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The Derived Category of a Locally Complete Intersection Ring

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THE DERIVED CATEGORY OF A LOCALLY COMPLETE INTERSECTION RING

by

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A DISSERTATION

Presented to the Faculty of

The Graduate College at the University of Nebraska

In Partial Fulfilment of Requirements

For the Degree of Doctor of Philosophy

Major: Mathematics

Under the Supervision of Professors Luchezar L. Avramov and Mark E. Walker

Lincoln, Nebraska

August, 2019

THE DERIVED CATEGORY OF A LOCALLY COMPLETE INTERSECTION RING

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University of Nebraska, 2019

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Let R be a commutative noetherian ring. A well-known theorem in commutative algebra states that R is regular if and only if every complex with finitely generated homology is a perfect complex. This homological and derived category characterization of a regular ring yields important ring theoretic information; for example, this characterization solved the well-known "localization problem" for regular local rings. The main result of this thesis is establishing an analogous characterization for when R is locally a complete intersection. Namely, R is locally a complete intersection if and only if each nontrivial complex with finitely generated homology can build a nontrivial perfect complex in the derived category using finitely many cones and retracts. This answers a question of Dwyer, Greenlees and Iyengar posed in 2006 and yields a completely triangulated category characterization of locally complete intersection rings. Moreover, this work gives a new proof that a complete intersection localizes.

DEDICATION

To my parents, John and Tess.

To my advisors, Lucho and Mark.

To all my dearest friends.

Thank you, thank you, thank you!

ACKNOWLEDGMENTS

First and foremost, I thank my thesis advisors Luchezar Avramov and Mark Walker. Regarding this work, I thank them for carefully reading and discussing preliminary drafts of [24]. In general, the support I have received from both of them has been invaluable for my development as a mathematician; a student could not have two better advisors.

I need to thank Srikanth Iyengar. We had several helpful conversations on [24], and related topics, while he hosted me during a week's visit in Utah. His passion for mathematics has also been a large influence on me mathematically.

The algebra group at UNL has been tremendous. I would like to thank Alexandra Seceleanu and Tom Marley for being readers of this thesis. Also, I appreciate them always having open doors so that we could discuss mathematics. Thank you, Roger and Sylvia Wiegand; whenever you were in town, Lincoln become that much better. I have learned so much from Nicholas Packaukas, Luigi Ferraro, Mohsen Gheibi, Seth Lindokken and Alessandro De Stefani. It was always a pleasure doing math with you and ending up at Happy Raven after CARS. I have appreciated discussing mathematics, as well as other things, with Eric Canton, Andrew Windle, Amadeus Martin, Taran Funk, Ben Drabkin, Matthew Bachmann and Michael DeBellevue.

Outside of Algebra, I would like to thank Marilyn Johnson for all of her help and knowledge. Also, thanks goes to Shari Stenberg for serving as my outside representative on my committee. An immeasurable amount of thanks is due to some amazing friends: Jessie Jamieson, Nathan Poppelreiter, Matthew Reichenbach, and Allison Dean. You brought a great deal of balance to my life.

Lastly, I need to thank my family and Rebecca Straub for their love, support and, most importantly, their understanding.

GRANT INFORMATION

The author was partly supported through NSF grant DMS 1103176.

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Chapter 1

Introduction

Preliminary Remark: This thesis is an extended version of the author's article [24]. Currently, [24] has been submitted for publication and is posted at arxiv.org under the reference number arXiv:1805.07627.

Historically, in commutative algebra, it has been effective to study the homological properties of a ring's category of modules to gather ring theoretic information. More recently, studying the structure of a ring's derived category has been a new and enlightening approach. The framework of the derived category allows one to draw from a wealth of ideas from homotopy theory and triangulated categories.

One of the first, and major, accomplishments of importing homological methods into commutative algebra is the celebrated theorem of Auslander-Buchsbaum and Serre. Recall that a local ring is regular if its unique maximal ideal is generated by a regular sequence. The Auslander-Buchsbaum and Serre theorem provides the following homological characterization of a regular local ring: *a local ring R is regular if and only if every finitely generated R-module has finite projective dimension.* This theorem solved the long open *localization problem* for regular rings. Said precisely, an immediate corollary of their result is that if R is a regular local ring, then $R_\mathfrak{p}$ is regular for every prime ideal $\mathfrak{p}$.

