! \, \alpha_{k+2r+2-2j}, \quad r, \; k \neq 0.$$

Since $\alpha_2 = \frac{1}{3}$, but $\beta_{00} = -\frac{1}{3} = 2\alpha_2 - 1$, we obtain

$$c_r(\eta) = 2^{r+1} \sum_{j=0}^{r} \frac{\gamma_j}{2^j} \{ \sum_{k=0}^{\infty} \frac{(\frac{k}{2} + r + 1 - j)!}{(\frac{k}{2})!} \alpha_{k+2r+2-2j} \eta^k \} - \gamma_r, \quad |\eta| < 2\sqrt{\pi}. \tag{4.2.8}$$

We now evaluate the inner sum over k in (4.2.8) and establish a connection with this sum to the coefficients $c_r(\eta)$ and the Stirling coefficients γ_r. Let us first take $j = r$ and consider the sum

$$\sum_{k=0}^{\infty} (\frac{k}{2} + 1)\alpha_{k+2}\eta^k, \quad |\eta| < 2\sqrt{\pi}. \tag{4.2.9}$$

Using the expansion of μ in powers of η in (2.8.10), we have

$$\mu - \eta = \alpha_2\eta^2 + \alpha_3\eta^3 + \cdots = \sum_{k=0}^{\infty} \alpha_{k+2}\eta^{2(\frac{k}{2}+1)}, \quad |\eta| < 2\sqrt{\pi}. \tag{4.2.10}$$

Taking the derivative of both sides in (4.2.10) with respect to η^2 gives

$$\frac{d}{d\eta^2}(\mu - \eta) = \sum_{k=0}^{\infty} (\frac{k}{2} + 1)\alpha_{k+2}\eta^k. \tag{4.2.11}$$

From the relation between μ and η given in (4.2.2), we find

$$\frac{d\mu}{d\eta} = \frac{(\mu + 1)\eta}{\mu},$$

so that from (4.2.3), we have

$$\frac{d}{d\eta^2}(\mu - \eta) = \frac{1}{2} + \frac{1}{2}c_0(\eta). \tag{4.2.12}$$

Substitution of (4.2.12) into (4.2.11), then yields

$$\sum_{k=0}^{\infty} (\frac{k}{2} + 1)\alpha_{k+2}\eta^k = \frac{1}{2} + \frac{1}{2}c_0(\eta). \tag{4.2.13}$$

Then the terms involving γ_r in (4.2.8) become

$$2\gamma_r \sum_{k=0}^{\infty} (\frac{k}{2} + 1)\alpha_{k+2}\eta^k - \gamma_r = \gamma_r c_0(\eta). \tag{4.2.14}$$

Using a similar reasoning as above, the sum of the second term in (4.2.8) involving γ_{r-1} can be written as

$$4\gamma_{r-1} \sum_{k=0}^{\infty} (1 + \frac{k}{2})(2 + \frac{k}{2})\alpha_{k+4}\eta^k. \tag{4.2.15}$$

Considering the expansion of μ in powers of η in (4.2.10) again, we have

$$\mu - \eta - \alpha_2\eta^2 - \alpha_3\eta^3 = \sum_{k=0}^{\infty} \alpha_{k+4}\eta^{2(\frac{k}{2}+2)}. \tag{4.2.16}$$

Taking the derivative of both sides in (4.2.16) with respect to η^2 leads to

$$\frac{d}{d\eta^2}(\mu - \eta - \alpha_2\eta^2 - \alpha_3\eta^3) = \sum_{k=0}^{\infty} (\frac{k}{2} + 2)\alpha_{k+4}\eta^{2(\frac{k}{2}+1)}. \tag{4.2.17}$$

Putting (4.2.12) into (4.2.17) gives

$$\frac{1}{2}(c_0(\eta) + 1) - \frac{d}{d\eta^2}(\alpha_2\eta^2 + \alpha_3\eta^3) = \sum_{k=0}^{\infty} (\frac{k}{2} + 2)\alpha_{k+4}\eta^{2(\frac{k}{2}+1)}$$

and differentiating both sides of the above equation with respect to η^2, we have

$$\frac{1}{2}\frac{d}{d\eta^2}c_0(\eta) - \frac{d}{d\eta^2}\frac{d}{d\eta^2}(\alpha_2\eta^2 + \alpha_3\eta^3) = \sum_{k=0}^{\infty} (\frac{k}{2} + 1)(\frac{k}{2} + 2)\alpha_{k+4}\eta^k. \tag{4.2.18}$$

From (4.2.4) and (4.2.7), we then find

$$\sum_{k=0}^{\infty} (1 + \frac{k}{2})(2 + \frac{k}{2})\alpha_{k+4}\eta^k = \frac{1}{4}\{c_1(\eta)\gamma_0 - c_0(\eta)\gamma_1\}, \tag{4.2.19}$$

so that

$$\gamma_{r-1} \sum_{k=0}^{\infty} \alpha_{k+4}(k+2)(k+4)\eta^k = \gamma_{r-1}\{c_1(\eta)\gamma_0 - c_0(\eta)\gamma_1\}. \tag{4.2.20}$$

Observing the pattern in equations (4.2.14) and (4.2.20), we conjecture that the sum of pth term involving γ_{r-p} in (4.2.8) for $p = 0, 1, \cdots, r$ can be written as

$$\begin{aligned}&\gamma_{r-p} \sum_{k=0}^{\infty} (k+2)\cdots(k+2p)(k+2p+2)\alpha_{k+2p+2}\eta^k \\ &= \gamma_{r-p}\{\gamma_0 c_p(\eta) - \gamma_1 c_{p-1}(\eta) + \cdots + (-)^p \gamma_p c_0(\eta)\}.\end{aligned} \tag{4.2.21}$$

This result is proved by induction in Appendix C.1. Thus, the sum of the last term in (4.2.8), which is a special case of (4.2.21) for $p = r$, is

$$\gamma_0 \sum_{k=0}^{\infty} (k+2)\cdots(k+2r)(k+2r+2)\alpha_{k+2r+2}\eta^k$$
$$= \gamma_0\{\gamma_0 c_r(\eta) - \gamma_1 c_{r-1}(\eta) + \cdots + (-)^r \gamma_r c_0(\eta)\}. \tag{4.2.22}$$

This result will form the basis of our method of determining the leading behaviour of $c_r(\eta)$ for $r \gg 1$. We assume that the left side of (4.2.22) is given by

$$\frac{\Gamma(r+\frac{1}{2})}{(2\pi)^{r+1}} F_r(\eta),$$

where the function $F_r(\eta)$ is defined as follows

$$F_r(\eta) = \frac{(2\pi)^{r+1}}{\Gamma(r+\frac{1}{2})} \sum_{k=0}^{\infty} \gamma_0 (k+2)\cdots(k+2r)(k+2r+2)\alpha_{k+2r+2}\eta^k, \tag{4.2.23}$$

and write the coefficient $c_r(\eta)$ in the form

$$c_r(\eta) = \frac{\Gamma(r+\frac{1}{2})}{(2\pi)^{r+1}} f_r(\eta). \tag{4.2.24}$$

Then (4.2.22) becomes

$$\begin{aligned} \frac{\Gamma(r+\frac{1}{2})}{(2\pi)^{r+1}} F_r(\eta) &= \gamma_0 \frac{\Gamma(r+\frac{1}{2})}{(2\pi)^{r+1}} f_r(\eta) - \gamma_1 \frac{\Gamma(r-\frac{1}{2})}{(2\pi)^{r}} f_{r-1}(\eta) + \cdots \\ &+ (-)^{r-1}\gamma_{r-1} c_1(\eta) + (-)^r \gamma_r c_0(\eta). \end{aligned} \tag{4.2.25}$$

In order to obtain the asymptotic form of the coefficients $c_r(\eta)$, we employ the following well-known asymptotics of γ_r for large r in (4.2.25) [Boyd, 1995]

$$(-)^r \gamma_r \sim \begin{cases} \frac{2(-)^{\frac{r-1}{2}}\Gamma(r)}{(2\pi)^{r+1}}, & r \text{ is odd}, \\ \frac{2(-)^{\frac{r}{2}}\gamma_1\Gamma(r-1)}{(2\pi)^{r}}, & r \text{ is even}. \end{cases} \tag{4.2.26}$$

The two cases of (4.2.25) are discussed separately in the following depending on whether r is odd or even.

Case 1: r is odd

When r is odd, putting the asymptotic of γ_r in (4.2.26) for odd r into (4.2.25), we have the following asymptotic relation

$$\begin{aligned} \frac{\Gamma(r+\frac{1}{2})}{(2\pi)^{r+1}} F_r(\eta) &\sim \gamma_0 \frac{\Gamma(r+\frac{1}{2})}{(2\pi)^{r+1}} f_r(\eta) - \gamma_1 \frac{\Gamma(r-\frac{1}{2})}{(2\pi)^{r}} f_{r-1}(\eta) + \cdots \\ &+ \frac{2(-)^{(r-1)/2}}{(2\pi)^{r-1}} \gamma_1 \Gamma(r-2) c_1(\eta) + \frac{2(-)^{(r-1)/2}}{(2\pi)^{r+1}} \Gamma(r) c_0(\eta). \end{aligned} \tag{4.2.27}$$

The dominant terms of the right side in (4.2.27) are the first and last terms for odd r. The asymptotic behaviour of $f_r(\eta)$ then takes the form

$$f_r(\eta) \sim F_r(\eta) + (-)^{(r+1)/2} \frac{2c_0(\eta)}{\sqrt{r}}. \tag{4.2.28}$$

It is noted that the above asymptotic form of $f_r(\eta)$ contains the first coefficient $c_0(\eta)$ of (4.2.1).

Case 2: r is even

Using the case with r even in (4.2.26) and substituting this asymptotic form into (4.2.25), we obtain

$$\begin{aligned}\frac{\Gamma(r+\frac{1}{2})}{(2\pi)^{r+1}} F_r(\eta) &= \gamma_0 \frac{\Gamma(r+\frac{1}{2})}{(2\pi)^{r+1}} f_r(\eta) - \gamma_1 \frac{\Gamma(r-\frac{1}{2})}{(2\pi)^r} f_{r-1}(\eta) + \cdots \\ &+ \frac{2(-)^{(r-2)/2}}{(2\pi)^r} \Gamma(r-1) c_1(\eta) + \frac{2(-)^{r/2}}{(2\pi)^r} \gamma_1 \Gamma(r-1) c_0(\eta). \end{aligned} \tag{4.2.29}$$

When r is even, the dominant terms of the right side of (4.2.29) are the first two and the last two terms. Then the asymptotic form of $f_r(\eta)$ is

$$f_r(\eta) \sim F_r(\eta) + \frac{2\pi\gamma_1}{(r-\frac{1}{2})} f_{r-1}(\eta) + (-)^{r/2} \frac{4\pi\Gamma(r-1)}{\Gamma(r+\frac{1}{2})} \{c_1(\eta)\gamma_0 - c_0(\eta)\gamma_1\}. \tag{4.2.30}$$

The asymptotic form (4.2.30) includes the function $f_{r-1}(\eta)$ whose behaviour when $r-1$ is odd is discussed in Case 1.

So far, we have obtained the asymptotic form for the function $f_r(\eta)$ which includes the function $F_r(\eta)$. We will discuss the asymptotic behaviour of the function $F_r(\eta)$ in the next section.

4.3 Asymptotic Expansion of $F_r(\eta)$ for Large r

4.3.1 Representation of $F_r(\eta)$ for Large r

In this subsection, the first and second leading terms of the function $F_r(\eta)$ will be developed for large r. The definition of $F_r(\eta)$ is given by (4.2.23) which can also be written as

$$F_r(\eta) = \frac{(4\pi)^{r+1}}{\Gamma(r+\frac{1}{2})} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{k}{2}+r+2)}{\Gamma(\frac{k}{2}+1)} \alpha_{k+2r+2} \eta^k. \tag{4.3.1}$$

The above expression for $F_r(\eta)$ contains α_{k+2r+2} and the asymptotic expansion of α_r for large r is [Diekmann, 1975]

$$\alpha_r = (\frac{-1}{2\sqrt{\pi}})^r \frac{2\sqrt{2}}{r\sqrt{r}}\{a_r + (\frac{3}{4}a_r + \frac{\pi}{3}b_r)\frac{1}{r} + O(\frac{1}{r^2})\}, \tag{4.3.2}$$

where

$$a_r = -\sin\frac{r-1}{4}\pi, \quad b_r = \cos\frac{r-1}{4}\pi.$$

Using (4.3.2), the asymptotic expansion of α_{k+2r+2} for large r is

$$\begin{aligned}\alpha_{k+2r+2} &= \frac{2\sqrt{2}}{(k+2r+2)\sqrt{k+2r+2}}(\frac{-1}{2\sqrt{\pi}})^{k+2r+2}\{-\sin\frac{k+2r+1}{4}\pi \\ &+(-\frac{3}{4}\sin\frac{k+2r+1}{4}\pi + \frac{\pi}{3}\cos\frac{k+2r+1}{4}\pi)\frac{1}{k+2r+2} \\ &+O(\frac{1}{r^2})\}.\end{aligned} \tag{4.3.3}$$

Substituting (4.3.3) into (4.3.1), we find

$$\begin{aligned}F_r(\eta) &= \frac{(4\pi)^{r+1}}{\Gamma(r+\frac{1}{2})}\sum_{k=0}^{\infty}\frac{\Gamma(\frac{k}{2}+r+2)}{\Gamma(\frac{k}{2}+1)}\frac{(-)^k\eta^k}{(2\sqrt{\pi})^{k+2r+2}(\frac{k}{2}+r+1)^{3/2}} \\ &\{-\sin\frac{k+2r+1}{4}\pi + (-\frac{3}{4}\sin\frac{k+2r+1}{4}\pi + \frac{\pi}{3}\cos\frac{k+2r+1}{4}\pi) \\ &\frac{1}{k+2r+2} + O(\frac{1}{r^2})\} \\ &= -\frac{1}{\Gamma(r+\frac{1}{2})}\sum_{k=0}^{\infty}\frac{\Gamma(\frac{k}{2}+r+1)}{(\frac{k}{2}+r+1)^{\frac{1}{2}}}\frac{(-)^k z^k}{\Gamma(\frac{k}{2}+1)}\{\sin\frac{k+2r+1}{4}\pi \\ &+\frac{3}{4(k+2r+2)}\sin\frac{k+2r+1}{4}\pi - \frac{\pi}{3(k+2r+2)}\cos\frac{k+2r+1}{4}\pi \\ &+O(\frac{1}{r^2})\}\end{aligned} \tag{4.3.4}$$

where $z = \eta/(2\sqrt{\pi})$ and $|z| < 1$.

The above equation includes the ratios of two gamma functions. Putting $z = \frac{k}{2} + r + 1$, $a = 0$ and $b = -\frac{1}{2}$ in (B.2.3) for large r, we have

$$\frac{\Gamma(\frac{k}{2}+r+1)}{(\frac{k}{2}+r+1)^{\frac{1}{2}}} = \Gamma(\frac{k}{2}+r+\frac{1}{2})\{1 - \frac{3}{8}\frac{1}{(\frac{k}{2}+r+1)} + O(\frac{1}{r^2})\}. \tag{4.3.5}$$

Substitution of (4.3.5) into (4.3.4), the final asymptotic expansion of $F_r(\eta)$ is

$$\begin{aligned}F_r(\eta) &= -\frac{1}{\Gamma(r+\frac{1}{2})}\sum_{k=0}^{\infty}(-)^k\frac{\Gamma(\frac{k}{2}+r+\frac{1}{2})}{\Gamma(\frac{k}{2}+1)}z^k \\ &\{\sin\frac{k+2r+1}{4}\pi - \frac{\pi}{3(k+2r+2)}\cos\frac{k+2r+1}{4}\pi + O(\frac{1}{r^2})\}.\end{aligned} \tag{4.3.6}$$

If we let

$$G_r(z) = \frac{1}{\Gamma(r+\frac{1}{2})}\sum_{k=0}^{\infty}(-)^k\frac{\Gamma(\frac{k}{2}+r+\frac{1}{2})}{\Gamma(\frac{k}{2}+1)}z^k, \quad |z|<1, \tag{4.3.7}$$

then the first leading term of $F_r(\eta)$ can be written as follows

$$\begin{aligned} F_r^1(\eta) &= -\frac{1}{\Gamma(r+\frac{1}{2})}\sum_{k=0}^{\infty}(-)^k\frac{\Gamma(\frac{k}{2}+r+\frac{1}{2})}{\Gamma(\frac{k}{2}+1)}z^k\sin\frac{k+2r+1}{4}\pi \\ &= \frac{1}{2i}\{e^{-\frac{(2r+1)}{4}\pi i}G_r(e^{-\frac{\pi}{4}i}z) - e^{\frac{(2r+1)}{4}\pi i}G_r(e^{\frac{\pi}{4}i}z)\}, \end{aligned} \tag{4.3.8}$$

and the second leading term of $F_r(\eta)$ is

$$\begin{aligned} F_r^2(\eta) &= -\frac{1}{\Gamma(r+\frac{1}{2})}\sum_{k=0}^{\infty}(-)^k\frac{\Gamma(\frac{k}{2}+r+\frac{1}{2})}{\Gamma(\frac{k}{2}+1)}z^k\{-\frac{\pi}{3(k+2r+2)}\cos\frac{k+2r+1}{4}\pi\} \\ &= \frac{\pi}{6\Gamma(r+\frac{1}{2})}\sum_{k=0}^{\infty}(-)^k\frac{\Gamma(\frac{k}{2}+r+\frac{1}{2})}{\Gamma(\frac{k}{2}+1)(\frac{k}{2}+r+1)}z^k\cos\frac{k+2r+1}{4}\pi\} \\ &= \frac{\pi\Gamma(r-\frac{1}{2})}{12\Gamma(r+\frac{1}{2})}\{e^{-\frac{(2r+1)}{4}\pi i}G_{r-1}(e^{-\frac{\pi}{4}i}z) + e^{\frac{(2r+1)}{4}\pi i}G_{r-1}(e^{\frac{\pi}{4}i}z)\} \\ &\sim \frac{\pi}{12(r-\frac{1}{2})}\{e^{-\frac{(2r+1)}{4}\pi i}G_{r-1}(e^{-\frac{\pi}{4}i}z) + e^{\frac{(2r+1)}{4}\pi i}G_{r-1}(e^{\frac{\pi}{4}i}z)\}. \end{aligned} \tag{4.3.9}$$

The functions $F_r^1(\eta)$ and $F_r^2(\eta)$ involve the function $G_r(z)$. We will develop a general form of $G_r(z)$ in the next subsection.

4.3.2 Representation of $G_r(z)$

In this subsection, the representation of $G_r(z)$ using the hypergeometric function will be given. The hypergeometric function is defined by

$$F(a,b,c,z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)n!}z^n.$$

From the point of view of its convergence, the above series can be defined as a function which is analytic when $|z|<1$.

Separation of the right side of (4.3.7) into two groups for odd and even k, since each group is a hypergeometric function, $G_r(z)$ can be written as

$$\begin{aligned} G_r(z) &= \frac{1}{\Gamma(r+\frac{1}{2})}\{\sum_{m=0}^{\infty}\frac{\Gamma(m+r+\frac{1}{2})}{\Gamma(m+1)}z^{2m} - z\sum_{m=0}^{\infty}\frac{\Gamma(m+r+1)}{\Gamma(m+\frac{3}{2})}z^{2m}\} \\ &= F(1,r+\frac{1}{2};1;z^2) - \frac{\Gamma(r+1)}{\Gamma(\frac{3}{2})\Gamma(r+\frac{1}{2})}zF(r+1,1;\frac{3}{2};z^2), \end{aligned} \tag{4.3.10}$$

We can use one hypergeometric function instead of two hypergeometric functions in (4.3.10) by using the following identity [Whittaker & Watson, 1965, p. 291],

$$\Gamma(c)\Gamma(a)\Gamma(b)\Gamma(c-a-b)F(a,b;a+b-c+1;z^2)$$
$$+\Gamma(c)\Gamma(c-a)\Gamma(c-b)\Gamma(a+b-c)(z^2)^{c-a-b}F(c-a,c-b;c-a-b+1;z^2)$$
$$=\Gamma(c-a)\Gamma(c-b)\Gamma(a)\Gamma(b)F(a,b;c;1-z^2), \tag{4.3.11}$$

where $|\arg z| < \frac{\pi}{2}$ and $|z| < 1$. Putting $a = \frac{1}{2}$, $b = r + \frac{1}{2}$, and $c = r + \frac{3}{2}$ into (4.3.11), we obtain

$$F(1, r+\frac{1}{2}; 1; z^2) - \frac{\Gamma(r+1)}{\Gamma(\frac{3}{2})\Gamma(r+\frac{1}{2})} zF(r+1, 1; \frac{3}{2}; z^2)$$
$$= \frac{\Gamma(r+1)}{\Gamma(\frac{1}{2})\Gamma(r+\frac{3}{2})} F(\frac{1}{2}, r+\frac{1}{2}; r+\frac{3}{2}; 1-z^2), \tag{4.3.12}$$

where $|\arg z| < \frac{\pi}{2}$ and $|z| < 1$. Substituting (4.3.12) into (4.3.10) then leads to

$$G_r(z) = \frac{\Gamma(r+1)}{\Gamma(\frac{1}{2})\Gamma(r+\frac{3}{2})} F(\frac{1}{2}, r+\frac{1}{2}; r+\frac{3}{2}; 1-z^2), \tag{4.3.13}$$

where $|\arg z| \leq \frac{\pi}{2}$ and can be extended to $|z| \geq 1$ by analytic continuation.

Replacing z by $-z$ in (4.3.10), we have

$$G_r(-z) = F(1, r+\frac{1}{2}; 1; z^2) + \frac{\Gamma(r+1)}{\Gamma(\frac{3}{2})\Gamma(r+\frac{1}{2})} zF(r+1, 1; \frac{3}{2}; z^2) \tag{4.3.14}$$

Adding (4.3.10) and (4.3.14) gives

$$G_r(z) = 2(1-z^2)^{-r-\frac{1}{2}} - G_r(-z). \tag{4.3.15}$$

This is the reflection formula for $G_r(z)$. Thus, if z is in the right-half plane including the boundary, the formula (4.3.13) is used to calculate the function $G_r(z)$. If z is in the left-half plane, the formula (4.3.15) is used to calculate the function $G_r(z)$.

4.4 Numerical Results of the Asymptotic Form of $f_r(\eta)$ for Large r

The structure of the asymptotic form of $c_r(\eta)$ for large r has been given in the above section. In this section, the numerical results of (4.2.28) and (4.2.30) will be presented.

For convenience, we rewrite the asymptotic results (4.2.28) and (4.2.30) here:

$$f_r(\eta) \sim F_r(\eta) + (-)^{(r+1)/2}\frac{2c_0(\eta)}{\sqrt{r}}, \quad r \text{ is odd}, \tag{4.4.1}$$

$$f_r(\eta) \sim F_r(\eta) + \frac{2\pi\gamma_1}{(r-\frac{1}{2})}f_{r-1}(\eta) + (-)^{r/2}\frac{4\pi\Gamma(r-1)}{\Gamma(r+\frac{1}{2})}\{c_1(\eta)\gamma_0 - c_0(\eta)\gamma_1\}, \quad r \text{ is even}. \tag{4.4.2}$$

Using the leading term $F_r^1(\eta)$ to approximate $F_r(\eta)$ in (4.4.1), the first and second leading terms $F_r^1(\eta) + F_r^2(\eta)$ to approximate $F_r(\eta)$ in (4.4.2), we can write $F_r(\eta)$ as

$$F_r(\eta) \sim \begin{cases} \frac{1}{2i}\{e^{-\frac{(2r+1)}{4}\pi i}G_r(e^{-\frac{\pi}{4}i}z) - e^{\frac{(2r+1)}{4}\pi i}G_r(e^{\frac{\pi}{4}i}z)\}, & \text{in (4.4.1)}, \\ \\ \frac{1}{2i}\{e^{-\frac{(2r+1)}{4}\pi i}G_r(e^{-\frac{\pi}{4}i}z) - e^{\frac{(2r+1)}{4}\pi i}G_r(e^{\frac{\pi}{4}i}z)\} & \\ +\frac{\pi}{12(r-\frac{1}{2})}\{e^{-\frac{(2r+1)}{4}\pi i}G_{r-1}(e^{-\frac{\pi}{4}i}z) + e^{\frac{(2r+1)}{4}\pi i}G_{r-1}(e^{\frac{\pi}{4}i}z)\}, & \text{in (4.4.2)}, \end{cases} \tag{4.4.3}$$

where

$$G_r(z) = \begin{cases} \frac{\Gamma(r+1)}{\Gamma(\frac{1}{2})\Gamma(r+\frac{3}{2})}F(\frac{1}{2}, r+\frac{1}{2}; r+\frac{3}{2}; 1-z^2), & |\arg z| \le \frac{\pi}{2}, \\ 2(1-z^2)^{-r-\frac{1}{2}} - \frac{\Gamma(r+1)}{\Gamma(\frac{1}{2})\Gamma(r+\frac{3}{2})}F(\frac{1}{2}, r+\frac{1}{2}; r+\frac{3}{2}; 1-z^2), & |\arg z| > \frac{\pi}{2}, \end{cases}$$

and $z = \eta/2\sqrt{\pi}$.

There are the functions $G_r(e^{\mp\frac{\pi}{4}i}z)$ and $G_{r-1}(e^{\mp\frac{\pi}{4}i}z)$ in the representation of the function $F_r(\eta)$. For simplicity in the presentation, we define $z_{1,2} = e^{\mp\frac{\pi}{4}i}z$, where the subscripts 1 and 2 correspond to the exponential functions $e^{-\frac{\pi}{4}i}z$ or $e^{+\frac{\pi}{4}i}z$, respectively. The different types of the formula (4.4.3) can be used to calculate the functions $G_r(e^{\mp\frac{\pi}{4}i}z)$ and $G_{r-1}(e^{\mp\frac{\pi}{4}i}z)$ depending on the location of z.

It is convenient to consider the mapping $\lambda \to \eta(\lambda)$ given in (4.2.2), which we write here

$$\lambda = z/a, \quad \mu = \lambda - 1, \quad \frac{1}{2}\eta^2 = \lambda - 1 - \log\lambda. \tag{4.4.4}$$

Let the half-lines l_ϕ be defined by $l_\phi = \{\lambda = \rho e^{i\phi},\ \rho > 0\}$, where ϕ is real and $|\phi| \le 2\pi$. Writing $\eta = \alpha + i\beta$ and considering (4.4.4), we have

$$\begin{aligned} \frac{1}{2}(\alpha^2 - \beta^2) &= \rho\cos\phi - 1 - \log\rho, \\ \alpha\beta &= \rho\sin\phi - \phi. \end{aligned} \tag{4.4.5}$$

Taking into account the convention about the choice of the square root (4.4.4), we obtain Fig. 4.1 for different ρ and different ϕ. Since the complete picture for

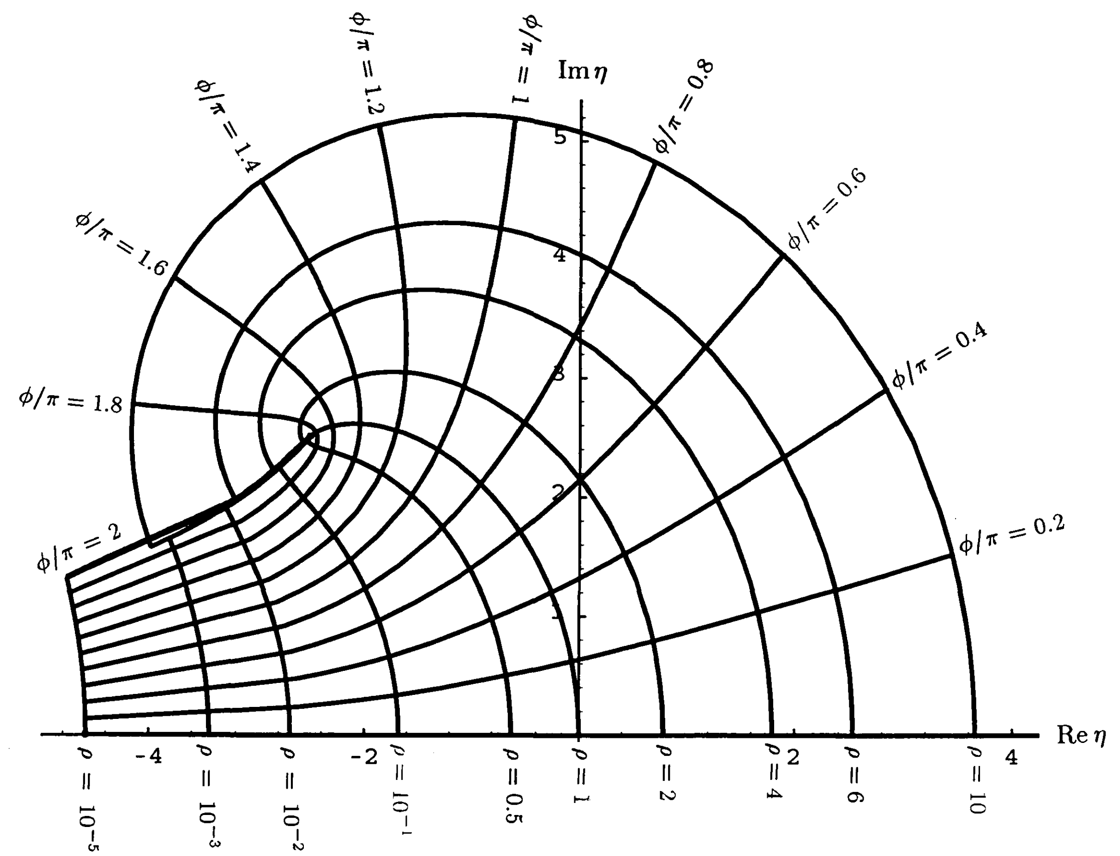


Figure 4.1: The upper half η plane for different ρ and different ϕ, where $\eta = \rho e^{i\phi}$

$-2\pi \leq \phi \leq 2\pi$ is symmetric with respect to the Re (η)-axis, only the upper half η plane is given.

