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Possibilistic preference networks[☆]

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This paper studies the use of product-based possibilistic networks for representing preferences in multidimensional decision problems. This approach uses symbolic possibility weights and defines a partial preference order among solutions to a set of conditional preference statements on the domains of discrete decision variables. In the case of Boolean decision variables, this partial ordering is shown to be consistent with the preference ordering induced by the *ceteris paribus* assumption adopted in CP-nets. Namely, by completing the possibilistic net ordering with suitable constraints between products of symbolic weights, all CP-net preferences can be recovered. Computing procedures for comparing solutions are provided. The flexibility and representational power of the approach is stressed.

1. Introduction

Modeling preferences is essential in any decision analysis task. However, getting these preferences becomes non trivial as soon as alternatives are described by a Cartesian product of multiple features. Indeed, the direct assessment of a preference relation between these alternatives is usually not feasible due to its combinatorial nature. Fortunately, the decision maker can express *contextual* preferences that exhibit some independence relations, which allows us to represent her/his preferences in a compact manner. Moreover, graphical representations facilitate preference elicitation, as well as the construction of an ordering from these contextual local preferences. This use of graphical preference representations has been inspired by the success of Bayesian networks as a computationally tractable uncertainty management device [1].

The use of possibilistic networks for representing conditional preference statements on discrete variables has been proposed only recently. The approach uses non-instantiated possibility weights to define conditional preference tables. Moreover, additional information about the relative strengths of these symbolic weights can be taken into account. The fact that at best we have some information about the relative values of these weights acknowledges the qualitative nature of preference specification. These conditional preference tables give birth to vectors of symbolic weights that reflect the preferences that are satisfied and those that are violated in a considered situation. The comparison of such vectors may rely on different orderings: the ones induced by the product-based, or the minimum-based chain rule underlying the possibilistic network. A thorough study of the relations between these orderings in presence of vector components that are symbolic rather than numerical is presented. In particular, we establish that the product-based ordering and the symmetric Pareto ordering co-

[☆] *Dedication*: This paper is dedicated to Janusz Kacprzyk on the occasion of his jubilee. The second and the fourth author, who have a long and fruitful research companionship with him, are glad to offer him this new piece of work.

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incide in presence of constraints comparing pairs of symbolic weights. The paper highlights the merits of product-based possibilistic networks for representing preferences, in which case they are called π -pref nets.

Possibilistic preference networks (π -pref nets) belong to the family of methods for the modeling of preference and decision that stem from the fuzzy set and possibility theory literature, variants of which are fuzzy Markovian decision processes studied very early by Kacprzyk [2]. Just like CP-nets may be used for flexible querying [3], it seems that π -pref nets might also serve this purpose (following ideas in [4]), a topic that has been also much investigated by Kacprzyk (e.g., [5]).

In this paper, we also discuss existing relationships between π -pref nets and some preference models that are related to them in some sense, namely, CP-nets [6], CP-theories [7], and OCF-nets [8]. Indeed, π -pref nets share the same preference specification and graphical structure as CP-nets, CP-theories are a generalization of CP-nets, while OCF-nets are based on an additive structure which parallels the one of π -pref nets.

The paper is organized as follows. Section 2 presents a symbolic graphical model for preferences based on possibility theory and possibilistic networks. Section 3 compares various ways of ordering the solutions to a decision problem expressed by a possibilistic preference network. Section 4 compares possibilistic preference networks with other qualitative graphical representations of conditional preference, especially CP-nets.

This paper has its roots in a conference paper [9] and somewhat borrows from another conference paper on the comparison between different orderings that can be defined between configurations [10], and to a lesser extent from a third conference paper providing a comparative overview of graphical preference structures [11].

2. Introducing possibilistic preference networks

Possibilistic conditional preference networks, π -pref nets for short, are a novel model for representing preferences. They are based on possibilistic networks [12,13]. The latter are a possibilistic counterpart of Bayesian networks [1] in the context of possibility theory [14,15], which offers a setting for preferences representation. We use a set of conditional preference tables expressing preferences about the values of variables conditional to the values of other variables. Here we assume that such conditional preferences are represented by conditional possibility distributions. Moreover, as it is difficult to directly quantify preference, we shall assume that possibility weights remain symbolic (i.e., non-instantiated) and that we may add appropriate preference constraints between such weights if they are available. In other words, π -pref nets, like CP-nets, are intended to be a qualitative preference representation framework.

2.1. Conditional possibility and possibilistic networks

Before describing π -pref nets in detail, we recall basic notions of possibility theory that will be useful in the sequel. Possibility theory relies on the notion of a possibility distribution π [15], which is a mapping from a universe of discourse Ω to the unit interval $[0, 1]$, or to any bounded totally ordered scale. This possibilistic scale can be the unit interval when possibility values are the result of a clear measurement procedure, or an ordinal scale when values only reflect a total preorder between the different elements of Ω . We assume that the possibility distribution is such that $\pi(\omega) = 1$ for some element of Ω ($\exists \omega \in \Omega$ such that $\pi(\omega) = 1$). The possibility distribution π is then said to be normalized. When used to represent uncertainty about some variable x taking values on Ω , the assignment $\pi(\omega) = 0$ means that ω is fully impossible as a value for x , while $\pi(\omega) = 1$ means that ω is fully possible, i.e., non-surprising.

The occurrence of an event $F \subseteq \Omega$ is then associated with the possibility measure $\Pi(F) = \sup_{\omega \in F} \pi(\omega)$ estimating its plausibility, and with the dual necessity measure $N(F) = 1 - \Pi(\bar{F}) = 1 - \sup_{\omega \notin F} \pi(\omega)$ estimating its certainty. The degree $\Pi(F)$ evaluates to what extent F is *consistent* with the knowledge represented by π , while $N(F)$ evaluates at what level F is certainly implied by π . See [14] for an introduction to possibility theory.

Conditioning in possibility theory is defined from the Bayesian-like equation

$$\Pi(F \cap G) = \Pi(F|G) \otimes \Pi(G),$$

where \otimes is associative, monotonically increasing in the wide sense and 1 represents the identity element such that $1 \otimes \alpha = \alpha$. In this paper, \otimes stands for the product in a quantitative setting (numerical), or for the minimum in a qualitative setting (ordinal). Namely,

- if \otimes is the product, we get a straightforward counterpart of conditional probability:

$$\Pi(F|G) = \frac{\Pi(F \cap G)}{\Pi(G)} \text{ provided that } \Pi(G) > 0;$$

- if \otimes is the minimum, we get a qualitative version of conditioning, that makes sense on a finite possibility scale:

$$\Pi(F|G) = \begin{cases} \Pi(F \cap G) & \text{if } \Pi(G) > \Pi(F \cap G); \\ 1 & \text{if } \Pi(G) = \Pi(F \cap G) > 0. \end{cases}$$

The two definitions of possibilistic conditioning lead to two variants of possibilistic networks: in the numerical context, we can express product-based networks, while in the qualitative context, we only have min-based networks (also known as qualitative possibilistic networks) [12].

Let $\mathcal{V} = \{A_1, \dots, A_n\}$ be a set of n variables. Each variable A_i has a finite domain D_{A_i} whose elements are $a_i \in D_{A_i}$. The universe of discourse $\Omega = \{\omega_1, \dots, \omega_{|\Omega|}\}$ is the Cartesian product $D_{A_1} \times \dots \times D_{A_n}$ of domains of variables in \mathcal{V} (so, $|\Omega| = |D_{A_1}| \times \dots \times |D_{A_n}|$, where $|T|$ denotes the cardinality of a finite set T). Each element $\omega \in \Omega$ will be called a configuration. It corresponds to a complete instantiation of the variables in \mathcal{V} . A possibilistic network has a definition similar to the one of a Bayesian network.

Definition 1 (Possibilistic networks). [12,16] A possibilistic network over a set of variables \mathcal{V} is characterized by two components:

- (i) a *graphical component* which is a Directed Acyclic Graph $\mathcal{G} = (V, E)$ where V is a set of nodes representing variables and E a set of directed edges $A_i \rightarrow A_j$ encoding conditional (in)dependencies between variables.
- (ii) a *valued component* associating a local normalized conditional possibility distribution $\pi(A_i | p(A_i))$ to each variable $A_i \in \mathcal{V}$ in the context of each instantiation $p(A_i)$ of its parents $\mathcal{P}(A_i) = \{A_j : A_j \rightarrow A_i \in E\}$.

We assume that $\pi(A_i | p(A_i)) > 0$ in order to avoid conditioning on a value of possibility 0. It also comes down to assuming that all configurations are somewhat possible. This assumption will be innocuous in the modeling of preferences.

Given a possibilistic network, we can compute a joint possibility distribution using the following chain rule:

$$\pi(A_1, \dots, A_n) = \otimes_{i=1, n} \Pi(A_i | \mathcal{P}(A_i)). \quad (1)$$

When \otimes is the product, and no configuration is impossible, the conditional tables can be retrieved from the joint possibility distribution obtained by the chain rule, using the same ordering of variables as in the original network. However this is no longer the case if \otimes is the minimum, as some conditional possibility values may be lost when computing the minimum in the chain rule.

Originally, possibilistic networks were meant to model uncertainty and to compute the impact of observations assigning values to some variables so as to predict the values of other variables of the network. In this paper, we advocate their interest in preference modeling rather than uncertainty management. Thus here $\pi(\omega)$ should be understood as the level of satisfaction when choosing configuration ω . For a set of configurations F , $\Pi(F)$ evaluates to what extent satisfying a constraint modeled by F is satisfactory, and $N(F)$ evaluates to what extent this constraint is imperative. As we shall see, beyond their graphical appeal, conditional preference possibilistic networks provide a natural encoding of preferences. In the following, we introduce the kind of preference information needed to construct π -pref nets. Then, we present the definition of π -pref nets and explain their representational power.