For the rest of the introduction, let R be a commutative noetherian ring and let $\mathsf{D}(R)$ denote the derived category of R. Basic information in $\mathsf{D}(R)$ is contained in its full subcategory consisting of complexes with finitely generated homology, denoted $\mathsf{D}^f(R)$. Studying $\mathsf{D}^f(R)$ as a subcategory of $\mathsf{D}(R)$ is analogous to examining the finitely generated R-modules when working in the category of R-modules. Another interesting class of objects in $\mathsf{D}(R)$ is the collection of small objects; these are the complexes isomorphic to bounded complexes of finitely generated projective R-modules. Loosely speaking, the small objects of $\mathsf{D}(R)$ are exactly the complexes of finite projective dimension. The Auslander-Buchsbaum and Serre theorem can be translated and strengthened to the following homotopical characterization of regular rings: *a commutative noetherian ring R is regular if and only if every object of $\mathsf{D}^f(R)$ is a small object of $\mathsf{D}(R)$.*

In many regards, the local rings that are closest to being regular are complete intersections. A local ring $(R, \mathfrak{m})$ is said to be a complete intersection if its $\mathfrak{m}$-adic completion is isomorphic to a regular local ring modulo a regular sequence. The main contribution of this thesis is establishing a homotopical characterization for a locally complete intersection ring.

In 2006, Dwyer, Greenlees and Iyengar laid the foundations for finding such a characterization for complete intersections [12, 9.4]. Namely, they proved the following result: *if R is a complete intersection, then every object of $\mathsf{D}^f(R)$ finitely builds a nontrivial small object*. Roughly speaking, a complex M finitely builds a complex N provided that N can be obtained from M using finitely many cones and retracts (see 3.1.7 for a precise statement). Moreover, they showed that if every nontrivial object of $\mathsf{D}^f(R)$ finitely builds a nontrivial small object then R is Gorenstein, but the converse does not hold (see [12, 9.11,9.13,9.14]). This led them to ask the following question in [12, 9.10]:

Question 1. *If every nontrivial object of $\mathsf{D}^f(R)$ finitely builds a nontrivial small object, is R a complete intersection?*

The main result of this thesis is the following:

Theorem 1. *Let R be a commutative noetherian local ring. The following are equivalent.*

1. *R is a complete intersection.*
2. *Every nontrivial object of $\mathsf{D}^f(R)$ finitely builds a nontrivial small object of $\mathsf{D}(R)$.*
3. *For each nontrivial object M of $\mathsf{D}^f(R)$, there exists a small object P of $\mathsf{D}(R)$ such that M finitely builds P and $\operatorname{Supp}_R M = \operatorname{Supp}_R P$.*

In particular, this theorem settles Question 1 in the affirmative. The implication "(1) $\implies$ (3)" is a strengthening of [12, 9.4] that exploits a construction of P. A. Bergh from [10, 3.2]. The support condition in (3), forces the small object P, built by the nontrivial object M, to also be nontrivial. Hence, "(3) $\implies$ (2)" holds immediately; combining this implication with "(1) $\implies$ (3)" provides a new proof of [12, 9.4]. A bulk of the work in this thesis is done to establish "(2) $\implies$ (1)", which answers the original question set forth by Dwyer, Greenlees and Iyengar.

Similar to the Auslander-Buchsbaum and Serre theorem discussed above, the structural characterization of a complete intersection's derived category in Theorem 1 yields ring theoretic information. Namely, condition (3) in Theorem 1 localizes. Hence, we recover a theorem of Avramov [4] as an immediate corollary of Theorem 1

Corollary. *If R is a complete intersection, then $R_{\mathfrak{p}}$ is a complete intersection for every $\mathfrak{p} \in \operatorname{Spec} R$.*

Finally, some extra work is done to establish a global characterization for rings that are locally complete intersections.

Theorem 2. *For a commutative noetherian ring R, R is locally a complete intersection ring if and only if nontrivial object of $\mathsf{D}^f(R)$ finitely builds a nontrivial small object.*

Outline. Much of this work relies heavily on DG algebra techniques. A major tool is DG homological algebra and studying the derived category of a DG algebra. Hence, Chapter 2 is devoted to providing the necessary background for the rest of the paper. Section 2.1 is comprised of notation and conventions that will be used throughout the paper. The reader should feel free to skip this section and refer to it as needed. Sections 2.2 and 2.3 contain foundational material involving DG homological algebra, triangulated categories, and the derived category of a DG algebra. These sections can be skipped be the experts.

