

ROBUST CONTROL
OF NONLINEAR SYSTEMS
IN THE PRESENCE OF UNCERTAINTIES

By **LIQUN CONG**

Robust Control of Nonlinear Systems in the Presence of Uncertainties



*A thesis submitted in partial fulfilment
of the requirements of Dundee Institute
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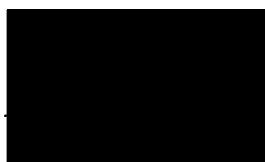
To

*my dear wife, Ying, my daughter, Xue, and my parents
for their continuing support, understanding and encouragement.*

☒ Declaration

I hereby declare that while registered as a candidate for the degree for which this thesis is presented I have not been a candidate for any other award. I further declare that, except where stated, the work contained in this thesis is original and was performed by myself.

Signed

A solid black rectangular box used to redact the signature of the candidate.

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10 Aug. 1993

✓ Abstract

Robust Control of Nonlinear Systems in the Presence of Uncertainties

Any mathematical model that is adopted for the purposes of design is, at best, an approximation to reality. However, despite the existence of such mismatch between the plant and its model, the engineering system should still be stable and achieve some prespecified performance. Different robustness measure bounds and synthesis techniques have been developed. A promising area is the so-called deterministic theory, where the uncertainties incorporated in the system are described only in terms of the bounds on their possible size, and the objective is to find a class of controller which can achieve some prescribed behaviour for all possible variations of the uncertainties within the prescribed bounds. This has found wide applications in such areas as robotics and aircraft control.

The results presented here cover various novel techniques, which can be roughly divided into two categories according to the concepts on which the techniques are based. One category uses feedback linearisation, in which, besides a basic feedback linearisation controller proposed for the nominal part of the system, additional control effort is introduced to compensate the uncertainties in the system. The other category uses a variable structure controller which is developed for the nominal part of the system, whilst a variable feedback gain is employed to attenuate the effect of the uncertainties. Both techniques can be applied to effectively deal with systems in the presence of nonlinearity and uncertainty, and some stability theory can be developed.

The techniques developed here are concerned with both robust stability control design and robust tracking control design for SISO and MIMO nonlinear uncertain systems where closed loop stability can be guaranteed and robustness is shown.

For illustrative purposes, a second order system, with uncertain pole location and non-minimum phase properties, is adopted to demonstrate the performance of the techniques. Some applications are also included in the thesis, and it is shown that the techniques developed here are an improvement on previously developed methods.

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✓ List of Symbols

\mathbb{R}	The reals
\mathbb{R}^n	Real vector space of n-vectors
$\forall x$	For all states x
\ni	Such that
\in	Belongs to
\Rightarrow	Implies
\triangleq	Defined as
$ \cdot $	Absolute value
$\ \cdot\ $	Euclidean norm of vectors or matrices
$\langle \cdot, \cdot \rangle$	Inner product
$\mathbf{I}_{1 \times m}, \mathbf{I}_{m \times 1}$	Vector of m elements with all entries 1
B_κ	An open ball with a finite radius κ
$p_A(\cdot)$	Characteristic polynomial of matrix A
$\text{tr}(\cdot)$	Trace of a matrix
$\det(\cdot)$	Determinant of a matrix
C^∞	Smooth, <i>i.e.</i> , infinitely differentiable
H_∞	Set of stable transfer functions G, with $\ G\ _\infty < \infty$
x^*	Equilibrium state
M_s	Symmetrised form of non-symmetric matrix M
M^T	Transpose of matrix M
f, ϕ	C^∞ vector field and one form on \mathbb{R}^n respectively
∇f	The gradient or Jacobian of scalar or vector fields
v	Relative order (degree) of a nonlinear system
$L_f h$	Lie derivative of h with respect to f
$\text{ad}_f(g)$	Lie bracket of g with respect to f

μ, λ	Open loop and closed loop poles respectively
$\varphi_M(\cdot), \varphi_m(\cdot)$	Maximum and minimum singular values of matrices
$\lambda_M(\cdot), \lambda_m(\cdot)$	Maximum and minimum eigenvalues respectively
Ψ, Ψ^{-1}	Coordinate transformation and its inverse
$\mathfrak{Z}, \mathfrak{Z}^{-1}$	Matrix transformation and its inverse
$V(\cdot), \dot{V}(\cdot)$	Lyapunov function, and its time derivative
P, Q	Matrices in Lyapunov equation
ν	Stability margin
ν_1, ν_2	Boundary functions of Lyapunov function
ν_3, ν_4	Constants for further requirement on Lyapunov function
x, z	State and transformed state of a system respectively
u, v	Control and transformed control of a system respectively
y	System output
z, ζ	States of external and internal dynamics respectively
$F(\cdot), f(\cdot)$	State mapping of nonlinear systems
$G(\cdot), g(\cdot)$	Input mapping of nonlinear systems
$H(\cdot), h(\cdot)$	Output mapping of nonlinear systems
$\Upsilon(\cdot), \bar{\Upsilon}$	Lumped uncertain element and its nominal value
$\xi(\cdot), \omega_\xi$	External disturbance and its bound
$\Delta f(\cdot)$	State mapping uncertainty in nonlinear systems
$\Delta g(\cdot)$	Input mapping uncertainty in nonlinear systems
$p(\cdot)$	State mapping uncertainty matching parameter
$q(\cdot)$	Input mapping uncertainty matching parameter
$\omega_{\Delta f}, \omega_{\Delta g}$	Bounds of mismatched uncertainties
ω_p, ω_q	Bounds of matched uncertainties
$\Omega_{\Delta f}, \Omega_{\Delta g}$	Generalised bounds of the system uncertainties
Φ_G, ϕ_G	Supremum and infimum bounds of $G(x, \gamma)$
δ, Δ	System uncertainties and their bounds in robust tracking analysis
q, p	Expressions for internal dynamics
$\rho, \rho_1, \rho_2, \rho'$	State feedback control gains
C_1, C_2	Constants of feedback control
$\sigma(\cdot), \Sigma(\cdot)$	Switching function and its diagonal matrix form

$u(\cdot), U(\cdot)$	Control and its diagonal matrix form
u_{eq}	Equivalent control in variable structure systems
f^1, f^2, g^2	Regular form expressions of a nonlinear system
ς, ι	Constants in multi-input variable structure controllers
ϑ, μ	Constants in internal dynamics analysis
$\beta_0, \beta_1, \beta_2$	Coefficients of the bounds for mismatched uncertainties
$e(\cdot)$	Tracking errors
$d(\cdot, \cdot)$	Parameter concerned with uniform boundedness
$T(\cdot, \cdot)$	Parameter concerned with uniform ultimate boundedness
r	Initial state
z^d, c	Ideal trajectory and its bound in robust tracking
$G(s)$	Transfer function
t	Time
Ω	Admissible domain of states
\bar{x}, x	Sets of states which are on and off the switching surface
\square	End of proof of theorems

✓ Preface

Motivated by the theoretical and practical importance of robust control in engineering, it is proposed to investigate the robust control problem for nonlinear uncertain systems, and also to seek to develop more robust and intuitive methodologies than those currently in use, and to relax some of the conditions imposed.

0.1 MAJOR CONTRIBUTIONS OF THE RESEARCH

The major contributions described here may be summarised as follows:

- ① Firstly, the so-called matching conditions have been relaxed further and further. Initially, the condition that the modulus of the input mapping matching parameter is less than unity is replaced by simply requiring that this parameter be greater than zero. This difference leads to a new control law which is related to both the bounds of the uncertainties in the system and to the nominal control component, so that the effect of the uncertainties can be effectively attenuated by the proposed control.
- ② Secondly the technique can be extended to more general cases where the matching conditions are not met. So a unified control can be found for the following cases:
 - The uncertainties satisfy the matching conditions, and the modulus of the input mapping matching parameter is less than unity or greater than zero;
 - Either the state mapping uncertainty or the input mapping uncertainty satisfies the matching conditions, but not both;
 - The uncertainties lie in the span of the input mapping, but neither a continuous input mapping matching parameter nor a continuous state mapping matching parameter exists;
 - None of the uncertainties satisfy matching conditions;
- ③ The results are intuitive, and the performance is robust. Two typical forms of controller are discussed in chapters 4 and 5, in which one, additive compensation, uses the idea of an additional control component to compensate both the effect of state mapping uncertainty and the effect of input mapping uncertainty via the nominal control. The

other, multiplicative compensation, adopts concepts from adaptive control where feedback gain is variable instead of constant. Both methods can be understood as employing extra control effort to compensate for the effect of uncertainties.

- ④ One of the most important results in this thesis is the application to multi-input systems. The technique developed for the single-input case has been extended to the multi-input case without further conditions being placed on either the system or the uncertainty. The principle is exactly the same except that it uses more mathematical concepts. The control law is similar to that developed for single-input systems. The regulation as well as the tracking problems for both single and multi-input systems are considered, and robust control laws are developed for both.
- ⑤ Robustness is demonstrated by simulation using a simple second order system, in which the uncertainty in pole location can be effectively controlled even for the case where the open loop pole is believed to be in the left half, but is in fact in the right half of the complex plane. Furthermore the well-known non-minimum phase problem is treated as a special kind of uncertainty.

0.2 PUBLICATIONS

The following conferences were attended during the performance of the work described here:

- ① *The IEE International Conference on Control*, Edinburgh, Scotland, 25-28 March, 1991
- ② *The 8th International Conference on System Engineering*, Coventry, England, 10-12 September, 1991
- ③ *The 6th IMA International Conference on Control: Modelling, Computation, Information*, Manchester, England, 2-4 September, 1992

The work described here has resulted in the writing of the following papers:

- ① L. Cong and P. H. Landers, An Improved Synthesis Technique for Nonlinear Uncertain Systems, *Proc. 8th ICSE*, Coventry, UK, 10-12 Sept., 1991, pp663-670
- ② L. Cong and P. H. Landers, Robust Control Design for Nonlinear Uncertain Systems with Incomplete Matching Conditions, *Trans. Inst. M&C*, Vol.15, No.1, 1993, pp46-52
- ③ L. Cong and P. H. Landers, Robust Stabilisation Control Design for Multivariable Nonlinear Uncertain Systems, *Proc. 6th IMA Int. Conf. on Control: Modelling, Computation, Information*, Manchester, England, 2-4 September, 1992, and submitted to *IMA Journal of Mathematics and its Applications*, 1993

- ④ P. H. Landers and L. Cong, Robust Control Synthesis for Both Matched and Mismatched Uncertain Nonlinear Systems. Submitted to *IEE Proc. Pt.D*, 1993
- ⑤ L. Cong and P. H. Landers, Robust Control Design for Nonminimum Phase Systems. Submitted to *Trans. Inst. M&C*, 1993
- ⑥ L. Cong, P. H. Landers and A. T. Sapeluk, A New Synthesis Technique for Nonlinear Systems with Mismatched Uncertainty. Submitted to *Automatica*, 1993
- ⑦ L. Cong and P. H. Landers, Robust Tracking Control of Nonlinear Uncertain Systems. *In preparation*, 1993

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Robust Control
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in the Presence of **Uncertainties**

Introduction



Overview

This chapter gives a general introduction to the main developments in robust control of nonlinear uncertain systems, and describes current knowledge.



Outline

- ✓ Current Research
- ✓ The Objective of the Research
- ✓ An Overview of the Thesis

1.1 CURRENT RESEARCH

THE objective of control design can be stated as follows: given a physical system to be controlled and performance specifications, construct a feedback control law to make the closed loop system display the desired behaviour.

In general, a physical plant has very complex dynamics, and is also affected by the environment in which it works. So, when a real plant is modelled, an assessment of the errors must be made. The causes of such errors in the open loop system are typically limited model information, modelling inaccuracy and disturbances. Unknown or varying parameters resulting from poorly understood physical phenomena are examples of model information uncertainty, while linear approximation, order reduction, and neglected coupling terms are examples of model inaccuracy. Any mathematical model adopted for control design therefore is, at best, an approximation to reality. However, despite the presence of such uncertainties, the system should still be stable. A critical property of a feedback system is its robustness, that is, its ability to reduce the sensitivity of the system to variations of system parameters and to unmodelled dynamics. In pure model-based control, the control law is based only on a nominal, linear in many cases, model of the physical system. How the control system will behave in the presence of model uncertainties and unmodelled dynamics is not clear at the design stage, and the stability of the closed loop system cannot be guaranteed. A nominal model based controller, or a linear controller based on inaccurate or obsolete values of the model parameters, or a nonlinear controller without consideration of the structure and size of uncertainties may exhibit significant performance degradation or even instability. Therefore, robust control of systems in the presence of nonlinearity and uncertainty is of great significance in practice, and many researchers and designers, from such broad areas as flight control, robotics,

process control, and biomedical engineering, have shown an interest in the development and applications of robust control methodologies for nonlinear uncertain systems.

The problem of robust design of control systems, otherwise described as reliable design in the presence of uncertainty, has been studied, for many years, without conspicuous success. Because of this, engineers have turned to techniques such as fuzzy logic and statistical metric spaces involving knowledge based systems. Little intuitive understanding of the process results. Even the linear problem isn't easy whilst the nonlinear problem with uncertain perturbation is made more difficult because systems with nonlinearity and uncertainty can exhibit more complex behaviour than linear systems, and many of the established techniques are based on the assumption of exactly known models and parameters. Quite apart from the undesirability of this, problems of modelling errors tend to become submerged in the overall technique.

The last two decades have seen major progress in the analysis and synthesis theory of systems with nonlinearity and/or uncertainty, utilising many advanced mathematical concepts, and different robustness measure bounds. These include stochastic control theory, if a prior statistical characterisation of the uncertainties is available, as well as deterministic methods, where such statistics are unavailable but precise bounds on uncertainties are known. Where deterministic theory is used, the objective is to find a class of controllers which can achieve some prescribed behaviour for all possible variations of the uncertainties within the prescribed bounds, often termed 'guaranteed performance', which indicates that the resulting closed loop system will exhibit certain desirable properties for all admissible uncertainties. When the bounds of the uncertainties are known, the controller guarantees that the states of the system enter a particular vicinity of the equilibrium state after a finite period of time and remain there, and also guarantees that the state trajectory will be kept arbitrarily close to the equilibrium point if started close to it.

One of the best developed techniques in the frequency domain is quantitative feedback theory, denoted QFT, which was first proposed by Horowitz and Sidi^[1] in 1972

for single-input single-output (SISO) single-loop linear time-invariant minimum phase plants with large uncertainty. The theory has been extended to other system types. These include linear time-variant, nonlinear, multi-input multi-output (MIMO) and non-minimum phase plants. The key tool is the conversion of the initial set of plants into an equivalent set of linear time-invariant SISO plants. Schauder's fixed point theorem is used to justify the equivalence. The principle of QFT is to use pointwise design, *i.e.*, repeat the design procedure in the same manner at sufficient frequency points separately to permit drawing a continuous curve of the bound, and to achieve the performance prespecified for large uncertainty. Pointwise design provides designers with the opportunity to make some tradeoffs between the loops, compensator complexity, and bandwidth economy, and between the extent of plant uncertainty, tolerances and feedback cost. But it also produces the problems that the size of the manipulated regions on the Nichols chart may be inconveniently large. Since QFT was proposed in 1972, many advances have been made, and many application examples have been published.

The development of H_∞ optimal control theory^[4,5] can be seen as a return to the ideas and principles established by Bode^[6] in the 1940s, but one which also led to considerable generalisations of these ideas. Notions such as the sensitivity function and stability margins, which were rather eclipsed by LQG theory, which dominated the 1960s and 1970s, have been re-established as central to the theory, and have been successfully extended to multivariable systems. The theoretical key to these extensions has been the introduction of the 'infinity norm' of a transfer function matrix G (written $\|G\|_\infty$) as a measure of its gain. The set of (linear) stable multivariable systems, whose infinity norms are finite, forms what mathematicians call a 'Hardy space', which has been given the name ' H_∞ ', and it is this which gives much of recent robust control theory its name. If all we know about some input (which may be a vector of inputs, and 'input' includes 'disturbance') is that it belongs to a specified set, and if we measure the size of output signals in similar ways, then the infinity norm of the transfer function relating the input to the output is the

worst-case gain between the two. The term ' H_∞ problem' arises from the fact that, for a closed loop system with plant P and feedback control K , we are minimising $\|G_1(P,K)\|_\infty$ over all $G_1(P,K)$ where F_1 represents the transfer function matrix of external inputs to the output errors, such that $G_1(P,K) \in H_\infty$ and the feedback combination of P and K is internally stable. Use of the infinity norm therefore makes it possible to formulate realistic 'worst-case' performance specifications as mathematical problems to which theoretical solutions can be found. The theory is of great interest because it gives solutions to realistic robust control problems, posed as H_∞ optimisation problems. The application of the theory to control problems originated with Zames^[7]. In fact recent developments have shown the theory to have remarkable similarities with the LQG theory. A consistent term for the LQG problem, which is sometimes used, is ' H_2 problem', since that requires the minimisation of $\|G_1(P,K)\|_2$ over all $G_1(P,K) \in H_2$, again with the constraint of internal stability. LQG problems can even be seen as special cases of H_∞ problems.

The most useful and general approach for studying nonlinear systems is Lyapunov stability theory, which was introduced in the late 19th century by the Russian mathematician Alexandr Mikhailovich Lyapunov. Basic Lyapunov theory consists of two methods, the indirect method and the direct method. The indirect method, often called the linearisation method, states the stability properties of a nonlinear system in the vicinity of an equilibrium point by analysing those of its linearised approximation, while the direct method draws conclusions from the original system directly by constructing a scalar function for the system and examining the function's time variation.

There are two major time domain techniques for the design of controllers for nonlinear systems displaying significant uncertainties. One is the variable structure control (VSC) approach, which can be applied to highly nonlinear systems, and results in robustness to model errors, parameter variations and unknown disturbances. The VSC approach was first proposed in the 1950's by Utkin, and has been developed over several decades, see Utkin^[8,9], Zak *et al*^[10,11], and Sira-Ramirez^[12]. Essentially, VSC uses a high-

speed switching control law to drive the nonlinear plant's state trajectory onto a specified and designer-chosen surface in the state space (called the sliding or switching surface), and to maintain the trajectory on this surface for all subsequent time. The plant dynamics restricted to this surface represent the control behaviour of the system. By properly choosing the switching surface, VSC attains the conventional goals of control such as stabilisation, tracking, and regulation. The main result is that the controlled system is insensitive to certain parameter variations and disturbances while the trajectory is on the switching surface. The variable structure technique is now well developed, Zak *et al*^[13,14], in that by properly choosing the switching surface, the original system can be decomposed, by a transformation, into two sub-systems, the fast one which describes the motion of the system off the switching surface, and the slow one which describes the motion on the switching surface, whilst the stability properties of the systems can be justified by Lyapunov theory.

Another method for synthesis of nonlinear uncertain systems based on Lyapunov stability theory was proposed by Gutman^[15], Leitmann *et al*^[16,17,18,19], and other authors^[20-26]. The design is based on the constructive use of Lyapunov stability theory. Roughly speaking, a Lyapunov function for a nominal system (*i.e.*, the certain part of the real system) is employed as a candidate Lyapunov function for the actual uncertain system with control, and a robust control strategy can be developed so that it can guarantee a negative derivative of the Lyapunov function along all possible solutions in the presence of uncertainties. The success of the method depends crucially on the satisfaction of additional *a priori* assumptions on the nature of the uncertainties. These assumptions essentially restrict the structure and/or size of the uncertainties in the system. In the case of many of the previous references, these restrictions have been appropriately referred to as matching conditions, which means that the uncertainties originate directly through the control variable, *i.e.*, the uncertainties lie in the span of the nominal input mapping. Such conditions make analysis and synthesis much easier, but they are often not met in practice.

Recently, a number of papers have appeared which take mismatched uncertainties into account, but do so in a variety of ways. One way^[18,19,20] is to decompose the system uncertainties into a matched part and a mismatched part, and treat them separately. Usually some limitation must be imposed on the mismatched part of the uncertainties, say, a critical 'mismatched threshold' on the allowable size. Another approach^[21] is to consider mismatched uncertainties in the state mapping but not in the input mapping, whilst a third way^[22,23] is to convert the mismatched uncertain system into a matched one by a change of basis and translation of the state.

Robust control differs from model-based control in that it is based not only on consideration of the nominal model, but also on some characterisation of the model uncertainties. By the nominal model is usually meant the model obtained by various identification techniques, the parameters of which are given by the nominal values. Such a model is not unique, as we might adopt different nominal models for easing the control design and for simplifying the uncertainty description.

1.2 THE OBJECTIVE OF THE RESEARCH

Motivated by the aforementioned theoretical and practical considerations, it is proposed to investigate some nonlinear design techniques already developed, and also to seek to develop more robust and intuitive methods for systems with nonlinearity and uncertainty. Intuitive methods are very important because feedback tends to be counter-intuitive and this makes design, which is inevitably interactive (between computer and operator), even more difficult.

The objective of this research is to study the synthesis problem of nonlinear uncertain systems in a deterministic way, the problem statement being similar to that of Leitmann *et al*^[16,17], but differing fundamentally in the control strategy.

Nonlinear uncertain systems can be generally represented as

$$\dot{x}(t) = f(x) + g(x)u(t) + \Delta f(x, \gamma, t) + \Delta g(x, \gamma, t)u(t) + \xi(t)$$

where $f(x) + g(x)u(t)$ is the certain part of the system, often termed the 'nominal system'; $\Delta f(x, \gamma, t)$, $\Delta g(x, \gamma, t)$ represent uncertainties incorporated in the system; $\xi(t)$ denotes the external disturbance; γ is a lumped uncertain element.

Roughly speaking, there are two ways of dealing with the robust design problem; one phase design and two phase design. One phase design is founded on the intuitive fact that any uncertainty should be effectively compensated by the designed controller. In order to achieve this, control effort must be introduced in addition to the main control component designed for the nominal part of the system. Based on this concept, the conventional controllers obtained for the nominal part of the system may be modified by introducing an extra control component, or employing a variable feedback gain instead of a constant one.

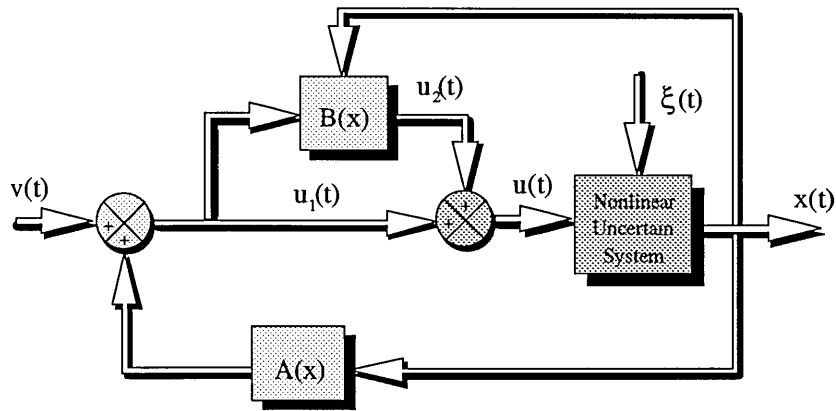


Fig 1.1 One phase design

For example, the controlled inputs may be of the form

$$u(t) = u_1(t) + u_2(t)$$

where $u_1(t)$ is obtained for the nominal system according to one of various design theories, without consideration of any uncertain element, (for example, to linearise the nominal system, and place the closed loop poles in desired positions); and $u_2(t)$ is the additional

feedback control to compensate for the effect of uncertainties in the system. The stability of the overall system can be guaranteed by this combined feedback control. Usually, one phase design synthesises a closed loop system with respect to the original nonlinear uncertain system directly. In general, $u_2(t)$ is related to the nominal control $u_1(t)$ as well as to the uncertainty bounds on Δf and Δg , because not only the effect of the uncertainty in the state mapping $\Delta f(x, \gamma, t)$, but also the effect caused by $u_1(t)$ through the uncertainty in the input mapping $\Delta g(x, \gamma, t)$ should be compensated.

The two phase method differs from the one phase method in that two feedback loops are included, and each of them is designed separately according to different theories, and will therefore meet different requirements. This design procedure usually involves the transformation of the original system to new coordinates and linearisation of this new system at the first stage. The development of controllers using various established synthesis techniques occurs at the second stage.

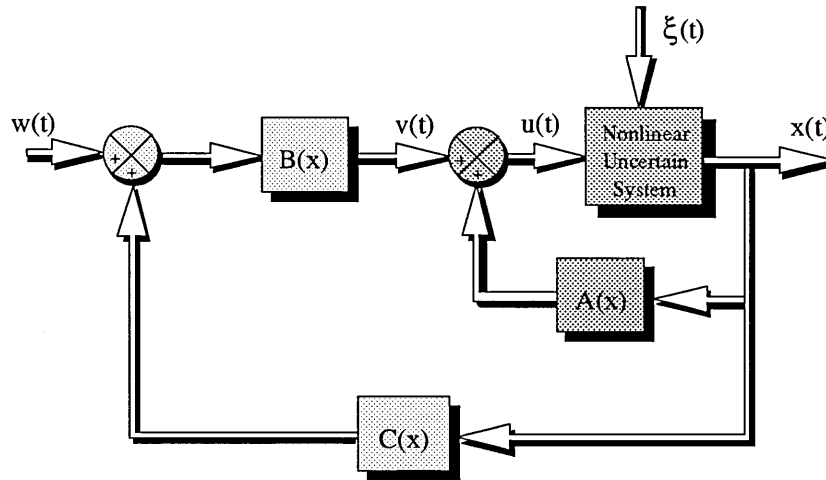


Fig 1.2 Two phase design

The feedback control is usually of the following form

$$u(t)=u(x,v); \quad v(t)=v(x,w)$$

where $u(t)$ is developed in the first phase. This may be done, for instance, according to the feedback linearisation technique. The system undergoes a coordinate transformation and is

then linearised, such that a new system with a linearised nominal part and some uncertainties, usually nonlinear, is obtained. In the second phase, the controller design will be carried on with respect to this new system, and the objective is to obtain a feedback controller such that the closed loop system performs in the desired manner.

The results presented in this thesis are concerned with both methodologies. At first, a simple case is considered, in which the structures of the uncertainties are assumed to satisfy the so-called matching conditions, but unlike the assumption made by Barmish *et al*^[17], it is not required that the uncontrolled nominal system should be stable or precompensated to be stable, and it is not required that the matched form of uncertainty bounds should be less than 1; instead a weaker and more flexible condition, is imposed. A set of robust feedback controllers are obtained by extending the feedback linearisation technique, and using Lyapunov stability theory, which results in a practically stabilised closed loop system, even for nonlinear systems with unstable nominal part, in the presence of significant parameter tolerances and external disturbances. Compared with the technique of Barmish *et al*^[17], some significant improvements have been made in that less severe matching conditions have been assumed. More importantly, such improvement enables us to extend the technique to more complicated systems in which the uncertainties do not meet the so-called matching conditions. This kind of uncertainty is considered throughout the rest of the work, and new control techniques are obtained. These may be applied to various cases, such as, where although the uncertainties lie in the span of the input mapping, so satisfying the generalised matching assumption, there are no continuous functions p and q such that the uncertainties are of the desired form, which has been assumed for the matching conditions, or where the uncertainties may only satisfy partial matching conditions, or even where the uncertainties do not satisfy any matching assumption. The techniques appear to represent a significant advance on previous results, with no restriction on the size of the uncertainty bounds, except for a weak and flexible condition imposed on the uncertainty in the input mapping.

A novel robust technique has also been proposed, where the problem statement is similar to that above, but differs fundamentally in the control strategy. The design procedure utilises concepts of sliding mode from the theory of variable structure systems, and concepts of practical stabilisation from the theory of Leitmann *et al*^[16,17,18], but shows obvious differences from them, in that, instead of the assumptions of pre-compensation on the nominal part of the system and matching conditions on the uncertainties, only a rather weak condition is imposed on uncertainties with no further assumptions. The proposed control is of variable structure, and can also be used to deal with nonlinear systems with both matched and mismatched uncertainties. This development results in some advantages in that it avoids the requirement for proper choice of some design constants, thus easing the design problem. It is also shown that the controller has the same structure as the one developed for the nominal system where no uncertainty is explicitly considered, the only difference being that the former has a variable controller gain, which depends on the known uncertainty bounds, and the latter has a constant one. The method has been extended to the multi-input case and this also is fully described in the sequel.

Finally, the robust tracking problem has been investigated for nonlinear uncertain systems. A robust stability controller is first proposed for SISO systems, and extended to the MIMO case. The proposed design method is divided into two phases. Firstly, the original nonlinear uncertain system is transformed into new coordinates using differential geometric theory, and a new system model, which has a linearised nominal part and nonlinear uncertainties, is obtained. Secondly, a robust variable-structure-like controller is developed, and the feedback gain is related to the uncertainty bounds. Stability of the closed loop system is justified by using Lyapunov stability theory. The results are obtained separately, for the cases where the uncertainties are assumed to satisfy the generalised matching assumption, as well as where they do not. It is also shown that the tracking errors depend crucially on the amplitudes of the mismatched part of the uncertainties. When only matched uncertainties are present, the tracking errors will converge to zero. However when

both matched and mismatched uncertainties are present, the tracking errors cannot converge to zero, but converge in a finite time, to a ball with a finite radius that depends only on the bounds of the mismatched uncertainties. The internal dynamics are also considered. For asymptotic minimum-phase systems, the internal states will also converge to a ball with the radius depending on the bound of the desired state trajectory.

Throughout the thesis, for simplicity, a typical second order linear system is utilised to illustrate the usage of the developed methodologies and some engineering control problems, such as uncertain pole locations, non-minimum phase, and parameter variations, are discussed. Moreover, applications of the proposed techniques to some more practical nonlinear uncertain systems are described, and simulation results are included to show the effectiveness of the proposed techniques.

1.3 AN OVERVIEW OF THE THESIS

The thesis consists of nine chapters, each of which starts with a short introduction providing the background for the main issues and techniques to be discussed, and a brief summary is also included in each chapter.

Chapter 2 introduces the class of nonlinear systems to be considered, and also describes the types of uncertainty which may occur. Chapter 3 describes the concepts of stability and boundedness, and presents the major analytical tools that are required subsequently. Chapters 4–7 are the main results of the thesis. Various proposed robustness techniques are presented one by one. In chapter 4, the techniques based on feedback linearisation are described for matched uncertainties first and mismatched ones thereafter. Chapters 5 and 6 describe the techniques of variable-structure-like control, chapter 5 providing the technique for single input systems, while chapter 6 is concerned with multi-input systems. These results are obtained by using the one phase design method. Unlike the

previous three chapters in which only the regulation problem is discussed, chapter 7 discusses the robust tracking control problem using the two phase design method. Chapter 8 presents some applications of the aforementioned techniques, to some practical nonlinear uncertain system models, and simulation results are given. The last chapter concludes the techniques developed, and some remarks on possible future work in this area are made from the point of view of the author.

The thesis concentrates on nonlinear uncertain systems in continuous-time form. Even though most controllers are implemented digitally, nonlinear physical systems are continuous in nature, while digital controllers may be treated as continuous-time systems in analysis and design if high sampling rates are used. The thesis also pays more attention to uncertainties than to nonlinearities, because robustness means the ability to reduce the sensitivity of the system to any uncertainties in the system, regardless of whether the system is linear or nonlinear.

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System Description



Overview

The class of nonlinear systems to be considered is introduced, and the types of uncertainties, with which the thesis is concerned, are described.



Outline

- ✓ Introduction
- ✓ Nonlinear Uncertain System Models
- ✓ Nonlinearities
- ✓ Uncertainties
- ✓ Summary

2.1 INTRODUCTION

PHYSICAL systems are inherently nonlinear. Thus, strictly speaking, all control systems are nonlinear to some extent. Nonlinear uncertain systems can be modelled by nonlinear differential equations. The nonlinear system may be reasonably approximated by a linearised system only when the operating range is small, and the nonlinearities are smooth. This is not always the case, and then nonlinear control techniques are necessary.

On the other hand, the mathematical models used to describe physical systems may also be imprecise. Model imprecision may come from actual uncertainty about the plant (*e.g.*, unknown plant parameters), or from the intentional choice of a simplified representation of the system's dynamics (*e.g.*, linear approximation, order reduction, and neglected coupling terms). Therefore discrepancies between the model and the real system exist. Any mathematical model adopted for control design is, at best, an approximation to reality. However, despite the presence of such uncertainties, the final design should still result in a stable system.

2.2 NONLINEAR UNCERTAIN SYSTEM MODELS

We consider a class of nonlinear systems modelled by the following equations

$$\dot{x}(t)=F(x)+G(x)u(t) \tag{2.1}$$

$$y(t)=H(x)$$

where $F(\cdot)$ and $G(\cdot)$ are C^∞ vector fields on \mathbb{R}^n , $H(\cdot)$ is a C^∞ scalar field on \mathbb{R}^n , and x , y and u are the state, output and admissible control having appropriate dimensions. It is assumed that the functions $F(\cdot)$ and $G(\cdot)$ are Caratheodory functions, *i.e.*, for all $t \in \mathbb{R}$ they are

continuous in $x \in \mathbb{R}^n$, and for all x they are Lebesgue measurable^[9] in t .

If some uncertain elements exist, we can then write

$$\begin{aligned}\dot{x}(t) &= F(x, \gamma, t) + G(x, \gamma, t)u(t) + \xi(t) \\ y(t) &= H(x)\end{aligned}\tag{2.2}$$

All the uncertainties in the system are represented by a lumped uncertain element $\gamma \in \mathbb{R}^{\gamma}$, which could be an element representing unknown constant parameters and inputs; or could be a function $\gamma(t): \mathbb{R} \rightarrow \mathbb{R}^{\gamma}$, representing unknown time varying parameters and inputs; or could be a function $\gamma(t, x, u): \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^{\gamma}$, representing nonlinear elements which are difficult to characterise exactly; and $\xi(\cdot)$ represents external disturbance which could be either deterministic or stochastic, but is normally stochastic.

For ease of design, the system model is usually decomposed into two parts; the certain part and the uncertain part, and then formulated as

$$\begin{aligned}\dot{x}(t) &= f(x) + g(x)u(t) + \Delta f(x, \gamma, t) + \Delta g(x, \gamma, t)u(t) + \xi(t) \\ y(t) &= h(x)\end{aligned}\tag{2.3}$$

where

$$\begin{aligned}F(x, \gamma, t) &= f(x) + \Delta f(x, \gamma, t) \\ G(x, \gamma, t) &= g(x) + \Delta g(x, \gamma, t)\end{aligned}$$

Here $f(\cdot)$ and $g(\cdot)$ represent the nominal part of the system, which is independent of the uncertain elements, and $\Delta f(\cdot, \cdot, \cdot)$ and $\Delta g(\cdot, \cdot, \cdot)$ indicate the uncertainties in the state and input mapping respectively. The system (2.1) is called the nominal version of (2.3).

Such a decomposition is not unique. One way to perform this decomposition is to choose the certain parts f and g such that the uncertain parts Δf and Δg satisfy some desirable conditions, as will be seen in the sequel. Moreover, the certain part, $f(x) + g(x)u(t)$, is not necessarily required to be a part of the actual dynamics, but could have been added for controller design purposes in the event of the absence of a suitable nominal portion for which some existing techniques can be applied or for which a Lyapunov

^[9] *Lebesgue Measurable:*

A set S is said to be Lebesgue measurable if the inner measure of S equals the outer measure.

A function $f(x)$ defined on a measurable set $S \subset \mathbb{R}$ is said to be Lebesgue measurable on S if, for each real number λ , the set of points $x \in S$ such that $f(x) > \lambda$ is measurable.

function could be readily found. Here $\gamma(t) \in \mathbb{R}^\gamma$ is a lumped uncertain parameter, such that $\Delta f(x, \gamma, t)$ and $\Delta g(x, \gamma, t)$ are bounded. These bounds are given $\forall (x, t) \in \mathbb{R}^n \times \mathbb{R}$ by

$$\omega_{\Delta f} \triangleq \left\{ \max_{\gamma(t) \in \mathbb{R}^\gamma} |\Delta f_k(x, \gamma, t)|_{k=1,2,\dots,n} \right\} \quad (2.4)$$

$$\omega_{\Delta g} \triangleq \left\{ \max_{\gamma(t) \in \mathbb{R}^\gamma} |\Delta g_k(x, \gamma, t)|_{(k=1,2,\dots,n)} \right\} \quad (2.5)$$

and

$$\omega_\xi \triangleq \left\{ \max_{t \geq 0} |\xi_k(t)|_{(k=1,2,\dots,n)} \right\} \quad (2.6)$$

where the functions $\omega_{\Delta f}$, $\omega_{\Delta g}$ and ω_ξ are presumed deterministic and known.

Note that the bounds $\omega_{\Delta f}$ and $\omega_{\Delta g}$ could either be functions of x and t , or acceptable constants satisfying conditions (2.4) and (2.5) if there is not enough information to define these functions explicitly.

Next we introduce some useful concepts to describe the characteristics of the system and the classification of the uncertainties in the system.

DEFINITION 2.1. (Index of a Vector)

Let the nominal system (*i.e.*, the certain part of the system) of (2.3) have relative order $v \leq n$ (as defined in appendix A). An uncertainty vector field $\Gamma(x, \gamma)$ is said to have an index $k \leq v$ with respect to the system if

$$\Gamma(x, \gamma) \in \text{Ker}\{dh(x), dL_f h(x), \dots, dL_f^{k-1} h(x)\} \quad (2.7)$$

The index of the uncertainty vector $\Gamma(x, \gamma)$ with respect to the nominal system is simply the number of times the system output must be differentiated with respect to time before the first appearance of the uncertainty terms.

2.3 NONLINEARITIES

Nonlinearities can be classified as inherent (natural) and intentional (artificial). Examples of inherent nonlinearities include centripetal forces in rotational motion, and

Coulomb friction between contacting surfaces. Usually, such nonlinearities have undesirable effects, and controllers have to properly compensate for them. Intentional nonlinearities, on the other hand, are artificially introduced by the designer. Nonlinear control laws, such as adaptive control laws and variable structure control laws, are typical examples of intentional nonlinearities. Nonlinearities can also be classified in terms of their mathematical properties, as continuous and discontinuous. Because discontinuous nonlinearities cannot be locally approximated by linear functions, they are also called 'hard' nonlinearities. Hard nonlinearities (*e.g.*, backlash and stiction) are commonly found in control systems, both in small range operation and large range operation. Whether a system in small range operation should be regarded as nonlinear or linear depends on the magnitude of the hard nonlinearities and on the extent of their effects on the system performance.

The behaviour of nonlinear systems, however, is much more complex than that of linear systems. Because of the lack of linearity and of the associated superposition property, nonlinear systems may respond to external inputs quite differently from linear systems. Nonlinear systems frequently have more than one equilibrium point, so different initial conditions could lead to different steady state conditions. Furthermore the stability of nonlinear systems may depend on the initial conditions. Nonlinear systems can display oscillations of fixed amplitude and fixed period without external excitation. These oscillations are called limit cycles, which are different from sustained oscillations in marginally stable linear systems, in that the amplitude of the self-sustained excitation is independent of the initial condition, and not easily affected by parameter changes. Nonlinear systems can also display a phenomenon called chaos, by which we mean that the system output is extremely sensitive to initial conditions. The essential feature of chaos is the unpredictability of the system output. Even if we have an exact model of a nonlinear system and an extremely accurate computer, the system's response in the long-run still cannot be well predicted. Some other interesting types of behaviour, such as jump

resonance, subharmonic generation, asynchronous quenching, and frequency-amplitude dependence of free vibrations, can also occur and are important in some system studies. The above description should provide ample evidence that nonlinear systems can exhibit considerably richer and more complex behaviour than linear systems.

2.4 UNCERTAINTIES

2.4.1 Description of Uncertainties

Uncertainties arise from practical control problems. A 'real world' physical plant contains very complex dynamics, and is also affected by the environment in which it works. When an attempt is made to control a plant, it is desirable to describe it from prior knowledge in mathematical terms. No nominal model should be considered without an assessment of its errors. This is because: ① Our knowledge of the physical mechanisms of the plant is limited, and it is not possible to obtain all the desired information about plant dynamics. ② Our ability to represent the physical mechanisms of the plant is so limited that we could not formulate all dynamics of the plant without any error. It is, for example, difficult to model the high-frequency dynamics of a plant. ③ It should also be considered, to what extent the model can be dealt with by theories presently available. It is common that a quite accurately modelled nonlinear element is treated as a linear one, or a quite complicated model is replaced by a simple one because our design techniques cannot deal with complex models effectively, and sometimes, we may deliberately choose to ignore various known dynamics in order to achieve a simple nominal model. We call these errors 'the model uncertainties'. The discrepancy between the plant and its model, *i.e.*, model error, is one of the most important uncertainties in control problems. From a control point of view, model inaccuracies can be classified into two kinds: structured (or parametric) uncertainties and unstructured uncertainties (or unmodelled dynamics). The first kind

corresponds to inaccuracies in the terms actually included in the model, while the second kind corresponds to inaccuracies in (*i.e.*, underestimation of) the system order for linear cases and to inaccuracies in the number and the type of terms of the model for nonlinear cases. Another kind of uncertainty arises from external disturbances. The variations of the plant environment will affect the plant dynamic characteristics. The disturbances are either deterministic but unknown, or stochastic, but most of them are not exactly measurable, hence unmodelled.

2.4.2 Requirements on the Uncertainties: Matching Conditions

The control of systems which contain uncertainties can in general be treated in two different ways: from a stochastic point of view or from a deterministic one. Where the deterministic technique is used the uncertainties are described only in terms of bounds, *i.e.*, the maximum and minimum values, and no assumptions are made concerning the statistics of the uncertain parameters. Instead, the uncertainties may satisfy some prespecified conditions, such as matching conditions, which require that they must lie in the span of the nominal input mapping $g(\cdot)$.

DEFINITION 2.2. (*Matching Assumption*)

For the nonlinear uncertain system of the form (2.3), if the uncertainty vector fields $\Delta f(x, \gamma, t)$ and $\Delta g(x, \gamma, t)$ satisfy

$$\Delta f(x, \gamma, t) \text{ and } \Delta g(x, \gamma, t) \in \text{span}\{g(x)\} \quad (2.8)$$

it is said that the system has matched uncertainties.

DEFINITION 2.3. (*Generalised Matching Assumption*)

Assume that the nominal part of the system (2.3) has relative order ν , and the uncertainties $\Delta f(x, \gamma, t)$ and $\Delta g(x, \gamma, t)$ are smooth vector fields with indices ν_1 and ν_2 . If

$$\min\{\nu_1, \nu_2\} \geq \nu - 1 \quad (2.9)$$

then it is said that the uncertainties satisfy generalised matching conditions.

REMARK 2.1:

- By uncertainties satisfying matching conditions, it is meant that they enter the system only through the nominal input mapping of the system. It is worthwhile to point out that definition 2.3 is a generalisation of the so-called matching assumption of definition 2.2.

In order to develop the results of the following chapters, we introduce the more intuitive form on matching conditions made by Barmish *et al*^[2], and some other relaxed versions made in this thesis.

DEFINITION 2.4. (*Matching Conditions*)

For the system of the form (2.3), suppose there exist continuous functions, $p(x, \gamma, t)$ and $q(x, \gamma, t)$, such that the uncertain vectors can be expressed as

$$\Delta g(x, \gamma, t) = g(x) \cdot q(x, \gamma, t) \quad (2.10)$$

$$\Delta f(x, \gamma, t) = g(x) \cdot p(x, \gamma, t) \quad (2.11)$$

The system is then said to satisfy the complete matching conditions.

DEFINITION 2.5. (*Incomplete Matching Conditions*)

For the system of the form (2.3), suppose there exists a continuous function $q(x, \gamma, t)$ such that

$$\Delta g(x, \gamma, t) = g(x) \cdot q(x, \gamma, t) \quad (2.12)$$

or $p(x, \gamma, t)$, such that

$$\Delta f(x, \gamma, t) = g(x) \cdot p(x, \gamma, t) \quad (2.13)$$

hold. The system is then said to meet incomplete matching conditions.

DEFINITION 2.6. (*Mismatched Uncertainties*)

For the system of the form (2.3), if there are no continuous functions $q(x, \gamma, t)$ and $p(x, \gamma, t)$ such that (2.10) or (2.11) holds. The system is then said to have mismatched uncertainties.

In order to achieve desired control, some further conditions must be imposed on the system uncertainties as follows.

ASSUMPTION 2.7. (Conditions on the Uncertainty in Input Mapping)

For the nonlinear uncertain system of the form (2.3), if the uncertainty in input mapping is matched, then it is also assumed that the function q satisfies either of the following conditions

$$|q(x, \gamma, t)| < 1 \quad (2.14)$$

$$q(x, \gamma, t) > 0 \quad (2.15)$$

Otherwise, if the uncertainty in input mapping is mismatched, then it is assumed that either of the following conditions

$$L_g V \cdot L_{\Delta g} V > 0 \quad (2.16)$$

or

$$|L_g V| > |L_{\Delta g} V| \quad (2.17)$$

holds.

REMARK 2.2:

- In assumption 2.7, it is clear that condition (2.15) can be regarded as the matched form of (2.16), because if Δg satisfies the matching conditions, then $\Delta g = g \cdot q$, and it follows that $L_g V \cdot L_{\Delta g} V = (L_g V)^2 \cdot q > 0$, so $q > 0$ holds.
- Similarly, if Δg satisfies the matching conditions, then condition (2.17) becomes $|L_g V| > |L_{\Delta g} V| = |L_g V| \cdot |q|$, implying that $|q| < 1$ holds. So condition (2.17) can be regarded as the mismatched extension of condition (2.14).

2.4.3 A Further Discussion of Matching Conditions and an Example

The matching conditions are the basis of robust control of uncertain systems at present. Most proposed control techniques for uncertain systems use these conditions. For clarity of exposition, we use a simple linear system to discuss these conditions further and make some important observations about the structure of the uncertainties.

Consider the following second order linear system with transfer function

$$G(s) = \frac{k_1 s + k_2}{(s + \mu_1)(s + \mu_2)} \quad (2.18)$$

The state variable form is as follows

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} u(t) \quad (2.19)$$

$$y(t) = x_1$$

where if

$$\alpha = \mu_1 + \mu_2 \quad (2.20)$$

$$\beta = \mu_1 \mu_2 \quad (2.21)$$

then

$$a_{11} + a_{22} = -\alpha \quad (2.22)$$

$$a_{12} a_{21} - a_{11} a_{22} = -\beta \quad (2.23)$$

$$a_{12} b_2 - a_{22} b_1 = k_2 \quad (2.24)$$

$$b_1 = k_1 \quad (2.25)$$

Case 1. If $a_{11}=0$; $a_{12}=1$; $k_1=0$, then $a_{21}=-\beta$, $a_{22}=-\alpha$, $b_1=0$, $b_2=k_2$, therefore

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\alpha & -\beta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ k_2 \end{pmatrix} u(t) \quad (2.26)$$

If the open loop pole assumed to be at $-\mu_1$ is in fact at $-\mu'_1$, and if the value of the numerator coefficient of s is k'_2 rather than k_2 , the system may be regarded as uncertain and of the following form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\alpha & -\beta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ k_2 \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ \Delta\alpha x_1 + \Delta\beta x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \Delta k_2 \end{pmatrix} u(t) \quad (2.27)$$

where $\Delta\alpha = (\alpha - \alpha')$, $\Delta\beta = (\beta - \beta')$, and $\Delta k_2 = k'_2 - k_2$.

Clearly, the uncertainties here do lie in the span of the input mapping $g(x)$, so that the existence of matched uncertainties can be concluded. But in order to deal with such a system with input mapping uncertainty, more restrictions described in assumption 2.7 must be placed on the uncertainties, *i.e.*, $|q| = |\Delta k_2 / k_2| < 1$ or $\Delta k_2 / k_2 > 0$ is required. We may satisfy the condition, $\Delta k_2 / k_2 > 0$, by expressing the input mapping as $k'_2 = k_2 + \Delta k_2$ as above, and choosing k_2 properly such that $q = \Delta k_2 / k_2 > 0$ holds.

Case 2. If $a_{12}=a_{21}=a \neq 0$, $\alpha^2 \geq 4(\beta+a^2)$ and $k_2=1$, then

$$a_{11} = \frac{-\alpha - \sqrt{\alpha^2 - 4(\beta + a^2)}}{2} \quad (2.28)$$

$$a_{22} = \frac{-\alpha + \sqrt{\alpha^2 - 4(\beta + a^2)}}{2} \quad (2.29)$$

$$b_1 = k_1 \quad (2.30)$$

$$b_2 = \frac{(k_2 + a_{22}k_1)}{a_{12}} \quad (2.31)$$

Again, uncertainty in one of the open loop pole positions results in uncertainty in f . Supposing $k_1=0$, we have

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a \\ a & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ b_2 \end{pmatrix} u(t) + \begin{pmatrix} \Delta a_{11}x_1 \\ \Delta a_{22}x_2 \end{pmatrix} \quad (2.32)$$

where a_{11} , a_{22} , b_2 are nominal values resulting in the nominal eigenvalue $-\mu_1$, a'_{11} , a'_{22} are real values resulting in the true eigenvalue $-\mu'_1$, and $\Delta a_{11}=a'_{11}-a_{11}$, $\Delta a_{22}=a'_{22}-a_{22}$ are the uncertain parameters.

If the assumed open loop pole position $-\mu_1$ is correct, but $k_1 < 0$, this results in a non-minimum phase control problem. It may be regarded as an uncertain problem and the following system results

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a \\ a & a_{21} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ b_2 \end{pmatrix} u(t) + \begin{pmatrix} \Delta b_1 \\ \Delta b_2 \end{pmatrix} u(t) \quad (2.33)$$

where $b_2=1/a$, $\Delta b_1=k_1$, and $\Delta b_2=a_{22}k_1/a$.

Clearly, in both cases, the uncertainties do not satisfy the matching conditions given in definitions 2.4, but each case does satisfy the conditions of definition 2.5. It is therefore said that the system has partially matched uncertainties.

If an open loop pole is not in the nominal position $-\mu_1$, and non-minimum phase occurs, *i.e.*, $k_1 < 0$, on the one hand, the system can be written as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a \\ a & a_{21} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} u(t) + \begin{pmatrix} \Delta a_{11}x_1 \\ \Delta a_{22}x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \Delta b_2 \end{pmatrix} u(t) \quad (2.34)$$

where $b_1=k_1$, $b_2=1/a$, and $\Delta b_2=a_{22}k_1/a$. On the other hand, it can also be expressed as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a \\ a & a_{21} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ b_2 \end{pmatrix} u(t) + \begin{pmatrix} \Delta a_{11} x_1 \\ \Delta a_{22} x_2 \end{pmatrix} + \begin{pmatrix} \Delta b_1 \\ \Delta b_2 \end{pmatrix} u(t) \quad (2.35)$$

where $b_2=1/a$, $\Delta b_1=k_1$, and $\Delta b_2=a_{22}k_1/a$.

Although the uncertainties in (2.34) lie in the range of the nominal input mapping $g(x)$ and generalised matched uncertainties can be concluded, there do not exist functions p and q such that the conditions (2.10) and (2.11) hold. However, they can be treated as mismatched uncertainties as in (2.35) if either of the conditions (2.16) and (2.17) given in assumption 2.7 is satisfied.

2.5 SUMMARY

The matching conditions play a key role in various robust synthesis techniques, and mismatched uncertainties are much more difficult to deal with than matched uncertainties. In terms of these definitions, the system (2.3) can be expressed in various forms as follows:

① Matched Uncertainties:

$$\dot{x}(t) = f(x) + g(x)u(t) + g(x) \cdot \{ p(x, \gamma, t) + q(x, \gamma, t)u(t) \} \quad (2.36)$$

② Partially Matched Uncertainties:

$$\dot{x}(t) = f(x) + g(x)u(t) + \Delta f(x, \gamma, t) + g(x) \cdot q(x, \gamma, t)u(t) \quad (2.37)$$

or
$$\dot{x}(t) = f(x) + g(x)u(t) + g(x) \cdot p(x, \gamma, t) + \Delta g(x, \gamma, t)u(t) \quad (2.38)$$

③ Mismatched Uncertainties:

$$\dot{x}(t) = f(x) + g(x)u(t) + \Delta f(x, \gamma, t) + \Delta g(x, \gamma, t)u(t) \quad (2.39)$$

The objective of robust control theory is to find a family of controllers for nonlinear uncertain systems, subject to various uncertainties either matched, partially matched or mismatched, which guarantees that any given system has desired stability properties for any initial condition $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$ and all uncertain elements $\gamma(t) \in \mathbb{R}^l$.

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Lyapunov Stability Theory



Overview

This chapter presents some basic results of Lyapunov stability theory, and gives some extended concepts on system boundedness.



Outline

- ✓ Introduction
- ✓ Lyapunov Stability Theory
- ✓ Extension of the Lyapunov Method
- ✓ Summary

3.1 INTRODUCTION

GIVEN a control system, the first and most important question about its various properties is whether it is stable, because an unstable control system is typically useless and potentially dangerous. Stability properties characterise how a system behaves if its state is initiated close to, but not precisely at, a given operating point. Qualitatively, a system is described as stable if, initiating the system somewhere away from, but near, its desired operating point, implies that it will stay around the point ever after, unless disturbed, in which case it will, after the effect of the disturbance has passed, tend to the region of the operating point.

The most useful and general approach for studying the stability of nonlinear systems is the theory introduced in the late 19th century by the Russian mathematician A.M.Lyapunov. Lyapunov's work, *The General Problem of Motion Stability*, introduces two methods for stability analysis (the so-called linearisation method and the direct method) and was first published in 1892. The linearisation method draws conclusions about a nonlinear system's local stability around an equilibrium point from the stability properties of its linear approximation, while the direct method is not restricted to local motion, and determines the stability properties of a system by constructing a scalar function for the system and examining the function's time variation. For over half a century, however, Lyapunov's pioneering work on stability received little attention outside Russia. Many refinements of Lyapunov's methods have since been made. Today, Lyapunov's linearisation method has come to represent the theoretical justification of linear system theory, whilst Lyapunov's direct method has become the most important tool for nonlinear system analysis and design. Together, the linearisation method and the direct method constitute the so-called Lyapunov stability theory.

The objective of this chapter is to provide the basic mathematical preliminaries for the development of the main results in the following chapters. To avoid excessive mathematical complexity, this chapter presents only the major concepts of Lyapunov stability theory, and some extended results on system stability frequently used in the analysis and design of nonlinear uncertain systems.

3.2 LYAPUNOV STABILITY THEORY

Basic Lyapunov theory consists of two methods, the indirect method and the direct method. The indirect method, or linearisation method, states that the stability properties of many nonlinear systems in the vicinity of an equilibrium point are essentially the same as those of its linearised approximation. The method serves as the theoretical justification for applying linear theory to physical systems, which are always inherently nonlinear. In using the direct method to analyse the stability of a nonlinear system, the idea is to construct a scalar 'energy-like' function (a Lyapunov function) for the system, and to see whether it decreases. The power of this method comes from its generality; it is applicable to all kinds of control systems, be they time-varying or time-invariant, finite dimensional or infinite dimensional. Conversely, the limitation of the method lies in the fact that it is often difficult to find a Lyapunov function for a given system, and that sufficient conditions are not generally necessary conditions.

3.2.1 Concepts of Stability

Some concepts of system stability and instability are now introduced.

DEFINITION 3.1. (*Equilibrium Point*)

Given a dynamic system of the form

$$\dot{x}(t) = F(x) \quad (3.1)$$

a state x^* is an equilibrium state (or equilibrium point) of the system if once $x(t)$ becomes equal to x^* , it remains equal to x^* for all future time.

DEFINITION 3.2. (*Autonomous and Non-autonomous Systems*)

A nonlinear system is said to be autonomous if it does not depend explicitly on time. Otherwise the system is called non-autonomous.

The fundamental difference between autonomous and non-autonomous systems lies in the fact that the state trajectory of an autonomous system is independent of the initial time, while that of a non-autonomous system is generally not. This clearly makes the stability analysis of non-autonomous systems more complicated than that of autonomous systems.

DEFINITION 3.3. (*Stability*^[1,3])

Given a non-autonomous system as follows

$$\dot{x}(t) = F(x, t) \quad F(0, t) = 0 \quad (3.2)$$

consider the stability problem in the vicinity of the equilibrium point $x=0$.

① If, for any $R>0$, there exists $r(R, t_0)>0$ depending only on R and t_0 , such that

$$\|x(0)\| < r(R, t_0) \Rightarrow \|x(t)\| < R \quad \forall t \geq t_0 \quad (3.3)$$

then the equilibrium is said to be stable.

② If $x=0$ is stable, and $r(R)>0$ is independent of initial time t_0 , then the equilibrium is said to be uniformly stable.

③ If $x=0$ is stable, and there exists $r(t_0)>0$ such that

$$\|x(0)\| < r(t_0) \Rightarrow \|x(t)\| \rightarrow 0 \text{ as } t \rightarrow \infty \quad (3.4)$$

then the equilibrium is said to be asymptotically stable.

④ If $x=0$ is stable, and there exists $r>0$ independent of initial time t_0 , such that

$$\|x(0)\| < r \Rightarrow \|x(t)\| \rightarrow 0 \text{ as } t \rightarrow \infty \quad (3.5)$$

then the equilibrium is said to be uniformly asymptotically stable.

⑤ If $x=0$ is stable, and for any initial state $x(0)=x_0$, such that

$$\|x(t)\| \rightarrow 0 \text{ as } t \rightarrow \infty \quad (3.6)$$

then the equilibrium is said to be globally uniformly asymptotically stable.

⑥ If there exists $\alpha > 0$, and also for any $R > 0$, there exists $r(R) > 0$ such that

$$\|x(0)\| < r(R) \Rightarrow \|x(t)\| \leq R e^{-\alpha(t-t_0)} \quad \forall t \geq t_0 \quad (3.7)$$

then the equilibrium is said to be exponentially stable.

⑦ If there exists $\alpha > 0$, and also for any $r > 0$, there exists $R(r) > 0$ such that

$$\|x(0)\| < r \Rightarrow \|x(t)\| \leq R(r) e^{-\alpha(t-t_0)} \quad \forall t \geq t_0 \quad (3.8)$$

then the equilibrium is said to be globally exponentially stable.

⑧ If, for some $R > 0$, there exists $r > 0$ no matter how small r is, such that

$$\|x(0)\| < r \Rightarrow \|x(t)\| \geq R \quad \forall t \geq t_0 \quad (3.9)$$

then the equilibrium is said to be unstable.

REMARK 3.1:

- Essentially, stability (also called Lyapunov stability) means that the system trajectory can be kept arbitrarily close to the origin by starting sufficiently close to it. Asymptotic stability means that the equilibrium point is stable, and that in addition, states started close to 0 actually converge to 0 as time t goes to infinity. An equilibrium point which is Lyapunov stable but not asymptotically stable is called marginally stable. Exponential stability means that the state vector of a system converges to 0 faster than a given exponential function with constants α and R .
- Uniform stability means that the stability property of a system is independent of the initial time t_0 , so the uniform stability of a non-autonomous system is equivalent to the stability of an autonomous one.

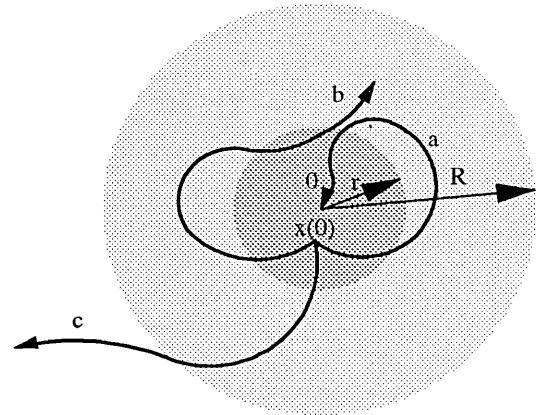


Fig. 3.1 Concepts of stability

- a — asymptotically stable
- b — marginally stable
- c — unstable

- Finally, global stability means that the stability property holds for any initial state x_0 , i.e., the whole state space. In contrast, local stability is only concerned with a finite domain around the equilibrium point $x=0$.

3.2.2 The Direct Method of Lyapunov

The direct method of Lyapunov attempts to make a statement on the stability of the equilibrium directly without any knowledge of the solutions of the system. The basic philosophy of Lyapunov's direct method is the mathematical extension of a fundamental physical observation; if the total energy of a system is continuously dissipated, then the system, whether linear or nonlinear, must eventually settle down to an equilibrium point. Thus we may infer the stability of a system by examining the variation of a single 'energy-like' scalar function without requiring explicit knowledge of solutions. This energy function has two properties. The first is a property of the function itself; it is strictly positive unless all state variables are zero. The second is a property associated with the system dynamics; the function is monotonically decreasing when the states vary along the system dynamics. The first property is formalised by the notion of positive definite functions, and the second is formalised by the so-called Lyapunov functions.

DEFINITION 3.4. (*Positive Definite Function*^[1,4])

If, for any vector x , a scalar continuous function $V(x)$ is such that

$$V(x) > 0 \quad \forall x \neq 0; \quad V(0) = 0 \quad (3.10)$$

then it is said to be positive definite (p.d.).

A few concepts, such as negative definite, positive (negative) semi-definite, can be defined similarly.

DEFINITION 3.5. (*Lyapunov Function*)

If the function $V(x)$ is positive definite and has continuous first partial derivatives with respect to x , and if its time derivative along any state trajectory of the system is

negative semi-definite, i.e.,

$$V(x) > 0 \quad \forall x \neq 0 \quad (3.11)$$

$$\dot{V}(x) \leq 0 \quad (3.12)$$

then $V(x)$ is said to be a Lyapunov function for the system.

THEOREM 3.6. (*The Direct Method of Lyapunov*)

Assume that there exists a scalar function $V(x)$

$$V(x) > 0 \quad \forall x \neq 0 \quad (3.13)$$

with continuous first partial derivatives. Then, $\forall x \neq 0$

① if the time derivative

$$\dot{V}(x) \leq 0 \quad (3.14)$$

it follows that the equilibrium at the origin is stable;

② if the time derivative

$$\dot{V}(x) < 0 \quad (3.15)$$

and

$$V(x, t) \rightarrow \infty \quad \text{as } \|x\| \rightarrow \infty \quad (3.16)$$

then the equilibrium at the origin is asymptotically stable in the large^①.

REMARK 3.2:

- Many Lyapunov functions may exist for the same system. For instance, if V is a Lyapunov function for a given system, so is $V' = b \cdot V^a$, where b is any strictly positive constant and a is any scalar (not necessarily an integer) greater than or equal to one. More importantly, for a given system, a specific Lyapunov function may yield more precise results than other choices.
- It is important to realise that the theorems of Lyapunov are all sufficiency theorems. If for a particular choice of Lyapunov function candidate V , the conditions on \dot{V} are not met, one cannot draw any conclusion on the stability or instability of the system.
- Usually, Lyapunov stability theorems have local and global versions. The local versions are concerned with stability properties in the vicinity of the equilibrium point and usually involve a

^① In the large means that the domain of definition is the entire state space.

locally positive definite function, whilst the global version satisfies all the conditions of the local versions, and needs additional requirements on the function, *i.e.*, $V(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$.

THEOREM 3.7. (Necessary and Sufficient Conditions for Exponential Stability⁽¹⁾)

Given a system of the form (3.2), if $F(x,t)$ has continuous and bounded first partial derivatives with respect to x and t , for x in a certain ball B_K centered at the origin, and all $t \geq 0$, then the equilibrium point at the origin is exponentially stable if and only if there exists a Lyapunov function $V(x,t)$ and some positive constants v_1, v_2, v_3, v_4 such that for $x \in B_K$ and $\forall t \geq 0$

$$v_1 \|x\|^2 \leq V(x,t) \leq v_2 \|x\|^2 \quad (3.17)$$

$$\dot{V}(x,t) \leq -v_3 \|x\|^2 \quad (3.18)$$

$$\left\| \frac{\partial V}{\partial x} \right\| \leq v_4 \|x\| \quad (3.19)$$

REMARK 3.3:

- In this theorem, the $v_i \|x\|$ can be replaced by class- k_∞ functions^② $v_i(\|x\|)$, and the system is still globally exponentially stable.
- The theorem provides us with necessary and sufficient conditions for a Lyapunov function to exist, so that it can be used as a converse theorem to examine the existence of a Lyapunov function. This means that, for an exponentially stable system, if there are class- k_∞ functions v_1, v_2, v_3, v_4 , then a Lyapunov function which satisfies conditions (3.17)–(3.19) exists.

3.2.3 Existence and Construction of Lyapunov Functions

All theorems of Lyapunov theory make a basic assumption; an explicit Lyapunov function is somehow known. The question is therefore how to find a Lyapunov function for a specific problem. Yet there is no general way of finding Lyapunov functions for

^② DEFINITION (*class- k_∞ function*)⁽¹⁾:

If $v(\cdot)$ is a strictly continuous non-decreasing function, and satisfies $v(0)=0$ and $\lim_{\epsilon \rightarrow \infty} v(\epsilon)=\infty$, then it can be written as $v(\cdot) \in k_\infty$ and called a class- k_∞ function.

nonlinear systems. This is a fundamental drawback of the direct method. Therefore, faced with specific systems, one has to use experience, intuition, and physical insights to search for an appropriate Lyapunov function.

THEOREM 3.8. (*Lyapunov Function for Linear Time-invariant Systems^[1,3,4]*)

Given a linear system of the form $\dot{x}=Ax$, a quadratic function

$$V(x) = x^T P x \quad (3.20)$$

is a Lyapunov function, if P is a symmetric positive definite matrix satisfying

$$A^T P + P A = -Q \quad (3.21)$$

where Q is a symmetric positive definite matrix.

Obviously, if P is positive definite, then $V(x) > 0 \forall x \neq 0$, and it follows that

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} = -x^T Q x < 0 \quad \forall x \neq 0 \quad (3.22)$$

if Q is positive definite. The global asymptotic stability of the linear system is therefore guaranteed. One way of constructing a Lyapunov function is to derive P from a chosen positive definite matrix Q . Any positive definite matrix Q can be used to determine the stability of a linear system. A simple choice of Q is the identity matrix.

In some circumstances, instead of choosing Q to be the identity matrix, a special form of matrix P may be assumed such that the chosen Lyapunov function can meet certain requirements. The following theorem states that a positive definite diagonal matrix P exists.

THEOREM 3.9. (*Lyapunov Function with a Diagonal Matrix P*)

For a linear system $\dot{x}=Ax$, a positive definite diagonal matrix P satisfying (3.21) can be found

① if A is generally of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \quad (3.23)$$

with all the diagonal elements non-zero; or

② if A is of controllable canonical form

$$A = \begin{pmatrix} 0 & 1 & \cdot & 0 \\ \vdots & \cdot & \cdot & \cdot \\ \vdots & 0 & \cdot & 1 \\ a_1 & a_2 & \cdot & \cdot & a_n \end{pmatrix} \quad (3.24)$$

a transformation \mathfrak{S} can be defined such that a new system with the state matrix

$$\tilde{A} = \mathfrak{S}A\mathfrak{S}^{-1} \quad (3.25)$$

can be obtained, where all the diagonal elements of \tilde{A} are non-zero. In both cases, a Lyapunov function of the form (3.20) can be defined.

Actually, a transformation of the following form

$$\mathfrak{S} = \begin{pmatrix} \tau_1 & & & 0 \\ \tau_1, \tau_2 & & & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \tau_1, \tau_2, \dots, \tau_n \end{pmatrix} \quad (3.26)$$

with inverse of the form

$$\mathfrak{S}^{-1} = \begin{pmatrix} \tau_1^{-1} & & & 0 \\ -\tau_2^{-1}, \tau_2^{-1} & & & \\ \cdot & \cdot & \cdot & \\ 0 & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ -\tau_n^{-1}, \tau_n^{-1} \end{pmatrix} \quad (3.27)$$

can be defined, such that a matrix of the form (3.24) can be transformed into

$$\tilde{A} = \mathfrak{S}A\mathfrak{S}^{-1} = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & & 0 \\ \tilde{a}_{21} & \tilde{a}_{22} & \tilde{a}_{23} & \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \tilde{a}_{n1} & \dots & \dots & \tilde{a}_{nn} \end{pmatrix} \quad (3.28)$$

where all diagonal elements $\tilde{a}_{ii} \neq 0$ ($i=1, 2, \dots, n$) by properly arranging the elements of \mathfrak{S} .

If the elements of \mathfrak{S} are chosen to be

$$\tau_i = a_i \tau_n / a_n \quad (i=1, 2, \dots, n-1) \quad (3.29)$$

where τ_n may be any positive constant, a diagonal matrix P can then be defined as

$$P_{ii} = -Q_{ii} / 2\tilde{a}_{ii} \quad (i=1, 2, \dots, n) \quad (3.30)$$

where Q is a symmetric positive definite matrix, whose diagonal elements are given by

$$Q_{jj} = -\frac{\tilde{a}_{ij} \cdot \tilde{a}_{jj}}{\tilde{a}_{ji} \cdot \tilde{a}_{ii}} \times Q_{ii} \quad (i=1, 2, \dots, n-1, j=i+1) \quad (3.31)$$

with Q_{11} a positive constant, such that P is positive definite and satisfies the Lyapunov equation (3.21).

The following example demonstrates the application of the theorem.

Given a 5th-order linear stable system with the state matrix of the form

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ a_1 & a_2 & a_3 & a_4 & a_5 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1800 & -2250 & -1168 & -293 & -32 \end{pmatrix}$$

define a transformation of the form (3.26) with the inverse (3.27) as follows

$$S = \begin{pmatrix} \tau_1 & 0 & 0 & 0 & 0 \\ \tau_1 & \tau_2 & 0 & 0 & 0 \\ \tau_1 & \tau_2 & \tau_3 & 0 & 0 \\ \tau_1 & \tau_2 & \tau_3 & \tau_4 & 0 \\ \tau_1 & \tau_2 & \tau_3 & \tau_4 & \tau_5 \end{pmatrix} \quad S^{-1} = \begin{pmatrix} 1/\tau_1 & 0 & 0 & 0 & 0 \\ -1/\tau_2 & 1/\tau_2 & 0 & 0 & 0 \\ 0 & -1/\tau_3 & 1/\tau_3 & 0 & 0 \\ 0 & 0 & -1/\tau_4 & 1/\tau_4 & 0 \\ 0 & 0 & 0 & -1/\tau_5 & 1/\tau_5 \end{pmatrix}$$

The original state matrix can be transformed into the following form

$$\begin{aligned} \tilde{A} = SAS^{-1} &= \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & 0 & 0 & 0 \\ \tilde{a}_{21} & \tilde{a}_{22} & \tilde{a}_{23} & 0 & 0 \\ \tilde{a}_{31} & \tilde{a}_{32} & \tilde{a}_{33} & \tilde{a}_{34} & 0 \\ \tilde{a}_{41} & \tilde{a}_{42} & \tilde{a}_{43} & \tilde{a}_{44} & \tilde{a}_{45} \\ \tilde{a}_{51} & \tilde{a}_{52} & \tilde{a}_{53} & \tilde{a}_{54} & \tilde{a}_{55} \end{pmatrix} \\ &= \begin{pmatrix} -\tau_1/\tau_2 & \tau_1/\tau_2 & & & \\ -\tau_1/\tau_2 & \tau_1/\tau_2 - \tau_2/\tau_3 & & & \\ -\tau_1/\tau_2 & \tau_1/\tau_2 - \tau_2/\tau_3 & & & \\ -\tau_1/\tau_2 & \tau_1/\tau_2 - \tau_2/\tau_3 & & & \\ -\tau_1/\tau_2 + (a_1/\tau_1 - a_2/\tau_2) & \tau_1/\tau_2 - \tau_2/\tau_3 + (a_2/\tau_2 - a_3/\tau_3) & & & \\ 0 & 0 & 0 & & \\ \tau_2/\tau_3 & 0 & 0 & & \\ \tau_2/\tau_3 - \tau_3/\tau_4 & \tau_3/\tau_4 & 0 & & \\ \tau_2/\tau_3 - \tau_3/\tau_4 & \tau_3/\tau_4 - \tau_4/\tau_5 & \tau_4/\tau_5 & & \\ \tau_2/\tau_3 - \tau_3/\tau_4 + (a_3/\tau_3 - a_4/\tau_4) & \tau_3/\tau_4 - \tau_4/\tau_5 + (a_4/\tau_4 - a_5/\tau_5) & \tau_4/\tau_5 + a_5 & & \end{pmatrix} \end{aligned}$$

It is possible to choose τ_i ($i=1,2,3,4,5$) such that the elements in the bracket of the above matrix are zero

$$\frac{a_1}{\tau_1} - \frac{a_2}{\tau_2} = \frac{a_2}{\tau_2} - \frac{a_3}{\tau_3} = \frac{a_3}{\tau_3} - \frac{a_4}{\tau_4} = \frac{a_4}{\tau_4} - \frac{a_5}{\tau_5} = 0$$

It follows that, for a positive constant τ_5 (here let $\tau_5=1$)

$$\tau_4 = a_4 \cdot \tau_5 / a_5 = 9.156$$

$$\tau_3 = a_3 \cdot \tau_4 / a_4 = a_3 / a_4 \times a_4 \cdot \tau_5 / a_5 = a_3 \cdot \tau_5 / a_5 = 36.500$$

$$\tau_2 = a_2 \cdot \tau_3 / a_3 = a_2 / a_3 \times a_3 \cdot \tau_5 / a_5 = a_2 \cdot \tau_5 / a_5 = 70.313$$

$$\tau_1 = a_1 \cdot \tau_2 / a_2 = a_1 / a_2 \times a_2 \cdot \tau_5 / a_5 = a_1 \cdot \tau_5 / a_5 = 56.250$$

The transformation is therefore obtained as follows

$$\mathfrak{S} = \begin{pmatrix} 56.250 & 0 & 0 & 0 & 0 \\ 56.250 & 70.313 & 0 & 0 & 0 \\ 56.250 & 70.313 & 36.500 & 0 & 0 \\ 56.250 & 70.313 & 36.500 & 9.156 & 0 \\ 56.250 & 70.313 & 36.500 & 9.156 & 1.000 \end{pmatrix}$$

$$\mathfrak{S}^{-1} = \begin{pmatrix} 0.018 & 0 & 0 & 0 & 0 \\ -0.014 & 0.014 & 0 & 0 & 0 \\ 0 & -0.027 & 0.027 & 0 & 0 \\ 0 & 0 & -0.109 & 0.109 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

by which the original system can be transferred to the following form

$$\tilde{A} = \mathfrak{S} A \mathfrak{S}^{-1} = \begin{pmatrix} -\tau_1/\tau_2 & \tau_1/\tau_2 & 0 & 0 & 0 \\ -\tau_1/\tau_2 & \tau_1/\tau_2 - \tau_2/\tau_3 & \tau_2/\tau_3 & 0 & 0 \\ -\tau_1/\tau_2 & \tau_1/\tau_2 - \tau_2/\tau_3 & \tau_2/\tau_3 - \tau_3/\tau_4 & \tau_3/\tau_4 & 0 \\ -\tau_1/\tau_2 & \tau_1/\tau_2 - \tau_2/\tau_3 & \tau_2/\tau_3 - \tau_3/\tau_4 & \tau_3/\tau_4 - \tau_4/\tau_5 & \tau_4/\tau_5 \\ -\tau_1/\tau_2 & \tau_1/\tau_2 - \tau_2/\tau_3 & \tau_2/\tau_3 - \tau_3/\tau_4 & \tau_3/\tau_4 - \tau_4/\tau_5 & \tau_4/\tau_5 + a_4 \end{pmatrix}$$

$$= \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & 0 & 0 & 0 \\ \tilde{a}_{11} & \tilde{a}_{22} & \tilde{a}_{23} & 0 & 0 \\ \tilde{a}_{11} & \tilde{a}_{22} & \tilde{a}_{33} & \tilde{a}_{34} & 0 \\ \tilde{a}_{11} & \tilde{a}_{22} & \tilde{a}_{33} & \tilde{a}_{44} & \tilde{a}_{45} \\ \tilde{a}_{11} & \tilde{a}_{22} & \tilde{a}_{33} & \tilde{a}_{44} & \tilde{a}_{55} \end{pmatrix}$$

$$= \begin{pmatrix} -0.800 & 0.800 & 0 & 0 & 0 \\ -0.800 & -1.126 & 1.926 & 0 & 0 \\ -0.800 & -1.126 & -2.060 & 3.986 & 0 \\ -0.800 & -1.126 & -2.060 & -5.170 & 9.156 \\ -0.800 & -1.126 & -2.060 & -5.170 & -22.874 \end{pmatrix}$$

From the results, all the diagonal elements of the above matrix \tilde{A} are non-zero. It is now possible to define a positive definite diagonal matrix P such that the Lyapunov equation (3.21) is satisfied.