2.2. Conditional preference statements

In qualitative preference models, users are supposed to express their preferences under the form of comparison statements between instantiations of a variable, conditioned by some other instantiated variables. For instance, in the particular case of Boolean variables, we deal with preferences of the form: “ a is preferred to $\neg a$ ” if the preference is unconditional, and for conditional statements, in the form “in the context where c is true, a is preferred to $\neg a$ ”, where c corresponds to the instantiation of (maybe several) other variables. More generally,

Definition 2. A *preference statement* $(A_i, p(A_i), \succeq)$ is a preference relation between values $a_{ik} \in D_{A_i}$ of a variable A_i , conditioned by the instantiation $p(A_i)$ of a set $\mathcal{P}(A_i)$ of other variables, in the form of a complete preorder \succeq on D_{A_i} . Namely $\forall a_{ik}, a_{im} \in D_{A_i}$, we have

- i) either $p(A_i): a_{ik} \succ a_{im}$, i.e., in the context $p(A_i)$, a_{ik} is preferred to a_{im} ,
- ii) or $p(A_i): a_{ik} \sim a_{im}$, i.e., in the context $p(A_i)$, one is indifferent between a_{im} and a_{ik} ,

where \succ is the strict part of \succeq , and \sim is the indifference part of \succeq . If $\mathcal{P}(A_i) = \emptyset$, then the preference statement about A_i is *unconditional*.

Note that we do not allow incomplete preference local specifications in the form $a_{ik} \succeq a_{im}$. On each variable domain D_{A_i} , the user must choose between $a_{ik} \succ a_{im}$, $a_{ik} \sim a_{im}$ and $a_{ik} \sim a_{im}$. It comes down to rating each possible instantiation of the variable A_i (whose domain can be nominal) on a local totally ordered ordinal value scale, which is a usual assumption in multicriteria decision making.

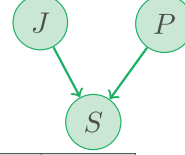
The running [Example 1](#), inspired from [6], illustrates such preference statements.

Example 1. Consider a preference specification about an evening suit over three decision variables $V = \{J, P, S\}$ standing for jacket, pants and shirt respectively, with values in $D_J = \{Red (j_r), Black (j_b)\}$, $D_P = \{White (p_w), Black (p_b)\}$ and $D_S = \{Black (s_b), Red (s_r), White (s_w)\}$. The conditional preferences are given in [Table 1](#). Preference statements (s_1) and (s_2) are unconditional. Note that the user is indifferent between the values of the color of the shirt if his jacket is black and his pants are white (in the context $j_b p_w$), which is not the case if he wears a red jacket and black pants. Indeed, he prefers red shirt to a black one (in the context $j_r p_b$).

Table 1
Conditional preference specification of [Example 1](#).

(s ₁) $j_b > j_r$
(s ₂) $p_b > p_w$
(s ₃) $j_b p_b: s_b > s_r > s_w$
(s ₄) $j_b p_w: s_w > s_b > s_r$
(s ₅) $j_r p_b: s_r > s_b > s_w$
(s ₆) $j_r p_w: s_b \sim s_r \sim s_w$

$\pi(j_b)$	1	$\pi(p_b)$	1
$\pi(j_r)$	α	$\pi(p_w)$	β



$\pi(\cdot \cdot)$	$j_b p_b$	$j_b p_w$	$j_r p_b$	$j_r p_w$
s_b	1	θ_1	λ_1	1
s_r	δ_1	θ_2	1	1
s_w	δ_2	1	λ_2	1

Fig. 1. A possibilistic preference network.

2.3. Introducing π -pref nets

Representing the preference statements in a graphical way means that each node in the graph represents a decision variable A_i which is associated to a set of local conditional preference statements, conditional to the values of variables that are its parent nodes in the graph. A (conditional) *preference network* can be defined as follows:

Definition 3. A preference network is a Directed Acyclic Network (DAG) (E, V) with nodes $A_i, A_j \in V$, s.t. each arc from $A_j \rightarrow A_i \in E$ expresses that the preference about A_i depends on A_j . Each node A_i is associated with a Conditional Preference Table CPT_i that associates preference statements $(A_i, p(A_i), \succeq)$ between the values of A_i , conditional to each possible instantiation $p(A_i)$ of the parents $\mathcal{P}(A_i)$ of A_i (if any).

In a possibilistic preference network, for each particular instantiation $p(A_i)$ of $\mathcal{P}(A_i)$, the preference order between the values of A_i stated by the user will be encoded by a local conditional possibility distribution expressed by means of *symbolic weights*. By a symbolic weight, we mean a symbol representing a *strictly* positive real number in $(0, 1]$ whose value is unspecified. We rely on symbolic weights in the absence of available numerical values. More formally, we have:

Definition 4. (Conditional Preference Possibilistic network (π -pref net)) A *possibilistic preference network* based on operation \otimes (\otimes - π -pref net) ΠG over a set $V = \{A_1, \dots, A_n\}$ of decision variables is a preference network where each local preference relation at node A_i is associated with a *symbolic* conditional possibility distribution (π_i -table for short), encoding the ordering between the values of A_i such that:

- (i) If $p(A_i) : a_i < a'_i$ then $\pi(a_i|p(A_i)) = \alpha, \pi(a'_i|p(A_i)) = \beta$ where α and β are symbolic weights, and $0 < \alpha < \beta \leq 1$;
- (ii) If $p(A_i) : a_i \sim a'_i$ then $\pi(a_i|p(A_i)) = \pi(a'_i|p(A_i)) = \alpha > 0$ where α is a symbolic weight such that $\alpha \leq 1$;
- (iii) For each instantiation $p(A_i)$ of $\mathcal{P}(A_i)$, $\exists a_i \in D_{A_i}$ such that $\pi(a_i|p(A_i)) = 1$.
- (iv) A symbolic degree of possibility is assigned to each configuration ω using the chain rule (1) based on \otimes .

Let \mathcal{C}_0 be the set storing the constraints between the symbolic possibility weights pertaining to each preference statement $(A_i, p(A_i), \succeq)$, encoding the complete preordering \succeq . In addition to these preferences encoded by a π -pref net, additional constraints can be taken into account. Such constraints, forming a set denoted by \mathcal{C}_1 , may express that some weights pertaining to one preference statement are equal to, or greater than, weights pertaining to another preference statement. Let $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1$ be the set of all constraints. In case one needs to compare two weights α and β , one check if there is any relation between them stored in \mathcal{C} , or if one can infer it by transitivity from \mathcal{C} ; otherwise they are considered as *incomparable*.

Example 2. Consider the preference specification about an evening suit of [Example 1](#). Its corresponding π -pref net and the conditional possibility weights are given by [Fig. 1](#). The graph is built based on [Definition 3](#). In fact, since $\mathcal{P}(J) = \mathcal{P}(P) = \emptyset$ the two variables J and P are independent, while S depends on J and P since the preference statements associated to S are conditioned by $\mathcal{P}(S) = \{J, P\}$. The constraints between symbolic weights inherent from the preference specification are represented by the set \mathcal{C}_0 such that $\mathcal{C}_0 = \{(\delta_1 > \delta_2), (\theta_1 > \theta_2), (\lambda_1 > \lambda_2)\}$.

Table 2

Vectors associated to each configuration of [Example 2](#).

configurations	Symbolic vectors		
	J	P	S
$j_b p_b S_b$	(1,	1,	1)
$j_b p_b S_r$	(1,	1,	δ_1)
$j_b p_b S_w$	(1,	1,	δ_2)
$j_b p_w S_b$	(1,	β ,	θ_1)
$j_b p_w S_r$	(1,	β ,	θ_2)
$j_b p_w S_w$	(1,	β ,	1)
$j_r p_b S_b$	(α ,	1,	λ_1)
$j_r p_b S_r$	(α ,	1,	1)
$j_r p_b S_w$	(α ,	1,	λ_2)
$j_r p_w S_b$	(α ,	β ,	1)
$j_r p_w S_r$	(α ,	β ,	1)
$j_r p_w S_w$	(α ,	β ,	1)

A set of conditional preference tables encoded as a π -pref net determines a partial order among configurations. Indeed, each configuration has a satisfaction level encoded by a possibility degree computed by means of the possibilistic chain rule (1). This leads us to the following definition of the induced preference ordering on configurations.

Definition 5 (Preference ordering). Consider a symbolic possibilistic preference network Π_G and a set \mathcal{C} of constraints between the symbolic weights. Let ω and ω' be two configurations in Ω , and $\pi_{\Pi_G}(\omega)$ (resp. $\pi_{\Pi_G}(\omega')$) be the symbolic possibility degree of ω (resp. ω') computed by (1). Then, configuration ω is weakly preferred to ω' , denoted by $\omega \succeq_{\otimes} \omega'$, iff $\pi_{\Pi_G}(\omega) \geq \pi_{\Pi_G}(\omega')$.

In the definition, $\pi_{\Pi_G}(\omega)$ is a combination of symbolic weights using \otimes . So, $\pi_{\Pi_G}(\omega) \geq \pi_{\Pi_G}(\omega')$ (resp. $\pi_{\Pi_G}(\omega) > \pi_{\Pi_G}(\omega')$, $\pi_{\Pi_G}(\omega) = \pi_{\Pi_G}(\omega')$) should be understood as follows: this inequality (resp. strict inequality, equality) holds whatever the numerical instantiations of the weights involved in the possibility values, in agreement with constraints in \mathcal{C} . This is respectively denoted by $\omega \succeq_{\otimes} \omega'$, $\omega \succ_{\otimes} \omega'$ and $\omega \sim_{\otimes} \omega'$. When it is not possible to prove an inequality between $\pi_{\Pi_G}(\omega)$ and $\pi_{\Pi_G}(\omega')$, because it is possible to have strict inequalities in both directions by substituting distinct numerical values, we interpret this situation in terms of incomparability as already said, and this is denoted by $\omega \pm_{\otimes} \omega'$.