Chapter 3 reviews the theory of support and localization in two settings. Section 3.1 is concerned with localization and support in the derived category of a commutative noetherian ring; while Section 3.2 focuses on homogeneous localization and the corresponding homogeneous/graded support for graded modules over a graded commutative noetherian ring. The only new result in this Chapter 3 is Lemma 3.1.8. The rest of Chapter 3 is background that will be applied later.

Chapter 4 contains a large portion of the technical work in this document. Section 4.1 reviews basic concepts regarding Koszul complexes. It is worth noting that in this thesis we are interested in Koszul complexes, not just as complexes, but as DG algebras. Hence, Chapter 4 is where many of the tools in Chapter 2 are applied. Section 4.2 has one of the key technical results used to establish Theorem 1. Theorem 4.2.1 exploits the theory of DG Γ-algebras. References are given in this section for the necessary background on DG Γ-algebras. Section 4.3 develops a theory of cohomology operators for pairs of DG modules over a Koszul complex. These operators are an extension of the operators defined over a complete intersection studied by Gulliksen in [16], Avramov and Buchweitz in [5] and [6], Eisenbud in [14], and many others. Importing the theory of support for graded modules from Section 3.2 allows us to define and study a theory of support varieties for DG modules over a Koszul complex in Section 4.4.

Chapter 5 discusses the content from the introduction. Sections 5.1 and 5.2 are mostly setup regarding thick subcategories, virtually small objects and proxy small objects. Finally, Section 5.3 contains the proofs of Theorem 1 and Theorem 2. The varieties defined in Section 4.4 are put to use in this final section to obtain the main results of this thesis.

Chapter 2

Preliminaries

2.1 Notation and Conventions

We fix a commutative noetherian ring Q.

2.1.1. By a graded Q-module, we mean a family of Q-modules $M = \{M_i\}_{i\in\mathbb{Z}}$. An element m of M has homological degree i, denoted $|m| = i$, provided that $m \in M_i$. Every Q-module can be regarded as a graded Q-module concentrated in degree zero. That is, we associate to each Q-module M the graded Q-module $\{M_i\}_{i\in\mathbb{Z}}$ given by

$$M_i := \begin{cases} M & i = 0 \\ 0 & i \neq 0 \end{cases}$$

Abusing notation, we write M for this graded Q-module, as well.

2.1.2. A graded A-module M can be graded cohomologically where $M = \{M^i\}_{i\in\mathbb{Z}}$ where $M^i := M_{-i}$ for all $i \in \mathbb{Z}$. An element m of M has cohomological degree i provided that $m \in M^i$.

2.1.3. A complex of Q-modules is a graded Q-module $M = \{M_i\}_{i\in\mathbb{Z}}$ equipped with a degree -1 Q-linear endomorphism $\partial^M = \{\partial_i^M : M_i \to M_{i-1}\}_{i\in\mathbb{Z}}$ satisfying $\partial_i^M \partial_{i+1}^M = 0$ for all $i \in \mathbb{Z}$. The map ∂^M is referred to as the differential of M. If $\partial^M = 0$, we say that M has trivial differential. Every graded Q-module can be regarded as a complex of Q-modules with trivial differential.

2.1.4. For a complex of Q-modules M, define $M^\natural$ to be the underlying graded Q-module. That is, $M^\natural = \{M_i\}_{i\in\mathbb{Z}}$ with trivial differential.

2.1.5. Let M be a complex of Q-modules. The boundaries and cycles of M are $\mathrm{B}(M) := \{\mathrm{Im}\,\partial^M_{i+1}\}_{i\in\mathbb{Z}}$ and $\mathrm{Z}(M) := \{\mathrm{Ker}\,\partial^M_i\}_{i\in\mathbb{Z}}$, respectively. The (co)homology of M is defined to be the graded Q-module

$$\mathrm{H}(M) := \mathrm{Z}(M)/\,\mathrm{B}(M) = \{\mathrm{H}_i(M)\}_{i\in\mathbb{Z}}.$$

2.1.6. Let M and N be complexes of Q-modules. We say that $\alpha : M \to N$ is a morphism of complexes provided α is a family of Q-linear maps $\{\alpha_i : M_i \to N_i\}_{i\in\mathbb{Z}}$ such that $\alpha_i\partial^M_{i+1} = \partial^N_{i+1}\alpha_{i+1}$ for all $i \in \mathbb{Z}$.