The half-lines $l_{\pm 2\pi}$ are mapped on part of the hyperbole $\alpha\beta = \mp 2\pi$. The points $\eta^{\pm} = e^{\pm 3\pi i/4} 2\sqrt{\pi}$ are singular points of the mapping. Other singular points are located in other Riemann sheets of the η-plane. Convenient branch-cuts for the function $\lambda(\eta)$ are the parts of the hyperbole $\alpha\beta = \pm 2\pi$ with $\alpha \leq -\sqrt{2\pi}$. With the η-plane cut along these curves, lines l_ϕ with the values of ϕ outside the interval $[-2\pi, 2\pi]$ can be traced.

To make it clear, we draw Fig. 4.2 to show the domains in the z plane, where $-\pi < \arg z \leq \pi$. The z-plane is cut along the wavy curve which corresponds to the cut in the η-plane [see Fig. 4.1], where $z = \eta / 2\sqrt{\pi}$. Since the figure is symmetrical in the real z axis, we only draw the upper-half z plane. The following cases will be considered:

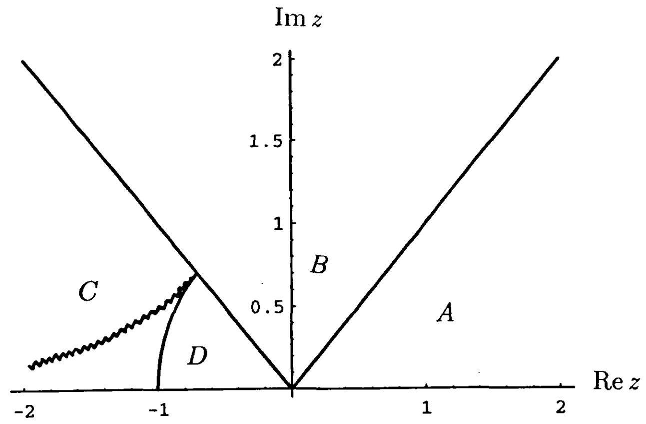


Figure 4.2: The complex z plane

Case 1: z is in the domain A, which is denoted by the domain $0 \leq \arg z \leq \pi/4$. In this case, we have $-\pi/4 \leq \arg z_1 \leq 0$ and $\pi/4 \leq \arg z_2 \leq \pi/2$. The following formulas for $G_r(z_{1,2})$ can be used

$$G_r(z_{1,2}) = \frac{\Gamma(r+1)}{\Gamma(\frac{1}{2})\Gamma(r+\frac{3}{2})} F(\frac{1}{2}, r+\frac{1}{2}; r+\frac{3}{2}; 1-z_{1,2}^2). \quad (4.4.6)$$

Case 2: z is in the domain B, which corresponds to $\pi/4 < \arg z \leq 3\pi/4$. We have $0 < \arg z_1 \leq \pi/2$, $\pi/2 < \arg z_2 \leq \pi$ and $\pi < \arg z_2^2 \leq 2\pi$. Thus

$$\begin{aligned}
G_r(z_1) &= \frac{\Gamma(r+1)}{\Gamma(\frac{1}{2})\Gamma(r+\frac{3}{2})} F(\frac{1}{2}, r+\frac{1}{2}; r+\frac{3}{2}; 1-z_1^2), \\
G_r(z_2) &= 2(1-z_2^2)^{-r-\frac{1}{2}} - G_r(-z_2) \\
&= 2(1-z_2^2)^{-r-\frac{1}{2}} - \frac{\Gamma(r+1)}{\Gamma(\frac{1}{2})\Gamma(r+\frac{3}{2})} F(\frac{1}{2}, r+\frac{1}{2}; r+\frac{3}{2}; 1-z_2^2). \quad (4.4.7)
\end{aligned}$$

Case 3: The domain C satisfies $3\pi/4 < \arg z$ and z is in the above the wavy curve, where $|z| > 1$, the z-plane cut along the wavy curve [see Fig. 4.2]. We find that $\pi/2 < \arg z_1 < 3\pi/4$, $-\pi < \arg z_2 < -3\pi/4$ and $-\pi < \arg z_1^2 < -\pi/2$, but z_2^2 is located in another Riemann sheet. When $|z_2| \geq 1$, the argument of $(1-z_2^2)^{-r-\frac{1}{2}}$ is changed as z_2^2 encircles the branch point at $z_2 = 1$. Thus,

$$\begin{aligned}
G_r(z_1) &= 2(1-z_1^2)^{-r-\frac{1}{2}} - G_r(-z_1) \\
&= 2(1-z_1^2)^{-r-\frac{1}{2}} - \frac{\Gamma(r+1)}{\Gamma(\frac{1}{2})\Gamma(r+\frac{3}{2})} F(\frac{1}{2}, r+\frac{1}{2}; r+\frac{3}{2}; 1-z_1^2),
\end{aligned}$$

$$
\begin{aligned}
G_r(z_2) &= -2(1-z_2^2)^{-r-\frac{1}{2}} - G_r(-z_2) \\
&= -2(1-z_2^2)^{-r-\frac{1}{2}} - \frac{\Gamma(r+1)}{\Gamma(\frac{1}{2})\Gamma(r+\frac{3}{2})} F(\frac{1}{2}, r+\frac{1}{2}; r+\frac{3}{2}; 1-z_2^2). \quad (4.4.8)
\end{aligned}
$$

Case 4: The domain D denotes the region above the negative z axis and below the wavy curve in the z plane. We have $\pi/2 < \arg z_1 < 3\pi/4$, $-\pi < \arg z_2 < -3\pi/4$ and $-\pi < \arg z_1^2 < -\pi/2$, z_2^2 is located in another Riemann sheet. When $|z_2| < 1$, the argument of $(1-z_2^2)^{-r-\frac{1}{2}}$ does not change as z_2^2 encircles the branch point at $z_2 = 1$. Other values of z are below the cut. Thus,

$$
\begin{aligned}
G_r(z_1) &= 2(1-z_1^2)^{-r-\frac{1}{2}} - G_r(-z_1) \\
&= 2(1-z_1^2)^{-r-\frac{1}{2}} - \frac{\Gamma(r+1)}{\Gamma(\frac{1}{2})\Gamma(r+\frac{3}{2})} F(\frac{1}{2}, r+\frac{1}{2}; r+\frac{3}{2}; 1-z_1^2), \\
G_r(z_2) &= 2(1-z_2^2)^{-r-\frac{1}{2}} - G_r(-z_2) \\
&= 2(1-z_2^2)^{-r-\frac{1}{2}} - \frac{\Gamma(r+1)}{\Gamma(\frac{1}{2})\Gamma(r+\frac{3}{2})} F(\frac{1}{2}, r+\frac{1}{2}; r+\frac{3}{2}; 1-z_2^2). \quad (4.4.9)
\end{aligned}
$$

We can also use the formula (4.3.10) to calculate the function $G_r(z_{1,2})$ and $G_{r-1}(z_{1,2})$; the details are omitted here.

The relation between η and μ is given in (4.2.2). In order to calculate the coefficients $c_0(\eta)$ and $c_1(\eta)$, the choice of μ for a given η is very important. For this purpose, we present Fig. 4.3 showing the μ plane. Some values of μ are presented in Appendix C.3.

The numerical results of the asymptotic forms of $f_r(\eta)$ (which are denote by $\tilde{f}_r(\eta)$) from (4.4.1) and (4.4.2) are given next for different ρ and ϕ, where $\eta = \rho e^{i\phi}$. The exact values of $f_r(\eta)$ are calculated from the convergent series (2.8.8) truncated after 80 terms when η is near the origin ($|\eta| < 2\sqrt{\pi}$) and from (2.3.2) when η is bounded away from the origin. An asterisk $*$ denotes the value of η below the cut.

The results of computation for the exact and the asymptotic values of $f_r(\eta)$ are shown in Tables 4.1-4.8 for large odd r and Tables 4.9-4.13 for large even r. Table 4.14 gives the results of computation for the exact values and the asymptotic form of $f_r(\eta)$ for large r when η is real.

It is shown that the accuracy is about 3 or 4 decimals. When $r = 15$, we choose ρ from 1/2 to 10; when $r = 20$, we choose 1/2 to 4 for different ϕ. When ρ is in the neighourbood of $2\sqrt{\pi}$ and ϕ is near 0.75, we found that the values of $\tilde{f}_r(\eta)$ become

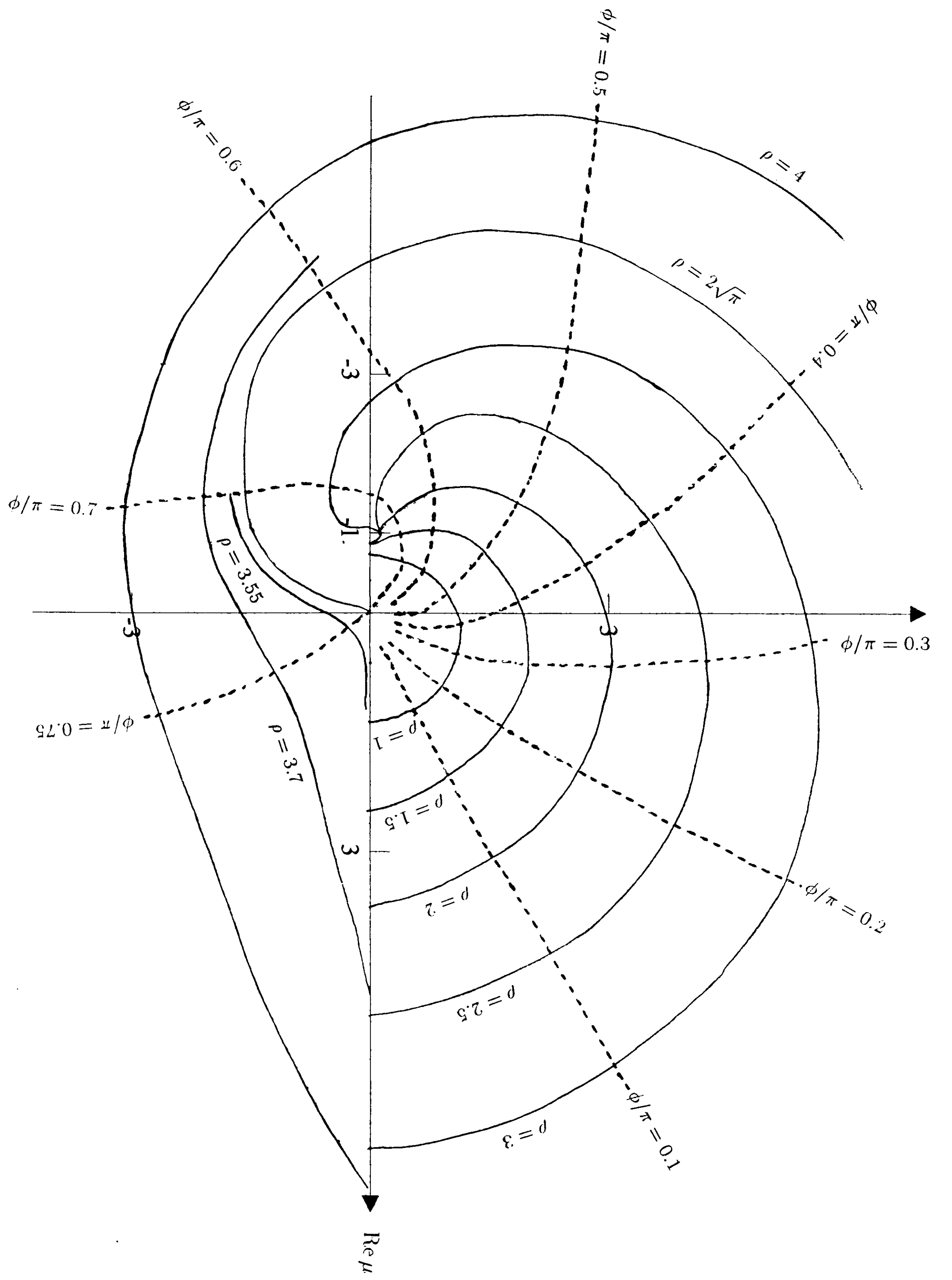


Figure 4.3: The complex μ plane for different ρ and different ϕ, where $\eta = \rho e^{i\phi}$

Table 4.1: Computations of $f_r(\eta)$ from (4.4.1) when $r = 15$ and $\rho = \frac{1}{2}$ for different ϕ, where $\eta = \rho e^{i\phi}$

ϕ/π	$f_r(\eta)$	$\tilde{f}_r(\eta) = F_r(\eta) + (-)^{\frac{r+1}{2}} \frac{2c_0(\eta)}{\sqrt{r}}$
0	0.43378645	0.43541751
0.1	0.44186522−0.04711756i	0.44350616−0.04720216i
0.2	0.46820429−0.09295623i	0.46987545−0.09312285i
0.3	0.51906205−0.13237808i	0.52078521−0.13262045i
0.4	0.60237353−0.14940435i	0.60417196−0.14971023i
0.5	0.71351568−0.10829783i	0.71528148−0.10864631i
0.6	0.79944565+0.03372461i	0.79569140+0.02930138i
0.7	0.74787459+0.24818316i	0.75000658+0.24786008i
0.8	0.51323692+0.37317880i	0.51547276+0.37293164i
0.9	0.24300228+0.27120493i	0.24530930+0.27107183i
1	0.12973941	0.13207096

Table 4.2: Computations of $f_r(\eta)$ from (4.4.1) when $r = 15$ and $\rho = 2$ for different ϕ

ϕ/π	$f_r(\eta)$	$\tilde{f}_r(\eta) = F_r(\eta) + (-)^{\frac{r+1}{2}} \frac{2c_0(\eta)}{\sqrt{r}}$
0	0.14077538	0.14184111
0.1	$0.13285351 - 0.05894854i$	$0.13392168 - 0.05911066i$
0.2	$0.10537287 - 0.12057512i$	$0.10644869 - 0.12090956i$
0.3	$0.04357736 - 0.18330318i$	$0.04466810 - 0.18383418i$
0.4	$-0.09473383 - 0.21366628i$	$-0.09361477 - 0.21444232i$
0.5	$-0.45375846 + 0.18256595i$	$-0.45282276 + 0.1801941i$
0.6	$-8.23079858 + 8.56059245i$	$-8.22937841 + 8.55889928i$
0.7	$41.6380992 - 225.836222i$	$41.6405501 - 225.838588i$
0.75	$268.038575 + 268.367921i$	$268.041945 + 268.365737i$
0.8	$-226.211008 + 41.9199850i$	$-226.207036 + 41.9185527i$
0.9	$8.14801887 - 7.98149542i$	$8.15175060 - 7.98175904i$
1	0.16796026	0.17146730

Table 4.3: Computations of $f_r(\eta)$ from (4.4.1) when $r = 15$ and $\rho = 3$ for different ϕ

ϕ/π	$f_r(\eta)$	$\tilde{f}_r(\eta) = F_r(\eta) + (-)^{\frac{r+1}{2}} \frac{2c_0(\eta)}{\sqrt{r}}$
0	0.07855512	0.07941339
0.1	$0.07099051 - 0.03759124i$	$0.07184490 - 0.03775110i$
0.2	$0.04725079 - 0.07146541i$	$0.04809223 - 0.07179422i$
0.3	$0.00587472 - 0.09402737i$	$0.00668900 - 0.09454546i$
0.4	$-0.04580917 - 0.09387424i$	$-0.04505106 - 0.09461873i$
0.5	$-0.14989709 - 0.10462998i$	$-0.13767771 - 0.09587447i$
0.6	$8.36162722 + 20.2853696i$	$8.36174919 + 20.2838773i$
0.7	-1.62670832×10^6 $-1.69344395i \times 10^6$	-1.62670832×10^6 $-1.69344395i \times 10^6$
0.75	$2.126804480268986 \times 10^8$ $+2.126804485786718i \times 10^8$	$2.126804480322793 \times 10^8$ $+2.126804485593489i \times 10^8$
0.8	-1.69344455×10^6 $-1.62670800i \times 10^6$	-1.69344453×10^6 $-1.62670800i \times 10^6$
0.9	$19.7890786 + 8.53088730i$	$19.7921994 + 8.53138306i$
1	-0.56611014	-0.56276072

very large. This results from the fact that the function $F_r(\eta)$ involves the factors $2(1-z_1^2)^{-r-\frac{1}{2}}$ or $2(1-z_2^2)^{-r-\frac{1}{2}}$. When η is in the domains $|1-e^{\mp\frac{\pi}{2}i}\eta^2/4\pi| < 1$ (see Fig. 4.4), $\tilde{f}_r(\eta)$ is very large for large r. Thus, there are two lobes in the complex η-plane situated symmetrically either side of the negative real η axis. Fig. 4.5 shows the modulus of the function $F_r(z)$ for $r = 15$, where the function $F_r(z)$ given in (4.4.3) and $z = \eta/2\sqrt{\pi}$.

When $\phi = 0$ (so that η is real), $\tilde{f}_r(\eta)$ given in (4.4.1) or (4.4.2) is real function. Fig. 4.6 shows the behaviour of $\tilde{f}_r(\eta)$ ($r = 15$, 20, 25, 30, 35, 40) on the real η axis. (On the scale of these graphs, $f_r(\eta)$ and $\tilde{f}_r(\eta)$ coincide.) It is noted that $\tilde{f}_r(\eta)$ tends to zero smoothly for $\eta \geq 0$ and possesses an oscillatory structure for $\eta < 0$ when passing between the two lobes before settling down to a constant value for large negative values of η. Table 4.15 shows the numerical results of $\tilde{f}_{15}(\eta)$ and $\tilde{f}_{35}(\eta)$ for real η in the oscillatory zone.

Table 4.4: Computations of $f_r(\eta)$ from (4.4.1) when $r = 15$ and $\rho = 2\sqrt{\pi}$ for different ϕ

ϕ/π	$f_r(\eta)$	$\tilde{f}_r(\eta) = F_r(\eta) + (-)^{\frac{r+1}{2}} \frac{2c_0(\eta)}{\sqrt{r}}$
0	0.06034976	0.06112514
0.1	$0.05372957 - 0.02997278i$	$0.05449894 - 0.03012723i$
0.2	$0.03362283 - 0.05569163i$	$0.03437248 - 0.05600789i$
0.3	$0.00063502 - 0.07090292i$	$0.01344967 - 0.07139643i$
0.4	$-0.04041534 - 0.068887434i$	$-0.03978125 - 0.06956842i$
0.5	$-0.08295276 - 0.04045044i$	$-0.08247630 - 0.04136202i$
0.6	$-3.93511259 + 2.34606401i$	$-3.93504789 + 2.34510156i$
0.65	$1713.52644 + 271.486871i$	$1713.52598 + 271.486523i$
0.7	-4.30949761×10^7 $-5.04577337i \times 10^7$	-4.30949767×10^7 $-5.04577337i \times 10^7$
0.72	$-1.738483412604396 \times 10^{11}$ $+1.919303555945985i \times 10^{10}$	$-1.738483412606959 \times 10^{11}$ $+1.191930355593097i \times 10^{10}$
0.74	$-1.992966928823569 \times 10^{18}$ $+3.764059523287501i \times 10^{18}$	$-1.992966928823527 \times 10^{18}$ $+3.764059523287475i \times 10^{18}$
0.78*	$1.919303555810603 \times 10^{10}$ $-1.738483412605253i \times 10^{11}$	$1.919303555861485 \times 10^{10}$ $-1.738483412602607i \times 10^{11}$
0.8*	-5.04577343×10^7 $-4.30949761i \times 10^7$	-5.04577344×10^7 $-4.30949759i \times 10^7$
0.85*	$270.897267 + 1713.69053i$	$270.899890 + 1713.67896i$
0.9*	$1.81292271 - 3.80627415i$	$1.81634087 - 3.80624082i$
0.95*	$-0.59797730 + 0.02535034i$	$-0.59477694 + 0.02545331i$
1*	-0.51203415	-0.50881612

Table 4.5: Computations of $f_r(\eta)$ from (4.4.1) when $r = 15$ and $\rho = 4$ for different ϕ

ϕ/π	$f_r(\eta)$	$\tilde{f}_r(\eta) = F_r(\eta) + (-)^{\frac{r+1}{2}} \frac{2c_0(\eta)}{\sqrt{r}}$
0	0.04952205	0.05023945
0.1	$0.04366020 - 0.02516217i$	$0.04437032 - 0.02531139i$
0.2	$0.02618401 - 0.04609942i$	$0.02687053 - 0.04640381i$
0.3	$-0.00164270 - 0.05745918i$	$-0.00100239 - 0.05793046i$
0.4	$-0.03572256 - 0.05373956i$	$-0.03516524 - 0.05439228i$
0.5	$-0.06999576 - 0.03029859i$	$-0.06958575 - 0.03112825i$
0.6	$0.12134919 - 0.31508798i$	$0.12156706 - 0.31591038i$
0.65	$51.3407349 + 71.7439203i$	$51.3411801 + 71.7434573i$
0.7	$261632.345 + 60298.6929i$	$261632.348 + 60298.6926i$
0.75	$-3.828775742367585 \times 10^8$ $+3.828775745010522i \times 10^8$	$-3.828775744940121 \times 10^8$ $+3.828775748594648i \times 10^8$
0.8	$-60298.4641 - 261632.388i$	$-60298.4452 - 261632.371i$
0.85	$-71.5865506 - 51.4575596i$	$-71.5978997 - 51.4366406i$
0.88*	$-1.97615147 - 2.32540922i$	$-1.97462952 - 2.32290768i$
0.9*	$-0.85913606 + 0.21790206i$	$-0.85647483 + 0.21844677i$
0.92*	$-0.48659359 + 0.06820424i$	$-0.48361847 + 0.06838583i$
0.94*	$-0.50315482 - 0.00445907i$	$-0.50007378 - 0.004437385i$
0.96*	$-0.52014250 - 0.00492942i$	$-0.51701437 - 0.00488370i$
0.98*	$-0.52172321 - 0.00075294i$	$-0.51857240 - 0.00073206i$
1*	-0.52109507	-0.51793731

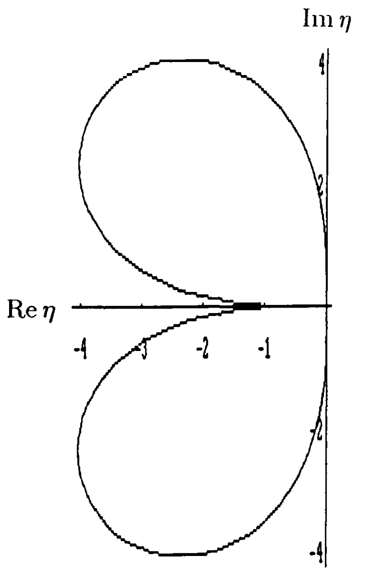


Figure 4.4: The η plane showing the domains $|1 - e^{\mp\frac{\pi}{2}i}\eta^2/4\pi| < 1$

Table 4.6: Computations of $f_r(\eta)$ from (4.4.1) when $r = 15$ and $\rho = 2\sqrt{2\pi}$ for different ϕ

ϕ/π	$f_r(\eta)$	$\tilde{f}_r(\eta) = F_r(\eta) + (-)^{\frac{r+1}{2}} \frac{2c_0(\eta)}{\sqrt{r}}$
0	0.03375843	0.03437330
0.1	$0.02927907 - 0.01777215i$	$0.02988498 - 0.01790940i$
0.2	$0.01626811 - 0.03183038i$	$0.01684550 - 0.03210826i$
0.3	$-0.00358315 - 0.03817671i$	$-0.00305904 - 0.03860095i$
0.4	$-0.02664304 - 0.03297259i$	$-0.02620566 - 0.03354738i$
0.5	$-0.04692927 + 0.01296323i$	$-0.04661577 + 0.01212942i$
0.6	$-0.05238260 + 0.02440188i$	$-0.05216601 + 0.02353899i$
0.7	$0.13820589 - 0.12927581i$	$0.13815131 - 0.130892455i$
0.75	$-0.68855381 + 0.77221738i$	$-0.68982640 + 0.770598955i$
0.8	$0.23691267 - 0.11355005i$	$0.23547997 - 0.11400742i$
0.85	$0.04018782 + 0.00742823i$	$0.03912192 + 0.00726345i$
0.9	$0.04021573 + 0.00318344i$	$0.03925251 + 0.00303588i$
0.91	$0.03877805 + 0.00121951i$	$0.03782504 + 0.00110343i$
0.94*	$-0.52036086 - 0.00004651i$	$-0.51721373 - 0.00012217i$
0.96*	$-0.52034043 + 0.00001521i$	$-0.51718165 - 0.00003207i$
0.98*	$-0.52031952 + 0.00000638i$	$-0.51715221 - 0.00001597i$
1*	-0.52032057	-0.51715018

Table 4.7: Computations of $f_r(\eta)$ from (4.4.1) when $r = 15$ and $\rho = 6$ for different ϕ

ϕ/π	$f_r(\eta)$	$\tilde{f}_r(\eta) = F_r(\eta) + (-)^{\frac{r+1}{2}} \frac{2c_0(\eta)}{\sqrt{r}}$
0	0.02461635	0.02515609
0.1	$0.02110613 - 0.01325800i$	$0.02163611 - 0.01338437i$
0.2	$0.01107095 - 0.02337246i$	$0.01157041 - 0.02362685i$
0.3	$-0.00379368 - 0.02721697i$	$-0.00334963 - 0.02760158i$
0.4	$-0.04303183 - 0.03124394i$	$-0.01983755 - 0.02258170i$
0.5	$-0.03275333 - 0.00627392i$	$-0.03251503 - 0.00691569i$
0.6	$-0.03265308 + 0.01867803i$	$-0.03256904 + 0.01788472i$
0.7	$-0.00947091 + 0.03923103i$	$-0.00978995 + 0.03821693i$
0.8	$0.02098053 + 0.02960519i$	$0.02009404 + 0.02901633i$
0.9	$0.02778993 + 0.00717156i$	$0.02701954 + 0.00677201i$
0.94	$0.02625235 + 0.00047682i$	$0.02543017 + 0.00037593i$
0.95*	$-0.52032477 - 0.00000009i$	$-0.51707812 - 0.00008227i$
0.97*	$-0.52032500 - 0.00000005i$	$-0.51707613 - 0.00004827i$
1*	-0.52032492	-0.51707460

Table 4.8: Computations of $f_r(\eta)$ from (4.4.1) when $r = 15$ and $\rho = 10$ for different ϕ

ϕ/π	$f_r(\eta)$	$\tilde{f}_r(\eta) = F_r(\eta) + (-)^{\frac{r+1}{2}} \frac{2c_0(\eta)}{\sqrt{r}}$
0	0.00962950	0.00999026
0.1	$0.00804634 - 0.00542181i$	$0.00839738 - 0.00515591i$
0.2	$0.00367623 - 0.00921337i$	$0.00399745 - 0.00939940i$
0.3	$-0.00233518 - 0.01000372i$	$-0.00206571 - 0.01027819i$
0.4	$-0.00810273 - 0.00704415i$	$-0.00790988 - 0.00739960i$
0.5	$-0.00112182 - 0.00074656i$	$-0.01113132 - 0.00116944i$
0.6	$-0.00953623 + 0.00666459i$	$-0.00959219 + 0.00620139i$
0.7	$-0.00304291 + 0.011125744i$	$-0.00327605 + 0.010081544i$
0.8	$0.00463366 + 0.01016379i$	$0.00424067 + 0.00983564i$
0.9	$0.00913935 + 0.00485258i$	$0.00866173 + 0.00469084i$
0.97	$0.00975851 + 0.00057731i$	$0.00926265 + 0.00053451i$
0.98*	-0.52032494	-0.51862576
1*	-0.52032494	-0.51682644

Table 4.9: Computations of $f_r(\eta)$ from (4.4.2) when $r = 20$ and $\rho = \frac{1}{2}$ for different ϕ

ϕ/π	$f_r(\eta)$	$\tilde{f}_r(\eta) = F_r(\eta) + \frac{2\pi\gamma_1}{(r-\frac{1}{2})} f_{r-1}(\eta)$ $+(-)^{r/2} \frac{4\pi\Gamma(r-1)}{\Gamma(r+\frac{1}{2})} \{c_1(\eta)\gamma_0 - c_0(\eta)\gamma_1\}$
0	-0.24446783	-0.24447742
0.1	$-0.24298584 + 0.08378269i$	$-0.24299540 + 0.08378328i$
0.2	$-0.23970815 + 0.17823013i$	$-0.23971767 + 0.17823139i$
0.3	$-0.24042627 + 0.29767298i$	$-0.24043577 + 0.29767511i$
0.4	$-0.26502120 + 0.461757765i$	$-0.26503086 + 0.46176095i$
0.5	$-0.36760290 + 0.68154043i$	$-0.36761321 + 0.68154522i$
0.6	$-0.63799990 + 0.89991920i$	$-0.63801177 + 0.89919837i$
0.7	$-1.08155276 + 0.93140263i$	$-1.08156733 + 0.93140996i$
0.8	$-1.46880745 + 0.66282337i$	$-1.46882548 + 0.66283010i$
0.9	$-1.60389038 + 0.28913400i$	$-1.60391142 + 0.28913809i$
1	-1.60747203	-1.60749426