Since we use symbolic weights, a definite preference between all configurations cannot be established (as long as we do not instantiate all symbolic weights). Each configuration $\omega = a_1 \dots a_n$ can also be associated with a vector $(\alpha_1, \dots, \alpha_n)$, where $\alpha_i = \pi(a_i | p(A_i))$ and $p(A_i) = \omega[\mathcal{P}(A_i)]$, where $\omega[\mathcal{P}(A_i)]$ is the restriction of the configuration ω to the parents of A_i . For instance, vectors associated to the preference possibilistic network of [Example 2](#) are given in [Table 2](#). Thus, comparing configurations amounts to comparing vectors of symbolic weights attached to configurations, and the use of the chain rule is just one way of comparing such vectors, among other ones as discussed in the next section. However, note that symbolic weights attached to a variable depend on the instantiations of its parents.

3. On various ways of ordering configurations induced by conditional preference networks

In the previous section, we have shown how to encode the preference specifications in a possibilistic network format and we have defined a partial ordering on configurations based on comparing expressions involving symbolic weights combined with operation \otimes . In contrast we may also compare vectors of symbolic weights representing the local satisfaction of the conditional tables for configurations. In this section we will first compare the partial order relations based on product and minimum defined above, and then classical vector comparison techniques such as Pareto and symmetric Pareto orderings with the purpose to use them to generate a meaningful ordering over configurations.

3.1. Ordering relations for symbolic vectors

A symbolic vector on $[0, 1]$ is of the form $\alpha = (\alpha_1, \dots, \alpha_n)$. Viewed as vectors of ratings of configurations $a_1 \dots a_n$, the range of each component $\alpha_i = \pi(a_i | p(A_i))$ is either the open interval $(0, 1)$, or reduces to $\{1\}$ (if $\pi(a_i | p(A_i)) = 1$). In other words, the vector contains 1's and/or symbols standing for unknown positive values strictly less than 1.

Definition 6. Ordering relations between symbolic vectors $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ based on an operation \otimes , that can be the minimum or the product, are defined as follows:

- $\alpha \succ_{\otimes} \beta$ iff $\otimes_{i=1}^n \alpha_i > \otimes_{i=1}^n \beta_i$
- $\alpha \succeq_{\otimes} \beta$ iff $\otimes_{i=1}^n \alpha_i \geq \otimes_{i=1}^n \beta_i$
- $\alpha \sim_{\otimes} \beta$ iff $\alpha \succeq_{\otimes} \beta$ and $\beta \succeq_{\otimes} \alpha$
- $\alpha \pm_{\otimes} \beta$ iff neither $\alpha \succeq_{\otimes} \beta$ nor $\beta \succeq_{\otimes} \alpha$ hold.

When $\otimes = \min$ (resp. product), we write $\alpha \succ_{\min} \beta$ (resp. $\alpha \succ_{\text{prod}} \beta$), and so on for the other relations.

In this definition, $\alpha \succ_{\otimes} \beta$ really means that the inequality $\otimes_{i=1}^n a_i > \otimes_{j=1}^n b_j$ holds for any choice of numerical values a_i and b_j ¹ in the respective ranges of α_i and β_j , and likewise for $\alpha \succeq_{\otimes} \beta$ using a weak inequality. Actually we can drop the terms α_i and β_j whose ranges are $\{1\}$. Note that the relation $\alpha \succ_{\otimes} \beta$ is then more demanding than the one defined by “ $\alpha \succeq_{\otimes} \beta$ and not $\beta \succeq_{\otimes} \alpha$ ”, since in the former we request a strict inequality for all instantiations by numerical values. The latter only requires $\alpha \succeq_{\otimes} \beta$ and $\exists a_i, b_j : \otimes_{i:\alpha_i \neq 1} a_i > \otimes_{j:\beta_j \neq 1} b_j$. Finally, $\alpha \pm_{\otimes} \beta$ stands for $\exists a_i, b_j, a'_i, b'_i : \otimes_{i:\alpha_i \neq 1} a_i > \otimes_{j:\beta_j \neq 1} b_j$ and $\otimes_{i:\alpha_i \neq 1} a'_i < \otimes_{j:\beta_j \neq 1} b'_j$.

We also consider the following classical relations, expressed in the symbolic case:

Definition 7 (Pareto).

- $\alpha \succeq_P \beta$ iff $\forall k, \alpha_k \geq \beta_k$.
- $\alpha \succ_P \beta$ iff $\alpha \succeq_P \beta$ and $\exists k, \alpha_k > \beta_k$.
- $\alpha \sim_P \beta$ iff $\alpha = \beta$.

Note that with the type of symbolic vectors that we use, $\alpha_k \geq \beta_k$ may hold only if either $\alpha_k = 1$, or $\alpha_k = \beta_k$, or $1 > \alpha_k > \beta_k$ in the same conditional preference table, and $\alpha_k > \beta_k$ may hold only if either $\alpha_k = 1$, $\beta_k \neq 1$, or $1 > \alpha_k > \beta_k$.

Definition 8 (Symmetric Pareto). $\alpha \succeq_{SP} \beta$ iff there exists a permutation σ of the components of α , yielding a vector α_{σ} , such that $\alpha_{\sigma} \succeq_P \beta$.

Similar definitions can be written for $\alpha \succ_{SP} \beta$, $\alpha \sim_{SP} \beta$. In the numerical setting it is easy to see that $\alpha \succ_P \beta$ implies $\alpha \succ_{SP} \beta$, which in turn implies $\alpha \succ_{\text{prod}} \beta$. Besides, $\alpha \succeq_P \beta$ implies $\alpha \succeq_{SP} \beta$ which implies both $\alpha \succeq_{\text{prod}} \beta$ and $\alpha \succeq_{\min} \beta$. But $\alpha \succ_{SP} \beta$ only implies $\alpha \succeq_{\min} \beta$: for instance, $(0.3, 0.8) \succ_{SP} (0.7, 0.3)$, while $\min(0.3, 0.8) = \min(0.7, 0.3)$. Things change when we consider vectors of symbolic weights.

3.2. Comparison of product-based and min-based orderings in the symbolic setting

In the following, we present the possible relations between the product-based and the minimum-based orderings in the particular case where the constraints known between the symbolic weights only pertain to the expression of conditional preferences, as allowed by [Definition 4](#). Thus, a constraint of this kind may only compare weights pertaining to the same component in the vectors, and we have $C_1 = \emptyset$. Namely, weights located in different components of a vector are assumed to be incomparable. Under this assumption, Pareto ordering and symmetric Pareto yield the same ordering. Indeed, for two vectors $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ encoding the symbolizing ratings of configurations, each symbolic weight $\alpha_i \neq 1$ of α can only be compared to the symbolic weight $\beta_i \neq 1$ of β . Thus, there is no need to permute components as the result would definitely be incomparable with another component weight since $C_1 = \emptyset$.

In the following, we denote by $\text{prod}(\alpha)$ the product $\prod_{i=1}^n \alpha_i$. Then, it can be shown that in this situation, the following equivalences hold:

Proposition 1. *The following equivalences hold when $C_1 = \emptyset$:*

- $\alpha \succ_{\text{prod}} \beta \Leftrightarrow \alpha \succ_P \beta$.
- $\alpha \sim_{\text{prod}} \beta \Leftrightarrow \alpha = \beta$
- $\alpha \succeq_{\text{prod}} \beta \Leftrightarrow \alpha \succeq_P \beta$

Proof: $\alpha \succ_{\text{prod}} \beta$ requires that each instantiation of $\text{prod}(\alpha)$ be at least as great as each instantiation of $\text{prod}(\beta)$. As by assumption, symbols of the form α_i and β_j are not comparable for $i \neq j$, unless one of them is equal to 1, the only way to have $\alpha \succ_{\text{prod}} \beta$ is to have a constraint of the form $\alpha_i > \beta_i$ (possibly $\alpha_i = 1$) for some i 's and $\alpha_j = \beta_j$ for the other components. Hence $\alpha \succ_P \beta$. The other cases follow using the same approach. \square

We now compare the different orderings induced by the use of product or minimum, depending on the chain rule applied to the possibilistic network. The product of symbolic vectors often has a discriminating power greater than the one of the minimum operator, in the sense that $\alpha \beta < \alpha$, while we only have $\min(\alpha, \beta) \leq \alpha$. However, with instantiated numerical values, both product and minimum lead to total orders that may also contradict each other: for instance $0.1 \times 0.9 > 0.2 \times 0.2$, while with the min we get $\min(0.1, 0.9) < \min(0.2, 0.2)$. The following example illustrates the difference between the partial orders obtained with product and minimum in case of symbolic weights.

Example 3. Let us consider the possibilistic preference network of [Example 2](#). Using the chain rule, we obtain the symbolic vectors presented in [Table 2](#), and the following symbolic joint possibility distribution: $\pi(j_b p_b s_b) = 1$, $\pi(j_b p_b s_r) = \delta_1$, $\pi(j_b p_b s_w) = \delta_2$, $\pi(j_b p_w s_b) = \beta \otimes \theta_1$, $\pi(j_b p_w s_r) = \beta \otimes \theta_2$, $\pi(j_b p_w s_w) = \beta$, $\pi(j_r p_b s_b) = \alpha \otimes \lambda_1$, $\pi(j_r p_b s_r) = \alpha$, $\pi(j_r p_b s_w) = \alpha \otimes \lambda_2$, $\pi(j_r p_w s_b) = \pi(j_r p_w s_r) = \pi(j_r p_w s_w) = \alpha \otimes \beta$. The product-based induced ordering based on inequality constraints in C_0 is represented by [Fig. 2](#). For instance, $j_b p_b s_r \succ_{\text{prod}} j_b p_b s_w$ because $\delta_1 > \delta_2$ is in C_0 and $j_b p_w s_w \succ_{\text{prod}} j_r p_w s_w$ because $\beta > \alpha \otimes \beta$. Now, if we use the min-based chain rule, we will not be able to get as many strict preferences as long as no

¹ Not to be confused with instantiations a_i of decision variables A_i , etc.