2.1.7. Let M and N be complexes of Q-modules. We define $M\otimes_Q N$ to be the complex of Q-modules

$$(M \otimes_Q N)_i := \bigoplus_{j\in\mathbb{Z}} M_j \otimes_Q N_{i-j}$$

and

$$\partial^{M\otimes_Q N} := \partial^M \otimes N + M \otimes \partial^N.$$

According to the sign-rule, the differential of $M\otimes_Q N$ is the Q-linear map determined by

$$\partial^{M\otimes_Q N}(m \otimes n) = \partial^M(m) \otimes n + (-1)^{|m|} m \otimes \partial^N(n).$$

2.1.8. Let M and N be complexes of Q-modules. We define $\mathrm{Hom}_Q(M, N)$ to be the complex of Q-modules

$$\mathrm{Hom}_Q(M,N)_i := \prod_{j\in\mathbb{Z}} \mathrm{Hom}_Q(M_j, N_{i+j})$$

and

$$\partial^{\mathrm{Hom}_Q(M,N)} := \mathrm{Hom}(M, \partial^N) - \mathrm{Hom}(\partial^M, N).$$

Again, according to the sign-rule the differential of $\mathrm{Hom}_Q(M,N)$ applied to an element of $\mathrm{Hom}_Q(M,N)$ is

$$\partial^{\mathrm{Hom}_Q(M,N)}(f) = \partial^N f - (-1)^{|f|} f \partial^M.$$

For $\alpha \in \mathrm{Hom}_Q(M,N)$, $\alpha \in \mathrm{Z}_0(\mathrm{Hom}_Q(M,N))$ if and only if α is a morphism of complexes.

2.1.9. A DG Q-algebra is a complex of Q-modules A equipped with two morphisms of complexes $\mu^A : A \otimes_Q A \to A$ and $\eta^A : Q \to A$ such the following diagrams commute:

$$\begin{array}{ccc} A\otimes_Q A\otimes_Q A & \xrightarrow{\mu^A\otimes A} & A\otimes_Q A \\ \Big\downarrow{\scriptstyle A\otimes\mu^A} & & \Big\downarrow{\scriptstyle \mu^A} \\ A\otimes_Q A & \xrightarrow{\mu^A} & A \end{array} \quad \text{and} \quad \begin{array}{ccccc} & \overset{\eta^A\otimes A}{\nearrow} & A\otimes_Q A & \overset{A\otimes\eta^A}{\nwarrow} & \\ & & \Big\downarrow{\scriptstyle \mu^A} & & \\ Q\otimes_Q A & \xrightarrow{\cong} & A & \xleftarrow{\cong} & A\otimes_Q Q \end{array}$$

We will write ab for $\mu^A(a\otimes b)$ and let 1 denote $\eta^A(1_Q)$. Because μ^A is a morphism of complexes, the *Leibniz-rule* holds:

$$\partial^A(ab) = \partial^A(a)b + (-1)^{|a|} a\partial^A(b)$$

for all $a, b \in A$. Furthermore, we will impose the condition that a DG Q-algebra is graded commutative. That is,

$$ab = (-1)^{|a||b|} ba$$

for all $a, b \in A$. A commutative Q-algebra is a DG Q-algebra concentrated in degree 0.

For the rest of the section, we fix a DG Q-algebra A.

2.1.10. A straightforward calculation shows that $\mathrm{Z}(A)$ is a graded Q-subalgebra of A and $\mathrm{B}(A)$ is a homogeneous ideal of $\mathrm{Z}(A)$. Thus, $\mathrm{H}(A)$ is a graded Q-algebra. In particular, $\mathrm{H}_0(A)$ is a Q-algebra and $\mathrm{H}_i(A)$ is a $\mathrm{H}_0(A)$-module.

2.1.11. Let A and A' be DG Q-algebras. A morphism of DG Q-algebras is a morphism of complexes $\varphi : A \to A'$ that is also a morphism of Q-algebras. Given a DG Q-algebra A where $A_i = 0$ for all $i < 0$, the canonical morphism of complexes $A \to \mathrm{H}_0(A)$ is a morphism of DG Q-algebras.