Table 4.10: Computations of $f_r(\eta)$ from (4.4.2) when $r = 20$ and $\rho = 2$ for different ϕ

ϕ/π	$f_r(\eta)$	$\tilde{f}_r(\eta) = F_r(\eta) + \frac{2\pi\gamma_1}{(r-\frac{1}{2})} f_{r-1}(\eta)$ $+(-)^{r/2} \frac{4\pi\Gamma(r-1)}{\Gamma(r+\frac{1}{2})} \{c_1(\eta)\gamma_0 - c_0(\eta)\gamma_1\}$
0	-0.01754067	-0.01754917
0.1	$-0.01202881 + 0.01347452i$	$-0.01204100 + 0.01344847i$
0.2	$0.00344426 + 0.01941309i$	$0.00338487 + 0.01937178i$
0.3	$0.02145975 + 0.00713151i$	$0.02147219 + 0.00711013i$
0.4	$0.00457359 - 0.02800090i$	$0.00440772 - 0.02806339i$
0.5	$-0.25502344 + 0.25482616i$	$-0.25506313 + 0.25480153i$
0.6	$-18.4396401 + 18.6328023i$	$-18.4395678 + 18.6327639i$
0.7	$1152.90603 - 655.176984i$	$1152.90613 - 655.176893i$
0.75	$-1823.70982 + 1823.70260i$	$-1823.70974 + 1823.702751i$
0.8	$655.172131 - 1152.90953i$	$655.171957 - 1152.90911i$
0.9	$-18.6155255 + 18.4318929i$	$-18.6136949 + 18.4271998i$
1	-0.52283761	-0.52295951

Table 4.11: Computations of $f_r(\eta)$ from (4.4.2) when $r = 20$ and $\rho = 3$ for different ϕ

ϕ/π	$f_r(\eta)$	$\tilde{f}_r(\eta) = F_r(\eta) + \frac{2\pi\gamma_1}{(r-\frac{1}{2})} f_{r-1}(\eta)$ $+(-)^{r/2} \frac{4\pi\Gamma(r-1)}{\Gamma(r+\frac{1}{2})} \{c_1(\eta)\gamma_0 - c_0(\eta)\gamma_1\}$
0	-0.00583168	-0.00584018
0.1	$-0.00375232 + 0.00442693i$	$-0.00376098 + 0.00442704i$
0.2	$0.00115827 + 0.00553747i$	$0.00114906 + 0.00553749i$
0.3	$0.00482932 + 0.00192809i$	$0.00481880 + 0.00192748i$
0.4	$0.00374849 - 0.00270487i$	$0.00373473 - 0.00270763i$
0.5	$-0.00933141 + 0.00727498i$	$-0.00935516 + 0.00726491i$
0.6	$22.469078 - 55.32856680i$	$22.4690064 - 55.3286171i$
0.7	$-2.109971157550034 \times 10^8$ $+1.629041594487618i \times 10^8$	$-2.109971157526371 \times 10^8$ $+1.62904159444827i \times 10^8$
0.75	$-1.155166335406257 \times 10^{11}$ $+1.155166335405910i \times 10^{11}$	$-1.155166335406207 \times 10^{11}$ $+1.155166335405916i \times 10^{11}$
0.8	$-1.629041594379282 \times 10^8$ $+2.109971157275783i \times 10^8$	$-1.629041594406363 \times 10^8$ $+2.109971157292130i \times 10^8$
0.9	$55.3383216 - 22.4750976i$	$55.3381562 - 22.4751261i$
1	-0.01046266	-0.01061320

Table 4.12: Computations of $f_r(\eta)$ from (4.4.2) when $r = 20$ and $\rho = 2\sqrt{\pi}$ for different ϕ

ϕ/π	$f_r(\eta)$	$\tilde{f}_r(\eta) = F_r(\eta) + \frac{2\pi\gamma_1}{(r-\frac{1}{2})} f_{r-1}(\eta)$ $+(-)^{r/2} \frac{4\pi\Gamma(r-1)}{\Gamma(r+\frac{1}{2})} \{c_1(\eta)\gamma_0 - c_0(\eta)\gamma_1\}$
0	-0.00369909	-0.00370747
0.1	$-0.00238920 + 0.00275437i$	$-0.00239777 + 0.00275468i$
0.2	$0.00059055 + 0.00340508i$	$0.00058130 + 0.00340561i$
0.3	$0.00274763 + 0.00135121i$	$0.00273677 + 0.00135178i$
0.4	$0.00212271 - 0.00140162i$	$0.00210790 - 0.00140124i$
0.5	$-0.00038711 - 0.00291950i$	$-0.00041371 - 0.00291845i$
0.6	$1.68523109 - 7.03404098i$	$1.68515164 - 7.03401745i$
0.65	$-3010.09179 + 19004.9472i$	$-3010.09198 + 19004.9473i$
0.7	$1.736654153977106 \times 10^{9}$ $-2.206628323640814i \times 10^{10}$	$1.736654153976220 \times 10^{9}$ $-2.206628323640646i \times 10^{10}$
0.72	$-6.926743526789312 \times 10^{14}$ $-2.617398267920689i \times 10^{14}$	$-6.926743526789350 \times 10^{14}$ $-2.617398267920746i \times 10^{14}$
0.74	$-2.613560458133926 \times 10^{24}$ $+3.480935708426168i \times 10^{24}$	$-2.613560458133863 \times 10^{24}$ $+3.480935708426143i \times 10^{24}$
0.78*	$2.617398267920686 \times 10^{14}$ $+6.926743526789230i \times 10^{14}$	$2.617398267920747 \times 10^{14}$ $+6.926743526789350i \times 10^{14}$
0.8*	$2.206628323638712 \times 10^{10}$ $-1.736654153836570i \times 10^{9}$	$2.206628323644719 \times 10^{10}$ $-1.736654153985935i \times 10^{9}$
0.85*	$-19004.9279 + 3010.08370i$	$-19004.9293 + 3010.08573i$
0.9*	$7.04712620 - 1.68907531i$	$7.04695813 - 1.68914249i$
0.95*	$-0.02216082 + 0.01131847i$	$-0.02230018 + 0.01130301i$
1*	0.01407211	0.01393284

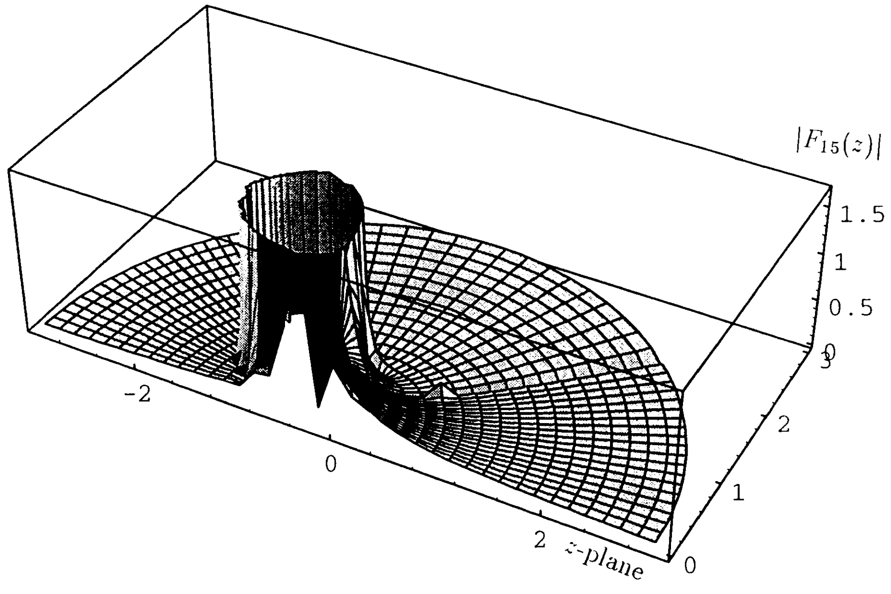


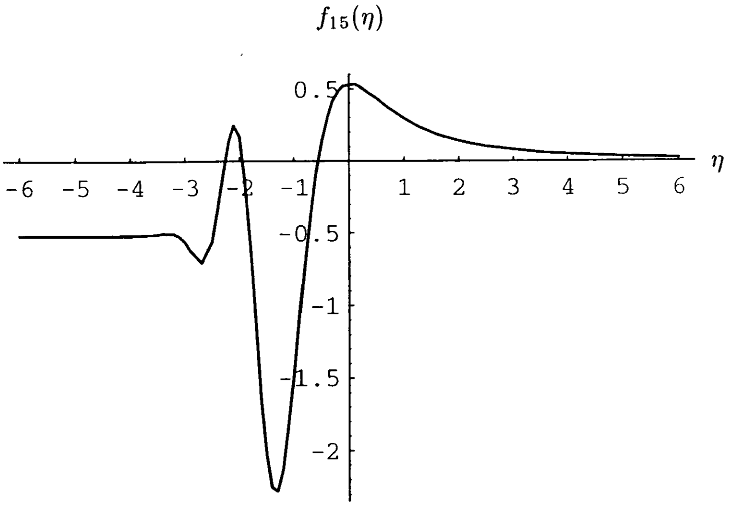
Figure 4.5: A modular plot of the function $F_r(z)$ for $r = 15$

Table 4.13: Computations of $f_r(\eta)$ from (4.4.2) when $r = 20$ and $\rho = 4$ for different ϕ

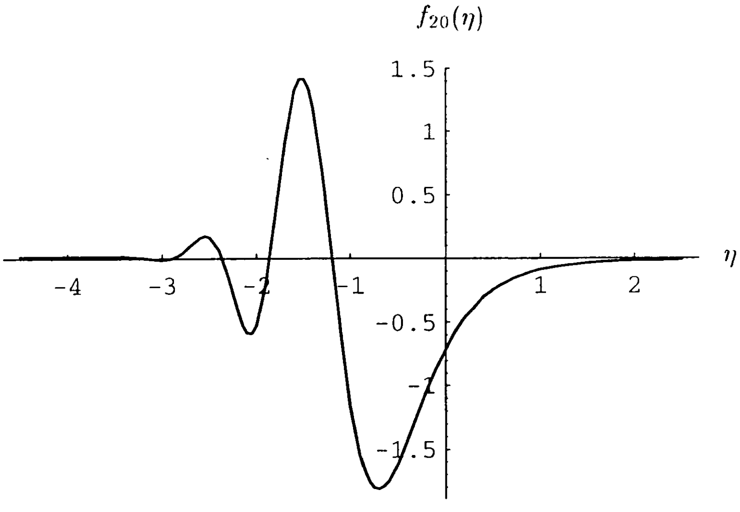
ϕ/π	$f_r(\eta)$	$\tilde{f}_r(\eta) = F_r(\eta) + \frac{2\pi\gamma_1}{(r-\frac{1}{2})} f_{r-1}(\eta)$ $+(-)^{r/2} \frac{4\pi\Gamma(r-1)}{\Gamma(r+\frac{1}{2})} \{c_1(\eta)\gamma_0 - c_0(\eta)\gamma_1\}$
0	-0.00369909	-0.00268096
0.1	$-0.00174464 + 0.00195623i$	$-0.00175307 + 0.00195669i$
0.2	$0.00033282 + 0.00242521i$	$0.00032370 + 0.00242614i$
0.3	$0.00183115 + 0.00106230i$	$0.00182041 + 0.00106379i$
0.4	$0.00145340 - 0.00076564i$	$0.00143897 - 0.00076296i$
0.5	$-0.00041860 - 0.00112833i$	$-0.00044221 - 0.00112039i$
0.6	$0.29312099 + 0.06068546i$	$0.29307972 + 0.06073234i$
0.65	$274.482590 + 254.429350i$	$274.482588 + 254.429486i$
0.7	-9.32189236×10^6 $-1.19330423i \times 10^7$	-9.32189236×10^6 $-1.19330423i \times 10^7$
0.75	$2.513866233872877 \times 10^{11}$ $+2.513866233872468i \times 10^{11}$	$2.513866233872798 \times 10^{11}$ $+2.513866233872473i \times 10^{11}$
0.8	-1.19330423×10^7 $-9.32189236i \times 10^6$	-1.19330423×10^7 $-9.32189237i \times 10^6$
0.85	$254.426043 + 274.486223i$	$254.419697 + 274.483012i$
0.88*	$3.61225657 + 1.20602510i$	$3.61247329 + 1.20629908i$
0.9*	$-0.04699096 - 0.29496104i$	$-0.04706185 - 0.29494463i$
0.92*	$-0.01574361 + 0.01741187i$	$-0.01585525 + 0.01739988i$
0.94*	$0.01584126 + 0.00336840i$	$0.01571823 + 0.00335671i$
0.96*	$0.01295898 - 0.00068433i$	$0.01283134 - 0.00069221i$
0.98*	$0.01231266 - 0.00008990i$	$0.01218298 - 0.00009378i$
1*	0.01242552	0.01229525

Table 4.14: Computations of $\tilde{f}_{15}(\eta)$ from (4.4.1) $\tilde{f}_{20}(\eta)$ from (4.4.2) for real η

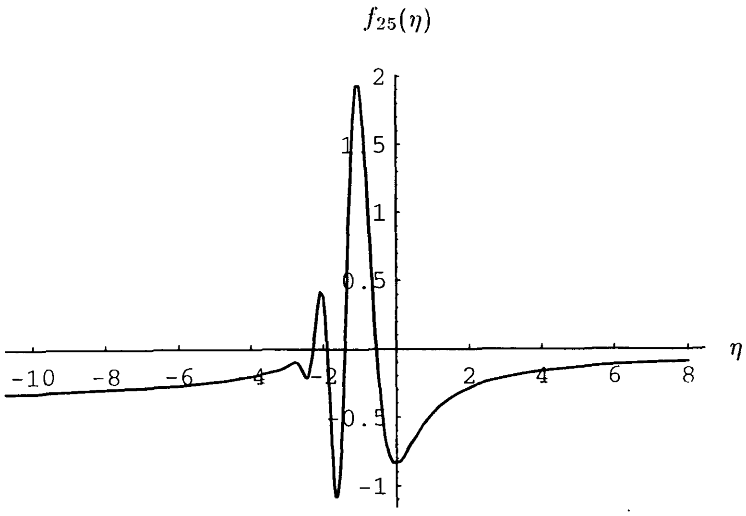
η	$f_{15}(\eta)$	$\tilde{f}_{15}(\eta)$	$f_{20}(\eta)$	$\tilde{f}_{20}(\eta)$
5.8	0.02614521	0.02669863	−0.00102212	−0.00102955
5.4	0.02965653	0.03023958	−0.00122326	−0.00123088
5.0	0.03391362	0.03452964	−0.00148870	−0.00149651
4.6	0.03914067	0.03979362	−0.00184772	−0.00185571
4.2	0.04565167	0.04634622	−0.00234739	−0.00235554
3.8	0.05389609	0.05463787	−0.00306646	−0.00307476
3.4	0.06453320	0.06532904	−0.00414292	−0.00415134
3.0	0.07855512	0.07941339	−0.00583168	−0.00584018
2.6	0.09749356	0.09842463	−0.00863294	−0.00864148
2.2	0.12376453	0.12478142	−0.01359800	−0.01360600
1.8	0.16120815	0.16232733	−0.02317046	−0.02310995
1.4	0.21994583	0.21698334	−0.04292872	−0.04293695
1.0	0.29519012	0.29658330	−0.08802241	−0.08803101
0.6	0.40372082	0.40529918	−0.19765328	−0.19766255
0.2	0.51412547	0.51593227	−0.46587876	−0.46588994
0	0.53303431	0.53497419	−0.70739648	−0.70740947
−0.2	0.47974588	0.48183235	−1.0379325	−1.03794823
−0.6	−0.09192874	−0.08950934	−1.74702331	−1.74704856
−1.0	−1.47913737	−1.47634627	−1.15946933	−1.15951220
−1.4	−2.25377257	−2.24818289	1.18247912	1.18240935
−1.8	−0.57383792	−0.57040725	0.35862030	0.35555834
−2.2	0.13603456	0.13957042	−0.41474534	−0.41488019
−2.6	−0.67891392	−0.67543917	0.16146425	0.16131345
−3.0	−0.56611014	−0.56276072	−0.01046266	−0.01061320
−3.4	−0.50408615	−0.50083973	0.01380467	0.01366201
−3.8	−0.52013697	−0.51695833	0.01267988	0.01254612
−4.2	−0.52077619	−0.51762996	0.01243555	0.01230793
−4.6	−0.52035465	−0.51720700	0.01245740	0.01233292
−5.0	−0.52032048	−0.51715102	0.01245639	0.01233319
−5.4	−0.52032386	−0.51712388	0.01245607	0.01233315
−5.8	−0.52032484	−0.51709141	0.01245604	0.01233291

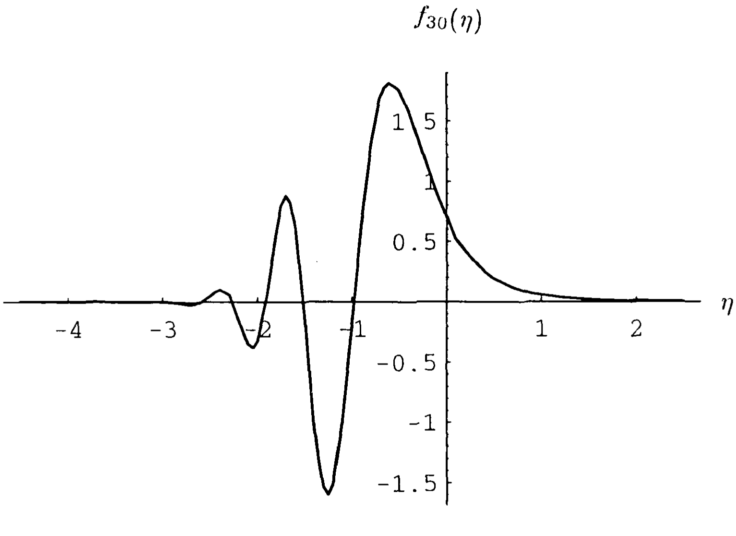


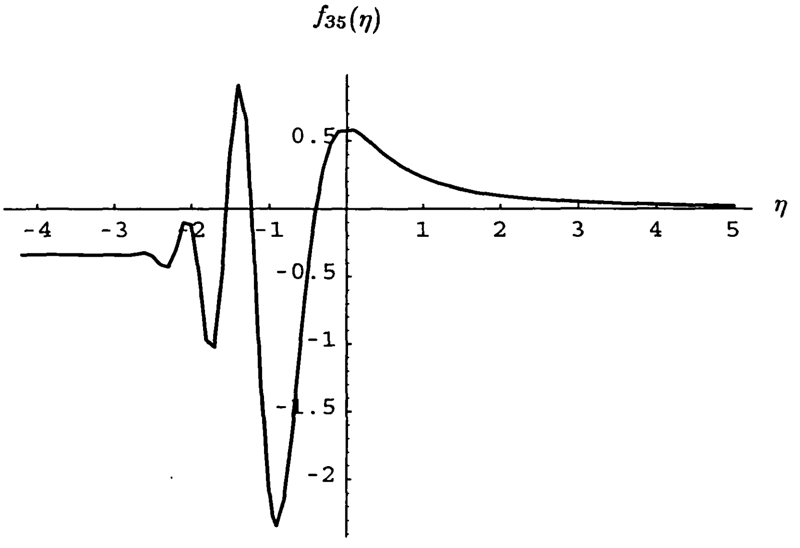
$f_{15}(\eta)$
0.5
-0.5
-1
-1.5
-2
-6 -5 -4 -3 -2 -1 1 2 3 4 5 6
η



$f_{20}(\eta)$
1.5
1
0.5
-0.5
-1
-1.5
-4 -3 -2 -1 1 2
η


$f_{25}(\eta)$
2
1.5
1
0.5
-0.5
-1
-10
-8
-6
-4
-2
2
4
6
8
η


$f_{30}(\eta)$
1.5
1
0.5
-0.5
-1
-1.5
-4
-3
-2
-1
1
2
η



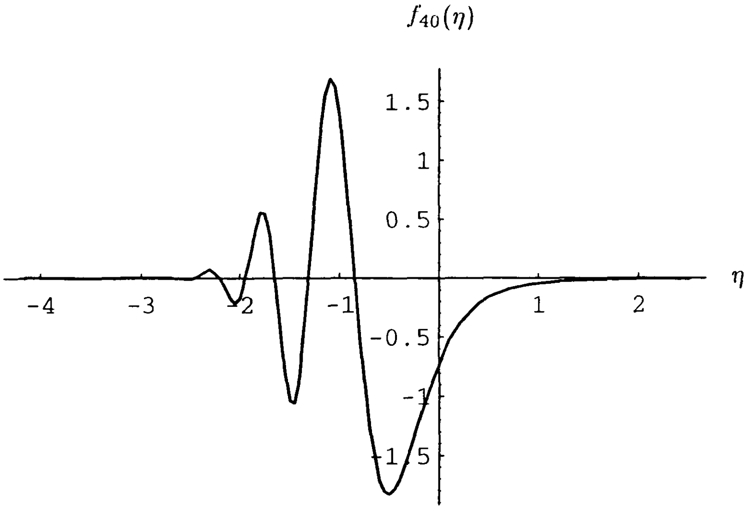


Figure 4.6: The graph of the functions $f_r(\eta)$ when $r = 15,\ 20,\ 25,\ 30,\ 35,\ 40$ for real η

Table 4.15: Computations of $\tilde{f}_{15}(\eta)$ and $\tilde{f}_{35}(\eta)$ from (4.4.1) for $-0.2 \leq \eta \leq 3.2$

η	$\tilde{f}_{15}(\eta)$	$\tilde{f}_{35}(\eta)$	η	$\tilde{f}_{15}(\eta)$	$\tilde{f}_{35}(\eta)$
−0.2	0.4818	0.4754	−0.3	0.4100	0.2878
−0.4	0.2963	−0.0235	−0.5	0.1321	−0.4727
−0.6	−0.0895	−1.0394	−0.7	−0.3711	−1.6461
−0.8	−0.7081	−2.1457	−0.9	−1.0857	−2.3422
−1	−1.4763	−2.0645	−1.1	−1.8391	−1.2865
−1.2	−2.1229	−0.2313	−1.3	−2.274	0.6509
−1.4	−2.2482	0.9062	−1.5	−2.0263	0.4139
−1.6	−1.6265	−0.4436	−1.7	−1.1097	−1.0267
−1.8	−0.5704	−0.9719	−1.9	−0.1147	−0.4957
−2	0.1715	−0.1159	−2.1	0.2493	−0.1033
−2.2	0.1396	−0.2990	−2.3	−0.0885	−0.4261
−2.4	−0.3453	−0.4094	−2.5	−0.5545	−0.3481
−2.6	−0.6754	−0.3238	−2.7	−0.7072	−0.3320
−2.8	−0.6765	−0.3418	−2.9	−0.6188	−0.3428
−3	−0.5628	−0.3406	−3.2	−0.5033	−0.3388

Table 4.15 and Fig. 4.6 demonstrate that the wavelength of the oscillations and the oscillatory zone of $\tilde{f}_r(\eta)$ decrease as r increases. A theoretical discussion of $\tilde{f}_r(\eta)$ is given in Section 4.7.

4.5 Asymptotic Behaviour of the Function $G_r(z)$

4.5.1 Asymptotic Formula for $G_r(z)$ Involving the Complementary Error Function

Since the functions $F_r^1(\eta)$ and $F_r^2(\eta)$ involve the function $G_r(z)$ defined in (4.3.13), we need to know the asymptotic behaviour of the function $G_r(z)$. In this section, the asymptotic behaviour of the function $G_r(z)$ for large r will be developed. The integral representation of the hypergeometric function is [Whittaker & Watson, 1965, p. 293]

$$\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}F(a,b;c;z) = \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt, \tag{4.5.1}$$

where $\mathrm{Re}(c) > \mathrm{Re}(a) > 0$. Using (4.5.1), the integral representation of $G_r(z)$ is

$$G_r(z) = \frac{\Gamma(r+1)}{\Gamma(\frac{1}{2})\Gamma(r+\frac{1}{2})} \int_0^1 t^{r-\frac{1}{2}}(1-(1-z^2)t)^{-\frac{1}{2}}dt, \tag{4.5.2}$$

where $|\arg z| \le \frac{\pi}{2}$.

We consider the integral in (4.5.2) for large r and $|\arg z| \le \frac{\pi}{2}$. Replacing the variable t by e^{-u} and considering the behaviour of the factor $e^{-(r+\frac{1}{2})u}$ for $u > 0$, we see that it tends to 1 as u tends to zero and to zero otherwise. Thus the main contribution of the integral comes from the neighbourhood of $u = 0$. Expanding the exponential factor e^{-u} in the neighbourhood of $u = 0$, the integral can be represented as

$$\begin{aligned} &\int_0^1 t^{r-\frac{1}{2}}(1-(1-z^2)t)^{-\frac{1}{2}}dt = \int_0^\infty e^{-(r+\frac{1}{2})u}(1-(1-z^2)e^{-u})^{-\frac{1}{2}}du \\ &\sim \int_0^\infty e^{-(r+\frac{1}{2})u}\{z^2+(1-z^2)(u-\frac{u^2}{2}+\frac{u^3}{3}-\cdots)\}^{-\frac{1}{2}}du. \end{aligned} \tag{4.5.3}$$

Here we only consider the first two main terms in the expansion of the exponential factor e^{-u} in (4.5.3). From (4.5.2), the function $G_r(z)$ can then be written as

$$\begin{aligned} G_r(z) &\sim \frac{\Gamma(r+1)}{\Gamma(\frac{1}{2})\Gamma(r+\frac{1}{2})} \int_0^\infty e^{-(r+\frac{1}{2})u}\{z^2+(1-z^2)(u-\frac{u^2}{2})\}^{-\frac{1}{2}}du \\ &\sim \frac{\Gamma(r+1)}{\Gamma(\frac{1}{2})\Gamma(r+\frac{1}{2})}(1-z^2)^{-\frac{1}{2}} \int_0^\infty e^{-(r+\frac{1}{2})u}(\kappa+u)^{-\frac{1}{2}}\{1-\frac{u^2}{2(\kappa+u)}\}^{-\frac{1}{2}}du \\ &\sim \frac{\Gamma(r+1)}{\Gamma(\frac{1}{2})\Gamma(r+\frac{1}{2})}(1-z^2)^{-\frac{1}{2}} \int_0^\infty e^{-(r+\frac{1}{2})u}(\kappa+u)^{-\frac{1}{2}}\{1+\frac{u^2}{4(\kappa+u)}\}du, \end{aligned} \tag{4.5.4}$$

where $\kappa = z^2/(1-z^2)$.

For convenience, we introduce the notation $G_r^{1,2}(z)$ to indicate the first and second leading terms of $G_r(z)$, respectively. From (4.5.4), we find

$$G_r^1(z) = \frac{\Gamma(r+1)}{\Gamma(\frac{1}{2})\Gamma(r+\frac{1}{2})}(1-z^2)^{-\frac{1}{2}} \int_0^\infty e^{-(r+\frac{1}{2})u}(\kappa+u)^{-\frac{1}{2}}du. \tag{4.5.5}$$

From the representation of $G_r^1(z)$ in (4.5.5), the following expression involving the complementary error function is obtained

$$\begin{aligned} G_r^1(z) &= \frac{\Gamma(r+1)}{\Gamma(\frac{1}{2})\Gamma(r+\frac{1}{2})}(r+\frac{1}{2})^{-\frac{1}{2}}(1-z^2)^{-\frac{1}{2}}e^{(r+\frac{1}{2})\kappa}\Gamma(\frac{1}{2},(r+\frac{1}{2})\kappa) \\ &= \frac{\Gamma(r+1)}{(r+\frac{1}{2})^{\frac{1}{2}}\Gamma(r+\frac{1}{2})}(1-z^2)^{-\frac{1}{2}}e^{(r+\frac{1}{2})\kappa}\mathrm{erfc}\sqrt{(r+\frac{1}{2})\kappa}. \end{aligned} \tag{4.5.6}$$

In the above derivation, we have used the formula $\Gamma(1/2, z^2) = \sqrt{\pi}\operatorname{erfc} z$.