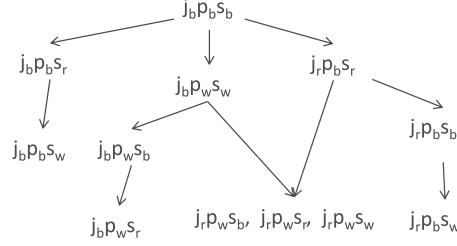


Fig. 2. Possibilistic product-based order relative to Example 3.

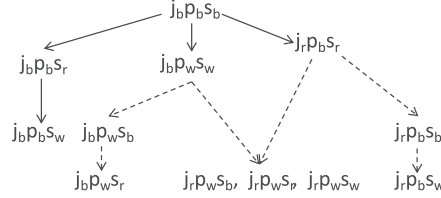


Fig. 3. Possibilistic minimum-based order relative to Example 3.

other constraint is added. In fact, the only strict ordering information we can get at that stage is that $j_b p_b s_b \succ_{\min} j_b p_b s_r \succ_{\min} j_b p_b s_w$; $j_b p_b s_b \succ_{\min} j_b p_w s_w$; and $j_b p_b s_b \succ_{\min} j_r p_b s_b$. But, for the rest, we only get weak inequalities such as $j_r p_w s_b \preceq_{\min} j_b p_w s_w$, since $\pi_{\min}(j_r p_w s_b) = \min(\alpha, \beta) \leq \pi_{\min}(j_b p_w s_w) = \beta$ (dotted arrows on Fig. 3 depicts this min-based ordering).

The following results can be observed [10]:

Proposition 2. When $C_1 = \emptyset$, $\alpha \sim_{\text{prod}} \beta \Leftrightarrow \alpha \sim_{\min} \beta$.

Indeed, $\alpha \sim_{\text{prod}} \beta$ if and only if $\alpha = \beta$ due to its coincidence with Pareto ordering and then $\alpha \sim_{\min} \beta$. Conversely, suppose $\alpha \sim_{\min} \beta$, i.e., $\min \alpha_i = \min \beta_i$. Suppose $\alpha_i \neq \beta_i$ for some i . Then either $\alpha_i = 1$ and it is easy to let $\min \beta_i$ as small as possible, fixing the values of other α_j 's and setting β_i to a very small value b_i , i.e., $\min \alpha_i > \min \beta_i$. We can do something similar if $\alpha_i > \beta_i$. As the weights are not comparable across components of vectors ($C_1 = \emptyset$), except for weights 1 that do not affect minimum nor product, we conclude that α and β contain the same weights in each component.

Proposition 3. When $C_1 = \emptyset$: $\alpha \pm_{\text{prod}} \beta \Leftrightarrow \alpha \pm_{\min} \beta$.

Proof. Indeed, $\alpha \pm_{\text{prod}} \beta$ indicates that the vector α contains symbolic values that are not comparable to symbolic values in β . In that case, $\alpha \pm_{\min} \beta$ as well. Conversely, note that no symbolic weight in α will be absorbed with the minimum operation since weights appearing in α (resp. β) are pairwise incomparable. So, in our situation where $C_1 = \emptyset$, $\alpha \pm_{\min} \beta$ holds in the same cases as when $\alpha \pm_{\text{prod}} \beta$ holds. \square

Example 4. Many cases of incomparability can be identified on Example 2 and the min-based (resp. product-based) ordering presented by Fig. 3 (resp. Fig. 2). For instance, we have $j_r p_b s_r \pm_{\text{prod}} j_b p_w s_w$ (resp. $j_r p_b s_r \pm_{\min} j_b p_w s_w$).

Moreover, using symbolic weights, product and minimum provide consistent orderings, in contrast with the numerical setting, in the sense that:

Proposition 4. When $C_1 = \emptyset$, $\alpha \succ_{\text{prod}} \beta \Leftrightarrow \alpha \succeq_{\min} \beta$ and $\alpha \neq \beta$.

Proof. It is obvious that $\alpha \succ_{\text{prod}} \beta$ implies $\alpha \neq \beta$. That it implies $\alpha \succeq_{\min} \beta$ comes from the equivalence between Pareto and product orderings when $C_1 = \emptyset$. Conversely since $C_1 = \emptyset$, the only possibility to get $\alpha \succeq_{\min} \beta$ is that $\forall i, \alpha_i \geq \beta_i$ since the weights are not comparable across components of vectors (e.g. α_i is not comparable to β_j for $i \neq j$). As $\alpha \neq \beta$, there must exist i such that $\alpha_i > \beta_i$, and for the other components, either $\alpha_k > \beta_k$ or $\alpha_k = \beta_k$. Hence, $\alpha \succ_{\text{prod}} \beta$. \square

As a consequence, the strict preference graph of \succ_{prod} will be the same as the weak preference graph of \succ_{\min} , as patent when comparing Figs. 2 and 3 for Example 2. And we can see that $\alpha \succeq_{\min} \beta \Rightarrow \alpha \succeq_{\text{prod}} \beta$.

Proposition 5. When $C_1 = \emptyset$, $\alpha \succ_{\min} \beta \Rightarrow \alpha \succ_{\text{prod}} \beta$.

Proof. This is because $\alpha \succ_{\min} \beta$ if and only if $\forall i$ such that $\beta_i \neq 1$, $\alpha_i > \beta_i$. Indeed suppose $\alpha_i \leq \beta_i$ for some i . Then we can always set both α_i and β_i to the same value close to 0 and possible so as to make $\alpha \succ_{\min} \beta$ fail. \square

For instance, all solid arrows in Fig. 3 also appear in Fig. 2 for Example 2. This indicates that \succ_{\min} is a strong form of Pareto ordering. Thus, the product ordering is a refinement of the minimum-based ordering in the symbolic case.

3.3. Constraints across components of symbolic vectors

As already mentioned, constraints between symbolic weights, beside those induced from the preference specification, can be added when available. In the following we will study the relations between the different ordering relations in the presence of such constraints. It has been shown above that, when there is no constraint between symbolic weights in the vectors, the ordering induced by the product-based chain rule corresponds exactly to the Pareto ordering. This result actually holds replacing Pareto by Symmetric Pareto *in the presence of inequality constraints between symbolic weights*.

Proposition 6. *Given any set of constraints C of the form $\alpha_i \geq \beta_j$ or $\alpha_i > \beta_j$ between symbolic weights, it holds that $\alpha \succ_{SP} \beta$ iff $\alpha \succ_{prod} \beta$ and $\alpha \succeq_{SP} \beta$ iff $\alpha \succeq_{prod} \beta$.*

The proof of this result is not trivial and can be found in [10]. It appears in the appendix for the sake of self-containedness.

Let us compare the minimum based-ordering and the product-based ordering (equivalently, SP). It is clear that we have as a corollary of the previous result:

Proposition 7. $\alpha \sim_{prod} \beta \Rightarrow \alpha \sim_{min} \beta$.

Proof. $\alpha \sim_{prod} \beta$ iff $\alpha \sim_{SP} \beta$, i.e., $\alpha \sim_P \alpha'_\sigma$ for some permutation σ . Thus, $\forall i, \alpha_i = \beta_{i\sigma}$, where $i \in \{1, \dots, n\}$. Therefore, $\min(\beta_1, \dots, \beta_n) = \min(\alpha_1, \dots, \alpha_n)$. \square

The last proposition shows that the strict min-based ordering can solve some incomparability cases for the Symmetric Pareto ordering. Indeed:

Proposition 8. *If $\alpha \succ_{min} \beta$ we may either have $\alpha \pm_{SP} \beta$ or $\alpha \succ_{SP} \beta$.*

Proof. From Proposition 7, if $\alpha \sim_{SP} \beta$ then $\alpha \sim_{min} \beta$. Moreover, if $\alpha \prec_{SP} \beta$ then by definition, $\forall i, \alpha_i \leq \beta_{\sigma(i)}$, thus it follows that $\min(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n) = \min(\alpha_1, \dots, \alpha_n)$, this proves that we cannot have $\alpha \succ_{min} \beta$ in this case. Proposition 8 follows. \square

It is important to notice that, in general, symmetric Pareto dominance (hence the product ordering) in the wide sense does not refine the minimum ordering, since the former may yield incomparability in some cases when minimum succeeds in comparing.

The extreme case is when assuming a total preorder between all symbolic weights. In that case, minimum ordering is total. However, in the presence of such rich constraints, symmetric Pareto ordering (i.e., product) with symbolic weights may still lead to incomparability. Indeed, the only case, where symmetric Pareto leads to a total ordering is when there are enough constraints between *subsets* of symbolic weights (corresponding to the comparison of subproducts).

3.4. Application to best solution and dominance queries in π -pref nets

In a preference model, two types of queries are commonly used: namely, optimization queries for finding the optimal configuration(s) (i.e., those which are not dominated by others) and dominance queries for comparing configurations.

Optimization. Since π -pref nets allow the user to express indifference, the optimization query may return more than one configuration. Clearly, the best configurations are those having a joint possibility degree equal to 1. Indeed, such a configuration always exists since the joint possibility distribution associated to the possibilistic network is normalized, thanks to the normalization of each conditional possibility table (indeed, for each variable A_i , for each instantiation $p(A_i)$ of $\mathcal{P}(A_i)$, we have: $\max(\pi(a_i | p(A_i)), \pi(-a_i | p(A_i))) = 1$ where $\{-a_i\} = D_{A_i} / \{a_i\}$ with $a_i \in D_{A_i}$). Thus, we can always find an optimal configuration, starting from the root nodes where we choose each time the most or one of the most preferred value(s) (i.e., with possibility equal to 1). Then, depending on the parents instantiation, each time we again choose an alternative with a conditional possibility equal to 1. At the end of the procedure, we get one or several completely instantiated configurations having a possibility equal to 1. Consequently, partial preference orders with incomparable maximal elements cannot be represented by a π -pref net.