According to (3.31), let $Q_{11}=3$. Then we have

$$Q_{22} = -\frac{\tilde{a}_{12} \cdot \tilde{a}_{22}}{\tilde{a}_{21} \cdot \tilde{a}_{11}} \times Q_{11} = 4.224 \quad Q_{33} = -\frac{\tilde{a}_{23} \cdot \tilde{a}_{33}}{\tilde{a}_{32} \cdot \tilde{a}_{22}} \times Q_{22} = 13.212$$

$$Q_{44} = -\frac{\tilde{a}_{34} \cdot \tilde{a}_{44}}{\tilde{a}_{43} \cdot \tilde{a}_{33}} \times Q_{33} = 64.163 \quad Q_{55} = -\frac{\tilde{a}_{45} \cdot \tilde{a}_{55}}{\tilde{a}_{54} \cdot \tilde{a}_{44}} \times Q_{44} = 502.117$$

Then, according to (3.30), the diagonal matrix P can be obtained

$$P = \begin{pmatrix} 1.875 & 0 & 0 & 0 & 0 \\ 0 & 1.875 & 0 & 0 & 0 \\ 0 & 0 & 3.207 & 0 & 0 \\ 0 & 0 & 0 & 6.205 & 0 \\ 0 & 0 & 0 & 0 & 10.990 \end{pmatrix}$$

Such a matrix satisfies the following Lyapunov equation

$$\begin{aligned} \tilde{A}^T P + P \tilde{A} &= \begin{pmatrix} 2\tilde{a}_{11}P_{11} & \tilde{a}_{12}P_{11} + \tilde{a}_{11}P_{22} & \tilde{a}_{11}P_{33} & \tilde{a}_{11}P_{44} & \tilde{a}_{11}P_{55} \\ \tilde{a}_{12}P_{11} + \tilde{a}_{11}P_{22} & 2\tilde{a}_{22}P_{22} & \tilde{a}_{23}P_{22} + \tilde{a}_{22}P_{33} & \tilde{a}_{22}P_{44} & \tilde{a}_{22}P_{55} \\ \tilde{a}_{11}P_{33} & \tilde{a}_{23}P_{22} + \tilde{a}_{22}P_{33} & 2\tilde{a}_{33}P_{33} & \tilde{a}_{34}P_{33} + \tilde{a}_{33}P_{44} & \tilde{a}_{33}P_{55} \\ \tilde{a}_{11}P_{44} & \tilde{a}_{22}P_{44} & \tilde{a}_{34}P_{33} + \tilde{a}_{33}P_{44} & 2\tilde{a}_{44}P_{44} & \tilde{a}_{45}P_{44} + \tilde{a}_{44}P_{55} \\ \tilde{a}_{11}P_{55} & \tilde{a}_{22}P_{55} & \tilde{a}_{33}P_{55} & \tilde{a}_{45}P_{44} + \tilde{a}_{44}P_{55} & 2\tilde{a}_{55}P_{55} \end{pmatrix} \\ &= \begin{pmatrix} -3.000 & 0 & -2.565 & -4.964 & -8.792 \\ 0 & -4.224 & 0 & -6.990 & -12.379 \\ -2.565 & 0 & -13.211 & 0 & -22.640 \\ -4.964 & -6.990 & 0 & -64.163 & 0 \\ -8.791 & -12.379 & -22.640 & 0 & -502.117 \end{pmatrix} \\ &= -Q \end{aligned}$$

where it is clear that Q is positive definite with eigenvalues: $\lambda_1=1.6967$, $\lambda_2=3.3675$, $\lambda_3=12.6755$, $\lambda_4=65.3476$, $\lambda_5=503.6378$, and hence P can be used as a candidate Lyapunov function.

THEOREM 3.10. (Lyapunov Function for Nonlinear Systems^[1,5])

For a given function

$$\dot{x} = f(x), \quad f(0)=0$$

if

$$\nabla f^T(x) + \nabla f(x) < 0 \quad (3.32)$$

a Lyapunov function for this system is then given by

$$V(x) = f^T(x) \cdot f(x) \quad (3.33)$$

Proof: From definition (3.33), obviously

$$V(x) = f^T(x) f(x) = \|f(x)\|^2 > 0 \quad \forall x \neq 0 \quad (3.34)$$

$$\begin{aligned} \dot{V}(x) &= \dot{f}^T(x) \cdot f(x) + f^T(x) \cdot \dot{f}(x) \\ &= f^T(x) (\nabla f^T + \nabla f) f(x) < 0 \end{aligned} \quad (3.35) \quad \square$$

3.2.4 System Analysis and Control Design Based on Lyapunov Functions

Lyapunov functions are primarily used for stability analysis of systems, but sometimes they can provide an estimate of the transient performance of stable systems. In particular, they can allow estimation of the convergence rate of linear or nonlinear systems which are asymptotically stable.

THEOREM 3.11. (*Convergence Rate Estimation^[1]*)

If, for a given system, a Lyapunov function $V(t)$ can be found that satisfies the following inequality

$$\dot{V}(t) + \alpha \cdot V(t) \leq 0 \quad (3.36)$$

then

$$V(t) \leq V(0) \cdot e^{-\alpha t} \quad (3.37)$$

The Lyapunov function $V(t)$ can be guaranteed to exponentially converge to zero at the convergence rate α .

The above theorem implies that, if $V(x)$ is a non-negative function, the satisfaction of (3.37) guarantees the exponential convergence of $V(x)$ to zero. The reciprocal of α can be regarded as the largest time constant of the system in some region in the state space. In using Lyapunov's direct method for stability analysis, it is often possible to manipulate $\dot{V}(t)$ into the form of (3.36). In such a case, the exponential convergence rate of the state may then be determined. For instance, let us consider a linear system. The Lyapunov function is $V(t) = x^T P x$, and the time derivative is

$$\dot{V}(t) = -x^T Q x \leq -\lambda_{\min}(Q) x^T x = -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} \lambda_{\max}(P) x^T x \leq -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} V(t) = -\alpha \cdot V(t)$$

then $V(t) \leq V(0) \cdot e^{-\alpha t}$

where $\alpha = \lambda_{\min}(Q) / \lambda_{\max}(P)$. This, together with the fact that $V(t) = x^T P x \geq \lambda_{\min}(P) x^T x$, implies that

$$\|x\| \leq \sqrt{V(0)/\lambda_{\min}(P)} \cdot e^{-\frac{\alpha}{2}t} \quad (3.38)$$

i.e., the state x converges to the origin at a rate of at least $\alpha/2$.

In using Lyapunov's direct method for system analysis, it has been implicitly assumed that certain control laws have been chosen for the system, and the problem is to justify the stability of the given system. However, in many control problems, the task is the converse; that is, to find an appropriate control law for a given system, such that the closed loop system is stable.

There are basically two ways of using Lyapunov's direct method for control design, and both involve trial and error. The first technique hypothesises one form of control law and then requires the finding of a Lyapunov function to justify the choice, while the second technique, conversely, requires hypothesising a Lyapunov function candidate and then finding a control law to make this candidate a true Lyapunov function.

The controller design methods to be described in the following chapters are all based on the second usage of Lyapunov's direct method.

3.3 EXTENSION OF THE LYAPUNOV METHOD

There are some systems for which the desired state of a system may be unstable in the sense of Lyapunov, and yet the system may behave sufficiently well near this state that this performance is acceptable, or the output of the system may not converge to the origin, but is nevertheless bounded. The boundedness of all solutions of a system is also a kind of stability, and of great importance in practice, particularly for robust control of nonlinear uncertain systems. These investigations are basically independent of Lyapunov theory, but the analogy to Lyapunov's direct method is obvious, and is emphasised by the fact that the boundedness of the states can be interpreted in the sense of a stability property of the trivial solutions.

As a simple example of an investigation of boundedness, La Salle^[2] studies van der Pol's equation to show that near the origin the damping is negative and the origin is unstable, but by selecting the parameters properly any degree of boundedness desired can be obtained.

Barmish *et al*^[7] discuss another very simple example

$$\dot{x}(t) = x(t) + \gamma(t) + u(t)$$

with $x(t_0)=1$ and uncertainty $\gamma(t)$ such that $|\gamma(t)| \leq 1$. Suppose the control is selected as a linear feedback of the form $u(t)=kx(t)$ with $k < -1$. Then, if a state $x(t) < -1/(1+k)$ is reached, an admissible uncertainty $\gamma(t) \equiv 1$ results in the final state away from zero. The system is therefore not asymptotically stable. Although uniform asymptotic stability cannot be guaranteed, it is nevertheless possible to drive the state to an arbitrarily small neighbourhood of the origin. A kind of stability is then achieved.

3.3.1 Boundedness

Stability and even asymptotic stability by themselves may not be suitable descriptions of the stability properties of a practical system. Consider now stability in the sense of Lagrange, or more simply boundedness, which has been commonly used in robust stability control.

DEFINITION 3.12. (*Boundedness*^[2,5,7])

Consider a system

$$\dot{x}(t) = F(x, t)$$

with any solution $x(\cdot):[t_0, t_1] \rightarrow \mathbb{R}^n$, $x(t_0)=x_0$, and any initial condition $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$

① If, for a given number $r > 0$, there exists a constant $R(r, t_0) > 0$ depending on r and initial time t_0 such that

$$\|x_0\| \leq r \quad \Rightarrow \quad \|x(t)\| \leq R(r, t_0) \quad \forall t \geq t_0 \quad (3.39)$$

then it is said that $x(t)$ is bounded.

② If, for $r>0$, there exists a constant $R(r)>0$ depending only on r such that

$$\|x_0\| \leq r \Rightarrow \|x(t)\| \leq R(r) \quad \forall t \geq t_0 \quad (3.40)$$

then it is said that $x(t)$ is uniformly bounded.

③ Given $r>0$, if there are positive numbers d and $T(r, t_0)$ which may depend on r and t_0 , such that

$$\|x_0\| \leq r \Rightarrow \|x(t)\| \leq d \quad \forall t \geq t_0 + T(r, t_0) \quad (3.41)$$

then it is said that $x(t)$ is ultimately bounded with the bound d .

④ If the system is ultimately bounded, and T can be chosen to possibly depend on r but not on t_0 , such that

$$\|x(0)\| \leq r \Rightarrow \|x(t)\| \leq d \quad \forall t \geq t_0 + T(r) \quad (3.42)$$

then it is said that $x(t)$ is uniformly ultimately bounded with the bound d .

REMARK 3.4:

- The concept of boundedness differs from the traditional Lyapunov-type stability. Lyapunov's theorems draw conclusions about system stability from the signs of the function $V(t)$ and its time derivative $\dot{V}(t)$ in the neighbourhood of the origin, whilst this method applies this idea to the case where the signs of the function $V(t)$ and its time derivative $\dot{V}(t)$ are considered not in the neighbourhood of the origin but exterior to a certain hyper-sphere. If it can be concluded that all the state trajectories penetrate those hyper-surfaces on which $V(t)$ is constant from the outside to the inside, then consequently, all solutions with bounded initial states are bounded themselves for sufficiently large time t . These results are analogous to Lyapunov's direct method in that the boundedness of the states can be interpreted in the sense of a stability property of the trivial solutions.
- In contrast to marginal stability, where given a number $R>0$ there exists a number $r(R)>0$ such that $\|x(t_0)\| < r \Rightarrow \|x(t)\| < R \quad \forall t > t_0$, boundedness is defined as: given a number $r>0$, there is a $R(r)>0$ such

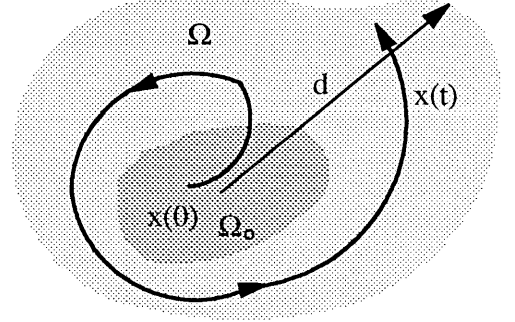


Fig. 3.2 The concept of boundedness

that $\|x(t_0)\| < r \Rightarrow \|x(t)\| < R \quad \forall t > t_0$. This difference could be very significant in describing the stability properties of a system.

To characterise the different types of boundedness by means of Lyapunov's direct method, a function $V(x,t)$, which has all the aforementioned properties, is introduced. In what follows, we also call the function $V(x,t)$ a Lyapunov function.

THEOREM 3.13. (*Boundedness*^[2])

A domain Ω containing the origin is defined as

$$\Omega: \|x(t)\| < R \quad \forall t \geq t_0 \quad (3.43)$$

Assume that, throughout the outside of Ω , a Lyapunov function $V(x,t)$ with the property

$$v_1(\|x\|) \leq V(x,t) \leq v_2(\|x\|) \quad (3.44)$$

exists, where v_1 and v_2 are continuous positive increasing functions.

If the time derivative of the Lyapunov function is such that

$$\textcircled{1} \quad \dot{V}(t) \leq 0 \quad (3.45)$$

it is said the solutions are uniformly bounded; or

$$\textcircled{2} \quad \dot{V}(t) \leq -v(\|x\|) \quad (3.46)$$

where $v(\|x\|)$ is a positive continuous function, it is said the solutions are uniformly ultimately bounded.

3.3.2 Practical Stabilisability

As already mentioned, in using the direct method of Lyapunov for robust control design, a candidate Lyapunov function is hypothesised and then a control law is developed to make this candidate a real Lyapunov function, so that the closed loop system is practically stable. The concept of practical stabilisability is now introduced.

DEFINITION 3.14. (*Practical Stabilisability*^[7])

A nonlinear uncertain system of form

$$\dot{x}(t) = F(x,\gamma,t) + G(x,\gamma,t)u(t) + \xi(t) \quad (3.47)$$

is said to be practically stabilisable if, given any $d>0$, any admissible uncertainty $\gamma(\cdot) \in R^\gamma$, and any initial condition $(x_0, t_0) \in R^n \times R$, there exists a control law $u(t): R^n \times R \rightarrow R$, for which the following properties hold:

- ① *Existence of solutions*: the closed loop system possesses a solution

$$x(\cdot): [t_0, t_1] \rightarrow R^n \quad x(t_0) = x_0$$

- ② *Uniform boundedness*: given any $r>0$, any solution $x(\cdot): [t_0, t_1] \rightarrow R^n$, $x(t_0) = x_0$, there is a constant $0 < d(r) < \infty$ such that

$$\|x_0\| \leq r \Rightarrow \|x(t)\| \leq d(r) \quad \forall t \in [t_0, t_1] \quad (3.48)$$

- ③ *Extension of solutions*: every solution $x(\cdot)$ can be continued over $t \in [t_0, \infty)$;

- ④ *Uniform ultimate boundedness*: given any $d' \geq d$, any $r > 0$, and any solution $x(\cdot): [t_0, t_1] \rightarrow R^n$, $x(t_0) = x_0$, there is a finite time $T(d', r) < \infty$, possibly dependent on r but not on t_0 , such that

$$\|x_0\| \leq r \Rightarrow \|x(t)\| \leq d' \quad \forall t \geq t_0 + T(d', r) \quad (3.49)$$

- ⑤ *Uniform stability*: given any $d' \geq d$ and any solution $x(\cdot): [t_0, t_1] \rightarrow R^n$, $x(t_0) = x_0$, there is a constant $r(d') > 0$, such that

$$\|x_0\| \leq r(d') \Rightarrow \|x(t)\| \leq d' \quad \forall t \geq t_0 \quad (3.50)$$

The problem then is to find a family of controllers for the nonlinear uncertain system, which guarantees that the system is practically stabilisable for any initial condition $(x_0, t_0) \in R^n \times R$ and all uncertain elements $\gamma(t) \in R^\gamma$.

3.4 SUMMARY

Stability is a fundamental issue in system analysis and control design. Various concepts of stability must be defined in order to accurately characterise stability in nonlinear uncertain systems. Since analytical solutions of nonlinear differential equations usually cannot be obtained, Lyapunov stability theory is of major importance in system

analysis and control design. However, asymptotic stability may not be applicable to some real systems, especially for some nonlinear systems in the presence of uncertainties. Practical stability is therefore defined and often used in robust design. Although slight differences exist theoretically between the two definitions, the procedure for use is the same, *i.e.*, construction of a Lyapunov function and examination of its time derivative. The direct method of Lyapunov is applicable to essentially all dynamic systems, but it suffers from the common difficulty of finding a Lyapunov function for a given system.

The controller design methods to be described in the following chapters are all actually established by constructive use of Lyapunov's direct method, and based on a fundamental concept: a system admits a control law such that the Lyapunov function for the nominal system (*i.e.*, the certain part of the system) is also a Lyapunov function candidate for the uncertain system (*i.e.*, the overall system with uncertainties).

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Robust Stability Control (I)



Overview

This chapter presents new synthesis techniques based on feedback linearisation and using Lyapunov stability theory. The techniques are obtained for nonlinear systems with either matched or partially matched as well as mismatched uncertainties. A second order system is used to demonstrate the techniques.



Outline

- ✓ Introduction
- ✓ Control Design Based on Feedback Linearisation
- ✓ Comments on System Performance
- ✓ Illustrative Example
- ✓ Summary

4.1 INTRODUCTION

As pointed out in chapter 1, the objective of control design can be stated as follows: **A** given a physical system to be controlled and the specifications of its desired behaviour, construct a feedback control law to make the closed loop system display the desired behaviour. In pure model-based nonlinear control, such as the basic feedback linearisation approach, the control law is based on a nominal model of the physical system. How the control system will behave in the presence of model uncertainties is not clear at the design stage. In robust nonlinear control, such as the techniques described here, the controller design is based on consideration of both the nominal model and some characterisation of the model uncertainties (such as knowledge of the load to be picked up by a robot). Robust nonlinear control techniques have proven very effective in a variety of practical control problems.

Although Lyapunov's direct method originated as a method of stability analysis, it can be used for other problems in system control. One important application is the design of various control strategies, and another is justification of system robustness when uncertainty is considered. The idea is to formulate a scalar positive function of the system states, and then choose a control law to make this function decrease. A nonlinear control system thus designed will be guaranteed to be stable despite the presence of some uncertain but bounded elements. Such a design approach has been used to solve many complex design problems, for instance, in robotics and aircraft control.

Feedback linearisation is an approach to nonlinear control design which has attracted a great deal of research interest in recent years. The central idea of the approach is to algebraically transform nonlinear system dynamics into a (full or partial) linear equivalent one of a simple form so that well-known and powerful linear control techniques

can be applied to complete the control design. More precisely, the nonlinearities in a system can be cancelled by properly chosen nonlinear feedback so that the closed loop dynamics are of linear normal form. The principle of feedback linearisation and the associated mathematical concepts from differential geometry are briefly reviewed in appendix A.

This chapter provides detailed discussion of robust stability control design for SISO nonlinear uncertain systems based on the feedback linearisation technique. Section 4.2 presents the major results of this chapter, in which the technique proposed by Barmish *et al*^[2] is first introduced. This is based on matching assumptions. Subsequently, an improved version is proposed where only matched uncertainties are considered, and then step by step, the technique is extended to systems with partially matched and then mismatched uncertainties. Whatever the uncertainties are, a unified result is achieved. Section 4.3 gives a brief description of system stability properties under the robust control laws developed. For illustrative purposes, a second order system is used to demonstrate the robustness of the techniques and simulation results are included in section 4.4. Finally, in section 4.5, a brief summary is made of the proposed techniques.

4.2 CONTROL DESIGN BASED ON FEEDBACK LINEARISATION

Nonlinear systems in the presence of uncertainties are now considered. In general, feedback linearisation relies on the system model both for the controller design and for the computation of a new set of states. If there are uncertainties in the model, *e.g.*, uncertainties regarding the values of parameters, they will cause errors in the computation of both the new state vector and of the control input. Robust control is now attempted by applying the aforementioned feedback linearisation technique to nonlinear systems with uncertainties of the form

$$\begin{aligned}\dot{x}(t) &= F(x, \gamma, t) + G(x, \gamma, t)u(t) \\ y(t) &= H(x)\end{aligned}\tag{4.1}$$

where γ is a lumped uncertain element. Let $\tilde{\gamma}$ denote the nominal value of γ . The following control strategy is proposed.

- ① *Transforming the original nonlinear system into a new one of linearisable nominal form;*
- ② *Designing a control law to linearise the nominal nonlinear system;*
- ③ *Placing the closed-loop poles of the linearised nominal system at prescribed positions;*
- ④ *Compensating the effect of uncertainties.*

Thus, according to the feedback linearisation theory in appendix A, a coordinate transformation

$$z = \psi(x, \tilde{\gamma}) \quad \Leftrightarrow \quad x = \psi^{-1}(z, \tilde{\gamma})$$

is defined with the choice of

$$\psi_k(x, \tilde{\gamma}) = L_F^{k-1} H(x) \quad (k=1, \dots, v)$$

where v is the relative order. Such a transformation leads to a system with the following external dynamics in the new coordinate z :

$$\begin{aligned}\dot{z}(t) &= f(z) + g(z)u(t) + \Delta f(z, \gamma, t) + \Delta g(z, \gamma, t)u(t) \\ y(t) &= h(z)\end{aligned}\tag{4.2}$$

where the k^{th} entries of vectors f and g are

$$\hat{f}_k(z) + \Delta \hat{f}_k(z, \gamma, t) = L_F^k H \circ \psi^{-1}(z, \tilde{\gamma})\tag{4.3}$$

$$\hat{g}_k(z) + \Delta \hat{g}_k(z, \gamma, t) = L_G L_F^{k-1} H \circ \psi^{-1}(z, \tilde{\gamma}) \quad (k=1, \dots, v)\tag{4.4}$$

where the certain part of the system, whose structure and parameters are precisely known, is represented by f , g and h , whilst Δf and Δg represent the uncertainties in the state and input mapping respectively.

This system is now in input-output linearisable form, and the feedback linearisation technique is applicable to the certain part.

In what follows, it is assumed that the given nonlinear uncertain system is of, or has been transformed to be of, the input-output linearisable form. Furthermore, it is also required that the given nonlinear system is asymptotically minimum phase if the relative order $v < n$, i.e., the internal dynamics of the system are asymptotically stable, so the stability of the internal dynamics is assumed.

In accordance with the stability theory described in chapter 3, the following definition, which will be employed to develop all the results in this section, is introduced.

DEFINITION 4.1. (Stability Margin)

For a given nonlinear system

$$\dot{z}(t) = f(z, u) = f(z) + g(z)u(t)$$

if a state feedback $u(t)$ can be found such that the following inequalities

$$v_1(\|z\|) \leq V(z) \leq v_2(\|z\|) \quad (4.5)$$

$$L_f V = \frac{\partial V(z)}{\partial z} \cdot \dot{z} < -v(\|z\|) \quad (\forall z \neq 0) \quad (4.6)$$

hold, where $V(z)$ is the Lyapunov function of the closed loop system, and $v_i(\cdot)$ satisfying

$$v_i(0) = 0 \quad (i=1,2) \quad (4.7)$$

$$\lim_{\varepsilon \rightarrow \infty} v_i(\varepsilon) = \infty \quad (4.8)$$

are strictly increasing continuous functions. Then the closed loop system is said to have stability margin v .

The theorems described in this section enable us to achieve our aims with only very weak conditions on the nature and size of the uncertainties. The condition on the size is only that the uncertainties are bounded and that the bounds are known. No limit is placed on the size of the bounds. The theorems also enable us to take advantage of any matching which may be present. They involve the extension of the Lyapunov function for the nonlinear, but now linearised, certain part of the system, to the overall nonlinear uncertain system. The control signal will generally be of the following form

$$u(t) = u_1(t) + u_2(t) \quad (4.9)$$

where

$$u_1(t) = \frac{-\sum_{k=0}^v \alpha_k \cdot L_f^k h(z)}{\alpha_v \cdot L_g L_f^{v-1} h(z)} \quad (4.10)$$

is state feedback obtained according to feedback linearisation theory with $v(t)=0$, that causes the closed loop system of the certain part of (4.2) to achieve a definite stability margin, and

$$u_2(t) = -\rho(z) \cdot L_g V(z) \quad (4.11)$$

is used to compensate for any uncertainty in the system.

A simple identity, which will be used throughout the rest of this section, is now introduced

$$a\xi - b\xi^2 = \frac{a^2}{4b} - \frac{b}{a^2} \left(a\xi - \frac{a^2}{2b} \right)^2 \leq \frac{a^2}{4b} \quad (b > 0) \quad (4.12)$$

4.2.1 A Preliminary Technique

For completeness, the technique developed by Barmish *et al*^[2] is described without proof.

THEOREM 4.2. (Matched Uncertainties: Case 1)

Consider a nonlinear uncertain system, incorporating some bounded uncertainties with matching conditions (2.10)–(2.11) of definition 2.4, and also condition (2.14) of assumption 2.7, rewritten as follows

$$|q(x, \gamma, t)| < 1 \quad (4.13)$$

of the form

$$\dot{z}(t) = f(z) + g(z)u(t) + g(z) \cdot \{ p(z, \gamma, t) + q(z, \gamma, t)u(t) \} \quad (4.14)$$

Suppose the uncontrolled nominal part of the system, $\dot{z}(t)=f(z)$, is stable or pre-stabilised.

Then the system (4.14) is stabilisable if the input is of the form

$$u(t) = -\rho(z) \cdot L_g V(z) \quad (4.15)$$

where, for a Lyapunov function $V(z)$ defined for $\dot{z}=f(z)$,

$$\rho(z) \geq \frac{\omega_p^2}{4(C_2 - C_1 L_f V)(1 - \omega_q)} > 0 \quad (4.16)$$

holds, where

$$L_g V \triangleq \frac{\partial V}{\partial z} \cdot g \quad (4.17)$$

$$L_f V \triangleq \frac{\partial V}{\partial z} \cdot f \quad (4.18)$$

are the Lie derivatives of the Lyapunov function $V(z)$ with respect to g and f , and

$$\omega_p \triangleq \max_{\gamma(t) \in R^Y} |p(z, \gamma, t)| \quad (4.19)$$

$$\omega_q \triangleq \max_{\gamma(t) \in R^Y} |q(z, \gamma, t)| < 1 \quad (4.20)$$

are the bounds of the uncertainties in the system (4.14), and

$$C_1 < 1;$$

$$\text{either } C_1 \neq 0 \text{ or } C_2 \neq 0;$$

$$C_2 \neq 0 \text{ whenever } \lim_{z \rightarrow 0} [\omega_p^2 / L_f V] \text{ does not exist;}$$

$$C_2 / (1 - C_1) < (v \circ v_2^{-1} \circ v_1)(d).$$

Proof: (See reference [2])

□

4.2.2 An Improved Technique with Matching Assumption

THEOREM 4.3. (*Matched Uncertainties: Case 2*)

Consider a nonlinear system, incorporating some bounded uncertainties with matching conditions (2.10)~(2.11) of definition 2.4, of the form

$$\dot{z}(t) = F(z, \gamma, t, u) = f(z) + g(z)u(t) + g(z) \{ p(z, \gamma, t) + q(z, \gamma, t)u(t) \} \quad (4.21)$$

as well as condition (2.15), *i.e.*,

$$q(x, \gamma, t) > 0 \quad (4.22)$$

Suppose the nominal system, *i.e.*, the certain part of (4.21), denoted as $f = f(z) + g(z)u_1(t)$, is stable under the feedback of (4.10), and the closed loop system has stability margin v , then

the overall nonlinear system (4.21) is also stabilisable if the control is as follows

$$u(t) = u_1(t) + u_2(t) = u_1(t) - \rho(z) \cdot L_g V(z) \quad (4.23)$$

and if a Lyapunov function $V(z)$, for $\dot{z} = f + gu_1$, can be found such that

$$\rho(z) \geq \frac{1}{4(C_2 - C_1 L_f V)} (\omega_p^2 + \omega_q \cdot u_1^2) > 0 \quad (4.24)$$

holds, where, for $z \neq 0$

$$L_g V \triangleq \frac{\partial V}{\partial z} \cdot g \neq 0 \quad (4.25)$$

$$L_f V \triangleq \frac{\partial V}{\partial z} \cdot f = \frac{\partial V}{\partial z} \cdot (f + gu_1) < -v(\|z\|) \quad (4.26)$$

are the Lie derivatives of the Lyapunov function $V(z)$ with respect to g and $f + gu_1$, and

$$\omega_p \triangleq \max_{\gamma(t) \in \mathbb{R}^r} |p(z, \gamma, t)| \quad (4.27)$$

$$\omega_q \triangleq \max_{\gamma(t) \in \mathbb{R}^r} |q(z, \gamma, t)| \quad (4.28)$$

are the bounds of the uncertainties in the system (4.21), and C_1, C_2 are any constants satisfying

$$C_1 < 1 \quad (4.29)$$

$$\text{either } C_1 \neq 0 \text{ or } C_2 \neq 0; \quad (4.30)$$

$$C_2 \neq 0 \text{ whenever } \lim_{z \rightarrow 0} [(\omega_p^2 + \omega_q \cdot u_1^2) / L_f V] \text{ does not exist;} \quad (4.31)$$

$$-C_1 v < C_2 < (1 - C_1) v \quad (4.32)$$

$$C_2 / (1 - C_1) < (v \circ v_2^{-1} \circ v_1)(d) \quad (4.33)$$

Proof: A Lyapunov function for the linearised nominal system can be defined by

$$V(z) = z^T P z > 0 \quad \forall z \neq 0 \quad \text{and} \quad V(0) = 0 \quad (4.34)$$

where P is the solution of the Lyapunov equation $A^T P + P A = -Q$, and $Az = f + gu_1$ is obtained by applying the state feedback (4.10) to linearise the nominal part of (4.21). Q is a positive definite symmetric real matrix.

If the closed loop system has stability margin v , *i.e.*, conditions (4.5), (4.6) of definition 4.1 hold, then for the system with uncertainties, let

$$L_F V = \frac{\partial V}{\partial z} \cdot \dot{z} = (L_f V + L_g V \cdot u_1) + L_g V \cdot u_2 + L_g V \cdot (p + q \cdot u) \quad (4.35)$$

Considering (4.26), then

$$L_F V = L_f V + \{L_g V \cdot u_2 + L_g V \cdot p + L_g V \cdot q \cdot u_1 + L_g V \cdot q \cdot u_2\}$$

Using (4.11), for $q > 0$,

$$\begin{aligned} L_F V &= L_f V + \{L_g V \cdot p - \rho \cdot (L_g V)^2 + q \cdot [L_g V \cdot u_1 - \rho \cdot (L_g V)^2]\} \\ &= L_f V + \left\{ \frac{p^2}{4\rho} - \frac{\rho}{p^2} \left(p \cdot L_g V - \frac{p^2}{2\rho} \right)^2 + \frac{q u_1^2}{4\rho} - \frac{q\rho}{u_1^2} \left(L_g V \cdot u_1 - \frac{u_1^2}{2\rho} \right)^2 \right\} \\ &\leq L_f V + \frac{\omega_p^2}{4\rho} + \frac{\omega_q \cdot u_1^2}{4\rho} \end{aligned} \quad (4.36)$$

Here ω_p and ω_q are defined by (4.27) and (4.28).

Note the identity (4.12) has been used here. For properly selected C_1 and C_2 , $C_2 - C_1 L_f V > 0$, let

$$\frac{\omega_p^2}{4\rho} + \frac{\omega_q \cdot u_1^2}{4\rho} \leq C_2 - C_1 L_f V \quad (4.37)$$

Then (4.24) holds, and so

$$L_F V \leq L_f V + C_2 - C_1 L_f V = (1 - C_1) L_f V + C_2 \quad (4.38)$$

If C_1 and C_2 are chosen according to (4.29)~(4.33), then, bearing in mind that $L_f V < -v$, it follows that

$$C_2 - C_1 L_f V > 0 \quad (4.39)$$

$$L_F V \leq -(1 - C_1)v(\|z\|) + C_2 < 0 \quad (4.40)$$

hold. The stability of the closed loop system of (4.21) is guaranteed. \square

REMARK 4.1:

- Note that this is different from theorem 4.2, in that the condition $|q| < 1$ has been replaced by $q > 0$.

Although the condition appears a restriction on the system, it can be met by properly expressing $g(x)$ and $\Delta g(x, \gamma, t)$ when decomposing uncertain systems. On the other hand, theorem 4.3 is a great advance over theorem 4.2 in that it provides us with an important basis from which to develop more general techniques.

- In (4.24), both ω_p^2 and $\omega_q u_1^2$ are included in $\rho(z)$. It follows that the technique compensates for the effect of the uncertainty in the state matrix Δf as well as uncertainty resulting from the effect of $u_1(t)$ acting through Δg , and stability of the closed loop system can be guaranteed for any bounded uncertainties.
- Compared with the approach of Barmish *et al* where $\omega_q < 1$ (and $\omega_q \ll 1$ is highly desirable), it should be noted that the factor $(1-\omega_q)$ has been removed from the denominator of $\rho(z)$. This is of great importance, because $\rho(z)$ becomes large when ω_q tends to 1 if $\rho(z)$ is dependent on such a term. Here this is not so and ω_q may even be equal to or greater than 1.

4.2.3 Control Techniques with Incomplete Matching Assumption

THEOREM 4.4. (*Partially Matched Uncertainties: Case 1*)

Consider a nonlinear system with mismatched uncertainty in the state mapping, but matched uncertainty in the input mapping, *i.e.*, the condition (2.12) of definition 2.5 is satisfied. The system is of the form

$$\dot{z}(t) = F(z, \gamma, t, u) = f(z) + g(z)u(t) + \Delta f(z, \gamma, t) + g(z) \cdot q(z, \gamma, t)u(t) \quad (4.41)$$

The uncertainty in the input mapping is assumed to satisfy condition (2.15), *i.e.*,

$$q(x, \gamma, t) > 0 \quad (4.42)$$

Suppose the nominal system, *i.e.*, the certain part of (4.41), is stable under the feedback of (4.10), and the closed loop system has stability margin v . Then the system (4.41) is stabilisable by feedback of the form

$$u(t) = u_1(t) + u_2(t) = u_1(t) - \rho(z) \cdot L_g V(z) \quad (4.43)$$

if a Lyapunov function $V(z)$, for $f = f + gu_1$, can be found such that

$$\rho(z) \geq \frac{1}{4(C_2 - C_1 L_f V)} \left(\frac{\Omega_{\Delta f}^2}{(L_g V)^2} + \omega_q \cdot u_1^2 \right) > 0 \quad (4.44)$$

holds, where $L_g V \neq 0$ and $L_f V$ are defined by (4.25) and (4.26), and

$$\Omega_{\Delta f} \triangleq \sum_{k=1}^n \left| \frac{\partial V}{\partial z_k} \right| \cdot \max_{\gamma(t) \in R^\gamma} |\Delta f_k(z, \gamma, t)| \triangleq \left| \frac{\partial V}{\partial z} \right| \omega_{\Delta f} \quad (4.45)$$

$$\omega_q \triangleq \max_{\gamma(t) \in \mathbb{R}^r} |q(z, \gamma, t)| \quad (4.46)$$

are the bounds on the uncertainties, and C_1, C_2 are given by (4.29)–(4.33).

Proof: Define a Lyapunov function $V(z)$ of form (4.34) for the linearised nominal system. If the closed loop system has a stability margin v , *i.e.*, conditions (4.5) and (4.6) hold, then for the system with uncertainties let

$$L_F V \triangleq \frac{\partial V}{\partial z} \cdot \dot{z} = (L_f V + L_g V \cdot u_1) + L_{\Delta f} V + L_g V \cdot u_2 + L_{\Delta g} V \cdot u \quad (4.47)$$

From partial matching condition (2.12)

$$L_F V = L_f V + L_{\Delta f} V + L_g V \cdot u_2 + L_g V \cdot q \cdot (u_1 + u_2)$$

Let

$$\rho = \rho' \cdot (L_{\Delta f} V)^2 > 0 \quad (4.48)$$

where $\rho' > 0$. Then

$$u_2(t) = -\rho' \cdot (L_{\Delta f} V)^2 \cdot L_g V \quad (4.49)$$

so that

$$\begin{aligned} L_F V &= L_f V + \left\{ L_{\Delta f} V - \rho' \cdot (L_{\Delta f} V)^2 \cdot (L_g V)^2 + q(L_g V \cdot u_1 - \rho' \cdot (L_{\Delta f} V)^2 \cdot (L_g V)^2) \right\} \\ &= L_f V + \left\{ \frac{1}{4\rho' \cdot (L_g V)^2} - \rho' \cdot (L_g V)^2 \left(L_{\Delta f} V - \frac{1}{2\rho' \cdot (L_g V)^2} \right)^2 \right. \\ &\quad \left. + q \left[\frac{u_1^2}{4\rho' \cdot (L_{\Delta f} V)^2} - \frac{\rho' \cdot (L_{\Delta f} V)^2}{u_1^2} \left(L_g V \cdot u_1 - \frac{u_1^2}{2\rho' \cdot (L_{\Delta f} V)^2} \right)^2 \right] \right\} \end{aligned}$$

where, once again, identity (4.12) has been used. Taking into account condition (4.42), it is then possible to write

$$L_F V \leq L_f V + \frac{1}{4\rho' \cdot (L_g V)^2} + \omega_q \cdot \frac{u_1^2}{4\rho' \cdot (L_{\Delta f} V)^2} \quad (4.50)$$

Selecting C_1 and C_2 according to (4.29)–(4.33) such that $C_2 - C_1 L_f V > 0$, and letting

$$\frac{1}{4\rho' \cdot (L_g V)^2} + \omega_q \cdot \frac{u_1^2}{4\rho' \cdot (L_{\Delta f} V)^2} \leq C_2 - C_1 L_f V$$

then

$$\rho' \geq \frac{1}{4(C_2 - C_1 L_f V)} \left(\frac{1}{(L_g V)^2} + \omega_q \cdot \frac{u_1^2}{(L_{\Delta f} V)^2} \right) \quad (4.51)$$

and so

$$\rho(z) = \rho' \cdot (L_{\Delta f} V)^2 \geq \frac{1}{4(C_2 - C_1 L_f V)} \left(\frac{(L_{\Delta f} V)^2}{(L_g V)^2} + \omega_q \cdot u_1^2 \right) \quad (4.52)$$

Because

$$L_{\Delta f} V \leq \left| \frac{\partial V}{\partial z} \Delta f(z, \gamma, t) \right| \leq \sum_{k=1}^n \left| \frac{\partial V}{\partial z_k} \right| \cdot |\Delta f_k(z, \gamma, t)| \leq \sum_{k=1}^n \left| \frac{\partial V}{\partial z_k} \right| \cdot \max_{\gamma(t) \in \mathbb{R}^\gamma} |\Delta f_k(z, \gamma, t)| = \Omega_{\Delta f}$$

if $\rho(z)$ is chosen to satisfy (4.44), it follows that inequality (4.52) holds, and

$$L_F V \leq (1 - C_1) L_f V + C_2 \leq -(1 - C_1) v(\|z\|) + C_2 \quad (4.53)$$

for all $(z, t) \in \mathbb{R}^n \times \mathbb{R}$. It may therefore be concluded that, if C_1 and C_2 are chosen according to (4.29)–(4.33) and bearing in mind that $L_f V < -v$, then

$$C_2 - C_1 L_f V > 0 \quad (4.54)$$

$$L_F V \leq -(1 - C_1) v(\|z\|) + C_2 < 0 \quad (4.55)$$

Therefore the system (4.41) has been stabilised. \square

THEOREM 4.5. (Partially Matched Uncertainties: Case 2)

Consider a nonlinear system with matched uncertainty in the state mapping, but mismatched uncertainty in the input mapping, *i.e.*, the condition (2.13) is satisfied. The system is of the form

$$\dot{z}(t) = F(z, \gamma, t, u) = f(z) + g(z)u(t) + g(z) \cdot p(z, \gamma, t) + \Delta g(z, \gamma, t)u(t) \quad (4.56)$$

It is also assumed that the mismatched uncertainty satisfies condition (2.16)

$$L_g V \cdot L_{\Delta g} V \geq 0 \quad (4.57)$$

Suppose the nominal system, *i.e.*, the certain part of (4.56), is stable with the control represented by (4.10), and has stability margin v . Then the system (4.56) is stabilisable by the control

$$u(t) = u_1(t) + u_2(t) = u_1(t) - \rho(z) \cdot L_g V(z) \quad (4.58)$$

if a Lyapunov function $V(z)$, for $f = f + gu_1$, can be found such that

$$\rho(z) \geq \frac{1}{4(C_2 - C_1 L_f V)} \left(\omega_p^2 + \frac{|\Omega_{\Delta g}|}{|L_g V|} \cdot u_1^2 \right) > 0 \quad (4.59)$$

holds, where $L_g V$ and $L_f V$ are defined by (4.25) and (4.26), and

$$\omega_p \triangleq \max_{\gamma(t) \in R^Y} |p(z, \gamma, t)| \quad (4.60)$$

$$\Omega_{\Delta g} \triangleq \sum_{k=1}^n \left| \frac{\partial V}{\partial z_k} \right| \cdot \max_{\gamma(t) \in R^Y} |\Delta g_k(z, \gamma, t)| \triangleq \left| \frac{\partial V}{\partial z} \right| \omega_{\Delta g} \quad (4.61)$$

are the bounds of the uncertainties in the system. C_1, C_2 are chosen according to conditions (4.29)~(4.33).

Proof: The procedure of proof is similar to that of Theorem 4.4.

For the Lyapunov function of form (4.34) and the stability margin v , let

$$\rho = \rho' \cdot |L_{\Delta g} V| > 0 \quad (4.62)$$

then

$$u_2(t) = -\rho' \cdot |L_{\Delta g} V| \cdot L_g V \quad (4.63)$$

Using condition (4.57), results in

$$|L_{\Delta g} V| \cdot L_g V \cdot L_{\Delta g} V = |L_g V| \cdot (L_{\Delta g} V)^2$$

It follows that

$$\begin{aligned} L_F V &= L_f V + \left\{ L_g V \cdot p - \rho' \cdot |L_{\Delta g} V| \cdot (L_g V)^2 + L_{\Delta g} V \cdot u_1 - \rho' \cdot |L_{\Delta g} V| \cdot L_g V \cdot L_{\Delta g} V \right\} \\ &= L_f V + \left\{ L_g V \cdot p - \rho' \cdot |L_{\Delta g} V| \cdot (L_g V)^2 + L_{\Delta g} V \cdot u_1 - \rho' \cdot |L_g V| \cdot (L_{\Delta g} V)^2 \right\} \\ &= L_f V + \left\{ \frac{p^2}{4\rho' \cdot |L_{\Delta g} V|} - \frac{\rho' \cdot |L_{\Delta g} V|}{p^2} \left(L_g V \cdot p - \frac{p^2}{2\rho' \cdot |L_{\Delta g} V|} \right)^2 \right. \\ &\quad \left. + \frac{u_1^2}{4\rho' \cdot |L_g V|} - \frac{\rho' \cdot |L_g V|}{u_1^2} \left(L_{\Delta g} V \cdot u_1 - \frac{u_1^2}{2\rho' \cdot |L_g V|} \right)^2 \right\} \\ &\leq L_f V + \frac{\omega_p^2}{4\rho' \cdot |L_{\Delta g} V|} + \frac{u_1^2}{4\rho' \cdot |L_g V|} \end{aligned}$$

Let

$$\frac{\omega_p^2}{4\rho' \cdot |L_{\Delta g} V|} + \frac{u_1^2}{4\rho' \cdot |L_g V|} \leq C_2 - C_1 L_f V$$

and the desired result follows

$$\rho(z) = \rho' \cdot |L_{\Delta g} V| \geq \frac{1}{4(C_2 - C_1 L_f V)} \left(\omega_p^2 + \frac{|L_{\Delta g} V|}{|L_g V|} \cdot u_1^2 \right) > 0 \quad (4.64)$$

because $L_{\Delta g} V \leq \Omega_{\Delta g}$. It follows that

$$C_2 - C_1 L_f V > 0 \quad (4.65)$$

$$L_F V \leq -(1 - C_1)v(\|z\|) + C_2 < 0 \quad (4.66)$$

The stabilisability of the system (4.56) is therefore guaranteed. \square

4.2.4 Control Techniques without Matching Assumptions

THEOREM 4.6. (Mismatched Uncertainties: Case 1)

Consider a nonlinear uncertain system, incorporating some bounded uncertainties which do not satisfy the matching conditions, of the form

$$\dot{z}(t) = F(z, \gamma, t, u) = f(z) + g(z)u(t) + \Delta f(z, \gamma, t) + \Delta g(z, \gamma, t)u(t) \quad (4.67)$$

with the condition (2.16) of assumption 2.7

$$L_g V \cdot L_{\Delta g} V \geq 0 \quad (4.68)$$

If the certain part of (4.67) is stabilised by the feedback control of (4.10) and has stability margin v , then the nonlinear system (4.67) is also stabilisable by feedback of the form (4.9)–(4.11), if a Lyapunov function $V(z)$, for $f = f + gu_1$, satisfying the conditions of definition 4.1, can be found such that the inequality

$$\rho(z) \geq \frac{1}{4(C_2 - C_1 L_f V)} \left(\frac{\Omega_{\Delta f}^2}{(L_g V)^2} + \frac{|\Omega_{\Delta g}|}{|L_g V|} u_1^2 \right) > 0 \quad (4.69)$$

holds, where $L_g V$ and $L_f V$ are the Lie derivatives of $V(z)$ with respect to g and $f + gu_1$ defined by (4.25) and (4.26), $\Omega_{\Delta f}$ and $\Omega_{\Delta g}$ are the uncertainty bounds defined by (4.45) and (4.61), respectively, and C_1, C_2 are any constants satisfying (4.29)–(4.33).

Proof: The result follows easily from theorems 4.4 and 4.5. Define a Lyapunov function $V(z)$ for the linearised nominal system as above which satisfies conditions (4.5)–(4.6). Suppose the closed loop system has stability margin v . In order to derive the desired results, the following notations are needed

$$L_{\Delta f} V = \frac{\partial V}{\partial z} \cdot \Delta f \quad L_{\Delta g} V = \frac{\partial V}{\partial z} \cdot \Delta g$$

Then for the nonlinear system with mismatched uncertainties, we have

$$L_F V = \frac{\partial V}{\partial z} \cdot \dot{z} = (L_f V + L_g V \cdot u_1) + L_{\Delta f} V + L_g V \cdot u_2 + L_{\Delta g} V \cdot u$$

$$\text{Let } \rho = \rho' \cdot (L_{\Delta f} V)^2 \cdot |L_{\Delta g} V| > 0 \quad (4.70)$$

where $\rho' > 0$. Considering condition (4.68) and using the identity (4.12), it follows that

$$\begin{aligned} L_F V &= L_f V + \{ L_{\Delta f} V - \rho' \cdot (L_{\Delta f} V)^2 \cdot |L_{\Delta g} V| \cdot (L_g V)^2 \\ &\quad + L_{\Delta g} V \cdot u_1 - \rho' \cdot (L_{\Delta f} V)^2 \cdot |L_g V| \cdot (L_{\Delta g} V)^2 \} \\ &\leq L_f V + \{ |L_{\Delta f} V| - \rho' \cdot (L_{\Delta f} V)^2 \cdot |L_{\Delta g} V| \cdot (L_g V)^2 \\ &\quad + |L_{\Delta g} V \cdot u_1| - \rho' \cdot (L_{\Delta f} V)^2 \cdot |L_g V| \cdot (L_{\Delta g} V)^2 \} \\ &= L_f V + \left\{ \frac{1}{4\rho' |L_{\Delta g} V| (L_g V)^2} - \rho' \cdot |L_{\Delta g} V| \cdot (L_g V)^2 \left(|L_{\Delta f} V| - \frac{1}{2\rho' |L_{\Delta g} V| (L_g V)^2} \right)^2 \right. \\ &\quad \left. + \frac{u_1^2}{4\rho' (L_{\Delta f} V)^2 |L_g V|} - \frac{\rho' (L_{\Delta f} V)^2 |L_g V|}{u_1^2} \left(|L_{\Delta g} V \cdot u_1| - \frac{u_1^2}{2\rho' (L_{\Delta f} V)^2 |L_g V|} \right)^2 \right\} \\ &\leq L_f V + \frac{1}{4\rho' |L_{\Delta g} V| (L_g V)^2} + \frac{u_1^2}{4\rho' (L_{\Delta f} V)^2 |L_g V|} \end{aligned} \quad (4.71)$$

Choosing C_1 and C_2 according to conditions (4.29)~(4.33), so that $C_2 - C_1 L_f V > 0$, now let

$$\frac{1}{4\rho' |L_{\Delta g} V| (L_g V)^2} + \frac{u_1^2}{4\rho' (L_{\Delta f} V)^2 |L_g V|} \leq C_2 - C_1 L_f V$$

then

$$\rho(z) = \rho' \cdot (L_{\Delta f} V)^2 \cdot |L_{\Delta g} V| \geq \frac{1}{4(C_2 - C_1 L_f V)} \left(\frac{(L_{\Delta f} V)^2}{(L_g V)^2} + \frac{|L_{\Delta g} V|}{|L_g V|} \cdot u_1^2 \right) > 0 \quad (4.72)$$

Because

$$\begin{aligned} L_{\Delta f} V &\leq \left| \frac{\partial V}{\partial z} \Delta f(z, \gamma, t) \right| \leq \sum_{k=1}^n \left| \frac{\partial V}{\partial z_k} \right| \cdot |\Delta f_k(z, \gamma, t)| \leq \sum_{k=1}^n \left| \frac{\partial V}{\partial z_k} \right| \cdot \max_{\gamma(t) \in R^{\gamma}} |\Delta f_k(z, \gamma, t)| = \Omega_{\Delta f} \\ L_{\Delta g} V &\leq \left| \frac{\partial V}{\partial z} \Delta g(z, \gamma, t) \right| \leq \sum_{k=1}^n \left| \frac{\partial V}{\partial z_k} \right| \cdot |\Delta g_k(z, \gamma, t)| \leq \sum_{k=1}^n \left| \frac{\partial V}{\partial z_k} \right| \cdot \max_{\gamma(t) \in R^{\gamma}} |\Delta g_k(z, \gamma, t)| = \Omega_{\Delta g} \end{aligned}$$

it follows that, if $\rho(z)$ is chosen by inequality (4.69) according to the known bounds given by (4.45) and (4.61), the inequality (4.72) holds obviously. We have

$$L_F V \leq (1 - C_1) L_f V + C_2 \leq -(1 - C_1) v(\|z\|) + C_2 < 0 \quad (4.73)$$

The inequality above shows that the closed loop system of (4.67) is stable. \square

The next theorem represents an extension of the preceding work to the problem of stability in the presence of disturbances.

THEOREM 4.7. (Mismatched Uncertainties: Case 2)

Consider the same nonlinear uncertain system as that of theorem 4.6, with external disturbance, as follows

$$\dot{z}(t) = F(z, \gamma, t, u) = f(z) + g(z)u(t) + \Delta f(z, \gamma, t) + \Delta g(z, \gamma, t)u(t) + \xi(t) \quad (4.74)$$

where $\xi(t)$ represents external disturbances impinging on the system. If the closed loop system of the nominal part of (4.74) is stable and has the stability margin ν , then the system (4.74) is also stabilisable by feedback of the form (4.9)~(4.11) if, for $f=f+gu_1$, a Lyapunov function $V(z)$ satisfying conditions (4.5)~(4.8) can be found such that

$$\rho(z) \geq \frac{1}{4(C_2 - C_1 L_f V)} \left(\frac{(\Omega_{\Delta f + \xi})^2}{(L_g V)^2} + \frac{|\Omega_{\Delta g}|}{|L_g V|} u_1^2 \right) > 0 \quad (4.75)$$

holds, where $L_g V$, $L_f V$, $\Omega_{\Delta f}$, $\Omega_{\Delta g}$, C_1 and C_2 are defined by (4.25), (4.26), (4.45), (4.61), and (4.29)~(4.33) respectively, and

$$\Omega_{\xi} \triangleq \sum_{k=1}^n \left| \frac{\partial V}{\partial z_k} \right| \cdot \max_{t \geq 0} |\xi(t)| \triangleq \left| \frac{\partial V}{\partial z} \right| \cdot \omega_{\xi} \quad (4.76)$$

indicates the bounds of the external disturbances $\xi(t)$ of the system (4.74).

Proof: Simply let

$$\rho(z) = \rho' \cdot (L_{\Delta f + \xi} V)^2 \cdot |L_{\Delta g} V| \quad (4.77)$$

The result is straightforward following the same procedure as that of theorem 4.6.

$$L_F V \leq L_f V + \frac{1}{4\rho' \cdot |L_{\Delta g} V| \cdot (L_g V)^2} + \frac{u_1^2}{4\rho' \cdot (L_{\Delta f + \xi} V)^2 \cdot |L_g V|}$$

if $\rho' > 0$. Let

$$\frac{1}{4\rho' \cdot |L_{\Delta g} V| \cdot (L_g V)^2} + \frac{u_1^2}{4\rho' \cdot (L_{\Delta f + \xi} V)^2 \cdot |L_g V|} \leq C_2 - C_1 L_f V$$

then

$$L_F V \leq (1 - C_1) L_f V + C_2 \leq -(1 - C_1) \nu(\|z\|) + C_2 < 0 \quad \square$$

REMARK 4.2:

- The result of theorem 4.7 is based on condition (2.16), where $L_g V \cdot L_{\Delta g} V$ is assumed to be non-negative. For any system which satisfies condition (2.16), stability of the closed loop system can be guaranteed by the control of (4.75).
- As a matter of fact, (2.16) is quite a strong condition, and hard to satisfy, because

$$L_g V \cdot L_{\Delta g} V = \frac{\partial V}{\partial z} (g \cdot \Delta g^T) \frac{\partial V^T}{\partial z}$$

where $g \cdot \Delta g^T$ is not generally symmetric and its symmetrised form may not be sign definite. The results developed may then only be applicable to some special cases. So, a more general control technique is needed to deal with the cases where condition (2.16) is not satisfied.

4.2.5 A Novel Control Algorithm for Mismatched Uncertainties

In what follows, we consider the case where condition (2.17) of assumption 2.7 is assumed to be satisfied, instead of condition (2.16).

THEOREM 4.8. (*Mismatched Uncertainties: Case 3*)

Consider a nonlinear uncertain system with mismatched uncertainties and disturbances as follows

$$\dot{z}(t) = f(z) + g(z)u(t) + \Delta f(z, \gamma, t) + \Delta g(z, \gamma, t)u(t) + \xi(t) \quad (4.78)$$

with the condition (2.17) of assumption 2.7

$$|L_g V| > |L_{\Delta g} V| \quad (4.79)$$

If the closed loop system of the certain part of (4.78) is stabilised by the feedback control of (4.10) and has stability margin v , then the nonlinear system (4.78) is also stabilisable by feedback of the form (4.9)~(4.11), if a Lyapunov function $V(z)$, for $f = f + gu_1$, satisfying conditions (4.5) and (4.6) of definition 4.1, can be found such that the feedback gain is of the form

$$\rho(z) \geq \frac{1}{4(C_2 - C_1 L_f V)} \left(\frac{(\Omega_{\Delta f + \xi})^2}{(L_g V)^2} + \frac{|\Omega_{\Delta g}|}{|L_g V|} u_1^2 \right) + \frac{1}{|L_g V| - \Omega_{\Delta g}} \left(\frac{\Omega_{\Delta f + \xi}}{|L_g V|} + \frac{\Omega_{\Delta g}}{|L_g V|} |u_1| \right) > 0 \quad (4.80)$$

where $L_g V$ and $L_f V$ are the Lie derivatives of the Lyapunov function $V(z)$ with respect to g and $f+gu_1$ defined by (4.25) and (4.26), $\Omega_{\Delta f}$, $\Omega_{\Delta g}$ and Ω_ξ are the uncertainty bounds given by (4.45), (4.61) and (4.76), respectively, and C_1 , C_2 are any constants satisfying (4.29)~(4.33).

Proof: Suppose a Lyapunov function satisfying conditions (4.5) and (4.6) can be found, and stability margin v is achieved. Then for the nonlinear system with mismatched uncertainties, let

$$\rho(z) = \rho' \cdot (L_{\Delta f + \xi} V)^2 \cdot |L_{\Delta g} V| \quad (4.81)$$

$$u_2(t) = -\rho' \cdot (L_{\Delta f + \xi} V)^2 \cdot |L_{\Delta g} V| \cdot L_g V \quad (4.82)$$

It follows that

$$\begin{aligned} L_F V &= L_f V + \{ L_{\Delta f + \xi} V - \rho' \cdot (L_{\Delta f + \xi} V)^2 \cdot |L_{\Delta g} V| \cdot (L_g V)^2 \\ &\quad + L_{\Delta g} V \cdot u_1 - \rho' \cdot (L_{\Delta f + \xi} V)^2 \cdot |L_{\Delta g} V| \cdot L_{\Delta g} V \cdot L_g V \} \\ &\leq L_f V + \{ |L_{\Delta f + \xi} V| - \rho' \cdot (L_{\Delta f + \xi} V)^2 \cdot |L_{\Delta g} V| \cdot (L_g V)^2 \\ &\quad + |L_{\Delta g} V \cdot u_1| - \rho' \cdot (L_{\Delta f + \xi} V)^2 \cdot |L_g V| \cdot (L_{\Delta g} V)^2 \} + 2\rho' \cdot (L_{\Delta f + \xi} V)^2 |L_g V| \cdot (L_{\Delta g} V)^2 \\ &= L_f V + \frac{1}{4\rho' |L_{\Delta g} V| (L_g V)^2} - \rho' \cdot |L_{\Delta g} V| \cdot (L_g V)^2 \left(|L_{\Delta f + \xi} V| - \frac{1}{2\rho' |L_{\Delta g} V| (L_g V)^2} \right)^2 \\ &\quad + \frac{u_1^2}{4\rho' (L_{\Delta f + \xi} V)^2 |L_g V|} - \frac{\rho' (L_{\Delta f + \xi} V)^2 |L_g V|}{u_1^2} \left(|L_{\Delta g} V \cdot u_1| - \frac{u_1^2}{2\rho' (L_{\Delta f + \xi} V)^2 |L_g V|} \right)^2 \\ &\quad + 2\rho' \cdot (L_{\Delta f + \xi} V)^2 \cdot |L_g V| \cdot (L_{\Delta g} V)^2 \end{aligned}$$

If the following inequality

$$\begin{aligned} &\rho' \cdot |L_{\Delta g} V| \cdot (L_g V)^2 \left(|L_{\Delta f + \xi} V| - \frac{1}{2\rho' |L_{\Delta g} V| (L_g V)^2} \right)^2 \\ &\quad + \frac{\rho' (L_{\Delta f + \xi} V)^2 |L_g V|}{u_1^2} \left(|L_{\Delta g} V \cdot u_1| - \frac{u_1^2}{2\rho' (L_{\Delta f + \xi} V)^2 |L_g V|} \right)^2 \\ &\geq 2\rho' \cdot (L_{\Delta f + \xi} V)^2 \cdot |L_g V| \cdot (L_{\Delta g} V)^2 \end{aligned} \quad (4.83)$$

holds, then

$$L_F V \leq L_f V + \frac{1}{4\rho' |L_{\Delta g} V| (L_g V)^2} + \frac{u_1^2}{4\rho' (L_{\Delta f + \xi} V)^2 |L_g V|} \quad (4.84)$$

From (4.83), we have

$$\begin{aligned}
 & |L_{\Delta g} V| \cdot (L_g V)^2 \left(|L_{\Delta f + \xi} V| - \frac{1}{2\rho' |L_{\Delta g} V| (L_g V)^2} \right)^2 \\
 & \quad + \frac{(L_{\Delta f + \xi} V)^2 |L_g V|}{u_1^2} \left(|L_{\Delta g} V \cdot u_1| - \frac{u_1^2}{2\rho' (L_{\Delta f + \xi} V)^2 |L_g V|} \right)^2 \\
 & = (L_{\Delta f + \xi} V)^2 (|L_{\Delta g} V| (L_g V)^2 + |L_g V| (L_{\Delta g} V)^2) \\
 & \quad + \frac{1}{4\rho'^2} \left(\frac{1}{|L_{\Delta g} V| (L_g V)^2} + \frac{u_1^2}{(L_{\Delta f + \xi} V)^2 |L_{\Delta g} V|} \right) - \frac{1}{\rho'} (|L_{\Delta f + \xi} V| + |L_{\Delta g} V \cdot u_1|) \\
 & \geq 2(L_{\Delta f + \xi} V)^2 \cdot |L_g V| \cdot (L_{\Delta g} V)^2 \tag{4.85}
 \end{aligned}$$

$$\begin{aligned}
 i.e., \quad & 4\rho'^2 \cdot (L_{\Delta f + \xi} V)^2 \cdot (|L_{\Delta g} V| (L_g V)^2 - |L_g V| (L_{\Delta g} V)^2) \\
 & - 4\rho' (|L_{\Delta f + \xi} V| + |L_{\Delta g} V \cdot u_1|) + \left(\frac{1}{|L_{\Delta g} V| (L_g V)^2} + \frac{u_1^2}{(L_{\Delta f + \xi} V)^2 |L_{\Delta g} V|} \right) \geq 0
 \end{aligned}$$

It is clear that if the following inequality

$$\rho' \cdot (L_{\Delta f + \xi} V)^2 \cdot (|L_{\Delta g} V| (L_g V)^2 - |L_g V| (L_{\Delta g} V)^2) - (|L_{\Delta f + \xi} V| + |L_{\Delta g} V \cdot u_1|) \geq 0$$

holds, then (4.83) and hence (4.84) are true. Bearing in mind (4.79), we can then say

$$\rho' \geq \frac{1}{|L_g V| - |L_{\Delta g} V|} \left(\frac{|L_{\Delta f + \xi} V|}{(L_{\Delta f + \xi} V)^2 \cdot |L_{\Delta g} V| |L_g V|} + \frac{|u_1|}{(L_{\Delta f + \xi} V)^2 \cdot |L_g V|} \right) \tag{4.86}$$

Again from (4.84), let

$$\frac{1}{4\rho' \cdot |L_{\Delta g} V| \cdot (L_g V)^2} + \frac{u_1^2}{4\rho' \cdot (L_{\Delta f + \xi} V)^2 \cdot |L_g V|} \leq C_2 - C_1 L_f V \tag{4.87}$$

where C_1, C_2 are any constants satisfying (4.29)~(4.33), so that $C_2 - C_1 L_f V > 0$. Then

$$\rho' \geq \frac{1}{4(C_2 - C_1 L_f V)} \left(\frac{1}{|L_{\Delta g} V| \cdot (L_g V)^2} + \frac{u_1^2}{(L_{\Delta f + \xi} V)^2 \cdot |L_g V|} \right) > 0 \tag{4.88}$$

Considering (4.83) and (4.84), one may choose

$$\rho' \geq \max \{ \rho'_1, \rho'_2 \}$$

where

$$\begin{aligned}
 \rho'_1 &= \frac{1}{|L_g V| - |L_{\Delta g} V|} \left(\frac{|L_{\Delta f + \xi} V|}{(L_{\Delta f + \xi} V)^2 \cdot |L_{\Delta g} V| |L_g V|} + \frac{|u_1|}{(L_{\Delta f + \xi} V)^2 \cdot |L_g V|} \right) \\
 \rho'_2 &= \frac{1}{4(C_2 - C_1 L_f V)} \left(\frac{1}{|L_{\Delta g} V| \cdot (L_g V)^2} + \frac{u_1^2}{(L_{\Delta f + \xi} V)^2 \cdot |L_g V|} \right)
 \end{aligned}$$

or simply let

$$\begin{aligned} \rho' \geq & \frac{1}{4(C_2 - C_1 L_f V)} \left(\frac{1}{|L_{\Delta g} V| \cdot (L_g V)^2} + \frac{u_1^2}{(L_{\Delta f + \xi} V)^2 \cdot |L_g V|} \right) \\ & + \frac{1}{|L_g V| - |L_{\Delta g} V|} \left(\frac{|L_{\Delta f + \xi} V|}{(L_{\Delta f + \xi} V)^2 \cdot |L_{\Delta g} V| |L_g V|} + \frac{|u_1|}{(L_{\Delta f + \xi} V)^2 \cdot |L_g V|} \right) \end{aligned}$$

so that

$$\begin{aligned} \rho(z) &= \rho' \cdot (L_{\Delta f + \xi} V)^2 \cdot |L_{\Delta g} V| \\ &\geq \frac{1}{4(C_2 - C_1 L_f V)} \left(\frac{(L_{\Delta f + \xi} V)^2}{(L_g V)^2} + \frac{|L_{\Delta g} V| \cdot u_1^2}{|L_g V|} \right) + \frac{1}{|L_g V| - |L_{\Delta g} V|} \left(\frac{|L_{\Delta f + \xi} V|}{|L_g V|} + \frac{|L_{\Delta g} V| \cdot |u_1|}{|L_g V|} \right) > 0 \end{aligned} \quad (4.89)$$

It follows that, if $\rho(z)$ is chosen by inequality (4.80) according to the known bounds given by (4.45), (4.61) and (4.76), the inequality (4.89) holds obviously. We have

$$L_F V \leq (1 - C_1) L_f V + C_2 \leq -(1 - C_1) v(\|z\|) + C_2 < 0 \quad (4.90)$$

The inequality above shows that the closed loop system of (4.78) is stable. \square

COROLLARY 4.9. (*Matched Uncertainties: Case 3*)

If the uncertainties in system (4.78) satisfy matching conditions, it implies that condition (4.79) can be written as

$$\omega_q < 1$$

the following feedback control exists

$$\rho(z) \geq \frac{1}{4(C_2 - C_1 L_f V)} (\omega_p^2 + \omega_q \cdot u_1^2) + \frac{1}{|L_g V| (1 - \omega_q)} (\omega_p + \omega_q \cdot |u_1|) > 0 \quad (4.91)$$

such that the nonlinear uncertain system is stabilisable.

REMARK 4.3:

- Ideally we wish to choose the control $u(t)$ so that the feedback controlled system is uniformly asymptotically stable about the equilibrium point. However to achieve uniform asymptotic stability of an uncertain system one sometimes has to resort to controllers that can deliver too large or even infinite control effort. To avoid such a control, the criterion has been relaxed from uniform asymptotic stability to practical stability.

- The results show that the proposed control cannot guarantee the asymptotic stability of the closed loop system because $L_F V < 0$ only for $\|z\| > 0$. However, as will be shown in the following section, practical stability is achieved. Therefore, the states of the system cannot go to zero as t increases, but will converge to a closed ball B_ε with finite radius $\varepsilon > 0$, where ε is a small positive constant.

4.3 COMMENTS ON SYSTEM PERFORMANCE

In the last section, several new robust design techniques have been developed for a rather general class of nonlinear system with either matched, partially matched, or mismatched uncertainties. The stability of the closed loop system can be guaranteed if these techniques are used. These results may be summarised, according to Lyapunov stability theory, by the following inequalities:

$$v_1(\|z\|) \leq V(z) \leq v_2(\|z\|) \quad (4.92)$$

$$\dot{V}(z) = L_F V \leq (1 - C_1)L_F V + C_2 \leq -(1 - C_1)v(\|z\|) + C_2 < 0 \quad (4.93)$$

subject to $C_2 - C_1 L_F V > 0$ with proper choice of C_1 and C_2 . Here v is called the stability margin achieved by using a feedback linearisation technique (4.10) to control the nominal part of the system (4.2).

4.3.1 Practical Stability

Having these results available now makes it possible to apply directly the results of Barmish *et al*^[2] to estimate the stability of the closed loop system in the sense of definition 3.14, *i.e.*, practical stability, even though the systems considered are members of a much broader class than those considered in Barmish *et al*^[2]. Here ultimate boundedness and uniform ultimate boundedness of the closed loop system are considered.

Consider again the nonlinear uncertain system of the form

$$\dot{z}(t) = f(z) + g(z)u(t) + \Delta f(z, \gamma, t) + \Delta g(z, \gamma, t)u(t) + \xi(t) \quad (4.94)$$

Suppose $z(\cdot): [t_0, t_1] \rightarrow \mathbb{R}^n$, $z(t_0) = z_0$ is a solution of (4.94) under the feedback of (4.9), with $\|z_0\| \leq r$. Then select

$$d(r) = \begin{cases} (v_1^{-1} \cdot v_2)(R) & \text{if } r \leq R \\ (v_1^{-1} \cdot v_2)(r) & \text{if } r > R \end{cases} \quad (4.95)$$

where

$$R = \frac{v^{-1} \cdot C_2}{(1 - C_1)} \quad (4.96)$$

Let $\tilde{r} \triangleq \max\{r, R\}$, so that $\|z_0\| \leq \tilde{r}$ and $R \leq \tilde{r}$. Also according to definition (4.95),

$$d(r) = (v_1^{-1} \cdot v_2)(\tilde{r})$$

so that $\tilde{r} \leq (v_1^{-1} \cdot v_2)(\tilde{r}) = d(r)$ because $v_1(\tilde{r}) \leq v_2(\tilde{r})$. Thus $\|z(t_0)\| = \|z_0\| \leq \tilde{r} \leq d(r)$.

Suppose there is a $t_3 > t_0$, such that $\|z(t_3)\| > d(r)$. Since $z(\cdot)$ is continuous and $\|z(t_0)\| \leq \tilde{r} \leq d(r) < \|z(t_3)\|$, there is a $t_2 \in [t_0, t_3]$, such that $\|z(t_2)\| = \tilde{r}$ and $\|z(t)\| \geq \tilde{r} \quad \forall t \in [t_2, t_3]$.

In view of (4.92) and (4.93)

$$\begin{aligned} v_1(\|z(t_3)\|) &\leq V(z(t_3)) \\ &= V(z(t_2)) + \int_{t_2}^{t_3} \dot{V}(z(\tau)) d\tau \\ &\leq v_2(\|z(t_2)\|) + \int_{t_2}^{t_3} [-(1 - C_1)v(z(\tau)) + C_2] d\tau \\ &\leq v_2(\tilde{r}) + \int_{t_2}^{t_3} [-(1 - C_1)v(R) + C_2] d\tau \\ &= v_2(\tilde{r}) \end{aligned}$$

Hence, $\|z(t_3)\| \leq (v_1^{-1} \cdot v_2)(\tilde{r}) = d(r)$. However this contradicts the supposition, hence

$$\|z(t)\| \leq d(r) \quad \forall t \in [t_0, t_1]$$

i.e., the system is uniformly bounded.

Again for a given number $d' > (v_1^{-1} \cdot v_2)(R)$, define

$$T(d', r) = \begin{cases} 0 & \text{if } r \leq (v_2^{-1} \cdot v_1)(d') \\ \frac{v_2(r) - (v_1 \cdot v_2^{-1} \cdot v_1)(d')}{(1 - C_1)(v \cdot v_2^{-1} \cdot v_1)(d') - C_2} & \text{otherwise} \end{cases} \quad (4.97)$$

with the proviso that

$$(1 - C_1)(v \cdot v_2^{-1} \cdot v_1)(d') - C_2 > 0 \quad (4.98)$$

where $\bar{R} = (v_2^{-1} \cdot v_1)(d')$, so that $\bar{R} > R$ and $d(\bar{R}) = (v_1^{-1} \cdot v_2)(\bar{R}) = d'$.

If $r \leq \bar{R}$, then $\|z_0\| \leq \bar{R}$, hence $\|z(t)\| \leq d(\bar{R}) = d' \quad \forall t \geq [t_0, \infty]$, so that $T(d', r) = 0$. If $r > \bar{R}$, and $\|z(t)\| > \bar{R} \quad \forall t \geq [t_0, t_1]$, then in view of (4.92) and (4.93)

$$\begin{aligned} v_1(\|z(t_1)\|) &\leq V(z(t_1)) = V(z(t_0)) + \int_{t_0}^{t_1} \dot{V}(z(\tau)) d\tau \\ &\leq v_2(\|z(t_0)\|) + \int_{t_0}^{t_1} [-(1 - C_1)v(z(\tau)) + C_2] d\tau \\ &\leq v_2(r) + T(d', r)[-(1 - C_1)v(\bar{R}) + C_2] \\ &= v_2(r) + \frac{v_2(r) - (v_1 \cdot v_2^{-1} \cdot v_1)(d')}{(1 - C_1)(v \cdot v_2^{-1} \cdot v_1)(d') - C_2} [-(1 - C_1)v(\bar{R}) + C_2] \\ &= v_1(\bar{R}) \end{aligned}$$

That is, $\|z(t_1)\| \leq \bar{R}$. But this contradicts the assumption above. Hence there must exist a $t_2 \in [t_0, t_1]$ such that $\|z(t_2)\| \leq \bar{R}$. Then, as a consequence of the uniform boundedness result $\|z(t)\| \leq d(\bar{R}) = d' \quad \forall t \geq t_2$. Hence

$$\|z(t)\| \leq d' \quad \forall t \geq t_1 = t_0 + T(d', r)$$

i.e., the system is uniformly ultimately bounded. \square

4.3.2 Remarks

- If C_1 and C_2 are chosen according to (4.29)~(4.33), it then follows that

$$L_F V < -(1 - C_1) \cdot v(\|z\|) + C_2 < 0 \quad (4.99)$$

holds; that is, $L_F V$ lies in the shaded triangular area D in Fig. 4.1. The nonlinear uncertain system (4.94) is therefore practically stabilisable by the family of

controllers of form (4.9) with the feedback gain given by (4.16), (4.24), (4.44), (4.59), (4.69), (4.75) or (4.80) whether the uncertainties satisfy the various matching assumptions or not. The range of acceptable values for C_2 is indicated in Fig. 4.1 for the case where $v(\|z\|)$ is a positive constant.

- $L_f V < 0$ implies that

$$L_f V < -v(\|z\|) < 0 \quad (\forall \|z\| \geq \varepsilon > 0)$$

is always true, so the constant C_2 satisfying (4.32) exists. On the other hand, using (4.32) a larger range of C_2 is possible if ε is allowed to be larger. This will allow greater tolerance on the uncertainty bounds.