We now consider the second leading term of $G_r(z)$. From (4.5.4), the uniform representation of the second term of $G_r(z)$ is

$$G_r^2(z) = \frac{\Gamma(r+1)}{\Gamma(\frac{1}{2})\Gamma(r+\frac{1}{2})}(1-z^2)^{-\frac{1}{2}} \int_0^\infty e^{-(r+\frac{1}{2})u} \frac{u^2}{4(\kappa+u)^{\frac{3}{2}}} du. \tag{4.5.7}$$

The uniformly in z of the second leading term of $G_r(z)$ can also be expressed in terms of the complementary error function as

$$\begin{aligned} G_r^2(z) &= \frac{\Gamma(r+1)}{4\Gamma(\frac{1}{2})\Gamma(r+\frac{1}{2})}(1-z^2)^{-\frac{1}{2}} \int_0^\infty e^{-(r+\frac{1}{2})u} \frac{(u+\kappa)^2 + \kappa^2 - 2u\kappa - 2\kappa^2}{(\kappa+u)^{\frac{3}{2}}} du \\ &= \frac{\Gamma(r+1)}{4\Gamma(\frac{1}{2})\Gamma(r+\frac{1}{2})}(1-z^2)^{-\frac{1}{2}} \{ \int_0^\infty e^{-(r+\frac{1}{2})u} (\kappa+u)^{\frac{1}{2}} du \\ &+\kappa^2 \int_0^\infty e^{-(r+\frac{1}{2})u}(\kappa+u)^{-\frac{3}{2}} du - 2\kappa \int_0^\infty e^{-(r+\frac{1}{2})u}(\kappa+u)^{-\frac{1}{2}} du \} \\ &= \frac{\Gamma(r+1)}{4\Gamma(\frac{1}{2})\Gamma(r+\frac{1}{2})}(1-z^2)^{-\frac{1}{2}} \{ \frac{1}{r+\frac{1}{2}} \kappa^{\frac{1}{2}} + 2\kappa^{\frac{3}{2}} + \sqrt{\pi} \\ & [\frac{1}{2(r+\frac{1}{2})} - 2\kappa^2(r+\frac{1}{2}) - 2\kappa] e^{(r+\frac{1}{2})\kappa} (r+\frac{1}{2})^{-\frac{1}{2}} \operatorname{erfc}\sqrt{(r+\frac{1}{2})\kappa} \}. \end{aligned} \tag{4.5.8}$$

When $|\arg z| > \frac{\pi}{2}$, the reflection formula (4.3.15) can be used to calculate the asymptotic behaviour of $G_r(z)$. To make it clear, the asymptotic behaviour of $G_r(z)$ for large r can be written as

$$G_r(z) \sim \begin{cases} G_r^1(z), & |\arg z| \le \frac{\pi}{2}, \\ 2(1-z^2)^{-r-\frac{1}{2}} - G_r^1(-z), & |\arg z| > \frac{\pi}{2}, \end{cases} \tag{4.5.9}$$

or, more accurately by

$$G_r(z) \sim \begin{cases} G_r^1(z) + G_r^2(z) & |\arg z| \le \frac{\pi}{2}, \\ 2(1-z^2)^{-r-\frac{1}{2}} - \{G_r^1(-z) + G_r^2(-z)\}, & |\arg z| > \frac{\pi}{2}. \end{cases}$$

The functions $G_r^1(z)$ and $G_r^2(z)$ contain the complementary error function which can be further expanded under certain conditions. Using the asymptotic expansion of the complementary error function, a further asymptotic representation of $G_r^1(z)$ and $G_r^2(z)$ will be given in the next subsection.

4.5.2 Further Asymptotic Formulas of $G_r^{1,2}(z)$

The functions $G_r^1(z)$ and $G_r^2(z)$ involve the complementary error functions. Using the asymptotic expansion of the complementary error function, the asymptotic approximation of the functions $G_r^1(z)$ and $G_r^2(z)$ can be developed. The asymptotic expansion of the complementary error function is [Olver, 1964, p. 67]

$$\operatorname{erfc} z \sim \frac{e^{-z^2}}{\sqrt{\pi} z} \sum_{s=0}^{\infty} (-)^s \frac{1 \cdot 3 \cdots (2s-1)}{(2z^2)^s}, \quad z \to \infty, \quad |\arg z| < \frac{3}{4}\pi. \tag{4.5.10}$$

If $\operatorname{erfc} z$ is approximated by the first term, (4.5.10) can be written as

$$\operatorname{erfc} z \sim \frac{e^{-z^2}}{\sqrt{\pi} z}, \quad z \to \infty, \quad |\arg z| < \frac{3}{4}\pi. \tag{4.5.11}$$

Obviously, the condition for the asymptotic expansion (4.5.11) to be valid is that z is large. In order to apply the above asymptotic expansion to the function $G_r^1(z)$, we assume that z is bounded away from the origin and r is very large. Substituting (4.5.11) into (4.5.6) gives the asymptotic behaviour of $G_r^1(z)$ as

$$G_r^1(z) \sim \frac{\Gamma(r+1)}{\sqrt{\pi}\Gamma(r+\frac{3}{2})z}, \tag{4.5.12}$$

If we choose the leading term $G_r^1(z)$ to approximate the function $G_r(z)$, we have the following asymptotic approximation when z is bounded away from the origin and r is very large

$$G_r(z) \sim \begin{cases} \frac{\Gamma(r+1)}{\sqrt{\pi}\Gamma(r+\frac{3}{2})z}, & |\arg z| \le \frac{\pi}{2}, \\ 2(1-z^2)^{-r-\frac{1}{2}} + \frac{\Gamma(r+1)}{\sqrt{\pi}\Gamma(r+\frac{3}{2})z}, & |\arg z| > \frac{\pi}{2}. \end{cases} \tag{4.5.13}$$

When z is near the origin and r is very large, we let $z \to 0$ in the formula (4.5.6) to find

$$G_r(z) \sim \begin{cases} \frac{\Gamma(r+1)}{(r+\frac{1}{2})^{\frac{1}{2}}\Gamma(r+\frac{1}{2})}, & |\arg z| \le \frac{\pi}{2}, \\ \\ 2 - \frac{\Gamma(r+1)}{(r+\frac{1}{2})^{\frac{1}{2}}\Gamma(r+\frac{1}{2})}, & |\arg z| > \frac{\pi}{2}, \end{cases} \tag{4.5.14}$$

Thus, $G_r(z)$ tends to 1 when z is near the origin for large r.

Now consider the second leading term $G_r^2(z)$ of $G_r(z)$. When z is bounded away from the origin and r is very large, the asymptotic expansion of the complementary error function in (4.5.10) can be truncated after the first three terms, since there is some cancellation in the calculation. Using the first three terms of (4.5.10) in (4.5.8), we find that the asymptotic form of $G_r^2(z)$ for large r and z bounded away from the origin is

$$\begin{aligned} G_r^2(z) &\sim \frac{\Gamma(r+1)}{4\Gamma(\frac{1}{2})\Gamma(r+\frac{1}{2})}(1-z^2)^{-\frac{1}{2}}\{\frac{1}{r+\frac{1}{2}}\kappa^{\frac{1}{2}}+2\kappa^{\frac{3}{2}} \\ &\quad -[\kappa+2\kappa^2(r+\frac{1}{2})+\frac{7}{4\kappa(r+\frac{1}{2})^2}]\frac{\kappa^{-\frac{1}{2}}}{r+\frac{1}{2}}\} \\ &\sim -\frac{7\Gamma(r+1)}{16\Gamma(\frac{1}{2})\Gamma(r+\frac{1}{2})(r+\frac{1}{2})^3}(1-z^2)^{-\frac{1}{2}}\kappa^{-\frac{3}{2}}. \end{aligned} \tag{4.5.15}$$

When z is near the origin and r is very large, the asymptotic formula for $G_r^2(z)$, from (4.5.8), can be written as

$$G_r^2(z) \sim \frac{\Gamma(r+1)}{8(r+\frac{1}{2})^{\frac{3}{2}}\Gamma(r+\frac{1}{2})}, \quad |\arg z| \leq \frac{\pi}{2}. \tag{4.5.16}$$

The second leading term $G_r^2(z)$ of the function $G_r(z)$ is $O(1/r)$ when r is large and so is small. The leading term $G_r^1(z)$ of the function $G_r(z)$ can be used to approximate the function $G_r(z)$. To show the accuracy of the asymptotic approximation $G_r^1(z)$, numerical results will be given in the next section.

4.6 Discussion and Numerical Results of (4.5.9)

In this section, numerical results are presented to compare the exact value of the function $G_r(z)$ with the leading approximation $G_r^1(z)$ for large r. When $|\arg z| \leq \frac{\pi}{2}$, the function $G_r(z)$ is defined by (4.3.10) or (4.3.13) and the reflection formula (4.3.15) can be used when $|\arg z| > \frac{\pi}{2}$. The approximation formula of $G_r(z)$ is defined by (4.5.9). In the calculation, we put $z = \rho e^{i\phi}$ and varied ϕ in the range $-\pi \leq \phi \leq \pi$ for different values of $\rho = \frac{1}{2}$, 2 when $r = 15$.

Tables 4.16-4.17 show the comparison between the leading asymptotic approximation $G_r^1(z)$ and the exact value of $G_r(z)$ for different ϕ when $\rho = \frac{1}{2}$, $\rho = 2$ and $r = 15$. It is noted that the values of $G_r(z)$ and $G_r^1(z)$ in Table 4.16 are large when ϕ near ± 1, but those in Table 4.17 are not .The reason for this is that the functions $G_r(z)$ and $G_r^1(z)$ include the factor $(1-z^2)^{-r-\frac{1}{2}}$ when $|\arg z| > \frac{\pi}{2}$; thus there is a lobe in which the values of $G_r(z)$ and $G_r^1(z)$ are large in the half-left plane z for large r. As r is increased, the values of $G_r(z)$ and $G_r^1(z)$ become larger.

Table 4.16: Computations of $G_r^1(z)$ from (4.5.9) when $r = 15$ and $\rho = \frac{1}{2}$ for different ϕ

ϕ/π	$G_r(z)$	$G_r^1(z)$
-0.5	$0.0314698591 - 0.3582607437i$	$0.0399696487 - 0.3571003208i$
-0.4	$0.1402775683 + 0.2872674606i$	$0.1434703080 + 0.2847836804i$
-0.3	$0.2026186128 + 0.2116271883i$	$0.2034901650 + 0.2094949698i$
-0.2	$0.2385882800 + 0.1385648572i$	$0.2383388020 + 0.1371104574i$
-0.1	$0.2574304867 + 0.0684495883i$	$0.2566551131 + 0.0677228395i$
0	0.2633097302	0.2623787499
0.1	$0.2574304867 - 0.0684495883i$	$0.2566551131 - 0.0677228395i$
0.2	$0.2385882800 - 0.1385648572i$	$0.2383388020 - 0.1371104574i$
0.3	$0.2026186128 - 0.2116271883i$	$0.2034901650 - 0.2094949698i$
0.4	$0.1402775683 - 0.2872674606i$	$0.1434703080 - 0.2847836804i$
0.5	$0.0314698591 - 0.3582607437i$	$0.0399696487 - 0.3571003208i$
0.6	$-0.1751963300 - 0.3823626852i$	$-0.1593333190 - 0.3335731828i$
0.7	$-0.6280894630 - 0.1140331883i$	$-0.6289610151 - 0.1119009697i$
0.8	$-3.2802696469 + 2.7947796243i$	$-3.2800201690 + 2.7962340242i$
0.9	$-48.950538649 - 16.102897181i$	$-48.949763275 - 16.102170432i$
1	172.55128336	172.55221434
-1	172.55128336	172.55221434
-0.9	$-48.950538649 + 16.102897181i$	$-48.949763275 + 16.102170432i$
-0.8	$-3.2802696469 - 2.7947796243i$	$-3.2800201690 - 2.7962340242i$
-0.7	$-0.6280894630 + 0.1140331883i$	$-0.6289610151 + 0.1119009697i$
-0.6	$-0.1720035903 + 0.3848464654i$	$-0.1751963300 + 0.3823626852i$

Table 4.17: Computations of $G_r^1(z)$ from (4.5.9) for $r = 15$ and $\rho = 2$ for different ϕ

ϕ/π	$G_r(z)$	$G_r^1(z)$
-0.5	$1.4654459541 \times 10^{-11}$	$1.8270972082 \times 10^{-6}$
	$-0.0741217143i$	$-0.0743939594i$
-0.4	$0.0232463092 + 0.0702360430i$	$0.0233711522 + 0.0704582101i$
-0.3	$0.0437931452 + 0.0591972325i$	$0.0439705572 + 0.0593238194i$
-0.2	$0.0596031143 + 0.0425476650i$	$0.0597730823 + 0.0425968379i$
-0.1	$0.0694663171 + 0.0221836670i$	$0.0696101644 + 0.0221948234i$
0	0.0728081906	0.0729395776
0.1	$0.0694663171 - 0.0221836670i$	$0.0696101644 - 0.0221948234i$
0.2	$0.0596031143 - 0.0425476650i$	$0.0597730823 - 0.0425968379i$
0.3	$0.0437931452 - 0.0591972325i$	$0.0439705572 - 0.0593238194i$
0.4	$0.0232463092 - 0.0702360430i$	$0.0233711522 - 0.0704582101i$
0.5	$1.4654459541 \times 10^{-11}$	$1.8270972083 \times 10^{-6}$
	$-0.0741217143i$	$-0.0743939594i$
0.6	$-0.0232463092 - 0.0702360431i$	$-0.0233711522 - 0.0704582102i$
0.7	$-0.0437931454 - 0.0591972325i$	$-0.0439705574 - 0.0593238193i$
0.8	$-0.0596031123 - 0.0425476654i$	$-0.0597730804 - 0.0425968382i$
0.9	$-0.0694663170 - 0.0221836431i$	$-0.0696101643 - 0.0221947996i$
1	-0.0728081906	-0.0729395776
-1	-0.0728081906	-0.0729395776
-0.9	$-0.0694663170 + 0.0221836431i$	$-0.0696101643 + 0.0221947996i$
-0.8	$-0.0596031123 + 0.0425476654i$	$-0.0597730804 + 0.0425968382i$
-0.7	$-0.0437931454 + 0.0591972325i$	$-0.0439705574 + 0.0593238193i$
-0.6	$-0.0232463092 + 0.0702360431i$	$-0.0233711522 + 0.0704582102i$

4.7 The Asymptotic Behaviour of $f_r(\eta)$ for Large r

In Section 4.6, the first and second leading terms of $G_r(z)$ were obtained for large r. We also have the asymptotic approximations for $f_r(\eta)$ in (4.4.1) and (4.4.2) involving the function $F_r(z)$, which contains the function $G_r(z)$. In this section, we will obtain the asymptotic behaviour of $f_r(\eta)$ for large r. Two cases for η will be discussed: one is η bounded away from the origin and the other is η near the origin. We first consider odd values of r and, referring to Fig. 4.2, a number of different cases are studied as follows.

Case 1: z is in the domain A

In this domain we have $z = \eta/2\sqrt{\pi}$ bounded away from the origin. The discussion is the same as case 1 in Section 4.4. Using the asymptotic formula for $G_r(z)$, which is defined in (4.5.13), we have

$$G_r(z_{1,2}) \sim \frac{\Gamma(r+1)}{\sqrt{\pi}\Gamma(r+\frac{3}{2})e^{\mp\frac{\pi}{4}i}z}, \tag{4.7.1}$$

where we recall that $z_{1,2} = e^{\mp\frac{\pi}{4}i}z$. Applying (4.7.1) to (4.4.1), we obtain

$$\begin{aligned}
f_r(\eta) &\sim \frac{1}{2i}\{e^{-\frac{(2r+1)}{4}\pi i}G_r(z_1) - e^{\frac{(2r+1)}{4}\pi i}G_r(z_2)\} + (-)^{(r+1)/2}\frac{2c_0(\eta)}{\sqrt{r}} \\
&\sim \frac{1}{2i}\{e^{-\frac{(2r+1)}{4}\pi i}\frac{\Gamma(r+1)}{\sqrt{\pi}\Gamma(r+\frac{3}{2})e^{-\frac{\pi}{4}i}z} - e^{\frac{(2r+1)}{4}\pi i}\frac{\Gamma(r+1)}{\sqrt{\pi}\Gamma(r+\frac{3}{2})e^{\frac{\pi}{4}i}z}\} \\
&\quad +(-)^{(r+1)/2}\frac{2c_0(\eta)}{\sqrt{r}} \\
&\sim \frac{\Gamma(r+1)}{\Gamma(r+\frac{3}{2})i\eta}\{e^{-\frac{r}{2}\pi i} - e^{\frac{r}{2}\pi i}\} + (-)^{(r+1)/2}\frac{2c_0(\eta)}{\sqrt{r}} \\
&\sim 2(-)^{(r+1)/2}\frac{1}{\sqrt{r}\eta} + 2(-)^{(r+1)/2}\frac{1}{\sqrt{r}}(\frac{1}{\mu} - \frac{1}{\eta}) \\
&\sim 2(-)^{(r+1)/2}\frac{1}{\sqrt{r}\mu}.
\end{aligned} \tag{4.7.2}$$

When z is near the origin, we use (4.5.14) to find

$$G_r(z_{1,2}) \sim \frac{\Gamma(r+1)}{(r+\frac{1}{2})^{\frac{1}{2}}\Gamma(r+\frac{1}{2})}. \tag{4.7.3}$$

Substituting (4.7.3) into (4.4.1), we obtain

$$f_r(\eta) \sim \frac{1}{2i}\{e^{-\frac{(2r+1)}{4}\pi i}G_r(z_1) - e^{\frac{(2r+1)}{4}\pi i}G_r(z_2)\} + (-)^{(r+1)/2}\frac{2c_0(\eta)}{\sqrt{r}}$$

$$\sim \frac{1}{2i}\{e^{-\frac{(2r+1)}{4}\pi i}\frac{\Gamma(r+1)}{(r+\frac{1}{2})^{\frac{1}{2}}\Gamma(r+\frac{1}{2})} - e^{\frac{(2r+1)}{4}\pi i}\frac{\Gamma(r+1)}{(r+\frac{1}{2})^{\frac{1}{2}}\Gamma(r+\frac{1}{2})}\}$$
$$+(-)^{(r+1)/2}\frac{2c_0(\eta)}{\sqrt{r}}$$
$$\sim (-)^{(r+1)/2}\sin\frac{\pi}{4} + (-)^{(r+1)/2}\frac{2c_0(\eta)}{\sqrt{r}}. \tag{4.7.4}$$

Case 2: z is in the domain B

When z is in the domain B and z is bounded away from the origin, using (4.4.7) and (4.5.13), we have

$$G_r(z_1) \sim \frac{\Gamma(r+1)}{\sqrt{\pi}\Gamma(r+\frac{3}{2})e^{-\frac{\pi}{4}i}z},$$
$$G_r(z_2) \sim 2(1-z_2^2)^{-r-\frac{1}{2}} + \frac{\Gamma(r+1)}{\sqrt{\pi}\Gamma(r+\frac{3}{2})e^{\frac{\pi}{4}i}z}. \tag{4.7.5}$$

Substituting (4.7.5) into (4.4.1), we find

$$f_r(\eta) \sim \frac{1}{2i}\{e^{-\frac{(2r+1)}{4}\pi i}G_r(z_1) - e^{\frac{(2r+1)}{4}\pi i}G_r(z_2)\} + (-)^{(r+1)/2}\frac{2c_0(\eta)}{\sqrt{r}}$$
$$\sim \frac{1}{2i}\{e^{-\frac{(2r+1)}{4}\pi i}\frac{\Gamma(r+1)}{\sqrt{\pi}\Gamma(r+\frac{3}{2})e^{-\frac{\pi}{4}i}z} - e^{\frac{(2r+1)}{4}\pi i}[2(1-z_2^2)^{-r-\frac{1}{2}}$$
$$+\frac{\Gamma(r+1)}{\sqrt{\pi}\Gamma(r+\frac{3}{2})e^{\frac{\pi}{4}i}z}]\} + (-)^{(r+1)/2}\frac{2c_0(\eta)}{\sqrt{r}}$$
$$\sim 2(-)^{(r+1)/2}\frac{1}{\sqrt{r\mu}} + ie^{\frac{(2r+1)}{4}\pi i}(1 - i\frac{\eta^2}{4\pi})^{-r-\frac{1}{2}}. \tag{4.7.6}$$

When z is near the origin, we use (4.5.14) to find

$$G_r(z_1) \sim \frac{\Gamma(r+1)}{(r+\frac{1}{2})^{\frac{1}{2}}\Gamma(r+\frac{1}{2})},$$
$$G_r(z_2) \sim 2 - \frac{\Gamma(r+1)}{(r+\frac{1}{2})^{\frac{1}{2}}\Gamma(r+\frac{1}{2})}. \tag{4.7.7}$$

Applying (4.7.7) to (4.4.1), we obtain

$$f_r(\eta) \sim \frac{1}{2i}\{e^{-\frac{(2r+1)}{4}\pi i}G_r(z_1) - e^{\frac{(2r+1)}{4}\pi i}G_r(z_2)\} + (-)^{(r+1)/2}\frac{2c_0(\eta)}{\sqrt{r}}$$
$$\sim \frac{1}{2i}\{e^{-\frac{(2r+1)}{4}\pi i}\frac{\Gamma(r+1)}{(r+\frac{1}{2})^{\frac{1}{2}}\Gamma(r+\frac{1}{2})} - e^{\frac{(2r+1)}{4}\pi i}[2 - \frac{\Gamma(r+1)}{(r+\frac{1}{2})^{\frac{1}{2}}\Gamma(r+\frac{1}{2})}]\}$$
$$+(-)^{(r+1)/2}\frac{2c_0(\eta)}{\sqrt{r}}$$
$$\sim (-)^{(r+1)/2}\sin\frac{\pi}{4} + (-)^{(r+1)/2}\frac{2c_0(\eta)}{\sqrt{r}}. \tag{4.7.8}$$

Case 3: z is in the domain C

When z is in the domain C, we also have z is bounded away from the origin. From (4.4.8) and (4.5.13), we then find

$$G_r(z_{1,2}) = \pm 2(1 - z_{1,2}^2)^{-r-\frac{1}{2}} + \frac{\Gamma(r+1)}{\sqrt{\pi}\Gamma(r+\frac{3}{2})e^{\mp\frac{\pi}{4}i}z}. \qquad (4.7.9)$$

Substitution of (4.7.9) to (4.4.1), yields

$$\begin{aligned} f_r(\eta) &\sim \frac{1}{2i}\{e^{-\frac{(2r+1)}{4}\pi i}G_r(z_1) - e^{\frac{(2r+1)}{4}\pi i}G_r(z_2)\} + (-)^{(r+1)/2}\frac{2c_0(\eta)}{\sqrt{r}} \\ &\sim \frac{1}{2i}\{e^{-\frac{(2r+1)}{4}\pi i}[2(1-z_1^2)^{-r-\frac{1}{2}} + \frac{\Gamma(r+1)}{\sqrt{\pi}\Gamma(r+\frac{3}{2})e^{-\frac{\pi}{4}i}z}] \\ &\quad -e^{\frac{(2r+1)}{4}\pi i}[-2(1-z_2^2)^{-r-\frac{1}{2}} + \frac{\Gamma(r+1)}{\sqrt{\pi}\Gamma(r+\frac{3}{2})e^{\frac{\pi}{4}i}z}]\} + (-)^{(r+1)/2}\frac{2c_0(\eta)}{\sqrt{r}} \\ &\sim 2(-)^{(r+1)/2}\frac{1}{\sqrt{r}\mu} \\ &\quad -i\{e^{\frac{(2r+1)}{4}\pi i}(1 - i\frac{\eta^2}{4\pi})^{-r-\frac{1}{2}} + e^{-\frac{(2r+1)}{4}\pi i}(1 + i\frac{\eta^2}{4\pi})^{-r-\frac{1}{2}}\}. \end{aligned} \qquad (4.7.10)$$

Since $|z| > 1$ in this domain, there is no need to consider the case $z \simeq 0$.

Case 4: z is in the domain D

If z is bounded away from the origin and is in the domain D, we obtain from (4.4.9) and (4.5.13),

$$G_r(z_{1,2}) = 2(1 - z_{1,2}^2)^{-r-\frac{1}{2}} + \frac{\Gamma(r+1)}{\sqrt{\pi}\Gamma(r+\frac{3}{2})e^{\mp\frac{\pi}{4}i}z}. \qquad (4.7.11)$$

Applying (4.7.11) to (4.4.1), we find

$$\begin{aligned} f_r(\eta) &\sim \frac{1}{2i}\{e^{-\frac{(2r+1)}{4}\pi i}G_r(z_1) - e^{\frac{(2r+1)}{4}\pi i}G_r(z_2)\} + (-)^{(r+1)/2}\frac{2c_0(\eta)}{\sqrt{r}} \\ &\sim \frac{1}{2i}\{e^{-\frac{(2r+1)}{4}\pi i}[2(1-z_1^2)^{-r-\frac{1}{2}} + \frac{\Gamma(r+1)}{\sqrt{\pi}\Gamma(r+\frac{3}{2})e^{-\frac{\pi}{4}i}z}] \\ &\quad -e^{\frac{(2r+1)}{4}\pi i}[2(1-z_2^2)^{-r-\frac{1}{2}} + \frac{\Gamma(r+1)}{\sqrt{\pi}\Gamma(r+\frac{3}{2})e^{\frac{\pi}{4}i}z}]\} + (-)^{(r+1)/2}\frac{2c_0(\eta)}{\sqrt{r}} \\ &\sim 2(-)^{(r+1)/2}\frac{1}{\sqrt{r}\mu} \\ &\quad +i\{e^{\frac{(2r+1)}{4}\pi i}(1 - i\frac{\eta^2}{4\pi})^{-r-\frac{1}{2}} - e^{-\frac{(2r+1)}{4}\pi i}(1 + i\frac{\eta^2}{4\pi})^{-r-\frac{1}{2}}\}. \end{aligned} \qquad (4.7.12)$$

we have

$$G_r(z_{1,2}) = 2 - \frac{\Gamma(r+1)}{(r+\frac{1}{2})^{\frac{1}{2}}\Gamma(r+\frac{1}{2})}. \qquad (4.7.13)$$

Applying (4.7.13) to (4.4.1), we obtain

$$
\begin{aligned}
f_r(\eta) &\sim \frac{1}{2i}\{e^{-\frac{(2r+1)}{4}\pi i}G_r(z_1) - e^{\frac{(2r+1)}{4}\pi i}G_r(z_2)\} + (-)^{(r+1)/2}\frac{2c_0(\eta)}{\sqrt{r}} \\
&\sim \frac{1}{2i}\{e^{-\frac{(2r+1)}{4}\pi i}[2 - \frac{\Gamma(r+1)}{(r+\frac{1}{2})^{\frac{1}{2}}\Gamma(r+\frac{1}{2})}] \\
&\quad -e^{\frac{(2r+1)}{4}\pi i}[2 - \frac{\Gamma(r+1)}{(r+\frac{1}{2})^{\frac{1}{2}}\Gamma(r+\frac{1}{2})}]\} + (-)^{(r+1)/2}\frac{2c_0(\eta)}{\sqrt{r}} \\
&\sim (-)^{(r+1)/2}\sin\frac{\pi}{4} + (-)^{(r+1)/2}\frac{2c_0(\eta)}{\sqrt{r}}. \qquad (4.7.14)
\end{aligned}
$$

When η is near the origin and r is odd, the above three cases with η in the domains A, B and D, yield the same result, namely

$$
f_r(\eta) \sim (-)^{(r+1)/2}\sin\frac{\pi}{4} + (-)^{(r+1)/2}\frac{2c_0(\eta)}{r^{1/2}}. \qquad (4.7.15)
$$

We know that $c_0(0) = -\frac{1}{3}$. Thus, the asymptotic approximation of $f_r(0)$ for large odd r is

$$
f_r(0) \sim (-)^{(r+1)/2}\sin\frac{\pi}{4} - (-)^{(r+1)/2}\frac{2}{3r^{1/2}}. \qquad (4.7.16)
$$

Next, we consider that the asymptotic behaviour of $f_r(\eta)$ defined in (4.4.2) for large even r. The analysis is divided into two cases: when η is bounded away from the origin and the other is when η near the origin.

For even r, the first and second leading terms $F_r^{1,2}(\eta)$ of $F_r(\eta)$ are used to approximate $F_r(\eta)$. The first two main terms of the asymptotic expansion (4.5.10) are used to approximate the complementary error function in $G_r^1(\eta)$ appearing in $F_r^1(\eta)$, and the asymptotic behaviour of $G_r(z)$ defined in (4.5.13) is used in $F_r^2(\eta)$. The asymptotic formula of $F_r(\eta)$ is developed next.