Example 5. Let us reconsider Example 2 and its product-based joint possibility degree depicted by Fig. 2. Then, $j_b p_b s_b$ is the preferred configuration since its joint possibility is equal to 1, and this is the only one.

This procedure is linear in the size of the network (using a forward sweep algorithm). A possible variant of the optimization problem is to compute the M most possible configurations using a variant of the Most Probable Explanation algorithm in [17]. This query can be interesting in π -pref nets even if the answer is not always obvious to obtain in presence of incomparable configurations.

Algorithm 1: Comparison between two joint symbolic possibility vectors.

```
Data:  $\alpha, \beta, C$ 
Result:  $R$ 
1 begin
2    $equality(\alpha, \beta, C)$ ;
3   if ( $empty(\alpha)$  and  $empty(\beta)$ ) then  $R \leftarrow \alpha = \beta$ ;
4   else  $s \leftarrow sort(\alpha, \beta, C)$ ;
5   if  $s = true$  then  $R \leftarrow \alpha > \beta$ ;
6   else  $s \leftarrow sort(\beta, \alpha, C)$ ;
7   if  $s = true$  then  $R \leftarrow \beta > \alpha$ ;
8   else  $R \leftarrow \alpha \pm \beta$ ;
9   return  $R$ 
```

Dominance. The comparison between symbolic possibility degrees can be found using [Algorithm 1](#) that takes as input a set of constraints C between symbolic weights and two rating vectors. Let us consider two configurations ω and ω' with simplified respective simplified vectors $\alpha^* = (\alpha_1, \dots, \alpha_k)$ and $\beta^* = (\beta_1, \dots, \beta_m)$ where the components equal to 1 have been deleted, with $k \leq m \leq n$. Then, the algorithm proceeds by first deleting all pairs of equal components, one in each vector, so to get totally different sets of components in each vector. Second, it checks if there exists an injective function $\varphi : \{1, \dots, k\} \rightarrow \{1, \dots, m\}$ such that $\forall i = 1, \dots, k, \alpha_i \geq \beta_{\varphi(i)}$ and $\exists \ell \in \{1, \dots, k\}, \alpha_\ell > \beta_{\varphi(\ell)}$ (otherwise they remain incomparable).

Thus the algorithm is based on the sequential application of:

- (1) The function *equality* that deletes the common values between α and β .
- (2) The function *sort* that returns *true* if an injection is found between α^* and β^* ensuring the above dominance condition.
- (3) The function *empty* that tests if a vector of weights α is empty or not.

Example 6. Let us consider the π -pref net ΠG of [Example 2](#). Using [Algorithm 1](#), the ordering between the configurations is shown in [Fig. 2](#) such that a link from ω to ω' means that ω is preferred to ω' . For instance, consider configuration $j_b p_w s_r$ such that $\pi(j_b p_w s_r) = \beta \cdot \theta_2$ and configuration $j_r p_w s_r$ such that $\pi(j_r p_w s_r) = \alpha \cdot \beta$. First, we should delete common values, namely the symbolic weight β . Then, we should check if C entails $\alpha < \theta_2$ or the converse. But here, α and θ_2 are not comparable. Thus, $j_b p_w s_r \pm_{prod} j_r p_w s_r$.

Clearly, for π -pref nets, the complexity is due to the comparison step in [Algorithm 1](#) (since the computation of the possibility degrees is a simple matter using the chain rule), and in particular to the *sort* function where the matching between the two vectors needs the definition of different possible arrangements, i.e., the algorithm is of time complexity $O(n!)$.

4. Comparison of π -pref nets with other graphical preference structures

We now compare π -pref nets with Conditional Preference networks (CP-nets) which deal with the same kind of conditional preference statements. Moreover we also discuss the OCF networks that are “semi-qualitative”. For the sake of simplicity, we restrict to the case where the decision variables are Boolean.

4.1. CP-nets

CP-nets, initially introduced in [\[6,18\]](#), are considered as an efficient model to manage qualitative preferences. They are based on a preferential independence property often referred to as a *ceteris paribus* assumption such that a partial configuration is preferred to another one, everything else being equal. Formally, it is defined as follows:

Definition 9 (Preferential independence). Let \mathcal{V} be a set of variables and \mathcal{W} be a subset of \mathcal{V} . \mathcal{W} is said to be preferentially independent from its complement $\mathcal{Z} = \mathcal{V} \setminus \mathcal{W}$ iff for any instantiations, z, z' of variables in \mathcal{Z} , and w, w' of variables in \mathcal{W} , it holds that $(w, z) > (w', z) \Leftrightarrow (w, z') > (w', z')$.

Preferential independence is asymmetric. Indeed, it might happen, e.g., for disjoint sets \mathcal{X}, \mathcal{Y} and \mathcal{Z} of variables that \mathcal{X} is preferentially independent ([Definition 9](#)) from \mathcal{Y} given \mathcal{Z} without having \mathcal{Y} preferentially independent from \mathcal{X} . This independence is at work in the graphical structure underlying CP-nets.

Definition 10 (CP-nets). A CP-net is a preference network in the sense of [Definition 3](#) where preference statements are interpreted by means of the *ceteris paribus* assumption, namely, the preference pertaining to each decision variable A_i at each node only depends on the parent(s) context $p(A_i)$, and is preferentially independent from the rest of variables.

Using the information in the CP-Tables and applying the *ceteris paribus* principle, we only obtain preferences between configurations differing by one flip, i.e., obtained by changing the value of a single variable. Indeed, when *flipping* a value of

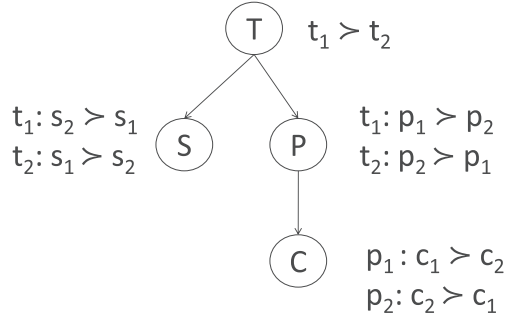


Fig. 4. Preference network for Example 7.

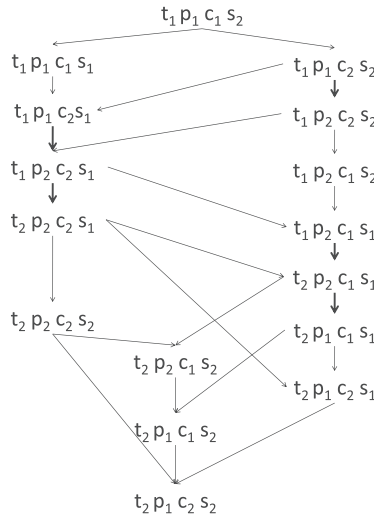


Fig. 5. Worsening flip graph for the CP-net of Fig. 4 up to transitive closure.

one variable in a configuration, one obtains either an improved configuration (improving flip), or a worse one (worsening flip). These pairs of configurations differing by one flip can be organized into a collection of worsening (directed) paths with a unique root corresponding to the best configuration and where the other path extremities are the worst ones. A CP-net is said to be satisfiable if there exists at least one partial order of configurations that satisfies it. Note that every acyclic CP-net is satisfiable.

Example 7. Consider a preference specification about a holiday house in terms of four decision variables $\mathcal{V} = \{T, S, P, C\}$ standing for type, size, place and car park respectively, with values $T \in \{flat(t_1), house(t_2)\}$, $S \in \{big(s_1), small(s_2)\}$, $P \in \{downtown(p_1), outskirts(p_2)\}$ and $C \in \{car(c_1), no-car(c_2)\}$. Preference on T is unconditional, while all the other preferences are conditional as follows: $t_1 > t_2$, $t_1: p_1 > p_2$, $t_2: p_2 > p_1$, $p_1: c_1 > c_2$, $p_2: c_2 > c_1$, $t_1: s_2 > s_1$, $t_2: s_1 > s_2$. Fig. 4 represents the corresponding CP-net, and its induced worsening flip graph is on Fig. 5.

Acyclic CP-nets have a unique optimal configuration. Finding it amounts to looking for a configuration where all the conditional preferences are best satisfied. It can be done by a simple forward sweeping procedure where, for each node, we assign the most preferred value according to the parents context. For acyclic CP-nets, this procedure is linear w.r.t. the number of variables [6]. In contrast, for cyclic ones answering this query needs an NP-hard algorithm and may lead to more than one optimal configuration [19]. Dominance queries are more complex. A configuration is preferred to another if there exists a chain (directed path) of worsening flips between them [18]. Note that if for any variable $A_i \in \mathcal{V}$, A_i is preferentially independent from $\mathcal{V} \setminus A_i$, then the CP-net graph is disconnected and many configurations cannot be compared. The complexity of dominance testing depends on the CP-net structure. Indeed for the case of acyclic CP-nets, Boutilier et al. [6] show that (i) in directed tree CP-nets, the complexity is quadratic in the number of variables, (ii) in polytree CP-nets, it is polynomial in the size of the CP-net description (variables and preference table sizes), (iii) in singly connected CP-nets, it is NP-complete. In multiply connected CP-nets, the problem is in NP or harder (it remains an open problem until now). For general CP-nets (allowing cycles) the problem is PSPACE-complete [19].