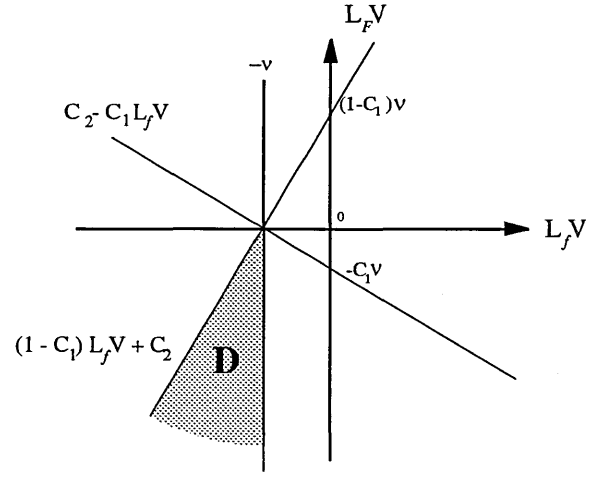


Fig. 4.1 Determination of Parameter C_2 for the case where v is a positive constant

- For systems of the form (4.74), if the various matching conditions are met, then the result derived in theorem 4.7, denoted by (4.75),

$$\rho(z) \geq \frac{1}{4(C_2 - C_1 L_f V)} \left(\frac{(\Omega_{\Delta f} + \xi)^2}{(L_g V)^2} + \frac{|\Omega_{\Delta g}|}{|L_g V|} u_1^2 \right) > 0$$

may be written in unified form as follows

$$\rho(z) \geq \frac{1}{4(C_2 - C_1 L_f V)} \left(\frac{\Omega_{\Delta f}^2}{(L_g V)^2} + \omega_q \cdot u_1^2 \right) > 0$$

$$\rho(z) \geq \frac{1}{4(C_2 - C_1 L_f V)} \left(\omega_p^2 + \frac{|\Omega_{\Delta g}|}{|L_g V|} u_1^2 \right) > 0$$

$$\rho(z) \geq \frac{1}{4(C_2 - C_1 L_f V)} (\omega_p^2 + \omega_q \cdot u_1^2) > 0$$

It can be seen that the condition for mismatched uncertainties generalises the conditions for partial matching and the conditions for partial matching generalise the condition for matched uncertainties. Therefore, the condition indicated by (4.75) in theorem 4.7 can be regarded as a generalisation of the results in the other

theorems, such as theorems 4.3~4.6, where less stringent assumptions are made. It is obvious that, when $\xi(t)=0$, we have the result of theorem 4.6; whenever incomplete or complete matching conditions are met, then either $\Omega_{\Delta}^2/(L_g V)^2=\omega_p^2$ is true, or $|\Omega_{\Delta g}|/|L_g V|=\omega_q$ is true, or both are true, and it follows that the results of theorem 4.4 or theorem 4.5 or theorem 4.3 hold. The condition (4.75) therefore represents a unified controller structure for nonlinear systems whether the matching conditions are completely satisfied, such as in theorem 4.3, partially satisfied, as in theorems 4.4 and 4.5, or not satisfied as in theorems 4.6 and 4.7. Similarly, from the result (4.80) of theorem 4.8, we can also derive other results subject to various matching conditions which may be considered. Therefore similar remarks can be made for the results obtained under the assumption of (2.17).

- The proposed techniques are a significant improvement over previous results, such as Barmish *et al*^[2]. In theorem 4.3, although the matching conditions have been assumed, the technique does not require that $\omega_q < 1$. Furthermore, in theorem 4.6, the matching requirement is totally removed. In theorem 4.8, although a similar assumption to that of Barmish *et al* has been made, the technique described here is a significant improvement in that it is applicable to nonlinear uncertain systems in the presence of mismatched uncertainties and disturbances.
- The fundamental idea behind all the theorems is that, by choosing a sufficient stability margin, which can be achieved by applying a certain control strategy (feedback linearisation) to the nominal system, then using sufficient feedback compensation defined by (4.75) or (4.80), it is possible to reduce the effect of the uncertainties on the overall system, so that the original stability margin is sufficient to guarantee stability of the overall nonlinear uncertain system.
- To achieve the results of theorems 4.6~4.8, it is required that $L_g V \neq 0 \quad \forall z \neq 0$. The requirement can be met by properly choosing a suitable Lyapunov function for the linearised nominal system. If a general form of the matrix P does not meet the

requirement, then theorem 3.9 provides another way to choose such a Lyapunov function. It is clear that, for a linear (or equivalently linearised) system $\dot{z}=f+gu=Az$ with relative order v , if the matrix P is chosen to be diagonal, the partial derivative of the Lyapunov function with respect to the states is then

$$\partial V/\partial z = 2[p_{11}z_1, p_{22}z_2, \dots, p_{vv}z_v]$$

and bearing in mind that, for the system with relative order v , the input mapping $g(z)$ must be of the form

$$g(z) = [0, 0, \dots, g_v(z)]^T$$

It follows that

$$\begin{aligned} L_g V &= \frac{\partial V}{\partial z} \cdot g(z) = 2[p_{11}z_1, p_{22}z_2, \dots, p_{vv}z_v] \cdot [0, 0, \dots, g_v(z)]^T \\ &= 2p_{vv} \cdot z_v \cdot g_v(z) \neq 0 \quad \forall z \neq 0 \end{aligned}$$

4.4 ILLUSTRATIVE EXAMPLE

For illustrative purposes, we consider the second-order linear system discussed in chapter 2 where

$$G(s) = \frac{k_1 s + k_2}{(s + \mu_1)(s + \mu_2)}$$

The state variable form is usually of the form $\dot{x}(t) = f(x) + g(x)u(t)$ with $f(x) = Ax(t)$ and $g(x) = B$, and the output $y(t) = Cx = x_1$ is chosen, where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \quad C = [1, 0]$$

4.4.1 Matched Uncertainties

Let $a_{11}=0$; $a_{12}=1$; $k_1=0$, so that $a_{21}=-\alpha$, $a_{22}=-\beta$, $b_1=0$, $b_2=k_2$, where $\alpha=\mu_1+\mu_2$, $\beta=\mu_1\mu_2$. Therefore

$$\dot{\mathbf{x}}(t) = \begin{pmatrix} 0 & 1 \\ -\alpha & -\beta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ k_2 \end{pmatrix} u(t)$$

Suppose now that uncertainty exists to the extent that μ_1 may be some other value μ_1' , and k_2 may be some other value k_2' . This results in a system with uncertainties

$$\dot{\mathbf{x}}(t) = \begin{pmatrix} 0 & 1 \\ -\alpha & -\beta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ k_2 \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ \Delta\alpha x_1 + \Delta\beta x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \Delta k_2 \end{pmatrix} u(t)$$

where $\Delta\alpha = (\alpha - \alpha')$, $\Delta\beta = (\beta - \beta')$, and $\Delta k_2 = k_2' - k_2$.

Clearly, the uncertainties here satisfy the matching conditions defined in definitions 2.4 and 2.5. The techniques of theorem 4.2 and 4.3 may then be applied. $u_1(t)$ can be designed via (4.10) as follows

$$L_f h(x) = CAx \quad L_g h(x) = CB = 0 \quad L_f^2 h(x) = CA^2x \quad L_g L_f h(x) = CAB \neq 0$$

Therefore the relative order $\nu = 2$. The state feedback is then of the form

$$u_1(t) = (\alpha_2 \cdot CAB)^{-1} \{ -C[\alpha_0 + \alpha_1 A + \alpha_2 A^2]x \}$$

The following values were selected for simulation purposes: $C_1 = 0.3$; $C_2 = 10$, and $\alpha_0 = 6$; $\alpha_1 = 5$; $\alpha_2 = 1$. This results in a closed loop system with poles $\lambda_1 = -2$ and $\lambda_2 = -3$, so that, by solving Lyapunov matrix equation $A^T P + PA = -Q$, a possible Lyapunov function for the given closed loop system $\dot{f} = Ax + gu_1$ is

$$V(x) = x^T P x = [x_1, x_2] \begin{pmatrix} 0.2028 & -0.1637 \\ -0.1637 & 0.4581 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and so $L_f V = 2x^T P (Ax + Bu_1)$, $L_g V = 2x^T P B$

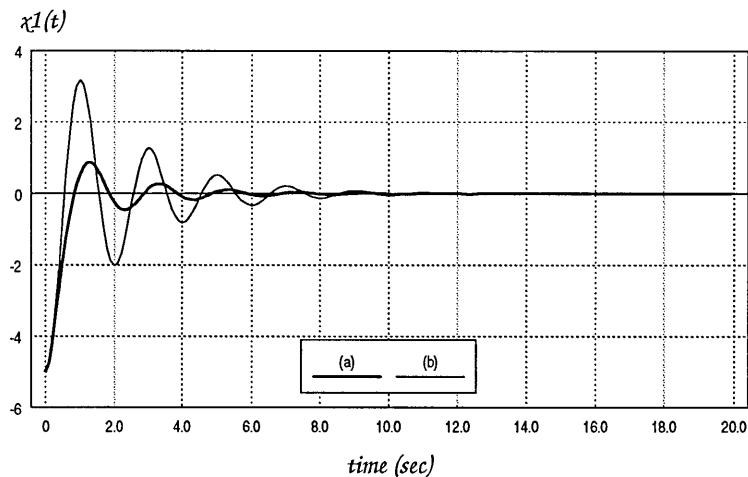
The uncertainty compensation terms for the techniques described in theorems 4.2 and 4.3 are respectively

$$u_2(t) = -\frac{2x^T P B}{4(1 - \omega_q)[C_2 - C_1 2x^T P (Ax + Bu_1)]} \{ \omega_p^2 \}$$

$$u_2(t) = -\frac{2x^T P B}{4[C_2 - C_1 2x^T P (Ax + Bu_1)]} \{ \omega_p^2 + \omega_q \cdot u_1^2 \}$$

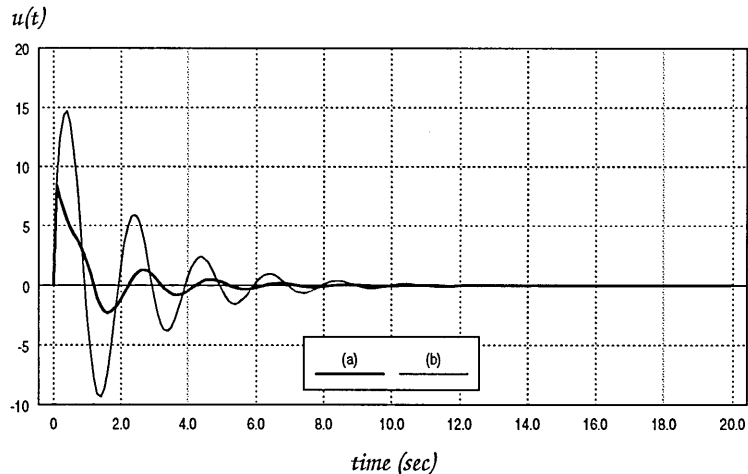
Simulation results are shown in Fig. 4.2, where a comparison of the technique of theorem 4.3 with that of theorem 4.2 is given.

(i)
Uncertain Parameters:
 $k_2=1, k'_2=5$
 $\mu_1=1; \mu_2=5$



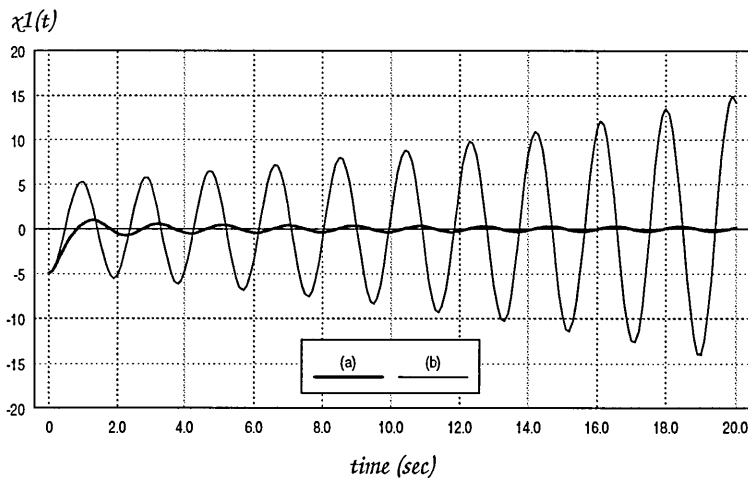
System Outputs

$\Delta f=0;$
 $\Delta g=\begin{pmatrix} 0 \\ 4 \end{pmatrix}$
 $\omega_p=0$
 $\omega_q=4$



Control Signals

(ii)
Uncertain Parameters:
 $k_2=1, k'_2=6;$
 $\mu_1=1; \mu_2=5;$



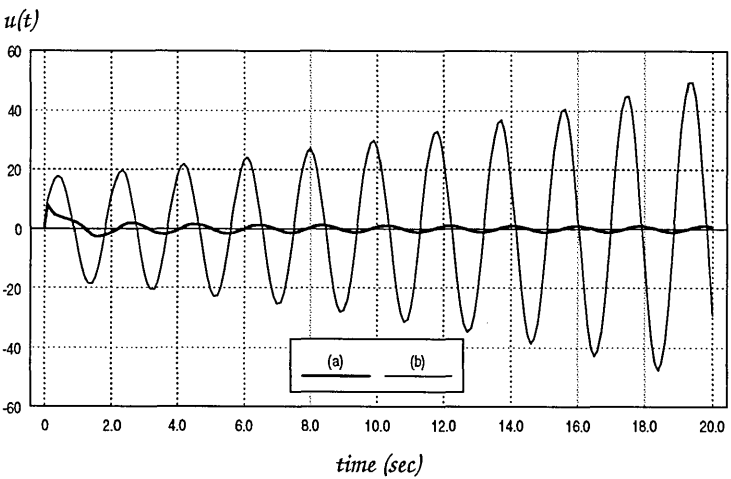
System Outputs

$\Delta f=0$

$\Delta g=\begin{pmatrix} 0 \\ 5 \end{pmatrix}$

$\omega_p=0$

$\omega_q=5$

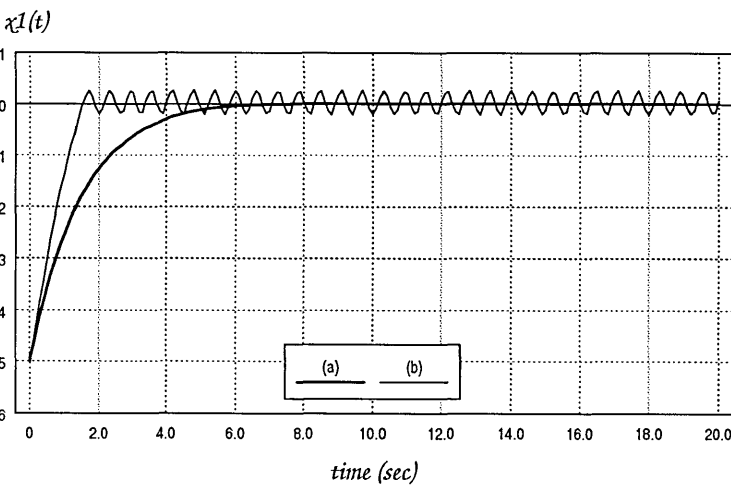


(iii)

Uncertain Parameters:

$k_2=1, k'_2=1.5;$

$\mu_1=1, \mu'_1=-1; \mu_2=5;$

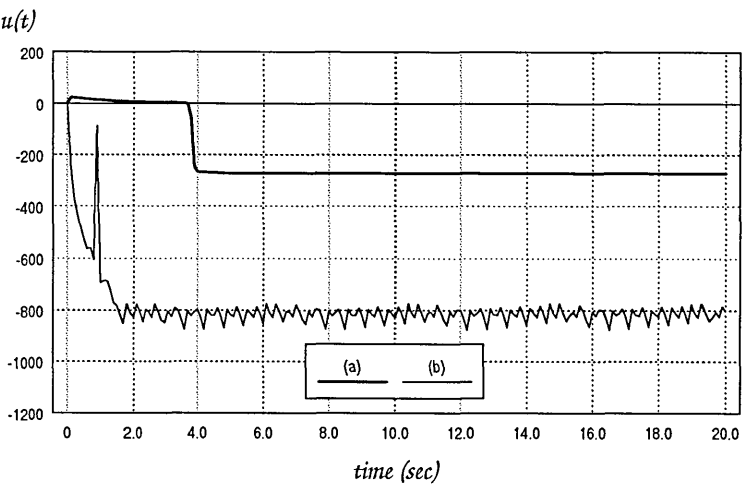


$\Delta f=\begin{pmatrix} 0 \\ 10x_1+2x_2 \end{pmatrix}$

$\Delta g=\begin{pmatrix} 0 \\ 0.5 \end{pmatrix}$

$\omega_p=160$

$\omega_q=0.5$



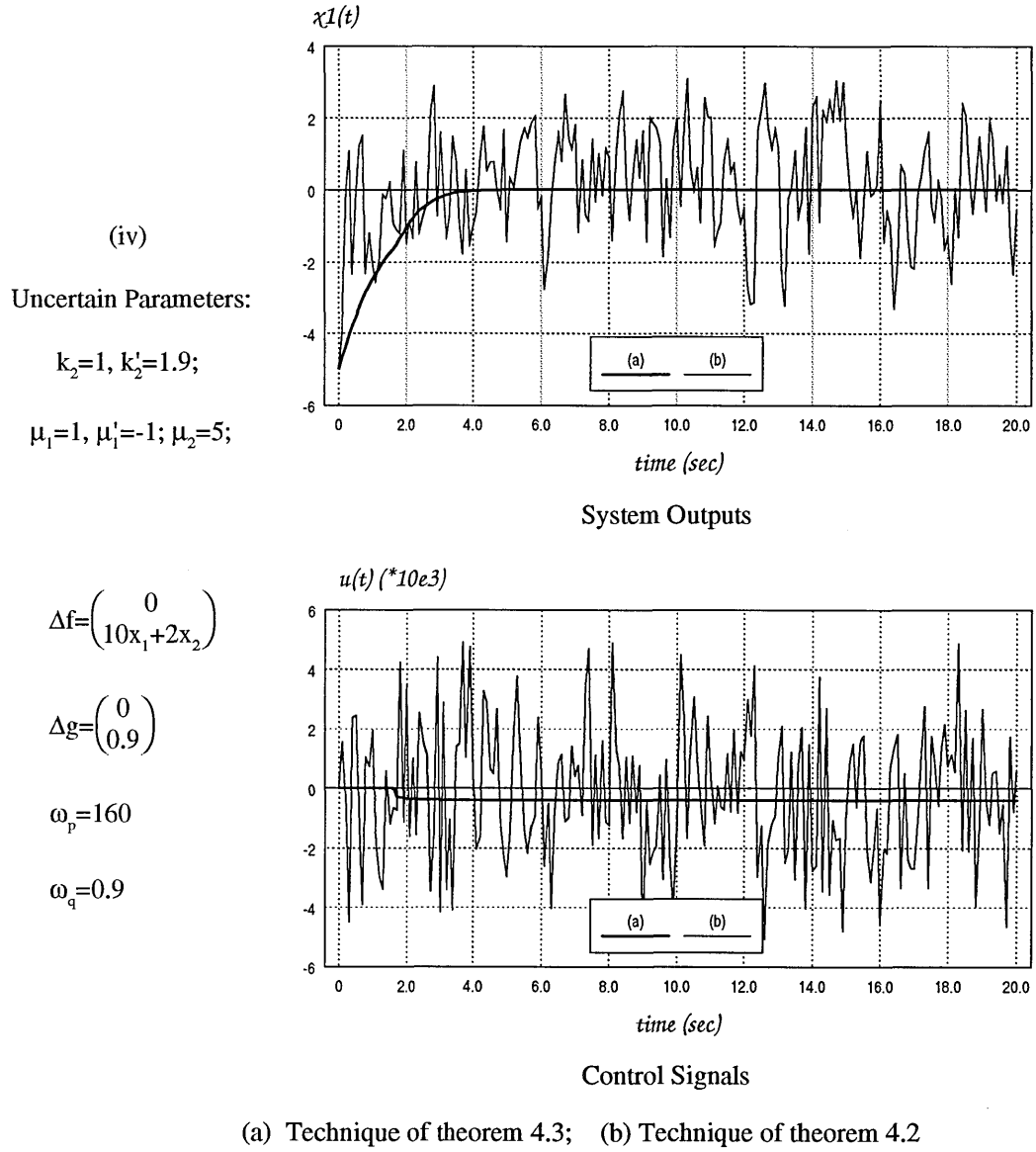


Fig. 4.2 Case 1: Comparison of the technique of theorem 4.3 with that of theorem 4.2

From the results, certain observations are easily made. Firstly, when only uncertainty on $g(z)$ occurs, the technique of Barmish *et al* fails, so that the closed loop response shows larger variations (i), or may become unstable, when the uncertainty becomes quite large (ii), but the technique of theorem 4.3 is clearly successful; secondly, when the uncertainty on both $f(z)$ and $g(z)$ occur, this technique results in better performance than that of Barmish *et al* because the term $\omega_q u_1^2$ included in the feedback control makes such compensation more effective, especially, when ω_q is close to unity (iv).

4.4.1 Partially Matched Uncertainties

Let $a_{12}=a_{21}=a \neq 0$, $\alpha^2 \geq 4(\beta+a^2)$ and $k_2=1$; then

$$a_{11} = \frac{-\alpha - \sqrt{\alpha^2 - 4(\beta+a^2)}}{2}, \quad a_{22} = \frac{-\alpha + \sqrt{\alpha^2 - 4(\beta+a^2)}}{2}, \quad b_1 = k_1, \quad b_2 = \frac{(k_2 + a_{22}k_1)}{a_{12}}$$

and then the system may be represented as

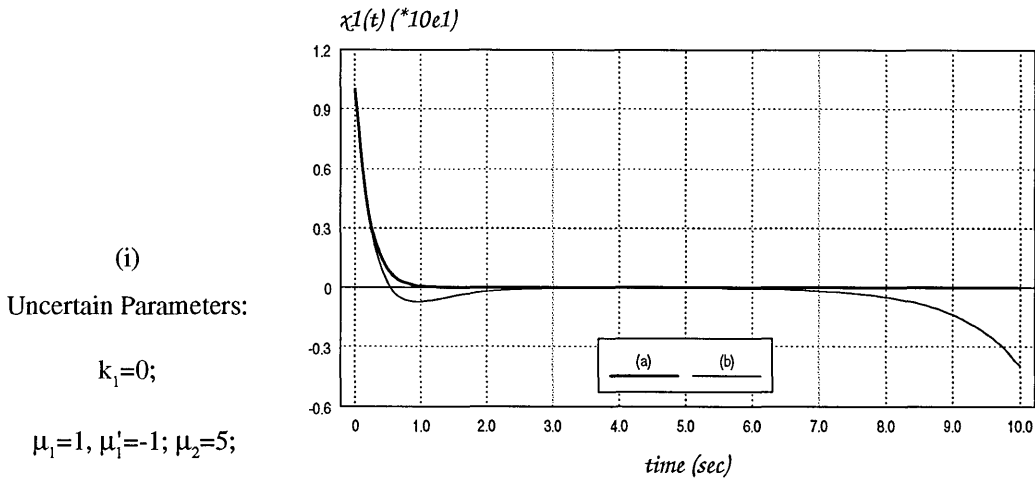
$$\dot{\mathbf{x}}(t) = \begin{pmatrix} a_{11} & a \\ a & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} u(t)$$

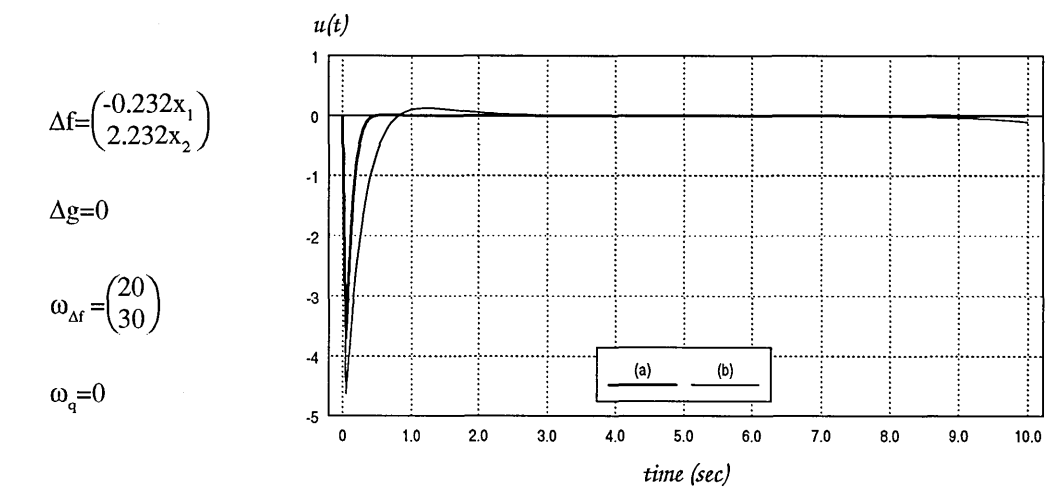
Similarly to case 1, if an open loop pole position is thought to be $-\mu_1$, but is in fact $-\mu'_1$, the uncertainty may be represented as

$$\Delta \mathbf{f} = \begin{pmatrix} \Delta a_{11} x_1 \\ \Delta a_{22} x_2 \end{pmatrix} \quad \Delta \mathbf{g} = \begin{pmatrix} 0 \\ \Delta b_2 \end{pmatrix}$$

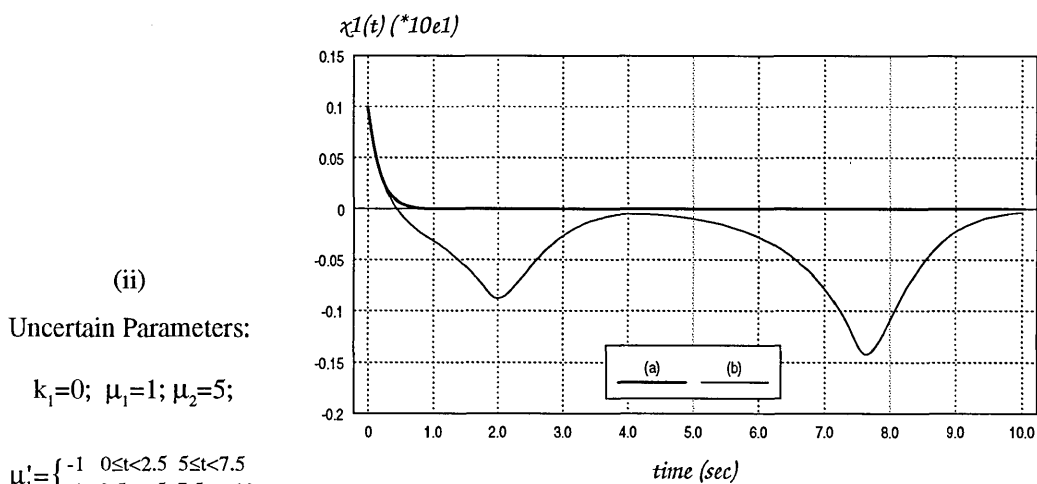
where a_{11} , a_{22} , b_1 , b_2 are nominal values depending on the nominal eigenvalue $-\mu_1$, a'_{11} , a'_{22} are real values depending on the true eigenvalue $-\mu'_1$, and $\Delta a_{11} = a'_{11} - a_{11}$, $\Delta a_{22} = a'_{22} - a_{22}$, and $\Delta b_2 = \Delta a_{22} k_1 / a$ are the uncertain parameters.

If, however, the open loop pole $-\mu_1$ is correct, but $k_1 < 0$, this results in a non-minimum phase problem. This difficult control problem is regarded as an uncertain problem with $\Delta \mathbf{g} = [\Delta b_1, \Delta b_2]^T$, where $b_2 = 1/a$, $\Delta b_1 = k_1$, and $\Delta b_2 = a_{22} k_1 / a$.

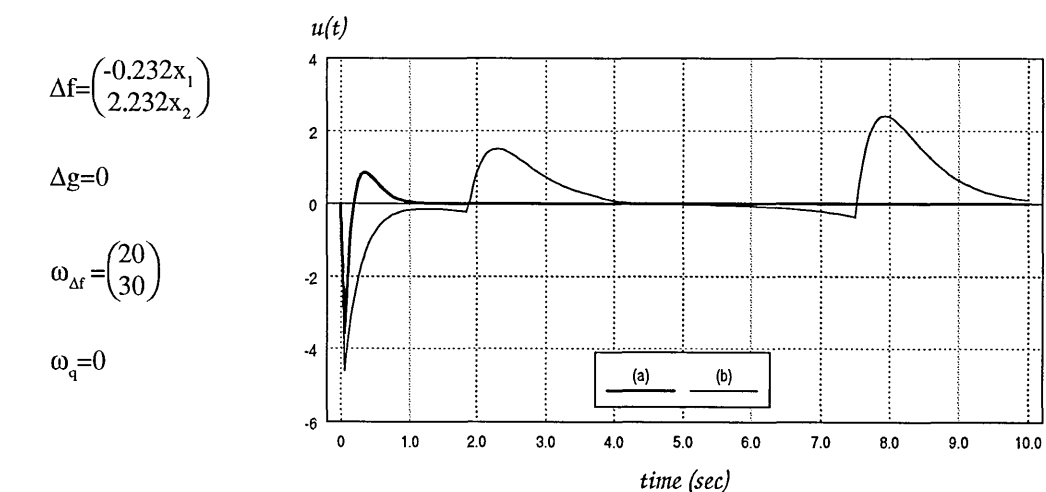




Control Signals



System Outputs



Control Signals

(a) The technique of theorem 4.4; (b) The technique of feedback linearisation alone
Fig. 4.3 Case 2: Mismatched uncertainty Δf caused by uncertainty in pole location

Clearly, in both cases, the uncertainties do not satisfy the complete matching conditions, but they do satisfy the incomplete matching conditions (2.12) and (2.13) respectively. It is therefore said that the system has partially matched uncertainties.

The feedback control can be designed via theorem 4.4 when the position of the pole is uncertain even if it is unstable, and the simulation results are shown in Fig. 4.3. The following values were selected: $C_1=0.3$; $C_2=10$, and $\alpha_0=6$; $\alpha_1=5$; $\alpha_2=1$. This results in a closed loop system with poles: $\lambda_1=-2$ and $\lambda_2=-3$.

For the non-minimum phase problem, the control can be designed via theorem 4.5. The following values were selected for simulation purposes: $C_1=0.3$; $C_2=10$; and $\alpha_0=10$; $\alpha_1=7$; $\alpha_2=1$. This results in a closed loop system with poles $\lambda_1=-2$ and $\lambda_2=-5$.

A possible Lyapunov function for the given closed loop system is

$$V(x) = x^T P x = [x_1, x_2] \begin{pmatrix} 0.1263 & -0.0577 \\ -0.0577 & 0.4449 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

by solving $A^T P + P A = -Q$, and condition (4.57) is of the form

$$\begin{aligned} L_g V \cdot L_{\Delta g} V &= 4x^T P g \cdot \Delta g^T P x = [x_1, x_2] \begin{pmatrix} 0.0623 & -0.1011 \\ -0.2430 & 0.3943 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= 0.0623x_1^2 + 0.3943x_2^2 - 0.3441x_1x_2 \end{aligned}$$

It is clear that $L_g V \cdot L_{\Delta g} V \geq 0$ holds as long as $x_1x_2 \leq 0$.

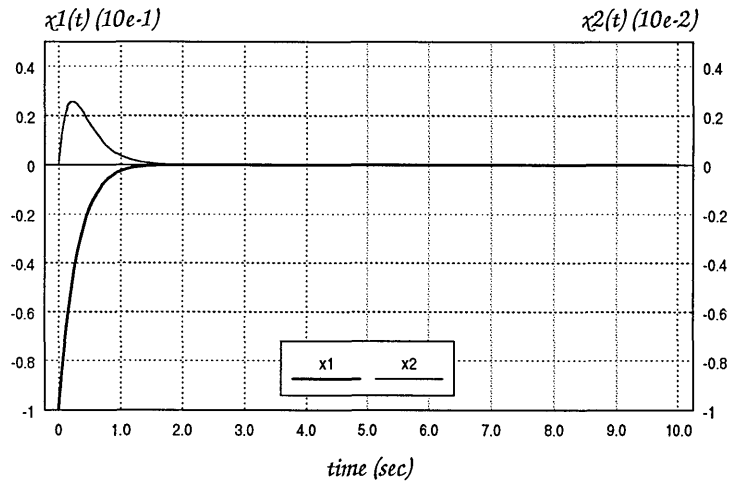
The simulation results are shown in Fig. 4.3 and 4.4. From the simulation results, it is clear that the applications of the techniques of theorem 4.4 and 4.5 result in satisfactory performance, compared with the performance resulting from the application of the feedback linearisation technique alone. For the feedback linearisation controller, although the output of the system $x_1(t)$ does appear stable (Fig.4.4), the internal dynamic $x_2(t)$ is highly unstable because of the presence of non-minimum phase, and hence an unstable system results. In contrast, the techniques of theorem 4.4 and 4.5 result in a stable closed loop system for both external and internal dynamics.

The simulation results of Fig. 4.4 show that theorem 4.5 is applicable and very effective for the control of this non-minimum phase problem when $x_1x_2 \leq 0 \forall t > 0$.

Uncertain Parameters:

$$k_1=0; k'_1=-2;$$

$$\mu_1=1; \mu_2=5$$



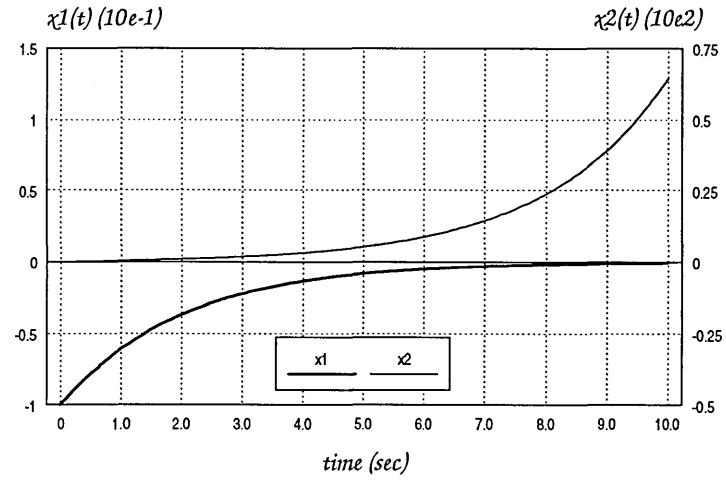
System states controlled by the technique of theorem 4.5

$$\Delta f=0$$

$$\Delta g = \begin{pmatrix} k'_1 \\ -1.122k'_1 \end{pmatrix}$$

$$\omega_p=0$$

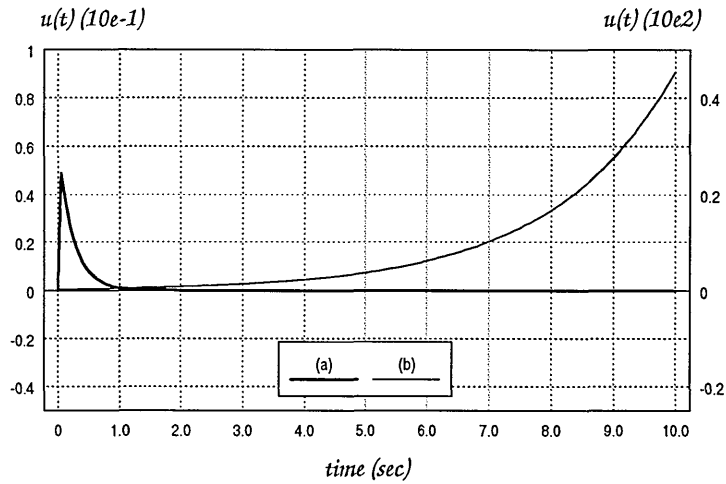
$$\omega_{\Delta g} = \begin{pmatrix} |k'_1| \\ 1.122|k'_1| \end{pmatrix}$$



System states controlled by feedback linearisation alone

(a) Technique of theorem 4.5;

(b) Technique of feedback linearisation alone



Control Signals

Fig. 4.4 Case 2: Mismatched uncertainty Δg caused by nonminimum phase dynamics, $L_g V \cdot L_{\Delta g} V > 0$

However, this is not a very practical case. For most practical systems, the trajectory of the closed loop system could be any value in the admissible region of state space, and may be unpredictable, particularly when some disturbances exist. Therefore, the general situation where (2.16) is not satisfied is now considered, but it is assumed that the uncertainty caused by non-minimum phase is such that condition (4.79) holds.

We now select $C_1=0.3$; $C_2=100$; $\lambda_1=-2$; $\lambda_2=-3$ and Lyapunov function

$$V(x) = x^T P x = [x_1, x_2] \begin{pmatrix} 0.2028 & -0.1638 \\ -0.1638 & 0.4580 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The simulation results are given in Fig.4.5, and the same conclusions as before can be drawn.

4.4.1 Mismatched Uncertainties

If the open loop poles are uncertain and non-minimum phase occurs, the system can be written as

$$\dot{x}(t) = \begin{pmatrix} a_{11} & a \\ a & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} u(t) + \begin{pmatrix} \Delta a_{11} x_1 \\ \Delta a_{22} x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \Delta b_2 \end{pmatrix} u(t)$$

where $b_1=k_1$, $b_2=(1+a_{22}k_1)/a$, and $\Delta b_2=\Delta a_{22}k_1/a$.

Although the uncertainties lie in the range of the input mapping $g(x)$, no functions p and q exist such that $\Delta f=g \cdot p$, and $\Delta g=g \cdot q$. Therefore they can only be treated as a special kind of mismatched uncertainty. The system can be expressed as

$$\dot{x}(t) = \begin{pmatrix} a_{11} & a \\ a & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ b_2 \end{pmatrix} u(t) + \begin{pmatrix} \Delta a_{11} x_1 \\ \Delta a_{22} x_2 \end{pmatrix} + \begin{pmatrix} \Delta b_1 \\ \Delta b_2 \end{pmatrix} u(t)$$

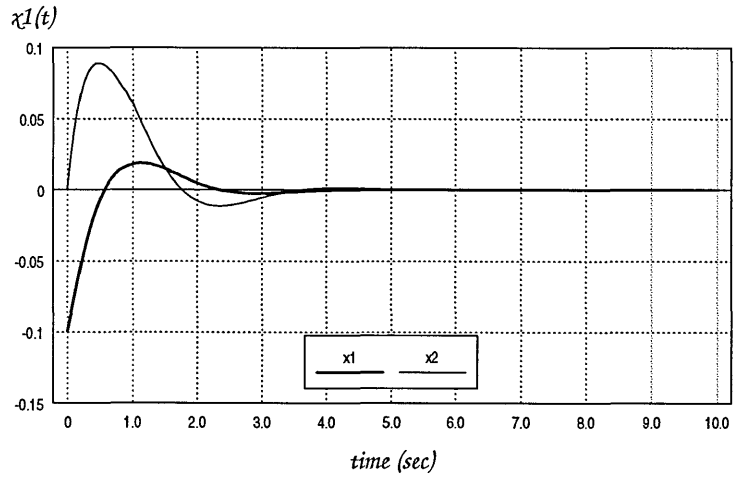
where $b_2=1/a$, $\Delta b_1=k_1$, and $\Delta b_2=(a_{22}+\Delta a_{22})k_1/a$. Thus the uncertainties, satisfying condition (2.16) or (2.17) in assumption 2.7, fall into the class of mismatched uncertainties.

The feedback control can be designed via theorem 4.8, where the following values were selected: $C_1=0.3$; $C_2=10$, and $\alpha_0=6$; $\alpha_1=5$; $\alpha_2=1$. This results in a closed loop system with poles $\lambda_1=-2$ and $\lambda_2=-3$. Simulation results are shown in Fig.4.6.

Uncertain Parameters:

$$k_1=0; k'_1=-0.5;$$

$$\mu_1=1; \mu_2=5$$



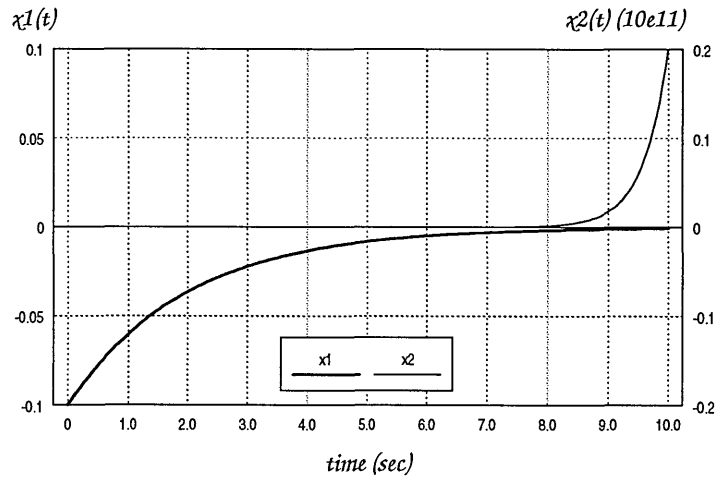
System states controlled by the technique of theorem 4.8

$$\Delta f=0$$

$$\Delta g = \begin{pmatrix} k'_1 \\ -1.122k'_1 \end{pmatrix}$$

$$\omega_p=0$$

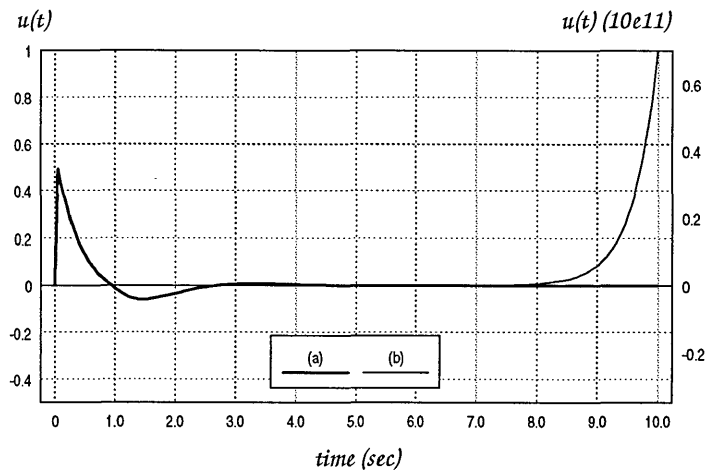
$$\omega_{\Delta g} = \begin{pmatrix} |k'_1| \\ 1.122|k'_1| \end{pmatrix}$$



System states controlled by feedback linearisation alone

(a) Technique of
theorem 4.8;

(b) Technique of
feedback linearisation
alone



Control Signals

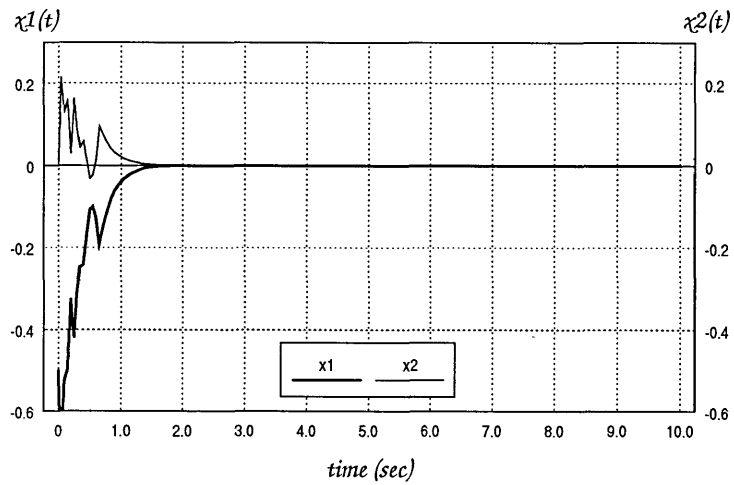
Fig. 4.5 Case 2: Mismatched uncertainty Δg caused by nonminimum phase dynamics, $|L_g V| > |L_{\Delta g} V|$

(i)

Uncertain Parameters:

$$k_1=0; k'_1=-0.5;$$

$$\mu_1=1, \mu'_1=-1; \mu_2=5;$$



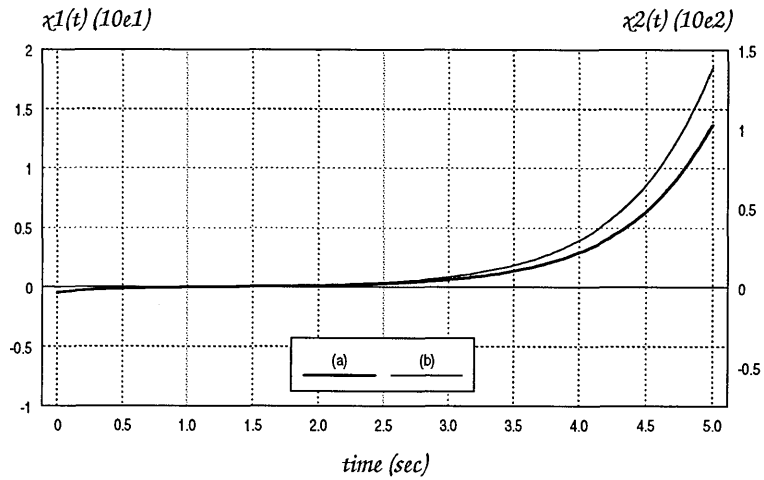
System states controlled by the technique of theorem 4.8

$$\Delta f = \begin{pmatrix} -1.646x_1 + 0.414x_2 \\ 0.414x_1 + 3.646x_2 \end{pmatrix}$$

$$\Delta g = \begin{pmatrix} k'_1 \\ -0.293 + 0.457k'_1 \end{pmatrix}$$

$$\omega_{\Delta f} = \begin{pmatrix} 30 \\ 50 \end{pmatrix}$$

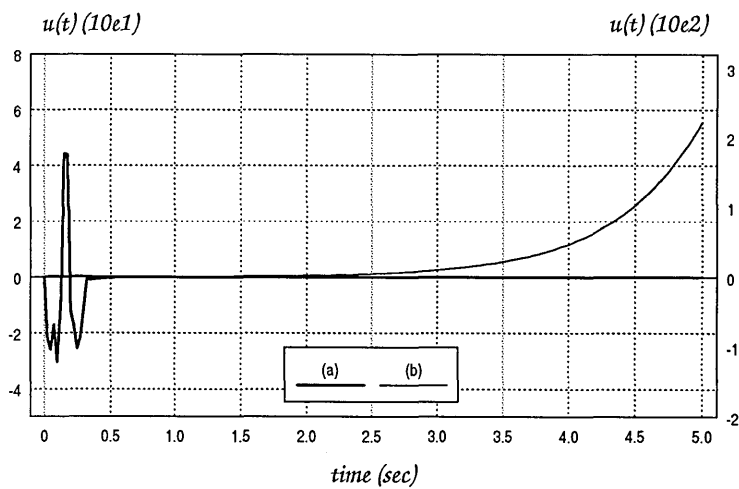
$$\omega_{\Delta g} = \begin{pmatrix} |k'_1| \\ 1 - 0.293|k'_1| + 0.457k'_1 \end{pmatrix}$$



System states controlled by feedback linearisation alone

(a) Technique of theorem 4.8;

(b) Technique of feedback linearisation alone

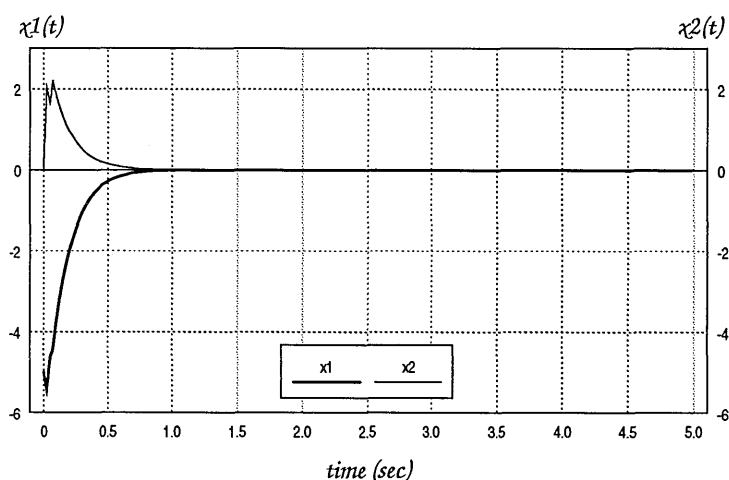


Control Signals

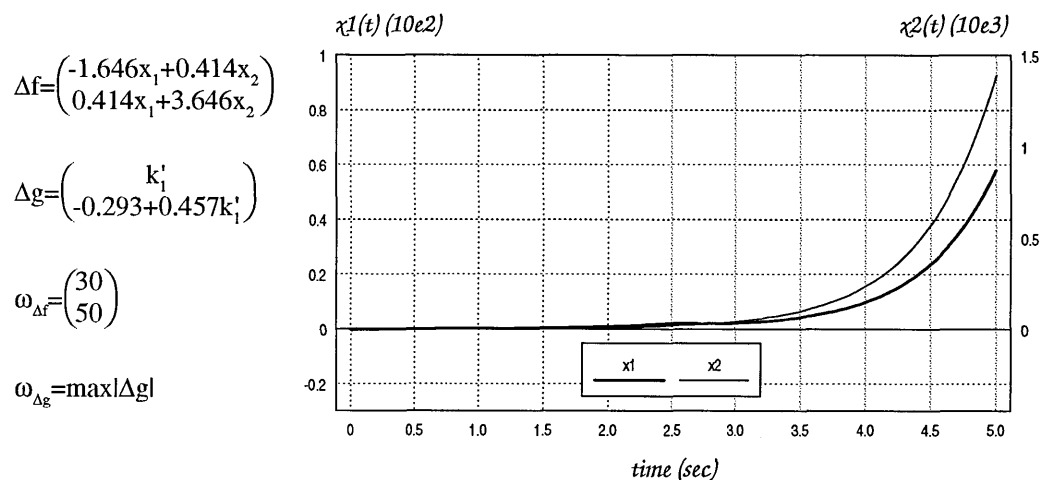
(ii)
Uncertain Parameters:

$$k_1=1; k'_1=\begin{cases} -0.3 & t=0 \\ -0.5 & t=2.5 \end{cases}$$

$$\mu_1=1, \mu'_1=-1; \mu_2=5;$$



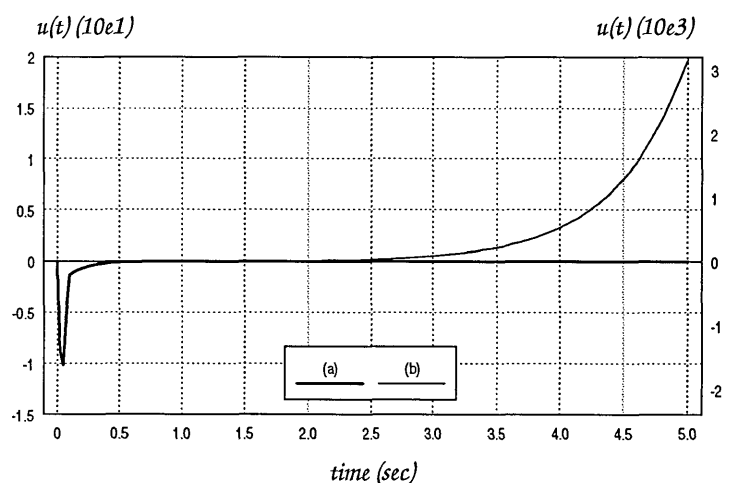
System states controlled by the technique of theorem 4.8



System states controlled by feedback linearisation alone

(a) Technique of
theorem 4.8;

(b) Technique of
feedback linearisation
alone



Control Signals

Fig. 4.6 Case 3: Mismatched uncertainties caused by uncertain pole position and nonminimum phase dynamics

4.5 SUMMARY

In this chapter, a rather general class of nonlinear uncertain systems has been considered, and robust feedback control laws have been obtained for different cases. The techniques can be summarised as follows:

Algorithm:

- ① Transform the original nonlinear uncertain system of form (4.1) into a linearisable form (4.2);
- ② Obtain u_1 from (4.10) to linearise the nominal system, i.e., the certain part of the nonlinear system (4.2);
- ③ Select parameters α_i ($i=0, \dots, \nu$) for linearisation feedback control (4.10) to place the nominal closed loop poles at desired positions;
- ④ Define a Lyapunov function $V(\cdot)$ for the linearised certain part of the system to be controlled;
- ⑤ Determine ρ from (4.75), (4.80) or their simplified versions and choose C_1, C_2 from (4.29)~(4.33) to construct $u_2(t)$ such that $L_2 V$ falls into the shaded area D in Fig.4.1.

Nonlinear state feedback, based on the feedback linearisation technique, is applied to the certain part of the system, such that a desired stability margin for the nominal closed loop system is achieved. Additional nonlinear feedback is introduced to compensate for uncertainties, such as parametric uncertainties, external disturbances, and stability is guaranteed via Lyapunov stability theory when some uncertainties are incorporated in the system regardless of whether matching conditions are satisfied. Compared to other developments based on Lyapunov theory, in particular that of Barmish *et al*^[2], significant improvements have been made, in that the techniques can achieve far better results because they can compensate not only for the effect of the uncertainty in the state matrix $\Delta f(x, \gamma, t)$,

but also for the effect caused by $u_1(t)$ through the uncertainty in the control matrix $\Delta g(z, \gamma, t)$, even for the case of $\omega_q > 1$ and for the case of mismatched uncertainty. The technique described in theorem 4.7 generalises the results of theorems 4.3~4.6, whilst retaining the concise statement of the algorithm, so that all results may be described in a unified fashion for the following cases: (i) uncertainties in both the state and the input mapping satisfying the matching conditions, (ii) only one of them satisfying the matching conditions, and (iii) mismatch in both.

Theorem 4.8 retains the same problem statement and achieves the same system performances as that of 4.7, but it does so by increasing the feedback gain for uncertainty compensation. However theorem 4.8 is more generally applicable.

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Robust Stability Control (II)



Overview

This chapter presents new synthesis techniques based on variable structure control and using Lyapunov stability theory. The techniques are designed for SISO systems with mismatched uncertainties. Simulation results for a second order system are included.



Outline

- ✓ Introduction
- ✓ Control Synthesis Based on Variable Structure Control
- ✓ Comments on System Performance
- ✓ Illustrative Example
- ✓ Summary

5.1 INTRODUCTION

THE approach to robust control considered in this chapter is that of variable structure control. It resembles adaptive control in that the structure of the controller varies in response to the changing state of the system in order to obtain the desired response. The controller is, however, synthesised in a deterministic way. This is accomplished by using a high speed switching feedback control which forces the trajectory of the system onto a prespecified hypersurface in state space, where it is maintained thereafter.

For the class of systems to which it applies, variable structure control design provides a systematic approach to the problem of maintaining stability and consistent performance in the presence of modelling uncertainties. Furthermore, by replacing a pure switching control by its smooth approximation, the relay chattering problem can be alleviated. Variable structure control has been successfully applied to robot manipulators, underwater vehicles, and power systems^[1,2].

The aim of this chapter is to investigate a synthesis problem of nonlinear uncertain systems in a deterministic way, in which the problem statement is the same as that in the last chapter, but a different type of controller is developed, so that different system performance may result. No requirements are imposed on the structure and size of the uncertainty, and no assumptions are made concerning precompensation of the nominal system. A set of robust feedback controllers can be obtained, which results in a practically stabilised closed loop system, even for nonlinear systems with unstable nominal part, in the presence of significant mismatched parameter tolerances and external disturbance. It is also shown that the controller has the same structure as that developed for the nominal system where no uncertainty is explicitly considered; the only difference is that the former employs a variable controller gain, which depends on the known uncertainty bounds, and

the latter has a constant one. The design procedure is based on Lyapunov theory.

The primary concepts of variable structure control are presented in appendix B, and form the basis for the development of the results of this chapter. Specifically, in section 5.2, the associated basic controller design for nominal systems is illustrated first, and then two robust variable structure controllers for nonlinear systems with uncertainties are developed. Section 5.3 describes the stability properties of the resulting controlled system, and section 5.4 presents an illustrative example with simulation results.

5.2 CONTROL SYNTHESIS BASED ON VARIABLE STRUCTURE CONTROL

In this section the robust stability control problem for nonlinear systems in the presence of uncertainties is still considered. The problem statement is the same as that of chapter 4, but the control synthesis is based on variable structure control.

Although the ideal sliding mode may not occur when nonlinear systems are subjected to uncertainties, the designs of this chapter guarantee the existence of a sliding mode within a vicinity of the switching surface. The following control strategy is proposed:

- ① *Design a switching surface to specify the closed loop system performance;*
- ② *Construct a control law with variable feedback gain to steer the state to the switching surface, and guarantee the existence of a sliding mode.*

The results of this section enable us to achieve our aims with only very weak conditions on the nature and size of the uncertainties. The technique is developed directly for nonlinear systems with mismatched uncertainties, but it is clearly applicable to other cases, such as partially matched or completely matched uncertainties. The control signal will be of the following form

$$u(t) = u_{eq} - [L_g \sigma]^{-1} \cdot \rho \cdot \text{sign}(\sigma) \quad (5.1)$$

where the feedback gain $\rho > 0$ is developed in the sequel, and u_{eq} is called the equivalent control, and is given by

$$u_{eq} = -[\nabla \sigma(x) \cdot g(x)]^{-1} \cdot \nabla \sigma(x) \cdot f(x) = -(L_g \sigma)^{-1} \cdot L_f \sigma \quad (5.2)$$

where $L_f \sigma$ and $L_g \sigma \neq 0$ are the Lie derivatives of $\sigma(x)$ with respect to f and g respectively.

DEFINITION 5.1. (*Generalised Lyapunov Function*)

A continuous function $V(t)$, which depends on the chosen switching function $\sigma(x)$, can be defined as a generalised candidate Lyapunov function, if

$$V(t) \triangleq \frac{1}{2} \cdot \sigma^2(x) > 0 \quad \forall (x, t) \ni \sigma(x) \neq 0 \quad \text{and} \quad V|_{\sigma(x)=0} = 0 \quad (5.3)$$

with continuous derivative, such that, for $X = \{x(t) \in \mathbb{R}^n | \sigma(x) \neq 0, x(t_0) = x_0\}$

$$\dot{V}(t) = \frac{1}{2} \cdot \frac{d}{dt} \cdot \sigma^2(x) < 0 \quad (5.4)$$

holds.

Also, some conditions on the Lyapunov function similar to those in theorem 3.7 are required. The following definition is therefore introduced.

DEFINITION 5.2. (*Conditions on the Generalised Lyapunov Function*)

We assume that the generalised Lyapunov function $V(t)$ defined in definition 5.1 satisfies the following conditions

$$v_1(\|\dot{x}(t)\|) \leq V(t) \leq v_2(\|x(t)\|) \quad (5.5)$$

$$\dot{V}(t) < -v_3(\|x(t)\|) + v_4(\|x(t)\|) < 0 \quad (5.6)$$

where $v_i(\cdot)$ ($i=1,2$) are continuous strictly increasing functions, with the properties $v_i(0)=0$ and $\lim_{\epsilon \rightarrow \infty} v_i(\epsilon) = \infty$, and v_3, v_4 are positive continuous functions such that $v_3 - v_4$ is positive, so $(v_3 - v_4)^{-1}$ is defined away from zero and is continuous.

5.2.1 Controller with Constant Feedback Gain

A well-known result in variable structure control for unperturbed nominal systems is first stated, and the proof is included for the sake of completeness.

THEOREM 5.3. (*VSC for Nonlinear Systems without Uncertainty^[2]*)

Consider a nonlinear system of the form

$$\dot{x}(t) = f(x) + g(x)u(t) \quad (5.7)$$

A set of states $x \in \bar{X}$, and a switching function $\sigma(x) = 0$ are defined to specify the desired response of the closed loop system. Then a feedback controller of the form

$$u(t) = -\frac{(L_f \sigma + \rho \cdot \text{sign}(\sigma))}{L_g \sigma} \quad (5.8)$$

exists such that the closed loop system is stable. Here ρ is any positive constant.

Proof: According to definition 5.1, consider a generalised Lyapunov function candidate of the form

$$V(t) \triangleq \frac{1}{2} \cdot \sigma(x)^2 > 0 \quad \forall (x, t) \ni \sigma(x) \neq 0 \quad (5.9)$$

The time derivative of $V(t)$ is then

$$\begin{aligned} \dot{V}(t) &= \sigma \cdot \dot{\sigma} = \sigma \cdot \frac{\partial \sigma}{\partial x} \{ f(x) + g(x)u(t) \} = \sigma \cdot \{ L_f \sigma + L_g \sigma \cdot u(t) \} \\ &= \sigma \cdot \left(L_f \sigma - L_g \sigma \cdot \frac{(L_f \sigma + \rho \cdot \text{sign}(\sigma))}{L_g \sigma} \right) \\ &= -\rho \cdot |\sigma| < 0 \end{aligned} \quad (5.10)$$

The closed loop system is therefore stable. \square

5.2.2 Controller with Variable Feedback Gain

The major result concerned with the robust control of nonlinear uncertain systems may be obtained in a similar fashion to that of theorem 5.3.

THEOREM 5.4. (*VSC for Nonlinear Systems with Uncertainty: Case 1*)

Consider a single input nonlinear system, incorporating some mismatched uncertainties and external disturbances, of the form

$$\dot{x}(t) = f(x) + g(x)u(t) + \Delta f(x, \gamma, t) + \Delta g(x, \gamma, t)u(t) + \xi(t) \quad (5.11)$$

where the uncertainties and disturbance are all bounded, and satisfy

$$L_g \sigma \cdot L_{\Delta g} \sigma \geq 0 \quad (5.12)$$

Defining a switching function $\{\sigma(x)=0 \mid x(t) \in \bar{X}\}$, a feedback controller

$$u(t) = -\frac{(L_f \sigma + \rho(x) \cdot \text{sign}(\sigma))}{L_g \sigma} \quad (5.13)$$

then exists such that the feedback system (5.11) is stable for any bounded uncertainties and disturbance, if the controller gain is chosen as

$$\rho(x) \geq \sqrt{(\Omega_{\Delta f} + \Omega_{\xi})^2 + \frac{1}{2} |\Omega_{\Delta g} \cdot L_g \sigma| \left(-\frac{L_f \sigma}{L_g \sigma}\right)^2} > 0 \quad (5.14)$$

where

$$L_g \sigma \triangleq \frac{\partial \sigma}{\partial x} \cdot g \neq 0 \quad (5.15)$$

$$L_f \sigma \triangleq \frac{\partial \sigma}{\partial x} \cdot f \quad (5.16)$$

are the Lie derivatives of $\sigma(x)$ with respect to $f(x)$ and $g(x)$, and, in general, $\Omega_{\Delta f}$, $\Omega_{\Delta g}$ and Ω_{ξ} are functions of (x, t) defined by

$$\Omega_{\Delta f} \triangleq \sum_{k=1}^n \left| \frac{\partial \sigma}{\partial x_k} \right| \cdot \max_{\gamma(t) \in \mathbb{R}^{\gamma}} |\Delta f_k(x, \gamma, t)| \triangleq \left| \frac{\partial \sigma}{\partial x} \right| \omega_{\Delta f} > 0 \quad (5.17)$$

$$\Omega_{\Delta g} \triangleq \sum_{k=1}^n \left| \frac{\partial \sigma}{\partial x_k} \right| \cdot \max_{\gamma(t) \in \mathbb{R}^{\gamma}} |\Delta g_k(x, \gamma, t)| \triangleq \left| \frac{\partial \sigma}{\partial x} \right| \omega_{\Delta g} > 0 \quad (5.18)$$

$$\Omega_{\xi} \triangleq \sum_{k=1}^n \left| \frac{\partial \sigma}{\partial x_k} \right| \cdot \max_{t \geq 0} |\xi_k(t)| \triangleq \left| \frac{\partial \sigma}{\partial x} \right| \omega_{\xi} > 0 \quad (5.19)$$

Proof: A continuous function $\sigma(x)=0$ can be defined where $x \in \bar{X}$ is a set of states used to prespecify the performance of the closed loop system. According to definition 5.1, a generalised Lyapunov function is of the form

$$V(t) \triangleq \frac{1}{2} \cdot \sigma(x)^2 > 0 \quad \forall (x, t) \ni \sigma(x) \neq 0$$

and satisfies inequality (5.5) for $\{x \in X \mid \sigma(x) \neq 0\}$, so that, with the feedback of (5.13), the time derivative of the Lyapunov function for the closed loop system obtained is given by

$$\dot{V}(t) = \sigma \cdot \dot{\sigma} = \sigma \cdot \frac{\partial \sigma}{\partial x} \{ f(x) + g(x)u(t) + \Delta f(x, \gamma, t) + \Delta g(x, \gamma, t)u(t) + \xi(t) \}$$

$$\begin{aligned}
&= \sigma \cdot \{ L_f \sigma + L_g \sigma \cdot u(t) + L_{\Delta f} \sigma + L_{\Delta g} \sigma \cdot u(t) + L_\xi \sigma \} \\
&= \sigma \left\{ L_f \sigma - L_g \sigma \frac{L_f \sigma + \rho(x) \cdot \text{sign}(\sigma)}{L_g \sigma} + (L_{\Delta f} \sigma + L_\xi \sigma) - L_{\Delta g} \sigma \frac{L_f \sigma + \rho(x) \cdot \text{sign}(\sigma)}{L_g \sigma} \right\} \\
&= \sigma \left\{ L_f \sigma - L_g \sigma \frac{L_f \sigma + \rho(x) \cdot \frac{\sigma}{|\sigma|}}{L_g \sigma} + (L_{\Delta f} \sigma + L_\xi \sigma) - L_{\Delta g} \sigma \frac{L_f \sigma + \rho(x) \cdot \frac{\sigma}{|\sigma|}}{L_g \sigma} \right\} \\
&= -\frac{\rho(x) \sigma^2}{2|\sigma|} + \left\{ \sigma \cdot (L_{\Delta f} \sigma + L_\xi \sigma) - \frac{\rho(x) \sigma^2}{2|\sigma|} + \sigma \cdot (-L_{\Delta g} \sigma \cdot \frac{L_f \sigma}{L_g \sigma}) - \frac{L_{\Delta g} \sigma \cdot \rho(x) \sigma^2}{L_g \sigma \cdot |\sigma|} \right\}
\end{aligned}$$

By using the identity (4.12), the second term in the above equality can be expressed as

$$\begin{aligned}
&\frac{(L_{\Delta f} \sigma + L_\xi \sigma)^2 \cdot |\sigma|}{2\rho(x)} - \frac{\frac{1}{2}\rho(x)}{(L_{\Delta f} \sigma + L_\xi \sigma)^2 \cdot |\sigma|} \left((L_{\Delta f} \sigma + L_\xi \sigma) \cdot \sigma - \frac{(L_{\Delta f} \sigma + L_\xi \sigma)^2 \cdot |\sigma|}{\rho(x)} \right)^2 \\
&+ \frac{(-L_{\Delta g} \sigma \cdot \frac{L_f \sigma}{L_g \sigma})^2 \cdot |\sigma|}{4\rho(x) \cdot \frac{L_{\Delta g} \sigma}{L_g \sigma}} - \frac{\rho(x) \cdot \frac{L_{\Delta g} \sigma}{L_g \sigma}}{(-L_{\Delta g} \sigma \cdot \frac{L_f \sigma}{L_g \sigma})^2 \cdot |\sigma|} \left((-L_{\Delta g} \sigma \cdot \frac{L_f \sigma}{L_g \sigma}) \sigma - \frac{(-L_{\Delta g} \sigma \cdot \frac{L_f \sigma}{L_g \sigma})^2 \cdot |\sigma|}{2\rho(x) \cdot \frac{L_{\Delta g} \sigma}{L_g \sigma}} \right)^2
\end{aligned}$$

Considering condition (5.12), we may write the inequality above as

$$\begin{aligned}
\dot{V}[\sigma(x)] &\leq -\frac{\rho(x)}{2|\sigma|} \cdot \sigma^2 + \frac{(L_{\Delta f} \sigma + L_\xi \sigma)^2 \cdot |\sigma|}{2\rho(x)} + \frac{(-L_{\Delta g} \sigma \cdot \frac{L_f \sigma}{L_g \sigma})^2 \cdot |\sigma|}{4\rho(x) \cdot \frac{L_{\Delta g} \sigma}{L_g \sigma}} \\
&\triangleq -v_3[x(t), \gamma(t)] + v_4[x(t), \gamma(t)] < 0
\end{aligned} \tag{5.20}$$

where we may identify

$$v_3[x(t), \gamma(t)] \triangleq \frac{\rho(x)}{2|\sigma|} \cdot \sigma^2 \tag{5.21}$$

$$v_4[x(t), \gamma(t)] \triangleq \frac{(L_{\Delta f} \sigma + L_\xi \sigma)^2 \cdot |\sigma|}{2\rho(x)} + \frac{(-L_{\Delta g} \sigma \cdot \frac{L_f \sigma}{L_g \sigma})^2 \cdot |\sigma|}{4\rho(x) \cdot \frac{L_{\Delta g} \sigma}{L_g \sigma}} \tag{5.22}$$

The problem is now to design a control such that the reachability condition (appendix B)

$$\sigma(x) \cdot \dot{\sigma}(x) < 0 \tag{5.23}$$

holds. Then the state trajectories will converge to $\sigma(x)=0$, and are restricted to it for all subsequent time. From (5.20) we have

$$\frac{(L_{\Delta f}\sigma + L_{\xi}\sigma)^2 \cdot |\sigma|}{2\rho(x)} + \frac{(-L_{\Delta g}\sigma \cdot \frac{L_f\sigma}{L_g\sigma})^2 \cdot |\sigma|}{4\rho(x) \cdot \frac{L_{\Delta g}\sigma}{L_g\sigma}} < \frac{\rho(x)}{2|\sigma|} \cdot \sigma^2$$

$$i.e., \quad \rho(x)^2 > (L_{\Delta f}\sigma + L_{\xi}\sigma)^2 + \frac{1}{2}L_{\Delta g}\sigma \cdot L_g\sigma \left(-\frac{L_f\sigma}{L_g\sigma}\right)^2$$

and so

$$\rho(x) > \sqrt{(L_{\Delta f}\sigma + L_{\xi}\sigma)^2 + \frac{1}{2}L_{\Delta g}\sigma \cdot L_g\sigma \left(-\frac{L_f\sigma}{L_g\sigma}\right)^2} > 0 \quad (5.24)$$

Because

$$L_{\Delta f}\sigma \leq \left| \frac{\partial \sigma}{\partial x} \cdot \Delta f(x, \gamma, t) \right| < \sum_{k=1}^n \left| \frac{\partial \sigma}{\partial x_k} \right| \cdot \max_{\gamma(t) \in \mathbb{R}^\gamma} |\Delta f_k(x, \gamma, t)| \leq \left| \frac{\partial \sigma}{\partial x} \right| \cdot \omega_{\Delta f} = \Omega_{\Delta f}$$

$$L_{\Delta g}\sigma \leq \left| \frac{\partial \sigma}{\partial x} \cdot \Delta g(x, \gamma, t) \right| < \sum_{k=1}^n \left| \frac{\partial \sigma}{\partial x_k} \right| \cdot \max_{\gamma(t) \in \mathbb{R}^\gamma} |\Delta g_k(x, \gamma, t)| \leq \left| \frac{\partial \sigma}{\partial x} \right| \cdot \omega_{\Delta g} = \Omega_{\Delta g}$$

$$L_{\xi}\sigma \leq \left| \frac{\partial \sigma}{\partial x} \cdot \xi(t) \right| < \sum_{k=1}^n \left| \frac{\partial \sigma}{\partial x_k} \right| \cdot \max_{t \geq 0} |\xi_k(t)| \leq \left| \frac{\partial \sigma}{\partial x} \right| \cdot \omega_{\xi} = \Omega_{\xi}$$

it follows that if $\rho(x)$ is chosen according to the known bounds given by (5.17), (5.18), and (5.19), the controller gain is of the form (5.14), whose entries are all deterministic and known. It is obvious that if

$$\rho(x) \geq \sqrt{(\Omega_{\Delta f} + \Omega_{\xi})^2 + \frac{1}{2}|\Omega_{\Delta g}L_g\sigma| \left(-\frac{L_f\sigma}{L_g\sigma}\right)^2}$$

then

$$\rho(x) \geq \sqrt{(L_{\Delta f}\sigma + L_{\xi}\sigma)^2 + \frac{1}{2}L_{\Delta g}\sigma \cdot L_g\sigma \left(-\frac{L_f\sigma}{L_g\sigma}\right)^2} > 0 \quad (5.25)$$

holds, and it follows that the inequality

$$\begin{aligned} \dot{V}(t) &< -\frac{\rho(x)}{2|\sigma|} \cdot \sigma^2 + \frac{(\Omega_{\Delta f} + \Omega_{\xi})^2 \cdot |\sigma|}{2\rho(x)} + \frac{(\Omega_{\Delta g} \cdot \frac{L_f\sigma}{L_g\sigma})^2 \cdot |\sigma|}{4\rho(x) \cdot \frac{\Omega_{\Delta g}}{|L_g\sigma|}} \\ &= -v_3[||x||] + v_4[||x||] < 0 \end{aligned} \quad (5.26)$$

is true. Thus the controller has the following form

$$u(t) = -\frac{1}{L_g\sigma} \left\{ L_f\sigma + \sqrt{(\Omega_{\Delta f} + \Omega_{\xi})^2 + \frac{1}{2}|\Omega_{\Delta g}L_g\sigma| \left(-\frac{L_f\sigma}{L_g\sigma}\right)^2} \cdot \text{sign}(\sigma) \right\} \quad (5.27)$$

and this results in a stable closed loop. \square

REMARK 5.1:

- The development of the results of theorem 5.4 is based on condition (5.12), where $L_g\sigma \cdot L_{\Delta g}\sigma$ is assumed to be positive. For any systems which satisfies condition (5.12), stability of the closed loop system is guaranteed by the feedback control of (5.13).
- Condition (5.12) may be satisfied by properly choosing the switching function $\sigma(x)$, especially for the most common type of switching function $\sigma(x)=Sx$. Then $\partial\sigma/\partial x=S$ and $L_g\sigma \cdot L_{\Delta g}\sigma=S(g \cdot \Delta g^T)S^T$. It is therefore possible to choose the elements of S such that condition (5.12) holds for the input mapping of the given nonlinear uncertain system.
- If, for any given nonlinear uncertain system, no suitable switching function $\sigma(x)$ exists such that both the prespecified system performance and condition (5.12) are satisfied, the result developed in theorem 5.4 is not applicable. The following theorem is an alternative version of theorem 5.4, which can be used to deal with the cases where condition (5.12) is not met.

5.2.3 Controller with Improved Variable Feedback Gain

THEOREM 5.5. (VSC for Nonlinear Systems with Uncertainty: Case 2)

Consider again the nonlinear uncertain system (5.11), and suppose the following condition

$$|L_g\sigma| > 2|L_{\Delta g}\sigma| \quad (5.28)$$

is satisfied. For a defined switching function $\{\sigma(x)=0 \mid x(t) \in \bar{X}\}$, a feedback controller

$$u(t) = -\frac{(L_f\sigma + \rho(x) \cdot \text{sign}(\sigma))}{L_g\sigma} \quad (5.29)$$

then exists such that the feedback system (5.11) is stable for any bounded uncertainties and disturbance, if the controller gain is chosen to satisfy

$$\rho(x) \geq \sqrt{(\Omega_{\Delta f+\xi})^2 + \frac{1}{2}|\Omega_{\Delta g} \cdot L_g\sigma| \left(\frac{L_f\sigma}{L_g\sigma}\right)^2} + \frac{\Omega_{\Delta f+\xi}|L_g\sigma| + \Omega_{\Delta g}|L_f\sigma|}{\frac{1}{2}|L_g\sigma| - \Omega_{\Delta g}} > 0 \quad (5.30)$$

where $L_g\sigma$, $L_f\sigma$ are the Lie derivatives of $\sigma(x)$ with respect to $f(x)$ and $g(x)$ defined by

(5.15) and (5.16), and $\Omega_{\Delta f}$, $\Omega_{\Delta g}$ and Ω_{ξ} are the bounds of the uncertainties and disturbance respectively, defined by (5.17), (5.18) and (5.19).