When z is away from the origin and $|\arg z| \leq \frac{\pi}{2}$ for large r, substitution of the first two terms in the asymptotic expansion (4.5.10) into (4.5.6), yields

$$
\begin{aligned}
G_r^1(z) &= \frac{\Gamma(r+1)}{(r+\frac{1}{2})^{\frac{1}{2}}\Gamma(r+\frac{1}{2})}(1-z^2)^{-\frac{1}{2}}e^{(r+\frac{1}{2})\kappa}\mathrm{erfc}(z\sqrt{\frac{(r+\frac{1}{2})}{1-z^2}}) \\
&\sim \frac{\Gamma(r+1)}{(r+\frac{1}{2})^{\frac{1}{2}}\Gamma(r+\frac{1}{2})}(1-z^2)^{-\frac{1}{2}}e^{(r+\frac{1}{2})\kappa}\frac{e^{-(r+\frac{1}{2})\kappa}}{\sqrt{\pi}z\sqrt{\frac{(r+\frac{1}{2})}{1-z^2}}}(1-\frac{(1-z^2)}{2(r+\frac{1}{2})z^2}) \\
&= \frac{\Gamma(r+1)}{\Gamma(r+\frac{3}{2})\sqrt{\pi}z}(1-\frac{(1-z^2)}{2(r+\frac{1}{2})z^2}). \qquad (4.7.17)
\end{aligned}
$$

Substituting (4.7.17) and (4.5.13) into $F_r^1(\eta)$ and $F_r^2(\eta)$, respectively, we obtain

$$
\begin{aligned}
F_r(\eta) &\sim F_r^1(\eta) + F_r^2(\eta) \\
&= \frac{1}{2i}\{e^{-\frac{(2r+1)}{4}\pi i}G_r(e^{-\frac{\pi}{4}i}z) - e^{\frac{(2r+1)}{4}\pi i}G_r(e^{\frac{\pi}{4}i}z)\} \\
&\quad + \frac{\pi}{12(r-\frac{1}{2})}\{e^{-\frac{(2r+1)}{4}\pi i}G_{r-1}(e^{-\frac{\pi}{4}i}z) + e^{\frac{(2r+1)}{4}\pi i}G_{r-1}(e^{\frac{\pi}{4}i}z)\} \\
&\sim -\frac{\Gamma(r+1)}{\Gamma(r+\frac{3}{2})\sqrt{\pi}z}\sin\frac{r}{2}\pi - \frac{\Gamma(r+1)}{2(r+\frac{1}{2})\Gamma(r+\frac{3}{2})\sqrt{\pi}z}\cos(\frac{r-1}{2})\pi \\
&\quad + \frac{\Gamma(r+1)}{2(r+\frac{1}{2})\Gamma(r+\frac{3}{2})\sqrt{\pi}z^3}\sin(\frac{r-1}{2})\pi \\
&\quad + \frac{\sqrt{\pi}\Gamma(r)}{6(r-\frac{1}{2})\Gamma(r+\frac{1}{2})z}\cos\frac{r}{2}\pi,
\end{aligned}
\tag{4.7.18}
$$

where $F_r^1(\eta)$ and $F_r^2(\eta)$ are defined in (4.3.8) and (4.3.9), respectively.

When r is even, the simplified form of (4.7.18) is

$$
\begin{aligned}
F_r(\eta) &\sim \frac{\Gamma(r+1)}{2(r+\frac{1}{2})\Gamma(r+\frac{3}{2})\sqrt{\pi}z^3}\sin(\frac{r-1}{2})\pi + \frac{\sqrt{\pi}\Gamma(r)}{6(r-\frac{1}{2})\Gamma(r+\frac{1}{2})z}\cos\frac{r}{2}\pi \\
&= \frac{\cos\frac{r}{2}\pi}{2\sqrt{r\pi}z}\{\frac{\pi}{3(r-\frac{1}{2})} - \frac{1}{(r+\frac{1}{2})z^2}\} \\
&\sim (-)^{r/2}\frac{\pi}{\sqrt{r\eta}}(\frac{1}{3r} - \frac{4}{r\eta^2}).
\end{aligned}
\tag{4.7.19}
$$

For the case when r is odd, the asymptotic expansion of $F_r(\eta)$ is, from (4.7.18),

$$
F_r(\eta) \sim -\frac{\Gamma(r+1)}{\Gamma(r+\frac{3}{2})\sqrt{\pi}z}\sin\frac{r}{2}\pi \sim 2(-)^{(r+1)/2}\frac{1}{\sqrt{r\eta}}.
$$

This is the same as the result given in (4.7.2) for large odd r.

Now we can obtain the asymptotic behaviour of $f_r(\eta)$ for large even r. Referring to Fig. 4.2 again and using a similar analysis as above, the discussion is given as follows. If $|\arg z| > \pi/2$ and $|\arg z^2| \le 2\pi$ or $|\arg z^2| > 2\pi$, $|z| < 1$, the following reflection formula is used

$$
G_r(z) = 2(1-z^2)^{-r-\frac{1}{2}} - G_r(-z). \tag{4.7.20}
$$

If $|\arg z^2| > 2\pi$ and $|z| \ge 1$, the following reflection formula is used

$$
G_r(z) = -2(1-z^2)^{-r-\frac{1}{2}} - G_r(-z), \tag{4.7.21}
$$

where $|\arg(-z)| \le \frac{\pi}{2}$.

Case 1: z is in the domain A

When z is bounded away from the origin, we recall that $z = \eta/2\sqrt{\pi}$. Applying (4.7.19) to (4.4.2) and using the result given in (4.7.2) for odd r, we have

$$\begin{aligned}
f_r(\eta) &\sim F_r(\eta) + \frac{2\pi\gamma_1}{(r-\frac{1}{2})} f_{r-1}(\eta) + (-)^{r/2} \frac{4\pi\Gamma(r-1)}{\Gamma(r+\frac{1}{2})} \{c_1(\eta)\gamma_0 - c_0(\eta)\gamma_1\} \\
&\sim \frac{1}{2i}\{e^{-\frac{(2r+1)}{4}\pi i} G_r(z_1) - e^{\frac{(2r+1)}{4}\pi i} G_r(z_2)\} \\
&\quad + \frac{\pi}{12(r-\frac{1}{2})}\{e^{-\frac{(2r+1)}{4}\pi i} G_{r-1}(z_1) + e^{\frac{(2r+1)}{4}\pi i} G_{r-1}(z_2)\} \\
&\quad + \frac{2\pi\gamma_1}{(r-\frac{1}{2})} f_{r-1}(\eta) + (-)^{r/2} 4\pi \frac{\Gamma(r-1)}{\Gamma(r+\frac{1}{2})} \{c_1(\eta)\gamma_0 - c_0(\eta)\gamma_1\} \\
&\sim (-)^{r/2} \frac{\pi}{\sqrt{r}\eta}\left(\frac{1}{3r} - \frac{4}{r\eta^2}\right) + (-)^{r/2} \frac{4\gamma_1\pi}{(r-\frac{1}{2})\sqrt{r-1}\mu} \\
&\quad + (-)^{r/2} \frac{4\pi\Gamma(r-1)}{\Gamma(r+\frac{1}{2})} \{c_1(\eta)\gamma_0 - c_0(\eta)\gamma_1\} \\
&\sim (-)^{(r+2)/2} \frac{4\pi}{r^{3/2}} \left(\frac{1+\mu+\frac{1}{12}\mu^2}{\mu^3}\right). \qquad (4.7.22)
\end{aligned}$$

When z is near the origin, applying (4.7.3) to (4.4.2) and using the result given in (4.7.4) for odd r, we have

$$\begin{aligned}
f_r(\eta) &\sim F_r(\eta) + \frac{2\pi\gamma_1}{(r-\frac{1}{2})} f_{r-1}(\eta) + (-)^{r/2} \frac{4\pi\Gamma(r-1)}{\Gamma(r+\frac{1}{2})} \{c_1(\eta)\gamma_0 - c_0(\eta)\gamma_1\} \\
&\sim \frac{1}{2i}\{e^{-\frac{(2r+1)}{4}\pi i} - e^{\frac{(2r+1)}{4}\pi i}\} + \frac{\pi}{12r}\{e^{-\frac{(2r+1)}{4}\pi i} + e^{\frac{(2r+1)}{4}\pi i}\} \\
&\quad + (-)^{r/2} \frac{2\pi\gamma_1}{r} \sin\frac{\pi}{4} + (-)^{r/2} \frac{4\pi}{r^{3/2}} c_0(\eta)\gamma_1 \\
&\quad + (-)^{r/2} 4\pi \frac{\Gamma(r-1)}{\Gamma(r+\frac{1}{2})} \{c_1(\eta)\gamma_0 - c_0(\eta)\gamma_1\} \\
&\sim -(-)^{r/2} \sin\frac{\pi}{4} + (-)^{r/2} \frac{4\pi}{r^{3/2}} c_1(\eta). \qquad (4.7.23)
\end{aligned}$$

Case 2: z is in the domain B

When z is bounded away from the origin, applying the reflection formula for z_2 in (4.7.20) and (4.7.19) to (4.4.2), and using the result given in (4.7.6), we obtain

$$\begin{aligned}
f_r(\eta) &\sim F_r(\eta) + \frac{2\pi\gamma_1}{(r-\frac{1}{2})} f_{r-1}(\eta) + (-)^{r/2} 4\pi \frac{\Gamma(r-1)}{\Gamma(r+\frac{1}{2})} \{c_1(\eta)\gamma_0 - c_0(\eta)\gamma_1\} \\
&\sim \frac{1}{2i}\{e^{-\frac{(2r+1)}{4}\pi i} G_r(z_1) - e^{\frac{(2r+1)}{4}\pi i}[2(1-z_2^2)^{-r-\frac{1}{2}} - G_r(-z_2)]\} \\
&\quad + \frac{\pi}{12(r-\frac{1}{2})}\{e^{-\frac{(2r+1)}{4}\pi i} G_{r-1}(z_1) + e^{\frac{(2r+1)}{4}\pi i}[2(1-z_2^2)^{-r+\frac{1}{2}} - G_{r-1}(-z_2)]\}
\end{aligned}$$

$$+\frac{2\pi\gamma_1}{(r-\frac{1}{2})}\{2(-)^{r/2}\frac{1}{\sqrt{r-1}\mu}+ie^{\frac{(2r-1)}{4}\pi i}(1-z_2^2)^{-r+\frac{1}{2}}\}$$

$$+(-)^{r/2}4\pi\frac{\Gamma(r-1)}{\Gamma(r+\frac{1}{2})}\{c_1(\eta)\gamma_0-c_0(\eta)\gamma_1\}$$

$$\sim\quad(-)^{(r+2)/2}\frac{4\pi}{r^{3/2}}(\frac{1+\mu+\frac{1}{12}\mu^2}{\mu^3})+ie^{\frac{(2r+1)}{4}\pi i}(1-z_2^2)^{-r-\frac{1}{2}}$$

$$+\frac{\pi}{6r}e^{\frac{(2r+1)}{4}\pi i}(1-z_2^2)^{-r+\frac{1}{2}}+\frac{2\pi\gamma_1}{r}ie^{\frac{(2r-1)}{4}\pi i}(1-z_2^2)^{-r+\frac{1}{2}}$$

$$\sim\quad(-)^{(r+2)/2}\frac{4\pi}{r^{3/2}}(\frac{1+\mu+\frac{1}{12}\mu^2}{\mu^3})+ie^{\frac{(2r+1)}{4}\pi i}(1-i\frac{\eta^2}{4\pi})^{-r-\frac{1}{2}}. \qquad (4.7.24)$$

When z is near the origin, applying (4.7.7) to (4.4.2) and using the result given in (4.7.8), we find

$$f_r(\eta)\quad\sim\quad F_r(\eta)+\frac{2\pi\gamma_1}{(r-\frac{1}{2})}f_{r-1}(\eta)+(-)^{r/2}\frac{4\pi\Gamma(r-1)}{\Gamma(r+\frac{1}{2})}\{c_1(\eta)\gamma_0-c_0(\eta)\gamma_1\}$$

$$\sim\quad\frac{1}{2i}\{e^{-\frac{(2r+1)}{4}\pi i}\frac{\Gamma(r+1)}{(r+\frac{1}{2})^{\frac{1}{2}}\Gamma(r+\frac{1}{2})}-e^{\frac{(2r+1)}{4}\pi i}[2-\frac{\Gamma(r+1)}{(r+\frac{1}{2})^{\frac{1}{2}}\Gamma(r+\frac{1}{2})}]\}$$

$$+\frac{\pi}{12(r-\frac{1}{2})}\{e^{-\frac{(2r+1)}{4}\pi i}\frac{\Gamma(r)}{(r-\frac{1}{2})^{\frac{1}{2}}\Gamma(r-\frac{1}{2})}+e^{\frac{(2r+1)}{4}\pi i}[2-\frac{\Gamma(r)}{(r-\frac{1}{2})^{\frac{1}{2}}\Gamma(r-\frac{1}{2})}]\}$$

$$+\frac{2\pi\gamma_1}{(r-\frac{1}{2})}[(-)^{r/2}\frac{2c_0(\eta)}{\sqrt{r-1}}+(-)^{r/2}\sin\frac{\pi}{4}]$$

$$+(-)^{r/2}4\pi\frac{\Gamma(r-1)}{\Gamma(r+\frac{1}{2})}\{c_1(\eta)\gamma_0-c_0(\eta)\gamma_1\}$$

$$\sim\quad-(-)^{r/2}\sin\frac{\pi}{4}+(-)^{r/2}\frac{4\pi}{r^{3/2}}c_1(\eta). \qquad (4.7.25)$$

Case 3: z is in the domain C

Since $|z|>1$, we have upon applying (4.7.20) for z_1, (4.7.21) for z_2 and (4.7.19) to (4.4.2), together with the result given in (4.7.10) for odd r,

$$f_r(\eta)\quad\sim\quad F_r(\eta)+\frac{2\pi\gamma_1}{(r-\frac{1}{2})}f_{r-1}(\eta)+(-)^{r/2}4\pi\frac{\Gamma(r-1)}{\Gamma(r+\frac{1}{2})}\{c_1(\eta)\gamma_0-c_0(\eta)\gamma_1\}$$

$$\sim\quad\frac{1}{2i}\{e^{-\frac{(2r+1)}{4}\pi i}[2(1-z_1^2)^{-r-\frac{1}{2}}-G_r(-z_1)]$$

$$-e^{\frac{(2r+1)}{4}\pi i}[-2(1-z_2^2)^{-r-\frac{1}{2}}-G_r(-z_2)]\}$$

$$+\frac{\pi}{12(r-\frac{1}{2})}\{e^{-\frac{(2r+1)}{4}\pi i}[2(1-z_1^2)^{-r+\frac{1}{2}}-G_{r-1}(-z_1)]$$

$$+e^{\frac{(2r+1)}{4}\pi i}[-2(1-z_2^2)^{-r+\frac{1}{2}}-G_{r-1}(-z_2)]\}$$

$$+\frac{2\pi\gamma_1}{(r-\frac{1}{2})}\{2(-)^{r/2}\frac{1}{\sqrt{r-1}\mu}-i[e^{-\frac{(2r-1)}{4}\pi i}(1-z_1^2)^{-r+\frac{1}{2}}$$

$$+e^{\frac{(2r-1)}{4}\pi i}(1-z_2^2)^{-r+\frac{1}{2}}]\}$$

$$
\begin{aligned}
&+(-)^{r/2}\frac{4\pi\Gamma(r-1)}{\Gamma(r+\frac{1}{2})}\{c_1(\eta)\gamma_0 - c_0(\eta)\gamma_1\}\\
\sim\ &(-)^{(r+2)/2}\frac{4\pi}{r^{3/2}}(\frac{1+\mu+\frac{1}{12}\mu^2}{\mu^3}) - i\{e^{\frac{(2r+1)}{4}\pi i}(1-z_2^2)^{-r-\frac{1}{2}}\\
&+e^{-\frac{(2r+1)}{4}\pi i}(1-z_1^2)^{-r-\frac{1}{2}}\}\\
&+\frac{\pi}{6r}\{-e^{\frac{(2r+1)}{4}\pi i}(1-z_2^2)^{-r+\frac{1}{2}} + e^{-\frac{(2r+1)}{4}\pi i}(1-z_1^2)^{-r+\frac{1}{2}}\}\\
&-\frac{2\pi\gamma_1}{r}i\{e^{\frac{(2r-1)}{4}\pi i}(1-z_2^2)^{-r+\frac{1}{2}} + e^{-\frac{(2r-1)}{4}\pi i}(1-z_1^2)^{-r+\frac{1}{2}}\}\\
\sim\ &(-)^{(r+2)/2}\frac{4\pi}{r^{3/2}}(\frac{1+\mu+\frac{1}{12}\mu^2}{\mu^3})\\
&-i\{e^{\frac{(2r+1)}{4}\pi i}(1-i\frac{\eta^2}{4\pi})^{-r-\frac{1}{2}} + e^{-\frac{(2r+1)}{4}\pi i}(1+i\frac{\eta^2}{4\pi})^{-r-\frac{1}{2}}\}. \qquad (4.7.26)
\end{aligned}
$$

Case 4: z is in the domain D

When z is bounded away from the origin, application of (4.7.20) for $z_{1,2}$ and (4.7.19) to (4.4.2) and use of the results given in (4.7.12) for odd r, yields

$$
\begin{aligned}
f_r(\eta)\ \sim\ &F_r(\eta) + \frac{2\pi\gamma_1}{(r-\frac{1}{2})}f_{r-1}(\eta) + (-)^{r/2}\frac{4\pi\Gamma(r-1)}{\Gamma(r+\frac{1}{2})}\{c_1(\eta)\gamma_0 - c_0(\eta)\gamma_1\}\\
\sim\ &\frac{1}{2i}\{e^{-\frac{(2r+1)}{4}\pi i}[2(1-z_1^2)^{-r-\frac{1}{2}} - G_r(-z_1)]\\
&-e^{\frac{(2r+1)}{4}\pi i}[2(1-z_2^2)^{-r-\frac{1}{2}} - G_r(-z_2)]\}\\
&+\frac{\pi}{12(r-\frac{1}{2})}\{e^{-\frac{(2r+1)}{4}\pi i}[2(1-z_1^2)^{-r+\frac{1}{2}} - G_{r-1}(-z_1)]\\
&+e^{\frac{(2r+1)}{4}\pi i}[2(1-z_2^2)^{-r+\frac{1}{2}} - G_{r-1}(-z_2)]\}\\
&+\frac{2\pi\gamma_1}{(r-\frac{1}{2})}\{2(-)^{r/2}\frac{1}{\sqrt{r-1}\mu} - i(e^{-\frac{(2r-1)}{4}\pi i}(1-z_1^2)^{-r+\frac{1}{2}}\\
&-e^{\frac{(2r-1)}{4}\pi i}(1-z_2^2)^{-r+\frac{1}{2}})\} + (-)^{r/2}\frac{4\pi\Gamma(r-1)}{\Gamma(r+\frac{1}{2})}\{c_1(\eta)\gamma_0 - c_0(\eta)\gamma_1\}\\
\sim\ &(-)^{(r+2)/2}\frac{4\pi}{r^{3/2}}(\frac{1+\mu+\frac{1}{12}\mu^2}{\mu^3}) + i\{e^{\frac{(2r+1)}{4}\pi i}(1-z_2^2)^{-r-\frac{1}{2}}\\
&-e^{-\frac{(2r+1)}{4}\pi i}(1-z_1^2)^{-r-\frac{1}{2}}\}\\
&+\frac{\pi}{6r}\{e^{\frac{(2r+1)}{4}\pi i}(1-z_2^2)^{-r+\frac{1}{2}} + e^{-\frac{(2r+1)}{4}\pi i}(1-z_1^2)^{-r+\frac{1}{2}}\}\\
&+\frac{2\pi\gamma_1}{r}i\{e^{\frac{(2r-1)}{4}\pi i}(1-z_2^2)^{-r+\frac{1}{2}} - e^{-\frac{(2r-1)}{4}\pi i}(1-z_1^2)^{-r+\frac{1}{2}}\}\\
\sim\ &(-)^{(r+2)/2}\frac{4\pi}{r^{3/2}}(\frac{1+\mu+\frac{1}{12}\mu^2}{\mu^3})\\
&+i\{e^{\frac{(2r+1)}{4}\pi i}(1-i\frac{\eta^2}{4\pi})^{-r-\frac{1}{2}} - e^{-\frac{(2r+1)}{4}\pi i}(1+i\frac{\eta^2}{4\pi})^{-r-\frac{1}{2}}\}. \qquad (4.7.27)
\end{aligned}
$$

When z is near the origin, we apply (4.7.13) to (4.4.2) and use the result given in (4.7.14) for odd r to obtain

$$\begin{aligned} f_r(\eta) &\sim F_r(\eta) + \frac{2\pi\gamma_1}{(r-\frac{1}{2})} f_{r-1}(\eta) + (-)^{r/2} \frac{4\pi\Gamma(r-1)}{\Gamma(r+\frac{1}{2})}\{c_1(\eta)\gamma_0 - c_0(\eta)\gamma_1\} \\ &\sim -(-)^{r/2}\sin\frac{\pi}{4} + (-)^{r/2}\frac{4\pi}{r^{3/2}}c_1(\eta). \end{aligned} \tag{4.7.28}$$

When η is near the origin for large even r, the above three cases yield the common result

$$f_r(\eta) \sim -(-)^{r/2}\sin\frac{\pi}{4} + (-)^{r/2}\frac{4\pi}{r^{3/2}}c_1(\eta). \tag{4.7.29}$$

Substitution of $c_1(0) = -\frac{1}{540}$ in (4.7.29). then gives, for large even r,

$$f_r(0) \sim -(-)^{r/2}\sin\frac{\pi}{4} - (-)^{r/2}\frac{\pi}{135r^{3/2}}. \tag{4.7.30}$$

A summary of the above results is shown in Tables 4.18 and 4.19, where we have defined

$$L_{\pm}(r,\eta) = -i\{e^{-\frac{(2r+1)}{4}\pi i}(1+i\frac{\eta^2}{4\pi})^{-r-\frac{1}{2}} \pm e^{\frac{(2r+1)}{4}\pi i}(1-i\frac{\eta^2}{4\pi})^{-r-\frac{1}{2}}\}.$$

The definition of the coefficients $c_r(\eta)$ in terms of $f_r(\eta)$ is given in (4.2.24).

Table 4.18: The asymptotic form of $f_r(\eta)$ for large odd r

	η bounded away from the origin
$\eta \in A$	$2(-)^{(r+1)/2}\frac{1}{\sqrt{r\mu}}$
$\eta \in B$	$2(-)^{(r+1)/2}\frac{1}{\sqrt{r\mu}} + ie^{\frac{(2r+1)}{4}\pi i}(1-i\frac{\eta^2}{4\pi})^{-r-\frac{1}{2}}$
$\eta \in C$	$2(-)^{(r+1)/2}\frac{1}{\sqrt{r\mu}} + L_+(r,\eta)$
$\eta \in D$	$2(-)^{(r+1)/2}\frac{1}{\sqrt{r\mu}} + L_-(r,\eta)$
	η near the origin
	$(-)^{(r+1)/2}\sin\frac{\pi}{4} + (-)^{(r+1)/2}\frac{2c_0(\eta)}{r^{1/2}}$
	$\eta = 0$
	$(-)^{(r+1)/2}\sin\frac{\pi}{4} - (-)^{(r+1)/2}\frac{2}{3r^{1/2}}$

When η is real and negative, so that η belongs to the domain D, we find that $L_-(r,\eta)$ has the simpler form

$$L_-(r,\eta) = -2(1+\frac{\eta^4}{16\pi^2})^{-\frac{(2r+1)}{4}}\sin\{\frac{2r+1}{4}\pi + (r+\frac{1}{2})\arctan\frac{\eta^2}{4\pi}\}. \tag{4.7.31}$$

Table 4.19: The asymptotic form of $f_r(\eta)$ for large even r

	η bounded away from the origin
$\eta \in A$	$(-)^{(r+2)/2}\frac{4\pi}{r^{3/2}}(\frac{1+\mu+\frac{1}{12}\mu^2}{\mu^3})$
$\eta \in B$	$(-)^{(r+2)/2}\frac{4\pi}{r^{3/2}}(\frac{1+\mu+\frac{1}{12}\mu^2}{\mu^3}) + ie^{\frac{(2r+1)}{4}\pi i}(1-i\frac{\eta^2}{4\pi})^{-r-\frac{1}{2}}$
$\eta \in C$	$(-)^{(r+2)/2}\frac{4\pi}{r^{3/2}}(\frac{1+\mu+\frac{1}{12}\mu^2}{\mu^3}) + L_+(r,\eta)$
$\eta \in D$	$(-)^{(r+2)/2}\frac{4\pi}{r^{3/2}}(\frac{1+\mu+\frac{1}{12}\mu^2}{\mu^3}) + L_-(r,\eta)$
	η near the origin
	$-(-)^{r/2}\sin\frac{\pi}{4} + (-)^{r/2}\frac{4\pi}{r^{3/2}}c_1(\eta)$
	$\eta = 0$
	$-(-)^{r/2}\sin\frac{\pi}{4} - (-)^{r/2}\frac{\pi}{135r^{3/2}}$

From the above representation of $L_-(r,\eta)$ and the asymptotic forms of $f_r(\eta)$ for large r when $\eta \in D$ in Tables 4.18-4.19, where we suppose that η is real and bounded away from the origin, we find

$$f_r(z) \sim \begin{cases} 2(-)^{(r+1)/2}\frac{1}{\sqrt{r\mu}} + L_-(r,\eta), & r \text{ odd}, \\ (-)^{(r+2)/2}\frac{4\pi}{r^{3/2}}(\frac{1+\mu+\frac{1}{12}\mu^2}{\mu^3}) + L_-(r,\eta), & r \text{ even}, \end{cases}$$

where $L_-(r,\eta)$ given in (4.7.31). We can see that $f_r(\eta)$ (η real) is more oscillatory for $\eta < 0$ between the two lobes as r increases; the oscillatory zone should decrease with increasing r [see Fig. 4.6].

4.8 Conclusions

The important and efficient asymptotic approximation of the coefficients $c_r(\eta)$ for large even and odd r is given in this Chapter. Extensive numerical results demonstrate the accuracy of these asymptotic results. From the numerical results, we found that the values of the coefficients $c_r(\eta)$ and their asymptotic approximation become large when η is in the domains $|1 - e^{\mp\frac{\pi}{2}i}\eta^2/4\pi| < 1$. We also analyse the reason for this local growth. It is found that the lobes are caused by the hypergeometric function $F(1, r+\frac{1}{2}; 1; e^{\pm\frac{\pi}{2}i}\eta^2/4\pi)$ appearing in the asymptotic approximation of $c_r(\eta)$. The domain of this lobe in the upper η plane is symmetrical with respect to the straight line joining 0 and $2\sqrt{\pi}e^{3\pi i/4}$; there is a symmetrical domain lying in the lower half η-plane.

A knowledge of the asymptotic behaviour of the coefficients $c_r(\eta)$ for large r is used to develop the asymptotic behaviour of the late terms in the expansion (2.7.5) in the next chapter.

CHAPTER 5

ASYMPTOTIC BEHAVIOUR OF THE LATE TERMS IN (2.4.4) AND (2.7.5)

5.1 Introduction

The asymptotic formula for $Z(t)$ is given in (2.4.4) and the modification of (2.4.4) in the neighbourhood of a discontinuity in N_t is given in (2.7.5). Since $E_m(t;N)$ and $E_m^*(t;N)$ are absolutely convergent series, the divergence of the correction terms in (2.4.4) and (2.7.5) as $m \to \infty$ must result from the divergence of the finite sums $\sum_{r=0}^{m-1}(\pi i/4)^r B_r(w)$ and $\sum_{r=0}^{m-1}(\pi i/4)^r B_r^*(w)$, where the coefficients $B_r(w)$ and $B_r^*(w)$ are defined in (2.4.12) and (2.8.3). In this Chapter, we shall examine the large-m behaviour of these coefficients to determine the nature of this divergence. For this purpose, the asymptotic behaviour of the coefficients $\alpha_k^{(r)}$ for large r is examined in Section 5.2, where $\alpha_k^{(r)}$ are given in (2.4.6). Then the asymptotic behaviour of the late terms in (2.4.4) and (2.7.5) are developed in Sections 5.3 and 5.4 respectively.

5.2 Asymptotic Behaviour of the Coefficients $\alpha_k^{(r)}$

In this section, we examine the large-r behaviour of the coefficients $\alpha_k^{(r)}$, where $k = 2r - p$, $p = 0, 1, 2, \cdots$, that appear in the polynomial $Q_r(\mu)$ in (2.4.6). For convenience, we recall the coefficients $c_r(\eta)$ in the uniform asymptotic expansion of $Q(a, z)$ given by

$$c_r(\eta) = (-)^r \frac{Q_r(\mu)}{\mu^{2r+1}} - \frac{D_r}{\eta^{2r+1}}, \tag{5.2.1}$$

where

$$Q_r(\mu) = \sum_{k=0}^{2r} \alpha_k^{(r)} \mu^k \tag{5.2.2}$$

and $D_r = (-2)^r \Gamma(r+\frac{1}{2})/\Gamma(\frac{1}{2})$. The recurrence relation satisfied by the $c_r(\eta)$ [see (4.2.3)] is

$$\begin{aligned} c_0(\eta) &= \frac{1}{\mu} - \frac{1}{\eta}, \\ c_r(\eta) &= \frac{1}{\eta}\frac{d}{d\eta} c_{r-1}(\eta) + \frac{\gamma_r}{\mu}, \quad r \geq 1, \end{aligned} \tag{5.2.3}$$

where the Stirling coefficients γ_r are defined in (2.3.12). Substitution of (5.2.1) into the recurrence relation (5.2.3) shows that $Q_r(\mu)$ satisfies the recursion

$$\begin{aligned} Q_0(\mu) &= 1, \\ Q_r(\mu) &= (1+\mu)\{(2r-1)Q_{r-1}(\mu) - \mu \frac{dQ_{r-1}(\mu)}{d\mu}\} + (-)^r \gamma_r \mu^{2r}, r \geq 1. \end{aligned} \tag{5.2.4}$$

Putting (5.2.2) into (5.2.4), we then find that the coefficients $\alpha_k^{(r)}$ satisfy the recurrence relation

$$\begin{aligned} \alpha_k^{(r)} &= (2r-k)\alpha_{k-1}^{(r-1)} + (2r-1-k)\alpha_k^{(r-1)}, \quad 0 \leq k \leq 2r-1, \\ \alpha_{2r}^{(r)} &= (-)^r \gamma_r, \qquad k = 2r, \end{aligned} \tag{5.2.5}$$

where $\alpha_{-1}^{(r-1)} \equiv 0$. If $2r - k = p$ in (5.2.5), the recurrence relation (5.2.5) becomes

$$\begin{aligned} \alpha_{2r-p}^{(r)} &= (p-1)\alpha_{2r-p}^{(r-1)} + p\alpha_{2r-p-1}^{(r-1)}, \quad 1 \leq p \leq 2r, \\ \alpha_{2r}^{(r)} &= (-)^r \gamma_r, \qquad p = 0. \end{aligned} \tag{5.2.6}$$

Letting $\mu = -1$ in the second equation of (5.2.4), we have

$$Q_r(-1) = (-)^r \gamma_r, \quad r \geq 1.$$

By the definition of $Q_r(\mu)$ in (5.2.2) and using the above equation together with the second part of (5.2.6), the coefficients $\alpha_k^{(r)}$ satisfy the condition

$$\sum_{k=0}^{2r-1} (-)^k \alpha_k^{(r)} = 0, \quad r \geq 1. \tag{5.2.7}$$

The coefficients $\alpha_k^{(r)}$ are presented in Table 5.1 for $0 \leq r \leq 10$; recurrence relations equivalent to (5.2.6) for these coefficients are given in (2.3.6).