In general, the ordering induced by a CP-net is strict and partial, since several configurations may remain incomparable (i.e., no worsening flips chain exists between them). Clearly, acyclic CP-nets cannot exhibit any ties. The *ceteris paribus* as-

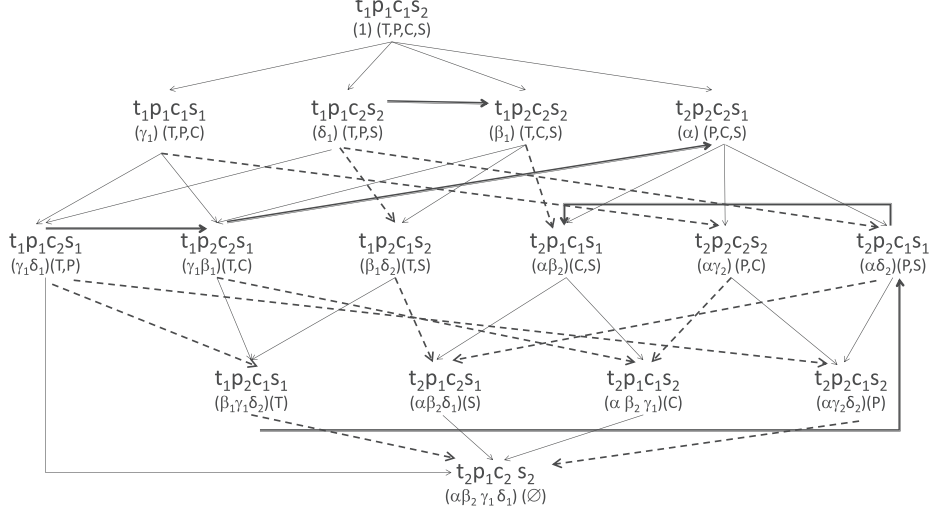


Fig. 6. Configuration graph of Ex. 7. Thin arrows reflect \succ_{prod} , dotted arrows compare sets $\mathcal{S}(\omega)$, and bold arrows reflect additional *ceteris paribus* comparisons, also in bold on Fig. 5.

sumption simplifies preference elicitation for CP-nets; the elicitation complexity is equal to $O(n^k)$ such that n is the number of nodes and k is the maximal number of parents [20].

However, in CP-nets, preference expressed in a parent node tends to be more important than the one expressed in a child one [21]. In other words, violating a preference associated with a father node is more important than violating a preference associated with a child one; this priority *implicitly* given by the application of *ceteris paribus* assumption may be debatable. For instance, in the CP-net of Fig. 4, configuration $t_1p_1c_2s_1$ is preferred to configuration $t_2p_2c_2s_2$ because there is a sequence of worsening flips from the former to the latter, as seen in the graph of Fig. 5. Moreover, this kind of priority is not transitive in the sense that CP-nets cannot always decide whether violating preferences of two children nodes is preferred to violating preferences associated with one child and one grandson node respectively (which might have been expected as being less damaging than violating two children preferences) [22]. This limitation is problematic. Generally, there are partial preference orderings that CP-nets cannot express, see [9] for a counterexample.

The expressive power of CP-nets is limited. In particular, we are unable to specify importance relations between variables, beside those implicitly imposed between parents and children. Tradeoffs-enhanced CP-nets (TCP-nets) [23] are an extension of CP-nets that adds a notion of importance between the variables by enriching the network with new arcs. TCP-nets obey the preference statements induced by *ceteris paribus* assumption, since the ordering obtained is a refinement of the CP-nets ordering. In fact, the refinement brought by TCP-nets cannot override the implicit priority in favor of parents nodes.

4.2. π -pref nets vs. CP-nets

In this section, we show that the configuration graph of any CP-net can be refined using a π -pref net without local indifference, based on the same preference network, provided some constraints on products of symbolic weights are added to the π -pref net, in order to restore some *ceteris paribus* assumption-based priorities. Precisely, the added constraints reflect the higher importance of father nodes with respect to their children.

The preferences expressed by the CP-nets can be represented by a π -pref net sharing the same graphical structure and where the conditional possibility distributions encode the local preferences.

Example 8. Consider the preference network of Fig. 4. Encoded as a possibilistic network it reads: $\pi(t_1) = 1$, $\pi(t_2) = \alpha$, $\pi(p_1|t_1) = \pi(p_2|t_2) = 1$, $\pi(p_2|t_1) = \beta_1$, $\pi(p_1|t_2) = \beta_2$, $\pi(s_1|t_1) = \gamma_1$, $\pi(s_2|t_2) = \gamma_2$, $\pi(s_2|t_1) = \pi(s_1|t_2) = 1$, $\pi(c_1|p_1) = \pi(c_2|p_2) = 1$, $\pi(c_2|p_1) = \delta_1$ and $\pi(c_1|p_2) = \delta_2$. Applying the product-based chain rule, we can compute the joint possibility distribution relative to T, P, C and S. Thin arrows in Fig. 6 represent the configuration graph induced from this joint possibility distribution, and bold arrows reflect additional *ceteris paribus* comparisons for the corresponding CP-nets. Clearly, the configuration $t_1p_1c_1s_2$ is the root (since it is the unique one with degree $\pi(t_1p_1c_1s_2) = 1$).

Given the ordinal nature of preference tables of CP-nets, and the fact that we restrict to Boolean variables, it also makes sense to characterize the quality of ω using the set $\mathcal{S}(\omega) = \{A_i : \alpha_i = 1\}$ of satisfied preference statements (one per variable), where $\alpha = (\alpha_1, \dots, \alpha_n)$. It is then clear that the Pareto ordering between configurations induced by the preference tables is refined by comparing these satisfaction sets:

$$\alpha \succ_{prod} \beta \Rightarrow \mathcal{S}(\omega') \subset \mathcal{S}(\omega) \quad (2)$$

since the only case when $\alpha_i > \beta_i$ is when $\alpha_i = 1$ if variables are Boolean and \succ_{prod} precisely coincides with the Pareto ordering, after Proposition 1.

Example 9. To see that this inclusion-based ordering is stronger than the π -pref net ordering, consider Fig. 6 where $\pi(t_1 p_2 c_1 s_2) = \beta_1 \delta_2$ with $S(t_1 p_2 c_1 s_2) = \{T, S\}$ and $\pi(t_2 p_1 c_2 s_1) = \alpha \beta_2 \delta_1$ with $S(t_2 p_1 c_2 s_1) = \{S\}$. We do have that $S(t_1 p_2 c_1 s_2) \supset S(t_2 p_1 c_2 s_1)$, but $\beta_1 \delta_2$ is not comparable with $\alpha \beta_2 \delta_1$. Dotted and thin arrows of Fig. 6 represent the configuration graph induced by comparing sets $S(\omega)$

The inclusion-based ordering $S(\omega') \subset S(\omega)$ does not depend on the parent variables contexts, but only on the fact that in the context of a configuration of its parents, a variable has a preferred value (we call “good”) or a less preferred value (we call “bad”).

In the following, we assume that the components of vector α are linearly ordered in agreement with the partial ordering of variables in the symbolic preference network, namely, if $i < j$ then A_i is not a descendent of A_j in the preference net (i.e., topological ordering). For instance in the preference net of Fig. 4, we can use the ordering (T, P, C, S) .

Let us first prove that, in the configuration graphs induced by a CP-net and the corresponding π -pref net, there cannot be any preference reversals between configurations. Let $\mathcal{C}(A)$ denote the children set of $A \in V$.

Lemma 1. *If $\omega \succ_{CP} \omega'$ and these configurations differ by one flip of some variable A_i , then the inclusion $S(\omega) \subset S(\omega')$ is not possible.*

Proof. Compare $S(\omega)$ and $S(\omega')$. It is clear that $A_i \notin S(\omega')$ (otherwise the flip would not be improving) and $S(\omega) = (S(\omega') \cup \{A_i\} \cup \mathcal{C}_+^+(A_i)) \setminus \mathcal{C}_+^-(A_i)$, where $\mathcal{C}_+^+(A_i)$ is the set of children variables that switch from a bad to a good value when going from ω' to ω , and $\mathcal{C}_+^-(A_i)$ is the set of children variables that switch from a good to a bad value when going from ω' to ω . It is clear that it can never be the case that $S(\omega) \subset S(\omega')$, indeed A_i is in $S(\omega)$ and not in $S(\omega')$ by construction. But $S(\omega')$ may contain variables not in $S(\omega)$ (those in $\mathcal{C}_+^-(A_i)$ if not empty). \square

In the following, given two configurations ω and ω' , let $\mathcal{D}^{\omega, \omega'}$ the set of variables which bear different values in ω and ω' .

Proposition 9. *If $\omega \succ_{CP} \omega'$ then $S(\omega') \subset S(\omega)$ is not possible.*

Proof. If $\omega \succ_{CP} \omega'$, then there is a chain of improving flips $\omega_0 = \omega' \prec_{CP} \omega_1 \prec_{CP} \dots \prec_{CP} \omega_k = \omega$. Applying the above Lemma 1, $S(\omega_i) = (S(\omega_{i-1}) \cup \{V_{i-1}\} \cup \mathcal{C}_+^+(V_{i-1})) \setminus \mathcal{C}_+^-(V_{i-1})$ for some variable $V_{i-1} = A_j$. By the above Lemma 1, we cannot have $\omega_i \prec_{\pi} \omega_{i-1}$. Suppose we choose the chain of improving flips by flipping at each step a top variable A_j in the preference net, among the ones to be flipped, i.e., $j = \min\{\ell : A_\ell \in \mathcal{D}^{\omega_{i-1}, \omega}\}$. It means that when following the chain of improving flips, the status of each flipped variable will not be questioned by later flips, as no flipped variable will be a child of variables flipped later on. So $S(\omega')$ will contain some variables not in $S(\omega)$, so $\omega \succ_{prod} \omega'$ is not possible. \square

The previous results show that it is impossible to have a preference reversal between CP-net ordering and the inclusion ordering, which implies that no preference reversal is possible between the CP-net ordering and the π -pref net ordering. It suggests that we can try to add *ceteris paribus* constraints to a π -pref net and so as to capture the preferences expressed by a CP-net. Define a preference relation \succ_{prod}^+ between configurations as follows

$$\omega \succ_{prod}^+ \omega' \iff \omega \succ_{prod} \omega' \text{ or } \omega \succ_{CP} \omega'.$$

As mentioned earlier, in CP-nets, parents preferences are more important than children ones. This property is not ensured by π -pref nets where all violations are considered having the same importance. In the following, we lay bare local constraints between products of symbolic weights, pertaining to each node and its children, that enable *ceteris paribus* assumption to be simulated.