Proof: Consider a generalised Lyapunov function of the form (5.3) satisfying inequality (5.5) for $\{x \in X | \sigma(x) \neq 0\}$, so that, with the feedback of (5.29), the time derivative of the Lyapunov function for the closed loop system obtained is given by

$$\begin{aligned}
 \dot{V}(t) &= \sigma \cdot \dot{\sigma} \\
 &= \sigma \cdot \{ L_f \sigma + L_g \sigma \cdot u(t) + L_{\Delta f} \sigma + L_{\Delta g} \sigma \cdot u(t) + L_{\xi} \sigma \} \\
 &= \sigma \cdot \left\{ L_f \sigma - L_g \sigma \cdot \frac{L_f \sigma + \rho \cdot \text{sign}(\sigma)}{L_g \sigma} + L_{\Delta f + \xi} \sigma - L_{\Delta g} \sigma \cdot \frac{L_f \sigma + \rho \cdot \text{sign}(\sigma)}{L_g \sigma} \right\} \\
 &= \sigma \cdot \left\{ L_f \sigma - L_g \sigma \cdot \frac{L_f \sigma + \rho \cdot \frac{\sigma}{|\sigma|}}{L_g \sigma} + L_{\Delta f + \xi} \sigma - L_{\Delta g} \sigma \cdot \frac{L_f \sigma + \rho \cdot \frac{\sigma}{|\sigma|}}{L_g \sigma} \right\} \\
 &\leq -\frac{\rho \sigma^2}{2|\sigma|} + 2 \frac{|L_{\Delta g} \sigma| \cdot \rho \sigma^2}{|L_g \sigma| \cdot |\sigma|} \\
 &\quad + \left\{ \sigma \cdot L_{\Delta f + \xi} \sigma - \frac{\rho \sigma^2}{2|\sigma|} + \sigma \left(-L_{\Delta g} \sigma \cdot \frac{L_f \sigma}{L_g \sigma} - \frac{|L_{\Delta g} \sigma| \cdot \rho \sigma^2}{|L_g \sigma| \cdot |\sigma|} \right) \right\} \quad (5.31)
 \end{aligned}$$

By using identity (4.12), the second term in the above equality can be expressed as

$$\begin{aligned}
 &\frac{(L_{\Delta f + \xi} \sigma)^2 \cdot |\sigma|}{2\rho} - \frac{\frac{1}{2}\rho}{(L_{\Delta f + \xi} \sigma)^2 \cdot |\sigma|} \left((L_{\Delta f + \xi} \sigma) \cdot \sigma - \frac{(L_{\Delta f + \xi} \sigma)^2 \cdot |\sigma|}{\rho} \right)^2 \\
 &+ \frac{(-L_{\Delta g} \sigma \cdot \frac{L_f \sigma}{L_g \sigma})^2 \cdot |\sigma|}{4\rho \cdot \frac{|L_{\Delta g} \sigma|}{|L_g \sigma|}} - \frac{\rho \cdot \frac{|L_{\Delta g} \sigma|}{|L_g \sigma|}}{(-L_{\Delta g} \sigma \cdot \frac{L_f \sigma}{L_g \sigma})^2 \cdot |\sigma|} \left((-L_{\Delta g} \sigma \cdot \frac{L_f \sigma}{L_g \sigma}) \sigma - \frac{(-L_{\Delta g} \sigma \cdot \frac{L_f \sigma}{L_g \sigma})^2 \cdot |\sigma|}{2\rho \cdot \frac{|L_{\Delta g} \sigma|}{|L_g \sigma|}} \right)^2
 \end{aligned}$$

We suppose that

$$\begin{aligned}
 &\frac{\frac{1}{2}\rho}{(L_{\Delta f + \xi} \sigma)^2 \cdot |\sigma|} \left((L_{\Delta f + \xi} \sigma) \cdot \sigma - \frac{(L_{\Delta f + \xi} \sigma)^2 \cdot |\sigma|}{\rho} \right)^2 \\
 &+ \frac{\rho \cdot \frac{|L_{\Delta g} \sigma|}{|L_g \sigma|}}{(-L_{\Delta g} \sigma \cdot \frac{L_f \sigma}{L_g \sigma})^2 \cdot |\sigma|} \left((-L_{\Delta g} \sigma \cdot \frac{L_f \sigma}{L_g \sigma}) \sigma - \frac{(-L_{\Delta g} \sigma \cdot \frac{L_f \sigma}{L_g \sigma})^2 \cdot |\sigma|}{2\rho \cdot \frac{|L_{\Delta g} \sigma|}{|L_g \sigma|}} \right)^2 \geq 2 \frac{|L_{\Delta g} \sigma| \cdot \rho \sigma^2}{|L_g \sigma| \cdot |\sigma|} \quad (5.32)
 \end{aligned}$$

and also considering (5.6) of definition 5.7, we have

$$\begin{aligned} \dot{V}[\sigma(x)] &\leq -\frac{\rho}{2|\sigma|} \cdot \sigma^2 + \frac{(L_{\Delta f+\xi}\sigma)^2 \cdot |\sigma|}{2\rho} + \frac{(-L_{\Delta g}\sigma \cdot \frac{L_f\sigma}{L_g\sigma})^2 \cdot |\sigma|}{4\rho \cdot \frac{|L_{\Delta g}\sigma|}{|L_g\sigma|}} \\ &\triangleq -v_3[x(t), \gamma(t)] + v_4[x(t), \gamma(t)] \end{aligned} \quad (5.33)$$

where $v_3[x(t), \gamma(t)] \triangleq \frac{\rho}{2|\sigma|} \cdot \sigma^2$ (5.34)

$$v_4[x(t), \gamma(t)] \triangleq \frac{(L_{\Delta f+\xi}\sigma)^2 \cdot |\sigma|}{2\rho} + \frac{(-L_{\Delta g}\sigma \cdot \frac{L_f\sigma}{L_g\sigma})^2 \cdot |\sigma|}{4\rho \cdot \frac{|L_{\Delta g}\sigma|}{|L_g\sigma|}} \quad (5.35)$$

Similarly to theorem 5.4, suppose the reachability condition $\dot{V}(t) = \sigma \cdot \dot{\sigma} < 0$ holds, i.e.,

$$\dot{V}[\sigma(x)] \leq -v_3[x(t), \gamma(t)] + v_4[x(t), \gamma(t)] < 0$$

Then the state trajectories will converge, and will be restricted to $\sigma(x)=0$. So we have

$$\begin{aligned} \frac{(L_{\Delta f+\xi}\sigma)^2 \cdot |\sigma|}{2\rho} + \frac{(-L_{\Delta g}\sigma \cdot \frac{L_f\sigma}{L_g\sigma})^2 \cdot |\sigma|}{4\rho \cdot \frac{|L_{\Delta g}\sigma|}{|L_g\sigma|}} &< \frac{\rho}{2|\sigma|} \cdot \sigma^2 \\ \rho &> \sqrt{(L_{\Delta f+\xi}\sigma)^2 + \frac{1}{2} L_{\Delta g}\sigma \cdot L_g\sigma \left(-\frac{L_f\sigma}{L_g\sigma}\right)^2} > 0 \end{aligned} \quad (5.36)$$

On the other hand, it can be shown that assumption (5.32) is true by developing the following inequality if (5.28) is satisfied. Extending the inequality (5.32), we have

$$\begin{aligned} &\frac{1}{2} \left(\sigma - \frac{|L_{\Delta f+\xi}\sigma| \cdot |\sigma|}{\rho} \right)^2 + \frac{|L_{\Delta g}\sigma|}{|L_g\sigma|} \left(\sigma - \frac{|L_{\Delta g}\sigma \cdot \frac{L_f\sigma}{L_g\sigma}| \cdot |\sigma|}{2\rho \cdot \frac{|L_{\Delta g}\sigma|}{|L_g\sigma|}} \right)^2 \\ &\geq \frac{1}{2} \sigma^2 \left\{ \left[1 + \frac{(L_{\Delta f+\xi}\sigma)^2}{\rho^2} - 2 \frac{|L_{\Delta f+\xi}\sigma|}{\rho} \right] \right\} + \frac{|L_{\Delta g}\sigma|}{|L_g\sigma|} \sigma^2 \left\{ \left[1 + \frac{(L_f\sigma)^2}{4\rho^2} - \frac{|L_f\sigma|}{\rho} \right] \right\} \\ &= \sigma^2 \left\{ \frac{1}{2\rho^2} [(L_{\Delta f+\xi}\sigma)^2 + \frac{|L_{\Delta g}\sigma|}{2|L_g\sigma|} (L_f\sigma)^2] + \left[\frac{1}{2} + \frac{|L_{\Delta g}\sigma|}{|L_g\sigma|} \right] - \frac{1}{\rho} [|L_{\Delta f+\xi}\sigma| + \frac{|L_{\Delta g}\sigma|}{|L_g\sigma|} \cdot |L_f\sigma|] \right\} \\ &\geq 2 \frac{|L_{\Delta g}\sigma| \sigma^2}{|L_g\sigma|} \end{aligned}$$

$$i.e., \quad \frac{1}{2\rho^2}[(L_{\Delta f+\xi}\sigma)^2 + \frac{|L_{\Delta g}\sigma|}{2|L_g\sigma|}(L_f\sigma)^2] + [\frac{1}{2} + \frac{|L_{\Delta g}\sigma|}{|L_g\sigma|}] - \frac{1}{\rho}[\frac{|L_{\Delta f+\xi}\sigma|}{|L_g\sigma|} + \frac{|L_{\Delta g}\sigma|}{|L_g\sigma|} \cdot |L_f\sigma|] \geq 0$$

Obviously, if

$$\left\{ \rho \left[\frac{1}{2} - \frac{|L_{\Delta g}\sigma|}{|L_g\sigma|} \right] - \left[\frac{|L_{\Delta f+\xi}\sigma|}{|L_g\sigma|} + \frac{|L_{\Delta g}\sigma|}{|L_g\sigma|} \cdot |L_f\sigma| \right] \right\} \geq 0$$

then the inequality (5.32) is true. We therefore obtain another feedback gain as follows

$$\rho \geq \frac{|L_{\Delta f+\xi}\sigma| \cdot |L_g\sigma| + |L_{\Delta g}\sigma| \cdot |L_f\sigma|}{\frac{1}{2}|L_g\sigma| - |L_{\Delta g}\sigma|} \quad (5.37)$$

Simply by letting

$$\rho \geq \sqrt{(L_{\Delta f+\xi}\sigma)^2 + \frac{1}{2}L_{\Delta g}\sigma \cdot L_g\sigma \left(-\frac{L_f\sigma}{L_g\sigma}\right)^2} + \frac{|L_{\Delta f+\xi}\sigma| \cdot |L_g\sigma| + |L_{\Delta g}\sigma| \cdot |L_f\sigma|}{\frac{1}{2}|L_g\sigma| - |L_{\Delta g}\sigma|} \quad (5.38)$$

both conditions (5.36) and (5.37) can then be satisfied. Because

$$L_{\Delta f}\sigma \leq \left| \frac{\partial \sigma}{\partial x} \cdot \Delta f(x, \gamma, t) \right| < \sum_{k=1}^n \left| \frac{\partial \sigma}{\partial x_k} \right| \cdot \max_{\gamma(t) \in R^{\gamma}} |\Delta f_k(x, \gamma, t)| \leq \left| \frac{\partial \sigma}{\partial x} \right| \cdot \omega_{\Delta f} = \Omega_{\Delta f}$$

$$L_{\Delta g}\sigma \leq \left| \frac{\partial \sigma}{\partial x} \cdot \Delta g(x, \gamma, t) \right| < \sum_{k=1}^n \left| \frac{\partial \sigma}{\partial x_k} \right| \cdot \max_{\gamma(t) \in R^{\gamma}} |\Delta g_k(x, \gamma, t)| \leq \left| \frac{\partial \sigma}{\partial x} \right| \cdot \omega_{\Delta g} = \Omega_{\Delta g}$$

$$L_{\xi}\sigma \leq \left| \frac{\partial \sigma}{\partial x} \cdot \xi(t) \right| < \sum_{k=1}^n \left| \frac{\partial \sigma}{\partial x_k} \right| \cdot \max_{t \geq 0} |\xi_k(t)| \leq \left| \frac{\partial \sigma}{\partial x} \right| \cdot \omega_{\xi} = \Omega_{\xi}$$

it follows that if ρ is chosen according to the known bounds given by (5.17), (5.18), and (5.19), we have the controller gain of the form (5.30), whose entries are all deterministic and known. It is obvious that if we set

$$\rho = \sqrt{(\Omega_{\Delta f+\xi})^2 + \frac{1}{2}|\Omega_{\Delta g}L_g\sigma| \left(-\frac{L_f\sigma}{L_g\sigma}\right)^2} + \frac{\Omega_{\Delta f+\xi}|L_g\sigma| + \Omega_{\Delta g}|L_f\sigma|}{\frac{1}{2}|L_g\sigma| - \Omega_{\Delta g}} > 0$$

then (5.38) holds. Also it follows that the inequality

$$\begin{aligned} \dot{V}(t) &< -\frac{\rho(x)}{2|\sigma|} \cdot \sigma^2 + \frac{(\Omega_{\Delta f} + \Omega_{\xi})^2 \cdot |\sigma|}{2\rho(x)} + \frac{(\Omega_{\Delta g} \cdot \frac{L_f\sigma}{L_g\sigma})^2 \cdot |\sigma|}{4\rho(x) \cdot \frac{\Omega_{\Delta g}}{|L_g\sigma|}} \\ &= -v_3[\|x(t)\|] + v_4[\|x(t)\|] < 0 \end{aligned} \quad (5.39)$$

is true. The closed loop system is therefore stable. \square

5.3 COMMENTS ON SYSTEM PERFORMANCE

The variable structure controller developed here results in a stable closed loop system when mismatched uncertainties are included in the system. These results may be summarised, according to Lyapunov stability theory, by the following inequalities:

$$v_1(\|x\|) \leq V(x) \leq v_2(\|x\|) \quad (5.40)$$

$$\dot{V}[\sigma(x)] \leq -v_3[\|x(t)\|] + v_4[\|x(t)\|] < 0 \quad (5.41)$$

5.3.1 Uniform Ultimate Boundedness

Having these results available now enables us to show that the system has the property of uniform ultimate boundedness in the sense of definition 3.14. Let us denote by

$$\delta[x(t, t_0, x_0), \sigma] \triangleq \delta(x, \sigma) = \inf \|x - \bar{x}\| \quad (5.42)$$

the distance of the point x from the surface $\sigma(x)=0$, where $x \in X$ are the states off the switching surface in admissible domain Ω , and $\bar{x} \in \bar{X}$ are the states on the switching surface.

In view of (5.40)

$$v_1[\delta(x, \sigma)] \leq V(x) \leq v_2[\delta(x, \sigma)] \quad (5.43)$$

Let R be the radius of the largest sphere in X , such that $V(x) > 0$ and $\dot{V}(x) < 0$.

Given a constant $r > 0$, we define

$$d(r) = (v_1^{-1} \cdot v_2)(\tilde{r}) \quad (5.44)$$

where $\tilde{r} \triangleq \max\{r, R\}$. Consider now a solution $x(t): [t_0, t_1] \rightarrow \mathbb{R}^n$, with $x(t_0) = x_0$ such that $\delta[x(t_0), \sigma(x_0)] \leq r$.

Suppose there is a $t_3 > t_0$ such that $x(t_3) = x_3$ and $\delta[x(t_3), \sigma(x_3)] > d(r)$. Since the solution $x(t)$ is continuous

$$\delta[x(t_0), \sigma(x_0)] < r \leq \tilde{r} < (v_1^{-1} \cdot v_2)(\tilde{r}) = d(r) < \delta[x(t_3), \sigma(x_3)]$$

Hence, there must exist a $t_2 \in [t_0, t_3]$, such that $\delta[x(t_2), \sigma(x_2)] = d(r)$ and $\delta[x(t), \sigma(x)] \geq d(r) \forall t \in [t_2, t_3]$.

In view of (5.40) and (5.41)

$$\begin{aligned} v_1[\delta(x(t_3), \sigma(x_3))] &\leq V(t_3) \\ &= V(t_0) + \int_{t_0}^{t_3} \dot{V}(\tau) d\tau \\ &\leq v_2[\delta(x(t_0), \sigma(x_0))] + \int_{t_0}^{t_3} [-v_3(\tau) + v_4(\tau)] d\tau \\ &\leq v_2(\tilde{r}) \end{aligned}$$

i.e., $\delta[x(t_3), \sigma(x_3)] \leq (v_1^{-1} \cdot v_2)(\tilde{r}) = d(r)$. However this contradicts the supposition above, hence

$$\delta[x(t), \sigma(x)] \leq d(r) \quad \forall t \in [t_0, t_1]$$

and the system is uniformly bounded.

Again if $x(t): [t_0, \infty) \rightarrow \mathbb{R}^n$, $x(t_0) = x_0$ is a solution of the system, such that $\delta[x(t_0), \sigma(x_0)] \leq r$, then for a given number $d' > (v_1^{-1} \cdot v_2)(R)$

$$\delta[x(t), \sigma(x)] \leq d' \quad \forall t \geq t_0 + T(d', r)$$

where

$$T(d', r) = \begin{cases} 0 & \text{if } r \leq \bar{R} \\ \frac{v_2(r) - v_1(\bar{R})}{v_3(\bar{R}) - v_4(\bar{R})} & \text{otherwise} \end{cases} \quad (5.45)$$

and $\bar{R} \triangleq (v_2^{-1} \cdot v_1)(d')$, so that $\bar{R} > R$ and $d(\bar{R}) = (v_1^{-1} \cdot v_2)(\bar{R}) = d' > (v_1^{-1} \cdot v_2)(R)$.

If $r \leq \bar{R}$, then $\delta[x(t_0), \sigma(x_0)] \leq \bar{R}$, hence, by the uniform boundedness result

$$\delta[x(t), \sigma(x)] \leq d(\bar{R}) = d' \quad \forall t \geq t_0$$

and obviously $T(d', r) = 0$.

If $r > \bar{R}$, and supposing that $\delta[x(t), \sigma(x)] > \bar{R} \quad \forall t \in [t_0, t_1]$, then bearing in mind conditions (5.40) and (5.41), we have

$$\begin{aligned} v_1[\delta(x(t_1), \sigma(x_1))] &\leq V(t_1) \\ &= V(t_0) + \int_{t_0}^{t_1} \dot{V}(\tau) d\tau \end{aligned}$$

$$\begin{aligned}
&\leq v_2[\delta(x(t_0), \sigma(x_0))] + \int_{t_0}^{t_1} [-v_3(\tau) + v_4(\tau)] d\tau \\
&\leq v_2(r) + T(d', r)[-v_3(\bar{R}) + v_4(\bar{R})] \\
&= v_2(r) + \frac{v_2(r) - v_1(\bar{R})}{v_3(\bar{R}) - v_4(\bar{R})} [-v_3(\bar{R}) + v_4(\bar{R})] \\
&= v_1(\bar{R})
\end{aligned}$$

That is, $\delta[x(t_1), \sigma(x_1)] \leq \bar{R}$. But this contradicts the assumption above. Hence there must exist a $t_2 \in [t_0, t_1]$ such that $\delta[x(t_2), \sigma(x_2)] \leq \bar{R}$. Then, as a consequence of the uniform boundedness result, $\delta[x(t), \sigma(x)] \leq d(\bar{R}) = d' \quad \forall t \geq t_2$. Hence

$$\delta[x(t), \sigma(x)] \leq d' \quad \forall t \geq t_1 = t_0 + T(d', r)$$

i.e., the system is uniformly ultimately bounded.

5.3.2 Remarks

- Comparing theorem 5.4 with theorem 5.3, the following fundamental conclusion is drawn. The nominal system (5.7) admits control action of form (5.8) such that the switching function $\sigma(x)=0$ is also a switching function for the uncertain nonlinear system (5.11), and the same structured controller can be employed to achieve a sliding mode along $\sigma(x)=0$ as long as the controller gain $\rho(x)$ is chosen according to (5.14) instead of being the constant of theorem 5.3.
- Compared with the techniques developed in chapter 4, the same design principle has been used, and similar assumptions have been made concerning the characteristics of the input mapping of the system. These conditions are as follow:

$$L_g V \cdot L_{\Delta g} V \geq 0 \quad \text{and} \quad L_g V \neq 0$$

$$L_g \sigma \cdot L_{\Delta g} \sigma \geq 0 \quad \text{and} \quad L_g \sigma \neq 0$$

It is necessary to choose a constant matrix S (for the case of linear switching function) such that the assumed conditions, $L_g \sigma \cdot L_{\Delta g} \sigma = S \cdot g \times S \cdot \Delta g \geq 0$ and $L_g \sigma = S \cdot g \neq 0$,

are true, whilst, in the former case, a special form of Lyapunov function is needed (in most cases a transformation must be made, as discussed in chapter 4, in order to find such a Lyapunov function) to guarantee $L_g V \neq 0$, and furthermore it may not be possible to make $L_g V \cdot L_{\Delta g} V \geq 0$ only through choice of a Lyapunov function. It is therefore concluded that the requirements here are less severe than those of chapter 4 and it is easier to implement the design.

- To apply the techniques developed in chapter 4, it is necessary to choose values of two parameters C_1 and C_2 so that

$$C_2 - C_1 L_f V > 0$$

and

$$L_F V \leq (1 - C_1) L_f V + C_2 < 0$$

where

$$L_f V = \frac{\partial V}{\partial x} \cdot f = L_f V + L_g V \cdot u_1 < -v$$

$$L_F V = \frac{\partial V}{\partial x} \cdot \dot{x} = L_f V + L_{\Delta f} V + L_g V \cdot u_2 + L_{\Delta g} V \cdot (u_1 + u_2)$$

The present development avoids the requirement for proper choice of C_1 and C_2 , thus easing the design problem further.

5.4 ILLUSTRATIVE EXAMPLE

We will consider the same example as the one in chapter 4 to illustrate the application of the techniques developed here. Both open loop pole uncertainty and non-minimum phase problems are considered. Although the uncertainties lie in the range of the input mapping $g(x)$, there do not exist functions p and q such that $\Delta f = g \cdot p$, $\Delta g = g \cdot q$, so they can only be treated as special kinds of mismatched uncertainties. The system can be expressed as

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \begin{pmatrix} a'_{11} & a'_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b'_1 \\ b'_2 \end{pmatrix} u(t) \\ &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ b_2 \end{pmatrix} u(t) + \begin{pmatrix} \Delta a_{11}x_1 + \Delta a_{12}x_2 \\ \Delta a_{21}x_1 + \Delta a_{22}x_2 \end{pmatrix} + \begin{pmatrix} \Delta b_1 \\ \Delta b_2 \end{pmatrix} u(t)\end{aligned}$$

where $\Delta a_{ij} = a'_{ij} - a_{ij} \quad (i, j=1, 2)$

$$b_2 = 1/a_{12}$$

$$\Delta b_1 = k_1$$

$$\Delta b_2 = 1/a'_{12} - 1/a_{12} + a'_{22}k_1/a'_{12}$$

Thus, the system falls into the class of systems with mismatched uncertainties.

As the nominal part of this system is already in regular form, it can be directly rewritten as

$$\begin{aligned}\dot{\mathbf{x}}^1(t) &= \mathbf{f}^1(\mathbf{x}^1, \mathbf{x}^2) \\ \dot{\mathbf{x}}^2(t) &= \mathbf{f}^2(\mathbf{x}^1, \mathbf{x}^2) + \mathbf{g}^2(\mathbf{x}^1, \mathbf{x}^2) \cdot u(t)\end{aligned}$$

such that two sets of new states $\mathbf{x}^1 = x_1$, $\mathbf{x}^2 = x_2$ result, and we therefore have

$$\mathbf{f}^1(\mathbf{x}) = a_{11}x_1 + a_{12}x_2 \quad \mathbf{f}^2(\mathbf{x}) = a_{21}x_1 + a_{22}x_2 \quad \mathbf{g}^2(\mathbf{x}) = b_2$$

The switching function, independent of any uncertain element in the system, is chosen as

$$\sigma(x) = \sigma_1(x_1) - x_2 = 0$$

such that a reduced order closed loop system of the form

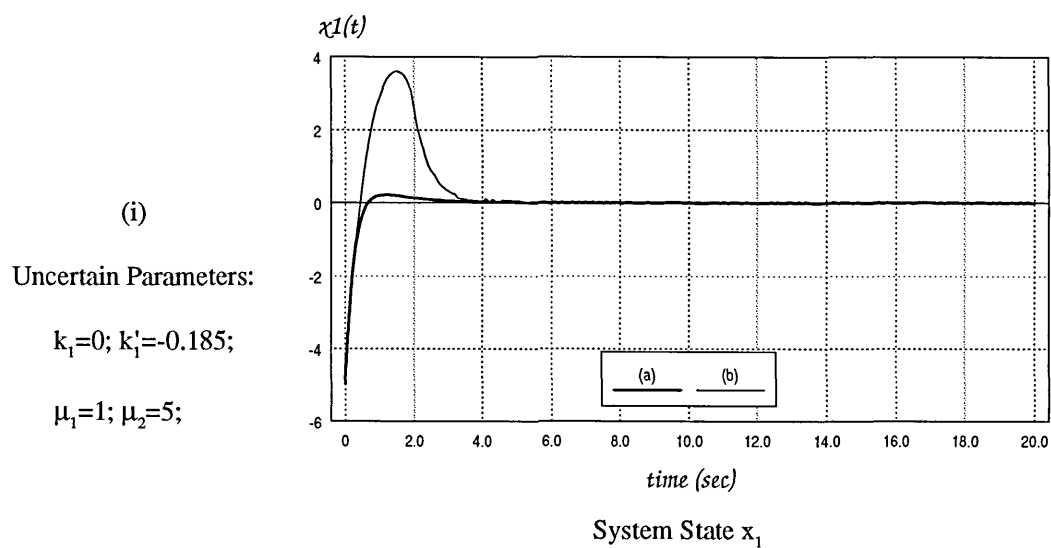
$$\dot{x}_1(t) = a_{11}x_1 + a_{12}x_2 = a_{11}x_1 + a_{12}\sigma_1(x_1) = \lambda x_1$$

results, where λ is the closed loop pole. Here a linear switching function is chosen, *i.e.*, $\sigma_1(x_1) = sx_1$, so $s = (\lambda - a_{11})/a_{12}$. From this, the closed loop pole may be placed at some desired location by appropriate choice of s , and also condition (5.12) is satisfied. The nominal system is chosen to be

$$\dot{\mathbf{x}}(t) = \begin{pmatrix} -4.732 & 1.000 \\ 1.000 & -1.268 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)$$

and $s = 2.2321$ was chosen for simulation purposes. This results in a reduced order closed loop system with pole $\lambda = -2.5$. For the chosen switching function

$$\sigma(x) = sx_1 - x_2 = Sx = 0$$



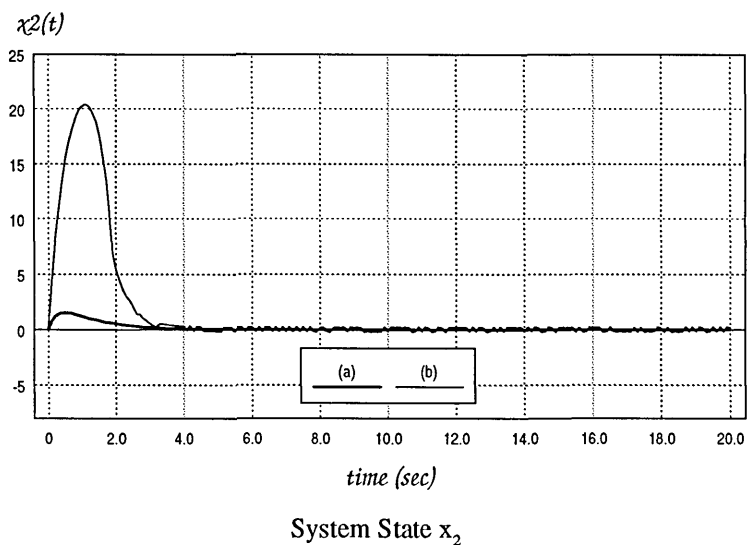
$$\Delta f = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Delta g = \begin{pmatrix} k'_1 \\ -1.268k'_1 \end{pmatrix}$$

$$\omega_{\Delta f} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\omega_{\Delta g} = \begin{pmatrix} |k'_1| \\ 1.268|k'_1| \end{pmatrix}$$

Constant Feedback
Gain: $\rho=17$

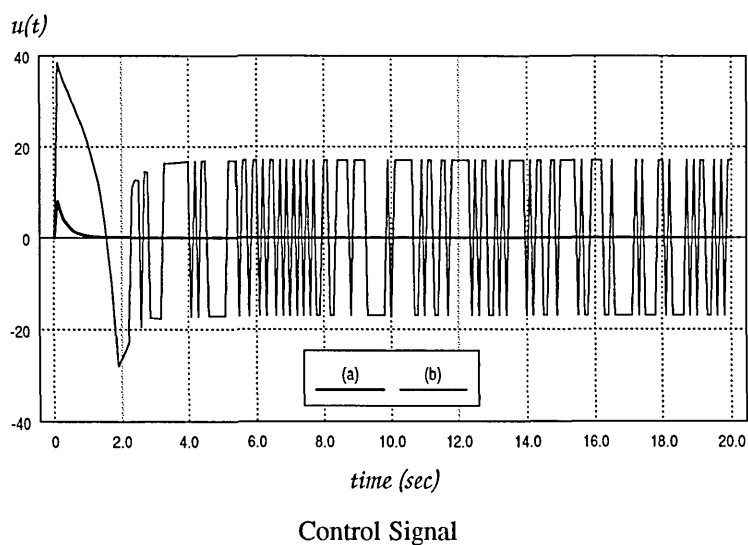


(a)

The controller with
variable feedback gain
of theorem 5.4;

(b)

The controller with
constant feedback gain
of theorem 5.3;

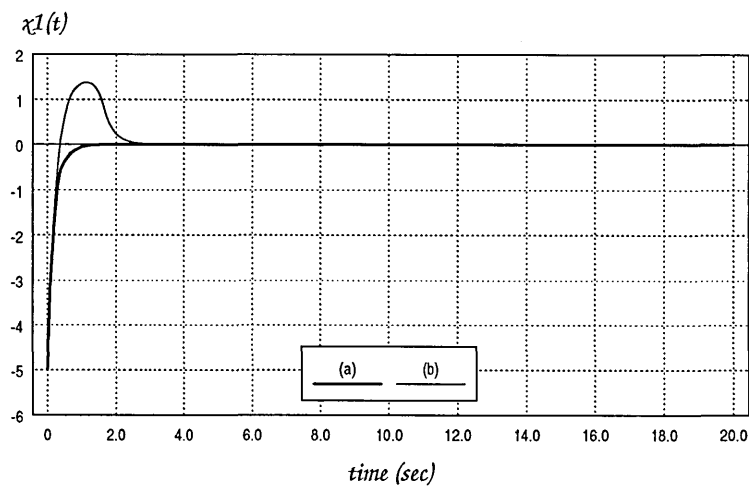


(ii)

Uncertain Parameters:

$$k_1=0;$$

$$\mu_1=1, \mu_1'=-1; \mu_2=5;$$



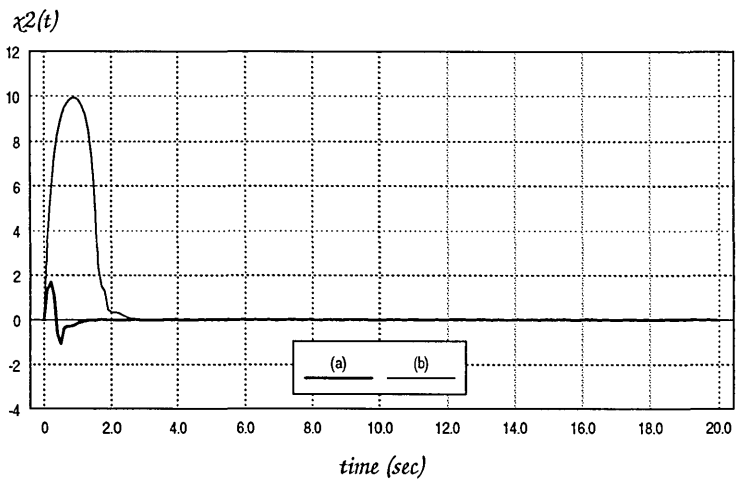
System State x_1

$$\Delta f = \begin{pmatrix} -0.183x_1 - 0.293x_2 \\ -0.293x_1 + 2.183x_2 \end{pmatrix}$$

$$\Delta g = \begin{pmatrix} 0 \\ 0.414 \end{pmatrix}$$

$$\omega_{\Delta f} = \begin{pmatrix} 3.299 \\ 13.846 \end{pmatrix}$$

$$\omega_{\Delta g} = \begin{pmatrix} 0 \\ 0.414 \end{pmatrix}$$



System State x_2

Constant Feedback

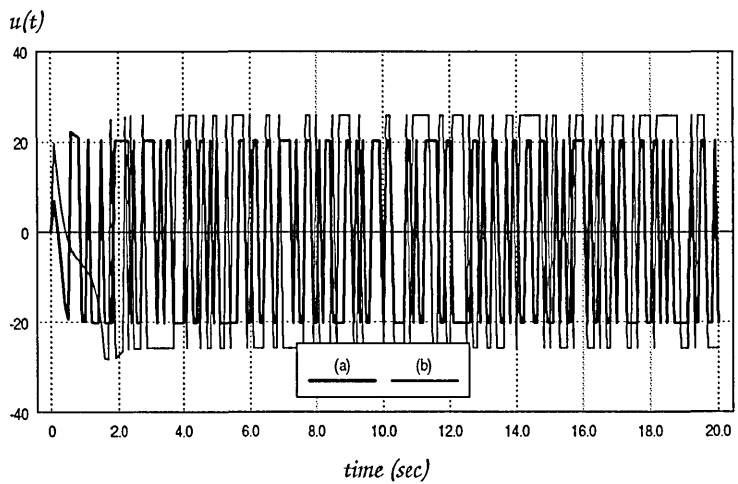
Gain: $\rho=27.1$

(a)

The controller with
variable feedback gain
of theorem 5.4;

(b)

The controller with
constant feedback gain
of theorem 5.3;



Control Signal

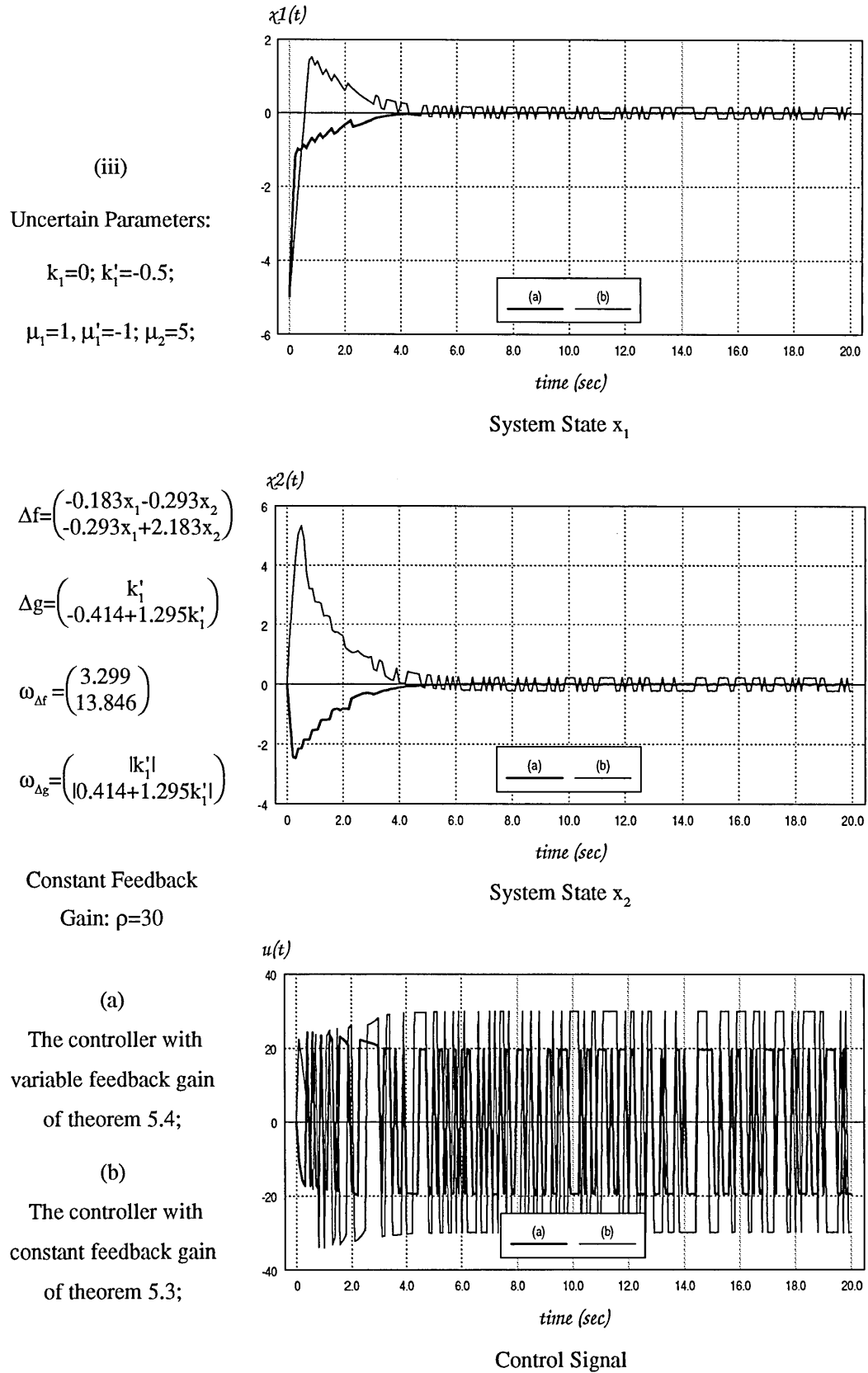


Fig.5.1 Case 1: Variable structure controller with variable feedback gain, $L_g \sigma \cdot L_{\Delta g} \sigma > 0$

the partial derivative with respect to x is $\partial\sigma/\partial x = S = [2.2321, -1]$, thus

$$L_g\sigma \cdot L_{\Delta g}\sigma = S \cdot g \times S \cdot \Delta g \geq 0$$

The controller with variable feedback gain, and for the sake of comparison, one with constant feedback gain, are designed in accordance with theorem 5.4 and 5.3 respectively

$$u(t) = -\frac{1}{L_g\sigma} \left\{ L_f\sigma + \sqrt{(\Omega_{\Delta f} + \Omega_{\xi})^2 + \frac{1}{2}|\Omega_{\Delta g}L_g\sigma| \left(-\frac{L_f\sigma}{L_g\sigma}\right)^2 \cdot \text{sign}(\sigma)} \right\}$$

$$u(t) = -\frac{1}{L_g\sigma} \left\{ L_f\sigma + \rho \cdot \text{sign}(\sigma) \right\}$$

Fig. 5.1 displays the results of simulation for the system. The responses of the system with feedback of both constant gain and variable gain are depicted for different parameter bounds. Use of the variable gain controller results not only in stable responses, but also in fairly small errors, whilst use of the constant gain controller results in large swings in the values of the states, and sometimes an unstable condition.

The second example is concerned with the case where condition (5.12) is not satisfied. The same nominal model as that of first example is considered, but the real system model is given by

$$\dot{x}(t) = \begin{pmatrix} -4.793 & 1.225 \\ 1.225 & 0.739 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -0.1 \\ 0.756 \end{pmatrix} u(t)$$

This implies some uncertainties in both state mapping and input mapping

$$\Delta f = \begin{pmatrix} -0.007 & 0.225 \\ 0.225 & 2.007 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Delta g = \begin{pmatrix} -0.1 \\ -0.244 \end{pmatrix}$$

such that for the following switching function

$$\sigma(x) = sx_1 - x_2 = Sx = [2.2321, -1]x = 0$$

$L_g\sigma \cdot L_{\Delta g}\sigma = -0.0208 < 0$, but $|L_g\sigma| > 2|L_{\Delta g}\sigma|$, so theorem 5.5 is applicable here.

The simulation results are shown in Fig. 5.2, and the same conclusions can be drawn.

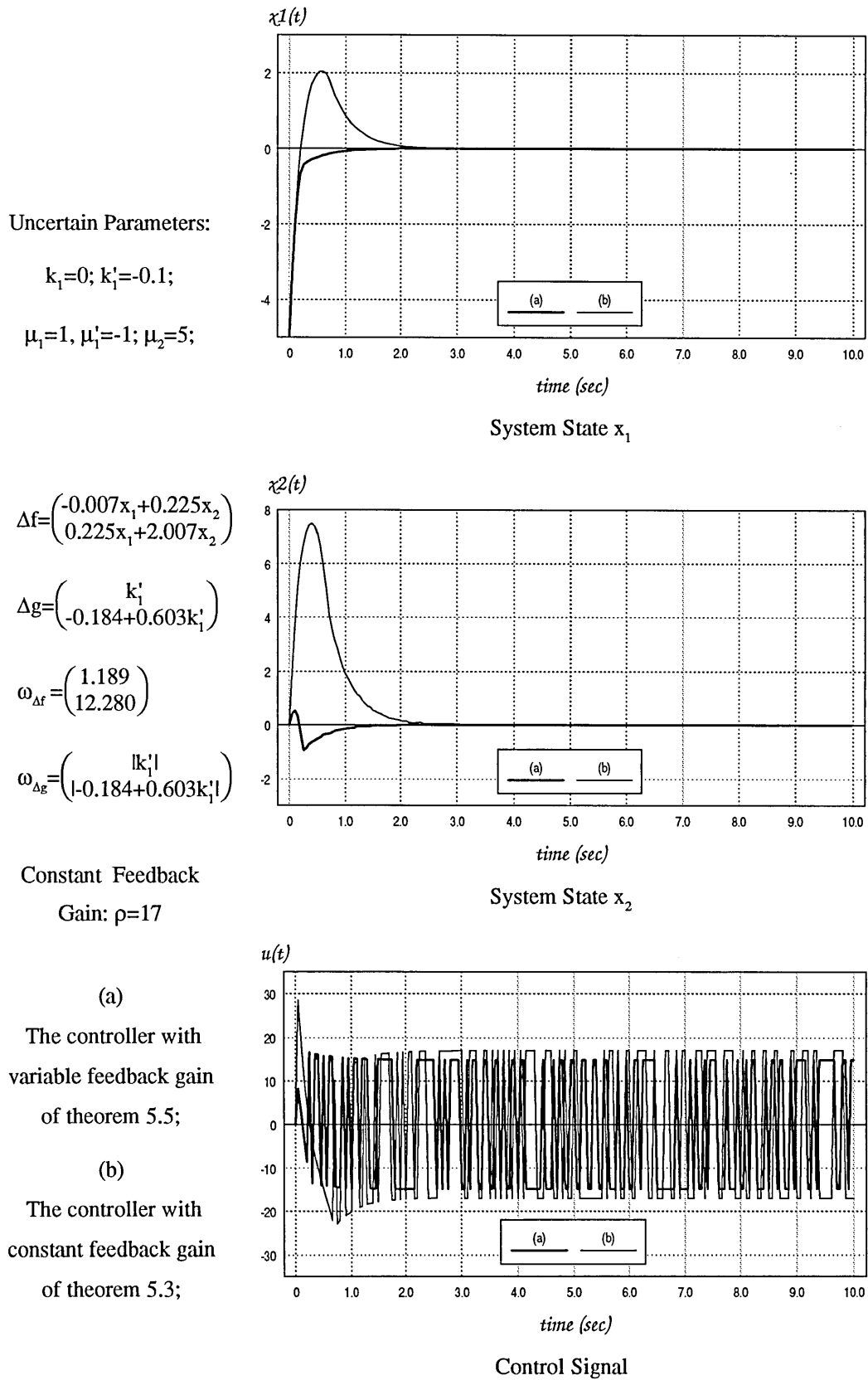


Fig.5.2 Case 2: Variable structure controller with variable feedback gain, $L_g \sigma \cdot L_{\Delta g} \sigma < 0$, $|L_g \sigma| > 2|L_{\Delta g} \sigma|$

5.5 SUMMARY

In this chapter, the same problem as that of chapter 4 has been addressed, but a different control strategy, variable structure robust control, is used to guarantee stability off the switching surface. The techniques are summarised as follows:

Algorithm:

- ① Transform the original nonlinear uncertain system into a regular form (see appendix B);
- ② Design a switching function $\sigma(x)$ such that either condition (5.12) or condition (5.28) is satisfied;
- ③ Obtain a feedback control of form (5.13) with variable feedback gain (5.14) subject to condition (5.12), or control (5.29) with feedback gain (5.30) subject to condition (5.28).

The design procedure does not require the nominal dynamics to be either stable or in some way precompensated, nor is there any requirement for the uncertainties to satisfy the assumption of matching conditions. The control law is directly applicable to nonlinear uncertain systems, even to the open loop unstable case, and the practical stability of the closed loop system is guaranteed. The simulation results show that the controller attenuates the effects of the uncertainty. On the other hand, the controller has the same structure as that developed for the case without consideration of uncertainty. The difference is that variable controller gains are employed, depending on the upper bounds of the uncertainty and disturbance.

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Robust Stability Control (III)



Overview

This chapter describes a new robust control technique developed for multi-input nonlinear systems with mismatched uncertainties. The proposed technique utilises variable structure theory, and the design is based on Lyapunov stability theory.



Outline

- ✓ Introduction
- ✓ Robust Control of Multi-input Nonlinear Uncertain Systems
- ✓ Illustrative Example
- ✓ Summary

6.1 INTRODUCTION

FEEDBACK control is now fairly well understood for large classes of nonlinear systems with single inputs or uncoupled multiple inputs. For general multi-input nonlinear systems, however, feedback control and especially robustness issues still represent difficult problems, the urgency of which has been rendered more acute by the recent development of systems with challenging nonlinear dynamics, such as robot manipulators, high performance aircraft, and advanced underwater and space vehicles.

Some methodologies have been developed to deal with the robust control of multi-input nonlinear systems in the time domain. One possibility is to decouple the system by properly choosing a state transformation so that large scale nonlinear systems can be decomposed into a number of sub-systems with only one input, and noninteracting controllers can be found to control the new transformed systems. Another is called generalised decentralised control where large scale nonlinear systems consist of a number of sub-systems which have only single input, whilst the interacting terms are treated artificially as uncertainties in the system. Both methods have some limitations, because decoupling of input-output is hard to implement for general nonlinear systems, particularly with uncertainties, while generalised decentralised control does not fully use the information concerning interacting terms so that conservative design results.

In this chapter, a new robust control technique for multivariable nonlinear systems in the presence of uncertainties and external disturbances is developed. In contrast to other methods, the method developed here avoids decoupling or decentralising the system into sub-systems, but synthesizes robust controllers directly with the original nonlinear uncertain dynamics, thereby easing the design problem and utilising all available system information. The proposed design technique does not require that the uncertainties should

satisfy matching conditions, nor does it require that the nominal system should be stable or pre-stabilised. Instead, only a rather weak condition is imposed on the uncertainties with no further assumptions, and strong robustness is obtained. The robust control strategy is still based on Lyapunov theory, and is established using concepts from variable structure theory but with certain extensions. The control possesses a quite simple structure, and can be used to effectively deal with MIMO nonlinear uncertain systems. A nonlinear example is considered and simulation results are presented.

6.2 ROBUST CONTROL OF MULTI-INPUT NONLINEAR UNCERTAIN SYSTEMS

To begin with, a general description of the system to be controlled is given, and an assumption is made which is a simple extension of that for the single-input case.

6.2.1 System Description

Consider a multivariable nonlinear system with mismatched uncertainties of the form

$$\dot{x}(t) = F(x, \gamma) + G(x, \gamma)u(t) + \xi(t) \quad (6.1)$$

where

$$F(x, \gamma) = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \quad G(x, \gamma) = \begin{pmatrix} g_{11} & \cdots & g_{1m} \\ \vdots & & \vdots \\ g_{n1} & \cdots & g_{nm} \end{pmatrix} \quad \xi(t) = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} \quad u(t) = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}$$

$F(\cdot, \cdot): \mathbb{R}^n \times \mathbb{R}^\gamma \rightarrow \mathbb{R}^n$, $G(\cdot, \cdot): \mathbb{R}^n \times \mathbb{R}^\gamma \rightarrow \mathbb{R}^{n \times m}$, $x(\cdot) \in \mathbb{R}^n$ is the state, and $u(\cdot) \in \mathbb{R}^m$ is the control input. All the uncertainties in the system are represented by the lumped uncertain elements $\gamma \in \mathbb{R}^\gamma$. $\xi(t)$ represents external disturbances which could be either deterministic or stochastic. The only information assumed here is the knowledge of the bounds of $\gamma(t)$ and $\xi(t)$. These bounds are given by

$$\Phi_F(x) \triangleq \{ \max_{\gamma(t) \in \mathbb{R}^r} |F_i(x, \gamma)| \mid (i=1, 2, \dots, n) \} \quad (6.2)$$

$$\Phi_\xi(t) \triangleq \{ \max_{i \geq 0} |\xi_i(t)| \mid (i=1, 2, \dots, n) \} \quad (6.3)$$

where Φ represents supremum bounds. Furthermore we define a matrix

$$\phi_G(x) \triangleq \{ \min_{\gamma(t) \in \mathbb{R}^r} G_{ij}(x, \gamma) \mid 1 \leq i \leq n, 1 \leq j \leq m \}$$

and assume that the following condition

$$\phi_G^T(x) \cdot G(x, \gamma) > \phi_G^T(x) \cdot \phi_G(x) > 0 \quad (6.4)$$

holds, where ϕ indicates the infimum bound of G , and the inequality of (6.4) means that the quadratic form of these matrices satisfies the above inequality.

Here by mismatched uncertainties, it is meant that it is not required to decompose the system (6.1) into the certain part and the uncertain part of the form

$$F(x, \gamma) = F(x, \tilde{\gamma}) + \Delta F(x, \gamma)$$

$$G(x, \gamma) = G(x, \tilde{\gamma}) + \Delta G(x, \gamma)$$

and that it is not necessary to represent the uncertainties by

$$\Delta F(x, \gamma) = G(x, \tilde{\gamma}) \delta f(x, \gamma)$$

$$\Delta G(x, \gamma) = G(x, \tilde{\gamma}) \delta g(x, \gamma)$$

where $\tilde{\gamma}$ is the nominal value of γ .

ASSUMPTION 6.1. (*Conditions on the Input Mapping*)

For a given system of form (6.1), it is assumed that the input mapping and its infimum bound satisfy the following conditions:

- ① all m non-zero eigenvalues of the following matrix

$$G(x, \gamma) \cdot \phi_G^T(x) \in \mathbb{R}^{n \times n} \quad (6.5)$$

are positive;

- ② the minimum non-zero eigenvalue of the above matrix is sufficiently large that the matrix

$$\phi_G^T(x) \cdot G(x, \gamma) \in \mathbb{R}^{m \times m} \quad (6.6)$$

is positive definite, *i.e.*, its symmetrised form is positive definite;

- ③ for a properly chosen switching surface

$$\sigma(x) = [\sigma_1(x), \sigma_2(x), \dots, \sigma_m(x)]^T$$

the following matrix

$$\nabla \sigma \cdot G(x, \gamma) \cdot \phi_G^T(x) \cdot \nabla \sigma^T = \omega_G(x, \gamma) \cdot \Omega_G^T(x) \in \mathbb{R}^{m \times m} \quad (6.7)$$

is positive definite, where $\nabla \sigma$ is the Jacobian of σ , and

$$\omega_G(x, \gamma) \triangleq \nabla \sigma \cdot G(x, \gamma) \quad (6.8)$$

and

$$\Omega_G(x) \triangleq \nabla \sigma \cdot \phi_G(x) \quad (6.9)$$

is a non-singular matrix, and is called the generalised infimum bound of $\omega_G(x, \gamma)$.

REMARK 6.1:

- This assumption is concerned mainly with the characteristics of the input mapping of the given system. In general, $\omega_G(x, \gamma) \cdot \Omega_G^T(x)$ is not symmetric, and its symmetrised form may not be sign-definite. Fortunately, the switching function $\sigma(x)$ can usually be chosen so that condition (6.7) holds.
- In most cases, linear switching functions $\sigma(x)$ of form

$$\sigma(x^1, x^2) = S_1 \cdot x^1 - x^2 = 0 \quad (6.10)$$

are adopted, and the partial derivative of $\sigma(x)$ with respect to x is simply a constant matrix given by $\nabla \sigma(x) = [S_1, -I]$. It is therefore possible to choose the elements of S_1 such that condition (6.7) holds for the input mapping of the given nonlinear uncertain system.

- More specifically, for the given nonlinear uncertain system of form (6.1), if a coordinate transformation $z = T(x)$ can be found, such that the system can be transformed into the following form

$$\begin{pmatrix} \dot{z}^1 \\ \dot{z}^2 \end{pmatrix} = \begin{pmatrix} f^1(z, \gamma) \\ f^2(z, \gamma) \end{pmatrix} + \begin{pmatrix} 0 \\ g^2(z, \gamma) \end{pmatrix} v(t) \quad (6.11)$$

where $z = [z^1, z^2]^T$ and $v(t)$ are respectively the state and input of the system in new coordinates, $f^1(z, \gamma) \in \mathbb{R}^{(n-m)}$, $f^2(z, \gamma) \in \mathbb{R}^m$, and $g^2(z, \gamma) \in \mathbb{R}^{m \times m}$ is non-singular. Then for the switching function (6.10), the matrix

$$\omega_G(z, \gamma) \cdot \Omega_G^T(z) = \nabla \sigma \cdot G(z, \gamma) \cdot \phi_G^T(z) \cdot \nabla \sigma^T$$

$$\begin{aligned}
&= [S_1, -I] \cdot \begin{pmatrix} 0 \\ g^2(z, \gamma) \end{pmatrix} \cdot (0, \phi_g^T(z)) \cdot [S_1, -I]^T \\
&= g^2(z, \gamma) \cdot \phi_g^T(z)
\end{aligned}$$

is required to be positive. From the discussion, it may be stated that if the signs of all elements of $G(x, \gamma)$ do not change for all admissible uncertainties, the condition (6.7) is met.

THEOREM 6.2.

For any matrices C , A , and any symmetric positive definite matrix B , if $C^T A$ is symmetric and $(A^T A)^{-1}$ exists, then

$$(C^T A - C^T B C) - \frac{1}{4}(A^T B^{-1} A) \quad (6.12)$$

is negative semidefinite.

Proof:

$$\begin{aligned}
C^T A - C^T B C &= \frac{1}{4}(A^T B^{-1} A) - [C^T A - \frac{1}{2}A^T B^{-1} A] (A^T B^{-1} A)^{-1} [C^T A - \frac{1}{2}A^T B^{-1} A]^T \\
&= \frac{1}{4}(A^T B^{-1} A) - \left\{ C^T A (A^T B^{-1} A)^{-1} A^T C + \frac{1}{4}(A^T B^{-1} A) (A^T B^{-1} A)^{-1} (A^T B^{-1} A)^T \right. \\
&\quad \left. - \frac{1}{2}C^T A (A^T B^{-1} A)^{-1} (A^T B^{-1} A)^T - \frac{1}{2}A^T B^{-1} A (A^T B^{-1} A)^{-1} (A^T C)^T \right\} \\
&= \frac{1}{4}(A^T B^{-1} A) - C^T A (A^T B^{-1} A)^{-1} A^T C - \frac{1}{4}(A^T B^{-1} A) + \frac{1}{2}C^T A + \frac{1}{2}A^T C
\end{aligned}$$

If $(A^T A)^{-1}$ exists, then

$$A(A^T B^{-1} A)^{-1} A^T = A(A^T B^{-1} A)^{-1} A^T B^{-1} (A A^T) (A A^T)^{-1} B = B$$

and so the above equality can be written as follows

$$(C^T A - C^T B C) - \frac{1}{4}(A^T B^{-1} A) = -[C^T A - \frac{1}{2}A^T B^{-1} A] (A^T B^{-1} A)^{-1} [C^T A - \frac{1}{2}A^T B^{-1} A]^T$$

i.e., $(C^T A - C^T B C) - \frac{1}{4}(A^T B^{-1} A)$ is negative semidefinite. \square

The result of theorem 6.2 enables it to be concluded that for any vector $z \neq 0$

$$z^T (C^T A - C^T B C) z \leq \frac{1}{4} z^T (A^T B^{-1} A) z$$

so it is possible to replace the right-hand side by the left-hand side in the development of the next section. Obviously, this theorem is an extension of identity (4.12) for scalar case.

In order to proceed, some definitions are now made and some new matrices are constructed by rearranging the elements of the existing matrices. Let

$$\omega_{F+\xi}(x, \gamma) = \begin{pmatrix} \sum_{j=1}^n \frac{\partial \sigma_1}{\partial x_j} \cdot (f_j + \xi_j) & & 0 \\ & \ddots & \\ 0 & & \sum_{j=1}^n \frac{\partial \sigma_m}{\partial x_j} \cdot (f_j + \xi_j) \end{pmatrix}$$

so that

$$L_{F+\xi} \sigma = \omega_{F+\xi}(x, \gamma) \times I_{m \times 1} \quad (6.13)$$

where $I_{m \times 1} \triangleq [1, \dots, 1]^T$, and also define $\omega_G(x, \gamma)$ by

$$\omega_G(x, \gamma) = L_G \sigma \quad (6.14)$$

Similarly, let

$$\Sigma(x) = \begin{pmatrix} \sigma_1(x) & & 0 \\ & \ddots & \\ 0 & & \sigma_m(x) \end{pmatrix}$$

$$\text{so } \sigma(x) = \Sigma(x) \times I_{m \times 1} \quad (6.15)$$

and finally, let

$$U(t) = \begin{pmatrix} u_1(t) & & 0 \\ & \ddots & \\ 0 & & u_m(t) \end{pmatrix}$$

$$u(t) = U(t) \times I_{m \times 1} \quad (6.16)$$

With these definitions, the vectors $L_{F+\xi} \sigma$, $\sigma(x)$ and $u(t)$ may be represented by diagonal matrices multiplied by a special kind of vector with all elements equal to unity.

6.2.2 Robust Control Synthesis

THEOREM 6.3.

For a matrix $G(x, \gamma)$ and its infimum bound $\phi_G(x) \in \mathbb{R}^{n \times m}$ ($n \geq m$), and its generalised form $\Omega_G(x) = \nabla \sigma \cdot \phi_G(x)$ defined in (6.9)

① For the matrix $A = \phi_G^T(x) \cdot G(x, \gamma)$, if its symmetric form $A_s = (A + A^T)/2 \in \mathbb{R}^{m \times m}$ is positive definite, then

$$\phi_G(x) \cdot \phi_G^T(x) \cdot G(x, \gamma) \cdot \phi_G^T(x) \in \mathbb{R}^{n \times n}$$

is non-negative definite.

② For the positive definite matrix $\omega_G \cdot \Omega_G^T$, i.e., $(\omega_G \cdot \Omega_G^T)_s$ p.d., if

$$\lambda_m[(\omega_G \cdot \Omega_G^T)_s] > \varsigma > 0 \quad (6.17)$$

then the matrix

$$\Phi_M^2(\Omega_G) \cdot \omega_G \cdot \Omega_G^T - \Omega_G \cdot \varsigma \cdot \Omega_G^T$$

is positive definite, i.e.,

$$z^T \{ \Phi_M^2(\Omega_G) \cdot \omega_G \cdot \Omega_G^T - \Omega_G \cdot \varsigma \cdot \Omega_G^T \} z > 0 \quad \forall z \neq 0 \quad (6.18)$$

where $\Phi_M(\cdot)$ and $\lambda_m(\cdot)$ indicate the spectral norm (greatest singular value) and minimum eigenvalue of the respective matrices, and ς is a positive constant satisfying (6.17).

Proof:

① Let $A = \Phi_G^T(x) \cdot G(x, \gamma)$, and $A_s = (A + A^T)/2$. Let $B = \Phi_G(x)$.

According to assumption 6.1, matrix $A \in \mathbb{R}^{m \times m}$ is positive definite, so it is obvious that matrix $BA_s B^T$ is non-negative definite.

② Knowing that, for any matrix $C \in \mathbb{R}^{n \times n}$, we have

$$\lambda_m(C) \|z\|^2 \leq z^T C z \leq \lambda_M(C) \|z\|^2 \quad \forall z \neq 0 \quad (6.19)$$

if $\lambda_m[(\omega_G \cdot \Omega_G^T)_s] > \varsigma > 0$, then $\lambda_m(\omega_G \cdot \Omega_G^T) > \lambda_m[(\omega_G \cdot \Omega_G^T)_s] > \varsigma > 0$, so

$$z^T \left\{ \omega_G \cdot \Omega_G^T - \frac{\Omega_G \cdot \varsigma \cdot \Omega_G^T}{\Phi_M^2(\Omega_G)} \right\} z > z^T \left\{ \lambda_m(\omega_G \cdot \Omega_G^T) - \frac{\varsigma \cdot \lambda_M(\Omega_G \Omega_G^T)}{\Phi_M^2(\Omega_G)} \right\} z > 0 \quad (6.20)$$

i.e.,
$$\Phi_M^2(\Omega_G) \cdot \omega_G \cdot \Omega_G^T - \Omega_G \cdot \varsigma \cdot \Omega_G^T$$

is positive definite. □

The problem now is, for a generalised Lyapunov function defined by

$$V(t) \triangleq \frac{1}{2} \sigma^T(x) \cdot \sigma(x) > 0 \quad \forall (x, t) \ni \sigma(x) \neq 0 \quad \text{and} \quad V|_{\sigma(x)=0} = 0$$

to find a feedback control $u(t)$ such that, for $X = \{x(t) \in \mathbb{R}^n | \sigma(x) \neq 0, x(t_0) = x_0\}$

$$\begin{aligned} \dot{V}(x) &= \sigma^T \cdot \dot{\sigma} = \sigma^T \cdot \nabla \sigma \cdot \{F(x, \gamma) + G(x, \gamma)u(t) + \xi(t)\} \\ &= \sigma^T \cdot \{L_F \sigma + L_G \sigma \cdot u(t) + L_\xi \sigma\} \\ &= I_{1 \times m} \cdot \Sigma^T(x) \{ \omega_{F+\xi}(x, \gamma) + \omega_G(x, \gamma)U(t) \} \cdot I_{m \times 1} < 0 \end{aligned} \quad (6.21)$$

That is, the matrix $\Sigma^T(x) \{ \omega_F(x, \gamma) + \omega_G(x, \gamma)U(t) \}$ is required to be negative definite, so that the system is stable. The following theorem solves this problem.

THEOREM 6.4.

The multivariable nonlinear uncertain system of form (6.1) admits a feedback control of form

$$U(t) = -(l^2+1)\rho(x)\Omega_G^T(x)\Sigma(x) \quad (6.22)$$

or written in vector form

$$u(t) = -(l^2+1)\rho(x)\Omega_G^T(x)\sigma(x) \quad (6.23)$$

where the feedback gain

$$\rho(x) = \frac{\varphi_M^2}{2l\zeta} [\Omega_G(x)]^{-1} \cdot \Omega_{F+\xi}(x) \cdot |\Sigma(x)|^{-1} [\Omega_G^T(x)]^{-1} \quad (6.24)$$

is a symmetric positive definite matrix, such that the matrix

$$\Sigma^T(x) \{ \omega_{F+\xi}(x, \gamma) + \omega_G(x, \gamma) U(t) \}$$

is negative definite (n.d.), i.e., the derivative of the Lyapunov function $\dot{V}(x) < 0 \quad \forall x \neq 0$, so the closed loop system is stable. Here

$$|\Sigma(x)| \triangleq \text{diag}(|\sigma_i(x)|) = \text{sign}[\Sigma(x)] \cdot \Sigma(x) \quad (6.25)$$

$$\Omega_{F+\xi}(x) \triangleq \begin{pmatrix} \sum_{j=1}^n \left| \frac{\partial \sigma_1}{\partial x_j} \right| \cdot (\Phi_{F_j}(x) + \Phi_{\xi_j}(t)) & & 0 \\ & \ddots & \\ 0 & & \sum_{j=1}^n \left| \frac{\partial \sigma_m}{\partial x_j} \right| \cdot (\Phi_{F_j}(x) + \Phi_{\xi_j}(t)) \end{pmatrix} \quad (6.26)$$

$$\Omega_G(x) \triangleq \begin{pmatrix} \sum_{j=1}^n \frac{\partial \sigma_1}{\partial x_j} \cdot \phi_{G_{j1}}(x) & \cdots & \sum_{j=1}^n \frac{\partial \sigma_1}{\partial x_j} \cdot \phi_{G_{jm}}(x) \\ \vdots & & \vdots \\ \sum_{j=1}^n \frac{\partial \sigma_m}{\partial x_j} \cdot \phi_{G_{j1}}(x) & \cdots & \sum_{j=1}^n \frac{\partial \sigma_m}{\partial x_j} \cdot \phi_{G_{jm}}(x) \end{pmatrix} \quad (6.27)$$

$$\varphi_M^2 = \lambda_{\max}(\Omega_G \cdot \Omega_G^T)$$

l and ζ are positive constants to be chosen by the designer.

$$\textbf{Proof: } \dot{V}(x) = \sigma^T \cdot \{ L_F \sigma + L_G \sigma \cdot u(t) + L_\xi \sigma \}$$

$$= I_{l \times m} \{ \Sigma^T(x) [\omega_{F+\xi}(x, \gamma) + \omega_G(x, \gamma) U(t)] \} I_{m \times l}$$

$$= I_{l \times m} \{ \Sigma^T(x) \omega_{F+\xi}(x, \gamma) - (l^2+1) \Sigma^T(x) \omega_G(x, \gamma) \rho(x) \Omega_G^T(x) \Sigma(x) \} I_{m \times l}$$

According to theorem 6.3, the inequality

$$I_{1 \times m} \{ \phi_M^2[\Omega_G(x)] \cdot \omega_G(x, \gamma) \cdot \Omega_G^T(x) - \Omega_G(x) \cdot \zeta \cdot \Omega_G^T(x) \} I_{m \times 1} \geq 0$$

holds, and also note that

$$I_{1 \times m} \{ \omega_G(x, \gamma) \rho(x) \Omega_G^T(x) \} I_{m \times 1} \geq \lambda_{\min}[\rho(x)] \cdot I_{1 \times m} \{ \omega_G(x, \gamma) \Omega_G^T(x) \} I_{m \times 1} \geq 0$$

if $\rho(x)$ and $\omega_G(x, \gamma) \cdot \Omega_G^T(x)$ are positive definite. Let $\rho(x) = \rho'(x) \cdot \phi_M^2$, so that

$$\begin{aligned} \dot{V}(x) &= I_{1 \times m} \{ \Sigma^T(x) \omega_{F+\xi}(x, \gamma) - (l^2 + 1) \Sigma^T(x) \phi_M^2 \omega_G(x, \gamma) \rho'(x) \Omega_G^T(x) \Sigma(x) \} I_{m \times 1} \\ &\leq I_{1 \times m} \{ \Sigma^T(x) \omega_{F+\xi}(x, \gamma) - (l^2 + 1) \Sigma^T(x) \zeta \Omega_G(x) \rho'(x) \Omega_G^T(x) \Sigma(x) \} I_{m \times 1} \\ &= I_{1 \times m} \{ -l^2 \Sigma^T(x) \zeta \Omega_G(x) \rho'(x) \Omega_G^T(x) \Sigma(x) \\ &\quad + \Sigma^T(x) \omega_{F+\xi}(x, \gamma) - \Sigma^T(x) \zeta \Omega_G(x) \rho'(x) \Omega_G^T(x) \Sigma(x) \} I_{m \times 1} \\ &\leq I_{1 \times m} \{ -l^2 \Sigma^T(x) \zeta \Omega_G(x) \rho'(x) \Omega_G^T(x) \Sigma(x) \\ &\quad + \frac{1}{4} \omega_{F+\xi}^T(x, \gamma) [\zeta \Omega_G(x) \rho'(x) \Omega_G^T(x)]^{-1} \omega_{F+\xi}(x, \gamma) \} I_{m \times 1} \end{aligned}$$

according to theorem 6.2. Obviously, if $\Omega_G(x) \rho'(x) \Omega_G^T(x)$ is positive definite, so is $[\Omega_G(x) \rho'(x) \Omega_G^T(x)]^{-1}$, so that we can choose $\rho'(x)$ as a positive definite symmetric matrix such that the following matrix

$$-l^2 \Sigma^T(x) \zeta \Omega_G(x) \rho'(x) \Omega_G^T(x) \Sigma(x) + \frac{1}{4} \omega_{F+\xi}^T(x, \gamma) [\zeta \Omega_G(x) \rho'(x) \Omega_G^T(x)]^{-1} \omega_{F+\xi}(x, \gamma)$$

is negative definite, and so $\dot{V}(x) < 0$. Let

$$\rho(x) = \frac{\phi_M^2}{2l\zeta} [\Omega_G(x)]^{-1} \Omega_{F+\xi}(x) \cdot |\Sigma(x)|^{-1} [\Omega_G^T(x)]^{-1}$$

Note here that $\Sigma(x)$, $\omega_F(x)$ and $|\Sigma(x)|$ are diagonal matrices, so they will commute with one another. Therefore

$$\begin{aligned} \dot{V}(x) &\leq -I_{1 \times m} \{ l^2 \Sigma^T(x) \zeta \Omega_G(x) \rho'(x) \Omega_G^T(x) \Sigma(x) \\ &\quad - \frac{1}{4} \omega_{F+\xi}^T(x, \gamma) [\zeta \Omega_G(x) \rho'(x) \Omega_G^T(x)]^{-1} \omega_{F+\xi}(x, \gamma) \} I_{m \times 1} \\ &= -I_{1 \times m} \{ \frac{l}{2} \Omega_F(x) |\Sigma(x)| - \frac{l}{2} \omega_{F+\xi}^T(x, \gamma) [\Omega_F(x) |\Sigma(x)|^{-1}]^{-1} \omega_{F+\xi}(x, \gamma) \} I_{m \times 1} \\ &= -I_{1 \times m} \{ \frac{l}{2} \Omega_{F+\xi}(x) |\Sigma(x)| [I - \omega_{F+\xi}^T(x, \gamma) \Omega_{F+\xi}^2(x) \omega_{F+\xi}(x, \gamma)] \} I_{m \times 1} \end{aligned}$$

where $|\Sigma(x)| \Omega_{F+\xi}(x)$ is positive definite.

For the i^{th} entry of the diagonal matrix $I - \omega_{F+\xi}^T(x) \Omega_{F+\xi}^2(x) \omega_{F+\xi}(x)$,

$$1 - \frac{[\omega_{F+\xi}^2(x, \gamma)]_i}{[\Omega_{F+\xi}^2(x)]_i} = 1 - \frac{\left(\sum_{j=1}^n \frac{\partial \sigma_i}{\partial x_j} (F_j(x, \gamma) + \xi_j(t)) \right)^2}{\left(\sum_{j=1}^n \left| \frac{\partial \sigma_i}{\partial x_j} \right| \cdot (\Phi_{F_j}(x) + \Phi_{\xi_j}(t)) \right)^2} > 0 \quad (6.28)$$

so, $I - \omega_{F+\xi}^T(x, \gamma) \Omega_{F+\xi}^2(x) \omega_{F+\xi}(x, \gamma)$ is positive definite. It follows that

$$- \left\{ \iota^2 \Sigma^T(x) \zeta \Omega_G(x) \rho'(x) \Omega_G^T(x) \Sigma(x) - \frac{1}{4} \omega_{F+\xi}^T(x, \gamma) [\zeta \Omega_G(x) \rho'(x) \Omega_G^T(x)]^{-1} \omega_{F+\xi}(x, \gamma) \right\}$$

is negative definite. Now the proof is completed and it is possible to conclude that

$$\dot{V}(x) < 0 \quad (6.29)$$

and the system is stable. \square

REMARK 6.2:

- The proposed control is of the form

$$u(t) = - \frac{(\iota^2 + 1) \varphi_M^2}{2\iota \zeta} [\Omega_G(x)]^{-1} \Omega_{F+\xi}(x) \cdot \text{sign}[\sigma(x)] \quad (6.30)$$

in which the constant ζ can be chosen by the designer to satisfy condition (6.17). For instance

① $\zeta = \varphi_M^2$; then

$$u(t) = - \frac{(\iota^2 + 1)}{2\iota} [\Omega_G(x)]^{-1} \Omega_{F+\xi}(x) \cdot \text{sign}[\sigma(x)] \quad (6.31)$$

② $\zeta = 1$; then

$$u(t) = - \frac{(\iota^2 + 1) \varphi_M^2}{2\iota} [\Omega_G(x)]^{-1} \Omega_{F+\xi}(x) \cdot \text{sign}[\sigma(x)] \quad (6.32)$$

6.3 ILLUSTRATIVE EXAMPLE

Consider the following simple nonlinear plant of the form:

$$\dot{x}(t) = F(x, \gamma) + G(x, \gamma)u(t)$$

where

$$F(x, \gamma) = \begin{pmatrix} a_{11}' \sin(x_1) + a_{12}' x_2 \\ a_{21}' x_1 + a_{23}' x_3 \\ a_{31}' x_1 + a_{33}' x_3 \end{pmatrix} \quad G(x, \gamma) = \begin{pmatrix} 0 & 0 \\ b_{21}' & 0 \\ 0 & b_{32}' \end{pmatrix}$$

in which the uncertainties have the following bounds:

$$\begin{aligned} a_{11}' &\in [-1, 1.2] & a_{12}' &\in [1, 2] & a_{21}' &\in [10, 15] & a_{31}' &\in [-20, -10] \\ a_{23}' &\in [-6, -5] & a_{33}' &\in [10, 20] & b_{21}' &\in [1, 2] & b_{32}' &\in [2, 10] \end{aligned}$$

The nominal matrices can then be chosen to be

$$\begin{aligned} F(x, \tilde{\gamma}) &= \begin{pmatrix} a_{11}\sin(x_1) + a_{12}x_2 \\ a_{21}x_1 + a_{23}x_3 \\ a_{31}x_1 + a_{33}x_3 \end{pmatrix} = \begin{pmatrix} 0.1\sin(x_1) + 1.5x_2 \\ 12.5x_1 - 5.5x_3 \\ -15x_1 + 15x_3 \end{pmatrix} \\ G(x, \tilde{\gamma}) &= \begin{pmatrix} 0 & 0 \\ b_{21} & 0 \\ 0 & b_{32} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1.5 & 0 \\ 0 & 6 \end{pmatrix} \end{aligned}$$

As this system model is already in regular form, it can be directly rewritten as

$$\begin{aligned} \dot{x}^1(t) &= F_1(x^1, x^2, \gamma) \\ \dot{x}^2(t) &= F_2(x^1, x^2, \gamma) + G_2(x^1, x^2, \gamma) \cdot u(t) \end{aligned}$$

such that two sets of new states $x^1 = x_1$, $x^2 = (x_2, x_3)^T$ result, and we therefore have

$$\begin{aligned} F_1(x, \gamma) &= a_{11}\sin(x_1) + [a_{12}, 0] \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \\ F_2(x, \gamma) &= \begin{pmatrix} a_{21} \\ a_{31} \end{pmatrix} x_1 + \begin{pmatrix} 0 & a_{23} \\ 0 & a_{33} \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \\ G_2(x, \gamma) &= \begin{pmatrix} b_{21} & 0 \\ 0 & b_{32} \end{pmatrix} \end{aligned}$$

A switching function is defined as follows

$$\sigma(x) = \sigma_1(x^1) - x^2 = 0$$

where
$$\sigma_1(x^1) = \begin{pmatrix} s_1 x_1 - \frac{a_{11}}{a_{12}} \sin(x_1) \\ s_2 x_1 \end{pmatrix}$$

The reduced order closed loop system (on the switching surface) is then

$$\dot{x}_1 = a_{11}\sin(x_1) + [a_{12}, 0] \cdot \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = a_{11}\sin(x_1) + [a_{12}, 0] \cdot \sigma_1(x^1) = \lambda \cdot x^1$$

so, $s_1 = \lambda/a_{12}$, and s_2 could be any value. Letting $s_1 = -1$, $s_2 = -0.7368$, results in a closed loop system with pole: $\lambda = -1.5$. The partial derivative of the switching function is given by

$$\nabla \sigma = \begin{pmatrix} -1 - \frac{a_{11}}{a_{12}} \cos(x_1) & \vdots & -1 & 0 \\ -0.7368 & \vdots & 0 & -1 \end{pmatrix}$$

$$\phi_G = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 2 \end{pmatrix}$$

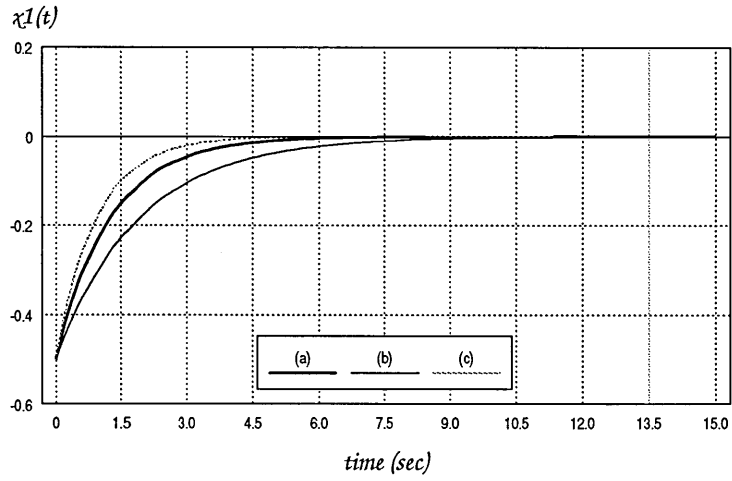
$$\Omega_G = \nabla \sigma \cdot \phi_G = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$$

$$\varphi_M^2 = \lambda_{\max}(\Omega_G \cdot \Omega_G^T) = 4$$

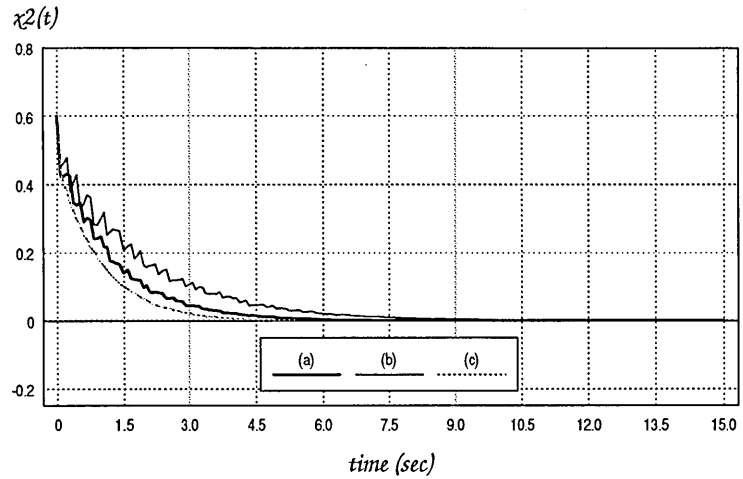
It is obvious that the matrix

$$\omega_G \cdot \Omega_G^T = \begin{pmatrix} -b'_{21} & 0 \\ 0 & -b'_{32} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$$

is positive definite for $b'_{21} \in [1, 2]$, $b'_{32} \in [2, 10]$, and satisfies condition (6.7), so the technique of theorem 6.4 is applicable here. We choose $\zeta < 1$.