An examination of the recurrence relation (5.2.6) reveals that $\alpha_{2r-p}^{(r)}$ $(0 \leq p \leq 2r)$ can be expressed as a linear combination of the Stirling coefficients γ_κ, $\gamma_{\kappa-1}$, $\cdots$, γ_{r-p}, where κ denotes the integer

$$\kappa = r - \text{int}\,[\frac{p+1}{2}] \tag{5.2.8}$$

and γ_j is to be interpreted as zero for $j < 0$. Table 5.2 shows the first few $\alpha_k^{(r)}$ expressed in terms of the Stirling coefficients.

From the well-known asymptotic expansion (4.2.26) of γ_r for large r, it then follows that the large-r behaviour of $\alpha_{2r-p}^{(r)}$ is determined by the Stirling coefficients with the largest indices. Thus we can express $\alpha_{2r-p}^{(r)}$ in the form

$$\alpha_{2r-p}^{(r)} \sim (-)^r (a_p \gamma_\kappa + b_p \gamma_{\kappa-1} + \cdots), \qquad r \to \infty; \ p = 0,\ 1,\ 2, \cdots, \tag{5.2.9}$$

where the constants a_p and b_p are independent of r. We give the first few constants a_p and b_p in Table 5.3.

Use of (4.2.26) in (5.2.9) then shows that

$$(-)^r \alpha_{2r-p}^{(r)} \sim \begin{cases} A_p \frac{(-)^{\frac{\kappa+1}{2}} \Gamma(\kappa)}{(2\pi)^\kappa}, & \kappa \text{ odd}, \\ B_p \frac{(-)^{\frac{\kappa}{2}} \Gamma(\kappa-1)}{(2\pi)^\kappa} & \kappa \text{ even}, \end{cases} \tag{5.2.10}$$

as $r \to \infty$ with $p = 0,\ 1,\ 2, \cdots$, where κ is defined by (5.2.8), p is finite and A_p, B_p are given by

$$A_p = \frac{a_p}{\pi}, \qquad B_p = 2b_p - \frac{a_p}{6}. \tag{5.2.11}$$

The asymptotic form of $(-)^r \alpha_{2r-p}^{(r)}$ for large r will be used to get the behaviour of the late terms in the expansions (2.4.4) and (2.7.5) in the next two sections.

Table 5.1: The coefficients $\alpha_k^{(r)}$ $(0 \le k \le 2r)$ for $0 \le r \le 10$.

$k \setminus r$	0	1	2	3	4	5	6
0	1	1	3	15	105	945	10395
1		1	5	35	315	3465	45045
2		$\frac{1}{12}$	$\frac{25}{12}$	$\frac{105}{4}$	$\frac{1365}{4}$	$\frac{19635}{4}$	$\frac{315315}{4}$
3			$\frac{1}{12}$	$\frac{77}{12}$	$\frac{1883}{12}$	$\frac{13321}{4}$	$\frac{283283}{4}$
4			$\frac{1}{288}$	$\frac{49}{288}$	$\frac{2513}{96}$	$\frac{102949}{96}$	$\frac{3278275}{96}$
5				$\frac{1}{288}$	$\frac{149}{288}$	$\frac{38291}{288}$	$\frac{797225}{96}$
6				$-\frac{139}{51840}$	$\frac{221}{51840}$	$\frac{35981}{17280}$	$\frac{2792933}{3456}$
7					$-\frac{139}{51840}$	$\frac{77}{10368}$	$\frac{108251}{10368}$
8					$-\frac{571}{2488320}$	$-\frac{2783}{497664}$	$\frac{715}{55296}$
9						$-\frac{571}{2488320}$	$-\frac{42887}{2488320}$
10						$\frac{163879}{209018880}$	$\frac{67951}{209018880}$
11							$\frac{163879}{209018880}$
12							$\frac{5246819}{75246796800}$

$k \setminus r$	7	8	9	10
0	135135	2027025	34459425	654729075
1	675675	11486475	218243025	4583103525
2	$\frac{5630625}{4}$	$\frac{111035925}{4}$	$\frac{2400673275}{4}$	$\frac{56524943475}{4}$
3	$\frac{6301295}{4}$	$\frac{148813665}{4}$	$\frac{3748930185}{4}$	$\frac{100794328635}{4}$
4	$\frac{32497465}{32}$	$\frac{962396435}{32}$	$\frac{29178284135}{32}$	$\frac{917537325705}{32}$
5	$\frac{35882275}{96}$	$\frac{1431239095}{96}$	$\frac{18236110035}{32}$	$\frac{692979802515}{32}$
6	$\frac{249151331}{3456}$	$\frac{561480777}{128}$	$\frac{29076114067}{128}$	$\frac{1399211644831}{128}$
7	$\frac{2196337}{384}$	$\frac{266722027}{384}$	$\frac{21196085911}{384}$	$\frac{462773826515}{128}$
8	$\frac{1155869}{18432}$	$\frac{851484491}{18432}$	$\frac{45229977793}{6144}$	$\frac{4567178250635}{6144}$
9	$-\frac{10673}{2488320}$	$\frac{364077389}{829440}$	$\frac{347763837967}{829440}$	$\frac{14128831080455}{165888}$
10	$-\frac{4735393}{69672960}$	$-\frac{25470029}{69672960}$	$\frac{6985191863}{1990656}$	$\frac{2803066279325}{663552}$
11	$\frac{531611}{209018880}$	$-\frac{9843493}{29859840}$	$-\frac{9031403}{1990656}$	$\frac{62794475543}{1990656}$
12	$\frac{123239699}{75246796800}$	$\frac{54058997}{3583180800}$	$-\frac{272680799}{143327232}$	$-\frac{7110853721}{143327232}$
13	$\frac{5246819}{75246796800}$	$\frac{10863221}{2149908480}$	$\frac{41125975}{429981696}$	$-\frac{608837881}{47775744}$
14	$-\frac{534703531}{902961561600}$	$-\frac{2335885}{5159780352}$	$\frac{2001631}{106168320}$	$\frac{383051837}{573308928}$
15		$-\frac{534703531}{902961561600}$	$-\frac{2295746687}{902961561600}$	$\frac{75936371527}{902961561600}$
16		$-\frac{4483131259}{86684309913600}$	$-\frac{107146209211}{86684309913600}$	$-\frac{401001785147}{28894769971200}$
17			$-\frac{4483131259}{86684309913600}$	$-\frac{47200698593}{12383472844800}$
18			$\frac{432261921612371}{514904800886784000}$	$\frac{379002322255451}{514904800886784000}$
19				$\frac{432261921612371}{514904800886784000}$
20				$\frac{6232523202521089}{86504006548979712000}$

Table 5.2: The coefficients $\alpha_k^{(r)}$ for $0 \le r \le 10$ and $0 \le k \le 5$

$r \setminus k$	0	1
0	γ_0	
1	γ_0	γ_0
2	$3\gamma_0$	$5\gamma_0$
3	$15\gamma_0$	$35\gamma_0$
4	$105\gamma_0$	$315\gamma_0$
5	$945\gamma_0$	$3465\gamma_0$
6	$10395\gamma_0$	$45045\gamma_0$
7	$135135\gamma_0$	$675675\gamma_0$
8	$2027025\gamma_0$	$11486475\gamma_0$
9	$34459425\gamma_0$	$218243025\gamma_0$
10	$654729075\gamma_0$	$4583103525\gamma_0$
$r \setminus k$	2	3
0		
1	$-\gamma_1$	
2	$2\gamma_0 - \gamma_1$	$-\gamma_1$
3	$26\gamma_0 - 3\gamma_1$	$6\gamma_0 - 5\gamma_1$
4	$340\gamma_0 - 15\gamma_1$	$154\gamma_0 - 35\gamma_1$
5	$4900\gamma_0 - 105\gamma_1$	$3304\gamma_0 - 315\gamma_1$
6	$78750\gamma_0 - 945\gamma_1$	$70532\gamma_0 - 3465\gamma_1$
7	$1406790\gamma_0 - 10395\gamma_1$	$1571570\gamma_0 - 45045\gamma_1$
8	$27747720\gamma_0 - 135135\gamma_1$	$37147110\gamma_0 - 675675\gamma_1$
9	$599999400\gamma_0 - 2027025\gamma_1$	$936275340\gamma_0 - 11486475\gamma_1$
10	$14128364250\gamma_0 - 34459425\gamma_1$	$25180395240\gamma_0 - 218243025\gamma_1$
$r \setminus k$	4	5
0		
1		
2	γ_2	
3	$-2\gamma_1 + \gamma_2$	γ_2
4	$24\gamma_0 - 26\gamma_1 + 3\gamma_2$	$-6\gamma_1 + 5\gamma_2$
5	$1044\gamma_0 - 340\gamma_1 + 15\gamma_2$	$120\gamma_0 - 154\gamma_1 + 35\gamma_2$
6	$33740\gamma_0 - 4900\gamma_1 + 105\gamma_2$	$8028\gamma_0 - 3304\gamma_1 + 315\gamma_2$
7	$1008980\gamma_0 - 78750\gamma_1 + 945\gamma_2$	$367884\gamma_0 - 70532\gamma_1 + 3465\gamma_2$
8	$29957620\gamma_0 - 1406790\gamma_1 + 10395\gamma_2$	$14777620\gamma_0 - 1571570\gamma_1 + 45045\gamma_2$
9	$909508600\gamma_0 - 27747720\gamma_1 + 135135\gamma_2$	$566780500\gamma_0 - 37147110\gamma_1 + 675675\gamma_2$
10	$28623034440\gamma_0 - 599999400\gamma_1 + 2027025\gamma_2$	$21577556000\gamma_0 - 936275340\gamma_1 + 11486475\gamma_2$

Table 5.3: The constants a_p and b_p for $0 \le p \le 15$

p	0	1	2	3	4	5	6	7
a_p	1	−1	−1	5	3	−35	−15	315
b_p	0	0	2	−6	−26	154	340	−3304
p	8	9	10	11	12	13	14	15
a_p	105	−3465	−945	45045	10395	−675675	−135135	11486475
b_p	−4900	70532	78750	−1571570	−1406790	37147110	27747720	−936275340

5.3 Behaviour of the Late Terms in the Expansion (2.4.4)

In this section, the leading behaviour of the late terms in the expansion (2.4.4) for large m is developed. The sum involving $B_r(w)$ in the expansion (2.4.4) is

$$\sum_{r=0}^{m-1} (\pi i/4)^r B_r(w), \tag{5.3.1}$$

where the coefficients $B_r(w)$ are defined in (2.4.12).

For convenience in presentation, we replace $m-1$ by m in (5.3.1). Using the definition of the coefficients $B_r(w)$ and carrying out a regrouping of the terms, we find that the sum (5.3.1) has the more conventional form

$$\begin{aligned}\sum_{r=0}^{m} (\pi i/4)^r B_r(w) &= \sum_{r=0}^{m} (\pi i/4)^r \sum_{k=0}^{2r} (-)^k \alpha_k^{(r)} (w/2)^{-k} S_{2r+1-k}(w) \\ &= \sum_{k=0}^{2m} (-)^k (w/2)^{-k} \sum_{r=k/2}^{m} \alpha_k^{(r)} (\pi i/4)^r S_{2r+1-k}(w) \\ &= \sum_{k=0}^{2m} (-)^k (w/2)^{-k} C_k^{(m)}(w),\end{aligned} \tag{5.3.2}$$

where

$$C_k^{(m)}(w) = \sum_{j=0}^{2m-k} \alpha_k^{(\frac{1}{2}k+\frac{1}{2}j)} (\pi i/4)^{(k+j)/2} S_{j+1}(w), \quad 0 \le k \le 2m. \tag{5.3.3}$$

The coefficients $\alpha_k^{(r)}$ are to be interpreted as zero for half-integer values of r and the functions $S_k(w)$ are defined in terms of $\operatorname{cosec} w$ and its derivatives in (2.4.11). It is then seen that the correction terms in (5.3.2) involve an expansion in descending powers of $w/2 \simeq \frac{1}{2}\pi(t/2\pi)^{1/2}$.

The leading behaviour of the late terms of the sum (5.3.2) is determined by the structure of the coefficients $C_k^{(m)}(w)$. It turns out that the structure of these coefficients becomes simpler as the index k increases. As a consequence, the last two coefficients $C_k^{(m)}(w)$ with $k = 2m$ and $k = 2m-1$ both consist of a single term given by

$$\begin{aligned}C_{2m}^{(m)}(w) &= \alpha_{2m}^{(m)} (\pi i/4)^m S_1(w) \\ &= (-)^m \gamma_m (\pi i/4)^m S_1(w),\end{aligned} \tag{5.3.4}$$

$$
\begin{aligned}
C_{2m-1}^{(m)}(w) &= \alpha_{2m-1}^{(m)}(\pi i/4)^m S_2(w) \\
&= (-)^{m-1}\gamma_{m-1}(\pi i/4)^m S_2(w),
\end{aligned} \tag{5.3.5}
$$

where we have employed the identities $\alpha_{2m}^{(m)} = (-)^m\gamma_m$, $\alpha_{2m-1}^{(m)} = (-)^{m-1}\gamma_{m-1}$ which can be obtained from (5.2.6).

Application of (5.3.4) and (5.3.5) into the late terms of the sum (5.3.2) for large m, then immediately shows that

$$
(w/2)^{-2m}C_{2m}^{(m)}(w) \sim \begin{cases} i\frac{\Gamma(m)}{(\pi t)^m}A_0S_1(w), & m \text{ odd}, \\ \frac{\Gamma(m-1)}{(\pi t)^m}B_0S_1(w), & m \text{ even}, \end{cases} \tag{5.3.6}
$$

and

$$
(-)^{2m-1}(w/2)^{-2m+1}C_{2m-1}^{(m)}(w) \sim \begin{cases} \frac{\pi i}{\sqrt{2}}\frac{\Gamma(m-2)}{(\pi t)^{m-\frac{1}{2}}}B_1S_2(w), & m \text{ odd} \\ -\frac{\pi}{\sqrt{2}}\frac{\Gamma(m-1)}{(\pi t)^{m-\frac{1}{2}}}A_1S_2(w), & m \text{ even} \end{cases} \tag{5.3.7}
$$

where the constants A_p, B_p are defined in (5.2.11), with $A_0 = -A_1 = 1/\pi$ and $B_0 = -B_1 = -1/6$.

The remaining high-order terms in the sum (5.3.2) can be similarly estimated using the asymptotic behaviour of the coefficients $\alpha_{2r-p}^{(r)}$ which is given in (5.2.10). For convenience, define the functions

$$
F_{1,2}^{(e)}(k,w) = \sum_{j=0}^{k}(-\pi i/4)^j \begin{Bmatrix} iA_{2j} \\ B_{2j} \end{Bmatrix} S_{2j+1}(w),
$$

$$
F_{1,2}^{(o)}(k,w) = \frac{\pi i}{\sqrt{2}}\sum_{j=0}^{k}(-\pi i/4)^j \begin{Bmatrix} iA_{2j+1} \\ B_{2j+1} \end{Bmatrix} S_{2j+2}(w),
$$

where the subscripts 1 or 2 correspond to the sum involving the coefficients A_p or B_p respectively.

Next, the leading behaviour of the coefficients $(-)^{2m-M}(w/2)^{-2m+M}C_{2m-M}^{(m)}(w)$ for $M = 0, 1, 2, \cdots$ when $m \to \infty$ is developed. It is found that these coefficients divide themselves into four different forms according to the value of M. We observe that the period is four for the presentation of the coefficients; thus we take $M = 4p$, $4p+1$, $4p+2$, $4p+3$, with $p = 0, 1, 2, \cdots$. The following cases are discussed.

Case 1: $M = 4p$

When $k = 2m - M$ and $M = 4p$, where $p = 0, 1, 2, \cdots$, we have from the definition of $C_k^{(m)}(w)$ in (5.3.3),

$$\begin{aligned} C_{2m-M}^{(m)}(w) &= \sum_{j=0}^{M} \alpha_{2(m-\frac{M}{2}+\frac{j}{2})-j}^{(m-\frac{1}{2}M+\frac{1}{2}j)} (\pi i/4)^{(2m-M+j)/2} S_{j+1}(w) \\ &= \sum_{j=0}^{2p} \alpha_{2(m-2p+j)-2j}^{(m-2p+j)} (\pi i/4)^{m-2p+j} S_{2j+1}(w), \end{aligned} \tag{5.3.8}$$

Substitution of (5.2.10) into (5.3.8) for odd κ and even κ, we then find for large m

$$C_{2m-M}^{(m)}(w) \sim \begin{cases} \sum_{j=0}^{2p} (-)^{(\kappa+j)} \frac{(-)^{\frac{\kappa+1}{2}}\Gamma(\kappa)}{(2\pi)^\kappa} A_{2j} (\pi i/4)^{\kappa+j} S_{2j+1}(w), & m \text{ odd}, \\ \sum_{j=0}^{2p} (-)^{(\kappa+j)} \frac{(-)^{\frac{\kappa}{2}}\Gamma(\kappa-1)}{(2\pi)^\kappa} B_{2j} (\pi i/4)^{\kappa+j} S_{2j+1}(w), & m \text{ even}, \end{cases} \tag{5.3.9}$$

where $\kappa = m - 2p$ and κ is defined in (5.2.8).

For simplicity in presentation, the coefficients $(-)^{2m-M}(w/2)^{-2m+M} C_{2m-M}^{(m)}(w)$ will be indicated by $\bar{C}_M^{(m)}(w)$ in the following study. Using the result $w/2 \simeq \frac{1}{2}\pi(t/2\pi)^{1/2}$ and the asymptotic approximation (5.3.9) for $C_{2m-M}^{(m)}(w)$, the asymptotic behaviour of the coefficients $\bar{C}_M^{(m)}(w)$ for large m is

$$\bar{C}_M^{(m)}(w) \sim \begin{cases} \frac{\Gamma(m-2p)}{(\pi t)^{m-2p}} \sum_{j=0}^{2p} (-\pi i/4)^j i A_{2j} S_{2j+1}(w), & m \text{ odd}, \\ \frac{\Gamma(m-2p-1)}{(\pi t)^{m-2p}} \sum_{j=0}^{2p} (-\pi i/4)^j B_{2j} S_{2j+1}(w), & m \text{ even}. \end{cases} \tag{5.3.10}$$

Case 2: $M = 4p + 1$

Using the definition of $C_k^{(m)}(w)$ in (5.3.3) when $k = 2m - M$, where $M = 4p+1$, we have

$$C_{2m-M}^{(m)}(w) = \sum_{j=0}^{2p} \alpha_{2(m-2p+j)-(2j+1)}^{(m-2p+j)} (\pi i/4)^{m-2p+j} S_{2j+2}(w), \tag{5.3.11}$$

Applying (5.2.10) to (5.3.11) for κ, we obtain for large m

$$C_{2m-M}^{(m)}(w) \sim \begin{cases} \sum_{j=0}^{2p} (-)^{(\kappa+j+1)} \frac{(-)^{\frac{\kappa}{2}}\Gamma(\kappa-1)}{(2\pi)^\kappa} B_{2j+1} (\pi i/4)^{\kappa+j+1} S_{2j+2}(w), & m \text{ odd}, \\ \sum_{j=0}^{2p} (-)^{(\kappa+j+1)} \frac{(-)^{\frac{\kappa+1}{2}}\Gamma(\kappa)}{(2\pi)^\kappa} A_{2j+1} (\pi i/4)^{\kappa+j+1} S_{2j+2}(w), & m \text{ even}, \end{cases} \tag{5.3.12}$$

where $\kappa = m - 2p - 1$.

Thus, the asymptotic behaviour of the coefficients $\bar{C}_M^{(m)}(w)$ for large m is

$$\bar{C}_M^{(m)}(w) \sim \begin{cases} \frac{\pi i}{\sqrt{2}} \frac{\Gamma(m-2p-2)}{(\pi t)^{m-2p-\frac{1}{2}}} \sum_{j=0}^{2p} (-\pi i/4)^j B_{2j+1} S_{2j+2}(w), & m \text{ odd}, \\ \frac{\pi i}{\sqrt{2}} \frac{\Gamma(m-2p-1)}{(\pi t)^{m-2p-\frac{1}{2}}} \sum_{j=0}^{2p} (-\pi i/4)^j i A_{2j+1} S_{2j+2}(w) & m \text{ even}. \end{cases} \tag{5.3.13}$$

Case 3: $M = 4p + 2$

Putting $k = 2m - M$ and $M = 4p+2$ into (5.3.3), the function $C^{(m)}_{2m-M}(w)$ can be written as

$$C^{(m)}_{2m-M}(w) = \sum_{j=0}^{2p+1} \alpha^{(m-2p-1+j)}_{2(m-2p-1+j)-2j} (\pi i/4)^{m-2p-1+j} S_{2j+1}(w). \tag{5.3.14}$$

For large m, applying (5.2.10) to (5.3.14) for κ, we find

$$C^{(m)}_{2m-M}(w) \sim \begin{cases} \sum_{j=0}^{2p+1} (-)^{(\kappa+j)} \frac{(-)^{\frac{\kappa}{2}}\Gamma(\kappa-1)}{(2\pi)^\kappa} B_{2j} (\pi i/4)^{\kappa+j} S_{2j+1}(w), & m \text{ odd}, \\ \sum_{j=0}^{2p+1} (-)^{(\kappa+j)} \frac{(-)^{\frac{\kappa+1}{2}}\Gamma(\kappa)}{(2\pi)^\kappa} A_{2j} (\pi i/4)^{\kappa+j} S_{2j+1}(w), & m \text{ even}, \end{cases} \tag{5.3.15}$$

where $\kappa = m - 2p - 1$.

The asymptotic behaviour of the coefficients $\bar{C}^{(m)}_M(w)$ for large m is

$$\bar{C}^{(m)}_M(w) \sim \begin{cases} \frac{\Gamma(m-2p-2)}{(\pi t)^{m-2p-1}} \sum_{j=0}^{2p+1} (-\pi i/4)^j B_{2j} S_{2j+1}(w) & m \text{ odd}, \\ \frac{\Gamma(m-2p-1)}{(\pi t)^{m-2p-1}} \sum_{j=0}^{2p+1} (-\pi i/4)^j i A_{2j} S_{2j+1}(w) & m \text{ even}. \end{cases} \tag{5.3.16}$$

Case 4: $M = 4p + 3$

When $k = 2m - M$ and $M = 4p+3$ in (5.3.3), the function $C^{(m)}_{2m-M}(w)$ becomes

$$C^{(m)}_{2m-M}(w) = \sum_{j=0}^{2p+1} \alpha^{(m-2p-1+j)}_{2(m-2p-1+j)-(2j+1)} (\pi i/4)^{m-2p-1+j} S_{2j+2}(w). \tag{5.3.17}$$

Using (5.2.10) into (5.3.17) for κ, we have for large m

$$C^{(m)}_{2m-M}(w) \sim \begin{cases} \sum_{j=0}^{2p+1} (-)^{(\kappa+j+1)} \frac{(-)^{\frac{\kappa+1}{2}}\Gamma(\kappa)}{(2\pi)^\kappa} A_{2j+1} (\pi i/4)^{\kappa+j+1} S_{2j+2}(w), & m \text{ odd}, \\ \sum_{j=0}^{2p+1} (-)^{(\kappa+j+1)} \frac{(-)^{\frac{\kappa}{2}}\Gamma(\kappa-1)}{(2\pi)^\kappa} B_{2j+1} (\pi i/4)^{\kappa+j+1} S_{2j+2}(w), & m \text{ even}, \end{cases} \tag{5.3.18}$$

where $\kappa = m - 2p - 1$.

The asymptotic behaviour of the coefficients $\bar{C}^{(m)}_M(w)$ for large m is

$$\bar{C}^{(m)}_M(w) \sim \begin{cases} \frac{\pi i}{\sqrt{2}} \frac{\Gamma(m-2p-2)}{(\pi t)^{m-2p-\frac{3}{2}}} \sum_{j=0}^{2p+1} (-\pi i/4)^j i A_{2j+1} S_{2j+2}(w), & m \text{ odd}, \\ \frac{\pi i}{\sqrt{2}} \frac{\Gamma(m-2p-3)}{(\pi t)^{m-2p-\frac{3}{2}}} \sum_{j=0}^{2p+1} (-\pi i/4)^j B_{2j+1} S_{2j+2}(w), & m \text{ even}. \end{cases} \tag{5.3.19}$$

The above results are summarized in Table 5.4 using the sums $F^{(e)}_{1,2}(k, w)$ and $F^{(o)}_{1,2}(k, w)$.

It is seen that the form of the leading approximation to these coefficients depends not only on the parity of m but also on the value of M.

Table 5.4: The leading behaviour of the coefficients $(-)^k(w/2)^{-k}C_k^{(m)}(w)$ as $m \to \infty$ when $k = 2m - M$, $M = 4p,\ 4p+1,\ 4p+2,\ 4p+3$ and $p = 0,\ 1,\ 2,\ \cdots$

	$M = 4p$	$M = 4p+1$
m odd	$\frac{\Gamma(m-2p)}{(\pi t)^{m-2p}} F_1^{(e)}(2p, w)$	$\frac{\Gamma(m-2p-2)}{(\pi t)^{m-2p-\frac{1}{2}}} F_2^{(o)}(2p, w)$
m even	$\frac{\Gamma(m-2p-1)}{(\pi t)^{m-2p}} F_2^{(e)}(2p, w)$	$\frac{\Gamma(m-2p-1)}{(\pi t)^{m-2p-\frac{1}{2}}} F_1^{(o)}(2p, w)$
	$M = 4p+2$	$M = 4p+3$
m odd	$\frac{\Gamma(m-2p-2)}{(\pi t)^{m-2p-1}} F_2^{(e)}(2p+1, w)$	$\frac{\Gamma(m-2p-2)}{(\pi t)^{m-2p-\frac{3}{2}}} F_1^{(o)}(2p+1, w)$
m even	$\frac{\Gamma(m-2p-1)}{(\pi t)^{m-2p-1}} F_1^{(e)}(2p+1, w)$	$\frac{\Gamma(m-2p-3)}{(\pi t)^{m-2p-\frac{3}{2}}} F_2^{(o)}(2p+1, w)$

The sum (5.3.1) is divergent and possesses the 'factorial divided by a power' dependence characteristic of an asymptotic series. Optimal truncation near the smallest term of this expansion is seen to correspond to approximately $m = \text{int}[\pi t]$. The leading behaviour of these coefficients is multiplied by the factors $F_{1,2}^{(e,o)}$, which consist of slowly oscillatory functions involving $\operatorname{cosec} w$ and its derivatives. A similar result has recently been derived for the Riemann-Siegel expansion by Berry [Berry, 1994] using formal arguments.

5.4 Behaviour of the Late Terms in the Expansion (2.7.5)

In this section, the asymptotic behaviour of the late terms in the expansion (2.7.5) is studied. Replacing $m-1$ by m in the sum involving $B_r^*(w)$ in the expansion (2.7.5) and using the definition of $B_r^*(w)$ in (2.8.3), we find

$$\sum_{r=0}^{m} (\pi i/4)^r B_r^*(w) = \sum_{r=0}^{m} (\pi i/4)^r \{\bar{B}_r^*(w) + (-)^{n_*+r-1}(w/2)^{-2r-1} c_r(\eta_*)\}, \qquad (5.4.1)$$

where $\bar{B}_r^*(w)$ and $c_r(\eta_*)$ are defined by (2.8.4) and (2.8.8) respectively.