Let ω, ω' differ by one flip, and such that none of $\omega \succ_{prod} \omega', \omega' \succ_{prod} \omega$ holds, and moreover, $\omega \succ_{CP} \omega'$. We must enforce the condition $\pi(\omega) > \pi(\omega')$. Suppose the flipping variable is A . Clearly, $A \in S(\omega)$, but $A \notin S(\omega')$. Let α be the possibility degree of A when it takes the bad value in context $p(A)$ (it is 1 when it takes the good value). When flipping A from a good to a bad value, only the quality of the children variables $\mathcal{C}(A)$ of A may change. $\mathcal{C}(A)$ can be partitioned into at most 4 sets, $\mathcal{C}_-^-(A)$ (resp. $\mathcal{C}_+^-(A), \mathcal{C}_+^+(A), \mathcal{C}_-^+(A)$), which represents the set of children of A whose values remain bad (resp. change from good to bad, from bad to good, and stay good) when flipping A . Then it can be easily checked that:

$$\begin{aligned} \pi(\omega) &= 1 \cdot \prod_{C_i \in \mathcal{C}_+^+(A)} \gamma_i \cdot \prod_{C_j \in \mathcal{C}_-^-(A)} \gamma_j \cdot \beta \\ \pi(\omega') &= \alpha \cdot \prod_{C_k \in \mathcal{C}_+^-(A)} \gamma_k \cdot \prod_{C_l \in \mathcal{C}_-^-(A)} \gamma_l \cdot \beta \end{aligned}$$

where β is a product of symbols, pertaining to nodes other than A and its children, that remain unchanged by the flip of A . Then the constraint $\pi(\omega) > \pi(\omega')$ comes down to the inequality:

$$\prod_{C_i \in \mathcal{C}_+^+(A)} \gamma_i > \alpha \cdot \prod_{C_k \in \mathcal{C}_+^-(A)} \gamma_k \tag{3}$$

where symbols appearing on one side do not appear on the other side. Such constraints are sufficient to retrieve the preferences of the CP-net. Note that the preferences $\omega \succ_{prod} \omega'$ and $\omega \succ_{CP} \omega'$ conjointly hold in both approaches whenever A has

no child node, and more generally whenever the down flip on A corresponds to no child variable moving from a bad to a good state, i.e. $C^\pm(A) = \emptyset$. Thus, adding constraints (3) to the π -pref net constraints, we get a refined configuration graph that includes preferences induced by the CP-net.

Example 10. The π -pref net configuration graph of Fig. 6 must be completed using the constraints (3) for the five bold arrows, corresponding to the *ceteris paribus* preferences not captured by the π -pref net, which come down to enforcing $\alpha < \beta_1\gamma_1$, $\beta_1 < \delta_1$ and $\beta_2 < \delta_2$. It turns out that we recover the CP-net ordering exactly. Especially, the dotted arrows (corresponding to the inclusion relations $\mathcal{S}(\omega') \supset \mathcal{S}(\omega)$) are obtained from the CP-net ordering by chains of worsening flips.

In the example, we exactly capture the preference graph of a CP-net using additional constraints between products of symbolic weights. This observation and the above considerations thus encourage us to study whether π -pref nets without constraints are *refined* by CP-nets, namely that the configuration graph of the former contains less strict preferences between configurations than the one of the latter, so that adding the constraints (3) are enough to exactly simulate a CP-net by a π -pref net with constraints. A sufficient condition would be that $\mathcal{S}(\omega) \subset \mathcal{S}(\omega')$ implies $\omega' \succ_{CP} \omega$. Note that if it were not the case, it would mean that CP-nets do not respect Pareto-ordering, which would cast a doubt on the rational nature of CP-nets. However, providing a formal proof looks tricky. This is a topic for further research.

4.3. π -pref nets vs. OCF-nets

Ordinal Conditional Functions (OCF) [24] are an uncertainty representation framework very close to possibility theory [25] that have been recently used for preference modeling [8]. They are functions $\kappa : \Omega \rightarrow \mathbb{N}$ such that $\kappa(\omega) = 0$ for some configuration ω , which means, in the preference setting that ω is fully acceptable, while the greater $\kappa(\omega)$, the less satisfactory it is. Besides it is assumed that $\kappa(U \cup V) = \min(\kappa(U), \kappa(V))$.

OCF-nets are like numerical possibilistic networks, except that they obey an additive chain rule of the form:

$$\kappa(\omega) = \sum_{i=1}^n \kappa(A_i | p(A_i)) \quad (4)$$

In [26], it was pointed out that the set-function $\pi_\kappa(A_i) = 2^{-\kappa(A_i)}$ is a possibility measure. The converse holds to some extent insofar as if $\pi(A_i) = \alpha$, the values $\kappa(A_i) = -\log_2(\alpha)$ are integer rank weights. In fact in the symbolic setting, we can indifferently use π -pref nets and OCF-nets, since the product of variables on $[0, 1]$ where 1 indicates full satisfaction, behaves exactly like the sum of variables on the positive integers, where 0 indicates full satisfaction.

Eichhorn et al. [8] proved that numerical OCF-nets can refine CP-net orderings. Precisely, OCF-nets will lead always to total orderings that are compatible with CP-nets. To do so they use a set of particular constraints to be imposed on their integer weights, which basically correspond to our constraints (3), albeit between numerical values. In contrast the use of symbolic weights in our approach preserves the partiality of the ordering, thus enabling the CP-net configuration graph to be exactly recovered. Moreover the use of symbolic weights does not commit us to the choice of particular numerical values.

4.4. π -pref nets vs. CP-theories

π -pref nets can also be compared with so-called CP-theories [27]. The latter interpret conditional preference statements assuming they hold *irrespectively* of the values of other variables. It means that any configuration ω where A takes a preferred value a^+ , according to a preference statement in the context $p(A)$, is preferred to any ω' where A takes a less preferred value a^- , in the same context $p(A)$. This constraint reads

$$\min_{\omega: \omega[\mathcal{P}(A) \cup \{A\}] = p(A)a^+} \pi(\omega) > \max_{\omega: \omega[\mathcal{P}(A) \cup \{A\}] = p(A)a^-} \pi(\omega).$$

In terms of possibility functions, it reads $\Delta(p(A) \wedge a^+) > \Pi(p(A) \wedge a^-)$, where $\Delta(F) = \min_{\omega \in F} \pi(\omega)$ [28]. In [27] are studied hybrid nets where some variables are handled using the *ceteris paribus* assumption, while the preference holds irrespectively of other variables. In π -pref nets preference statements are interpreted by $\pi(a^+ | p(A)) > \pi(a^- | p(A))$ which is provably equivalent to $\Pi(p(A) \wedge a^+) > \Pi(p(A) \wedge a^-)$, i.e., comparing best configurations. It is clear that if $\omega \succ_{CP} \omega'$ holds, then $\omega \succ \omega'$ holds in a CP-theory, where conditional preference holds irrespectively of other variables, because the CP-theory generates more preference constraints between configurations, including the ones induced by the *ceteris paribus* assumption. Constraints induced by CP theories can thus be captured in π -pref nets by adding more constraints between products of symbolic weights.

5. Conclusion

This paper presents an approach to preference modeling based on joint possibility distributions obtained from a conditional preference network, albeit without using numerical possibility values to represent preference intensity. We have used uninstantiated symbols taking values on the unit interval and constraints between them to describe local relative preferences. Then we compute the product thereof to assign symbolic possibility values to complete configurations (solutions) using the product chain rule of possibilistic networks. The preference graph between configurations is then obtained

by comparing composite symbolic possibility values. We have shown the connections existing between π -pref nets and CP-nets, showing that while π -pref nets capture the Pareto ordering between configurations described by vectors of local satisfaction degrees, the *ceteris paribus* assumption of CP-nets can be modeled by adding new constraints between products of symbols appearing in the π -pref net preference tables. In some sense, π -pref nets are a more flexible approach to preference modeling than CP-nets. In particular π -pref nets can express conditional indifference as well (as, e.g., in [Example 1](#)). Besides, as possibilistic networks, their contents can be put equivalently under a logical form [12]. Lastly, we have laid bare the question of determining whether the CP-net preference ordering always refines the one induced by π -pref nets, which remains an open problem.

Appendix. Proof of Proposition 6

We proceed in several steps. First recall that the implications: $\alpha \succ_{SP} \beta$ implies $\alpha \succ_{prod} \beta$ and $\alpha \succeq_{SP} \beta$ implies $\alpha \succeq_{prod} \beta$ are obvious since the product is symmetric and monotonically increasing.

For the converse, we must basically show that if α is SP-incomparable with β then they are also incomparable wrt the product ordering. We use several lemmas.

Lemma 2. *If Proposition 6 holds for a set of constraints C , it holds a fortiori for any subset of constraints in C .*

Indeed taking away constraints from C yields more freedom in the choice of values for the coefficients in order to ensure the incomparability of the symbolic product expressions associated to each vector. As a consequence of this lemma, the result should be established with the maximal amount of constraints, namely assuming a (non-trivial) complete pre-ordering of the symbolic coefficients appearing in the two vectors.