System State $x_1(t)$



System State $x_2(t)$

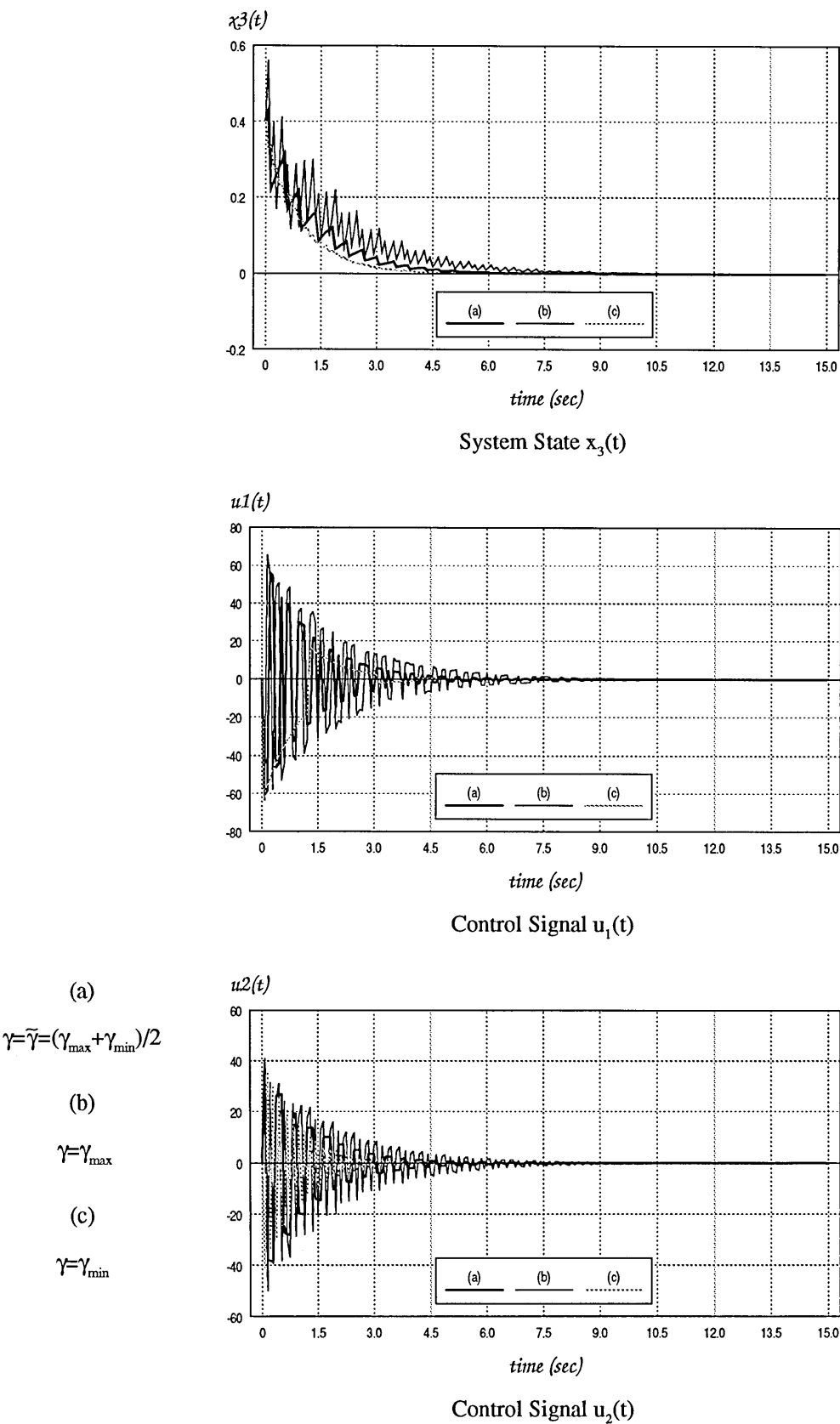


Fig. 6.1 Simulation results for the illustrative example

The simulation results are shown in Fig. 6.1. From the results it can be seen that although there are significant uncertainties in the system, the system has been stabilised and good closed loop system performance has been achieved.

6.4 SUMMARY

In this chapter, the robust control problem for a class of multivariable nonlinear systems in the presence of mismatched uncertainties has been addressed, and robust control techniques have been developed. In contrast to previous work on the problem, there is no requirement for decoupling the nonlinear uncertain system or decentralising the whole system into several subsystems, no requirement for the nominal dynamics to be either stable or in some way precompensated, and no requirement for matching assumptions on uncertainties.

The design method is summarised as follows:

Algorithm:

- ① Transform the original nonlinear uncertain system into a regular form;
- ② Construct matrices for the supremum bounds of $F(x, \gamma)$, $\xi(t)$ and infimum bound of $G(x, \gamma)$ satisfying conditions (6.2), (6.3) and (6.4);
- ③ Check that conditions (6.5) and (6.6) hold;
- ④ Design a switching function $\sigma(x)$ such that condition (6.7) is satisfied;
- ⑤ Calculate the spectral norm of Ω_σ , and choose constants ι and ς satisfying condition (6.17);
- ⑥ Construct new matrices of form (6.25), (6.26) and (6.27);
- ⑦ Obtain feedback control of form (6.23) with feedback gain (6.24).

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Robust Tracking Control



Overview

This chapter discusses the robust tracking problem. The behaviour of the closed loop system has been investigated and some important conclusions have been drawn. Simulation results are also included.



Outline

- ✓ Introduction
- ✓ Robust Tracking of SISO Systems
- ✓ Robust Tracking of MIMO Systems
- ✓ Illustrative Example
- ✓ Summary

7.1 INTRODUCTION

ONLY the regulator problem⁹ has been considered in previous chapters, the aim having been to compensate uncertainties and drive the states of the system to zero. Another important control aspect is the servo problem, *i.e.*, trajectory tracking. The objective is to make the states and outputs follow desired trajectories. In order to achieve this, an ideal trajectory x^d is introduced, and the control aims at driving the errors, $e = x - x^d$, towards zero.

In this chapter, the robust tracking control problem for a class of nonlinear systems in the presence of uncertainties is investigated, and robust controllers are developed. The proposed design procedure consists of two phases. Firstly, the original nonlinear uncertain system is transformed into a new coordinate system using the feedback linearisation technique such that a system with linearised nominal part is obtained. Secondly, a robust variable-structure-like controller is developed based on Lyapunov stability theory, and the feedback gain obtained is only related to uncertainty bounds. Results are obtained for the cases where the uncertainties satisfy the generalised matching assumption as well as where they do not. The controller possesses the same structure in each case, but the tracking errors may be larger when mismatched uncertainties occur. It is also shown that the tracking errors will converge to zero when only matched uncertainties are present, or to a finite open ball with a finite radius in a finite time when mismatched uncertainties are present, the radius of the ball depending only on the bounds of the mismatched uncertainties. The internal dynamics are also considered, and under the assumption of minimum phase, the internal dynamics will converge to a ball with finite radius which depends on the bound of the desired trajectory.

⁹ The regulator problem is sometimes referred to as the stability problem, whilst the servo problem is called the tracking problem.

7.2 ROBUST TRACKING OF SISO SYSTEMS

We now consider SISO nonlinear uncertain systems of the form

$$\begin{aligned}\dot{x}(t) &= F(x, \gamma) + G(x, \gamma)u(t) \\ y(t) &= H(x)\end{aligned}\tag{7.1}$$

where $F(x, \gamma): \mathbb{R}^n \times \mathbb{R}^\gamma \rightarrow \mathbb{R}^n$, $G(x, \gamma): \mathbb{R}^n \times \mathbb{R}^\gamma \rightarrow \mathbb{R}^n$, x , y and u are the state, output and admissible control respectively, having appropriate dimensions, and $\gamma(t)$ is a set of lumped uncertain elements. It is assumed that the state and input mappings $F(x, \gamma)$ and $G(x, \gamma)$ are bounded, and that the bounds are deterministic and known. These bounds will be described later.

In what follows, in order to investigate the stability properties and design the feedback control, the following generalised Lyapunov function candidate is considered

$$V(t) = V_1(t) + V_2(t) = e^T P e + \sigma^2(t)/2\tag{7.2}$$

where $\sigma(t)$ is the chosen switching function, and P is obtained by solving Lyapunov matrix equation $A^T P + P A = -Q$, where A is the state matrix of linearised nominal system of (7.1), P and Q are positive definite matrices having appropriate dimensions.

We define the notations

$$\lambda_{M(m)}(\cdot) = \max(\min)\{\lambda(\cdot)\}\tag{7.3}$$

to indicate the maximum (minimum) eigenvalue of a square matrix.

THEOREM 7.1. (*Uniform Ultimate Boundedness of SISO Nonlinear Uncertain Systems*)

For the SISO nonlinear uncertain system represented by (7.1), if the uncertainties are bounded, then a variable structure controller can be found such that the output response of the system will track a given desired trajectory, and the closed loop system is uniformly ultimately bounded. Moreover, the tracking errors will ① converge to zero in a finite time T and remain there when only matched uncertainties are present; or ② enter a ball B_κ with radius κ in a finite time $T(r, \kappa)$ and remain there when mismatched uncertainties are

present. Here r is the bound of the initial state, and the radius κ depends only on the bounds of the mismatched uncertainties.

The following sixteen pages are concerned with the proof of this theorem.

7.2.1 The Case of Matched Uncertainties

According to theorem A.6 of appendix A, a coordinate transformation can be found such that a given nonlinear uncertain system of form (7.1), with relative order $v \leq n$, in the presence of only matched uncertainties, can be transformed into the following form:

$$\begin{aligned} \dot{z}_1(t) &= z_2(t) \\ &\vdots \\ \dot{z}_{v-1}(t) &= z_v(t) \\ \dot{z}_v(t) &= a(z, \zeta) + b(z, \zeta) \cdot u(t) + \tilde{\delta}_1(z, \zeta, \gamma) + \tilde{\delta}_2(z, \zeta, \gamma) \cdot u(t) \end{aligned} \quad (7.4)$$

$$y(t) = h(z)$$

$$\dot{\zeta}(t) = q(z, \zeta)$$

$$\text{Let } u(t) = \frac{1}{b(z, \zeta)} \{ -a(z, \zeta) + v(t) \} \quad (7.5)$$

then

$$\begin{aligned} \dot{z}_1(t) &= z_2(t) \\ &\vdots \\ \dot{z}_{v-1}(t) &= z_v(t) \\ \dot{z}_v(t) &= v(t) + \delta_1(z, \zeta, \gamma) + \delta_2(z, \zeta, \gamma) \cdot v(t) \end{aligned} \quad (7.6)$$

$$y(t) = h(z)$$

$$\dot{\zeta}(t) = q(z, \zeta)$$

where

$$\delta_1(z, \zeta, \gamma) = \tilde{\delta}_1(z, \zeta, \gamma) - \tilde{\delta}_2(z, \zeta, \gamma) \cdot \frac{a(z, \zeta)}{b(z, \zeta)} \quad (7.7)$$

$$\delta_2(z, \zeta, \gamma) = \tilde{\delta}_2(z, \zeta, \gamma) \cdot \frac{1}{b(z, \zeta)} \quad (7.8)$$

are uncertainties in the system which clearly satisfy the generalised matching assumption of definition 2.3, and it is also required that $\delta_2 > 0$ (as assumed in assumption 2.7).

Denote the tracking errors by the difference between the real state trajectory $z(t)$ and the given ideal trajectory $z^d(t)$

$$e_i(t) = z_i(t) - z_i^d(t) \quad i=1,2,\dots,v \quad (7.9)$$

then we have a new system with the tracking errors $e(t)$ as the states and $v(t)$ as the input

$$\begin{aligned} \dot{e}_1(t) &= e_2(t) \\ &\vdots \\ \dot{e}_{v-1}(t) &= e_v(t) \\ \dot{e}_v(t) &= v(t) + \delta_1(z, \zeta, \gamma) + \delta_2(z, \zeta, \gamma) \cdot v(t) - \dot{z}_v^d(t) \\ \dot{\zeta}(t) &= q(z, \zeta) \end{aligned} \quad (7.10)$$

Define the following polynomial

$$b(\lambda) = \lambda^{v-1} + a_1 \lambda^{v-2} + \dots + a_{v-1}$$

where a_i ($i=1,2,\dots,v-1$) are chosen such that $b(\lambda)$ is Hurwitz^②. The switching function can therefore be defined as follows

$$\sigma(t) = e_v(t) + \sum_{k=1}^{v-1} a_k \cdot e_{v-k}(t) \quad (7.11)$$

Using (7.10)

$$\begin{aligned} \dot{e}_{v-1}(t) &= e_v(t) + \sum_{k=1}^{v-1} a_k \cdot e_{v-k}(t) - \sum_{k=1}^{v-1} a_k \cdot e_{v-k}(t) \\ &= - \sum_{k=1}^{v-1} a_k \cdot e_{v-k}(t) + \sigma(t) \end{aligned}$$

and the time derivative of the switching function (7.11) is

$$\begin{aligned} \dot{\sigma}(t) &= \dot{e}_v(t) + \sum_{k=1}^{v-1} a_k \cdot \dot{e}_{v-k}(t) \\ &= v(t) + \delta_1(z, \zeta, \gamma) + \delta_2(z, \zeta, \gamma) \cdot v(t) - \dot{z}_v^d(t) + \sum_{k=1}^{v-1} a_k \cdot \dot{e}_{v-k}(t) \end{aligned} \quad (7.12)$$

The feedback control is chosen to be of the following form

$$\begin{aligned} v(t) &= \dot{z}_v^d(t) - \sum_{k=1}^{v-1} a_k \cdot e_{v-k+1}(t) - \rho_1 \cdot \sigma(t) - \rho_2 \cdot \text{sgn}[\sigma(t)] \\ &= \dot{z}_v^d(t) - \sum_{k=1}^{v-1} a_k \cdot \dot{e}_{v-k}(t) - \rho_1 \cdot \sigma(t) - \rho_2 \cdot \text{sgn}[\sigma(t)] \end{aligned} \quad (7.13)$$

^② A Hurwitz polynomial is a polynomial having only roots with negative real part.

where $\text{sgn}(\sigma)$ is the sign function of $\sigma(t)$. Therefore, from (7.12)

$$\begin{aligned}\dot{\sigma}(t) &= \left\{ \dot{z}_v^d(t) - \sum_{k=1}^{v-1} a_k \cdot \dot{e}_{v-k}(t) - \rho_1 \cdot \sigma(t) - \rho_2 \cdot \text{sgn}[\sigma(t)] \right\} \\ &\quad + \delta_1(z, \zeta, \gamma) + \delta_2(z, \zeta, \gamma) v(t) - \dot{z}_v^d(t) + \sum_{k=1}^{v-1} a_k \cdot \dot{e}_{v-k}(t) \\ &= -\rho_1 \cdot \sigma(t) - \rho_2 \cdot \text{sgn}[\sigma(t)] + \delta_1(z, \zeta, \gamma) + \delta_2(z, \zeta, \gamma) \cdot v(t)\end{aligned}$$

It is concluded that, for the robust tracking problem, the original nonlinear uncertain system of form (7.1), subject to the generalised matching assumption, can be linearised and transferred into a new system of form

$$\begin{aligned}\dot{e}_1(t) &= e_2(t) \\ &\vdots \\ \dot{e}_{v-2}(t) &= e_{v-1}(t) \\ \dot{e}_{v-1}(t) &= -\sum_{k=1}^{v-1} a_k \cdot e_{v-k}(t) + \sigma(t)\end{aligned}\tag{7.14}$$

$$\dot{\sigma}(t) = -\rho_1 \cdot \sigma(t) - \rho_2 \cdot \text{sgn}[\sigma(t)] + \delta_1(z, \zeta, \gamma) + \delta_2(z, \zeta, \gamma) \cdot v(t)\tag{7.15}$$

or written in compact form

$$\dot{e}(t) = A e(t) + B \sigma(t)\tag{7.16}$$

$$\dot{\sigma}(t) = -\rho_1 \cdot \sigma(t) - \rho_2 \cdot \text{sgn}[\sigma(t)] + \delta_1(z, \zeta, \gamma) + \delta_2(z, \zeta, \gamma) \cdot v(t)\tag{7.17}$$

where

$$A = \begin{pmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ \vdots & 0 & & \vdots & 1 \\ -a_{v-1} & -a_{v-2} & \dots & -a_1 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix} \quad e = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_{v-1} \end{pmatrix}$$

The new system can be regarded as two subsystems, where (7.16) has $e(t)$ as the state and $\sigma(t)$ as the input, and (7.17) has $\sigma(t)$ as the state and $v(t)$ as the input.

All uncertainties (7.7), (7.8) are assumed bounded, and the bounds are given by

$$\Delta_1 \geq \max_{\gamma(t) \in R^\gamma} |\delta_1(z, \zeta, \gamma)| \geq 0\tag{7.18}$$

$$\Delta_2 \geq \max_{\gamma(t) \in R^\gamma} |\delta_2(z, \zeta, \gamma)| \geq 0\tag{7.19}$$

where Δ_1 and Δ_2 , which could either be functions of tracking errors $e(t)$ and time t or simply constants, are presumed deterministic and known.

§1. Stability on the Switching Surface

By stability on the switching surface, it is meant that $\sigma(t)=0$. The Lyapunov function, according to (7.2), is therefore given by

$$V(t) = e^T P e$$

Differentiating $V(t)$ along $e(t)$ and considering $\sigma(t)=0$, gives

$$\dot{V}(t) = \dot{e}^T P e + e^T P \dot{e} = -e^T Q e + 2e^T P B \sigma(t) \leq -\lambda_m(Q) e^T e < 0$$

Obviously, whenever $e^T e = \|e(t)\|^2 > 0$, $\dot{V}(t) < 0 \forall t > 0$, because

$$\lambda_m(Q) > 0 \quad (7.20)$$

It is therefore concluded that the tracking error will converge to zero, *i.e.*, the system is asymptotically stable.

§2. Stability off the Switching Surface

Initial conditions will not necessarily be on the switching surface, so the state trajectory must be considered for $\sigma(t) \neq 0$. The Lyapunov function is of the form

$$V(t) = V_1(t) + V_2(t) = e^T P e + \frac{1}{2} \sigma^2(t)$$

$$\begin{aligned} \dot{V}(t) &= \dot{V}_1(t) + \dot{V}_2(t) = \dot{e}^T P e + e^T P \dot{e} + \sigma(t) \dot{\sigma}(t) \\ &= \{ \dot{e}^T P e + e^T P \dot{e} - \lambda_M(P) \sigma^2(t) \} + \{ \sigma(t) \dot{\sigma}(t) + \lambda_M(P) \sigma^2(t) \} \\ &= \dot{V}_1'(t) + \dot{V}_2'(t) \end{aligned} \quad (7.21)$$

$$\text{where} \quad \dot{V}_1'(t) = \dot{V}_1(t) - \lambda_M(P) \sigma^2(t) \quad (7.22)$$

$$\dot{V}_2'(t) = \dot{V}_2(t) + \lambda_M(P) \sigma^2(t) \quad (7.23)$$

The two portions are now considered separately. Firstly

$$\dot{V}_1'(t) = \dot{e}^T P e + e^T P \dot{e} - \lambda_M(P) \sigma^2(t) = -e^T Q e + 2e^T P B \sigma(t) - \lambda_M(P) \sigma^2(t)$$

Note here that $\|B\|=1$, and

$$2e^T P B \sigma(t) \leq 2\lambda_M(P) \|e\| \cdot \|\sigma\| \leq \lambda_M(P) \|e\|^2 + \lambda_M(P) \|\sigma\|^2$$

$$\begin{aligned} \text{so} \quad \dot{V}_1'(t) &\leq -\lambda_m(Q) \|e\|^2 + \lambda_M(P) \|e\|^2 + \lambda_M(P) \|\sigma\|^2 - \lambda_M(P) \sigma^2(t) \\ &= -[\lambda_m(Q) - \lambda_M(P)] \|e\|^2 \end{aligned} \quad (7.24)$$

Again for the second term of (7.21)

$$\begin{aligned}
 \dot{V}_2'(t) &= \sigma(t)\dot{\sigma}(t) + \lambda_M(P)\sigma^2(t) \\
 &= \sigma(t) \left\{ -\rho_1\sigma - \rho_2\text{sgn}(\sigma) + \delta_1 + \delta_2[w(t) - \rho_1\sigma - \rho_2\text{sgn}(\sigma)] \right\} + \lambda_M(P)\sigma^2(t) \\
 &= -\left(\rho_1 + \frac{\rho_2}{\|\sigma\|}\right)\sigma^2 + \delta_1\sigma + \delta_2\left[w(t)\sigma - \left(\rho_1 + \frac{\rho_2}{\|\sigma\|}\right)\sigma^2\right] + \lambda_M(P)\sigma^2(t)
 \end{aligned}$$

where

$$w(t) = \dot{z}_v^d(t) - \sum_{k=1}^{v-1} a_k \cdot e_{v-k+1}(t) \quad (7.25)$$

and Δ_1, Δ_2 are defined by (7.18), (7.19). Therefore

$$\begin{aligned}
 \dot{V}_2'(t) &\leq -\left(\rho_1 + \frac{\rho_2}{\|\sigma\|}\right)\sigma^2 + \delta_1\sigma + \delta_2\left[w(t)\sigma - \left(\rho_1 + \frac{\rho_2}{\|\sigma\|}\right)\sigma^2\right] + (1 + \Delta_2)\lambda_M(P)\sigma^2 \\
 &\leq -\beta\left[\rho_1 + \frac{\rho_2}{\|\sigma\|} - \lambda_M(P)\right]\sigma^2 + \left\{ \delta_1\sigma - (1 - \beta)\left[\rho_1 + \frac{\rho_2}{\|\sigma\|} - \lambda_M(P)\right]\sigma^2 \right\} \\
 &\quad + \Delta_2\left|w(t)\sigma - \left[\rho_1 + \frac{\rho_2}{\|\sigma\|} - \lambda_M(P)\right]\sigma^2\right| \\
 &\leq -\beta\left[\rho_1 + \frac{\rho_2}{\|\sigma\|} - \lambda_M(P)\right]\sigma^2 + \frac{\frac{\Delta_1^2}{1 - \beta} + \Delta_2 w^2}{4\left[\rho_1 + \frac{\rho_2}{\|\sigma\|} - \lambda_M(P)\right]} \quad (7.26)
 \end{aligned}$$

where $0 < \beta < 1$ is a constant. Note that the identity (4.12) has been used here. Then the choice of

$$\rho_2 + [\rho_1 - \lambda_M(P)]\|\sigma\| \geq \frac{1}{2} \sqrt{\frac{\Delta_1^2}{(1 - \beta)\beta} + \frac{\Delta_2 w^2}{\beta}} \geq 0 \quad (7.27)$$

implies $\dot{V}_2'(t) \leq 0$, so that

$$\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) \leq \dot{V}_1'(t) \leq -[\lambda_m(Q) - \lambda_M(P)]\|e\|^2 \quad (7.28)$$

It is easy to see from (7.28) that, for any non-zero tracking error $\|e\| > 0$, we have $\dot{V}(t) < 0$ if

$$\lambda_m(Q)/\lambda_M(P) > 1 \quad (7.29)$$

This means that the error will tend to zero as time increases.

Now considering (7.27), let

$$\rho_1 = \lambda_M(P) > 0 \quad (7.30)$$

Then

$$\rho_2 \geq \frac{1}{2} \sqrt{\frac{\Delta_1^2}{(1-\beta)\beta} + \frac{\Delta_2 w^2}{\beta}} \geq 0 \quad (7.31)$$

Such a choice of control v guarantees that the closed loop system is asymptotically stable.

REMARK 7.1:

- The results obtained show that the closed loop response of the nonlinear system (7.10) is asymptotically stable with the choice of feedback gains (7.30) and (7.31) when the uncertainties satisfy the matching conditions.

7.2.2 The Case of Both Matched and Mismatched Uncertainties

Applying the same coordinate transformation to a system where both matched and mismatched uncertainties are present results in

$$\begin{aligned} \dot{e}_1(t) &= e_2(t) + \delta_{0,1}(z, \zeta, \gamma, v) \\ &\vdots \\ \dot{e}_{v-1}(t) &= e_v(t) + \delta_{0,v-1}(z, \zeta, \gamma, v) \\ \dot{e}_v(t) &= v(t) + \delta'_1(z, \zeta, \gamma) + \delta'_2(z, \zeta, \gamma)v(t) - \dot{z}^d(t) \\ y(t) &= h(x) \\ \dot{\zeta}(t) &= q(z, \zeta) \end{aligned} \quad (7.32)$$

where $\delta'_1(z, \zeta, \gamma)$ and $\delta'_2(z, \zeta, \gamma)$ represent the matched part of the uncertainties, and

$$\delta_{0,k}(z, \zeta, \gamma, v) = \delta_{0,k}^1(z, \zeta, \gamma) + \delta_{0,k}^2(z, \zeta, \gamma) \cdot v(t) \quad (k=1, 2, \dots, v-1) \quad (7.33)$$

indicates mismatched uncertainties in the system. According to definition (7.11)

$$\sigma(t) = e_v(t) + \sum_{k=1}^{v-1} a_k \cdot e_{v-k}(t)$$

so

$$\dot{e}_{v-1}(t) = e_v(t) + \delta_{0,v-1} + \sum_{k=1}^{v-1} a_k \cdot e_{v-k}(t) - \sum_{k=1}^{v-1} a_k \cdot e_{v-k}(t) = - \sum_{k=1}^{v-1} a_k \cdot e_{v-k}(t) + \sigma(t) + \delta_{0,v-1}$$

and hence

$$\dot{\sigma}(t) = \dot{e}_v(t) + \sum_{k=1}^{v-1} a_k \cdot \dot{e}_{v-k}(t)$$

$$\begin{aligned}
&= v(t) + \delta_1'(z, \zeta, \gamma) + \delta_2'(z, \zeta, \gamma) \cdot v(t) - \dot{z}_v^d(t) + \sum_{k=1}^{v-1} a_k \cdot \dot{e}_{v-k}(t) \\
&= \left\{ \dot{z}_v^d(t) - \sum_{k=1}^{v-1} a_k \cdot e_{v-k+1}(t) - \rho_1 \cdot \sigma(t) - \rho_2 \cdot \text{sgn}[\sigma(t)] \right\} \\
&\quad + \delta_1'(z, \zeta, \gamma) + \delta_2'(z, \zeta, \gamma) \cdot v(t) - \dot{z}_v^d(t) + \sum_{k=1}^{v-1} a_k \cdot [e_{v-k+1}(t) + \delta_{0,v-k}(z, \zeta, \gamma, v)] \\
&= -\rho_1 \cdot \sigma(t) - \rho_2 \cdot \text{sgn}[\sigma(t)] + \delta_1'(z, \zeta, \gamma) + \delta_2'(z, \zeta, \gamma) \cdot v(t) + \sum_{k=1}^{v-1} a_k \cdot \delta_{0,v-k}(z, \zeta, \gamma, v) \\
&= -\rho_1 \cdot \sigma(t) - \rho_2 \cdot \text{sgn}[\sigma(t)] \\
&\quad + [\delta_1'(z, \zeta, \gamma) + \sum_{k=1}^{v-1} a_k \cdot \delta_{0,v-k}^1(z, \zeta, \gamma)] + [\delta_2'(z, \zeta, \gamma) + \sum_{k=1}^{v-1} a_k \cdot \delta_{0,v-k}^2(z, \zeta, \gamma)] \cdot v(t)
\end{aligned}$$

So, the system can be written in the following form

$$\dot{e}(t) = Ae(t) + B\sigma(t) + \delta_0(z, \zeta, \gamma, v) \quad (7.34)$$

$$\dot{\sigma}(t) = -\rho_1 \cdot \sigma(t) - \rho_2 \cdot \text{sgn}[\sigma(t)] + \delta_1(z, \zeta, \gamma) + \delta_2(z, \zeta, \gamma) \cdot v(t) \quad (7.35)$$

where A, B and e are the same as those of (7.16) and (7.17) and

$$\delta_0 = [\delta_{0,1}, \dots, \delta_{0,v-1}]^T$$

and the matched uncertainties are

$$\delta_1(z, \zeta, \gamma) = \delta_1'(z, \zeta, \gamma) + \sum_{k=1}^{v-1} a_k \cdot \delta_{0,v-k}^1(z, \zeta, \gamma) \quad (7.36)$$

$$\delta_2(z, \zeta, \gamma) = \delta_2'(z, \zeta, \gamma) + \sum_{k=1}^{v-1} a_k \cdot \delta_{0,v-k}^2(z, \zeta, \gamma) \quad (7.37)$$

The uncertainties are still assumed to be bounded. The bounds of the matched part δ_1 , δ_2 are of the same form as (7.18), (7.19), and the bound of the mismatched part δ_0 is given by

$$\delta_0(z, \zeta, \gamma, v) \leq c_1 \|z\| + c_2 \|v\| \leq c_0 + c_1 \|e\| + c_2 \|v\| \quad (7.38)$$

where $c_0 = c_1 \|z^d\|$, c_1 and c_2 are all positive constants, and presumed known. It is reasonable to assume that, for any properly designed robust control, the state outputs $z(t)$ and the controls $v(t)$ are bounded. From (7.13)

$$\begin{aligned}
v(t) &= \dot{z}_v^d(t) - a_1 e_v - a_2 e_{v-1} - \dots - a_{v-1} e_2 - \rho_1 \cdot \sigma(t) - \rho_2 \cdot \text{sgn}[\sigma(t)] \\
&= \dot{z}_v^d(t) - \tilde{a}e + \tilde{a}e - a_1 e_v - a_2 e_{v-1} - \dots - a_{v-1} e_2 - \rho_1 \cdot \sigma(t) - \rho_2 \cdot \text{sgn}[\sigma(t)]
\end{aligned}$$

§1. Stability on the switching surface

From (7.2), $V(t) = e^T P e$ as $\sigma(t) = 0$. Differentiating $V(t)$ along $e(t)$, we have

$$\begin{aligned}\dot{V}(t) &= \dot{e}^T P e + e^T P \dot{e} = -e^T Q e + 2e^T P \delta_0 + 2e^T P B \sigma(t) \\ &\leq -\lambda_m(Q) e^T e + 2\lambda_M(P) \cdot \|e\| \{ \beta_0 + \beta_1 \|e\| \}\end{aligned}$$

bearing in mind (7.41). It is also true that

$$2\lambda_M(P) \beta_0 \|e\| \leq \lambda_M(P) \beta_0 \|e\|^2 + \lambda_M(P) \beta_0$$

so

$$\dot{V}(t) \leq -\{ \lambda_m(Q) - \lambda_M(P) [\beta_0 + 2\beta_1] \} \|e\|^2 + \lambda_M(P) \beta_0 \quad (7.45)$$

To make $\dot{V}(t) < 0$, it is required that

$$\frac{\lambda_m(Q)}{\lambda_M(P)} > \beta_0 + 2\beta_1 \quad (7.46)$$

and it is then concluded from (7.45) that whenever the tracking errors

$$\|e\|^2 > \frac{\beta_0}{\lambda_m(Q)/\lambda_M(P) - [\beta_0 + 2\beta_1]} > 0 \quad (7.47)$$

$\dot{V}(t) < 0$, and the system is stable.

REMARK 7.2:

- The result obtained above means that any tracking error such that $\|e\|^2$ is greater than

$$\varphi = \frac{\beta_0}{\lambda_m(Q)/\lambda_M(P) - [\beta_0 + 2\beta_1]}$$

makes $\dot{V}(t) < 0$, so that the system is stable. These tracking errors will converge and be arbitrarily close to φ .

- Let $\kappa = \varphi + \varepsilon$, where ε is an arbitrarily small positive constant. Then it is easy to see that the tracking errors $\|e\|^2$ will converge to a ball B_κ with radius of κ , which depends only on the bounds of the mismatched uncertainties in the system.

§2. Stability off the switching surface

The case of $\sigma(t) \neq 0$ is now considered. Here the state trajectories are not on the switching surface.

$$V(t) = V_1(t) + V_2(t) = e^T P e + \frac{1}{2} \sigma^2(t)$$

and

$$\begin{aligned} \dot{V}(t) &= \dot{V}_1(t) + \dot{V}_2(t) = \dot{e}^T P e + e^T P \dot{e} + \sigma(t) \dot{\sigma}(t) \\ &= \{ \dot{e}^T P e + e^T P \dot{e} - 2\lambda_M(P) \sigma^2(t) \} + \{ \sigma(t) \dot{\sigma}(t) + 2\lambda_M(P) \sigma^2(t) \} \\ &= \dot{V}_1'(t) + \dot{V}_2'(t) \end{aligned} \quad (7.48)$$

where

$$\dot{V}_1'(t) = \dot{V}_1(t) - 2\lambda_M(P) \sigma^2(t) \quad (7.49)$$

$$\dot{V}_2'(t) = \dot{V}_2(t) + 2\lambda_M(P) \sigma^2(t) \quad (7.50)$$

For the first term

$$\begin{aligned} \dot{V}_1'(t) &= \dot{e}^T P e + e^T P \dot{e} - 2\lambda_M(P) \sigma^2 = -e^T Q e + 2e^T P \delta_0 + 2e^T P B \sigma(t) - 2\lambda_M(P) \sigma^2 \\ &\leq -\lambda_m(Q) \|e\|^2 + 2\lambda_M(P) \|e\| \{ \beta_0 + \beta_1 \|e\| + \beta_2 \|\sigma\| \} + 2\lambda_M(P) \|e\| \cdot \|\sigma\| - 2\lambda_M(P) \sigma^2 \end{aligned}$$

where $\|B\|=1$ has been used. Now, using

$$2\lambda_M(P) \|e\| \beta_0 \leq \lambda_M(P) \beta_0 \|e\|^2 + \lambda_M(P) \beta_0$$

$$2\lambda_M(P) \beta_2 \|e\| \|\sigma\| \leq \lambda_M(P) \beta_2^2 \|e\|^2 + \lambda_M(P) \|\sigma\|^2$$

$$2\lambda_M(P) \|e\| \|\sigma\| \leq \lambda_M(P) \|e\|^2 + \lambda_M(P) \|\sigma\|^2$$

it follows that

$$\begin{aligned} \dot{V}_1'(t) &\leq -\lambda_m(Q) \|e\|^2 + \lambda_M(P) [1 + \beta_0 + 2\beta_1 + \beta_2^2] \|e\|^2 + \lambda_M(P) \beta_0 + 2\lambda_M(P) \|\sigma\|^2 - 2\lambda_M(P) \sigma^2 \\ &= -[\lambda_m(Q) - \lambda_M(P)(1 + \beta_0 + 2\beta_1 + \beta_2^2)] \|e\|^2 + \lambda_M(P) \beta_0 \end{aligned} \quad (7.51)$$

Again for the second term, we have

$$\begin{aligned} \dot{V}_2'(t) &= \sigma(t) \dot{\sigma}(t) + 2\lambda_M(P) \sigma^2(t) \\ &= \sigma(t) \{ -\rho_1 \cdot \sigma - \rho_2 \cdot \text{sgn}(\sigma) + \delta_1 + \delta_2 [w(t) - \rho_1 \cdot \sigma - \rho_2 \cdot \text{sgn}(\sigma)] \} + 2\lambda_M(P) \sigma^2(t) \\ &= -(\rho_1 + \frac{\rho_2}{\|\sigma\|}) \sigma^2(t) + \delta_1 \sigma(t) + \delta_2 [w(t) \sigma(t) - (\rho_1 + \frac{\rho_2}{\|\sigma\|}) \sigma^2] + 2\lambda_M(P) \sigma^2(t) \end{aligned}$$

where

$$w(t) = \dot{z}_v^d(t) - \sum_{k=1}^{v-1} a_k e_{v-k+1}(t)$$

Using definition (7.18) and (7.19), and bearing in mind identity (4.12),

$$\begin{aligned}
\dot{V}_2'(t) &\leq -\left(\rho_1 + \frac{\rho_2}{\|\sigma\|}\right)\sigma^2(t) + \delta_1\sigma + \delta_2[w(t)\sigma(t) - \left(\rho_1 + \frac{\rho_2}{\|\sigma\|}\right)\sigma^2(t)] + 2(1+\Delta_2)\lambda_M(P)\sigma^2(t) \\
&\leq -\beta\left[\rho_1 + \frac{\rho_2}{\|\sigma\|} - 2\lambda_M(P)\right]\sigma^2(t) + \left\{ \delta_1\sigma - (1-\beta)\left[\rho_1 + \frac{\rho_2}{\|\sigma\|} - 2\lambda_M(P)\right]\sigma^2(t) \right\} \\
&\quad + \Delta_2|w(t)\sigma(t) - \left[\rho_1 + \frac{\rho_2}{\|\sigma\|} - 2\lambda_M(P)\right]\sigma^2(t)| \\
&\leq -\beta\left[\rho_1 + \frac{\rho_2}{\|\sigma\|} - 2\lambda_M(P)\right]\sigma^2(t) + \frac{\frac{\Delta_1^2}{1-\beta} + \Delta_2 w^2}{4\left[\rho_1 + \frac{\rho_2}{\|\sigma\|} - 2\lambda_M(P)\right]} \quad (7.52)
\end{aligned}$$

where $0 < \beta < 1$ is a positive constant. The choice of

$$\rho_2 + \left[\rho_1 - 2\lambda_M(P)\right]\|\sigma\| \geq \frac{1}{2} \sqrt{\frac{\Delta_1^2}{(1-\beta)\beta} + \frac{\Delta_2 w^2}{\beta}} \geq 0 \quad (7.53)$$

implies that $\dot{V}_2'(t) \leq 0$. It follows that

$$\begin{aligned}
\dot{V}(t) &= \dot{V}_1(t) + \dot{V}_2(t) \leq \dot{V}_1'(t) \\
&\leq -\left[\lambda_m(Q) - \lambda_M(P)(1+\beta_0+2\beta_1+\beta_2^2)\right]\|e\|^2 + \lambda_M(P)\beta_0 < 0 \quad (7.54)
\end{aligned}$$

where the following condition

$$\lambda_m(Q)/\lambda_M(P) > 1 + \beta_0 + 2\beta_1 + \beta_2^2 \quad (7.55)$$

is required to be true. It is therefore concluded that whenever

$$\|e\|^2 > \frac{\beta_0}{\lambda_m(Q)/\lambda_M(P) - (1+\beta_0+2\beta_1+\beta_2^2)} > 0 \quad (7.56)$$

then $\dot{V}(t) < 0$, i.e., the system is stable. We can choose

$$\rho_1 = 2\lambda_M(P) > 0 \quad (7.57)$$

$$\rho_2 \geq \frac{1}{2} \sqrt{\frac{\Delta_1^2}{(1-\beta)\beta} + \frac{\Delta_2 w^2}{\beta}} \geq 0 \quad (7.58)$$

REMARK 7.3:

- From the discussion above, similar results to those for the case of only matched uncertainties are obtained, and the control possesses the same structure as that developed for the case of matched uncertainty. The difference is that the closed loop system cannot be asymptotically stable, but only ultimately bounded.

7.2.3 Internal Dynamics

No consideration of internal dynamics

$$\dot{\zeta}(t) = q(z, \zeta) \quad (7.59)$$

has been made so far. They may be of great importance and will now be investigated.

Suppose that the system is exponentially minimum-phase in some domain Ω , so that the zero dynamics of the system are exponentially stable in Ω . According to theorem 3.7, there exists a positive definite function $V_0(\zeta)$ satisfying the following inequalities

$$v_1 \|\zeta\|^2 < V_0(\zeta) < v_2 \|\zeta\|^2 \quad (7.60)$$

$$\frac{\partial V_0}{\partial \zeta} \cdot q(0, \zeta) < -v_3 \|\zeta\|^2 \quad (7.61)$$

$$\left| \frac{\partial V_0}{\partial \zeta} \right| < v_4 \|\zeta\| \quad (7.62)$$

for some positive constants v_1, v_2, v_3 and v_4 . Differentiating $V_0(\zeta)$ along ζ yields

$$\begin{aligned} \dot{V}_0(\zeta) &= \frac{\partial V_0}{\partial \zeta} \cdot q(z, \zeta) = \frac{\partial V_0}{\partial \zeta} \cdot q(0, \zeta) + \frac{\partial V_0}{\partial \zeta} \{ q(z, \zeta) - q(0, \zeta) \} \\ &\leq -v_3 \|\zeta\|^2 + v_4 \|\zeta\| \cdot \{ \|q(z, \zeta) - q(0, \zeta)\| \} \end{aligned}$$

It is also assumed that q is a Lipschitz vector function[⊗] because the states z, ζ , the state mapping f , and the coordinate transformation ψ are all smooth, *i.e.*, infinitely differentiable. This implies that

$$\|q(z, \zeta) - q(0, \zeta)\| \leq \vartheta \cdot \|z\|$$

$$\text{i.e.,} \quad \vartheta = \sup_{(z, \zeta) \in \Omega} \frac{\|q(z, \zeta) - q(0, \zeta)\|}{\|z\|} \quad (7.63)$$

where ϑ is the Lipschitz constant. So

$$\dot{V}_0(\zeta) \leq -v_3 \|\zeta\|^2 + v_4 \|\zeta\| \cdot \vartheta \cdot \|z\|$$

[⊗] DEFINITION (Lipschitz Condition)^[4]:

If the function $f(x, t)$ is continuous in t , and if there exists a strictly positive constant L such that

$$\|f(x_2, t) - f(x_1, t)\| \leq L \|x_2 - x_1\|$$

for all x_2 and x_1 in a finite neighbourhood of the origin and all t in the interval $[t_0, t_0 + T]$, then $f(x, t)$ is a Lipschitz function. The equation $\dot{x} = f(x, t)$ has a unique solution $x(t)$ for sufficiently small initial states and in a sufficiently short time interval.

In order to distinguish from the vector $e = (e_1, e_2, \dots, e_{v-1})^T$, define a new vector

$$e' = z - z^d = (e_1, e_2, \dots, e_v)^T \quad (7.64)$$

where $z = e' + z^d$, $\|z\| \leq \|e'\| + \|z^d\|$, and the following row vectors

$$a = (a_{v-1}, \dots, a_1), \quad b = (b_{v-1}, \dots, b_1)$$

where $b_i = a_i - 1$ ($i = 1, \dots, v-1$). We therefore have

$$\|e'\| = \|e_v + ae - be\| = \|\sigma - be\| \leq \|\sigma\| + b\|e\| \quad (7.65)$$

where $\sigma = e_v + ae = e_v + \sum_{k=1}^{v-1} a_k e_{v-k}(t)$, and $b = \|b\|$. So we have

$$\|z\| \leq \|e'\| + \|z^d\| \leq \|\sigma\| + b\|e\| + c \quad (7.66)$$

where $\|z^d\| \leq c$ is bounded.

$$\begin{aligned} \dot{V}_0(\zeta) &\leq -v_3 \|\zeta\|^2 + v_4 \|\zeta\| \cdot \vartheta \|z\| \leq -v_3 \|\zeta\|^2 + v_4 \|\zeta\| \cdot \vartheta \{ \|\sigma\| + b\|e\| + c \} \\ &\leq -v_3 \|\zeta\|^2 + v_4 \vartheta \|\zeta\| \cdot \|\sigma\| + v_4 \vartheta b \|\zeta\| \cdot \|e\| + v_4 \vartheta \|\zeta\| c \end{aligned} \quad (7.67)$$

Now we consider a Lyapunov function candidate of the following form

$$V(t) = V_1(t) + V_2(t) + \mu V_0(t) \quad (7.68)$$

where μ is a strictly positive constant to be determined. The time derivative of V along the trajectories of the system is

$$\begin{aligned} \dot{V}(t) &= \dot{V}_1(t) + \dot{V}_2(t) + \mu \dot{V}_0(t) \\ &= \dot{V}_1(t) - 2\lambda_M(P)\sigma^2 + \lambda_M(P)\|e\|^2 + \dot{V}_2(t) + 3\lambda_M(P)\sigma^2 + \mu \dot{V}_0(t) - \lambda_M(P)\|e\|^2 - \lambda_M(P)\sigma^2 \\ &= \dot{V}_1'(t) + \dot{V}_2'(t) + \dot{V}_0'(t) \end{aligned} \quad (7.69)$$

where

$$\dot{V}_1'(t) = \dot{V}_1(t) - 2\lambda_M(P)\sigma^2(t) + \lambda_M(P)\|e\|^2 \quad (7.70)$$

$$\dot{V}_2'(t) = \dot{V}_2(t) + 3\lambda_M(P)\sigma^2(t) \quad (7.71)$$

$$\dot{V}_0'(t) = \mu \dot{V}_0(t) - \lambda_M(P)\|e\|^2 - \lambda_M(P)\sigma^2(t) \quad (7.72)$$

It follows that

$$\begin{aligned} \dot{V}_0'(t) &\leq \{ -\mu v_3 \|\zeta\|^2 + \mu v_4 \|\zeta\| \cdot \vartheta \|z\| \} - \lambda_M(P)\|e\|^2 - \lambda_M(P)\sigma^2 \\ &\leq \{ -\mu v_3 \|\zeta\|^2 + \mu v_4 \|\zeta\| \cdot \vartheta [\|\sigma\| + b\|e\| + c] \} - \lambda_M(P)\|e\|^2 - \lambda_M(P)\sigma^2 \\ &= \{ -v_3 \mu \|\zeta\|^2 + \mu v_4 \vartheta \|\zeta\| \cdot \|\sigma\| + \mu v_4 \vartheta \|\zeta\| \cdot b\|e\| + \mu v_4 \vartheta \|\zeta\| c \} - \lambda_M(P)\|e\|^2 - \lambda_M(P)\sigma^2 \end{aligned}$$

Because

$$\begin{aligned}\mu v_4 \vartheta \|\zeta\| \cdot \|\sigma\| &\leq \frac{1}{4\lambda_M(P)} \mu^2 v_4^2 \vartheta^2 \|\zeta\|^2 + \lambda_M(P) \|\sigma\|^2 \\ \mu v_4 \vartheta b \|\zeta\| \cdot \|e\| &\leq \frac{1}{4\lambda_M(P)} \mu^2 v_4^2 \vartheta^2 b^2 \|\zeta\|^2 + \lambda_M(P) \|e\|^2\end{aligned}$$

then

$$\dot{V}_0'(t) \leq -\mu \left\{ v_3 - \frac{1}{4\lambda_M(P)} \mu v_4^2 \vartheta^2 (1+b^2) \right\} \|\zeta\|^2 + \mu v_4 \vartheta \|\zeta\| c < 0 \quad (7.73)$$

therefore

$$v_3 > \frac{1}{4\lambda_M(P)} \mu v_4^2 \vartheta^2 (1+b^2) > 0 \quad (7.74)$$

i.e.,

$$0 < \mu < \frac{4v_3\lambda_M(P)}{v_4^2\vartheta^2(1+b^2)} \quad (7.75)$$

and

$$\|\zeta\| > \frac{v_4 \vartheta c}{v_3 - \frac{1}{4\lambda_M(P)} \mu v_4^2 \vartheta^2 (1+b^2)} > 0 \quad (7.76)$$

Let

$$\mu = \frac{4\alpha v_3 \lambda_M(P)}{v_4^2 \vartheta^2 (1+b^2)} \quad (0 < \alpha < 1) \quad (7.77)$$

then

$$\|\zeta\| > \frac{v_4 \vartheta c}{(1-\alpha)v_3} > 0 \quad (7.78)$$

REMARK 7.4:

- The result obtained here implies that there exists an open ball B_κ with finite radius

$$\kappa = \frac{v_4 \vartheta c}{(1-\alpha)v_3}$$

which depends on the bounds of the desired trajectory z^d ($\|z^d\| < c$), such that the state of the internal dynamics will arbitrarily converge towards B_κ . So it is concluded that ζ is bounded as long as z^d is bounded.

- After considering the internal dynamics, the total Lyapunov function is given by (7.70)–(7.72)

instead of (7.49) and (7.50). Following the same procedure, it is therefore straightforward to extend (7.55)~(7.58) to

$$\lambda_m(Q)/\lambda_M(P) > 2 + \beta_0 + 2\beta_1 + \beta_2^2 > 0 \quad (7.79)$$

$$\|e\|^2 > \frac{\beta_0}{\lambda_m(Q)/\lambda_M(P) - (2 + \beta_0 + 2\beta_1 + \beta_2^2)} > 0 \quad (7.80)$$

$$\rho_1 = 3\lambda_M(P) > 0 \quad (7.81)$$

$$\rho_2 \geq \frac{1}{2} \sqrt{\frac{\Delta_1^2}{(1-\beta)\beta} + \frac{\Delta_2 w^2}{\beta}} \geq 0 \quad (0 < \beta < 1) \quad (7.82)$$

7.2.4 Estimate of Uniform Ultimate Boundedness

It is now possible to estimate the boundedness properties of the closed loop system. It can be shown that the closed loop system is uniformly ultimately bounded when either matched or mismatched uncertainties are present. Theorem 3.11 is required here.

The form of Lyapunov function to be used is

$$V(t) = V_1(t) + V_2(t) + \mu V_0(t) \quad (7.83)$$

Differentiating $V(t)$ along the system trajectory results in

$$\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) + \mu \dot{V}_0(t) = \dot{V}_1'(t) + \dot{V}_2'(t) + \dot{V}_0'(t) \quad (7.84)$$

Considering the results (7.77)~(7.82), we have $\dot{V}_0'(t) < 0$ and $\dot{V}_2'(t) < 0$. It follows that

$$\dot{V}(t) \leq \dot{V}_1'(t) \leq -[\lambda_m(Q) - \lambda_M(P)(2 + \beta_0 + 2\beta_1 + \beta_2^2)]\|e\|^2 + \lambda_M(P)\beta_0 < 0 \quad (7.85)$$

For simplicity, denote

$$\Phi_1 = \lambda_m(Q) - \lambda_M(P)(2 + \beta_0 + 2\beta_1 + \beta_2^2) > 0 \quad (7.86)$$

$$\Phi_0 = \lambda_M(P)\beta_0 > 0 \quad (7.87)$$

then

$$\dot{V}(t) \leq -\Phi_1\|e\|^2 + \Phi_0 < 0 \quad (7.88)$$

i.e.,

$$\|e\|^2 > \frac{\Phi_0}{\Phi_1} > 0 \quad (7.89)$$

so there exists a closed ball with radius

$$\kappa = \frac{\Phi_0}{\Phi_1} + \varepsilon \quad (7.90)$$

where ε is an arbitrarily small positive number such that the tracking error $\|e\|^2$ will enter the ball and remain in it thereafter. So we say the output of the closed loop system is ultimately bounded.

Furthermore, if the error $\|e\|^2$ enters the ball B_κ in a finite time, it is said that the system is uniformly ultimately bounded. Now let us try to find the time period required for the tracking errors $\|e\|$ to reach the surface of the ball B_κ .

$$\dot{V}_1'(t) = \frac{d[e^T P e]}{dt} \leq -\Phi_1 \|e\|^2 + \Phi_0 < 0 \quad (7.91)$$

So we have

$$\frac{d\|e\|^2}{dt} \leq \frac{-\Phi_1 \|e\|^2 + \Phi_0}{\lambda_m(P)} = -\Phi_1' \|e\|^2 + \Phi_0' < 0 \quad (7.92)$$

where

$$\Phi_1' = \Phi_1 / \lambda_m(P) \quad (7.93)$$

$$\Phi_0' = \Phi_0 / \lambda_m(P) \quad (7.94)$$

The solution of this differential inequality is

$$\|e\|^2 \leq \frac{\Phi_0'}{\Phi_1'} + \left[\|e_0\|^2 - \frac{\Phi_0'}{\Phi_1'} \right] e^{-\Phi_1' (t-t_0)} \quad (7.95)$$

An estimate of the time required for the trajectory to enter the ball B_κ is $T(r, \kappa)$, where

$$T(r, \kappa) \leq \frac{1}{\Phi_1'} \text{Ln} \left(\frac{\Phi_1' \cdot r - \Phi_0'}{\Phi_1' \cdot \kappa - \Phi_0'} \right) = \frac{1}{\Phi_1'} \text{Ln} \left(1 + \frac{r - \kappa}{\varepsilon} \right) \quad (7.96)$$

and $\|e\|^2 \leq r$ is the bound of initial values of $e(t)$.

So, it is concluded that, under the feedback control of (7.13), the closed loop output response of the nonlinear uncertain system will follow the given ideal trajectories, and the tracking errors $e(t)$ are uniformly ultimately bounded, *i.e.*, converge to a ball B_κ with radius of κ within a finite period of time $T(r, \kappa)$ where r is the bound of initial states. This completes the proof of theorem 7.1.

7.3 ROBUST TRACKING OF MIMO SYSTEMS

We now consider MIMO nonlinear uncertain systems of the form

$$\begin{aligned}\dot{x}(t) &= F(x, \gamma) + G(x, \gamma)u(t) \\ y(t) &= H(x)\end{aligned}\tag{7.97}$$

where $F(x, \gamma): \mathbb{R}^n \times \mathbb{R}^\gamma \rightarrow \mathbb{R}^n$, $G(x, \gamma): \mathbb{R}^n \times \mathbb{R}^\gamma \rightarrow \mathbb{R}^{n \times m}$, $H(x): \mathbb{R}^n \rightarrow \mathbb{R}^m$, x , y and u are the state, output and admissible control, respectively, having appropriate dimensions, all the uncertainties represented by the lumped uncertain elements $\gamma \in \mathbb{R}^\gamma$ are assumed to be bounded, and the bounds are presumed deterministic and known.

7.3.1 Transformation of MIMO Nonlinear Uncertain Systems

For the MIMO case, we consider the transformation of square systems, *i.e.*, systems with the same numbers of inputs and outputs. Applying the results of theorems A.13 (relative order) and A.14 (coordinate transformation) in appendix A to a MIMO nonlinear uncertain system, the system equations can be put into a nominal form, with (z, ζ) as new coordinates. Specifically, the external dynamics of the i^{th} subsystem with relative order $v_i (i=1, 2, \dots, m)$ can be expressed as follows

$$\begin{aligned}\dot{z}_{i,1}(t) &= z_{i,2}(t) \\ &\vdots \\ \dot{z}_{i,v_i-1}(t) &= z_{i,v_i}(t) \\ \dot{z}_{i,v_i}(t) &= a_i(z, \zeta) + \sum_{j=1}^m b_{ij}(z, \zeta) \cdot u_j(t) + \tilde{\delta}_{i,1}(z, \zeta, \gamma) + \sum_{j=1}^m \tilde{\delta}_{i,2j}(z, \zeta, \gamma) \cdot u_j(t)\end{aligned}\tag{7.98}$$

$$y_i(t) = h_i(z) \quad (i=1, \dots, m)\tag{7.99}$$

where $\tilde{\delta}_{i,1}$ and $\tilde{\delta}_{i,2}$ are matched uncertainties, and the internal dynamics are of the form

$$\dot{z}(t) = q(z, \zeta) + p(z, \zeta)u(t)\tag{7.100}$$

with $k=1, \dots, n-v$ and $i=1, \dots, m$

$$q_k(z, \zeta) = L_F \zeta_k(x) \quad p_{k,i}(z, \zeta) = L_{G_i} \zeta_k(x)$$

The feedback control

$$u(t) = -\Pi^{-1} \Lambda + \Pi^{-1} v(t) \quad (7.101)$$

renders the $(n-v)$ states ζ unobservable. Here

$$\Lambda = \left\{ L_F^{v_i} H_i \circ \Psi^{-1}(z) \Big|_{i=1, \dots, m} \right\} \in \mathbb{R}^m$$

$$\Pi = \left\{ L_{G_j} L_F^{v_i-1} H_i \circ \Psi^{-1}(z) \Big|_{i=1, \dots, m, j=1, \dots, m} \right\} \in \mathbb{R}^{m \times m}$$

We define a m dimensional vector switching function as follows

$$\sigma(t) = [\sigma_1, \sigma_2, \dots, \sigma_m]^T$$

where

$$\sigma_i(t) = e_{i,v_i}(t) + \sum_{k=1}^{v_i-1} a_{i,k} \cdot e_{i,v_i-k}(t) \quad (i=1, 2, \dots, m) \quad (7.102)$$

Then

$$\dot{e}_{i,v_i-1}(t) = e_{i,v_i}(t) + \sum_{k=1}^{v_i-1} a_{i,k} \cdot e_{i,v_i-k}(t) - \sum_{k=1}^{v_i-1} a_{i,k} \cdot e_{i,v_i-k}(t) = - \sum_{k=1}^{v_i-1} a_{i,k} \cdot e_{i,v_i-k}(t) + \sigma_i(t)$$

and the time derivative of the switching function (7.102) is given by

$$\begin{aligned} \dot{\sigma}_i(t) &= \dot{e}_{i,v_i}(t) + \sum_{k=1}^{v_i-1} a_{i,k} \cdot \dot{e}_{i,v_i-k}(t) \\ &= v_i(t) + \delta_{i,1}(z, \zeta, \gamma) + \sum_{j=1}^m \delta_{i,2j}(z, \zeta, \gamma) \cdot v_j(t) - \dot{z}_{i,v_i}^d(t) + \sum_{k=1}^{v_i-1} a_{i,k} \cdot \dot{e}_{i,v_i-k}(t) \end{aligned} \quad (7.103)$$

where $\delta_{i,1}, \delta_{i,2j}$ are of the form (7.7) and (7.8). We adopt feedback control of the form

$$\begin{aligned} v(t) &= \left\{ \dot{z}_{i,v_i}^d(t) - \sum_{k=1}^{v_i-1} a_{i,k} \cdot e_{i,v_i-k+1}(t) \right\} - \rho_1 \cdot \sigma(t) - \rho_2 \cdot \text{sgn}[\sigma(t)] \\ &= w(t) - \rho_1 \cdot \sigma(t) - \rho_2 \cdot \text{sgn}[\sigma(t)] \end{aligned} \quad (7.104)$$

where $w(t)$ is a m dimensional vector, and ρ_1, ρ_2 are square matrices of the form

$$\rho_1 = \begin{pmatrix} \rho_1^{(1)} & & 0 \\ & \ddots & \\ 0 & & \rho_m^{(1)} \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} \rho_{11}^{(2)} & \cdots & \rho_{1m}^{(2)} \\ \vdots & & \vdots \\ \rho_{m1}^{(2)} & \cdots & \rho_{mm}^{(2)} \end{pmatrix}$$

and $\text{sgn}(\sigma)$ is the sign function of $\sigma(t)$. From (7.103), we therefore have

$$\begin{aligned} \dot{\sigma}_i(t) &= \left\{ \dot{z}_{i,v_i}^d(t) - \sum_{k=1}^{v_i-1} a_{i,k} \cdot \dot{e}_{i,v_i-k}(t) - \rho_i^{(1)} \sigma_i(t) - \sum_{j=1}^m \rho_{ij}^{(2)} \cdot \text{sgn}[\sigma(t)] \right\} \\ &\quad + \delta_{i,1}(z, \zeta, \gamma) + \sum_{j=1}^m \delta_{i,2j}(z, \zeta, \gamma) \cdot v_j(t) - \dot{z}_{i,v_i}^d(t) + \sum_{k=1}^{v_i-1} a_{i,k} \cdot \dot{e}_{i,v_i-k}(t) \end{aligned}$$

$$= -\rho_i^{(1)}\sigma_i(t) - \sum_{j=1}^m \rho_{ij}^{(2)} \cdot \text{sgn}[\sigma(t)] + \delta_{i,1}(z, \zeta, \gamma) + \sum_{j=1}^m \delta_{i,2j}(z, \zeta, \gamma) \cdot v_j(t)$$

It is straightforward, following the procedure for the SISO case, to write the system, with tracking errors as new states, as follows:

$$\begin{aligned} \dot{e}_{i,1}(t) &= e_{i,2}(t) \\ &\vdots \\ \dot{e}_{i,v_i-1}(t) &= e_{i,v_i}(t) \\ \dot{e}_{i,v_i}(t) &= -\sum_{k=1}^{v_i-1} a_{i,k} \cdot e_{i,v_i}(t) + \sigma(t) \end{aligned} \quad (7.105)$$

$$\dot{\sigma}_i(t) = -\rho_i^{(1)}\sigma_i(t) - \sum_{j=1}^m \rho_{ij}^{(2)} \cdot \text{sgn}[\sigma(t)] + \delta_{i,1}(z, \zeta, \gamma) + \sum_{j=1}^m \delta_{i,2j}(z, \zeta, \gamma) \cdot v_j(t) \quad (7.106)$$

The whole system can be written compactly as

$$\dot{E}(t) = AE(t) + B\sigma(t) \quad (7.107)$$

$$\dot{\sigma}(t) = -\rho_1 \cdot \sigma(t) - \rho_2 \cdot \text{sgn}[\sigma(t)] + \delta_1(z, \zeta, \gamma) + \delta_2(z, \zeta, \gamma) \cdot v(t) \quad (7.108)$$

where

$$\begin{aligned} A &= \text{diag}\{A_i\} & B &= \text{diag}\{B_i\} & E &= [E_1, \dots, E_m]^T \\ A_i &= \begin{pmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ \vdots & 0 & & 1 \\ -a_{i,v_i-1} & -a_{i,v_i-2} & \dots & -a_{i,1} \end{pmatrix} & B_i &= \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 1 \end{pmatrix} & E_i &= \begin{pmatrix} e_{i,1} \\ e_{i,2} \\ \vdots \\ e_{i,v_i-1} \end{pmatrix} \end{aligned}$$

The elements of ρ_1 and ρ_2 are to be determined later.

When mismatched uncertainties are present, the system becomes

$$\dot{E}(t) = AE(t) + B\sigma(t) + \omega(z, \zeta, \gamma, v) \quad (7.109)$$

$$\dot{\sigma}(t) = -\rho_1 \cdot \sigma(t) - \rho_2 \cdot \text{sgn}[\sigma(t)] + \delta_1(z, \zeta, \gamma) + \delta_2(z, \zeta, \gamma) \cdot v(t) \quad (7.110)$$

where ω represents the mismatched uncertainties, whilst δ_1 and δ_2 , the matched uncertainties, may be of different forms to those in (7.108). Alternatively, (7.110) may be expressed as

$$\dot{\sigma}(t) = -\rho \cdot \sigma(t) + \delta_1(z, \zeta, \gamma) + \delta_2(z, \zeta, \gamma) \cdot [w - \rho \cdot \sigma(t)] \quad (7.111)$$

$$\text{where } \rho = \rho_1 + \rho_2 \cdot |\Sigma|^{-1} \quad (7.112)$$

$$\Sigma = \text{diag}[\sigma_i(t)]$$

$$w = [w_1, w_2, \dots, w_m]^T$$

$$w_i(t) = \dot{z}_{i,v_i}^d(t) - \sum_{k=1}^{v_i-1} a_{i,k} \cdot e_{i,v_i-k+1}(t) \quad (i=1,2,\dots,m) \quad (7.113)$$

$$v(t) = w - \rho \cdot \sigma(t) \quad (7.114)$$

Therefore

$$\begin{aligned} \dot{\sigma}(t) &= [\delta_1(z, \zeta, \gamma) + \delta_2(z, \zeta, \gamma)w] - [I + \delta_2(z, \zeta, \gamma)] \cdot \rho \cdot \sigma(t) \\ &= \omega_1(z, \zeta, \gamma) - \omega_2(z, \zeta, \gamma) \cdot \rho \cdot \sigma(t) \end{aligned} \quad (7.115)$$

where $\omega_1 = \delta_1 + \delta_2 w \in \mathbb{R}^m$, $\omega_2 = I + \delta_2 \in \mathbb{R}^{m \times m}$. It is required that all uncertainties in the system are bounded, *i.e.*,

$$\Delta_1 \triangleq \left\{ \max_{\gamma(t) \in \mathbb{R}^r} |\delta_{1,i}(z, \zeta, \gamma)| \mid i=1,2,\dots,m \right\} \quad (7.116)$$

$$\Delta_2 \triangleq \left\{ \max_{\gamma(t) \in \mathbb{R}^r} |\delta_{2,ij}(z, \zeta, \gamma)| \mid i=1,2,\dots,m, j=1,2,\dots,m \right\} \quad (7.117)$$

where Δ_1 and Δ_2 , which could either be functions of $e(t)$ and t or only constant scalars, are positive definite matrices and assumed to be deterministic and known, such that

$$\Omega_1 = [\Delta_1 + \Delta_2 \cdot \|w\|] \quad (7.118)$$

$$\Omega_2 = I - \Delta_2 \quad (7.119)$$

are positive definite. Furthermore, the mismatched uncertainty is also bounded, *i.e.*,

$$\omega(z, \zeta, \gamma, v) \leq \beta_0 + \beta_1 \|e\| + \beta_2 \|\sigma\| \quad (7.120)$$

where $\beta_0, \beta_1, \beta_2$ are positive constants.

7.3.2 The Case of Both Matched and Mismatched Uncertainties

THEOREM 7.2. (*Uniform Ultimate Boundedness of MIMO Nonlinear Uncertain Systems*)

For MIMO nonlinear systems in the presence of uncertainties, if the uncertainties are bounded, then a variable structure controller can be found such that the output response of the closed loop system will track a given desired trajectory, and furthermore the closed loop system is uniformly ultimately bounded. Moreover, the tracking errors will ① converge to zero in a finite time and remain there when matched uncertainties only are

present; or ② enter a ball B_κ with radius κ in a finite time $T(\kappa, r)$ and remain there should mismatched uncertainties be present, where the radius κ depends only on the bound of the mismatched uncertainties.

A Lyapunov function, $V(t) = V_1(t) + V_2(t) = e^T P e + \frac{1}{2} \sigma^T \sigma$, is considered, and the derivative of $V(t)$ is given by

$$\begin{aligned}\dot{V}(t) &= \dot{V}_1(t) + \dot{V}_2(t) = \dot{e}^T P e + e^T P \dot{e} + \sigma^T \dot{\sigma} \\ &= \{ \dot{e}^T P e + e^T P \dot{e} - 2\lambda_M(P) \sigma^T \sigma \} + \{ \sigma^T \dot{\sigma} + 2\lambda_M(P) \sigma^T \sigma \} \\ &= \dot{V}_1'(t) + \dot{V}_2'(t)\end{aligned}\quad (7.121)$$

where

$$\dot{V}_1'(t) = \dot{V}_1(t) - 2\lambda_M(P) \sigma^T \sigma \quad (7.122)$$

$$\dot{V}_2'(t) = \dot{V}_2(t) + 2\lambda_M(P) \sigma^T \sigma \quad (7.123)$$

The first part can be written

$$\begin{aligned}\dot{V}_1'(t) &= \dot{e}^T P e + e^T P \dot{e} - 2\lambda_M(P) \sigma^T \sigma \\ &= (e^T A^T + \sigma^T B^T + \omega^T) P e + e^T P (A e + B \sigma + \omega) - 2\lambda_M(P) \sigma^T \sigma \\ &= (e^T A^T P e + e^T P A e) + (\sigma^T B^T P e + e^T P B \sigma) + (\omega^T P e + e^T P \omega) - 2\lambda_M(P) \sigma^T \sigma \\ &= -e^T Q e + 2e^T P B \sigma + (\omega^T P e + e^T P \omega) - 2\lambda_M(P) \sigma^T \sigma\end{aligned}$$

Note that $\|B\|=1$, so that

$$2e^T P B \sigma \leq 2\lambda_M(P) \cdot \|B\| \cdot e^T \sigma \leq 2\lambda_M(P) \cdot \left[\frac{1}{2} e^T e + \frac{1}{2} \sigma^T \sigma \right] = \lambda_M(P) \cdot [e^T e + \sigma^T \sigma]$$

$$\omega^T P e + e^T P \omega = 2e^T P \omega \leq 2\lambda_M(P) \cdot \|e\| \cdot [\beta_0 + \beta_1 \|e\| + \beta_2 \|\sigma\|]$$

where (7.120) has been used, and

$$2\lambda_M(P) \|e\| \beta_0 \leq \lambda_M(P) \beta_0 \|e\|^2 + \lambda_M(P) \beta_0$$

$$2\lambda_M(P) \beta_2 \|e\| \|\sigma\| \leq \lambda_M(P) \beta_2^2 \|e\|^2 + \lambda_M(P) \|\sigma\|^2$$

so
$$\omega^T P e + e^T P \omega \leq \lambda_M(P) \cdot [\beta_0 + 2\beta_1 + \beta_2^2] e^T e + \lambda_M(P) \beta_0 + \lambda_M(P) \cdot \sigma^T \sigma$$

Then

$$\begin{aligned}\dot{V}_1'(t) &\leq -\lambda_m(Q) \|e\|^2 + \lambda_M(P) [1 + \beta_0 + 2\beta_1 + \beta_2^2] \|e\|^2 + \lambda_M(P) \beta_0 + 2\lambda_M(P) \sigma^T \sigma - 2\lambda_M(P) \sigma^T \sigma \\ &= -[\lambda_m(Q) - \lambda_M(P)(1 + \beta_0 + 2\beta_1 + \beta_2^2)] \|e\|^2 + \lambda_M(P) \beta_0\end{aligned}\quad (7.124)$$

In order to proceed, the following notations are defined

$$\begin{aligned}\Sigma &= \text{diag}\{\sigma_j\} \\ \sigma &= \Sigma \times I_{m \times 1}\end{aligned}\tag{7.125}$$

$$\begin{aligned}\omega_\delta &= \text{diag}\{\omega_1\} \\ \omega_1 &= \omega_\delta \times I_{m \times 1}\end{aligned}\tag{7.126}$$

$$\begin{aligned}\Omega_\delta &= \text{diag}\{\Omega_1\} \\ \Omega_1 &= \Omega_\delta \times I_{m \times 1}\end{aligned}\tag{7.127}$$

where $I_{m \times 1} \triangleq [1, \dots, 1]^T$.