2.1.12. A (left) DG A-module M is a complex of Q-modules equipped a morphism of complexes $\mu^M : A \otimes_Q M \to M$ such the following diagrams commute

$$\begin{array}{ccc} A \otimes_Q A \otimes_Q M & \xrightarrow{\mu^A \otimes M} & A \otimes_Q M \\ \Big\downarrow{\scriptstyle A\otimes\mu^M} & & \Big\downarrow{\scriptstyle \mu^M} \\ A \otimes_Q M & \xrightarrow{\mu^M} & M \end{array} \quad \text{and} \quad \begin{array}{ccc} & & A \otimes_Q M \\ & \nearrow{\scriptstyle \eta^A \otimes M} & \Big\downarrow{\scriptstyle \mu^M} \\ Q \otimes_Q M & \xrightarrow{\cong} & M \end{array}$$

We write am instead of $\mu^M(a \otimes m)$. Again, the *Leibniz-rule* holds:

$$\partial^M(am) = \partial^A(a)m + (-1)^{|a|} a \partial^M(m)$$

for all $a \in A$ and all $m \in M$. A DG A-module M is determined by its underlying graded Q-module $M^\natural$, its differential ∂^M and its A-action μ^M.

For the rest of the section, we let M and N be DG A-modules.

2.1.13. The action of A on M induces a graded $\mathrm{H}(A)$-module structure on $\mathrm{H}(M)$.

Explicitly, for each $a \in \mathrm{Z}(A)$ and $z \in \mathrm{Z}(M)$,

$$[a] \cdot [z] := [az]$$

makes $\mathrm{H}(M)$ a well-defined graded $\mathrm{H}(A)$-module.

2.1.14. For each $i \in \mathbb{Z}$, $\Sigma^i M$ is the DG A-module given by

$$(\Sigma^i M)_n := M_{n-i}, \ \partial^{\Sigma^i M} := (-1)^i \partial^M, \quad \text{and } a \cdot m := (-1)^{|a|i} am.$$

2.1.15. A degree d-map, $\alpha : M \to N$, from M to N is a family of Q-linear maps $\alpha = \{\alpha_i : M_i \to N_{i+d}\}_{i \in \mathbb{Z}}$ such that

$$\alpha(am) = (-1)^{d|a|} a\alpha(m)$$

for all $a \in A$ and $m \in M$.

2.1.16. We define $\mathrm{Hom}_A(M, N)$ to be the DG A-module with

$$\begin{aligned} \mathrm{Hom}_A(M,N)_d &: = \{\alpha : M \to N : \alpha \text{ is a degree } d \text{ map}\}, \\ \partial^{\mathrm{Hom}_A(M,N)} &: = \mathrm{Hom}(M, \partial^N) - \mathrm{Hom}(\partial^M, N), \text{ and} \\ a \cdot \alpha &: = a\alpha(-) = (-1)^{d|a|}\alpha(a \cdot -). \end{aligned}$$

We remark that $\mathrm{Hom}_A(M, N)$ is a subcomplex of $\mathrm{Hom}_Q(M, N)$.

2.1.17. Let $\alpha \in \mathrm{Hom}_A(M, N)_0$. We say that α is a morphism of DG A-modules if $\alpha \in \mathrm{Z}_0(\mathrm{Hom}_A(M, N))$. Spelling this out, α is a morphism of DG A-modules provided that α is morphism of complexes satisfying

$$\alpha(am) = a\alpha(m)$$

for all $a \in A$ and $m \in M$.

2.1.18. Let M and N be DG A-modules. We say that degree d maps α and β from M to N are homotopic, denoted $\alpha \sim \beta$, if $\alpha - \beta \in \mathrm{B}_d(\mathrm{Hom}_A(M, N))$. A morphism of DG A-modules $\alpha : M \to N$ is a homotopy equivalence if there exists a morphism of DG A-modules $\beta : N \to M$ such that

$$\beta\alpha \sim \mathrm{id}^M \text{ and } \alpha\beta \sim \mathrm{id}^N .$$

A morphism of DG A-modules $\alpha : M \to N$ is a quasi-isomorphism if $\mathrm{H}(\alpha) : \mathrm{H}(M) \to \mathrm{H}(N)$ is an isomorphism of graded $\mathrm{H}(A)$-modules.

2.1.19. Homotopic maps induce the same maps in homology. Hence, every homotopy equivalence is a quasi-isomorphism.

2.2 Semiprojective DG Modules and Ext

Fix a DG Q-algebra A.