The first term $\sum_{r=0}^{m} (\pi i/4)^r \bar{B}_r^*(w)$ of (5.4.1) can be written as

$$\sum_{r=0}^{m} (\pi i/4)^r \bar{B}_r^*(w) = \sum_{k=0}^{2m} (-)^k (w/2)^{-k} C_k^{(m*)}(w), \qquad (5.4.2)$$

where

$$C_k^{(m*)}(w) = \sum_{j=0}^{2m-k} \alpha_k^{(\frac{1}{2}k+\frac{1}{2}j)} (\pi i/4)^{(k+j)/2} S_{j+1}^*(w), \quad 0 \le k \le 2m. \tag{5.4.3}$$

The sum (5.4.2) is the same form as the sum (5.3.2) except that $S_{j+1}(w)$ is replaced by $S_{j+1}^*(w)$; thus, the late terms of the sum $\sum_{r=0}^{m} (\pi i/4)^r \bar{B}_r^*(w)$ in (5.4.2) for large m possess the same behaviour as Table 5.4, except that $S_k(w)$ is replaced by the deleted sum $S_k^*(w)$ defined in (2.8.5).

The coefficients $B_r^*(w)$ are given in terms of $\bar{B}_r^*$ and $c_r(\eta_*)$. In the analysis of the behaviour of the late terms in (5.4.1), it only remains to study the behaviour of the late terms in the sum

$$\sum_{r=0}^{m} (-)^{n_*+r-1} (\pi i/4)^r (w/2)^{-2r-1} c_r(\eta_*). \tag{5.4.4}$$

In order to get the same terms in the sums (5.4.2) and (5.4.4), the sum (5.4.4) is changed to

$$(-)^{n_*-1} \sum_{k=0}^{2m} (-)^{k/2} (\pi i/4)^{k/2} (w/2)^{-k-1} c_{\frac{k}{2}}(\eta_*), \tag{5.4.5}$$

where we interpret $c_r(\eta_*)$ as zero for half-integer values of r. Next we investigate the leading behaviour of the late terms in (5.4.5) for large m. They are given by

$$(-)^{n_*-1+k/2} (\pi i/4)^{k/2} (w/2)^{-k-1} c_{\frac{k}{2}}(\eta_*), \tag{5.4.6}$$

where $k = 2m - M$, $M = 4p$, $4p+1$, $4p+2$, $4p+3$ and $p = 0, 1, 2, \cdots$. When $M = 4p+1$, $4p+3$, the terms (5.4.6) correspond to 0. From (4.2.24), the coefficients $c_{k/2}(\eta_*)$ can be written as

$$c_{\frac{k}{2}}(\eta_*) = \frac{\Gamma(\frac{k}{2}+\frac{1}{2})}{(2\pi)^{\frac{k}{2}+1}} f_{\frac{k}{2}}(\eta_*). \tag{5.4.7}$$

Substituting (5.4.7) into (5.4.6) and using the approximation $w/2 \simeq \frac{1}{2}\pi(t/2\pi)^{\frac{1}{2}}$, we have

$$\begin{aligned} &(-)^{n_*-1+k/2} (\pi i/4)^{k/2} (w/2)^{-k-1} c_{\frac{k}{2}}(\eta_*) \\ \sim\ &(-)^{n_*+m-1-M/2}\, i^{m-M/2} \frac{\sqrt{2}}{\pi} \frac{\Gamma(m-\frac{M}{2}+\frac{1}{2})}{(\pi t)^{m-M/2+\frac{1}{2}}} f_{m-\frac{M}{2}}(\eta_*). \end{aligned} \tag{5.4.8}$$

Define

$$G_{1,2}^{(j)}(k,\eta_*) = (-)^{\frac{j}{2}+n_*+p}\frac{\sqrt{2}}{\pi}\left\{\begin{array}{c} i \\ \\ 1 \end{array}\right\} f_{m-2p-k}(\eta_*),$$

where the subscripts 1 or 2 correspond to the sum involving i or 1, respectively. The leading behaviour of the coefficients $(-)^{n_*-1+k/2}(\pi i/4)^{k/2}(w/2)^{-k-1}c_{k/2}(\eta_*)$ when $k = 2m - M$ and $M = 4p,\ 4p+1,\ 4p+2,\ 4p+3$ with $p = 0,\ 1,\ 2,\ \cdots$ for large m is summarised in Table 5.5.

Table 5.5: The leading behaviour of the coefficients $(-)^{n_*-1+k/2}(\pi i/4)^{k/2}(w/2)^{-k-1}$ $c_{k/2}(\eta_*)$ when $k = 2m - M$, $M = 4p,\ 4p+1,\ 4p+2,\ 4p+3$ and $p = 0,\ 1,\ 2,\ \cdots$ for large m

	$M = 4p$	$M = 4p+1$
m odd	$\frac{\Gamma(m-2p+\frac{1}{2})}{(\pi t)^{m-2p+\frac{1}{2}}}G_1^{(m-1)}(0,\eta_*)$	0
m even	$\frac{\Gamma(m-2p+\frac{1}{2})}{(\pi t)^{m-2p+\frac{1}{2}}}G_2^{(m-2)}(0,\eta_*)$	0
	$M = 4p+2$	$M = 4p+3$
m odd	$\frac{\Gamma(m-2p-\frac{1}{2})}{(\pi t)^{m-2p-\frac{1}{2}}}G_2^{(m+1)}(1,\eta_*)$	0
m even	$\frac{\Gamma(m-2p-\frac{1}{2})}{(\pi t)^{m-2p-\frac{1}{2}}}G_1^{(m-2)}(1,\eta_*)$	0

To summarise Tables 5.4 and 5.5, the leading behaviour of the coefficients $(-)^k\ (w/2)^{-k}C_k^{(m*)}(w) + (-)^{n_*-1+k/2}(\pi i/4)^{k/2}(w/2)^{-k-1}c_{k/2}(\eta_*)$ is shown in Table 5.6. In this Table, $F_{1,2}^{(e*)}(k,w)$, $F_{1,2}^{(o*)}(k,w)$ are the same as $F_{1,2}^{(e)}(k,w)$, $F_{1,2}^{(o)}(k,w)$, except that $S_k(w)$ is replaced by the deleted sum $S_k^*(w)$.

The asymptotic behaviour of the functions $f_{m-2p}(\eta_*)$ and $f_{m-2p-1}(\eta_*)$ appearing in $G_1^{(m-1)}(0,\eta_*)$, $G_2^{(m-2)}(0,\eta_*)$, $G_1^{(m-2)}(1,\eta_*)$ and $G_2^{(m+1)}(1,\eta_*)$ for large m are studied in Chapter 4, Section 4.7. The results of Tables 4.18-4.19 for large m can be used for the functions $f_{m-2p}(\eta_*)$ and $f_{m-2p-1}(\eta_*)$.

5.5 Conclusions

In this Chapter, the structure of the late terms in (2.4.4) and (2.7.5) has been investigated. It is shown that the leading behaviour of these late terms possesses

Table 5.6: The leading behaviour of the coefficients $(-)^k(w/2)^{-k}C_k^{(m*)}(w)$ $+(-)^{n_*-1+k/2}$ $(\pi i/4)^{k/2}(w/2)^{-k-1}c_{k/2}(\eta_*)$ when $k = 2m - M$, $M = 4p$, $4p+1$, $4p+2$, $4p+3$ and $p = 0, 1, 2, \cdots$ for large m

m	$M = 4p$	$M = 4p+1$
odd	$\frac{\Gamma(m-2p)}{(\pi t)^{m-2p}}F_1^{(e*)}(2p,w) + \frac{\Gamma(m-2p+\frac{1}{2})}{(\pi t)^{m-2p+\frac{1}{2}}}G_1^{(m-1)}(0,\eta_*)$	$\frac{\Gamma(m-2p-1)}{(\pi t)^{m-2p-\frac{1}{2}}}F_2^{(o*)}(2p,w)$
even	$\frac{\Gamma(m-2p-1)}{(\pi t)^{m-2p}}F_2^{(e*)}(2p,w) + \frac{\Gamma(m-2p+\frac{1}{2})}{(\pi t)^{m-2p+\frac{1}{2}}}G_2^{(m-2)}(0,\eta_*)$	$\frac{\Gamma(m-2p-2)}{(\pi t)^{m-2p-\frac{1}{2}}}F_1^{(o*)}(2p,w)$
m	$M = 4p+2$	$M = 4p+3$
odd	$\frac{\Gamma(m-2p-2)}{(\pi t)^{m-2p-1}}F_2^{(e*)}(2p+1,w) + \frac{\Gamma(m-2p-\frac{1}{2})}{(\pi t)^{m-2p-\frac{1}{2}}}G_2^{(m+1)}(1,\eta_*)$	$\frac{\Gamma(m-2p-2)}{(\pi t)^{m-2p-\frac{3}{2}}}F_1^{(o*)}(2p+1,w)$
even	$\frac{\Gamma(m-2p-1)}{(\pi t)^{m-2p-1}}F_1^{(e*)}(2p+1,w) + \frac{\Gamma(m-2p-\frac{1}{2})}{(\pi t)^{m-2p-\frac{1}{2}}}G_1^{(m-2)}(1,\eta_*)$	$\frac{\Gamma(m-2p-3)}{(\pi t)^{m-2p-\frac{3}{2}}}F_2^{(o*)}(2p+1,w)$

the 'factorial divided by a power' dependence characteristic of an asymptotic series combined with a slowing varying multiplier function. A similar result for the Riemann-Siegel expansion has recently been found by Berry [Berry, 1994].

CHAPTER 6

CONCLUSIONS

The major contributions of this thesis are summarised as follows:

1. The representation of $Z(t)$ for large t is given in (2.4.4). This asymptotic approximation, which is derived from the expansion of $\zeta(s)$ in terms of incomplete gamma functions in (2.2.10), consists of the original Dirichlet series defining $\zeta(s)$ in $\mathrm{Re}\,(s) > 1$ smoothed by a modified complementary error function together with a correction term. The correction term involves an expansion in descending powers of $w/2 \simeq \frac{1}{2}\pi(t/2\pi)^{\frac{1}{2}}$ multiplied by a factor of $O(t^{-\frac{1}{4}})$. The coefficients in this expansion can be given explicitly to any order in terms of $\operatorname{cosec} w$ and its derivatives. In addition, it is found that there is an intimate connection between these coefficients and certain coefficients appearing in the uniform asymptotics of the incomplete gamma function. The numerical results demonstrate that the asymptotic expansion (2.4.4) is very accurate and is comparable with the Riemann-Siegel and Berry-Keating formulas. However, a disadvantage appears when we attempt to compute $Z(t)$ for values of t which make w lie close to an integer multiple of π. Although the coefficients $A_r(s)$ are not singular for such critical t values, there will be a loss of accuracy due to round-off error when computing with fixed decimal arithmetic. In order to deal with this problem, the modified expression for $Z(t)$ is given by (2.7.5) associated with the correction term which replaces (2.4.4) for t values in the neighbourhood of a discontinuity in N_t. From numerical computations it is

found that it is better to employ (2.7.5) when t is very close to a critical value.

2. Another asymptotic formula in (3.3.8) for the Riemann zeta function has been presented when s is in the critical strip. This asymptotic formula involves a smoothing of the main sum (the original Dirichlet series) by the confluent hypergeometric function $M(a, b, -(n/K)^{2p})$, where a and b are free to be chosen. The choice $b = a + 1$ in (3.3.8) leads to the same asymptotic formula as in (3.2.11) which involves the original Dirichlet series smoothed by the incomplete gamma function. Others obvious choices are $a = 1$ and $a = \frac{1}{2}$, which lead to the particularly simple smoothing by the exponential factor and complementary error function. In order for the finite sum to possess an asymptotic character, the formula (3.3.8) requires the choice $K \sim t/2\pi$, (3.3.8) consequently has the character of a Gram-type formula, rather than that of the Riemann-Siegel-type formulas discussed in Chapter 2.

   We also considered the bound on the remainder term R_M in the formula (3.2.11). It was found numerically to be no longer realistic once $M \geq \text{int}\,[t/2]$. Numerical results indicate enormous accuracy as the truncation index M increases, until M reaches to the optimal truncation point.

   The Berry-Keating formula, for which the leading term is given by (1.3.7), also involves the main sum smoothed by a complementary error function, whose argument depends logarithmically on n/N_t. This difference in the n-dependence of the error function argument, however, results in the main sum cutting off after roughly $n^* = N_t \simeq (t/2\pi)^{\frac{1}{2}}$ terms, so that their formula is of the more powerful Riemann-Siegel type. The expansion (3.2.11) has been shown to produce a Gram-type expansion when $a = O(1)$, but a Riemann-Siegel-type expansion when $a = s/2$ in (3.2.11). It would be of interest to explore the domain of a values corresponding to the transition between these two categories of expansion.

3. The behaviour of the contribution to (2.7.5) from the coefficients $c_r(\eta)$ which appear in the uniform asymptotic expansion of the incomplete gamma function is considered. We found that $c_r(\eta)$ behave like $\{\Gamma(r + \frac{1}{2})/(2\pi)^{r+1}\}f_r(\eta)$ for

large r. In addition, we developed the asymptotic form of $f_r(\eta)$ for large r, this asymptotic form involves the hypergeometric function. Extensive numerical results and Fig. 4.5 show that the value of $f_r(\eta)$ is large and becomes larger as r increases for fixed η when η is in the domains $|1 - e^{\mp \frac{\pi}{2} i}\eta^2/4\pi| < 1$. Moreover, the function $f_r(\eta)$ is slowly decaying for real $\eta > 0$ and has an oscillatory domain for $\eta < 0$.

4. We examined the large-m behaviour of the late terms in (2.4.4) and (2.7.5). It has been shown that the behaviour of the late terms in (2.4.4) possesses the familiar 'factorial divided by a power' multiplied by a factor, which consists of slowly oscillatory functions involving $S_k(w)$. The first part of the late terms in (2.7.5) possessed the same behaviour as that given in (2.4.4), except that $S_k(w)$ is replaced by the deleted sums $S_k^*(w)$. The second part of the late terms in (2.7.5) possessed the 'factorial divided by a power' multiplied by the function $f_r(\eta_*)$, where $f_r(\eta_*)$ is a slowly varying function of η close to the real η-axis.

The contribution of this work to the advancement of knowledge is a deeper understanding of the behaviour of $\zeta(s)$ high up on the critical line. The zeta function is such an important mathematical object that any new way of understanding it is of great interest. The expansions for $\zeta(s)$ which are considered in this project are very recent, having been developed over the past two-year period, while the standard formulas (Riemann-Siegel and Gram formulas) have been known for the order of a century.

Appendix A

A.1 The Proof of (2.4.11)

The definition of $S_k(w)$ given by (2.4.10), which is

$$S_k(w) = 2^{k-1} \sum_{n=-\infty}^{\infty} (-)^n \frac{w^k}{(w^2 - \pi^2 n^2)^k}, \qquad k = 1, 2, \cdots. \tag{A.1.1}$$

The derivative of $S_k(w)$ with respect to w in (A.1.1) is

$$\frac{dS_k(w)}{dw} = \sum_{n=-\infty}^{\infty} (-)^n \{2^{k-1} \frac{k}{w} \frac{w^k}{(w^2 - \pi^2 n^2)^k} - 2^k k \frac{w^{k+1}}{(w^2 - \pi^2 n^2)^{k+1}}\}. \tag{A.1.2}$$

Then, the recursive relation satisfied by $S_k(w)$ is:

$$S_{k+1}(w) = \frac{1}{w} S_k(w) - \frac{1}{k} \frac{dS_k(w)}{dw}, \qquad k = 1, 2, \cdots$$

and

$$S_1(w) = \operatorname{cosec} w.$$

A.2 Estimation of the Imaginary Part of η_*

The relation between η_* and λ_* was given in (2.3.3) as

$$\frac{1}{2}\eta_*^2 = \lambda_* - 1 - \log\lambda_*, \tag{A.2.1}$$

where $\lambda_* = \pi^2 n_*^2 / w^2$, $n_* = \operatorname{int}[(t/2\pi)^{1/2}]$ and $w^2 = \pi t/2 - \pi i/4$.

For small $|\mu|$, we have

$$\eta_* \simeq \mu_* = \lambda_* - 1. \tag{A.2.2}$$

Let

$$n_* = \sqrt{t/2\pi} - \epsilon(t), \qquad T = \frac{t}{2\pi},$$

where $\epsilon(t)$ is a function of t. From (A.2.2), we find

$$\begin{aligned}\eta_* &\simeq \frac{\pi^2 n_*^2}{w^2} - 1 \\ &= (1 + \frac{i}{2t - i})(1 - \epsilon(t)T^{-\frac{1}{2}})^2 - 1 \\ &= -2\epsilon(t)T^{-\frac{1}{2}} + \epsilon^2 T^{-1} + \frac{i/2t}{1 - i/2t}(1 - \epsilon(t)T^{-\frac{1}{2}})^2. \end{aligned} \quad (A.2.3)$$

Hence

$$\mathrm{Re}\,\eta_* \simeq -2\epsilon(t)\sqrt{2\pi/t} = O(t^{-\frac{1}{2}}), \quad \mathrm{Im}\,\eta^* \simeq 1/2t = O(t^{-1}).$$

The imaginary part of η_* is approximately constant in the neighbourhood of a critical value (given by $\epsilon(t) = 0$) and is small like $O(t^{-1})$. The real part of η_* scales like $t^{-\frac{1}{2}}$ and varies linearly with the function $\epsilon(t)$.

A.3 The Computation of the Coefficients A_r

The following asymptotic formula has been given in [Paris, 1994]

$$\begin{aligned} Z(t) &= 2\sum_{n=1}^{N} \frac{\cos(\vartheta(t) - t\log n)}{\sqrt{n}} + 2\mathrm{Re}\{e^{i\vartheta(t)} E_m(t;N) \\ &+ \frac{\pi^{\frac{1}{4}} e^{\frac{1}{2} i\phi s}}{|\Gamma(\frac{s}{2})|}(\sum_{r=0}^{m-1} (\frac{s}{2})^{-r-1} A_r \frac{\Gamma^*(\frac{s}{2})}{\Gamma^*_{m-r}(\frac{s}{2})} - \frac{1}{s} + (\frac{s}{2})^{-m-1} R_m)\}, \end{aligned} \quad (A.3.1)$$

where the coefficients A_r and the remainder R_m are defined by

$$\begin{aligned} C_r(\lambda) &= (\frac{\lambda}{\lambda - 1}\frac{d}{d\lambda})^r \frac{1}{(\lambda - 1)} \\ A_r &= \sum_{n=1}^{\infty} e^{-\pi n^2 e^{i\phi}} C_r(\lambda), \quad r = 0, 1, 2, \cdots, \\ R_m &= \Gamma^*(\frac{s}{2}) \sum_{n=1}^{\infty} e^{-\pi n^2 e^{i\phi}} G_m(\frac{s}{2}, \eta), \end{aligned} \quad (A.3.2)$$

and the function $\Gamma^*(z)$ denotes the "scaled" gamma function.

The factor $\pi^{\frac{1}{4}} e^{\frac{1}{2} i\phi s}/|\Gamma(\frac{s}{2})|$ contains the exponential factor $\exp(\frac{1}{4}\pi - \frac{1}{2}\phi)t$. Thus if $\phi < \frac{\pi}{2}$, the factor $\pi^{\frac{1}{4}} e^{\frac{1}{2} i\phi s}/|\Gamma(\frac{s}{2})|$ becomes large for large t. In order to avoid this numerically large term, we choose $\phi = \frac{\pi}{2}$. This leads to the coefficients A_r and the remainder R_m being given by

$$A_r = \sum_{n=1}^{\infty} (-)^n C_r(\lambda) \quad (A.3.3)$$

$$R_m = \Gamma^*(\frac{s}{2}) \sum_{n=1}^{\infty} (-)^n G_m(\frac{s}{2}, \eta) \quad (A.3.4)$$

where $w^2 = \pi(t - \frac{i}{2})/2$ and $\lambda = \pi^2 n^2/w^2$.

To simplify the presentation, let

$$\bar{S}_j(w) = \sum_{n=1}^{\infty} (-)^n \frac{w^{2j}}{(\pi^2 n^2 - w^2)^j}. \tag{A.3.5}$$

Substitution of (A.3.5) into (A.3.3) then gives

$$A_0 = \sum_{n=1}^{\infty} (-)^n C_0(\lambda) = \sum_{n=1}^{\infty} (-)^n \frac{1}{\lambda - 1} = w^2 \sum_{n=1}^{\infty} (-)^n \frac{1}{\pi^2 n^2 - w^2} = \bar{S}_1(w), \tag{A.3.6}$$

and

$$\begin{aligned} A_1 &= \sum_{n=1}^{\infty} (-)^n C_1(\lambda) = \sum_{n=1}^{\infty} (-)^n (\frac{\lambda}{\lambda - 1} \frac{d}{d\lambda})(\frac{1}{\lambda - 1}) = \sum_{n=1}^{\infty} (-)^n \frac{-\lambda}{(\lambda - 1)^3} \\ &= -w^4 \sum_{n=1}^{\infty} (-)^n \frac{1}{(\pi^2 n^2 - w^2)^2} - w^6 \sum_{n=1}^{\infty} (-)^n \frac{1}{(\pi^2 n^2 - w^2)^3} \\ &= -\bar{S}_2(w) - \bar{S}_3(w). \end{aligned} \tag{A.3.7}$$

Using the same procedure as the above, the calculation of A_r for $r = 2, 3, \cdots$ is

$$\begin{aligned} A_2 &= 2\bar{S}_3(w) + 5\bar{S}_4(w) + 3\bar{S}_5(w), \\ A_3 &= -6\bar{S}_4(w) - 26\bar{S}_5(w) - 35\bar{S}_6(w) - 15\bar{S}_7(w), \\ A_4 &= 24\bar{S}_5(w) + 154\bar{S}_6(w) + 340\bar{S}_7(w) + 315\bar{S}_8(w) + 105\bar{S}_9(w), \qquad \text{(A.3.8)} \\ A_5 &= -120\bar{S}_6(w) - 1044\bar{S}_7(w) - 3304\bar{S}_8(w) - 4012\bar{S}_9(w) - 3466\bar{S}_{10}(w) - 617\bar{S}_{11}(w), \\ &\vdots \qquad \text{(A.3.9)} \end{aligned}$$

The above equations can be written in the form

$$A_r = \sum_{j=r+1}^{2r+1} C_j \bar{S}_j(w), \qquad r = 0, 1, 2 \cdots,$$

where the coefficients C_j are shown in Table A.1.

The recursive relation for the sum $\bar{S}_j(w)$ is

$$\begin{aligned} \bar{S}_{j+1}(w) &= \frac{w}{2j} \frac{d\bar{S}_j(w)}{dw} - \bar{S}_j(w), \qquad k = 1, 2, \cdots, \\ \bar{S}_1(w) &= \frac{1}{2} - \frac{w}{2} \operatorname{cosec} w. \end{aligned} \tag{A.3.10}$$

The proof of the recursive relation (A.3.10) is the same as the proof of (2.4.11) by differentiation of $\bar{S}_j$ in (A.3.5) with respect to w [see A.1].

Table A.1: The coefficients C_j of A_r for $0 \leq r \leq 5$

$r \setminus C_j$	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}
0	1										
1		−1	−1								
2			2	5	3						
3				−6	−26	−35	−15				
4					24	154	340	315	105		
5						−120	−1044	−3304	−4012	−3466	−617

Paris (1994) gave the first four coefficients A_r in the form

$$\begin{aligned}
A_0(w) &= \sum_{n=1}^{\infty} (-)^n \Psi_n, \\
A_1(w) &= -\frac{1}{4}\Theta \sum_{n=1}^{\infty} (-)^n \Psi_n^2, \\
A_2(w) &= \frac{1}{16}\Theta(\Theta - \frac{2}{3}) \sum_{n=1}^{\infty} (-)^n \Psi_n^3, \\
A_3(w) &= -\frac{1}{64}\Theta^2(\Theta - 2) \sum_{n=1}^{\infty} (-)^n \Psi_n^4,
\end{aligned} \tag{A.3.11}$$

where $\Psi_n = w^2/(\pi^2 n^2 - w^2)$ and Θ denotes the operator $w\, d/dw$.

These coefficients are exactly the same as $A_r = \sum_{j=r+1}^{2r+1} C_j \bar{S}_j(w)$. We only demonstrate the correspondence of $A_1(w)$ here. This proceeds as follows:

$$\begin{aligned}
A_1(w) &= -\frac{1}{4}\Theta \sum_{n=1}^{\infty} (-)^n \Psi_n^2 = -\frac{1}{4} w \frac{d}{dw} \sum_{n=1}^{\infty} (-)^n \frac{w^4}{(\pi^2 n^2 - w^2)^2} \\
&= -\frac{1}{4} w \sum_{n=1}^{\infty} (-)^n \frac{4w^3}{(\pi^2 n^2 - w^2)^2} - \frac{1}{4} w \sum_{n=1}^{\infty} (-)^n \frac{4w^5}{(\pi^2 n^2 - w^2)^3} \\
&= -\bar{S}_2(w) - \bar{S}_3(w).
\end{aligned} \tag{A.3.12}$$

Appendix B

B.1 Derivation of the Mellin-Barnes Representation of the Incomplete Gamma Function $\Gamma(a, z)$

The formula for the Whittaker function $W_{k,\mu}(z)$ given in [Gradshteyn & Ryzhik, 1965 p. 659] will be used to develop the Mellin-Barnes integral representation of the incomplete gamma function. This formula is given by

$$\int_{-i\infty}^{i\infty} \frac{\Gamma(\lambda+\mu-s+\frac{1}{2})\Gamma(\lambda-\mu-s+\frac{1}{2})}{\Gamma(\lambda-s-k+1)} z^s ds = 2\pi i z^{\lambda} e^{-\frac{z}{2}} W_{k,\mu}(z), \tag{B.1.1}$$

where

$$\operatorname{Re}\lambda > |\operatorname{Re}\mu| - \frac{1}{2}, \qquad |\arg z| < \frac{\pi}{2}, \tag{B.1.2}$$

and the definition of the Whittaker function is

$$W_{k,\mu}(z) = \frac{z^{\mu+\frac{1}{2}} e^{-\frac{z}{2}}}{\Gamma(\mu-k+\frac{1}{2})} \int_0^{\infty} e^{-zt} t^{\mu-k-\frac{1}{2}} (1+t)^{\mu+k-\frac{1}{2}} dt. \tag{B.1.3}$$

Let

$$k = \frac{a-1}{2}, \qquad \mu = \frac{a}{2}, \qquad \lambda = \frac{a-1}{2} + c. \tag{B.1.4}$$

where a and c are selected as positive values. Thus the parameters k, μ and λ satisfy the condition (B.1.2).

Substitution of (B.1.4) and (B.1.3) into formula (B.1.1) leads to

$$\begin{aligned}
\int_{-i\infty}^{i\infty} \frac{\Gamma(c+a-s)\Gamma(c-s)}{\Gamma(c+1-s)} z^s ds &= 2\pi i z^c z^{\frac{a-1}{2}} e^{-\frac{z}{2}} W_{\frac{a-1}{2},\frac{a}{2}}(z) \\
&= 2\pi i z^c e^{-z} \int_0^{\infty} e^{-w} (z+w)^{a-1} dw \\
&= 2\pi i z^c \Gamma(a, z).
\end{aligned} \tag{B.1.5}$$

Changing the variable of integration on the left-hand side of (B.1.5), the Mellin integral of the incomplete gamma function $\Gamma(a, z)$ can be represented as

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(a+s)}{s}z^{-s}ds = \Gamma(a, z), \tag{B.1.6}$$

where

$$c > 0, \quad a \geq 0, \quad |\arg z| < \frac{\pi}{2}.$$

If a is negative and $|\arg z| < \frac{\pi}{2}$, the condition on c can be replaced by $c+\mathrm{Re}(a) > 0$, or by a suitable indentation of the path of integration.

B.2 The Necessary Condition for $G(y)$ to be a Monotonically Decreasing Function

For positive c, the function $\bar{H}(y)$ is defined by

$$\bar{H}(y) = \left|\frac{\Gamma(c+iy)}{\Gamma(\frac{1}{2}-c+iy)}\right| \tag{B.2.1}$$

and the scaled gamma function ratio is

$$G(y) = y^{\frac{1}{2}-2c}\bar{H}(y). \tag{B.2.2}$$

We will prove that $c > \frac{1}{2}$ is a necessary condition for $G(y)$ to be a monotonically decreasing function.

The asymptotic expansion of $\Gamma(z+a)/\Gamma(z+b)$ for fixed a, b and large z is [Olver, 1974, p. 119]

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim \sum_{n=0}^{\infty} z^{a-b}\frac{G_s}{z^s}, \tag{B.2.3}$$

where the first few coefficients G_s are given by

$$\begin{aligned}
G_0 &= 1, \\
G_1 &= \frac{1}{2}(a-b)(a+b-1), \\
G_2 &= \frac{1}{24}(a-b)(a-b-1)\{3(a+b)^2 - 7a - 5b + 2\}.
\end{aligned}$$

Applying (B.2.3) to (B.2.2), we find

$$\begin{aligned} G(y) &= \left|1 + \frac{G_1}{iy} + \frac{G_2}{(iy)^2} + O(y^{-3})\right| \\ &= \{(1 - \frac{G_2}{y^2} + O(y^{-3}))^2 + (\frac{G_1}{y})^2\}^{\frac{1}{2}} \\ &= 1 + \frac{G_1^2 - 2G_2}{2y^2} + O(y^{-4}). \quad y \to +\infty. \end{aligned} \tag{B.2.4}$$

When $a = c$, $b = \frac{1}{2} - c$, the coefficient of y^{-2} in $G(y)$ is

$$A = \frac{1}{2}\{G_1^2 - 2G_2^2\} = \frac{1}{3}c(c - \frac{1}{4})(c - \frac{1}{2}).$$

Clearly, $c > \frac{1}{2}$ is a necessary (but not sufficient) condition for $G(y)$ to be a monotonically decreasing function of y for $y > 0$.