The second part can be dealt with as follows

$$\begin{aligned}\dot{V}_2'(t) &= \sigma^T(t) \dot{\sigma}(t) + 2\lambda_M(P) \sigma^T \sigma \\ &= \sigma^T \left\{ \omega_1 - \omega_2 \cdot \rho \cdot \sigma(t) \right\} + 2\lambda_M(P) \sigma^T \sigma \\ &= I_{1 \times m} \Sigma^T [\omega_\delta - \omega_2 \cdot \rho \cdot \Sigma] I_{m \times 1} + 2I_{1 \times m} \Sigma^T \cdot \lambda_M(P) \cdot \Sigma \cdot I_{m \times 1}\end{aligned}\tag{7.128}$$

the notations (7.125)~(7.127) having been used here. Let

$$\rho = \rho' \cdot [I - \Delta_2]^T = \rho' \cdot \Omega_2^T\tag{7.129}$$

and suppose that it is positive definite, and that, for matrix $\omega_2 \cdot \Omega_2^T$, its symmetrised form is also positive definite. Then, for a constant ς satisfying the following condition

$$\lambda_m[(\omega_2 \cdot \Omega_2^T)_s] > \varsigma > 0$$

the matrix

$$\lambda_M[\Omega_2 \cdot \Omega_2^T] \cdot \omega_2 \cdot \Omega_2^T - \Omega_2 \cdot \varsigma \cdot \Omega_2^T$$

is positive definite. Define $\varphi_M^2 = \lambda_M(\Omega_2 \cdot \Omega_2^T)$, so that we can substitute $(\varsigma \cdot \Omega_2 \cdot \rho' \cdot \Omega_2^T) / \varphi_M^2$ for $\omega_2 \cdot \rho' \cdot \Omega_2^T$ in $\dot{V}_2'(t)$. It therefore follows that

$$\begin{aligned}\dot{V}_2'(t) &\leq I_{1 \times m} \left\{ \Sigma^T \omega_\delta - \Sigma^T \frac{\varsigma \cdot \Omega_2 \cdot \rho' \cdot \Omega_2^T}{\varphi_M^2} \Sigma \right\} I_{m \times 1} + 2I_{1 \times m} \Sigma^T \cdot \lambda_M(P) \cdot \Sigma \cdot I_{m \times 1} \\ &= I_{1 \times m} \left\{ \Sigma^T \omega_\delta - \Sigma^T \left[\frac{\varsigma \cdot \Omega_2 \cdot \rho' \cdot \Omega_2^T}{\varphi_M^2} - 2\lambda_M(P) \right] \Sigma \right\} I_{m \times 1} \\ &= I_{1 \times m} \left\{ \Sigma^T \omega_\delta - \Sigma^T \left[\frac{\varsigma}{\varphi_M^2} \Omega_2 \left(\rho' - \frac{\varphi_M^2}{\varsigma} \Omega_2^{-1} 2\lambda_M(P) \Omega_2^T \right) \Omega_2^T \right] \Sigma \right\} I_{m \times 1}\end{aligned}$$

$$\text{Let } \rho' - \frac{\varphi_M^2}{\varsigma} \Omega_2^{-1} 2\lambda_M(P) \Omega_2^T = \rho'' (1^2 + 1)\tag{7.130}$$

$$i.e., \quad \rho' = \rho''(l^2+1) + \frac{\varphi_M^2}{\zeta} \Omega_2^{-1} 2\lambda_M(P) \Omega_2^{-T} \quad (7.131)$$

$$\begin{aligned} \dot{V}_2'(t) &\leq I_{1 \times m} \left\{ \Sigma^T \omega_\delta - \Sigma^T \left(\frac{\zeta}{\varphi_M^2} \Omega_2 \rho''(l^2+1) \Omega_2^T \right) \Sigma \right\} I_{m \times 1} \\ &= I_{1 \times m} \left\{ -l^2 \Sigma^T \frac{\zeta}{\varphi_M^2} \Omega_2 \rho'' \Omega_2^T \Sigma + \Sigma^T \omega_\delta - \Sigma^T \left(\frac{\zeta}{\varphi_M^2} \Omega_2 \rho'' \Omega_2^T \right) \Sigma \right\} I_{m \times 1} \\ &\leq I_{1 \times m} \left\{ -l^2 \Sigma^T \frac{\zeta}{\varphi_M^2} \Omega_2 \rho'' \Omega_2^T \Sigma + \omega_\delta^T \left(4 \frac{\zeta \Omega_2 \rho'' \Omega_2^T}{\varphi_M^2} \right)^{-1} \omega_\delta \right\} I_{m \times 1} \end{aligned}$$

Using the results of theorem 6.2, it is easy to show that the choice of

$$\rho'' = \frac{\varphi_M^2}{2l\zeta} \Omega_2^{-1} \Omega_\delta |\Sigma|^{-1} \Omega_2^{-T} \quad (7.132)$$

makes the above inequality negative as follows

$$\begin{aligned} \dot{V}_2'(t) &\leq I_{1 \times m} \left\{ -l^2 \Sigma^T \frac{\zeta}{\varphi_M^2} \Omega_2 \rho'' \Omega_2^T \Sigma + \omega_\delta^T \left(4 \frac{\zeta \Omega_2 \rho'' \Omega_2^T}{\varphi_M^2} \right)^{-1} \omega_\delta \right\} I_{m \times 1} \\ &= I_{1 \times m} \left\{ -l^2 \Sigma^T \frac{\zeta}{\varphi_M^2} \Omega_2 \frac{\varphi_M^2}{2l\zeta} \Omega_2^{-1} \Omega_\delta |\Sigma|^{-1} \Omega_2^{-T} \Sigma \right\} I_{m \times 1} \\ &\quad + I_{1 \times m} \left\{ \omega_\delta^T \left(4 \frac{\zeta \Omega_2 \frac{\varphi_M^2}{2l\zeta} \Omega_2^{-1} \Omega_\delta |\Sigma|^{-1} \Omega_2^{-T} \Omega_2^T}{\varphi_M^2} \right)^{-1} \omega_\delta \right\} I_{m \times 1} \\ &= I_{1 \times m} \left\{ -\frac{l}{2} \Omega_\delta |\Sigma| + \omega_\delta^T \left(\frac{2}{l} \Omega_\delta |\Sigma|^{-1} \right)^{-1} \omega_\delta \right\} I_{m \times 1} \\ &= -I_{1 \times m} \left\{ \frac{l}{2} \Omega_\delta |\Sigma| (I - \omega_\delta^T \Omega_\delta^{-2} \omega_\delta) \right\} I_{m \times 1} < 0 \end{aligned} \quad (7.133)$$

For the i^{th} entry of the matrix $I - \omega_\delta^T \Omega_\delta^{-2} \omega_\delta$, it is clear that

$$1 - [\omega_\delta^T(x, \gamma)]_i / [\Omega_\delta^2(x)]_i > 0 \quad (i=1, 2, \dots, m) \quad (7.134)$$

so, the matrix is positive definite. It follows that

$$\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) = \dot{V}_1'(t) + \dot{V}_2'(t) < 0 \quad (7.135)$$

and the system is stable. Furthermore, if

$$\lambda_m(Q) / \lambda_M(P) > 1 + \beta_0 + 2\beta_1 + \beta_2^2 \quad (7.136)$$

then

$$\dot{V}(t) \leq \dot{V}_1'(t) \leq -[\lambda_m(Q) - \lambda_M(P)(1 + \beta_0 + 2\beta_1 + \beta_2^2)] \|e\|^2 + \lambda_M(P) \beta_0 < 0 \quad (7.137)$$

and so

$$\|e\|^2 > \frac{\beta_0}{\lambda_m(Q)/\lambda_M(P) - (1 + \beta_0 + 2\beta_1 + \beta_2^2)} > 0 \quad (7.138)$$

The same conclusion as that of the SISO case can now be drawn. The tracking errors will converge to an open ball B_κ with radius

$$\kappa = \frac{\beta_0}{\lambda_m(Q)/\lambda_M(P) - (1 + \beta_0 + 2\beta_1 + \beta_2^2)} \quad (7.139)$$

We can further show that the closed loop system is uniformly ultimately bounded, with a finite time period

$$T(r, \kappa) \leq \frac{1}{\Phi_1'} \text{Ln} \left(\frac{\Phi_1' \cdot r - \Phi_0'}{\Phi_1' \cdot \kappa - \Phi_0'} \right) = \frac{1}{\Phi_1'} \text{Ln} \left(1 + \frac{r - \kappa}{\varepsilon} \right) \quad (7.140)$$

$$\text{where } \Phi_1' = \Phi_1 / \lambda_m(P) \quad (7.141)$$

$$\Phi_0' = \Phi_0 / \lambda_m(P) \quad (7.142)$$

So we have

$$\begin{aligned} \rho &= \left\{ \frac{\Phi_M^2}{2l\zeta} \Omega_2^{-1} \Omega_\delta |\Sigma|^{-1} \Omega_2^{-T} (t^2 + 1) + \frac{\Phi_M^2}{\zeta} \Omega_2^{-1} 2\lambda_M(P) \Omega_2^{-T} \right\} \cdot \Omega_2^T \\ &= \frac{\Phi_M^2}{\zeta} \Omega_2^{-1} \left\{ \frac{t^2 + 1}{2l} \Omega_\delta |\Sigma|^{-1} + 2\lambda_M(P) I \right\} \end{aligned} \quad (7.143)$$

$$\begin{aligned} v(t) &= w(t) - \rho(x)\sigma \\ &= w(t) - \frac{\Phi_M^2}{\zeta} \Omega_2^{-1} \left\{ \frac{t^2 + 1}{2l} \Omega_\delta |\Sigma|^{-1} + 2\lambda_M(P) I \right\} \sigma \\ &= w(t) - \rho_1 \cdot \sigma(t) - \rho_2 \cdot \text{sgn}[\sigma(t)] \end{aligned} \quad (7.144)$$

where

$$\rho_1 = \frac{\Phi_M^2}{\zeta} \Omega_2^{-1} 2\lambda_M(P) \quad (7.145)$$

$$\rho_2 = \frac{(t^2 + 1)\Phi_M^2}{2l\zeta} \Omega_2^{-1} \Omega_\delta \quad (7.146)$$

and

$$|\Sigma|^{-1} \sigma = \begin{pmatrix} |\sigma_1|^{-1} & & 0 \\ & \ddots & \\ 0 & & |\sigma_m|^{-1} \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_m \end{pmatrix} = \text{sgn}[\sigma(t)]$$

□

7.3.3 Consideration of Internal Dynamics

Unlike the SISO case, the internal dynamics of MIMO systems are usually related to the control $u(t)$. Using the nomenclature of appendix A, they are of the form

$$\dot{\zeta}(t) = q(z, \zeta) + p(z, \zeta)u(t) \quad (7.147)$$

where $q(z, \zeta) \in \mathbb{R}^{n-v}$, $p(z, \zeta) \in \mathbb{R}^{(n-v) \times m}$. In order to take the internal dynamics into account, it is required that the internal dynamics be of the form

$$\dot{\zeta}(t) = q(z, \zeta)$$

$$i.e., \quad p_{ki}(z, \zeta) = L_{G_i} \zeta_k(x) = 0 \quad (1 \leq i \leq m \quad 1 \leq k \leq n-v) \quad (7.148)$$

According to the feedback linearisation theory of appendix A, if, for the given system, the input mapping

$$G(x, \gamma) = \begin{pmatrix} g_{11} & \cdots & g_{1m} \\ \vdots & & \vdots \\ g_{n1} & \cdots & g_{nm} \end{pmatrix} = [g_1, \dots, g_m]$$

is involutive, then condition (7.148) holds.

In this case, we may take the internal dynamics into account by assuming that the system is exponentially minimum-phase, *i.e.*, the zero dynamics of the system is exponentially stable, and that conditions (7.60)~(7.62) hold. The same form of the Lyapunov function as that for the SISO case of form (7.68) is adopted here. Following the same procedure as that of the SISO case, similar results are obtained.

Assume that $q(z, \zeta)$ is a Lipschitz vector function, and the Lipschitz constant is defined as follows

$$\vartheta = \sup_{(z, \zeta) \in \Omega} \frac{\|q(z, \zeta) - q(0, \zeta)\|}{\|z\|}$$

Differentiating $V_0(\zeta)$ along ζ yields

$$\begin{aligned} \dot{V}_0(\zeta) &= \frac{\partial V_0}{\partial \zeta} \cdot q(z, \zeta) = \frac{\partial V_0}{\partial \zeta} \cdot q(0, \zeta) + \frac{\partial V_0}{\partial \zeta} \{q(z, \zeta) - q(0, \zeta)\} \\ &\leq -v_3 \|\zeta\|^2 + v_4 \|\zeta\| \cdot \{\|q(z, \zeta) - q(0, \zeta)\|\} \leq -v_3 \|\zeta\|^2 + v_4 \|\zeta\| \cdot \vartheta \cdot \|z\| \end{aligned}$$

Define a new vector $e' = z - z^d = (e_1, e_2, \dots, e_v)^T$, so that we have, for $\|z^d\| \leq c$

$$\begin{aligned}\|e'\| &= \|e_v + ae - be\| = \|\sigma - be\| \leq \|\sigma\| + b\|e\| \\ \|z\| &\leq \|e'\| + \|z^d\| \leq \|\sigma\| + b\|e\| + c \\ \dot{V}_0(\zeta) &\leq -v_3\|\zeta\|^2 + v_4\|\zeta\|\vartheta\|z\| \leq -v_3\|\zeta\|^2 + v_4\|\zeta\| \cdot \vartheta\{\|\sigma\| + b\|e\| + c\} \\ &\leq -v_3\|\zeta\|^2 + v_4\vartheta\|\zeta\| \cdot \|\sigma\| + v_4\vartheta b\|\zeta\| \cdot \|e\| + v_4\vartheta\|\zeta\|c\end{aligned}\quad (7.149)$$

Considering the Lyapunov function (7.68), the time derivative is given by

$$\begin{aligned}\dot{V}(t) &= \dot{V}_1(t) + \dot{V}_2(t) + \mu\dot{V}_0(t) \\ &= \dot{V}_1(t) - 2\lambda_M(P)\sigma^2 + \lambda_M(P)\|e\|^2 + \dot{V}_2(t) + 3\lambda_M(P)\sigma^2 + \mu\dot{V}_0(t) - \lambda_M(P)\|e\|^2 - \lambda_M(P)\sigma^2 \\ &= \dot{V}_1'(t) + \dot{V}_2'(t) + \dot{V}_0'(t)\end{aligned}\quad (7.150)$$

$$\text{where } \dot{V}_1'(t) = \dot{V}_1(t) - 2\lambda_M(P)\sigma^2(t) + \lambda_M(P)\|e\|^2 \quad (7.151)$$

$$\dot{V}_2'(t) = \dot{V}_2(t) + 3\lambda_M(P)\sigma^2(t) \quad (7.152)$$

$$\dot{V}_0'(t) = \mu\dot{V}_0(t) - \lambda_M(P)\|e\|^2 - \lambda_M(P)\sigma^2(t) \quad (7.153)$$

Form (7.149) and (7.153), it follows that the same results as those for the SISO case can be obtained as follows:

$$\mu = \frac{4\alpha v_3 \lambda_M(P)}{v_4^2 \vartheta^2 (1+b^2)} \quad (0 < \alpha < 1) \quad (7.154)$$

$$\|\zeta\| > \frac{v_4 \vartheta c}{(1-\alpha)v_3} > 0 \quad (7.155)$$

Again considering (7.151) and (7.152), it is straightforward to extend the results without consideration of the internal dynamics obtained previously as follows:

$$\|e\|^2 > \frac{\beta_0}{\lambda_m(Q)/\lambda_M(P) - (2 + \beta_0 + 2\beta_1 + \beta_2^2)} > 0 \quad (7.156)$$

where

$$\lambda_m(Q)/\lambda_M(P) > 2 + \beta_0 + 2\beta_1 + \beta_2^2 > 0 \quad (7.157)$$

and

$$\rho_1 = 3 \frac{\varphi_M^2(\Omega_2)}{\varsigma} \Omega_2^{-1} \lambda_M(P) > 0 \quad (7.158)$$

$$\rho_2 = \frac{(t^2 + 1)\varphi_M^2(\Omega_2)}{2t\varsigma} \Omega_2^{-1} \Omega_\delta \geq 0 \quad (7.159)$$

7.4 RESULTS AND REMARKS

The results developed for the case of mismatched uncertainties are now compared with those for the case of only matched uncertainties, and some comments are made.

7.4.1 Main Results for SISO Systems

	<i>In the Presence of Matched Uncertainties</i>	<i>In the Presence of Mismatched Uncertainties</i>
<i>On the Switching Surface</i> $\sigma(t)=0$	$\lambda_m(Q) > 0$ $\ e\ ^2 > 0$	$\frac{\lambda_m(Q)}{\lambda_M(P)} > \beta_0 + 2\beta_1 > 0$ $\ e\ ^2 > \frac{\beta_0}{\lambda_m(Q)/\lambda_M(P) - [\beta_0 + 2\beta_1]} > 0$
<i>Off the Switching Surface</i> $\sigma(t) \neq 0$	$\frac{\lambda_m(Q)}{\lambda_M(P)} > 1$ $\ e\ ^2 > 0$ $\rho_1 = \lambda_M(P) > 0$ $\rho_2 \geq \frac{1}{2} \sqrt{\frac{\Delta_1^2}{(1-\beta)\beta} + \frac{\Delta_2 w^2}{\beta}} \geq 0 \quad (0 < \beta < 1)$	$\frac{\lambda_m(Q)}{\lambda_M(P)} > 1 + \beta_0 + 2\beta_1 + \beta_2^2 > 0$ $\ e\ ^2 > \frac{\beta_0}{\lambda_m(Q)/\lambda_M(P) - (1 + \beta_0 + 2\beta_1 + \beta_2^2)} > 0$ $\rho_1 = 2\lambda_M(P) > 0$ $\rho_2 \geq \frac{1}{2} \sqrt{\frac{\Delta_1^2}{(1-\beta)\beta} + \frac{\Delta_2 w^2}{\beta}} \geq 0 \quad (0 < \beta < 1)$
<i>Off the Switching Surface with internal dynamics</i> $\sigma(t) \neq 0$	$\frac{\lambda_m(Q)}{\lambda_M(P)} > 2$ $\ e\ ^2 > 0$ $\rho_1 = 2\lambda_M(P) > 0$ $\rho_2 \geq \frac{1}{2} \sqrt{\frac{\Delta_1^2}{(1-\beta)\beta} + \frac{\Delta_2 w^2}{\beta}} \geq 0 \quad (0 < \beta < 1)$ $\mu = \frac{4\alpha v_3 \lambda_M(P)}{v_4^2 \vartheta^2 (1+b^2)} \quad (0 < \alpha < 1)$ $\ \zeta\ > \frac{v_4 \vartheta c}{(1-\alpha)v_3} > 0$	$\frac{\lambda_m(Q)}{\lambda_M(P)} > 2 + \beta_0 + 2\beta_1 + \beta_2^2 > 0$ $\ e\ ^2 > \frac{\beta_0}{\lambda_m(Q)/\lambda_M(P) - (2 + \beta_0 + 2\beta_1 + \beta_2^2)} > 0$ $\rho_1 = 3\lambda_M(P) > 0$ $\rho_2 \geq \frac{1}{2} \sqrt{\frac{\Delta_1^2}{(1-\beta)\beta} + \frac{\Delta_2 w^2}{\beta}} \geq 0 \quad (0 < \beta < 1)$ $\mu = \frac{4\alpha v_3 \lambda_M(P)}{v_4^2 \vartheta^2 (1+b^2)} \quad (0 < \alpha < 1)$ $\ \zeta\ > \frac{v_4 \vartheta c}{(1-\alpha)v_3} > 0$

7.4.2 Main Results for MIMO Systems

	<i>In the Presence of Matched Uncertainties</i>	<i>In the Presence of Mismatched Uncertainties</i>
<i>On the Switching Surface</i> $\sigma(t)=0$	$\lambda_m(Q) > 0$ $\ e\ ^2 > 0$	$\frac{\lambda_m(Q)}{\lambda_M(P)} > \beta_0 + 2\beta_1 > 0$ $\ e\ ^2 > \frac{\beta_0}{\lambda_m(Q)/\lambda_M(P) - [\beta_0 + 2\beta_1]} > 0$
<i>Off the Switching Surface</i> $\sigma(t) \neq 0$	$\frac{\lambda_m(Q)}{\lambda_M(P)} > 1$ $\ e\ ^2 > 0$ $\rho_1 = \frac{\varphi_M^2(\Omega_2)}{\varsigma} \Omega_2^{-1} \lambda_M(P) > 0$ $\rho_2 = \frac{(\iota^2 + 1)\varphi_M^2(\Omega_2)}{2\iota\varsigma} \Omega_2^{-1} \Omega_8 \geq 0$	$\frac{\lambda_m(Q)}{\lambda_M(P)} > 1 + \beta_0 + 2\beta_1 + \beta_2^2 > 0$ $\ e\ ^2 > \frac{\beta_0}{\lambda_m(Q)/\lambda_M(P) - (1 + \beta_0 + 2\beta_1 + \beta_2^2)} > 0$ $\rho_1 = 2 \frac{\varphi_M^2(\Omega_2)}{\varsigma} \Omega_2^{-1} \lambda_M(P) > 0$ $\rho_2 = \frac{(\iota^2 + 1)\varphi_M^2(\Omega_2)}{2\iota\varsigma} \Omega_2^{-1} \Omega_8 \geq 0$
<i>Off the Switching Surface with internal dynamics</i> $\sigma(t) \neq 0$	$\frac{\lambda_m(Q)}{\lambda_M(P)} > 2$ $\ e\ ^2 > 0$ $\rho_1 = 2 \frac{\varphi_M^2(\Omega_2)}{\varsigma} \Omega_2^{-1} \lambda_M(P) > 0$ $\rho_2 = \frac{(\iota^2 + 1)\varphi_M^2(\Omega_2)}{2\iota\varsigma} \Omega_2^{-1} \Omega_8 \geq 0$ $\mu = \frac{4\alpha v_3 \lambda_M(P)}{v_4^2 \vartheta^2 (1 + b^2)} \quad (0 < \alpha < 1)$ $\ \zeta\ > \frac{v_4 \vartheta c}{(1 - \alpha)v_3} > 0$	$\frac{\lambda_m(Q)}{\lambda_M(P)} > 2 + \beta_0 + 2\beta_1 + \beta_2^2 > 0$ $\ e\ ^2 > \frac{\beta_0}{\lambda_m(Q)/\lambda_M(P) - (2 + \beta_0 + 2\beta_1 + \beta_2^2)} > 0$ $\rho_1 = 3 \frac{\varphi_M^2(\Omega_2)}{\varsigma} \Omega_2^{-1} \lambda_M(P) > 0$ $\rho_2 = \frac{(\iota^2 + 1)\varphi_M^2(\Omega_2)}{2\iota\varsigma} \Omega_2^{-1} \Omega_8 \geq 0$ $\mu = \frac{4\alpha v_3 \lambda_M(P)}{v_4^2 \vartheta^2 (1 + b^2)} \quad (0 < \alpha < 1)$ $\ \zeta\ > \frac{v_4 \vartheta c}{(1 - \alpha)v_3} > 0$

7.4.3 Remarks

- When mismatched uncertainties are present in the system, the size of tracking errors $\|e\|$ becomes larger than that when only matched uncertainties are present, and the

feedback control gain ρ_1 is also larger in order to overcome these additional uncertainties. The feedback gain of the discontinuous part ρ_2 is expressed in the same form, but does not have the same value because the uncertainty bounds are different in this case.

- The size of the tracking error $\|e\|$ depends on the bounds of the mismatched uncertainties, but not on the matched part. It also depends on the ratio of the minimum eigenvalue of the matrix Q and the maximum eigenvalue of P , which are only related to the control design of the nominal system. They can be regarded, in a sense, as a kind of 'stability margin' for the nominal system.
- When only matched uncertainties are present, the size of the tracking error $\|e\|$ can be made zero, because for any $\|e\|^2 > 0$, $\dot{V}(t) < 0$, meaning that the chosen Lyapunov function guarantees that any non-zero tracking error will converge to zero in finite time, *i.e.*, the closed loop system is asymptotically stable. When both matched and mismatched uncertainties are present, the tracking errors usually become larger than those when only matched uncertainties occur, and cannot be made zero at any time.

However, there exists a closed ball B_κ with radius

$$\kappa = \frac{\beta_0}{\frac{\lambda_m(Q)}{\lambda_M(P)} - (1 + \beta_0 + 2\beta_1 + \beta_2^2)} + \varepsilon > 0 \quad (\varepsilon > 0)$$

such that whenever $\|e\|^2 \geq \kappa$, $\dot{V}(t) < 0$, implying that the tracking error will converge to and enter the ball B_κ and remain in it thereafter. Thus the closed loop system is uniformly ultimately bounded.

- It is assumed that the mismatched uncertainties in the system are bounded, *i.e.*, β_0 , β_1 , β_2 are finite positive scalars, and that the bound should be sufficiently small such that the condition $\lambda_m(Q)/\lambda_M(P) > 1 + \beta_0 + 2\beta_1 + \beta_2^2 > 0$ can be satisfied. This condition is a sufficient condition for theorems 7.1 and 7.2, which state that the measure of mismatch must be less than the critical mismatch threshold $\lambda_m(Q)/\lambda_M(P)$. (See Leitmann *et al*^[4,5])

7.5 ILLUSTRATIVE EXAMPLE

The same SISO second order linear system as that used in the previous three chapters is considered here to illustrate the application of the technique developed in this chapter. Similarly, the effects of open loop pole location uncertainty and non-minimum phase are considered. The state space model of the system is of the form

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \begin{pmatrix} a'_{11} & a'_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b'_1 \\ b'_2 \end{pmatrix} u(t) \\ &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ b_2 \end{pmatrix} u(t) + \begin{pmatrix} \Delta a_{11}x_1 + \Delta a_{12}x_2 \\ \Delta a_{21}x_1 + \Delta a_{22}x_2 \end{pmatrix} + \begin{pmatrix} \Delta b_1 \\ \Delta b_2 \end{pmatrix} u(t)\end{aligned}$$

$$y(t) = h(x) = x_1$$

where $\Delta a_{ij} = a'_{ij} - a_{ij} \quad (i,j=1,2)$

$$b_2 = k_2/a_{12}$$

$$\Delta b_1 = k_1$$

$$\Delta b_2 = (k_2 + a'_{22}k_1)/a'_{12} - k_2/a_{12}$$

The required coordinate transformation may be defined as follows:

$$\begin{aligned}z_1 &= h(x) = x_1 & \Leftrightarrow & \quad x_1 = z_1 \\ z_2 &= L_f h(x) = a_{11}x_1 + a_{12}x_2 & \Leftrightarrow & \quad x_2 = (-a_{11}z_1 + z_2)/a_{12}\end{aligned}$$

Such a transformation enables us to obtain a system with new coordinates as follows

$$\begin{aligned}\dot{z}_1(t) &= z_2 + \tilde{\delta}_0(z, \gamma, u) \\ \dot{z}_2(t) &= a(z) + b(z)u(t) + \tilde{\delta}_1(z, \gamma) + \tilde{\delta}_2(z, \gamma) \cdot u(t)\end{aligned}$$

where $a(z) = (a_{12}a_{21} - a_{11}a_{22})z_1 + (a_{11} + a_{22})z_2$

$$b(z) = a_{12}b_2$$

and $\tilde{\delta}_0(z, \gamma, u) = (\Delta a_{11} - \frac{a_{11}}{a_{12}} \Delta a_{12})z_1 + \frac{\Delta a_{12}}{a_{12}}z_2 + \Delta b_1 u(t) = \tilde{\delta}_0^1(z, \gamma) + \tilde{\delta}_0^2(z, \gamma)u(t)$

$$\tilde{\delta}_1(z, \gamma) = [a_{11}(\Delta a_{11} - \frac{a_{11}}{a_{12}} \Delta a_{12}) + a_{12}(\Delta a_{21} - \frac{a_{11}}{a_{12}} \Delta a_{22})]z_1 + (\frac{a_{11}}{a_{12}} \Delta a_{12} + \Delta a_{22})z_2$$

$$\tilde{\delta}_2(z, \gamma) = a_{11}\Delta b_1 + a_{12}\Delta b_2$$

Feedback control of the form

$$u(t) = \frac{1}{b(z)} \{ -a(z) + v(t) \}$$

converts the system into the following form

$$\dot{z}_1(t) = z_2 + \delta_0(z, \gamma, v)$$

$$\dot{z}_2(t) = v(t) + \delta_1'(z, \gamma) + \delta_2'(z, \gamma)v(t)$$

where
$$\delta_0(z, \gamma, v) = [\tilde{\delta}_0^1(z, \gamma) - \tilde{\delta}_0^2(z, \gamma) \cdot \frac{a(z)}{b(z)}] + \frac{1}{b(z)} \tilde{\delta}_0^2(z, \gamma)v(t)$$

$$\delta_1'(z, \gamma) = \tilde{\delta}_1(z, \gamma) - \tilde{\delta}_2(z, \gamma) \cdot \frac{a(z)}{b(z)}$$

$$\delta_2'(z, \gamma) = \tilde{\delta}_2(z, \gamma) \cdot \frac{1}{b(z)}$$

Thus, in general, the uncertainty δ_0 , which does not satisfy the generalised matching assumption, is the mismatched part, while the uncertainties δ_1 , δ_2 which satisfy the matching conditions, represent the matched part of the uncertainties. The system therefore falls into the class of systems with mismatched uncertainties.

Let the ideal trajectory to be tracked be $y^d(t) = \pi(t)$. Then $z_1^d(t) = y^d(t) = \pi(t)$, $z_2^d(t) = \dot{y}^d(t) = \dot{\pi}(t)$, and the tracking errors become

$$e_1(t) = z_1(t) - z_1^d(t) = z_1(t) - \pi(t)$$

$$e_2(t) = z_2(t) - z_2^d(t) = z_2(t) - \dot{\pi}(t)$$

The system model can be written as

$$\dot{e}_1(t) = e_2 + \delta_0(z, \gamma, v)$$

$$\dot{e}_2(t) = v(t) + \delta_1'(z, \gamma) + \delta_2'(z, \gamma)v(t) - \dot{z}_2^d(t)$$

Define a switching surface

$$\sigma(t) = e_2(t) + a_1 \cdot e_1(t) = 0$$

so that

$$v(t) = \dot{z}_2^d(t) - a_1 \cdot e_2(t) - \rho_1 \cdot \sigma(t) - \rho_2 \cdot \text{sgn}[\sigma(t)]$$

where a_1 will be determined according to the polynomial

$$p(\lambda) = \lambda + a_1 = 0$$

which is the characteristic equation of the reduced order closed loop system. We therefore have

$$\begin{aligned}\dot{e}_1(t) &= -a_1 \cdot e_1 + \sigma(t) + \delta_0(z, \gamma, v) \\ \dot{\sigma}(t) &= -\rho_1 \cdot \sigma(t) - \rho_2 \cdot \text{sgn}[\sigma(t)] + \delta_1(z, \gamma) + \delta_2(z, \gamma) \cdot v(t)\end{aligned}$$

i.e., $A = -a_1$. Here

$$\begin{aligned}\delta_1(z, \gamma) &= \delta_1'(z, \gamma) + a_1 \cdot \delta_0^1(z, \gamma) = [\tilde{\delta}_1(z, \gamma) + a_1 \cdot \tilde{\delta}_0^1(z, \gamma)] - [\tilde{\delta}_2(z, \gamma) + a_1 \cdot \tilde{\delta}_0^2(z, \gamma)] \cdot \frac{a(z)}{b(z)} \\ \delta_2(z, \gamma) &= \delta_2'(z, \gamma) + a_1 \cdot \delta_0^2(z, \gamma) = \frac{1}{b(z)} [\tilde{\delta}_2(z, \gamma) + a_1 \cdot \tilde{\delta}_0^2(z, \gamma)]\end{aligned}$$

The solution of the Lyapunov equation $A^T P + P A = -Q$ is that $P = 1/2a_1$ for $Q = 1$, and therefore $\lambda_{\max}(P) = 1/2a_1$ and $\lambda_{\min}(Q) = 1$. Now we can choose the feedback gains to be

$$\rho_1 = 2\lambda_{\max}(P) = 1/a_1 > 0 \quad \rho_2 \geq \sqrt{\Delta_1^2 + \frac{1}{2}\Delta_2 w^2} \geq 0 \quad (\text{for } \beta = 0.5)$$

where Δ_1 and Δ_2 are the bounds on the matched uncertainties, presumed deterministic and known. The bounds on the mismatched uncertainties, on which the tracking errors depend, are given by

$$\begin{aligned}\delta_0(z, \gamma, v) &= [\tilde{\delta}_0^1(z, \gamma) - \tilde{\delta}_{02}(z, \gamma) \cdot \frac{a(z)}{b(z)}] + \frac{1}{b(z)} \tilde{\delta}_0^2(z, \gamma) v(t) \\ &= [(\Delta a_{11} - \frac{a_{11}}{a_{12}} \Delta a_{12}) z_1 + \frac{\Delta a_{12}}{a_{12}} z_2 - \Delta b_1 \cdot \frac{(a_{12} a_{21} - a_{11} a_{22}) z_1 + (a_{11} + a_{22}) z_2}{a_{12} b_2}] + \frac{\Delta b_1}{a_{12} b_2} \cdot v \\ &= [(\Delta a_{11} - \frac{a_{11}}{a_{12}} \Delta a_{12} - \Delta b_1 \frac{a_{12} a_{21} - a_{11} a_{22}}{a_{12} b_2}) z_1 + (\frac{\Delta a_{12}}{a_{12}} - \Delta b_1 \frac{a_{11} + a_{22}}{a_{12} b_2}) z_2] + \frac{\Delta b_1}{a_{12} b_2} v \\ &\leq c_0 + c_1 \|e\| + c_2 \|v\| \\ &\leq \beta_0 + \beta_1 \|e\| + \beta_2 \|\sigma\|\end{aligned}$$

where

$$\begin{aligned}c_0 &= c_1 \|z^d\| \\ c_1 &= \max \left(\left| \Delta a_{11} - \frac{a_{11}}{a_{12}} \Delta a_{12} - \Delta b_1 \frac{a_{12} a_{21} - a_{11} a_{22}}{a_{12} b_2} \right|, \left| \frac{\Delta a_{12}}{a_{12}} - \Delta b_1 \frac{a_{11} + a_{22}}{a_{12} b_2} \right| \right) \\ c_2 &= \left| \frac{\Delta b_1}{a_{12} b_2} \right|\end{aligned}$$

$$\beta_0 = c_0 + c_2(d + p_2)$$

$$\beta_1 = c_1 + c_2 \|\tilde{a}\| \quad (\tilde{a} = -a_1^2)$$

$$\beta_2 = c_2 |p_1 + a_1|$$

and the bounds of the matched uncertainties, on which the feedback gains are based, are

$$\begin{aligned} \delta_1(z, \gamma) &= [a_{11}(\Delta a_{11} - \frac{a_{11}}{a_{12}} \Delta a_{12}) + a_{12}(\Delta a_{21} - \frac{a_{11}}{a_{12}} \Delta a_{22})] z_1 + (\frac{a_{11}}{a_{12}} \Delta a_{12} + \Delta a_{22}) z_2 \\ &\quad - (a_{11} \Delta b_1 + a_{12} \Delta b_2) \frac{(a_{12} a_{21} - a_{11} a_{22}) z_1 + (a_{11} + a_{22}) z_2}{a_{12} b_2} \\ &= [a_{11}(\Delta a_{11} - \frac{a_{11}}{a_{12}} \Delta a_{12}) + a_{12}(\Delta a_{21} - \frac{a_{11}}{a_{12}} \Delta a_{22}) - (a_{11} \Delta b_1 + a_{12} \Delta b_2) \frac{a_{12} a_{21} - a_{11} a_{22}}{a_{12} b_2}] z_1 \\ &\quad + (\frac{a_{11}}{a_{12}} \Delta a_{12} + \Delta a_{22} - (a_{11} \Delta b_1 + a_{12} \Delta b_2) \frac{a_{11} + a_{22}}{a_{12} b_2}) z_2 \leq \Delta_1 \\ \delta_2(z, \gamma) &= (a_{11} \Delta b_1 + a_{12} \Delta b_2) \frac{\Delta b_2}{a_{12} b_2} \leq \Delta_2 \end{aligned}$$

It is clear that $\delta_2(z, \gamma) > 0$ because $a_{11} < 0$, $\Delta b_1 = k_1 \leq 0$, $\Delta b_2 \geq 0$, $a_{12} > 0$ and $b_2 > 0$.

7.5.1 Matched Uncertainties

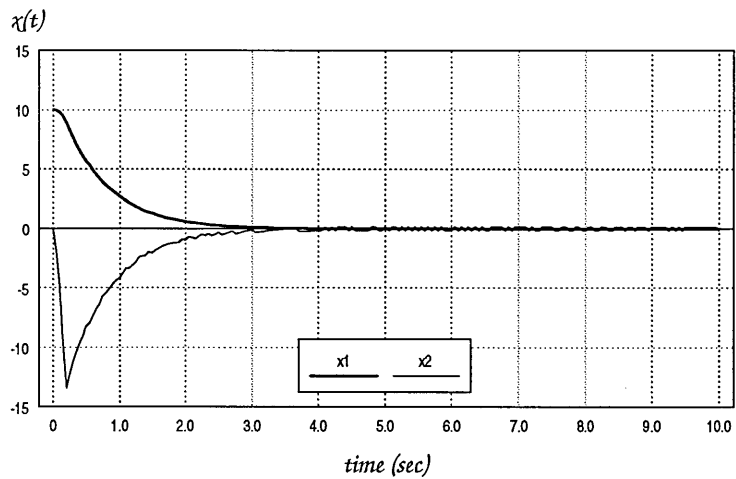
Let $a_{11} = 0$; $a_{12} = 1$; then $a_{21} = -\alpha$, $a_{22} = -\beta$, $b_1 = 0$, $b_2 = k_2$, where $\alpha = \mu_1 + \mu_2$, $\beta = \mu_1 \mu_2$, and $-\mu_1$, $-\mu_2$ are the assumed locations of the open loop poles. We first consider the case of minimum phase, so let $k_1 = 0$. One of the open loop poles however is assumed to be at $-\mu_1$ in the complex plane but is in fact at $-\mu_1'$, whilst the parameter k_2 is also assumed uncertain, having an actual value k_2' . This therefore results in a system with only matched uncertainties of the form

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -\alpha & -\beta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ k_2 \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ \Delta \alpha x_1 + \Delta \beta x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \Delta k_2 \end{pmatrix} u(t)$$

where $\Delta \alpha = (\alpha - \alpha')$, $\Delta \beta = (\beta - \beta')$, and $\Delta k_2 = k_2' - k_2$, and $\Delta a_{11} = \Delta a_{12} = \Delta b_1 = 0$.

Fig. 7.1 displays the simulation results where only matched uncertainties occur. The responses of the system are depicted for both stability (regulator) and tracking (servo) problems.

(i)
Regulator Problem:
 $y^d(t)=0$



System States

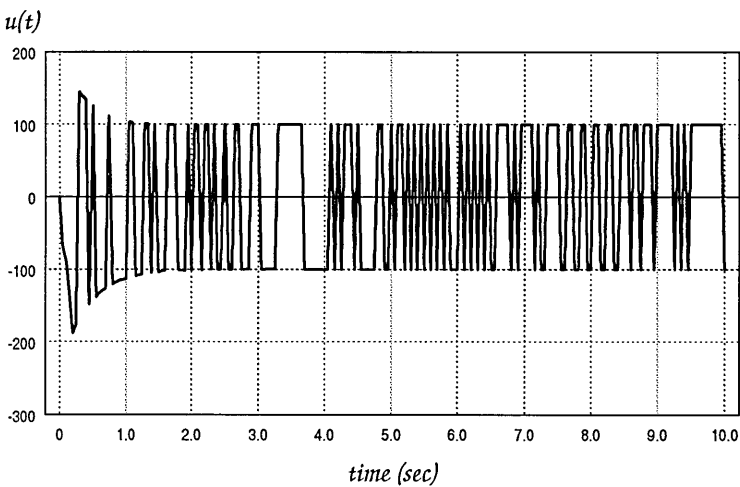
Uncertain Parameters:

$$k_2=1; k_2'=2;$$

$$\mu_1=1, \mu_1'=-1;$$

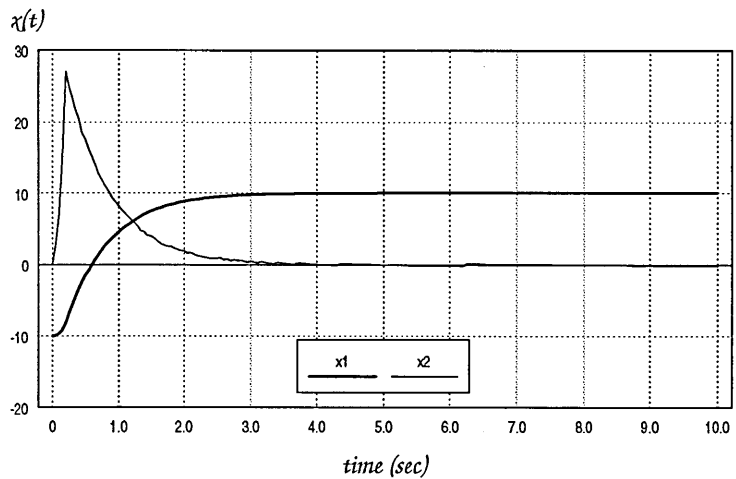
$$\mu_2=5, \mu_2'=5;$$

$$a_1=1.5;$$



Control Signal

(ii)
Tracking Problem:
 $y^d(t)$: a step function



System States

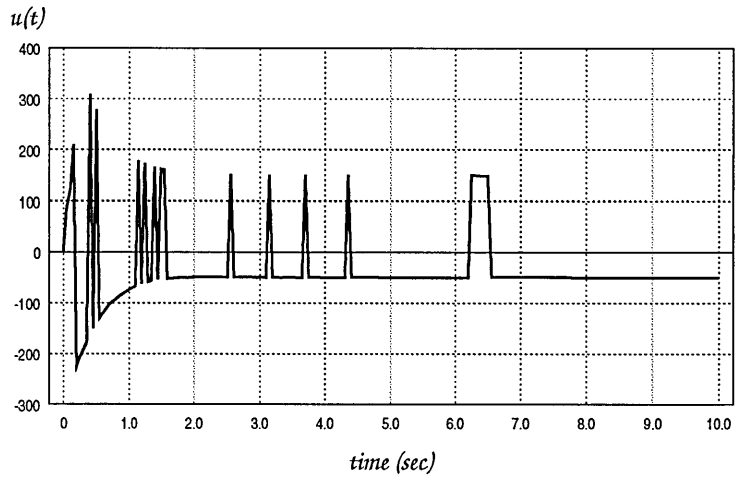
Uncertain Parameters:

$$k_2=1; k'_2=1.5;$$

$$\mu_1=1, \mu'_1=-1;$$

$$\mu_2=5, \mu'_2=5;$$

$$a_1=1.5;$$

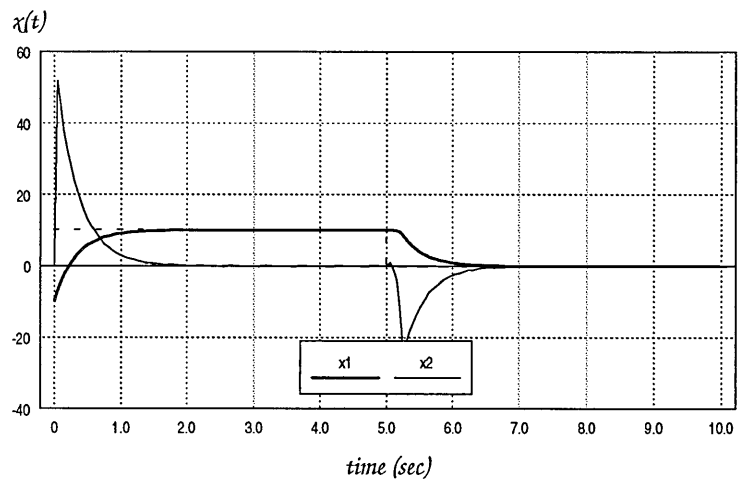


Control Signal

(iii)

Tracking Problem:

$y^d(t)$: a square wave



System States

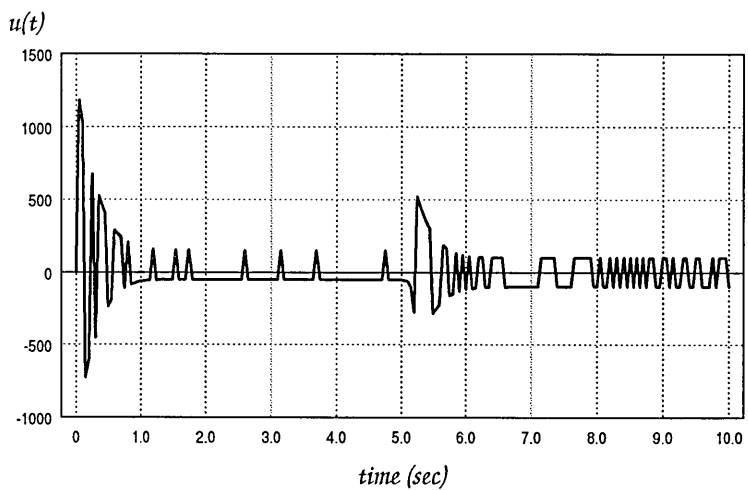
Uncertain Parameters:

$$k_2=1; k'_2=1.5;$$

$$\mu_1=1, \mu'_1=-1;$$

$$\mu_2=5, \mu'_2=5;$$

$$a_1=3;$$



Control Signal

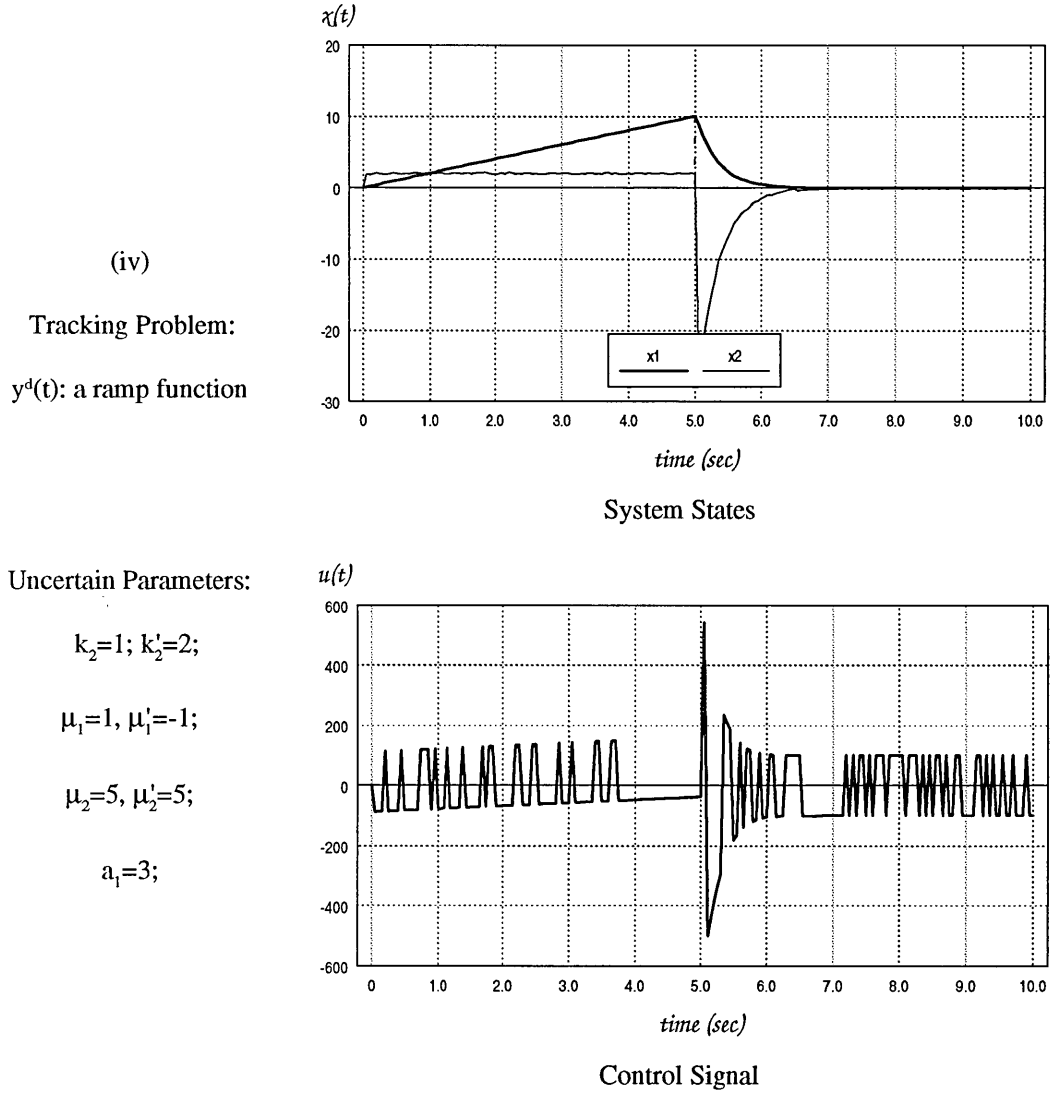


Fig.7.1 Case 1: Simulation results for robust tracking with matched uncertainties

7.5.2 Mismatched Uncertainties

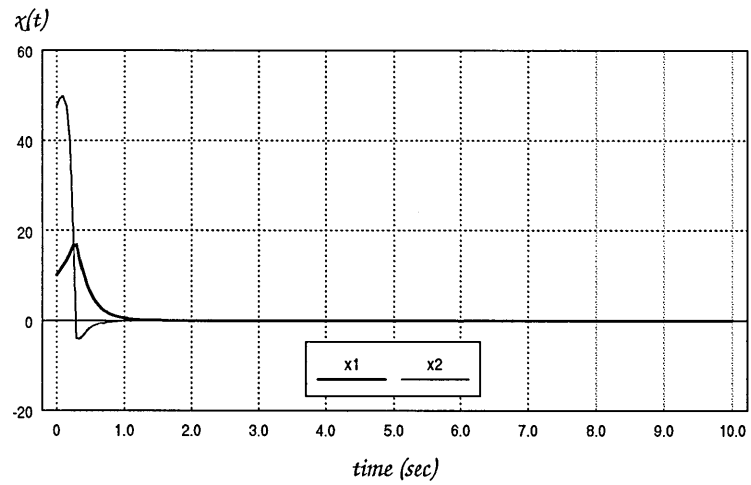
The second case to be considered is non-minimum phase, and also one of the open loop poles has location $-\mu'_1$ while it is assumed to be $-\mu_1$. The system is

$$\dot{x}(t) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ b_2 \end{pmatrix} u(t) + \begin{pmatrix} \Delta a_{11}x_1 + \Delta a_{12}x_2 \\ \Delta a_{21}x_1 + \Delta a_{22}x_2 \end{pmatrix} + \begin{pmatrix} \Delta b_1 \\ \Delta b_2 \end{pmatrix} u(t)$$

Let $a_{12}=a_{21}=a \neq 0$, $\alpha^2 \geq 4(\beta+a^2)$ and $k_2=1$. Then $b_1=k_1$, $b_2=(k_2+a_{22}k_1)/a_{12}$, and

$$a_{11} = [-\alpha - \sqrt{\alpha^2 - 4(\beta+a^2)}]/2 \quad a_{22} = [-\alpha + \sqrt{\alpha^2 - 4(\beta+a^2)}]/2$$

(i)
Regulator Problem:
 $y^d(t)=0$



System States

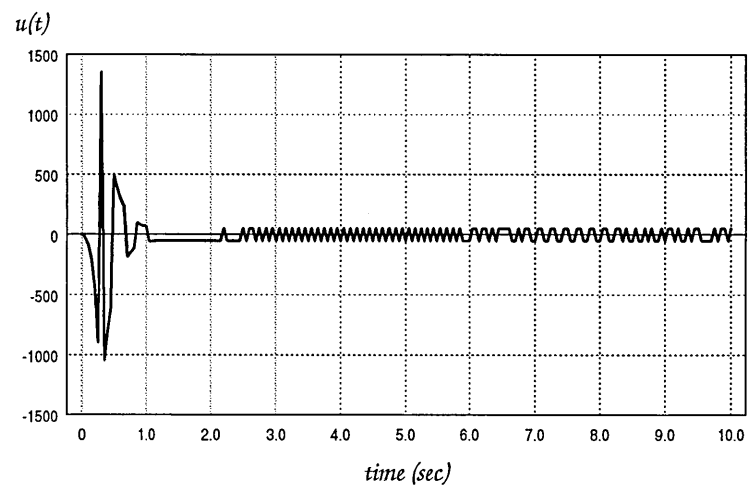
Uncertain Parameters:

$$k_1=0; k'_1=-0.1;$$

$$\mu_1=1, \mu'_1=-1;$$

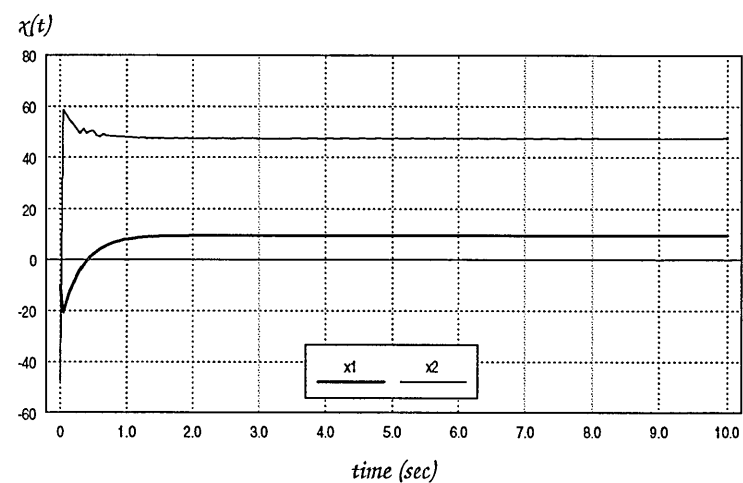
$$\mu_2=5, \mu'_2=5;$$

$$a_1=5;$$



Control Signal

(ii)
Tracking Problem:
 $y^d(t)$: a step function



System States

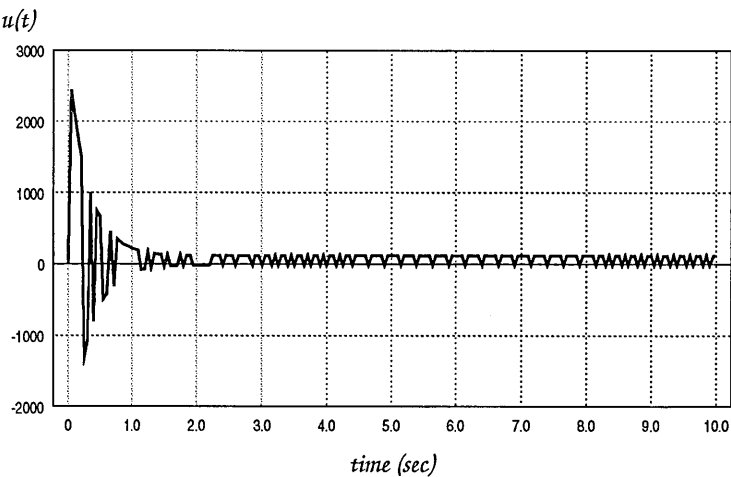
Uncertain Parameters:

$$k_1=0; k'_1=-0.25;$$

$$\mu_1=1, \mu'_1=2;$$

$$\mu_2=5, \mu'_2=4;$$

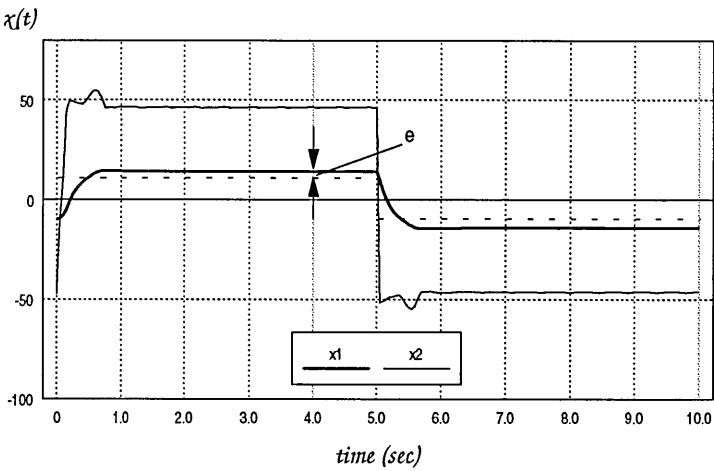
$$a_1=5;$$



(iii)

Tracking Problem:

$y^d(t)$: a square wave



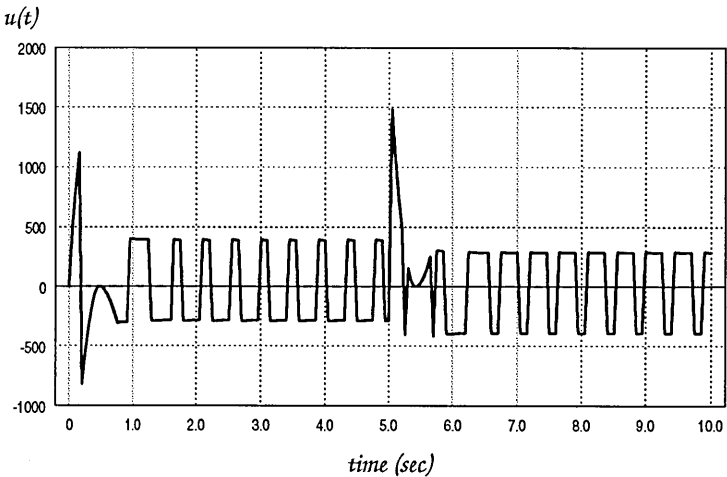
Uncertain Parameters:

$$k_1=1; k'_1=-0.01;$$

$$\mu_1=1, \mu'_1=-1;$$

$$\mu_2=5, \mu'_2=5;$$

$$a_1=5;$$



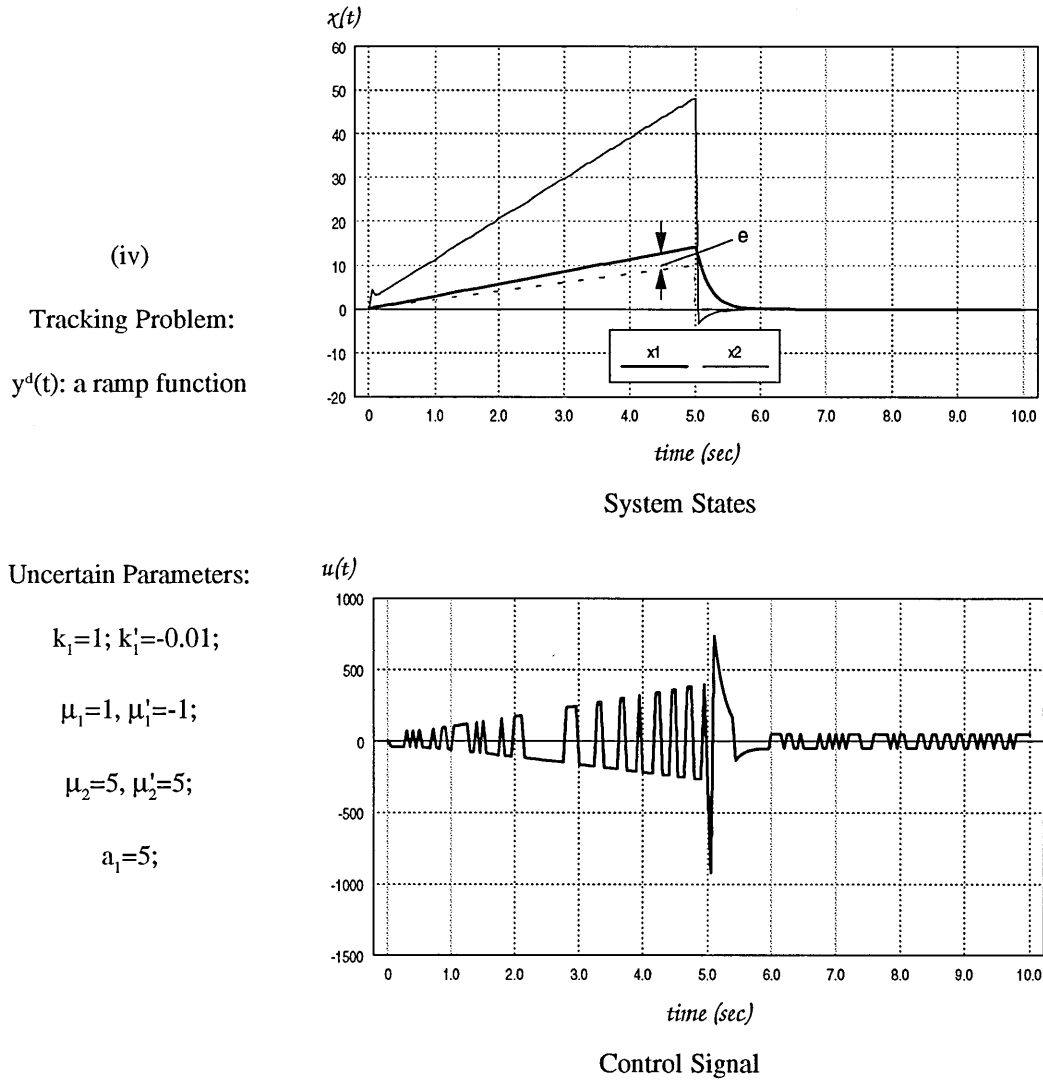


Fig.7.2 Case 2: Simulation results for robust tracking with mismatched uncertainties

Fig.7.2 displays the results of simulation for the system with mismatched uncertainties, where both regulator and tracking problems are considered.

Observe that the closed loop system maintains stability in every circumstance, regardless of the presence of uncertainties resulting from open loop pole position uncertainty and non-minimum phase dynamics. It can be seen that the controllers obtained via theorem 7.1 and 7.2 attenuate the effect of the uncertainties effectively, and the responses of the closed loop system do follow the given trajectories. Another interesting fact is that the tracking errors of the closed loop system converge to zero in the first case,

where only matched uncertainties occur (Fig.7.1), whilst the tracking errors cannot reach zero in the second case, where mismatched uncertainties are present (Fig.7.2).

7.6 SUMMARY

In this chapter, the robust tracking problem for a class of SISO and MIMO nonlinear systems in the presence of matched and mismatched uncertainties has been addressed, and robust tracking techniques have been developed.

The algorithm for the SISO case can be summarised as follows, and the similar algorithm for the MIMO case can be obtained according to the discussion of section 7.3.

Algorithm for SISO Systems:

- ① Transform the original nonlinear uncertain system into the form of (7.4);
- ② Design a switching function $\sigma(x)$ such that the regular form (7.16) and (7.17) can be obtained;
- ③ Construct bounds for the matched uncertainties of the form (7.18) and (7.19);
- ④ Calculate the ideal trajectory $y^d(t)$ to be tracked;
- ⑤ Obtain a feedback control of the form (7.5) where $v(t)$ is given by (7.13) with feedback gains of the form (7.30) and (7.31);
- ⑥ Check the performance of the closed loop system by considering the open ball with radius (7.90) and the time to reach the ball (7.96);

It is concluded that the tracking errors will converge to zero in the matched uncertainty case, whilst the errors cannot reach zero in the mismatched uncertainty case. However, the techniques proposed guarantee that the responses of the closed loop system follow the prescribed trajectory, and the tracking errors are uniformly ultimately bounded whenever matched or mismatched uncertainties are present.

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Applications



Overview

This chapter presents applications of the techniques developed. Strong robustness is indicated via simulation of four linear or nonlinear uncertain systems.



Outline

- ✓ Introduction
- ✓ Simulations
- ✓ Summary

8.1 INTRODUCTION

IN this chapter it is shown how, on the basis of the concepts introduced and developed in the previous four chapters, a number of relevant synthesis problems, such as practical stabilisation and robust trajectory tracking, can be solved for some real systems in the presence of uncertainties and disturbances under the mild assumptions that have been made in the previous chapters. Four application examples, which are either linear or nonlinear and are highly affected by either matched or mismatched uncertainties, are given here: a simple one link robot arm containing uncertain parameters and unknown disturbances; a crane system lifting an unknown load; a six-plate gas-absorber system with mismatched uncertain parameters; and a two-link robot manipulator subject to uncertain load mass. Both practical stabilisation and robust trajectory tracking problems are considered.

The behaviour of the systems is investigated by simulation and shown to be of the desired form.

8.2 SIMULATIONS

8.2.1 A Simple Robot Arm

The first example to be considered is that of a simple robot arm which is assumed to be one link. This is a commonly chosen example and our method may then be compared with other techniques. The system is shown in Fig. 8.1.

Assume that m and l represent the mass and, respectively, the length of the mass centre of the link subjected to a control moment delivered by a DC motor, where the DC

motor is armature controlled, and the motor inertia is negligible compared with the link inertia. The DC motor may be modelled as follows. The torques delivered by the motor and applied to the robot arm are

$$T_m = K_m \cdot I \quad (8.1)$$

$$T_p = N \cdot T_m = N \cdot K_m \cdot I \quad (8.2)$$

respectively, and the dynamical equation for the motor is

$$V = L \cdot \dot{I} + R \cdot I + K_m \cdot N \cdot \dot{\theta} \quad (8.3)$$

where I is the armature current, N is the gear ratio, θ is the angular position of the link, and K_m is the motor constant.

The dynamics of the robot arm can be described by the equation

$$T_p = -l^2 \cdot m \cdot \ddot{\theta} + l \cdot mg \cdot \sin(\theta) \quad (8.4)$$

Now, the following state variables are introduced: $x_1 = \theta$ the angular position, $x_2 = \dot{\theta}$ the angular velocity, $x_3 = I$ the armature current, and so the following equations describe the system

$$\dot{x}(t) = \begin{pmatrix} x_2 \\ K_1 \sin(x_1) + K_2 x_3 \\ K_3 x_2 + K_4 x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ K_5 \end{pmatrix} u(t) \quad (8.5)$$

$$y(t) = h(x) = x_1$$

The parameters of the nominal system are given by

$$K_1 = \frac{g}{l}$$

$$K_2 = \frac{10K_m}{l^2 m}$$

$$K_3 = -\frac{10K_m}{L}$$

$$K_4 = -\frac{R}{L}$$

$$K_5 = \frac{1}{L}$$

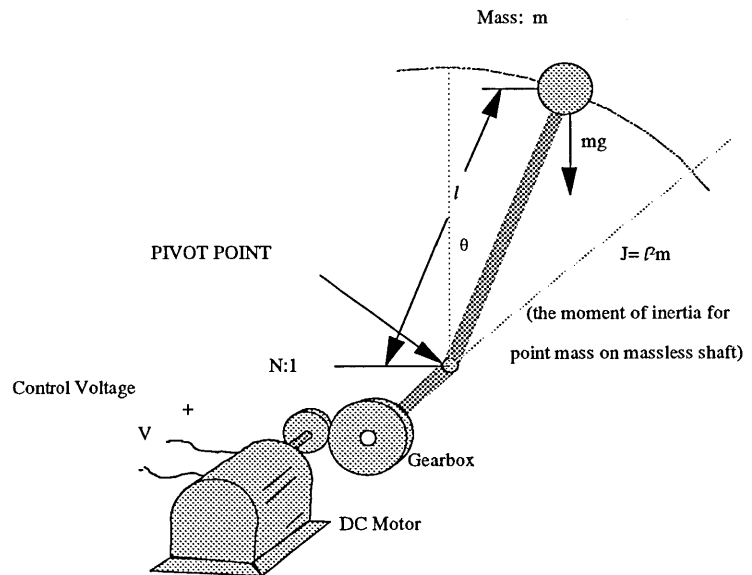


Fig. 8.1 One link robot arm

§1. Parameter Uncertainties

It is assumed that the robot arm is modelled with only one link, but in fact it has two degrees of freedom, *i.e.*, it can be rotated and it can also be extended or retracted. Such a problem is dealt with here by applying feedback control only to rotation and considering translation as a perturbation. The mass will vary, depending on the load carried, and the position of the centre of mass will change if the link is extended or retracted. The design must of course accommodate these uncertainties. The parameters K_1 and K_2 , which depend on the mass m and the length l , are then uncertain, and are denoted as $K_1 = K_1^o + \Delta k_1$ and $K_2 = K_2^o + \Delta k_2$. The system model is first transformed into new coordinates z by a coordinate transformation of the form

$$z_k = \psi_k(x) = L_f^{k-1} h(x) \quad (k=1,2,3)$$

and a new state space model is obtained as follows

$$\dot{z}(t) = \begin{pmatrix} z_2 \\ z_3 \\ [K_1^o \cos(z_1) + K_2^o K_3] z_2 \\ + K_4 [z_3 - K_1^o \sin(z_1)] \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ K_2^o K_5 \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ (\Delta k_1 - \frac{K_1^o}{K_2^o} \Delta k_2) \sin(z_1) + \frac{\Delta k_2}{K_2^o} z_3 \\ 0 \end{pmatrix}$$

$$y(t) = z_1 \quad (8.6)$$

The system model is of the regular form. It is obvious that a nonlinear system with mismatched uncertainties results, and therefore the technique described in theorem 4.4 is applicable here.

The following values were chosen for simulation purposes: $l=1\sim 1.2\text{m}$, $m=1\sim 1.8\text{kg}$, which are uncertain but bounded, and

$$g=9.8\text{m/s}^2, \quad K_m=0.1\text{Nm/A}=0.1\text{Vs/rad}, \quad R=1\Omega, \quad L=5\text{mH}$$

The nominal values of the parameters are then

$$K_1^o=8.909, \quad K_2^o=0.590, \quad K_3=-200, \quad K_4=-200, \quad K_5=200$$

The uncertain parameters are $K_1=K_1^o+\Delta k_1 \in [8.167, 9.8]$ and $K_2=K_2^o+\Delta k_2 \in [0.386, 1]$.

The closed loop poles of the system are chosen as: $\lambda_1=-0.8+j2$, $\lambda_2=-0.8-j2$ and

$\lambda_3 = -8$, resulting in the following controller parameters: $\alpha_0 = 37.12$, $\alpha_1 = 17.44$, $\alpha_2 = 9.6$, $\alpha_3 = 1$.

By solving Lyapunov equation $A^T P + P A = -Q$, a Lyapunov function may be obtained

$$V(z) = z^T P z = [z_1, z_2, z_3] \begin{pmatrix} 5.1575 & 3.3060 & 0.0050 \\ 3.3060 & 3.3806 & 0.0476 \\ 0.0050 & 0.0476 & 0.0322 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \quad (8.7)$$

The simulation results are shown in Fig. 8.2 and 8.3. Comparisons of the technique via theorem 4.4 with the feedback linearisation technique alone are given. The control based on theorem 4.4 clearly results in better performance than that resulting from the application of feedback linearisation alone.

Simulation Parameters:

Nominal Values:

$$l_0 = 1.1\text{m};$$

$$m_0 = 1.4\text{kg};$$

Real Values:

$$t = 0\text{sec}, l = 1.2\text{m};$$

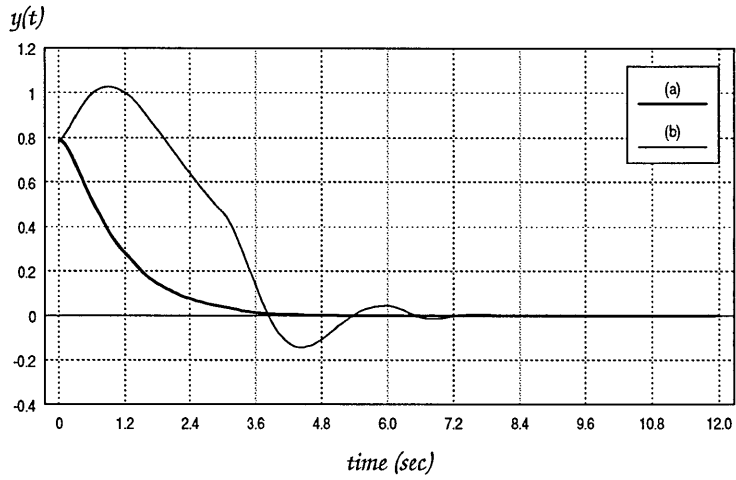
$$m = 1.8\text{kg}$$

$$t = 3\text{sec}, l = 1.1\text{m};$$

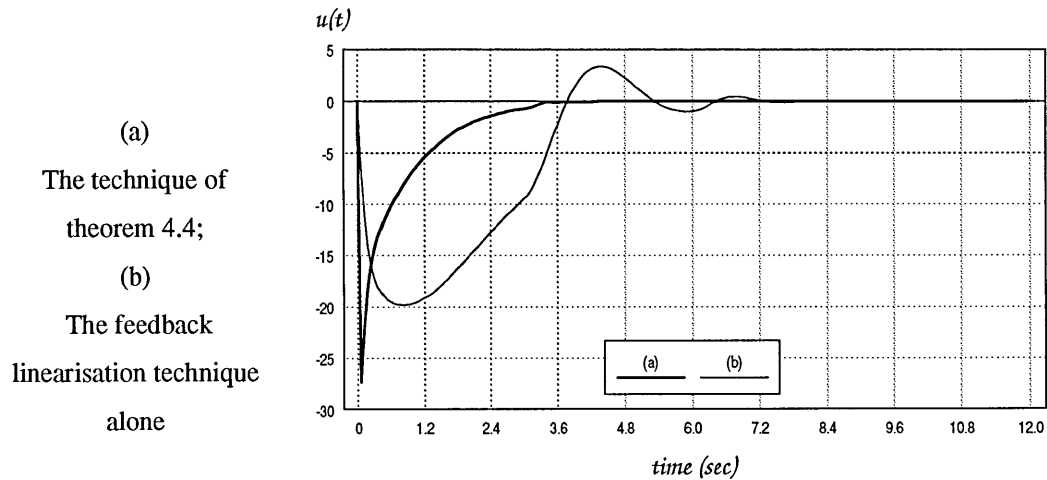
$$m = 1.4\text{kg}$$

$$t \geq 6\text{sec}, l = 1\text{m};$$

$$m = 1\text{kg}$$



System Outputs



Control Signals

Fig. 8.2 Result 1: Comparison of the present technique with the feedback linearisation technique alone

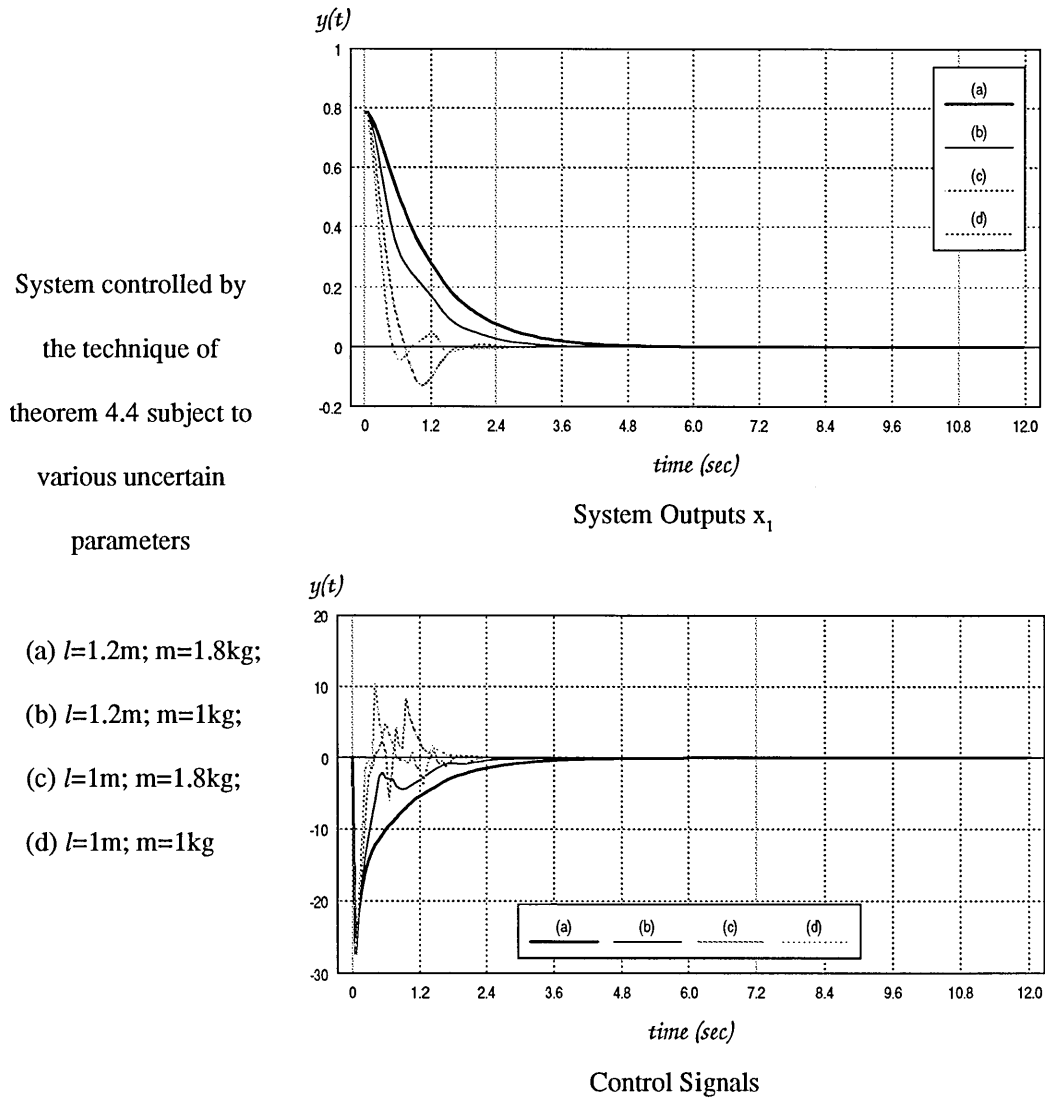


Fig. 8.3 Result 2: Control of the one link robot arm subject to different uncertainties

From Fig.8.3, it can be seen that, although the system is subject to significant uncertainty, the system outputs are stable and good performance is indicated. The design is therefore robust in the sense implied here.

§2. Uncertain Disturbances

We consider the same system model subject to an uncertain disturbance as follows

$$\dot{x}(t) = \begin{pmatrix} x_2 \\ K_1 \sin(x_1) + K_2 x_3 \\ K_3 x_2 + K_4 x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ K_5 \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ K_6 \cos(5t) \cos(x_1) \\ K_7 x_2 + K_8 x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ K_9 \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ \xi(t) \\ 0 \end{pmatrix}$$

where the uncertain parameters are assumed to be of the form

$$K_6 = -\frac{\hat{a}}{l} \quad K_7 = -10K_9K_m \quad K_8 = -K_9R \quad K_9 = -\Delta\left(\frac{1}{L}\right)$$

Here \hat{a} is a constant equal to the amplitude of the uncertainty, $\Delta(1/L)$ indicates the variations of L ; $\xi(t)$ is band limited white noise.