Lemma 3 ([29]). *Consider two symbolic vectors $\alpha = (\alpha_1, \dots, \alpha_n)$ such that C enforces $\alpha_1 \leq \dots \leq \alpha_n$ and $\beta = (\beta_1, \dots, \beta_n)$, where the symbols are totally ordered in the sense of a complete preordering. Let τ be a permutation of the components of β such that $\beta_{\tau(1)} \leq \dots \leq \beta_{\tau(n)}$ and β_τ the corresponding reordered vector. Then $\alpha \succ_{SP} \beta$ if and only if $\alpha \succ_P \beta_\tau$.*

In the totally ordered setting it gives a constructive way of expressing the SP-ordering as applying the Pareto ordering to the increasingly reordered vectors.

Without loss of generality, due to [Lemma 3](#), we can assume that vectors are increasingly ordered. Now we can try to prove that if α and β are SP-incomparable then they are so for product. If they are SP-incomparable then there are $i \neq j$ such that $\alpha_i > \beta_i$ and $\alpha_j < \beta_j$. The most constrained case is when there is one constraint of the form $\alpha_i > \beta_i$, while all the other ones are of the form $\alpha_j < \beta_j$. In that case $\prod_{j \neq i} \beta_j > \prod_{j \neq i} \alpha_j$ and denoting by α_{-i} the vector α deprived of component i , we also have $\alpha_{-i} \succ_{SP} \beta_{-i}$.

Let us show that this strong prerequisite does not enforce an inequality between $\prod_{j=1}^n \beta_j$ and $\prod_{j=1}^n \alpha_j$. First, if α_i and β_j are very close, then $\prod_{j=1}^n \beta_j > \prod_{j=1}^n \alpha_j$. Now, for the reversed inequality, replace α_j by α_1 for $j < i$, and by α_i for $j > i$, β_j by β_i for $j < i$ and by β_r for $j > i$. The inequality $(\alpha_1)^{i-1} \cdot (\alpha_i)^{n-i+1} > (\beta_i)^i \cdot (\beta_r)^{n-i}$ is more demanding than the inequality $\prod_{j=1}^n \alpha_j > \prod_{j=1}^n \beta_j$. Let us show we can satisfy the former because $\alpha_i > \beta_i$, even if $\alpha_1 < \alpha_i$, $\beta_i < \beta_n$. To see it, we can write $\alpha_1 = \alpha_i/p$ and $\beta_r = q\beta_i$ with $p, q > 1$. It is easy to see that the inequality now writes $\frac{\alpha_i}{\beta_i} > p^{\frac{i-1}{n}} q^{\frac{n-i}{n}} > 1$. It is clear that we can set $p, q > 1$ and find $\alpha_i > \beta_i \in [0, 1]$ that verifies this inequality.

References

- [1] J. Pearl, Probabilistic Reasoning in Intelligent Systems: Networks of Plausible Inference, Morgan Kaufmann, 1988.
- [2] J. Kacprzyk, Multistage Decision-Making Under Fuzziness, Verlag TÜV Rheinland, Köln, 1983.
- [3] R.I. Brafman, C. Domshlak, Database Preference Queries Revisited, Technical Report TR2004-1934, Cornell University, Computing and Information Science, 2004.
- [4] D. Dubois, H. Prade, F. Touazi, A possibilistic logic approach to conditional preference queries, in: H.L. Larsen, M.J. Martín-Bautista, M.A. Vila, T. Andersen, H. Christiansen (Eds.), Flexible Query Answering Systems - Proceedings of 10th International Conference, FQAS'13, Granada, Sept. 18–20, LNCS, 8132, Springer, 2013, pp. 376–388.
- [5] J. Kacprzyk, S. Zadrozny, A. Ziolkowski, FQUERY III +: a “human-consistent” database querying system based on fuzzy logic with linguistic quantifiers, Inf. Syst. 14 (6) (1989) 443–453.
- [6] C. Boutilier, R. Brafman, C. Domshlak, H. Hoos, D. Poole, CP-nets: a tool for representing and reasoning with conditional *ceteris paribus* preference statements, J. Artif. Intell. Res. 21 (2004) 135–191.
- [7] N. Wilson, Computational techniques for a simple theory of conditional preferences, Artif. Intell. 175 (7–8) (2011) 1053–1091.
- [8] C. Eichhorn, M. Fey, G. Kern-Isberner, CP-and OCF-networks—a comparison, Fuzzy Sets Syst. 298 (2016) 109–127.
- [9] N. Ben Amor, D. Dubois, H. Gouider, H. Prade, Possibilistic conditional preference networks, in: Proceedings of 13th European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU'15), Compiègne, France, Jul. 15–17, in: LNCS, 9161, Springer, 2015, pp. 36–46.
- [10] N. Ben Amor, D. Dubois, H. Gouider, H. Prade, Preference modeling with possibilistic networks and symbolic weights: A theoretical study, in: G.A. Kaminka, M. Fox, P. Bouquet, E. Hillermeier, V. Dignum, F. Dignum, F. van Harmelen (Eds.), Proceedings of 22nd European Conference on Artificial Intelligence (ECAI'16), The Hague, Aug. 29 - Sept. 2, Frontiers in AI and Appl., 285, IOS Press, 2016, pp. 1203–1211.
- [11] N. Ben Amor, D. Dubois, H. Gouider, H. Prade, Graphical models for preference representation: an overview, in: S. Schockaert, P. Senellart (Eds.), Proceedings of 10th International Conference on Scalable Uncertainty Management (SUM'16), Nice, Sept. 21–23, LNCS, 9858, Springer, 2016, pp. 96–111.
- [12] S. Benferhat, D. Dubois, L. García, H. Prade, On the transformation between possibilistic logic bases and possibilistic causal networks, Int. J. of Approx. Reasoning 29 (2) (2002) 135–173.
- [13] N. Ben Amor, D. Dubois, H. Gouider, H. Prade, Possibilistic networks: a new setting for modeling preferences, in: Proceedings of 8th International Conference on Scalable Uncertainty Management (SUM'14), (U. Straccia, A. Cali, eds.), Oxford, Sept.15–17, in: LNCS, 8720, Springer, 2014, pp. 1–7.
- [14] D. Dubois, H. Prade, Possibility Theory: An Approach to Computerized Processing of Uncertainty, Plenum Press, 1988.

- [15] L.A. Zadeh, Fuzzy sets as a basis for a theory of possibility, *Fuzzy Sets Syst.* 1 (1978) 3–28.
- [16] N. Ben Amor, S. Benferhat, K. Mellouli, Anytime propagation algorithm for min-based possibilistic graphs, *Soft Comput.* 8 (2) (2003) 150–161.
- [17] D. Nilsson, An efficient algorithm for finding the m most probable configurations in probabilistic expert systems, *Stat Comput.* 8 (2) (1998) 159–173.
- [18] C. Boutilier, Brafman, I. Ronen, H. Hoos, D. Poole, Reasoning with conditional ceteris paribus preference statements, in: K. Laskey, H. Prade (Eds.), *Proceedings of 15th Conf. on Uncertainty in Artificial Intelligence (UAI'99)*, 1999, pp. 71–80.
- [19] J. Goldsmith, J. Lang, M. Truszczynski, N. Wilson, The computational complexity of dominance and consistency in CP-nets, *J. Artif. Intell. Res.* (2008) 403–432.
- [20] F. Koriche, B. Zanuttini, Learning conditional preference networks, *Artif. Intell.* 174 (11) (2010) 685–703.
- [21] D. Dubois, A. Hadjali, H. Prade, F. Touazi, Erratum to: database preference queries - a possibilistic logic approach with symbolic priorities, *Ann. Math. Artif. Intell.* 73 (3–4) (2015) 359–363.
- [22] D. Dubois, H. Prade, F. Touazi, Conditional preference-nets, possibilistic logic, and the transitivity of priorities, in: M. Bramer, M. Petridis (Eds.), *Research and Development in Intelligent Syst. XXX, Proceedings of 33rd SGAI International Conference on Innovative Techniques and Applications of Artificial Intelligence (AI'2013)*, Cambridge, UK, Dec. 10–12, 2013, pp. 175–184.
- [23] R.I. Brafman, C. Domshlak, TCP-nets for preference-based product configuration, in: *The Forth Workshop on Configuration (ECAI)*, July 22–23, 2002, pp. 101–106.
- [24] W. Spohn, Ordinal conditional functions: a dynamic theory of epistemic states, in: W.L. Harper, B. Skyrms (Eds.), *Causation in Decision, Belief Change, and Statistics*, 2, D. Reidel, 1988, pp. 105–134.
- [25] D. Dubois, H. Prade, Qualitative and semi-quantitative modeling of uncertain knowledge—a discussion, in: C. Beierle, G. Brewka, M. Thimm (Eds.), *Computational Models of Rationality Essays Dedicated to Gabriele Kern-Isberner on the occasion of Her 60th Birthday*, College Publications, 2016, pp. 280–296.
- [26] D. Dubois, H. Prade, Epistemic entrenchment and possibilistic logic, *Artif. Intell.* 50 (2) (1991) 223–239.
- [27] N. Wilson, Extending CP-nets with stronger conditional preference statements, in: A.G. Cohn (Ed.), *Proceedings of 19th National Conference on Artificial Intelligence (AAAI'04)*, San Jose, July 25–29, 2004, pp. 735–741.
- [28] D. Dubois, H. Prade, Possibility theory and its applications: where do we stand?, in: J. Kacprzyk, W. Pedrycz (Eds.) *Springer Handbook of Computational Intelligence*, Springer, 2015, pp. 31–60.
- [29] C. Cayrol, D. Dubois, F. Touazi, Ordres Partiels entre Sous-Ensembles d'un Ensemble Partiellement Ordonné, Research report RR-2014-02-FR, IRIT, Université Paul Sabatier, Toulouse, 2014. www.irit.fr/publis/ADRIA/PapersCayrol/Rapport-IRIT-CDT-2014-02.pdf.