B.3 Monotonic Functions

B.3.1 The Monotonically Increasing Function $H(y)$

It is convenient to introduce the definition

$$H(y) = \left|\frac{\Gamma(\alpha + iy)}{\Gamma(\beta + iy)}\right|. \tag{B.3.1}$$

Next we will prove that the function $H(y)$ is a monotonically increasing function of $y > 0$ when $\alpha > |\beta|$.

The gamma function has the property [Abramowitz & Stegun, 1968, p. 256]

$$|\Gamma(\alpha + iy)| = |\Gamma(\alpha)| \prod_{n=0}^{\infty} \{1 + \frac{y^2}{(\alpha + n)^2}\}^{-\frac{1}{2}}. \tag{B.3.2}$$

From (B.3.2), we find

$$H(y) = \frac{|\Gamma(\alpha)|}{|\Gamma(\beta)|} \prod_{n=0}^{\infty} \left\{ \frac{1 + \frac{y^2}{(\beta+n)^2}}{1 + \frac{y^2}{(\alpha+n)^2}} \right\}^{\frac{1}{2}}. \tag{B.3.3}$$

Differentiating the function $\log H(y)$ respect to y in (B.3.3) then leads to

$$\frac{dH(y)}{dy} = \sum_{n=0}^{\infty} \frac{y(\alpha - \beta)(2n + \alpha + \beta)}{((n + \beta)^2 + y^2)((n + \alpha)^2 + y^2)} H(y). \tag{B.3.4}$$

In view of (B.3.4), we obtain $dH(y)/dy > 0$ if α and β satisfy the condition $\alpha > |\beta|$ for $y > 0$. Therefore the function $H(y)$ is a monotonically increasing function of y when $\alpha > |\beta|$ for $y > 0$.

For the special cases $\alpha = c$, $\beta = \frac{1}{2} - c$, $H(y)$ becomes $\bar{H}(y)$ and $\bar{H}(y)$ is a monotonically increasing function of y for $y > 0$ when $c > \frac{1}{4}$.

If $\alpha = \frac{1-\sigma}{2} + p\delta$, $\beta = \frac{\sigma}{2} - p\delta$, where $\delta = M + a - \frac{1}{2}$, $0 < \sigma < 1$, the function $F(y)$ defined in (3.6.3) is

$$F(y) = \left| \frac{\Gamma(\frac{1-\sigma}{2} + p\delta + iy)}{\Gamma(\frac{\sigma}{2} - p\delta + iy)} \right|. \tag{B.3.5}$$

Then $F(y)$ is a monotonically increasing function of $y > 0$ when $\frac{1}{2} - \sigma + 2p\delta > 0$

B.3.2 The Monotonically Decreasing Function $G(y)$

We rewrite the function $G(y)$ defined in (B.2.2)

$$G(y) = y^{\frac{1}{2}-2c}\bar{H}(y), \tag{B.3.6}$$

where $\bar{H}(y)$ is defined in (B.2.1). If we take $c = \frac{1-\sigma}{2} + p\delta$, where $\delta = M + a - \frac{1}{2}$, then $G(y)$ becomes $G_M(y)$, which is defined in (3.6.8).

We will show that $G(y)$ is a monotonically decreasing function for the special values $c = \frac{m}{2} + \frac{1}{4}$, $m = 1, 2, \cdots$. Differentiating the function $G(y)$ in (B.3.6) with respect to y, we obtain

$$\frac{dG(y)}{dy} = \{\frac{\frac{1}{2} - 2c}{y} + y\sum_{n=0}^{\infty}[\frac{1}{(n + \frac{1}{2} - c)^2 + y^2} - \frac{1}{(n + c)^2 + y^2}]\}G(y). \tag{B.3.7}$$

We introduce the inequality

$$\sum_{n=0}^{\infty}\frac{1}{(n + \frac{1}{2} - c)^2 + y^2} - \sum_{n=0}^{\infty}\frac{1}{(n + c)^2 + y^2} < \frac{2c - \frac{1}{2}}{y^2}. \tag{B.3.8}$$

In view of (B.3.7), we observe that $G(y)/dy \leq 0$ is equivalent to the inequality (B.3.8). Let the function $F_c(y)$ denote the left-hand side of the inequality (B.3.8). We only deal with the special case when $c = \frac{m}{2} + \frac{1}{4}$, $m = 1, 2, \cdots$. m cannot equal zero. If m equals zero, c must equal $\frac{1}{4}$, and it then follows that $F_c(y) = 0$. In order to satisfy the inequality (B.3.8), c must be greater than $\frac{1}{4}$; this is a contradiction.

With $c = \frac{m}{2} + \frac{1}{4}$, $m = 1, 2, \cdots$. we can write

$$\begin{aligned} F_c(y) &= \sum_{n=0}^{\infty} \frac{1}{(n + \frac{1}{4} - \frac{m}{2})^2 + y^2} - \sum_{n=0}^{\infty} \frac{1}{(n + \frac{1}{4} + \frac{m}{2})^2 + y^2} \\ &= \sum_{n=0}^{m-1} \frac{1}{(n + \frac{1}{4} - \frac{m}{2})^2 + y^2} \\ &< \frac{m}{y^2}. \end{aligned} \tag{B.3.9}$$

This implies that the inequality (B.3.8) is valid when $c = \frac{m}{2} + \frac{1}{4}$, $m = 1, 2, \cdots$. Thus the function $G(y)$ is a monotonically decreasing function of $y > 0$ when $c = \frac{m}{2} + \frac{1}{4}$, $m = 1, 2, \cdots$.

Next we will prove that $G(y)$ is a monotonically decreasing function of y for $y > 0$ when $c \geq 1$.

The duplication formula is

$$\Gamma(2z) = (2\pi)^{-\frac{1}{2}} 2^{2z - \frac{1}{2}} \Gamma(z) \Gamma(z + \frac{1}{2}).$$

Putting $z = c + iy$ in the above formula, we have

$$|\Gamma(c + iy)| = (2\pi)^{\frac{1}{2}} 2^{\frac{1}{2} - 2c} \left| \frac{\Gamma(2c + 2iy)}{\Gamma(c + \frac{1}{2} + iy)} \right|. \tag{B.3.10}$$

Substitution of (B.3.10) into (B.3.6) and using the formula $\Gamma(z)\Gamma(1-z) = \pi \csc \pi z$, we have

$$\begin{aligned} G(y) &= (2\pi)^{\frac{1}{2}} 2^{\frac{1}{2} - 2c} y^{\frac{1}{2} - 2c} \frac{|\Gamma(2c + 2iy)|}{|\Gamma(\frac{1}{2} + c + iy)||\Gamma(\frac{1}{2} - c + iy)|} \\ &= \sqrt{\frac{2}{\pi}} (2y)^{\frac{1}{2} - 2c} |\Gamma(2c + 2iy)| |\sin \pi(\frac{1}{2} - c + iy)|. \end{aligned} \tag{B.3.11}$$

Considering the formula (B.3.2) and $|\sin(x + iy)| = (\sin^2 x + \sinh^2 y)^{\frac{1}{2}}$, we find

$$G(y) = \sqrt{\frac{2}{\pi}} (2y)^{\frac{1}{2} - 2c} \Gamma(2c) \prod_{n=0}^{\infty} \{1 + \frac{4y^2}{(n + 2c)^2}\}^{-\frac{1}{2}} \{\sin^2 \pi(c + \frac{1}{2}) + \sinh^2 \pi y\}^{\frac{1}{2}}. \tag{B.3.12}$$

Differentiating the function $\log G(y)$ with respect to y in (B.3.12) then leads to

$$\frac{G'(y)}{2G(y)} = \{\frac{\frac{1}{2} - 2c}{2y} - 2y \sum_{n=0}^{\infty} \frac{1}{(n + 2c)^2 + 4y^2} + \frac{\pi}{2} \coth \pi y [1 - \frac{\sin^2 \pi(c + \frac{1}{2})}{\sin^2 \pi(c + \frac{1}{2}) + \sinh^2 \pi y}]\}. \tag{B.3.13}$$

We can write (B.3.13) as the inequality

$$\frac{G'(y)}{2G(y)} \leq \{\frac{\frac{1}{2}-2c}{2y} - 2y\sum_{n=0}^{\infty}\frac{1}{(n+2c)^2+4y^2} + \frac{\pi}{2}\coth \pi y\}. \tag{B.3.14}$$

Making use of the integral

$$\frac{2y}{(n+2c)^2+4y^2} = \int_0^{\infty} \sin 2yt\, e^{-(n+2c)t}dt$$

to express the sum appearing in (B.3.14) in the form

$$\sum_{n=0}^{\infty}\frac{2y}{(n+2c)^2+4y^2} = \int_0^{\infty} \sin 2yt\, e^{-2ct}\frac{dt}{1-e^{-t}} \tag{B.3.15}$$

and the identity [Gradshteyn & Ryzhik, 1965, p. 481]

$$\frac{\pi}{2}\coth \pi y = \frac{1}{2y} + 2\int_0^{\infty}\frac{2yt}{e^{2t}-1}dt, \tag{B.3.16}$$

we obtain

$$\frac{G'(y)}{2G(y)} \leq \frac{\frac{3}{2}-2c}{2y} - \int_0^{\infty} \sin 2yt\, (\frac{e^{-2ct}}{1-e^{-t}} - \frac{2}{e^{2t}-1})dt. \tag{B.3.17}$$

Let

$$H(t) = \frac{e^{-2ct}}{1-e^{-t}} - \frac{2}{e^{2t}-1},$$

so that

$$\frac{G'(y)}{2G(y)} \leq \frac{\frac{3}{2}-2c}{2y} - \frac{H(0)}{2y} + \frac{1}{4y^2}\int_0^{\infty} \sin 2yt\, H''(t)dt.$$

Using $H(0) = \frac{3}{2} - 2c$ in the above inequality, we have

$$\frac{G'(y)}{2G(y)} \leq \frac{1}{4y^2}\int_0^{\infty} \sin 2yt\, H''(t)dt. \tag{B.3.18}$$

In fact, $H''(t)$ is a negative increasing function of t for $t > 0$ when $c \geq 1$. Fig. B.1 shows this statement is true. Thus, the right-hand side of (B.3.18) is negative and consequently $G(y)$ is a monotonically decreasing function of y for $y > 0$ when $c \geq 1$. Numerical results demonstrate that the function $G(y)$ is a monotonically decreasing function of y for $y > 0$ when $c \geq c_0 = 0.63605$.

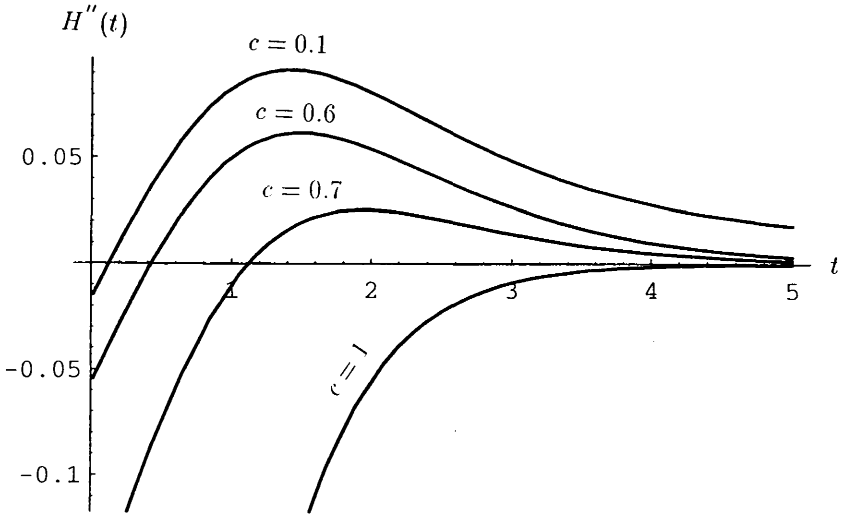


Figure B.1: $H''(t)$ as a function of t for different c

B.4 The Function $f(\xi)$ is a Continuous Decreasing Function of ξ.

We will prove that the function $f(\xi)$ defined in (3.8.4) is a continuous decreasing function of ξ.

Let $\mu(\xi) = \tau(\xi) - 1$ so that (3.8.2) becomes

$$\mu - \log(\mu + 1) = \frac{1}{2}\xi^2. \tag{B.4.1}$$

Differentiating $f(\xi)$ with respect to ξ, we obtain

$$\begin{aligned} f'(\xi) &= \frac{\mu(\xi) - \mu'(\xi)\xi}{\mu^2(\xi)} \\ &= \frac{\mu(\xi) - \frac{\mu(\xi)+1}{\mu(\xi)}\xi^2}{\mu^2(\xi)} \\ &= \frac{-\mu^2(\xi) - 2\mu(\xi) + 2(\mu(\xi)+1)\log(1+\mu(\xi))}{\mu^3(\xi)}. \end{aligned} \tag{B.4.2}$$

Let

$$g(\mu) = -\mu^2 - 2\mu + 2(\mu + 1)\log(1 + \mu).$$

Differentiating $g(\mu)$ with respect to μ, we find

$$g^{'}(\mu) = -2(\mu - \log(1+\mu)), \quad g(0) = 0$$

and

$$g^{''}(\mu) = \frac{-2\mu}{1+\mu}, \quad g^{'}(0) = 0.$$

For $\mu > 0$, $g^{''}(\mu) < 0$. Thus $g^{'}(\mu)$ is a decreasing function and $g^{'}(\mu) < 0$ for $\mu > 0$. Similarly, $g(\mu)$ is a decreasing function and $g(\mu) < 0$ for $\mu > 0$. Since $\mu > 0$ is equivalent to $\xi > 0$, we find $f^{'}(\xi) < 0$ for $\xi > 0$. From (B.4.2), we also can prove that $f^{'}(\xi) < 0$ for $\xi < 0$ using a similar argument. Thus, the function $f(\xi)$ is a decreasing function. It is noted that $\xi = 0$ is a removable singularity.

From (B.4.1), ξ regarded as a function of μ $(\mu \to 0)$, has two branches

$$\xi = \pm\mu(1 - \frac{2}{3}\mu + \frac{2}{4}\mu^2 - \cdots)^{\frac{1}{2}}.$$

Taking the upper sign, we have [Copson, 1965, p. 55]

$$\xi = \mu(1 - \frac{1}{3}\mu + \cdots). \tag{B.4.3}$$

From (3.8.4) and (B.4.3), we find $f(\xi) \sim 1$, when $\xi \sim 0$.

We define

$$f(\xi) = \begin{cases} \frac{\xi}{\tau(\xi)-1} = \frac{\xi}{\mu(\xi)} & \mu(\xi) \neq 0 \\ 1 & \mu(\xi) = 0. \end{cases} \tag{B.4.4}$$

The function $f(\xi)$ is a continuous decreasing function of ξ.

Appendix C

C.1 The Proof of (4.2.21)

For convenience, we rewrite (4.2.21) here

$$\sum_{k=0}^{\infty} \gamma_{r-p}(k+2)\cdots(k+2p)(k+2p+2)\alpha_{k+2p+2}\eta^k$$
$$= \gamma_{r-p}\{\gamma_0 c_p(\eta) - \gamma_1 c_{p-1}(\eta) + \cdots + (-)^p \gamma_p c_0(\eta)\}, \tag{C.1.1}$$

where $p = 0, 1, 2, \cdots$ and γ_r are the Stirling coefficients and the coefficients $c_r(\eta)$ are defined in (4.2.1). A recursive induction method is used to prove (C.1.1). Assuming that the equation (C.1.1) is true for the $(p+1)$th term in (4.2.8), we will prove that the equation (C.1.1) is true for the $(p+2)$th term in (4.2.8); that is

$$\sum_{k=0}^{\infty} \gamma_{r-(p+1)}(k+2)(k+4)\cdots(k+2p+4)\alpha_{k+2p+4}\eta^k$$
$$= \gamma_{r-(p+1)}\{\gamma_0 c_{p+1}(\eta) - \gamma_1 c_p(\eta) + \cdots + (-)^{p+1}\gamma_{p+1} c_0(\eta)\}. \tag{C.1.2}$$

Replacing $k+2$ by k on the left side of (C.1.2), we have

$$\sum_{k=0}^{\infty} \gamma_{r-(p+1)}(k+2)(k+4)\cdots(k+2p+4)\alpha_{k+2p+4}\eta^k$$
$$= \gamma_{r-(p+1)} \sum_{k=2}^{\infty} k(k+2)\cdots(k+2p+2)\alpha_{k+2p+2}\eta^{k-2}$$
$$= \gamma_{r-(p+1)} \frac{1}{\eta}\frac{d}{d\eta}\{\sum_{k=0}^{\infty}(k+2)(k+4)\cdots(k+2p+2)\alpha_{k+2p+2}\eta^k$$
$$-2\cdot 4\cdots(2p+2)\alpha_{2p+2} - 3\cdot 5\cdots(2p+2+1)\alpha_{2p+2+1}\eta\}. \tag{C.1.3}$$

Substituting (C.1.1) into (C.1.3) and using the relation between α_r and γ_r given in (4.2.7), yields

$$\sum_{k=0}^{\infty} \gamma_{r-(p+1)}(k+2)(k+4)\cdots(k+2p+4)\alpha_{k+2p+4}\eta^k$$

$$= \gamma_{r-(p+1)} \frac{d}{\eta d\eta}\{\gamma_0 c_p(\eta) + \cdots + (-)^p \gamma_p c_0(\eta)$$
$$-2\cdots(2p+2)\alpha_{2p+2} - (-)^{p+1}\gamma_{p+1}\eta\}$$
$$= 2\gamma_{r-(p+1)}\{\gamma_0 \frac{d}{d\eta^2}c_p(\eta) - \gamma_1 \frac{d}{d\eta^2}c_{p-1}(\eta)\cdots + (-)^p\gamma_p \frac{d}{d\eta^2}c_0(\eta)$$
$$-\frac{1}{2}(-)^{p+1}\gamma_{p+1}\frac{1}{\eta}\}. \quad \text{(C.1.4)}$$

Using (4.2.4) and the identity satisfied by the Stirling coefficients

$$\gamma_0\gamma_{p+1} - \gamma_1\gamma_p + \cdots + (-)^p\gamma_p\gamma_1 + (-)^{p+1}\gamma_{p+1}\gamma_0 \equiv 0, \quad p = 0,\ 1,\ \cdots, \quad \text{(C.1.5)}$$

on the right side of (C.1.4), we obtain the result (C.1.1) with p replaced by $p+1$. The proof of (C.1.5) will be given in C.2. We have proved that (C.1.1) is true in Section 4.2 when $p = 0,\ 1$. Hence, by induction, (C.1.1) is true for $p = 0,\ 1, \cdots,\ r$.

C.2 Proof of the Identity (C.1.5)

The well-known asymptotic expansions of the scaled gamma function for $z \to \infty$ are [Temme, 1979]

$$\Gamma^*(z) \sim \sum_{r=0}^{\infty}(-)^r\gamma_r z^{-r}, \quad \text{(C.2.1)}$$

$$\frac{1}{\Gamma^*(z)} \sim \sum_{r=0}^{\infty}\gamma_r z^{-r}. \quad \text{(C.2.2)}$$

The first few coefficients are given in Section 2.3. Multiplication of (C.2.1) and (C.2.2), yields

$$1 \sim \gamma_0^2 + (\gamma_0\gamma_1 - \gamma_1\gamma_0)z^{-1} + (\gamma_0\gamma_2 - \gamma_1^2 + \gamma_2\gamma_1)z^{-2} + \cdots$$
$$+(\gamma_0\gamma_{p+1} - \gamma_1\gamma_p + \cdots + (-)^p\gamma_p\gamma_1 + (-)^{p+1}\gamma_{p+1}\gamma_0)z^{-p-1} + \cdots . \quad \text{(C.2.3)}$$

Since the coefficients of the terms z^{-p-1} $(p \geq 0)$ must vanish, we have the identity

$$\gamma_0\gamma_{p+1} - \gamma_1\gamma_p + \cdots + (-)^p\gamma_p\gamma_1 + (-)^{p+1}\gamma_{p+1}\gamma_0 \equiv 0, \quad p = 0,\ 1, \cdots . \quad \text{(C.2.4)}$$

C.3 The Values of μ for a Given η in (4.2.2)

In order to calculate the values of μ for a given η, the following cases will be discussed. If $\operatorname{Im}\mu \geq 0$, we use $\frac{1}{2}\eta^2 = \mu - \log(\mu+1)$ to find the root μ for a given η. If $\operatorname{Im}\mu < 0$,

$\frac{1}{2}\eta^2 = \mu - \log(\mu + 1) - 2\pi i$ is used. For η situated below the cut [see Fig. 4.2], we use the formula $e^{\mu - \frac{1}{2}\eta^2} = 1 + \mu$. The values of μ for different ρ and ϕ ($\eta = \rho e^{i\phi}$) are given in the following Tables C.1-C.6. An asterisk $*$ denotes values of η below the cut.

Table C.1: The value of μ for $\rho = \frac{1}{2}$ and different ϕ

ϕ/π	$\mu : \{\frac{1}{2}\eta^2 = \mu - \log(1+\mu),\ \eta = \rho e^{i\phi}\}$
0	$0.5865820454591966218562272$
0.1	$0.5449155488423928225125663 + 0.2060874934658910296866496i$
0.2	$0.4293663002494485790011126 + 0.3763131859165835302757568i$
0.3	$0.2650267625866322227410579 + 0.4849644263724993676334538i$
0.4	$0.0842174590836771874636153 + 0.5226903390508890668465705i$
0.5	$-0.0835657140776722468715843 + 0.4965352105665538065856920i$
0.6	$-0.2191961518447978968926551 + 0.4242841016561129942025331i$
0.7	$-0.3161535516010442860618509 + 0.3261839276326963556959944i$
0.8	$-0.3774903977752089359254800 + 0.2180770055960077910104088i$
0.9	$-0.4102230448450046151273576 + 0.1085618189542768910131423i$
1	$-0.4203764666960421326229056$

Table C.2: The value of μ for $\rho = 2$ and different ϕ

ϕ/π	$\mu : \{\frac{1}{2}\eta^2 = \mu - \log(1+\mu),\ \eta = \rho e^{i\phi}\}$
0	$3.5052414957928833669986244$
0.1	$3.0930576914430573847830011 + 1.5341897566261285766382192i$
0.2	$2.0000857813713439203650145 + 2.6199804326027421472226687i$
0.3	$0.6006313022363321629742596 + 2.9800058181507543211785446i$
0.4	$-0.6486882588454711819057332 + 2.6127057203017146218351211i$
0.5	$-1.3949790827072933876246343 + 1.7881880413836294208945657i$
0.6	$-1.5452902130690552932813138 + 0.9269638180031942431998413i$
0.7	$-1.3215744388465391184077102 + 0.3760942478256077166206827i$
0.75	$-1.1887442839887226507021164 + 0.2390792742422237296198492i$
0.8	$-1.0859132018111665693484127 + 0.1604050111542008267187471i$
0.9	$-0.9761875123337327835006833 + 0.0708071649344370153511440i$
1	$-0.9475309025422851275901264$

Table C.3: The value of μ for $\rho = 2\sqrt{\pi}$ and different ϕ

ϕ/π	$\mu : \{\frac{1}{2}\eta^2 = \mu - \log(1+\mu),\ \eta = \rho e^{i\phi}\}$
0	$8.5385240711477301387098125$
0.1	$7.3125725562873337554960062 + 4.1568671736654617017713798i$
0.2	$4.0915681445517192696207037 + 6.9115414501154533615126830i$
0.3	$0.0706018087564943524311206 + 7.4028359434474956690377224i$
0.4	$-3.2767415059752599010723816 + 5.6471905993109355723918283i$
0.5	$-4.7683854814413921067910081 + 2.5471982656607517976672443i$
0.6	$-3.9813062912309964310712377 - 0.4136901048312328320518651i$
0.65	$-2.8655663474451782413154888 - 1.3242963126122162531572712i$
0.7	$-1.4601755923052754221346786 - 1.5515938809876079316591816i$
0.72	$-0.8646812275063693158613641 - 1.3603570127080419366057093i$
0.74	$-0.2783962674773301871965049 - 0.8606574116880522382655985i$
0.78*	$-0.8722641225587381659450134 - 0.0164001673283547143180032i$
0.8*	$-0.9472801077200644072001810 - 0.0177816150409193889558628i$
0.85*	$-0.9967443598763978346353329 - 0.0085913640676997977520580i$
0.9*	$-1.0019402652456221544486977 - 0.0011906025057812116788394i$
0.95*	$-1.0003392259556652667392050 + 0.0008702500092088675517868i$
1*	$-0.9993125337640284211188922$

Table C.4: The value of μ for $\rho = 4$ and different ϕ

ϕ/π	$\mu : \{\frac{1}{2}\eta^2 = \mu - \log(1+\mu),\ \eta = \rho e^{i\phi}\}$
0	$10.436839697575611225438066$
0.1	$8.884670906369080519152674 2 + 5.185408247742383766181438 4i$
0.2	$4.8106586639256986418801632 + 8.5841931979515794053026152i$
0.3	$-0.2607720110760312664642375 + 9.0981764185036847724615204i$
0.4	$-4.4472400864827727736291438 + 6.7455200178786720808598207i$
0.5	$-6.2298385798412922564636813 + 2.6696135913571067367206188i$
0.6	$-5.0315213198269386766885245 - 1.2581816443885088101718623i$
0.65	$-3.4446877594768281867430632 - 2.5283279222645333974184423i$
0.7	$-1.3613032322430148488611959 - 3.0153172178811299966600985i$
0.72	$-0.3986950147956399529943892 - 2.9444634340826753229685120i$
0.75	$1.2246477900657849542710563 - 2.5750978121508233673763166i$
0.8	$4.1608635930001630174396001 - 1.6314449417569871509745566i$
0.85	$6.7504207079445760690693171 - 0.2169332035415579242825522i$
0.88*	$-0.9992536132498180892299320 - 0.0007802696823004111546361i$
0.9*	$-1.0000060712142250266617789 - 0.0005686763947351802060893i$
0.92*	$-1.0002512124506911891465126 - 0.0002168957976468212385516i$
0.94*	$-1.0002122243646869515375844 + 0.0000422591546998810000764i$
0.96*	$-1.0000645323066259958624420 + 0.0001449463431801525052496i$
0.98*	$-0.9999292820431348930540334 + 0.0001108122544016578533114i$
1*	$-0.9998765749631136638245162$

Table C.5: The value of μ for $\rho = 2\sqrt{2\pi}$ and different ϕ

ϕ/π	$\mu : \{\frac{1}{2}\eta^2 = \mu - \log(1+\mu),\ \eta = \rho e^{i\phi}\}$
0	$15.3612887147251106624989 22$
0.1	$12.94051663478677026308192 6 + 7.902005517332876374713470 0i$
0.2	$6.594595646964880660043280 8 + 12.99317566741810466969130 21i$
0.3	$-1.277173222830064801706629 6 + 13.54258891128875497477948 71i$
0.4	$-7.708043353978999651802486 1 + 9.568554537817204269721987 7i$
0.5	$-10.29238892065070054988919 1 + 2.844535644583170902869313 6i$
0.6	$-8.084572104990176535988490 2 - 3.757122723255470754275257 8i$
0.7	$-1.880325501761411941323870 1 - 7.358015380346059423576179 7i$
0.75	$2.088843015613044504846807 5 - 7.461489285654253231650923 5i$
0.8	$6.143997740668299932505428 9 - 6.398552321724497226291353 3i$
0.85	$9.841409781485599954911368 5 - 4.257419373997526543037897 0i$
0.9	$12.79436975619383379131709 8 - 1.189133708403375760710381 3i$
0.91	$13.26878734588075393505882 1 - 0.484129005575696444445565 2i$
0.94*	$-1.000000267566074394494257 5$
0.96*	$-1.000001903697702221957008 4$
0.98*	$-1.000000005933874809236112 1$
1*	$0.999998717076796933444599 0$

Table C.6: The value of μ for $\rho = 6$ and different ϕ

ϕ/π	$\mu : \{\frac{1}{2}\eta^2 = \mu - \log(1+\mu),\ \eta = \rho e^{i\phi}\}$
0	$21.09536799350179599755366 8$
0.1	$17.63978857140701350658673 6 + 11.11794945501759634964855 7i$
0.2	$8.586376543537135893429617 2 + 18.20513234394495694497507 3i$
0.3	$-2.625980382808779906892257 9 + 18.77619608251866165934689 7i$
0.4	$-11.74403657157545933882225 2 + 12.84740600329426490143664 5i$
0.5	$-15.31785511153625227203152 7 + 2.939128123374187546722937 4i$
0.6	$-11.99938551954715895998323 7 - 6.879615768543990731202381 9i$
0.7	$-3.018579707021369688335157 4 - 12.56590694923913923144262 3i$
0.8	$8.267398255971328919540784 4 - 11.73832309342317414590467 8i$
0.9	$17.50978202668182015555449 5 - 4.537341089684814131075019 0i$
0.94	$19.76961855264639340767980 2 - 0.360407530955864083941517 2i$
0.95*	$-0.999999989842633221193565 5$
0.97*	$-1.000000007501554921945677 8$
1*	$-0.999999994397203531071404 3$

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