The variable structure controller of theorem 5.4, with variable feedback gain, is employed here. Suppose it is required that the closed loop system behave as a linear (reduced order) system. If the switching function is chosen to be

$$\sigma(x) = \sigma_1(x_1, x_2) - x_3 = 0$$

the reduced order dynamics are

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ K_1 \sin(x_1) + K_2 \sigma_1(x_1, x_2) \end{pmatrix} = \begin{pmatrix} x_2 \\ -\alpha_1 x_1 - \alpha_2 x_2 \end{pmatrix} \quad (8.8)$$

Thus

$$\sigma_1(x) = -\frac{K_1}{K_2} \sin(x_1) - \frac{\alpha_1}{K_2} x_1 - \frac{\alpha_2}{K_2} x_2$$

$$\sigma(x) = -\frac{K_1}{K_2} \sin(x_1) - \frac{\alpha_1}{K_2} x_1 - \frac{\alpha_2}{K_2} x_2 - x_3 = -\frac{K_1}{K_2} \sin(x_1) - S \cdot x = 0$$

where

$$S = \begin{pmatrix} \frac{\alpha_1}{K_2} & \frac{\alpha_2}{K_2} & -1 \end{pmatrix}$$

and therefore

$$\frac{\partial \sigma}{\partial x} = \begin{pmatrix} -\frac{K_1}{K_2} \cos(x_1) - \frac{\alpha_1}{K_2} & -\frac{\alpha_2}{K_2} & -1 \end{pmatrix}$$

The uncertainty bounds can be determined as follows

$$\Omega_{\Delta f} = \left| \frac{\alpha_2}{K_2} \right| |K_6| + \mu \quad \Omega_{\Delta g} = |K_9| \quad \Omega_{\xi} = \left| \frac{\alpha_2}{K_2} \right| \cdot \hat{e}$$

where $\mu > \max |K_7 x_2 + K_8 x_3|$, $\hat{e} > \max |\xi(t)|$. The controller gain is then given by

$$\rho(x, t, \sigma) = \sqrt{\left(\left| \frac{\alpha_2}{K_2} \right| \cdot |K_6| + \mu + \left| \frac{\alpha_2}{K_2} \right| \cdot \hat{e} \right)^2 + \frac{1}{2} |K_9| \cdot |L_g \sigma| \left(\frac{L_f \sigma}{L_g \sigma} \right)^2} > 0 \quad (8.9)$$

The following values were chosen for simulation purposes: $l=1\text{m}$, $m=1\text{kg}$, $g=9.8\text{m/s}^2$, $J=1\text{Nms}^2/\text{rad}$, $K_m=0.1\text{Nm/A}$, $\alpha=0.1\text{Vs/rad}$, $R=1\Omega$, $L=5\text{mH}$.

(i)

$$\sigma(x) = -K_1 \sin(x_1) - Sx = 0$$

$$S = [2, 3, 1]$$

Uncertain Parameters:

$$\Omega_{\Delta f} = 50;$$

$$\Omega_{\Delta g} = 5;$$

$$\Omega_{\xi} = 6;$$

Constant Gain:

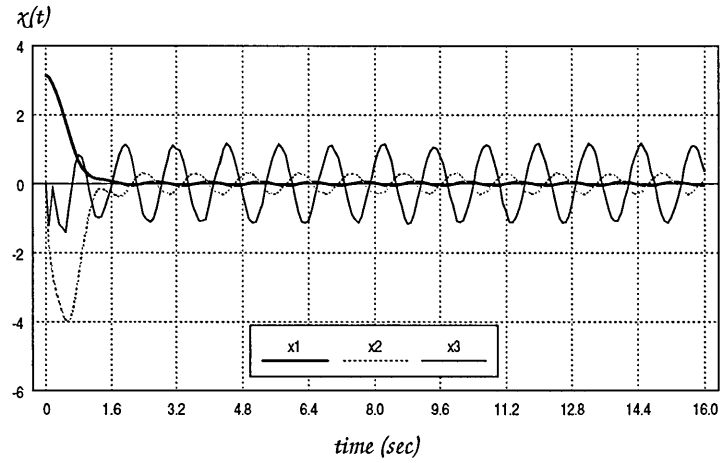
$$K = 10$$

(a)

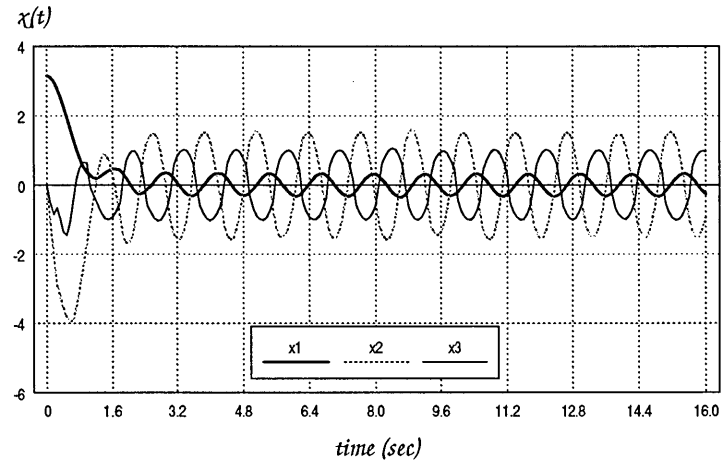
Variable structure
control with
feedback gain
of theorem 5.4;

(b)

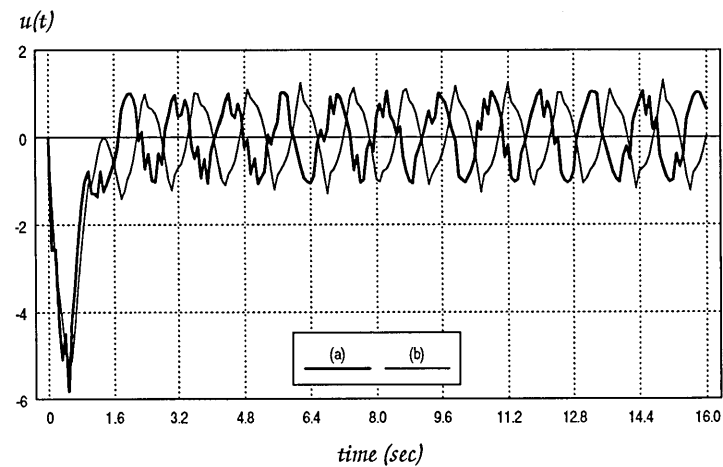
Variable structure
control with
feedback gain of
theorem 5.3



System states controlled by variable gain VSC



System states controlled by constant gain VSC



Control Signals

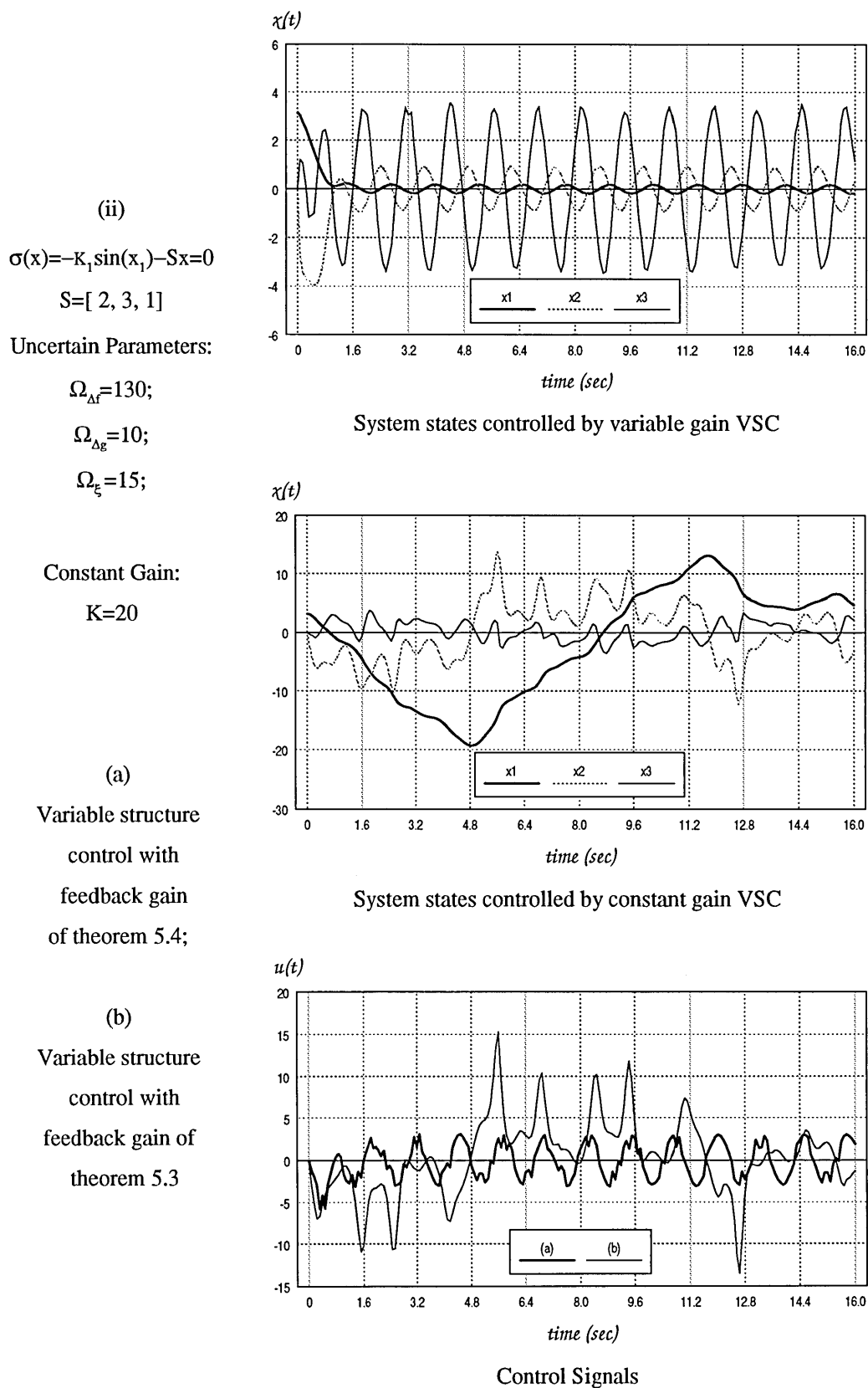


Fig. 8.4 Result 3: Comparison of variable structure controllers of constant gain with that of variable gain

Fig. 8.4 displays the results of simulation for the system. From the results, the responses of the system with feedback of both variable gain and constant gain are depicted for different parameter bounds. It can be seen that the controller obtained from theorem 5.4 works well, in contrast to the constant gain controller of theorem 5.3, which leads to large tolerances in the first case (i), and even an unstable response in the second case (ii).

8.2.2 A Crane System

The second example to be considered is concerned with the application of the technique of theorem 4.6 to a crane system.

It is likely that the mass to be lifted by a crane will vary greatly from time to time and may not be precisely known, and this uncertainty must be accommodated by the design. Furthermore the effective shaft stiffness will vary during operation because when large loads

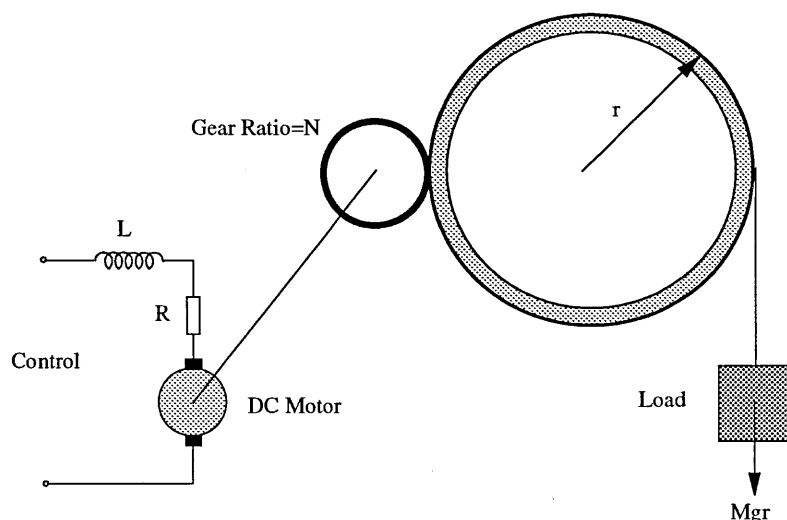


Fig. 8.5 A crane system to lift an unknown load controlled by a series-wound DC motor

are encountered the whole mounting tends to flex. Attempts to measure shaft stiffness are affected by the state of the system and typically quite significant differences in the measured value results. Also the motor constant depends upon the relationship between field strength and motor current and this varies considerably between low and high currents because of magnetic saturation. It should be noted that cranes employ series-wound DC motors so the field and armature current is the same. Finally armature

resistance is far from constant, not only because of heating effects, but also because it represents both eddy current and hysteresis losses in addition to the ohmic resistance. This variation is predictable to some degree where motors operate at constant speed, but in servo applications this is not the case. Here we only assume that the load to be lifted is unknown, ignoring other uncertainties. The system, subject to unknown load mass, may be described as follows

$$J_l \frac{d^2\theta_l}{dt^2} = T_l - T_d \quad (8.10)$$

$$T_l = k_s(\omega_m/N - \omega_l) \quad (8.11)$$

$$J_m \frac{d\omega_m}{dt} = T_m - \frac{T_l}{\alpha N} \quad (8.12)$$

$$T_m = k_m I \quad (8.13)$$

$$L \frac{dI}{dt} + R \cdot I + k_m \omega_m = V \quad (8.14)$$

where J_l , J_m and ω_l , ω_m are the moments of inertia and the angular velocities of the load rotating mechanism and the motor rotor respectively; k_s is the shaft stiffness; N and α are constants denoting the gear ratio and gearbox efficiency per unit respectively; L and R represent the combined field and leakage inductance and the resistance of the motor armature respectively, and

$$k_m = k \cdot I \quad (8.15)$$

where k represents the motor constant, so that

$$T_m = k_m I = k \cdot I^2 \quad (8.16)$$

The model may be rewritten as

$$\dot{x}(t) = \begin{pmatrix} x_2 \\ a_1 x_1 + a_2 x_3 \\ x_4 \\ a_3 x_1 + a_4 x_3 + a_5 x_5^2 \\ a_6 x_5 + a_7 x_4 x_5 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ b_5 \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ d_2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \xi \quad (8.17)$$

$$y(t) = x_1$$

by choosing the states $(x_1, x_3)=(\theta_l, \theta_m)$ and $(x_2, x_4)=(\dot{\theta}_l, \dot{\theta}_m)$ the angular positions and angular velocities, and $x_5=I$ the armature current. The system parameters are given by

$$\begin{aligned} a_1 &= -\frac{k_s}{J_l} & a_2 &= \frac{k_s}{J_l N} & a_3 &= \frac{k_s}{J_l \alpha N} & a_4 &= -\frac{k_s}{J_m \alpha N^2} \\ a_5 &= \frac{k}{J_m} & a_6 &= -\frac{R}{L} & a_7 &= -\frac{k}{L} & b_5 &= \frac{1}{L} & d_2 &= \frac{1}{J_l} \end{aligned}$$

The system model is first transformed to new coordinates according to the coordinate transformation defined in chapter 4, and a new state space model is obtained as follows

$$\begin{aligned} \dot{z}(t) &= \begin{pmatrix} z_2 \\ z_3 \\ z_4 \\ z_5 \\ \alpha_1 z_1 + \alpha_2 z_3 + \alpha_3 z_5 + \alpha_4 z_1 z_2 + \alpha_5 z_2 z_3 \\ \quad + \alpha_6 z_2 z_5 + \alpha_7 z_1 z_4 + \alpha_8 z_3 z_4 + \alpha_9 z_4 z_5 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \beta \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ d_2 \\ 0 \\ d_4 \\ 0 \end{pmatrix} \xi \\ y(t) &= z_1 \end{aligned} \quad (8.18)$$

which is of linearisable form with new state $z(t)$ and input $v(t)$, where α_i ($i=1, \dots, 9$) and β are transformed coefficients depending on the coefficients a_i ($i=1, \dots, 7$) and b_5 of the original system, $d_2=a_1$, $d_4=a_1^2$, and $v(t)$ is the new input.

It is obvious that the uncertainty, $\xi=M \cdot g \cdot r$, does not satisfy the matching conditions of definition 2.4 or 2.5. In order to apply the results of theorem 4.7 to this problem, the transformation of the form

$$\begin{aligned} \mathfrak{Z} &= \begin{pmatrix} 56.250 & 0 & 0 & 0 & 0 \\ 56.250 & 70.313 & 0 & 0 & 0 \\ 56.250 & 70.313 & 36.500 & 0 & 0 \\ 56.250 & 70.313 & 36.500 & 9.156 & 0 \\ 56.250 & 70.313 & 36.500 & 9.156 & 1.000 \end{pmatrix} \\ \mathfrak{Z}^{-1} &= \begin{pmatrix} 0.018 & 0 & 0 & 0 & 0 \\ -0.014 & 0.014 & 0 & 0 & 0 \\ 0 & -0.027 & 0.027 & 0 & 0 \\ 0 & 0 & -0.109 & 0.109 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} \end{aligned}$$

is introduced, so a diagonal matrix P is obtained according to theorem 3.9 as follows

$$P = \begin{pmatrix} 1.875 & 0 & 0 & 0 & 0 \\ 0 & 1.875 & 0 & 0 & 0 \\ 0 & 0 & 3.267 & 0 & 0 \\ 0 & 0 & 0 & 6.205 & 0 \\ 0 & 0 & 0 & 0 & 10.990 \end{pmatrix}$$

A Lyapunov function is then defined, and therefore the technique described in theorem 4.7 is applicable.

The following values were chosen for simulation purposes:

$$J_l = 1000 \text{ kg m}^2, \quad J_m = 0.2 \text{ kg m}^2, \quad k_s = 0.6 \times 10^8 \text{ Nm/rad}, \quad k = 0.25$$

$$N = 500, \quad \alpha = 0.8, \quad R = 10 \Omega, \quad L = 20 \text{ mH}, \quad g = 9.81 \text{ m/s}^2, \quad r = 0.25 \text{ m}$$

and the load mass to be lifted is

$$M = 0 \sim 2500 \text{ kg}$$

Simulation results are given in the following figure.

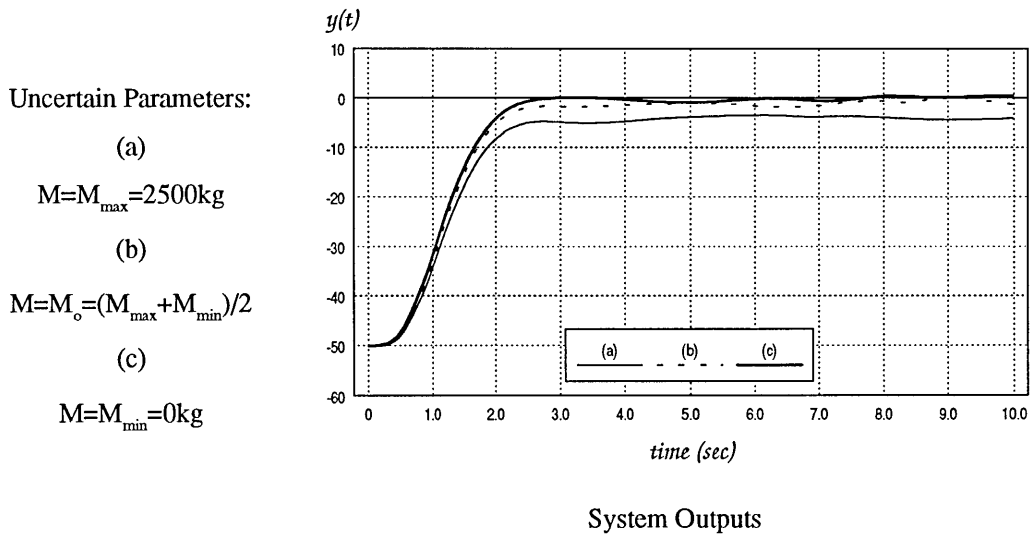


Fig.8.6 Result 4: Control of the crane system subject to various loads

From the results, although there are some static errors for the system output which depend on the amplitudes of disturbances, here the unknown load, satisfactory performance is achieved when the system is subjected to large variations of the load mass, here from 0 kg to 2500 kg.

8.2.3 Six-Plate Gas-absorber System

A gas absorber tower is an important element in several chemical processes. A typical gas absorber system consists of a number of vertically arranged plates enclosed within a chemical tower, as shown diagrammatically in Fig.8.7.

The chemical reactions which take place in the tower are affected by the inlet feed compositions corresponding to a downward liquid stream and an upward vapour stream. These reactions may give rise to instability if the inlet feed compositions are not properly chosen, and therefore stabilisation and control of such reactions is an important problem.

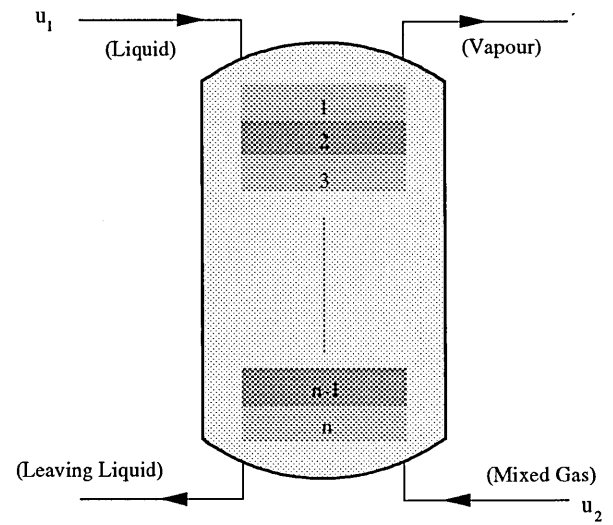


Fig. 8.7 Gas absorber Tower

A six-plate gas-absorber system is considered. A detailed description of such a system may be found in Darwish *et al*^[2].

The system is modelled by

$$\dot{x} = F(x, \gamma) + G(x, \gamma)u(t) \quad (8.19)$$

where

$$F(x, \gamma) = Ax$$

$$A = \begin{pmatrix} -d_2(1+d_1) & d_2 & 0 & 0 & 0 & 0 \\ d_1d_2 & -d_2(1+d_2) & d_2 & 0 & 0 & 0 \\ 0 & d_1d_2 & -d_2(1+d_1) & d_2 & 0 & 0 \\ 0 & 0 & d_1d_2 & -d_2(1+d_1) & d_2 & 0 \\ 0 & 0 & 0 & d_1d_2 & -d_2(1+d_1) & d_2 \\ 0 & 0 & 0 & 0 & d_1d_2 & -d_2(1+d_1) \end{pmatrix} \quad (8.20)$$

$$G(x,\gamma) = B = \begin{pmatrix} d_1 d_2 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & d_2/a \end{pmatrix} \quad (8.21)$$

where d_1 and d_2 depend on the inlet vapour and liquid hold up on each plate (h_r, h_e), the flow rates of inlet liquid absorbent and inlet gas stream (L_f, L_g), and also the ratio of liquid/vapour compositions a ^[2]. These parameters cannot be calculated with sufficient accuracy to be used in online controllers. Consequently, we consider that the parameters d_1 and d_2 undergo 25% variation about their nominal values^[3] which are chosen to be

$$d_1 = 0.849 \quad d_2 = 0.634$$

with $a = 0.72$. Then

$$A(\gamma) = \begin{pmatrix} -1.17+\gamma_1 & 0.63+\gamma_2 & 0 & 0 & 0 & 0 \\ 0.54+\gamma_3 & -1.17+\gamma_1 & 0.63+\gamma_2 & 0 & 0 & 0 \\ 0 & 0.54+\gamma_3 & -1.17+\gamma_1 & 0.63+\gamma_2 & 0 & 0 \\ 0 & 0 & 0.54+\gamma_3 & -1.17+\gamma_1 & 0.63+\gamma_2 & 0 \\ 0 & 0 & 0 & 0.54+\gamma_3 & -1.17+\gamma_1 & 0.63+\gamma_2 \\ 0 & 0 & 0 & 0 & 0.54+\gamma_3 & -1.17+\gamma_1 \end{pmatrix} \quad (8.22)$$

$$B(\gamma) = \begin{pmatrix} 0.54+\gamma_4 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0.88+\gamma_5 \end{pmatrix} \quad (8.23)$$

where the uncertain parameters are given by

$$\begin{aligned} -0.46 \leq \gamma_1 \leq 0.39 & & -0.235 \leq \gamma_4 \leq 0.303 \\ -0.158 \leq \gamma_2 \leq 0.158 & & -0.219 \leq \gamma_5 \leq 0.221 \\ -0.235 \leq \gamma_3 \leq 0.303 & & \end{aligned} \quad (8.24)$$

A transformation of the form

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (8.25)$$

is defined, such that the regular form of the system may be written as

$$\begin{aligned} \dot{\mathbf{x}}^1(t) &= F_1(\mathbf{x}^1, \mathbf{x}^2, \gamma) \\ \dot{\mathbf{x}}^2(t) &= F_2(\mathbf{x}^1, \mathbf{x}^2, \gamma) + G_2(\mathbf{x}^1, \mathbf{x}^2, \gamma) \cdot u(t) \end{aligned} \quad (8.26)$$

where the new states each represents a set of states $\mathbf{x}^1 = (x_2, x_3, x_4, x_5)^T$, $\mathbf{x}^2 = (x_1, x_6)^T$, and the state and input mapping are

$$F_1(\mathbf{x}, \gamma) = A_{11}\mathbf{x}^1 + A_{12}\mathbf{x}^2$$

$$F_2(\mathbf{x}, \gamma) = A_{21}\mathbf{x}^1 + A_{22}\mathbf{x}^2$$

where

$$A_{11} = \begin{pmatrix} -1.17 + \gamma_1 & 0.63 + \gamma_2 & 0 & 0 \\ 0.54 + \gamma_3 & -1.17 + \gamma_1 & 0.63 + \gamma_2 & 0 \\ 0 & 0.54 + \gamma_3 & -1.17 + \gamma_1 & 0.63 + \gamma_2 \\ 0 & 0 & 0.54 + \gamma_3 & -1.17 + \gamma_1 \end{pmatrix}$$

$$A_{12} = \begin{pmatrix} 0.54 + \gamma_3 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0.63 + \gamma_2 \end{pmatrix}$$

$$A_{21} = \begin{pmatrix} 0.63 + \gamma_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.54 + \gamma_3 \end{pmatrix}$$

$$A_{22} = \begin{pmatrix} -1.17 + \gamma_1 & 0 \\ 0 & -1.17 + \gamma_1 \end{pmatrix}$$

and

$$G_2 = \begin{pmatrix} 0.54 + \gamma_4 & 0 \\ 0 & 0.88 + \gamma_5 \end{pmatrix}$$

respectively. This is a linear system with uncertain parameters which do not satisfy matching conditions. A switching function is designed as

$$\sigma(x) = \sigma_1(x^1) - x^2 = 0$$

where

$$\sigma_1(x^1) = S \cdot x^1 = \begin{pmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{21} & s_{22} & s_{23} & s_{24} \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \quad (8.27)$$

The reduced order closed loop system resulting from the above switching function is

$$\dot{x}^1 = A_{11}x^1 + A_{12}x^2 = [A_{11}, A_{12}] \begin{pmatrix} x^1 \\ Sx^1 \end{pmatrix} = A_\sigma x^1$$

Let
$$S = \begin{pmatrix} -2.495 & -0.849 & -0.789 & 0 \\ 0 & -0.929 & -1.178 & -0.547 \end{pmatrix}$$

resulting in a closed loop system with poles $\lambda_1 = -2.6958$, $\lambda_2 = -1.7271$, $\lambda_{3,4} = -0.979 \pm j0.292$.

The partial derivative of the switching function is given by

$$\nabla \sigma = \begin{pmatrix} -2.495 & -0.849 & -0.789 & 0 & \vdots & -1 & 0 \\ 0 & -0.930 & -1.178 & -0.547 & \vdots & 0 & -1 \end{pmatrix}$$

$$\phi_G = \begin{pmatrix} 0.54 + \min(\gamma_4) & 0 \\ 0 & 0.88 + \min(\gamma_5) \end{pmatrix}$$

$$\sigma_M^2 = \lambda_{\max}\{\Omega_G \cdot \Omega_G^T\} = \lambda_{\max}\{\nabla \sigma \cdot \phi_G \cdot \phi_G^T \cdot \nabla \sigma^T\} = 0.4369$$

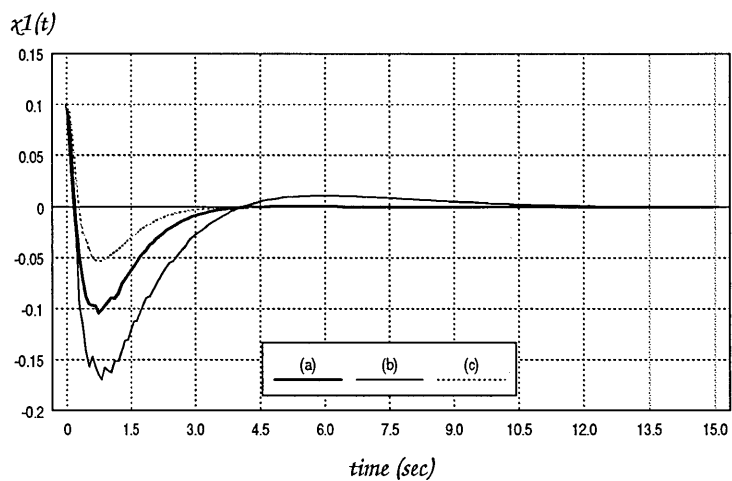
We check that

$$\begin{aligned} \omega_G \cdot \Omega_G^T &= \begin{pmatrix} 0.54 + \gamma_4 & 0 \\ 0 & 0.88 + \gamma_5 \end{pmatrix} \begin{pmatrix} 0.54 + \min(\gamma_4) & 0 \\ 0 & 0.88 + \min(\gamma_5) \end{pmatrix} \\ &= \begin{pmatrix} (0.538 + \gamma_4)(0.303) & 0 \\ 0 & (0.88 + \gamma_5)(0.661) \end{pmatrix} \end{aligned}$$

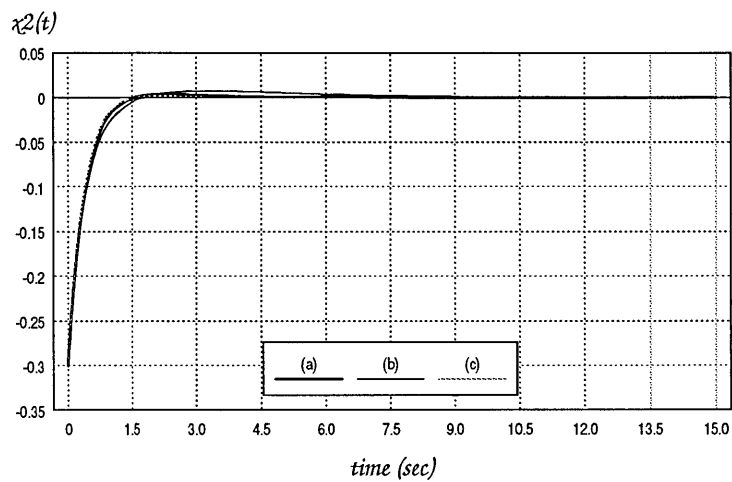
is positive definite, so condition (6.17) is satisfied, and choose

$$\varsigma = 0.09 < \lambda_{\min}\{\omega_G \cdot \Omega_G^T\} = 0.0918$$

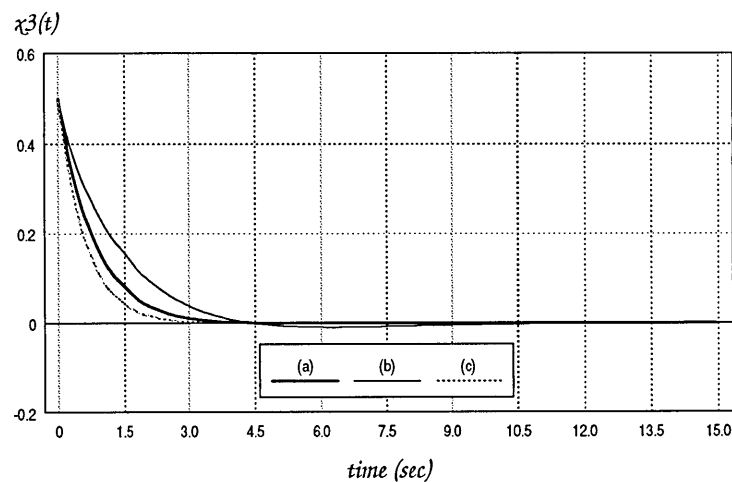
The simulation results are as follows. Good closed loop system performance is clearly indicated.



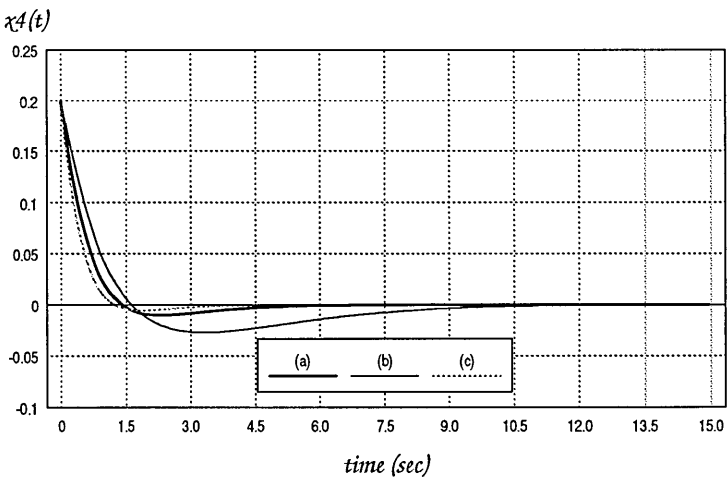
System States $x_1(t)$



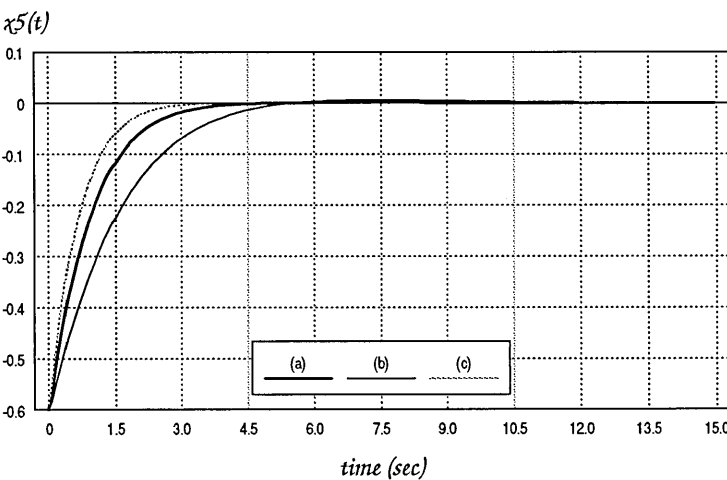
System States $x_2(t)$



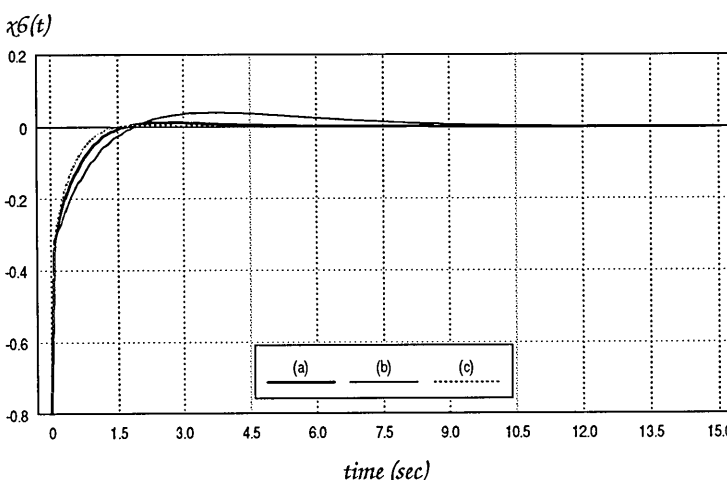
System States $x_3(t)$



System States $x_4(t)$



System States $x_5(t)$



System States $x_6(t)$

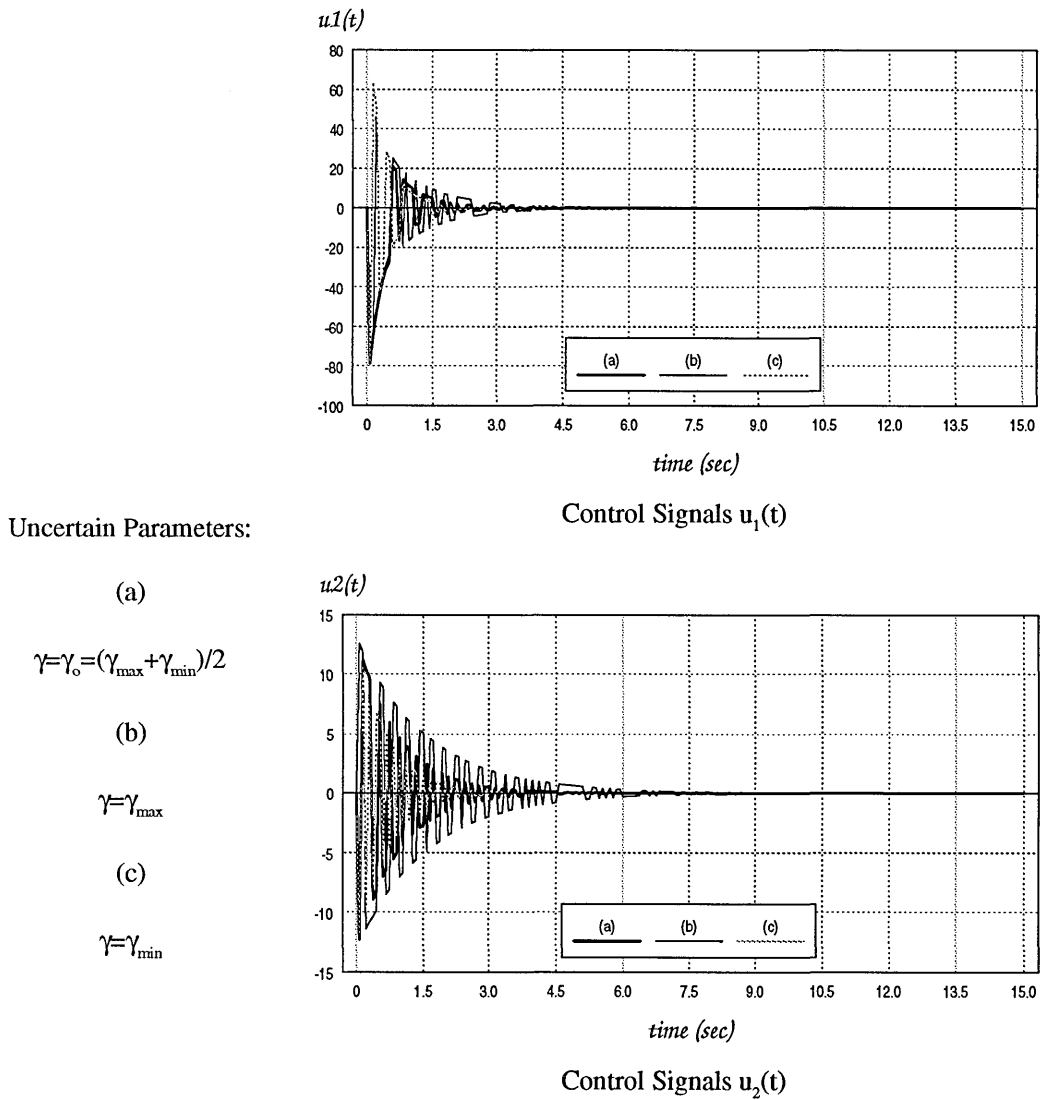


Fig. 8.8 Result 5: Control of the six-plate gas-absorber system

8.2.4 Two-Degree-of-Freedom Manipulator

Robot manipulators are familiar examples of trajectory-controllable mechanical systems. However, their nonlinear dynamics present a challenging control problem, and it is even harder when significant uncertainty is present.

Consider, for instance, a planar, two-link articulated manipulator, whose position can be described by a 2-vector of the polar coordinates, and whose actuator inputs consist of a 2-vector of torques applied at the manipulator joints. The dynamics of this simple

manipulator are strongly nonlinear, and include uncertainties caused by the load mass to be carried, which is not accurately known.

The control objective is to force the load with uncertain mass, whose trajectory is indicated by polar coordinates, to follow a prescribed trajectory in the Cartesian (y_1, y_2) -plane. The planar tracking problem of a two-degrees-of-freedom manipulator^[4] can be modelled by

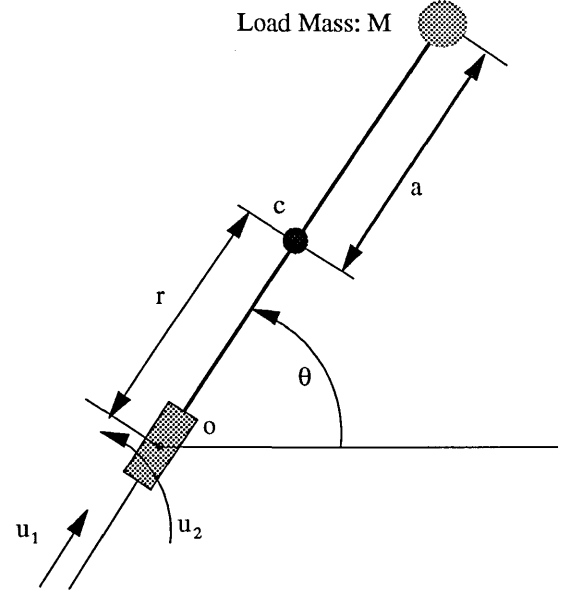


Fig. 8.9 Manipulator with two degrees of freedom

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ \frac{\mu x_1 + M(x_1 + a)x_2^2}{\mu + M} \\ x_4 \\ \frac{-2[\mu x_1 + M(x_1 + a)x_2 x_4]}{J_1 + J_2 + \mu x_1^2 + M(x_1 + a)^2} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \frac{1}{\mu + M} & 0 \\ 0 & 1 \\ 0 & \frac{1}{J_1 + J_2 + \mu x_1^2 + M(x_1 + a)^2} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad (8.28)$$

where $(x_1, x_3) = (r, \theta)$ are the polar coordinates of the mass centre of the arm, and $(x_2, x_4) = (\dot{r}, \dot{\theta})$; μ is the mass of the arm; M is the mass of the load; a is the distance from arm mass centre to the load; J_1 the moment of inertia of the rotation mechanism about the vertical axis through O ; J_2 the moment of inertia of the arm about the vertical axis through the arm mass centre.

For the purposes of illustration, it is presumed that all parameters in the model are precisely known with the exception of the constant load mass M which is subject to bounds: $0 \leq M_{\min} \leq M \leq M_{\max}$, where M_{\min} and M_{\max} are known constants.

We therefore have a nonlinear uncertain system, and the technique developed for MIMO systems in chapter 7 may be applied to the synthesis of this robust tracking problem.

The system is already in the regular form, so it can be expressed, with z as new coordinate, as follows

$$\begin{aligned}\dot{z}_{i,1}(t) &= z_{i,2}(t) \\ \dot{z}_{i,2}(t) &= a_i(z) + b_i(z) \cdot u_i(t) + \tilde{\delta}_{i,1}(z, \gamma) + \tilde{\delta}_{i,2}(z, \gamma) \cdot u_i(t) \quad (i=1,2) \\ y_i(t) &= z_{i,1}(t)\end{aligned} \quad (8.29)$$

where the new states are $z_{1,1}=x_1$, $z_{1,2}=x_2$, $z_{2,1}=x_3$, $z_{2,2}=x_4$, the outputs $y_1=x_1$, $y_2=x_3$, the uncertainties caused by unknown load mass $\tilde{\delta}_{1,1}(z, \gamma)$, $\tilde{\delta}_{1,2}(z, \gamma)$, and

$$a_1(z) = \frac{\mu x_1 + M_0(x_1 + a)x_4^2}{\mu + M_0} \quad (8.30)$$

$$b_1(z) = \frac{1}{\mu + M_0} \quad (8.31)$$

$$a_2(z) = \frac{-2[\mu x_1 + M_0(x_1 + a)x_2x_4]}{J_1 + J_2 + \mu x_1^2 + M_0(x_1 + a)^2} \quad (8.32)$$

$$b_2(z) = \frac{1}{J_1 + J_2 + \mu x_1^2 + M_0(x_1 + a)^2} \quad (8.33)$$

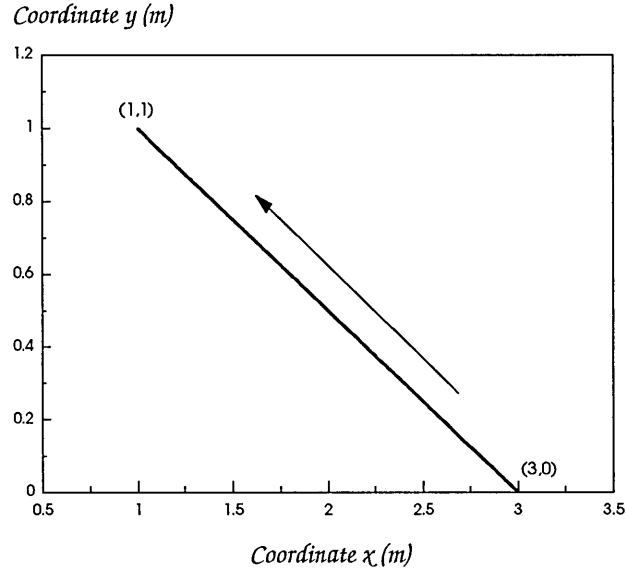
where the nominal value of unknown mass is $M_0 = (M_{\max} + M_{\min})/2$. The feedback linearisation is therefore of the form

$$u(t) = -\Pi^{-1}\Lambda + \Pi^{-1}v(t) = \begin{pmatrix} b_1(z) & 0 \\ 0 & b_2(z) \end{pmatrix}^{-1} \begin{pmatrix} -a_1(z) + v_1 \\ -a_2(z) + v_2 \end{pmatrix} \quad (8.34)$$

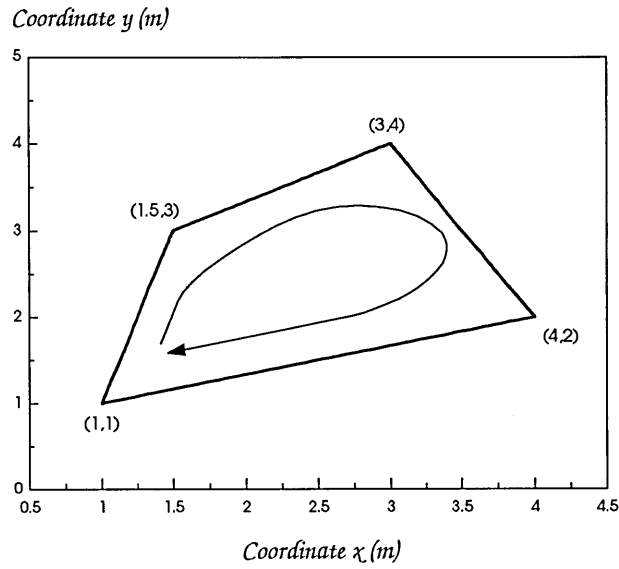
The ideal trajectories denoted by $y_{1,c}^d(t)$ and $y_{2,c}^d(t)$ are defined as a straight line path AB in the Cartesian (y_1, y_2) -plane, from the initial rest position A, with coordinate $(y_{1,A}, y_{2,A})$, to prescribed final rest position B, with coordinates $(y_{1,B}, y_{2,B})$, in a prescribed time T. A pair of Cartesian coordinate functions which characterises a straight line path from point A to point B is given by

$$y_{i,c}^d(t) = \begin{cases} y_{i,A} & t < t_0 \\ y_{i,A} + k_i(t-t_0)^3 & t_0 \leq t < t_0 + T/4 \\ y_{i,A} + k_i[(t-t_0)^3 - 2(t-t_0-T/4)^3] & t_0 + T/4 \leq t < t_0 + 3T/4 \\ y_{i,A} + k_i[(t-t_0)^3 - 2(t-t_0-T/4)^3 + 2(t-t_0-3T/4)^3] & t_0 + 3T/4 \leq t < t_0 + T \\ y_{i,B} & t > t_0 + T \end{cases} \quad (8.35)$$

for $i=1,2$, where $k_i=16[y_{i,B}-y_{i,A}]/(3T^3)$.



(i) A straight line load path



(ii) A combined straight line load path

Fig. 8.10 Ideal trajectories to be tracked by the two joint manipulator

The corresponding polar coordinate form is then given by

$$y_1^d(t) = z_{1,1}^d = \sqrt{(y_{1,c}^d)^2 + (y_{2,c}^d)^2} \quad (8.36)$$

$$y_2^d(t) = z_{2,1}^d = \tan^{-1}(y_{2,c}^d / y_{1,c}^d) \quad (8.37)$$

and the tracking errors are defined as

$$e_{ij}(t) = z_{ij} - z_{ij}^d \quad (i,j=1,2)$$

The variable structure controller is therefore of the form

$$v(t) = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \dot{z}_{1,2}^d(t) - a_{1,1} \cdot e_{1,2}(t) \\ \dot{z}_{2,2}^d(t) - a_{2,1} \cdot e_{2,2}(t) \end{pmatrix} - \begin{pmatrix} \rho_1^{(1)} & 0 \\ 0 & \rho_2^{(1)} \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} - \begin{pmatrix} \rho_{1,1}^{(2)} & \rho_{1,2}^{(2)} \\ \rho_{2,1}^{(2)} & \rho_{2,2}^{(2)} \end{pmatrix} \text{sgn} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}$$

We define a vector of switching functions as follows

$$\sigma(t) = \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} e_{1,2}(t) + a_{1,1} e_{1,1}(t) \\ e_{2,2}(t) + a_{2,1} e_{2,1}(t) \end{pmatrix} = 0 \quad (8.38)$$

resulting in a closed loop system of the form

$$\dot{E}(t) = AE(t) + B\sigma(t) \quad (8.39)$$

$$\dot{\sigma}(t) = -\rho_1 \cdot \sigma(t) - \rho_2 \cdot \text{sgn}[\sigma(t)] + \delta_1(z, \zeta, \gamma) + \delta_2(z, \zeta, \gamma) \cdot v(t) \quad (8.40)$$

where

$$A = \begin{pmatrix} -a_{1,1} & 0 \\ 0 & -a_{2,1} \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad E(t) = \begin{pmatrix} e_{1,1} \\ e_{2,1} \end{pmatrix}$$

The feedback gains ρ_1, ρ_2 can be obtained according to theorem 7.2 as follows

$$\rho_1 = 2 \frac{\lambda_M(\Omega_2^2)}{\varsigma} \cdot \Omega_2^{-1} \lambda_M(P) > 0 \quad (8.41)$$

$$\rho_2 = \frac{(1^2+1)\lambda_M(\Omega_2^2)}{21\varsigma} \cdot \Omega_2^{-1} \Omega_\delta \geq 0 \quad (8.42)$$

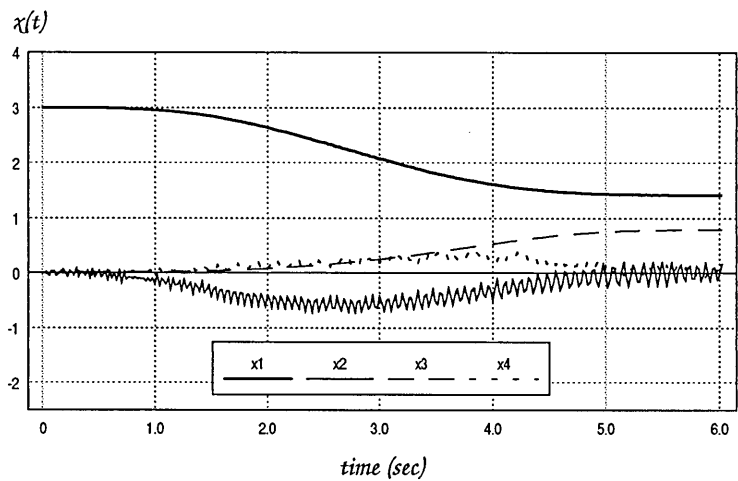
where Ω_1 and Ω_2 are matrices depending on the uncertainty bounds Δ_1, Δ_2 and also $\|w\|$, $\varphi_M^2(\Omega_2) = \lambda_M(\Omega_2^2)$ the spectral norm of Ω_2 , and $\lambda_M(P)$ is given by $\max\{1/2a_{1,1}, 1/2a_{2,1}\}$.

The following numerical values are taken throughout the simulation: $\mu=100$ kg, $J_1=J_2=100$ kg m², $a=1$ m.

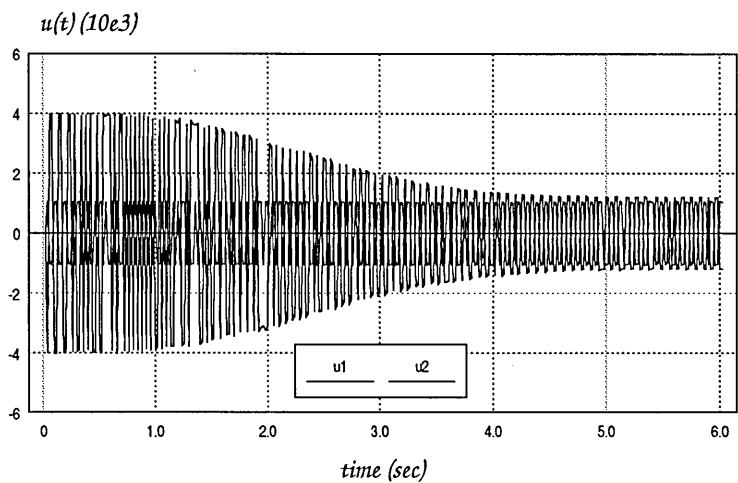
The tracking errors are measured by the norm

$$\|e(t)\| = \sqrt{[y_{1,c}^d - (z_{1,1} + a)\cos(z_{2,1})]^2 + [y_{2,c}^d - (z_{1,1} + a)\sin(z_{2,1})]^2} \quad (8.43)$$

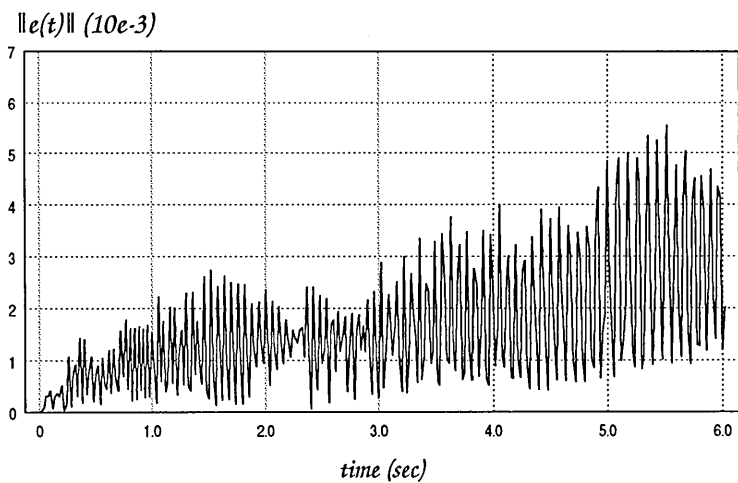
The results are shown in Fig. 8.11 and Fig. 8.12, where in case 1, the straight line path of Fig.8.10(i) is tracked with the mass $0\text{kg} \leq M \leq 100\text{kg}$; and in case 2, the combined straight lines path of Fig.8.10(ii) is tracked with the mass $0\text{kg} \leq M \leq 200\text{kg}$.



States



Control Signals

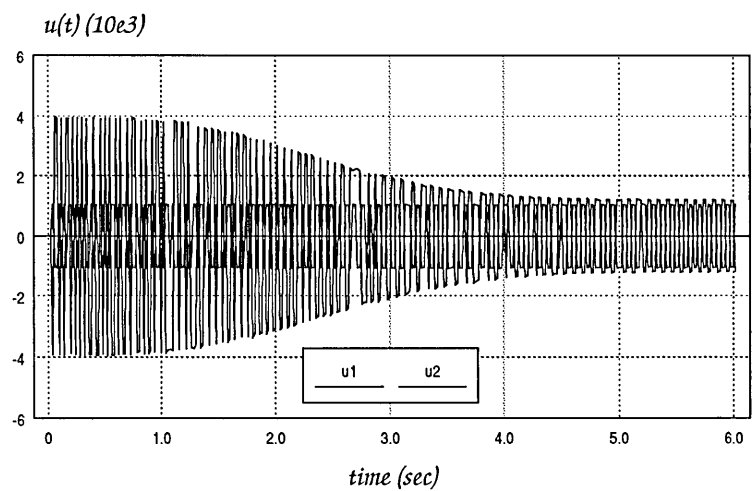
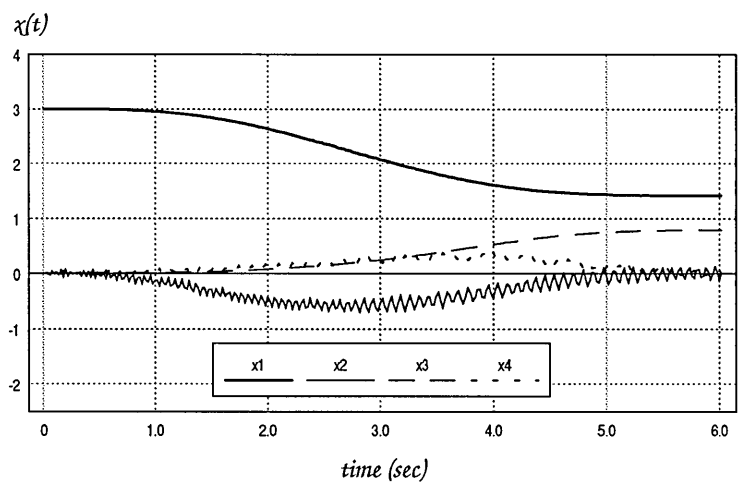


Norm of position error under feedback control

(i)

Uncertain Parameter:

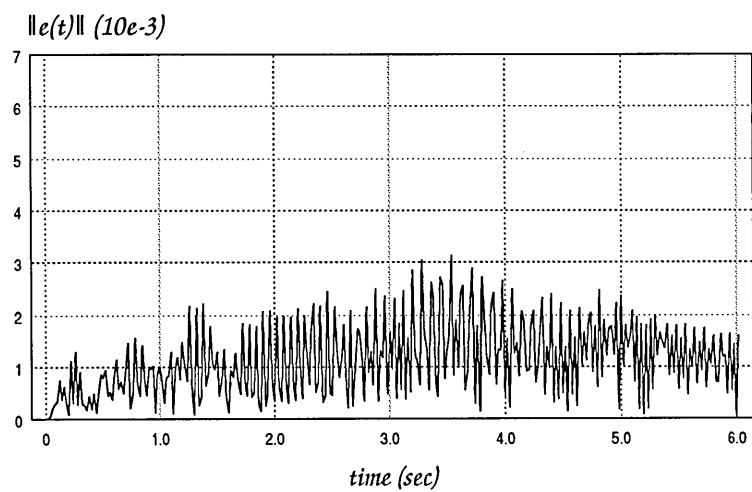
$$M=M_{\min}$$



(ii)

Uncertain Parameter:

$$M=M_0=(M_{\max}+M_{\min})/2$$



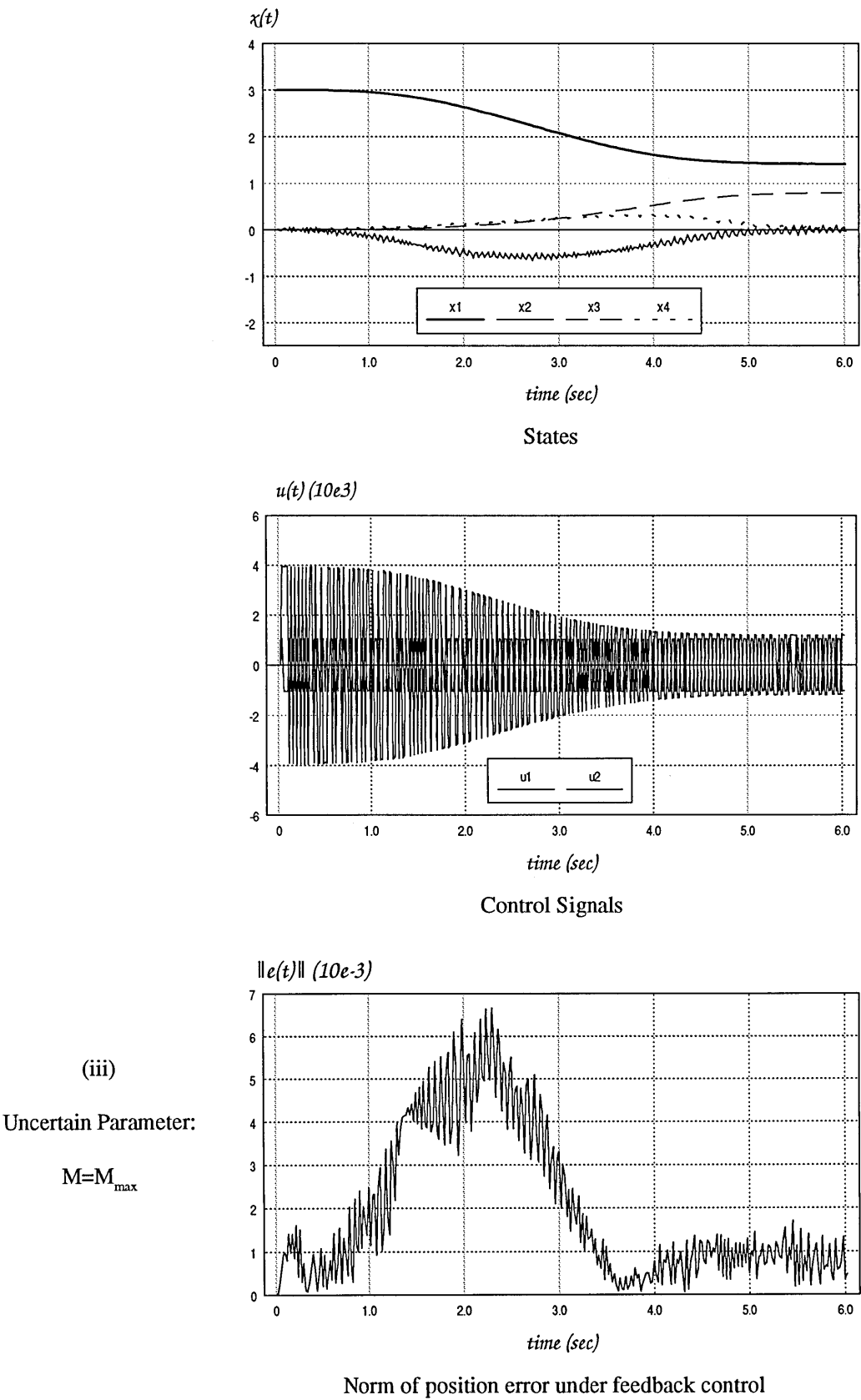
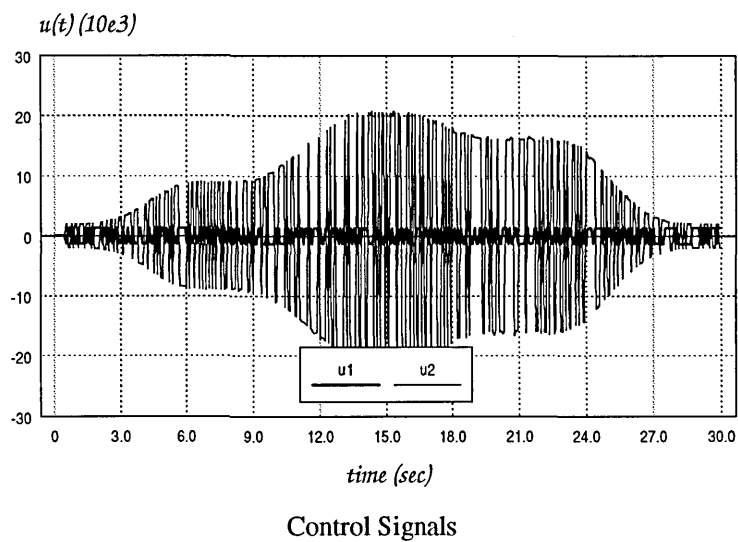
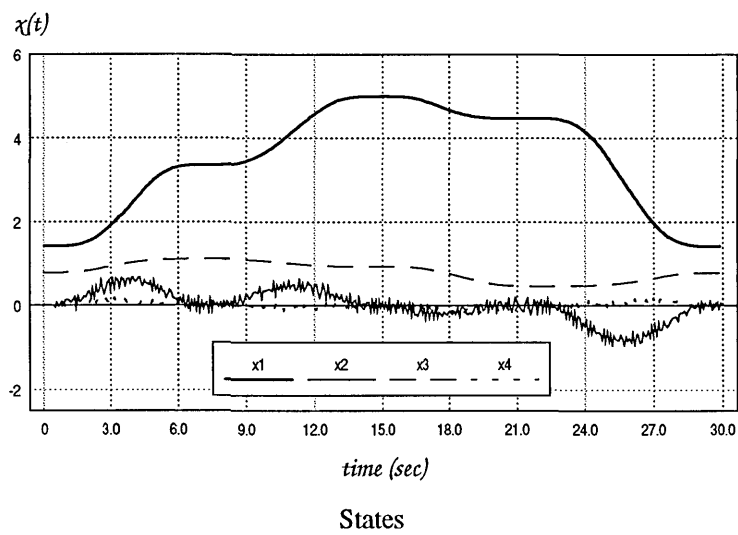


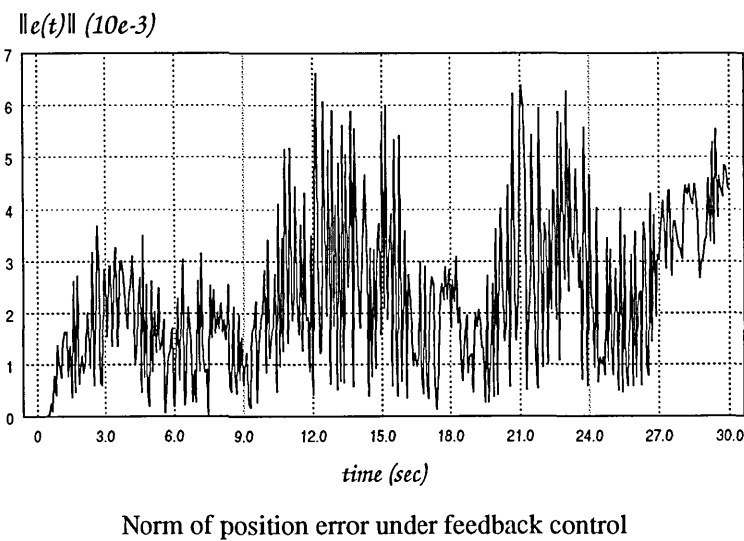
Fig. 8.11 Result 6: Robust tracking of straight line trajectory

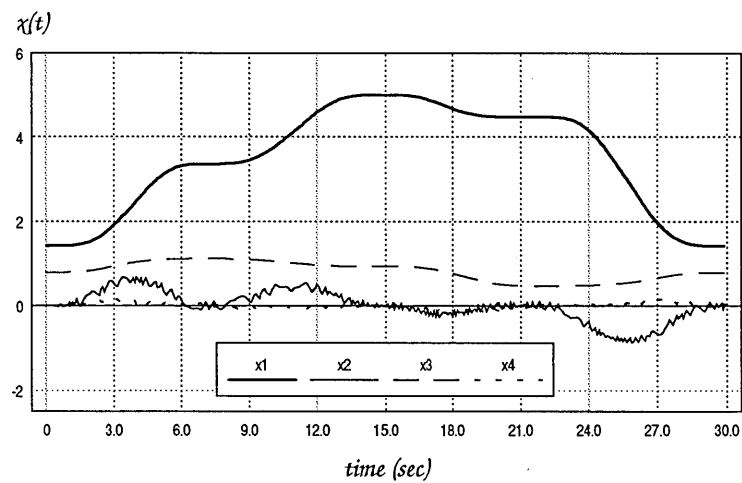


(i)

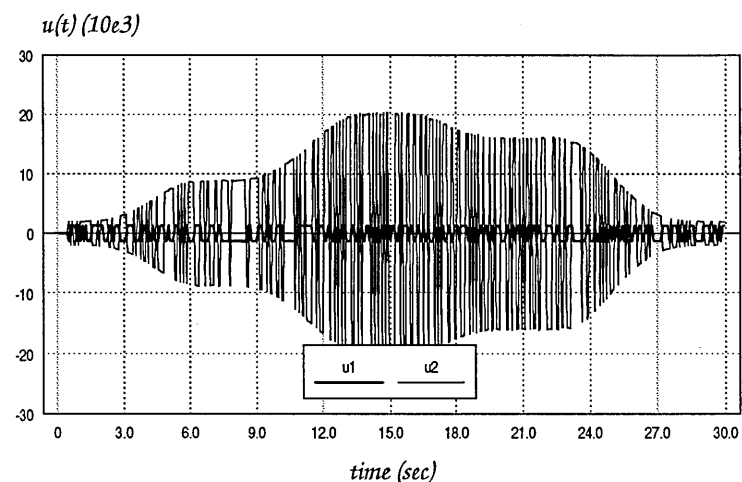
Uncertain Parameter:

$M=M_{\min}$





States

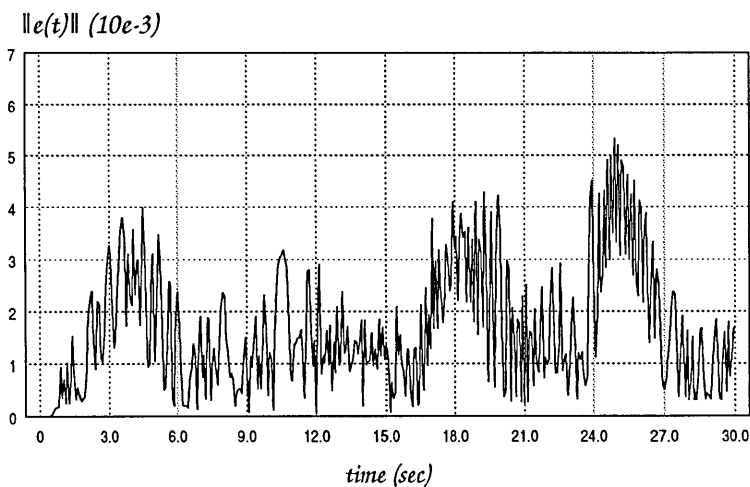


Control Signals

(ii)

Uncertain Parameter:

$$M=M_0=(M_{\max}+M_{\min})/2$$



Norm of position error under feedback control

(iii)
Uncertain Parameter:
 $M=M_{\max}$

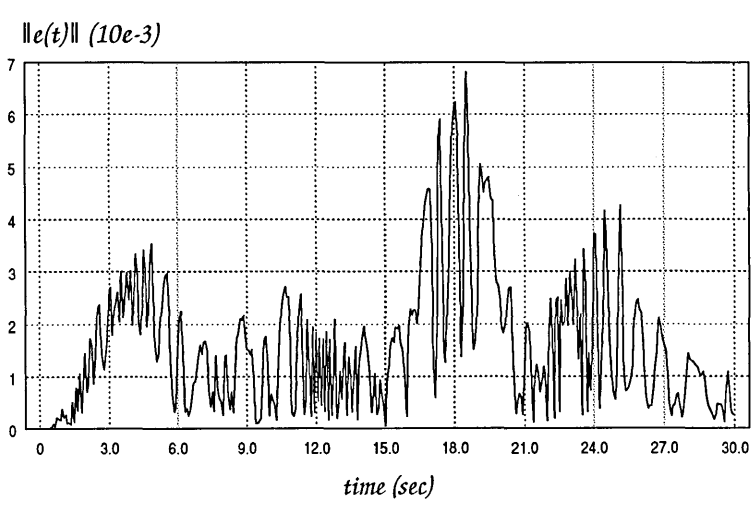
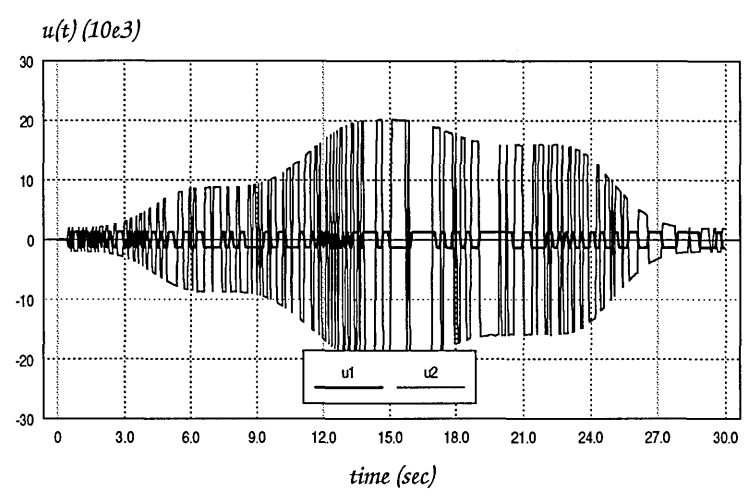
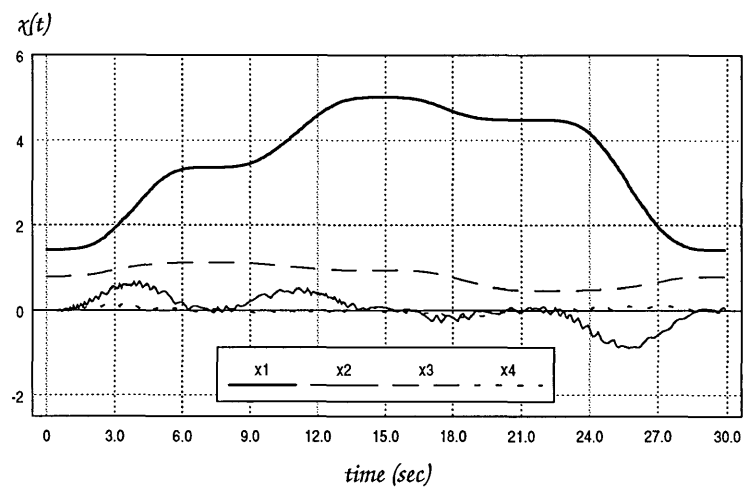


Fig. 8.12 Result 7: Robust tracking of the combined straight line trajectory

8.3 SUMMARY

In this chapter, the robust control techniques developed in previous chapters have been applied to the control of four different systems, where both stability and tracking problems are considered. The first two examples are SISO nonlinear systems in the presence of uncertainties which do not satisfy matching conditions. The third example is a chemical process which is MIMO, assumed linear, but highly uncertain and mismatched, and also open loop unstable. The last example is concerned with robust tracking of a two-degree of freedom manipulator with some uncertainties caused by unknown load mass. The simulation results show the great robustness of the techniques to the various uncertainties in the systems. The control techniques guarantee the stability of the closed loop systems and also achieve good performance both in regulation, for instance, examples 1, 2 and 3, and in tracking, for instance, example 4. In contrast to previous work on the problem, the main emphasis here is that, firstly, there is no requirement for the nominal dynamics to be either stable or in some way precompensated, and secondly, neither is it required that matching assumptions be met. The simulation results show that the controller attenuates the effects of the uncertainties and the stability of the closed loop system is guaranteed.

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Conclusions and Further Work



Overview

In this chapter the techniques described in earlier chapters are discussed. Also suggestions for further work are made.



Outline

- ✓ Conclusions
- ✓ Suggestions for Further Work

9.1 CONCLUSIONS

IN pure model-based control, the control law is based on a nominal model of the physical system, *i.e.*, the model used is assumed precisely known. How the control system will actually behave in the presence of parametric uncertainties and unmodelled dynamics is not clear at the design stage, and the stability of the closed loop system cannot be guaranteed. Robust control requires that the controller is based on consideration of both the nominal model and some characterisation of the uncertainties in the system. Despite the presence of such uncertainties, the system should still be stable and achieve some prescribed performance. By robust control, we usually mean two different but related aspects; stability robustness and performance robustness. A critical property of a feedback system is its robustness, particularly with respect to stability; *i.e.*, its ability to reduce the sensitivity of the system to any mismatch between the plant model and the real plant. But stability alone is insufficient and some performance criteria must be met. Therefore, the robust control of nonlinear systems in the presence of uncertainties is of great significance in practice.

Motivated by this crucial requirement, a rather general class of nonlinear uncertain systems has been investigated, where the systems are described by differential equations which contain parameters whose values are not precisely known. Robust feedback control laws have been derived whose structures depend on the known bounds of the uncertainties, where the control laws are based on Lyapunov stability theory. The objective of the design is, firstly, to guarantee the stability of the closed loop system, *i.e.*, stability robustness, and secondly to achieve some desired performance, *i.e.*, performance robustness.

All the techniques described in this thesis are based on Lyapunov stability theory, so that stability is the central result even where large uncertainty tolerance is required. On

the other hand, performance robustness is also achieved by the design. More specifically, when designing a control law, a nominal control is obtained first, which guarantees the closed loop behaviour of the nominal system, and then an extra control effort is introduced to counter the effect of uncertainties. In this case, two situations may occur: the first is that the control may fully compensate the uncertainty such that the output of the closed loop system finally achieves the desired performance prescribed for the nominal control, whilst the second is that the output of the closed loop system may not finally reach the ideal performance prescribed for the nominal control, but settle down in the vicinity of it. This is called boundedness. In this work, it has been shown that all techniques can achieve a system with uniformly ultimately bounded behaviour. Boundedness is also a kind of performance robustness in the sense that if, for a particular control problem, the bound is sufficiently small throughout the control process so that it is acceptable, it can also be concluded that performance robustness can be guaranteed. For instance, in using the variable structure control of chapter 5, the crucial problem is to ensure the stability of the states to the chosen switching surface. Once the states reach the surface, the motions to the equilibrium point can be guaranteed by the switching surface in the sense of the sliding mode. It is clear that stability robustness is guaranteed by the motion of the first part, from anywhere off the switching surface to the switching surface, whilst performance robustness is guaranteed by the motion of the second part, from anywhere on the switching surface to the equilibrium point.

Several new concepts are developed here. These are additive compensation and multiplicative compensation, indicating two different types of controller. By additive compensation it is meant that based on a nominal control, an extra control term is added in order to compensate uncertainties, for instance, the methods of chapter 4, whilst by multiplicative compensation it is meant that an extra feedback control gain is used to replace one in the nominal control, for instance, the methods of chapter 5. These two different concepts lead to different control strategies.

Further important concepts are one phase and two phase design. The results presented in this thesis are concerned with both methodologies, so that the regulation problem and the servo problem are solved, where the former is based on one phase design, and the latter on two phase design. It should be noted that the robust tracking algorithms have been successfully used for regulation problems.

The following concluding remarks can therefore be made. Firstly, it should be emphasised here that there is no requirement for the nominal dynamics to be either stable or in some way pre-stabilised. The synthesis can be applied directly to the original system no matter whether the open loop system is stable or not. Secondly, there is no requirement for the uncertainties to satisfy the matching conditions. These conditions have been relaxed so that the condition $|q(x,\gamma,t)| < 1$ is replaced by $q(x,\gamma,t) > 0$. This difference results in a new control law which depends both on the bounds of the uncertainties in the system and on the nominal control component. Such a relaxed condition enables the technique to be extended to the following more general cases: ① where the uncertainties satisfy the matching conditions, but $q(x,\gamma) > 0$; ② where only one of the uncertainties $\Delta f(x,\gamma)$ and $\Delta g(x,\gamma)$ satisfies matching conditions; ③ where the uncertainties lie in the span of the input mapping (matching assumption), but there are no continuous functions $p(x,\gamma)$ and $q(x,\gamma)$, such that the uncertainties are of the form $\Delta f(x,\gamma) = g(x) \cdot p(x,\gamma)$, $\Delta g(x,\gamma) = g(x) \cdot q(x,\gamma)$; ④ where no matching conditions are satisfied.

Two typical forms of controller are discussed in chapters 4 and 5, in which one uses the idea of an additive control component to compensate the effect of $\Delta f(x,\gamma)$ and of the nominal control $u_1(t)$ through $\Delta g(x,\gamma)$, called here additive compensation, while the other adopts concepts from adaptive control where feedback gain is variable instead of constant, called here multiplicative compensation. Both methods can be understood as employing extra control effort to compensate for the effect of uncertainties.

One of the most important results in this thesis is the technique applied to multi-input systems. The technique developed for the single-input case has been extended to the

multi-input case without further restriction on the nature of the system or the uncertainties. To be precise, there is no requirement for decoupling the nonlinear uncertain system or decentralising the whole system into several subsystems, and no requirement for decomposing the system model into a nominal part and an uncertain part. The control law is similar to that for single-input systems, and the principle is exactly the same though more mathematical concepts are used.

The robust tracking problem is also discussed in detail for both single-input and multi-input systems, and significant developments are made. The proposed control guarantees the uniform ultimate boundedness of the closed loop system. When only matched uncertainties are present, the tracking errors can be rendered zero within a finite time, whilst when both matched and mismatched uncertainties are present, the tracking errors cannot be made zero, but converge to an open ball B_κ , and remain there.

The robustness of the proposed methods has been shown by simulation using a simple second order linear system, in which uncertainty in open loop pole location can be effectively treated even for the case where the open loop poles are assumed negative but are in fact positive, and more interestingly, the well-known non-minimum phase problem has been considered as a special kind of uncertainty and effectively controlled by the proposed techniques, particularly the techniques of chapters 5 and 7.

9.2 SUGGESTIONS FOR FURTHER WORK

In the last two decades, many researchers and designers, from such broad areas as aircraft and spacecraft control, process control, robotics, and biomedical engineering, have been concerned with the development and applications of robust control methodologies, and robustness measure bounds and synthesis techniques have been developed in the time domain as well as in the frequency domain.

The techniques described are based on the fundamental assumption that the bounds of the uncertainties are precisely known. This is not always the case. There may be some physical systems which contain uncertainties with unknown bounds or the bounds may vary from time to time depending on the working environment. In the first case, the methods cannot be used, whilst in the second case, although the largest possible bounds can be used to develop the controller, too conservative control may result. In these circumstances, an adaptive mechanism is advisable, using measured states or output values of the system, to identify the bounds of the uncertainties, and hence to determine the control feedback gains according to the methods developed here. This idea is not new, but here we only estimate the bounds of the uncertainties not the parameters of the system, resulting in easier implementation.

Another possible area of investigation is the use of output feedback alone. In many cases, although the states of systems are physically meaningful, they aren't measurable. It is therefore possible to use the following two techniques: one is the state observer, in which estimates of the system states can be obtained from measured output values by using an online state estimator, and the other is output feedback control which could be done following a similar procedure to that of state feedback control but with a proper description of the relationship between the output and the input of the system.

Finally, it may also be interesting to refine the robust tracking control strategies, which only include information about the bounds of the matched uncertainties in the control, but not that of the bounds of the mismatched part, so that the closed loop response cannot reach zero when mismatched uncertainties are present. It is possible to do this by considering the mismatched uncertainties when constructing the control, so that the effect of mismatched uncertainties on the response of the closed loop system may be reduced, and hence the open ball, to which the output will be restricted, is reduced in size.

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Appendices



Overview

In these appendices, some preliminary results, which have been used in the thesis, are provided. For simplicity, most results are mentioned in the form of theorems without proof, and some commonly used references are listed. A software package, which has been developed for simulation purposes during research, is also introduced here.



Outline

- ✓ Feedback Linearisation
- ✓ Variable Structure Control
- ✓ Matrix Theory
- ✓ A Simulation Software Package

A FEEDBACK LINEARISATION

Feedback linearisation is an approach to nonlinear control design which has attracted a great deal of research interest in recent years. The central idea of the approach is to algebraically transform nonlinear system dynamics into a (full or partial) linear equivalent of a simple form, so that well-known and powerful linear control techniques can be applied to complete the control design. More precisely, the nonlinearities in a system can be cancelled by properly chosen nonlinear feedback so that the closed loop dynamics are of linear normal form. Within this framework, the technique includes two major parts: input-state and input-output linearisation.

The feedback linearisation approach, based on differential geometric theory, is one of the most systematically developed areas in nonlinear control theory. The primary idea can be found in Porter^[1], Tokumaru *et al*^[2], Krener^[3], Brockett^[4], and significant contributions to this area were made by Su^[5], Hunt *et al*^[6], Isidori^[7], and Vidyasagar^[8]. The distinctive feature of the method is that it allows one to develop nonlinear versions of several well-known results for linear systems, such as controllability, observability *etc.* The basic tools of the method are vector fields and their derivatives.

Feedback linearisation has been successfully applied to important classes of nonlinear systems (so-called input-state linearisable minimum phase systems). There are, however, a number of shortcomings and limitations associated with the feedback linearisation approach; for instance, it does not guarantee robustness in the presence of parameter uncertainty or disturbance.

A.1 Intuitive Concepts and Mathematical Tools

Some mathematical tools from differential geometry are now introduced. To limit the conceptual and notational complexity, we discuss these concepts directly in the context of nonlinear dynamic systems (instead of general topological spaces).

A.1.1 Some Definitions of Lie Algebra

In describing the mathematical tools, we shall call a vector function $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ a vector field, which is a column vector on \mathbb{R}^n , *i.e.*,

$$f(x) = \begin{pmatrix} f_1(x_1, x_2, \dots, x_n) \\ \vdots \\ f_n(x_1, x_2, \dots, x_n) \end{pmatrix} \in \mathbb{R}^n$$

Similarly, a one form $\phi(x)$ on \mathbb{R}^n is defined as $\phi: \mathbb{R}^n \rightarrow \mathbb{R}^n$, which is a row vector, *i.e.*,

$$\phi(x) = [\phi_1(x_1, x_2, \dots, x_n), \phi_2(x_1, x_2, \dots, x_n), \dots, \phi_n(x_1, x_2, \dots, x_n)] \in \mathbb{R}^n$$

We shall only be interested in smooth vector fields (or one forms), by which we mean that each component of the function f (or ϕ) has continuous partial derivatives of any required order or is infinitely differentiable, denoted C^∞ . Evidently, the product of a one form and a vector field

$$\langle \phi, f \rangle = \sum_{k=1}^n \phi_k(x) \cdot f_k(x) \quad (\text{A.1})$$

is a scalar field of the arguments $x=(x_1, x_2, \dots, x_n)$, called the inner product.

Given a smooth scalar field $h(x)$ of the state, the gradient of h is represented by a row vector (one form) and denoted by ∇h

$$\nabla h = \frac{\partial h}{\partial x} = \left(\frac{\partial h}{\partial x_1} \quad \dots \quad \frac{\partial h}{\partial x_n} \right) \in \mathbb{R}^n \quad (\text{A.2})$$

Similarly, given a vector field $f(x)$ on \mathbb{R}^n , the Jacobian of f is represented by a $m \times n$ matrix and denoted by ∇f

$$\nabla f = \frac{\partial f}{\partial x} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \dots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \in \mathbb{R}^{m \times n} \quad (\text{A.3})$$

DEFINITION A.1: (Lie Derivative of a Scalar Field)

Given f , a C^∞ vector field on \mathbb{R}^n , and h , a C^∞ scalar field on \mathbb{R} , the Lie derivative of h with respect to f is defined as

$$L_f h(x) = \langle \nabla h(x), f(x) \rangle \quad (\text{A.4})$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product, i.e.,

$$\langle \nabla h(x), f(x) \rangle = \sum_{k=1}^n \frac{\partial h}{\partial x_k} \cdot f_k(x)$$

The Lie derivative is also a C^∞ scalar field on \mathbb{R} . Thus, one can inductively define higher order Lie derivatives as follows:

$$L_f^k h(x) = L_f[L_f^{k-1} h(x)] = \langle \nabla L_f^{k-1} h(x), f(x) \rangle \quad (k=1, 2, \dots) \quad (\text{A.5})$$

Writing $L_f^0 h(x) = h(x)$, then

$$\begin{aligned} L_f^1 h(x) &= L_f h(x) \\ &\vdots \\ L_f^k h(x) &= L_f L_f^{k-1} h(x) \end{aligned} \quad (\text{A.6})$$

Similarly, if g is another vector field, then we may define another Lie derivative as

$$L_g L_f h(x) = \langle \nabla(L_f h), g \rangle \quad (\text{A.7})$$

DEFINITION A.2: (Lie Derivative of a Vector Field)

Given f, g C^∞ vector fields on \mathbb{R}^n , the Lie bracket is defined as

$$[f, g] = \text{ad}_f(g) = \nabla g \cdot f - \nabla f \cdot g \quad (\text{A.8})$$

This is also called the Lie bracket and is also a C^∞ vector field on \mathbb{R}^n . Successive Lie brackets can be defined as follows:

$$[f, [f, [f, \dots, [f, g]]] = \text{ad}_f^k(g) \quad (k=1, 2, \dots) \quad (\text{A.9})$$

Therefore, writing $\text{ad}_f^0(g)=g$, we have

$$\begin{aligned} \text{ad}_f^1(g) &= [f, g] \\ &\vdots \\ \text{ad}_f^k(g) &= [f, \text{ad}_f^{k-1}(g)] \end{aligned} \quad (\text{A.10})$$

A.1.2 Diffeomorphisms and State Transformations

The concept of diffeomorphism in differential geometry can be viewed as a generalisation of the familiar concept of coordinate transformation.

DEFINITION A.3: (Differentiable Map with Differentiable Inverse)

A function $\psi(x): \mathbb{R}^n \rightarrow \mathbb{R}^n$, defined in a region Ω on \mathbb{R}^n , is called a diffeomorphism if it is smooth, and if its inverse ψ^{-1} exists and is smooth. Furthermore, if the Jacobian matrix of $\nabla\psi$ is non-singular at every point x in Ω , then $\psi(x)$ defines a local diffeomorphism in Ω . If the region Ω is the whole space \mathbb{R}^n , then $\psi(x)$ is a global diffeomorphism.

A global diffeomorphism is rare, and therefore one often looks for a local diffeomorphism. A diffeomorphism can be used to transform a nonlinear system into another system, which may be nonlinear or linear, in terms of a new set of states.

DEFINITION A.4: (Relative Order)

Consider a SISO nonlinear system described by a set of differential equations of the form

$$\begin{aligned} \dot{x}(t) &= F(x) + G(x)u(t) \\ y(t) &= H(x) \end{aligned} \quad (\text{A.11})$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $y \in \mathbb{R}$ are state, control and output of the system respectively, with $F(x)$ and $G(x)$ being smooth vector fields, $H(x)$ a smooth scalar field, and $F(0)=0$. If there exists a positive integer $v \leq n$ such that

$$L_G L_F^k H(x) = 0 \quad (k=0, 1, \dots, v-2) \quad (\text{A.12})$$

$$L_G L_F^{v-1} H(x) \neq 0 \quad (\text{A.13})$$

then it is said that the system has relative order (or relative degree) v .

The relative degree v of a linear system can be interpreted as the excess of poles over finite zeros in the transfer function. In particular, any linear system in which v is strictly less than n has finite zeros in its transfer function. If however $v=n$, the transfer function has no finite zeros. For nonlinear systems, the relative order simply means the number of differentiation of output $y(t)$ required for the input $u(t)$ to appear.

THEOREM A.5: (Full State Transformation)

An n^{th} -order nonlinear system of form (A.11) with relative order $v=n$, can be transformed into input-state linearisable form by a diffeomorphism defined by

$$z_k = \psi_k(x) \Leftrightarrow x = \psi_k^{-1}(z) \quad (k=1,2, \dots, n) \quad (\text{A.14})$$

with the choice of

$$\psi_k(x) = L_F^{k-1} H(x) \quad (\text{A.15})$$

A new system of the following normal form

$$\begin{aligned} \dot{z}_1(t) &= z_2(t) \\ &\vdots \\ \dot{z}_{n-1}(t) &= z_n(t) \\ \dot{z}_n(t) &= a(z) + b(z)u(t) \\ y(t) &= h(z) \end{aligned} \quad (\text{A.16})$$

results, where z is the new state representation of the system, and

$$a(z) = L_F^n H \circ \psi^{-1}(z) \quad (\text{A.17})$$

$$b(z) = L_G L_F^{n-1} H \circ \psi^{-1}(z) \quad (\text{A.18})$$

Proof: A set of diffeomorphisms of the form (A.15), $z_k = \psi_k(x) = L_F^{k-1} H(x)$ ($k=1,2,\dots,n$), exists for systems with relative order $v=n$, such that the gradient of ψ is given by

$$d\psi_k = \nabla L_F^{k-1} H(x)$$

thus

$$\dot{z}_k = \sum_{j=1}^n \frac{\partial \psi_k}{\partial x_j} \cdot [F_j(x) + G_j(x)u] \quad (k=1,2, \dots, n) \quad (\text{A.19})$$

Since $\psi(x)$ is independent of u and the system has relative order $v=n$, using notation $\langle \cdot, \cdot \rangle$, it is concluded that $\langle d\psi_k, G \rangle = 0$. Therefore

$$\dot{z}_k = \langle d\psi_k, F \rangle = L_F^k H \circ \psi^{-1}(z) = \psi_{k+1} \quad (k=1,2, \dots, n-1) \quad (\text{A.20})$$

and

$$\begin{aligned} \dot{z}_n &= \langle d\psi_n, F + Gu \rangle = \langle d\psi_n, F \rangle + \langle d\psi_n, G \rangle \cdot u \\ &= L_F^n H(x) + L_G L_F^{n-1} H(x) \cdot u(t) \\ &= L_F^n H \circ \psi^{-1}(z) + L_G L_F^{n-1} H \circ \psi^{-1}(z) \cdot u(t) \\ &= a(z) + b(z)u(t) \end{aligned} \quad (\text{A.21})$$

Thus, if ψ_1 is known, then ψ_2, \dots , and ψ_n can be found by Lie differentiation, and the system can be transformed to the linearisable nominal form (A.16). \square

THEOREM A.6: (Partial State Transformation)

For an n^{th} -order nonlinear system of form (A.11) with relative order $v < n$, define a coordinate transformation

$$(z, \zeta) = \psi(x) \Leftrightarrow x = \psi^{-1}(z, \zeta) \quad (\text{A.22})$$

which results in a new system of the following normal form:

$$\begin{aligned} \dot{z}_1(t) &= z_2(t) \\ &\vdots \\ \dot{z}_{v-1}(t) &= z_v(t) \end{aligned}$$

$$\dot{z}_v(t) = b(z, \zeta) + a(z, \zeta)u(t) \quad (\text{A.23})$$

$$\dot{\zeta}(t) = q(z, \zeta)$$

$$y(t) = h(z)$$

with the choice of the transformation ψ as follows

$$z_k = \psi_k(x) = L_F^{k-1}H(x) \quad (k=1, 2, \dots, v) \quad (\text{A.24})$$

where z and ζ are the new state representations of the system, q indicates the internal dynamics, and

$$a(z, \zeta) = L_F^n H \circ \psi^{-1}(z, \zeta) \quad (\text{A.25})$$

$$b(z, \zeta) = L_G L_F^{n-1} H \circ \psi^{-1}(z, \zeta) \quad (\text{A.26})$$

Note that, in the transformed system with (z, ζ) as new state coordinates, the first v equations are in companion form, while the last $n-v$ equations are not related to the system input u . To show that the nonlinear system can indeed be transformed into this normal form usually involves showing that the components z_k ($k=1, \dots, v$) are independent (and thus are eligible to serve as a subset of the state vector), and that $n-v$ other variables z_k ($k=v+1, \dots, n$) can be found to complete the state vector. The formal proof can be found in many references, for instance Isidori^[7].

A.2 Linearisation of SISO Nonlinear Systems

The linearisation problem for single-input single-output nonlinear systems is now considered. By linearisation we mean that a linear differential relation between the states or output and a new input v can be generated by proper design of the control law. Note that the input-state linearisation problem is usually concerned with how to define a function $\eta(x)$ such that all the states of the given nonlinear system can be completely linearised with $\eta(x)$ as the output of the system. Here we will not discuss this general problem, but only discuss the method by which a given nonlinear system with a prespecified output function $H(x)$ is linearised.

A.2.1 Input-State Linearisation

In order to proceed with a detailed study of input-state linearisation, a formal definition of this concept is necessary:

DEFINITION A.7: (Input-State Linearisation)

A single-input single-output nonlinear system of form (A.11) with $F(x)$ and $G(x)$ smooth vector fields, is said to be input-state linearisable if there exist a region Ω in \mathbb{R}^n , a diffeomorphism $\psi: \Omega \rightarrow \mathbb{R}^n$, and a nonlinear feedback control law

$$u = \alpha(x) + \beta(x)v \quad (\text{A.27})$$

such that the new state variables $z = \psi(x)$ and the new input v satisfy a linear time-invariant relation of the form

$$\dot{z} = Az + bv \quad (\text{A.28})$$

where

$$A = \begin{pmatrix} 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \vdots & 0 & \cdot & \cdot & \cdot & \vdots \\ -\alpha_0 & -\alpha_1 & \cdot & \cdot & \cdot & -\alpha_{n-1} \end{pmatrix} \quad b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \alpha_n \end{pmatrix}$$

z and v are called the linearising state and control respectively, and α_k ($k=0,1,\dots, n$) are constants to be chosen such that A is Hurwitz.

The objective now is to find a set of diffeomorphisms $z_k = \psi_k(x)$ ($k=1,2,\dots,n$) for the nonlinear system (A.11) such that the system can be transformed to be of the linearisable form, and furthermore to find a feedback control such that the system is linearised. Two questions arise when such transformations are considered; what classes of nonlinear systems can be input-state linearised, and how can a transformation be found?

THEOREM A.8: (Sufficient and Necessary Condition for Input-State Linearisation)

An n^{th} -order nonlinear system of form (A.11) is input-state linearisable if, and only if, the system has relative order $v=n$ with $H(x)$ as the output of the system.

THEOREM A.9: (Input-State Linearisation)

The nonlinear system in the form (A.11), with relative order $v=n$, can be transformed into a linearisable nominal form (A.16), and furthermore the system can be exactly linearised by state feedback of the form

$$u(t) = \frac{-\sum_{k=0}^n \alpha_k \cdot L_F^k H \circ \psi^{-1}(z) + \alpha_n v(t)}{\alpha_n \cdot L_G L_F^{n-1} H \circ \psi^{-1}(z)} \quad (\text{A.29})$$

where α_k ($k=0,1,\dots,n$) are constants with $\alpha_n=1$, such that the system will be converted to a linear one with characteristic equation

$$\sum_{k=0}^n \alpha_k \cdot \lambda^k = 0 \quad (\text{A.30})$$

where λ_k ($k=1,\dots,n$) are the eigenvalues of the linearised system.

Proof: Note here that according to the definition of the diffeomorphism (A.15)

$$L_F^{k-1} H \circ \psi^{-1}(z) = z_k \quad (k=1,2, \dots, n) \quad (\text{A.31})$$

the system is transformed to new coordinates z . It is obvious that the control (A.29) is of the form

$$u(t) = \frac{1}{\alpha_n \cdot b(z)} \left\{ -\sum_{k=0}^{n-1} \alpha_k \cdot z_{k+1} + \alpha_n \cdot [-a(z) + v(t)] \right\} \quad (\text{A.32})$$

such that the resulting closed loop system is governed by the equations

$$\begin{aligned} \dot{z}_1(t) &= z_2(t) \\ &\vdots \\ \dot{z}_{n-1}(t) &= z_n(t) \\ \dot{z}_n(t) &= -\alpha_0 z_1 - \alpha_1 z_2 - \dots - \alpha_{n-1} z_n + v(t) \\ y(t) &= h(z) \end{aligned} \quad (\text{A.33})$$

i.e., is linear and controllable. Thus it is concluded that any nonlinear system with relative order $v=n$ in Ω can be transformed into a system which, in the region Ω , can be exactly linearised by the state feedback (A.29). \square

If we choose $\alpha_k=0$ ($k=0, \dots, n-1$) in (A.29), the feedback control is then of the form

$$u(t) = \frac{-L_F^v H \circ \Psi^{-1}(z) + v(t)}{L_G L_F^{n-1} H \circ \Psi^{-1}(z)} = \frac{-a(z) + v(t)}{b(z)}$$

and therefore an n^{th} -order integrator results (Fig. A.1).

REMARK A.1:

- The input-state linearisation is achieved by a combination of a state transformation and an input transformation, with state

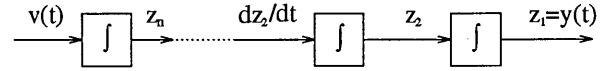


Fig. A.1 Exactly linearised system

feedback used for both. Thus it is a linearisation by feedback, or feedback linearisation, and is exact linearisation. This is fundamentally different from a Jacobian linearisation for small range operation on which linear control is based, where a curve is replaced artificially by a straight line under some presumed conditions. Such an approximation is only useful in a small neighbourhood around the operating point.

- In order to implement the control law, the new state components z_k must be available. If they are not physically meaningful or cannot be measured directly, the original state must be measured and used to compute them from (A.14).

A.2.2 Input-Output Linearisation

The problem of input-output linearisation differs from that of input-state linearisation in that it is not necessary to define a set of diffeomorphisms to transform the original nonlinear system into a new one. The linearising operation is carried out directly with the original nonlinear system, and a linear differential relation is created only between the output y and the new input v , regardless of the nonlinear relationship between states and the input of the system.

THEOREM A.10: (*Input-Output Linearisation*^[9,10])

The nonlinear system of the form (A.23) can be input-output linearised by state feedback

$$u_1(t) = \frac{-\sum_{k=0}^v \alpha_k \cdot L_F^k H(x) + \alpha_v v(t)}{\alpha_v \cdot L_G L_F^{v-1} H(x)} \quad (\text{A.34})$$

if and only if the nonlinear system has relative order $1 \leq v \leq n$. The closed loop system will be a linear differential relation between the output $y(t)$ and the new input $v(t)$. The linearised system has v eigenvalues λ_k ($k=1, 2, \dots, v$) satisfying the following characteristic equation

$$\sum_{k=0}^v \alpha_k \cdot \lambda^k = 0 \quad (\text{A.35})$$

where $\alpha_v=1$, together with $n-v$ unobservable eigenvalues λ_k ($k=v+1, \dots, n$).

Proof: The basic approach is simply to repeatedly differentiate the output function $y(t)$ until it is explicitly related to the input $u(t)$. Suppose the relative order of the system is $1 \leq v < n$, then denoting

$$y(t) = H(x) = L_F^0 H(x) \quad (A.36)$$

we differentiate the output function v times

$$\dot{y}(t) = \frac{\partial H}{\partial x} (F + Gu) = L_F H(x) + L_G H(x) u(t) = L_F^1 H(x) \quad (A.37)$$

where $L_G H(x) = 0$, and

$$\ddot{y}(t) = \frac{\partial}{\partial x} \cdot L_F^1 H(x) \cdot (F + Gu) = L_F^2 H(x) + L_G L_F H(x) \cdot u(t) = L_F^2 H(x) \quad (A.38)$$

where $L_G L_F H(x) = 0$, until

$$y^{(v)}(t) = \frac{\partial}{\partial x} \cdot L_F^{v-1} H(x) \cdot (F + Gu) = L_F^v H(x) + L_G L_F^{v-1} H(x) \cdot u(t) \quad (A.39)$$

where $L_G L_F^{v-1} H(x) \neq 0$. Then the control law of the form

$$u(t) = \frac{1}{L_G L_F^{v-1} H(x)} [-L_F^v H(x) - \sum_{k=0}^{v-1} \alpha_k \cdot L_F^k H(x) + v(t)] \quad (A.40)$$

yields the linear differential equation

$$y^{(v)}(t) + \sum_{k=0}^{v-1} \alpha_k \cdot y^{(k)}(t) = v(t) \quad (A.41)$$

The characteristic equation is therefore given by

$$\sum_{k=0}^v \alpha_k \cdot \lambda^k = 0$$

□

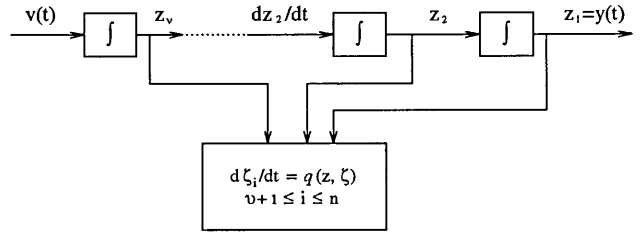


Fig. A.2 Input-Output linearised system

A.3 Zero Dynamics and Minimum Phase of Nonlinear Systems

We now introduce and discuss an important concept, zero dynamics, that in many circumstances plays a role exactly similar to that of the 'zeros' of the transfer function in a linear system.

We have already seen from theorem A.9 that, if the relative order $v=n$, a nonlinear system is completely input-state linearisable. This is not often the case in practice, particularly for the system with a prespecified output function $H(x)$. If the relative order $v < n$, this linearisation can only be achieved partially, *i.e.*, only some of the states are linearly related to the input after coordinate transformation.

The states of the original system are decomposed into two parts, z and ζ , by the transformation (A.22), in which z represents the states that are to be controlled to achieve desired output performance, and ζ represents the states that cannot be directly controlled by feedback. They are often referred to as external and internal dynamics respectively. Clearly, the stability properties of the internal dynamics are crucial because a closed loop system which appears stable may include

unstable internal dynamics. Since for linear systems the stability of the internal dynamics is simply determined by the locations of the zeros, it is interesting to see whether this relation can be extended to nonlinear systems. To do this it is necessary first to extend the concept of zero to nonlinear systems, and then to determine the relation of the internal dynamics to this extended concept of zero. A way to approach this is to define so-called zero-dynamics for a nonlinear system.

DEFINITION A.11: (Zero Dynamics)

The zero dynamics of the nonlinear system (A.11) are the dynamics of the system subject to the constraint that the external dynamics z be identically zero, *i.e.*,

$$\dot{\zeta}(t) = q(0, \zeta) \quad (\text{A.42})$$

REMARK A.2:

- The zero dynamics are an intrinsic feature of a nonlinear system, which do not depend on the choice of control law or the desired trajectories.
- Examining the stability of the zero dynamics is much easier than examining that of the internal dynamics, because the zero dynamics only involve the internal states (whilst the internal dynamics are coupled to the external dynamics).

DEFINITION A.12: (Minimum Phase)

A nonlinear system is said to be (asymptotically) minimum phase if its zero dynamics are (asymptotically) stable.

REMARK A.3:

- If the relative degree v associated with input-output linearisation is the same as the order of the system, the nonlinear system is fully linearised and this procedure leads to a satisfactory controller (assuming that the model is accurate). If the relative degree is smaller than the system order, then the nonlinear system is only partially linearised, and whether the controller can be applied depends on the stability of the internal dynamics.
- The study of the stability of the internal dynamics can be simplified locally by study of the zero dynamics instead. If the zero dynamics are unstable, different control strategies should be sought, only simplified by the fact that the transformed dynamics is partly linear.
- To summarise, control design based on input-output linearisation can be done in three steps: ① differentiate the output $y(t)$ until the input $u(t)$ appears; ② choose $u(t)$ to cancel the nonlinearities and guarantee the stability of the system; ③ study the stability of the internal dynamics.

A.4 Linearisation of MIMO Nonlinear Systems

The concepts discussed previously for SISO systems, such as input-state linearisation, input-output linearisation, normal form, and zero dynamics, can be extended to MIMO systems. For the

MIMO case, we consider the transformation of square systems, *i.e.*, systems with the same numbers of inputs and outputs. Such a transformation is now briefly discussed.

DEFINITION A.13: (*Relative Order of MIMO Systems*)

For the multivariable nonlinear system of form

$$\dot{x}(t) = F(x) + G(x)u(t) \quad (\text{A.43})$$

$$y_1(t) = H_1(x)$$

$$\vdots$$

$$y_m(t) = H_m(x)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$. The system is said to have relative order (v_1, v_2, \dots, v_m) if

$$L_{G_j} L_F^k H_i(x) = 0 \quad (k=0, 1, \dots, v_i - 2 \quad 1 \leq i \leq m, 1 \leq j \leq m) \quad (\text{A.44})$$

and the following matrix

$$\Pi = \begin{pmatrix} L_{G_1} L_F^{v_1-1} H_1(x) & \dots & L_{G_m} L_F^{v_1-1} H_1(x) \\ \vdots & & \vdots \\ L_{G_1} L_F^{v_m-1} H_m(x) & \dots & L_{G_m} L_F^{v_m-1} H_m(x) \end{pmatrix}$$

is non-singular. The total relative order of the system is defined by

$$v = \sum_{k=1}^m v_k \leq n \quad (\text{A.45})$$

How a normal form can be obtained for the system in a manner similar to the SISO case, is now shown.

THEOREM A.14: (*Coordinate Transformation^[5,8]*)

For the multivariable nonlinear system of form (A.43), if the system has relative order v where $1 \leq v \leq n$, then there exists a coordinate transformation

$$z = \psi(x) \quad \Leftrightarrow \quad x = \psi^{-1}(z)$$

Such a transformation leads to a new system representation with coordinates (z, ζ) , where, related to m inputs, z can be decomposed into m sets z_i , and each of them consists of v_i states of form

$$z_{i,k}(t) = \psi_{i,k}(x) = L_F^{k-1} H_i \circ \psi^{-1}(z) \quad (k=1, 2, \dots, v_i, i=1, 2, \dots, m) \quad (\text{A.46})$$

Specifically

$$\begin{array}{ccccccc} z_{1,1} = H_1(x) & \dots & \dots & \dots & z_{1,v_1} = L_F^{v_1-1} H_1(x) \\ \vdots & & & & \vdots \\ z_{m,1} = H_m(x) & \dots & \dots & \dots & z_{m,v_m} = L_F^{v_m-1} H_m(x) \end{array}$$

These are simply the outputs y_i and their derivatives up to v_i ($i=1, \dots, m$). Such a choice of new state vectors enables us to write the external dynamics of the system as follows:

$$\begin{aligned} \dot{z}_{i,1}(t) &= z_{i,2}(t) \\ &\vdots \\ \dot{z}_{i,v_i-1}(t) &= z_{i,v_i}(t) \\ \dot{z}_{i,v_i}(t) &= a_i(z, \zeta) + \sum_{j=1}^m b_{i,j}(z, \zeta) \cdot u_j(t) \end{aligned} \quad (\text{A.47})$$

$$y_i(t) = h_i(z) \quad (\text{A.48})$$

where

$$\begin{aligned} a_i(z, \zeta) &= L_F^{v_i} H_i \circ \Psi^{-1}(z) \\ b_{i,j}(z, \zeta) &= L_{G_j} L_F^{v_i-1} H_i \circ \Psi^{-1}(z) \end{aligned} \quad (i=1, 2, \dots, m)$$

The internal dynamics are of the form

$$\dot{z}(t) = q(z, \zeta) + p(z, \zeta)u(t) \quad (\text{A.49})$$

with $k=1, \dots, n-v$ and $i=1, \dots, m$

$$\begin{aligned} q_k(z, \zeta) &= L_F \zeta_k(x) \\ p_{k,i}(z, \zeta) &= L_{G_i} \zeta_k(x) \end{aligned}$$

DEFINITION A.15: (*The Involutive Property*)

The set of m linearly independent vector fields $g(x)$ is said to be a m -dimensional distribution, and if it is possible to write

$$[g_i, g_j](x) = \sum_{k=1}^m \gamma_{i,j,k}(x) \cdot g_k(x) \quad (1 \leq i, j, k \leq m) \quad (\text{A.50})$$

then the m -dimensional distribution is said to be involutive.

The concept of involution implies the solvability of a set of partial differential equations.

THEOREM A.16: (*Condition on Internal Dynamics*^[7,11])

The internal dynamics of multivariable nonlinear systems are usually of the form

$$\dot{z}(t) = q(z, \zeta) + p(z, \zeta)u(t) \quad (\text{A.51})$$

If the vector fields g_1, \dots, g_m are involutive, then

$$p_{k,i}(z, \zeta) = L_{G_i} \zeta_k(x) = 0 \quad (1 \leq i \leq m, 1 \leq k \leq n-v)$$

hold. It follows that the internal dynamics are of the form

$$\dot{\zeta}(t) = q(z, \zeta)$$

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B VARIABLE STRUCTURE CONTROL

The variable structure control approach was first introduced in the 1950's by the Russian scientist, Utkin, and surveyed in Utkin^[1,2], and has been well developed during the last three decades by many contributors, see Zak^[3,4,5]. The fundamental feature of this methodology is based on the fact that once the system trajectory reaches a prespecified surface, $\sigma(x) \in \mathbb{R}^m$ in the state space, the system will move or slide towards the equilibrium point along this surface. Here the term 'surface' represents a manifold in state space of lower dimensionality than the state space itself. The performance of the system therefore depends only on the structure of the surface, and remains insensitive to parameter variations and disturbances off the surface. All that is needed during the design is to choose a desired switching surface and to guarantee that the system output converges to this surface from anywhere in the admissible region Ω of state space, and to guarantee that the desired sliding motion exists, under the proposed control.

Two crucial problems arise: ① how to construct a continuous function, which is accessible, with unique desired equilibrium point, such that the system behaves according to some properties prescribed by the function; ② how to design a controller with switched feedback gain, such that the state can be driven towards the chosen surface from anywhere in the admissible region Ω of state space; *i.e.*, the stability of the state trajectory to the switching function is required.

We will consider nonlinear systems of the form

$$\dot{x}(t) = f(x) + g(x)u(t) \quad (\text{B.1})$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ are the state and control of the system respectively.

B.1 Sliding Mode

An important feature of variable structure systems is the sliding mode, by which we mean that, under some circumstances, the state trajectory of the system slides over a demanded surface despite disturbances acting on the system.

B.1.1 Switching Surface

A switching function $\sigma(x)=0$ is therefore required.

DEFINITION B.1: (*Switching Surface*)

For the system of (B.1), if there exists a surface

$$\sigma(x) = \begin{pmatrix} \sigma_1(x_1, x_2, \dots, x_n) \\ \vdots \\ \sigma_m(x_1, x_2, \dots, x_n) \end{pmatrix} \in \mathbb{R}^m$$

independent of any uncertain elements in the system, such that it is accessible by the states of the system from either side of it under the proposed control, then it is called a switching function or a switching surface.

By properly choosing a switching surface, which may be either linear or nonlinear, desired behaviour of the closed loop system consisting of a set of states

$$\bar{X} = \{x | \sigma(x, x_2, \dots, x_n) = 0\} \quad (B.2)$$

results. Here \bar{X} is used to indicate a set of states which are on the switching surface $\sigma(x)=0$, in order to distinguish them from another set X , consisting of all states off the switching surface, i.e., $\sigma(x) \neq 0$.

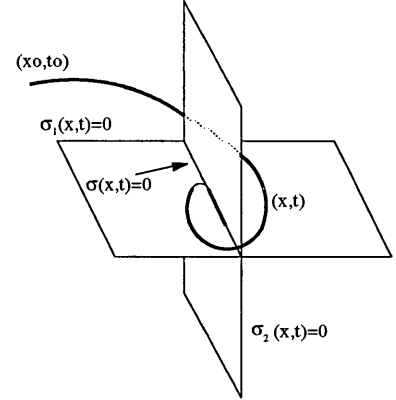


Fig. B.1 Illustration of the intersection of two switching surfaces

B.1.2 Sliding Mode

After switching surface design, the next important aspect of variable structure control is guaranteeing the existence of a sliding mode. Under variable structure control, the real trajectory of the closed loop system is obtained by composing a desired trajectory from the parts of trajectories of different structures corresponding to different control actions. Such a motion along $\sigma(x)=0$, which is not a trajectory of any of the structures, is called the sliding mode. A sliding mode exists, if, in the vicinity of the switching surface, $\sigma(x)=0$, the tangent or velocity vectors of the state trajectory always point toward the switching surface. Then if the state trajectory intersects the switching surface, it remains within a neighbourhood of the region $\{x | \sigma(x)=0\}$.

DEFINITION B.2: (Sliding Mode Domain)

A domain Ω in $\sigma(x)=0$ is a sliding mode domain if for each $\epsilon > 0$, a $\delta > 0$ exists such that any motion starting within an n -dimensional δ -vicinity of Ω may leave the n -dimensional ϵ -vicinity of Ω only through the boundary of the n -dimensional ϵ -vicinity intersected with Ω .

Existence of a sliding mode requires stability of the state trajectory to the sliding surface $\sigma(x)=0$ at least in a neighbourhood of $\{x | \sigma(x)=0\}$, i.e., the representative point must approach the surface at least asymptotically. The largest such neighbourhood is called the domain of attraction. Consequently, whenever the state trajectory intersects the switching surface, if the value of the state trajectory remains within an ϵ neighbourhood of $\bar{X} = \{x | \sigma(x)=0\}$, then a sliding mode occurs. If a sliding mode exists on $\sigma(x)=0$, then it is termed a sliding surface.

B.1.3 Reachability Condition

The existence problem of sliding mode resembles a generalised stability problem, hence the Lyapunov direct method provides a natural setting for analysis. Specifically, stability to the

switching surface requires selecting a generalised Lyapunov function $V(t)$ which is positive definite and has a negative time derivative in the region of attraction.

DEFINITION B.3: (Generalised Lyapunov Function)

A continuous function $V(t)$, which depends on the chosen switching function $\sigma(x)$, can be defined as a generalised candidate Lyapunov function, if

$$V(t) \triangleq \frac{1}{2} \cdot \sigma^T(x) \cdot \sigma(x) > 0 \quad \forall (x, t) \ni \sigma(x) \neq 0 \text{ and } V|_{\sigma(x)=0} = 0 \quad (\text{B.3})$$

with continuous derivative, such that, for $x = \{x(t) \in \mathbb{R}^n \mid \sigma(x) \neq 0, x(t_0) = x_0\}$

$$\dot{V}(t) = \frac{1}{2} \cdot \frac{d}{dt} \sigma^T(x) \cdot \sigma(x) = \sigma^T(x) \cdot \dot{\sigma}(x) < 0 \quad (\text{B.4})$$

holds.

DEFINITION B.4: (Reachability Condition)

For any accessible continuous function $\sigma(x)=0$, a sliding mode exists if, and only if, for $x = \{x(t) \in \mathbb{R}^n \mid \sigma(x) \neq 0, x(t_0) = x_0\}$

$$\sigma^T(x) \cdot \dot{\sigma}(x) < 0 \quad (\text{B.5})$$

or

$$\begin{cases} \dot{\sigma}_i(x) < 0 & \sigma_i(x) > 0 \\ \dot{\sigma}_i(x) > 0 & \sigma_i(x) < 0 \end{cases} \quad (i=1, \dots, m) \quad (\text{B.6})$$

holds, where

$$\dot{\sigma}(x) \triangleq \frac{d\sigma(x)}{dt} = \nabla \sigma(x) \cdot \dot{x}(t)$$

This is called the reachability condition.

Graphically, off the switching surface, if $V(t) > 0$ and $\dot{V}(t) = \sigma \cdot \dot{\sigma} < 0$, the reachability condition holds. The trajectory can therefore move while still pointing towards the surface until reaching it. This guarantees that the system state trajectory will approach the switching surface and tend to remain there. From the above discussion, it becomes clear that variable structure control

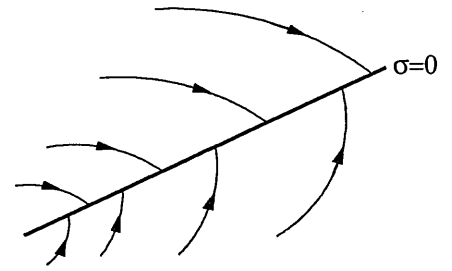


Fig. B.2 Illustration of sliding conditions

design can be divided into two phases. In phase one, the switching surface is constructed so that the system restricted to the switching surface produces the desired behaviour. Phase two entails constructing switched feedback gains which drive the system state trajectory to the switching surface and maintain it there.

B.2 Design of the Switching Surface

We now consider the problem of switching surface construction for nonlinear systems of the form (B.1).

B.2.1 Equivalent Control

The method of equivalent control is a means for determining the system motion restricted to the switching surface $\sigma(x)=0$.

DEFINITION B.5: (Equivalent Control)

For a chosen switching surface $\sigma(x)=0$, a feedback control of the form

$$u_{eq}(t) = -[\nabla\sigma(x) \cdot g(x)]^{-1} \cdot \nabla\sigma(x) \cdot f(x) \quad (B.7)$$

is said to be the equivalent control to the system (B.1) in sliding mode, if $\nabla\sigma \cdot g$ is non-singular. Here $\nabla\sigma$ is the Jacobian of σ .

The existence of the sliding mode implies both $\sigma(x)=0$ and $\dot{\sigma}(x)=0 \forall t \geq t_0$. Therefore

$$\dot{\sigma} = \nabla\sigma(x) \cdot \dot{x} = \nabla\sigma(x) \cdot [f(x) + g(x) \cdot u(t)] = 0$$

Clearly (B.7) solves this equation, and it is this which gives it the name equivalent control. It can also be expressed in terms of the Lie derivative as follows:

$$u_{eq}(t) = -(L_g \sigma)^{-1} \cdot L_f \sigma$$

B.2.2 Reduction of Order

Although general nonlinear switching surfaces are possible, it may be appropriate to seek linear ones in design. Moreover, for a large class of systems, design of linear switching surfaces proves amenable to classical linear control techniques. Thus for clarity, convenience, and simplicity, we may consider switching surfaces of the form

$$\sigma(x) = S \cdot x(t) = 0 \quad (B.8)$$

where S is a $m \times n$ matrix.

In sliding mode, the equivalent system must satisfy not only the n -dimensional state dynamics (B.1), but also the m algebraic equations $\sigma(x)=0$. The use of both constraints reduces the system dynamics from an n^{th} order model to an $(n-m)^{\text{th}}$ order one. Specifically, suppose the nonlinear system is of the form (B.1) subjected to $\sigma(x)=0$, then, from $\sigma(x)=S \cdot x(t)=0$, it is possible to solve for m of the state variables in terms of the remaining $n-m$, if $\text{rank}[S]=m$. To obtain the solution, substitute these relations into the remaining $n-m$ equations and the equations corresponding to the m state variables. The resultant $(n-m)^{\text{th}}$ order system fully describes the equivalent system subject to the restriction of $\sigma(x)=0$.

B.2.3 Regular Form and Reduced Order Dynamics

The regular form of the nonlinear system (B.1) is defined by^[3,5]

$$\begin{aligned} \dot{x}^1(t) &= f^1(x^1, x^2) \\ \dot{x}^2(t) &= f^2(x^1, x^2) + g^2(x^1, x^2) \cdot u(t) \end{aligned} \quad (B.9)$$

where $x^1 \in \mathbb{R}^{n-m}$ and $x^2 \in \mathbb{R}^m$ are subsets of the system states x , f^1 , f^2 are $n-m$ and m smooth vectors

respectively, and g^2 is a $m \times m$ non-singular smooth matrix. This regular form can normally be obtained by using a properly chosen transformation to rearrange the order of the original states and hence the state and input mappings of the original system.

THEOREM B.6: (Switching Surface)

For the nonlinear system of regular form (B.9), the switching surface can be generally defined as

$$\sigma(x^1, x^2) = \sigma_1(x^1) - S_2 \cdot x^2 = 0 \quad (B.10)$$

so that an $(n-m)^{\text{th}}$ (reduced) order closed loop system results under the equivalent control

$$u_{eq} = [S_2 \cdot \tilde{g}^2(x^1)]^{-1} \cdot [\nabla \sigma_1(x^1) \cdot \tilde{f}^1(x^1) - S_2 \cdot \tilde{f}^2(x^1)] \quad (B.11)$$

Here S_2 is an $m \times m$ non-singular matrix, and $\sigma_1(x^1)$ is a smooth function to be chosen by the designer such that the $(n-m)^{\text{th}}$ reduced order system has desired dynamics.

Proof: For the system of (B.9) on the switching surface, we have

$$\sigma(x^1, x^2) = \sigma_1(x^1) - S_2 \cdot x^2 = 0$$

$$x^2 = S_2^{-1} \cdot \sigma_1(x^1)$$

Therefore on the switching surface, i.e., $\sigma(x^1, x^2)=0$, the system can be written as

$$\dot{x}^1(t) = f^1(x^1, S_2^{-1} \cdot \sigma_1(x^1)) \triangleq \tilde{f}^1(x^1)$$

$$\dot{x}^2(t) = f^2[x^1, S_2^{-1} \cdot \sigma_1(x^1)] + g^2[x^1, S_2^{-1} \cdot \sigma_1(x^1)] \cdot u(t) \triangleq \tilde{f}^2(x^1) + \tilde{g}^2(x^1) \cdot u(t)$$

and again we have

$$\dot{\sigma}(x^1, x^2) = \nabla \sigma_1(x^1) \dot{x}^1 - S_2 \cdot \dot{x}^2 = \nabla \sigma_1(x^1) \cdot \tilde{f}^1(x^1) - S_2 \cdot [\tilde{f}^2(x^1) + \tilde{g}^2(x^1) \cdot u(t)] = 0$$

So the equivalent control is

$$u_{eq} = [S_2 \cdot \tilde{g}^2(x^1)]^{-1} \cdot [\nabla \sigma_1(x^1) \cdot \tilde{f}^1(x^1) - S_2 \cdot \tilde{f}^2(x^1)]$$

This results in a closed loop system of the form

$$\dot{x}^1(t) = f^1[x^1, S_2^{-1} \cdot \sigma_1(x^1)] \triangleq \tilde{f}^1(x^1) \quad (B.12)$$

We can specify the performance of this closed loop system by properly choosing the matrix S_2 . Suppose now that we want the system, when restricted to the switching surface $\sigma(x)=0$, to behave in a linear (reduced order) fashion. The reduced order dynamics are

$$\dot{x}^1(t) = \tilde{f}^1(x^1) = A_{n-m} \cdot x^1 \quad (B.13)$$

where

$$A_{n-m} = \begin{pmatrix} 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & 0 & \cdot & \cdot & \cdot & 1 \\ -\alpha_1 & -\alpha_2 & \cdot & \cdot & \cdot & -\alpha_{n-m} \end{pmatrix}$$

Then $\sigma_1(x^1)$ can be solved according to the equation above such that the desired system dynamics are achieved, and the switching surface is therefore

$$\sigma(x^1, x^2) = \sigma_1(x^1) - S_2 \cdot x^2 = 0 \quad (B.14) \quad \square$$

B.3 Synthesis of Controller with Switched Feedback Gain

The objective of the control is to make the state trajectory of the system converge to a chosen switching surface and remain there so that a sliding mode occurs. The state trajectory of a variable structure system will, in general, consist of two parts: a trajectory which is off the switching surface but approaches it, and one on it. The designed control must guarantee that both parts of the trajectories show satisfactory performance. More specifically, the control must first force the trajectory, in a desired manner, to approach the switching surface whenever the states are off the switching surface; on the other hand, it should also guarantee that the trajectory 'slides along' this surface to the equilibrium point once the trajectory reaches the surface. The first task can be achieved by applying a properly designed control to the system such that stability to the switching surface exists, while the second task can be achieved by defining a desired switching surface such that the trajectory will approach the unique equilibrium point whilst remaining on the switching surface.

An ideal sliding mode exists only when the state trajectory of the controlled system satisfies $\sigma(x)=0 \quad \forall t \geq t_0$. This requires infinitely fast switching in order to account for the presence of uncertainties. This, of course, is not possible because of such things as delay, hysteresis, *etc.*, which cause switching to occur at a finite rate. The trajectory may then not exactly rest on the switching surface, but swings across it within a small region. This oscillation is called chattering. Chattering is, in general, highly undesirable. This situation can be remedied by smoothing out the control discontinuities in a boundary layer neighbouring the switching surface. On the other hand, any disturbance acting on the system or parameter uncertainty may also make the states not exactly rest on the switching surface, such that the actual trajectory does not move along the switching surface perfectly but moves across it within a vicinity of it. Therefore, in actual variable structure control, the sliding mode represents an idealisation.

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C MATRIX THEORY

We now present some definitions and preliminary mathematics which are used in chapters 6 and 7 to develop the main results. For simplicity, we only mention the results and avoid the proofs which can be found in the references^[1,2,3].

DEFINITION C.1: (*Definiteness of Matrices*^[2])

Let $\langle \cdot, \cdot \rangle$ be an inner product. Then matrix $A \in \mathbb{R}^{n \times n}$ is

- ① positive definite (p.d.) or negative definite (n.d.) with respect to $\langle \cdot, \cdot \rangle$ if

$$\operatorname{Re} \langle z, Az \rangle > 0 \text{ or } < 0 \quad \forall z \neq 0 \quad (\text{C.1})$$

- ② positive semidefinite or negative semidefinite with respect to $\langle \cdot, \cdot \rangle$ if

$$\operatorname{Re} \langle z, Az \rangle \geq 0 \text{ or } \leq 0 \quad \forall z \neq 0 \quad (\text{C.2})$$

Note that this definition, differing from the usual form for definiteness of matrices in most references, applies to general matrices that are not necessarily Hermitian. Particularly to matrices that are real but not symmetric, we have the following theorem:

THEOREM C.2: (*Definiteness of Square Matrices*^[3])

Any real square $n \times n$ matrix A can be expressed as the sum of a symmetric matrix and a skew-symmetric matrix

$$A = (A + A^T)/2 + (A - A^T)/2 \quad (\text{C.3})$$

- ① The quadratic form associated with a skew-symmetric matrix is always zero;
 ② The quadratic form of any square matrix A can be represented by that of a symmetric matrix.

In what follows, by saying that a square matrix is positive definite, we always mean that the quadratic form of its symmetrised form is positive definite.

DEFINITION C.3: (*Spectral Norm of Matrices*^[4])

For any matrix A , the spectral norm (greatest singular value) is defined by

$$\phi_M(A) = \|A\|_s = [\lambda_{\max}(AA^T)]^{1/2} \quad (\text{C.4})$$

where $\lambda(A)$ indicates the eigenvalues of A . When A is a symmetric matrix, all the eigenvalues of A are real and

$$\phi_M(A) = \|A\|_s = [\lambda_{\max}(AA^T)]^{1/2} = \lambda_{\max}(A) \quad (\text{C.5})$$

When A is a symmetric positive definite matrix, then all the eigenvalues of A are positive and real, thus

$$\phi_M(A) = \|A\|_s = [\lambda_{\max}(AA^T)]^{1/2} = \lambda_{\max}(A) \quad (\text{C.6})$$

$$\phi_M(A) = \|A^{-1}\|_s = 1/\lambda_{\min}(A) \quad (\text{C.7})$$

The singular values of a matrix have many analogies with the eigenvalues of a Hermitian matrix. The square of the singular value of A is the maximum eigenvalue of $A^T A$. Unlike eigenvalues, singular values can be used to study rectangular matrices; they are also always real, and less sensitive to parameter variation than are eigenvalues.

LEMMA C.4:

For matrices $A, B \in \mathbb{R}^{n \times n}$

- ① if A is positive definite, then $\text{tr}(A)$, $\det(A)$, $\lambda(A)$, and all principal minors are positive;
- ② if A is positive definite, then A^{-1} exists and is also positive definite;
- ③ if A, B and $A-B$ are positive definite, then $B^{-1}-A^{-1}$ exists and is also positive definite;
- ④ if B is positive definite and A is any non-singular matrix, then $(A^T B A)^{-1}$ exists and is positive definite;
- ⑤ if B is symmetric, then $(A^T B A)^{-1}$ is also symmetric.

LEMMA C.5:

For a positive definite matrix $A \in \mathbb{R}^{n \times n}$ and a Hermitian matrix $B \in \mathbb{R}^{n \times n}$,

- ① the product $A \cdot B$ is a diagonalizable matrix, all of whose eigenvalues are real;
- ② the matrix $A \cdot B$ has the same number of positive, negative and zero eigenvalues as B , i.e., $A \cdot B$ has the same inertia as B

$$\text{In}(AB) = \{i_+(B), i_-(B), i_0(B)\} \quad (\text{C.8})$$

LEMMA C.6:

For matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times m}$ with $m \leq n$, $BA \in \mathbb{R}^{n \times n}$ has the same eigenvalues as $AB \in \mathbb{R}^{m \times m}$, counting multiplicity, together with additional $n-m$ eigenvalues equal to zero; that is,

$$p_{BA}(\lambda) = \lambda^{n-m} \cdot p_{AB}(\lambda) \quad (\text{C.9})$$

where $p(\cdot)$ is the characteristic polynomial of the matrix.

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D A SIMULATION SOFTWARE PACKAGE

A brief introduction to simulation software, which was written for the purposes of simulation of single-input single-output systems, is given here.

D.1 Introduction

Simulation tools are the most widely developed and available aids for CAD in control systems. Although there are several software packages available for simulation purposes, for instance MATLAB, they are not always appropriate. This package is developed for simulation purposes. It is written on an IBM PC in the C language. It can be used to simulate single-input single-output nonlinear uncertain systems with general analytic mathematical models. Several control strategies have been included in the package, and new techniques can also be included in the package by slightly changing the program. It is convenient to employ it as a tool when determining the parameters of controllers and comparing the performances of different control techniques.

The features can be summarised as follows:

- ① Multi-menus are adopted to set-up all simulation parameters, and to view the structure and parameters of the model;
- ② A special type file is used to describe the system models. The package can parse the model file and translate it into program code which can be understood and executed by the package;
- ③ A small editor is included in the package so that the model file can be revised on-line and the model parameters can be changed during simulation;
- ④ Several control strategies have been included in the package so that comparisons may be made;
- ⑤ Uncertainty bounds can be set to any values before starting simulation, and can be reset at any time, in order to view the robustness of the selected controllers;
- ⑥ Graphics can be shown simultaneously when the simulation is running so that the transient process of the simulated system can be viewed at any time;
- ⑦ A data file will be created on disk once the simulation is finished. It can be used for other purposes, for instance, drawing a graph either using other graphics software packages or this package.

D.2 The Structure of the Package

The package consists of four parts which will be introduced as follows:

- | | |
|-------------------------|------------------------------------|
| 1. menu system; | 2. model editor and parser; |
| 3. graphics subroutine; | 4. closed loop system computation. |

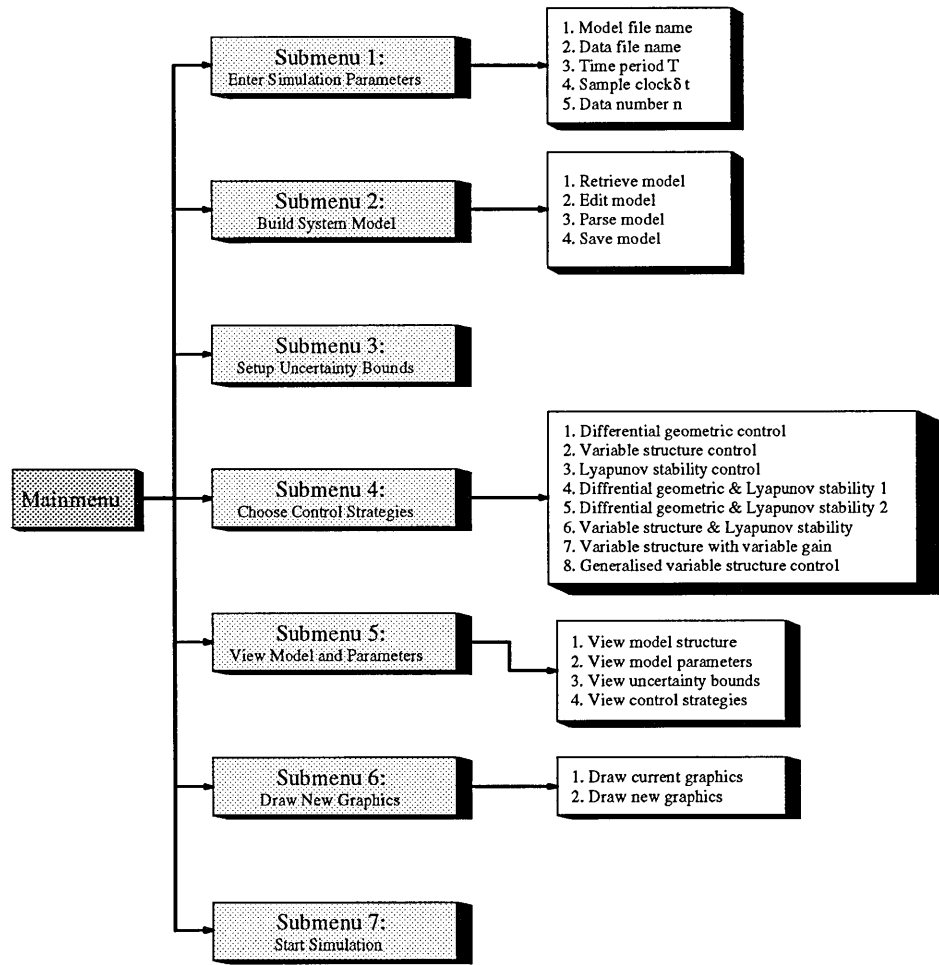


Fig. D.1 The menu system structure of the package

The package has been tested using several illustrative examples. Although there is sometimes a memory problem which remains to be solved, it does work correctly, and it has been found to be useful.

Fig. D.1, D.2 and D.3 show the structure of the menu system of this package.

D.2.1 Menu Structure

The menu system consists of two levels of menus: mainmenu and submenus, which are always shown on screen and provide a convenient means to establish the simulation environment at the start. There are 7 sets of commands, which are organised as submenus.

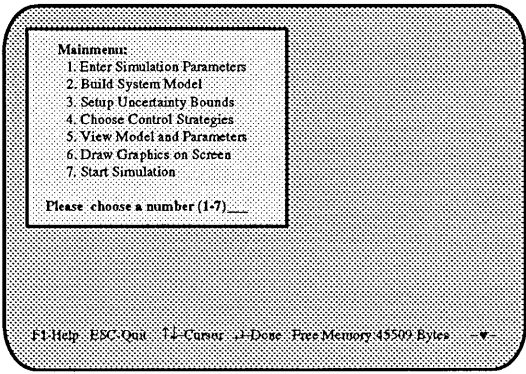


Fig. D.2 Main menu

The usage and features of these menus are now introduced.

SUBMENU1: Enter simulation parameters

First of all, a model file must be indicated so that the package can load it from disk. The model files usually have an extension of .mod. A data file name should also be given so that a data file will be created after simulation is finished. For default, the data file has the same name as the model file with an extension of .dat. Besides, before starting simulation, we should give the following parameters: how long the simulation will last (time period T), how big the computing time interval will be (sample clock δt), and how many data points will be picked up to be recorded in the data file.

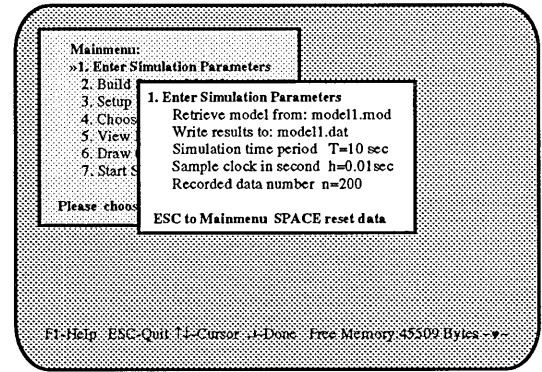


Fig. D.3 (i). Enter simulation parameters

SUBMENU2: Build system model

Load a system model from the model file which may be edited by using any word processor such as PCWRITE, Turbo C editor, etc., and show the model structure after loading successfully. The model structure and parameters can also be changed using a small built-in editor if necessary. The correct model can be saved on disk to update the original one. Once the model has been built up correctly, it should be parsed into executable codes. A small parsing program is already included in the package to translate the model file into a special kind of program which is executable during simulation. Any parsing information will be shown on screen.

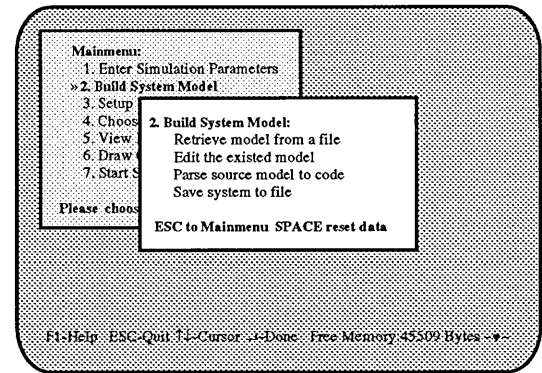


Fig. D.3 (ii). Build system model

SUBMENU3: Set up uncertainty bounds

The uncertainty bounds can be set up before starting simulation under this menu, which includes the bounds of uncertainties both in the state mapping and the input mapping, as well as the bound on external disturbances, which are of the form

$$\omega_{\Delta f} = \max |\Delta f(x, \gamma, t)|$$

$$\omega_{\Delta g} = \max |\Delta g(x, \gamma, t)|$$

$$\omega_{\xi} = \max |\xi(t)|$$

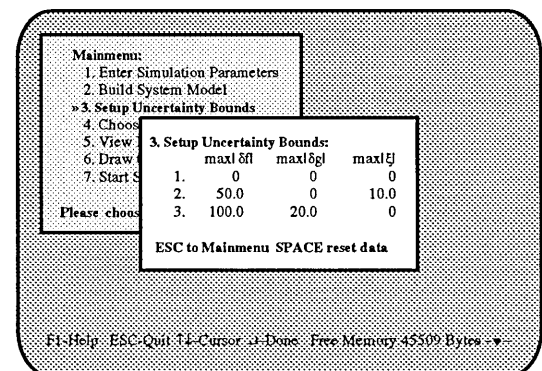


Fig. D.3 (iii). Setup uncertainty bounds

SUBMENU4: Choose control strategies

There are currently 8 control strategies available in the package. All of them are designed for simulation of single-input single-output systems. Four of them are related to differential geometric control theory and the improved versions, and the others to variable structure control theory. Any one of them can be selected to carry out the simulation, and can be compared with others. In principle, any other control techniques in the time domain can also be included by slightly changing the program.

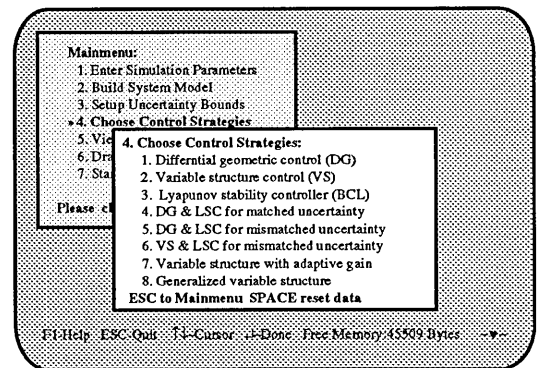


Fig. D.3 (iv). Choose control strategies

SUBMENU5: View model and parameters

After setting up all parameters, we have a chance to view the model which has been established previously before starting simulation. Model structure, model parameters, uncertainty bounds, and control strategies can be checked, in order to make sure that all parameters are correctly given.

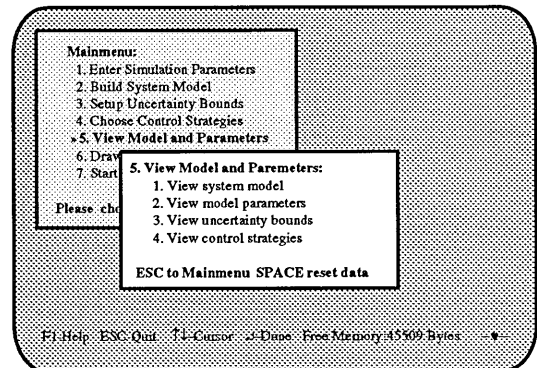


Fig. D.3 (v). View model and parameters

SUBMENU6: Draw graphics on screen

This submenu provides us with a tool in graphics mode either to review the results recorded on the data file earlier or the simulation results currently obtained. If there are more than one series of data, we can choose any number of curves and points for each curve to redraw graphics on screen.

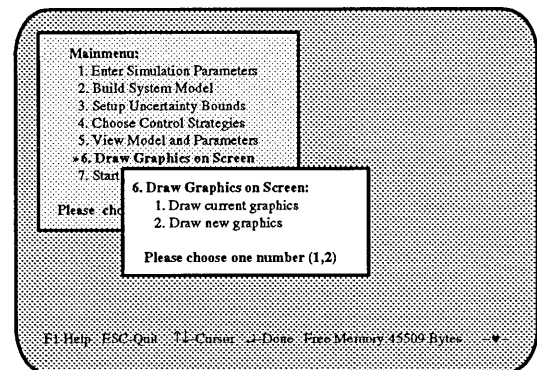


Fig. D.3 (vi). Draw graphics on screen

SUBMENU7: Start simulation

After setting up the system model and all parameters required, simulation is started from here. The package will carry out simulation according to the model, the parameters of the model, and the control strategies which have been chosen.

D.2.2 Model Editor and Parser

Model editing and parsing are two of the most important features of the package.

With the help of this, the package can be used for the simulation of any kind of nonlinear uncertain system model and any order system model, assuming there are no memory limitations.

The model file can be written in the form shown in Table D.1, using any word processor which can produce text files, for instance, PCWRITE, Turbo Editor. After loading the given model file, we can edit it using a small built-in editor in order to change the model structure or model parameter values. The uncertainty bounds can also be changed at this stage. It is especially convenient if we want to carry out simulation for a newly developed controller, because it is easy to change the parameters and compare the behaviour of the controller for different parameters.

The model file parser is another important feature of this package. It will translate the model file, which consists generally of many constants, variables, functions and expressions, into executive code, a special type of character string, which can only be recognised by the computation subroutine afterwards.

The translation process actually decomposes the normal expressions into many small elements which indicate one of the following simple operations between two operands, addition, subtraction, multiplication, division, exponentiation and assignment of values. The executable codes are shown in Table D.2.

When parsing the model, the memory can be allocated dynamically according to the size and complexity of the system model loaded into memory.

Table D.1

```
1.Model_dimension:
m=2
2.Initial_values:
x0={5,0}
3.Uncertainty_bounds:
dfmax={0,20.0}
dgmax={0.0,2}
Dmax={0,4}
4.Lyapunov_matrix:
lyap={15.0,3,3,3}
phai={2,3,1}
5.Model_parameters:
k_1=2
.
.
6.System_model_matrix:
f(1)=x2
.
.
7.Switching_Function:
.
.
8.Control_action:
.
.
```

Table D.2

```
&2>#0;&1@$14>#17;
-#20*#17>#1;0>#2;
&21>#3;&22*#30>#4;
5.0*&0>#18;
#18@$6>#19;
&1@$6>#20;
&23*#19*#20>#5;
&24>#6;&25>#7;
&26*#30>#8;
&27*#30>#9;
-&28*&1>#21;
&29*&2>#22;
#21-#22>#10;
-&28>#11;
-&29>#12;&1>#13;
&2>#14;&1@$14>#23;
-&20*#23>#15;
&21>#16;!
```

D.2.3 Graphics Subroutine

The package provides a function to show graphics when simulation is in progress. This makes it easier to understand what is happening. The slightly difficult task in this stage is to find a correct graphics ratio during simulation because we have no prior knowledge of the value range of the system output. This subroutine possesses the capacity to find a suitable ratio for the fixed graphics box at any simulation time to guarantee that graphics can always be drawn properly.

D.2.4 Closed Loop System Computation

This is used to do simulation according to the parameters set up. If we think of the executive codes as a special kind of program, then this part can be regarded as a small interpreter, which tells the computer what to do and where to put the results. After simulation is finished, it saves the results in a prespecified data file which can be used either by the package or by other graphics packages, for example, HG, FL., to draw graphics and so on.

D.3 The Block Diagram of the Package

The block diagram of this simulation package is shown in Fig. D.4.

D.4 Example

A 3rd-order nonlinear uncertain system is considered here for illustrative purposes.

D.4.1 System Model

$$\dot{x}(t) = \begin{pmatrix} x_2 \\ K_1 \sin(x_1) + K_2 x_3 \\ K_3 x_2 + K_4 x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ K_5 \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ K_6 \cos(5t) \cos(x_1) \\ K_7 x_2 + K_8 x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ K_9 \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ e(t) \\ 0 \end{pmatrix}$$

Initial values: $x_0 = \{\pi, 0, 0\}$

Uncertainty bounds:

$$\max |\Delta f(\gamma)| = [0, 50, 100]^T$$

$$\max |\Delta g(\gamma)| = [0, 0, 20]^T$$

$$\max |\xi(t)| = [0, 10, 0]^T$$

Switching function:

$$\sigma(x) = -K_1 \sin x_1 - a_1 x_1 - a_2 x_2 - a_3 x_3 = 0$$

$$(a_1=2, a_2=3, a_3=1)$$

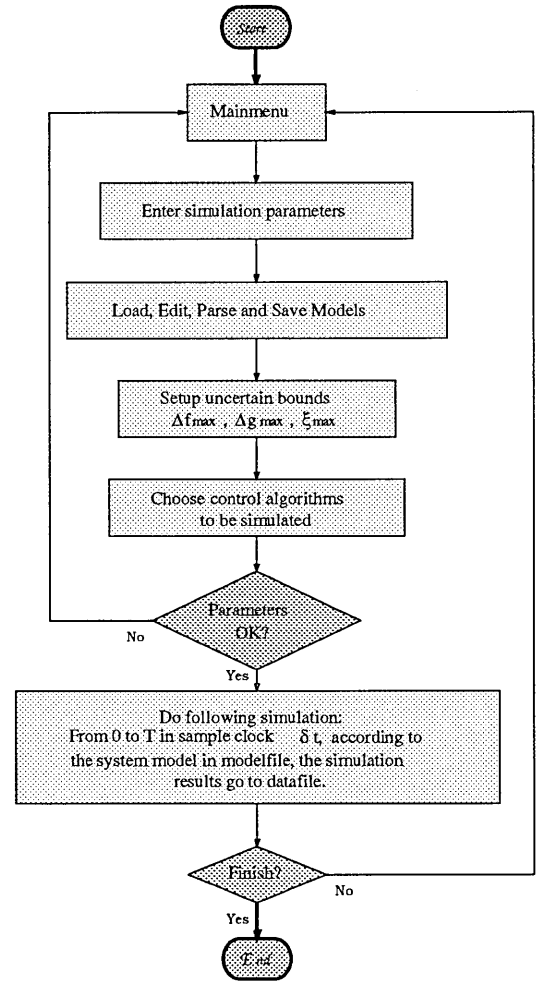


Fig. D.4. The block diagram of the package

Lyapunov function:

$$V(x) = x^T P x = [x_1, x_2, x_3] \begin{pmatrix} 6.3835 & 3.1384 & 0.0036 \\ 3.1384 & 4.1066 & 0.0204 \\ 0.0036 & 0.0204 & 0.0100 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

D.4.2 Model File

```

1. Model_dimension:                                alpha3=1
                                                    r(t)=0.0
    m=3

2. Initial_values:
    x0={-3.1415926,0,0}

3. Uncertainty_bounds:
    dfmax={0,50.0,100.0}
    dgmax={0,0,20.0}
    Dmax={0,10.0,0}

4. Lyapunov_matrix:
    lyap={6.384,3.139,0.004,
           0.139,4.107,0.020,
           0.004,0.020,0.010}
    phai={276.89,202.7689,
           102.0,1}
    C={0.1,100}

5. Model_parameters:
    k_1=9.8
    k_2=10
    k_3=-10
    k_4=-10
    k_5=10
    k_6=50
    k_7=20
    k_8=20
    k_9=20
    k_a=0
    k_b=10
    k_c=0
    alpha1=2
    alpha2=5

6. System_model_matrix:
    f(1)=x2
    f(2)=k_1*sin(x1)+k_2*x3
    f(3)=k_3*x2+k_4*x3
    g(1)=0
    g(2)=0
    g(3)=k_5
    df(1)=0
    df(2)=k_6*cos(5*time)*cos(x1)
    df(3)=-k_7*x2-k_8*x3
    dg(1)=0
    dg(2)=0
    dg(3)=k_9
    D(1)=k_a*r(t)
    D(2)=k_b*r(t)
    D(3)=k_c*r(t)

7. Switching_function:
    Sigma=-k_1*sin(x1)-alpha1*x1
    -alpha2*x2-alpha3*x3
    dsigma(1)=-k_1*cos(x1)-alpha1
    dsigma(2)=-alpha2
    dsigma(3)=-alpha3

8. Control_action:
    h(x)=x1
    Lh(x)=x2
    LLh(x)=k_1*sin(x1)+k_2*x3
    LLLh(x)=k_1*x2*cos(x1)
    +k_2*k_3*x2+k_2*k_4*x3
    LgLLh(x)=k_2*k_5

```

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