2.2.1. A DG A-module P is *semiprojective* if for every morphism of DG A-modules $\alpha : P \to N$ and each surjective quasi-isomorphism of DG A-modules $\gamma : M \to N$ there exists a unique up to homotopy morphism of DG A-modules $\beta : P \to M$ such that $\alpha = \gamma\beta$. Equivalently, $P^\natural$ is a projective graded $A^\natural$-module and $\mathrm{Hom}_A(P, -)$ preserves quasi-isomorphisms.

2.2.2. A *semiprojective resolution* of a DG A-module M is a surjective quasi-isomorphism of DG A-modules $\epsilon : P \to M$ where P is a semiprojective DG A-module. Semiprojective resolutions exist and any two semiprojective resolutions of M are unique up to homotopy equivalence [1, 5.1] or [15, 6.6].

2.2.3. For DG A-modules M and N, define

$$\operatorname{Ext}_A^*(M,N) := \mathrm{H}(\operatorname{Hom}_A(P,N))$$

where P is a semiprojective resolution of M over A. Since any two semiprojective resolutions of M are homotopy equivalent, $\operatorname{Ext}_A^*(M,N)$ is independent of choice of P. An element $[\alpha]$ of $\operatorname{Ext}_A^*(M,N)$ is the class of a morphism of DG A-modules

$$\alpha : P \to \Sigma^{|\alpha|} N.$$

Moreover, given $[\alpha]$ and $[\beta]$ in $\operatorname{Ext}_A^*(M,N)$, $[\alpha] = [\beta]$ if and only if α and β are homotopic morphisms of DG A-modules.

2.2.4. Fix a morphism of DG Q-algebras $\varphi : A' \to A$. Let M and N be DG A-modules, $\epsilon : P \to M$ be a semiprojective resolution of M over A, and $\epsilon' : P' \to M$ a semiprojective resolution of M over A'. There exists a unique up to homotopy morphism of DG A'-modules $\alpha : P' \to P$ such that $\epsilon' = \epsilon\alpha$. Define $\operatorname{Hom}_\varphi(\alpha, N)$ to be the composition

$$\operatorname{Hom}_A(P,N) \xrightarrow{\operatorname{Hom}_\varphi(P,N)} \operatorname{Hom}_{A'}(P,N) \xrightarrow{\operatorname{Hom}_{A'}(\alpha,N)} \operatorname{Hom}_{A'}(P',N).$$

This induces a map in cohomology

$$\operatorname{Ext}_\varphi^*(M,N) : \operatorname{Ext}_A^*(M,N) \to \operatorname{Ext}_{A'}^*(M,N)$$

given by $\operatorname{Ext}_\varphi^*(M,N) = \mathrm{H}(\operatorname{Hom}_\varphi(\alpha,N))$; it is independent of choice of α, P, and P'.

2.2.5. Let $\varphi : A' \to A$ be a morphism of DG Q-algebras and let M and N be DG A-modules. If φ is a quasi-isomorphism, then $\operatorname{Ext}_\varphi^*(M,N)$ is an isomorphism [15, 6.10].

2.3 The Derived Category of a DG Algebra

Throughout this thesis we use the theory of triangulated categories. See [21], [23, Chapter 1] or [27, Chapter 10] for standard references on triangulated categories.

2.3.1. Let T denote a triangulated category. We will use Σ to denote the suspension functor and will display exact triangles as

$$X \to Y \to Z \to .$$

A full subcategory T' of T is called triangulated if it is closed under suspensions and has the two out of three property on exact triangles. If, in addition, T' is closed under direct summands, we say that T' is a thick subcategory of T. That is, a full subcategory T' of T is a thick subcategory provided that

1. $\Sigma^n X$ is an object of T' for all $n \in \mathbb{Z}$ and all objects X of T',

2. if $X' \to X \to X'' \to$ is an exact triangle in T with two of X', X, X'' being objects of T', then so is third, and

3. if X is an object of T' and $X = X' \coprod X''$, then X' and X'' are objects of T'.

2.3.2. Let T denote a triangulated category and suppose that X is an object of T. Define the thick closure of X in T, denoted $\mathsf{Thick}_{\mathsf{T}} X$, to be the intersection of all thick subcategories of T containing X. Since an intersection of thick subcategories is a thick subcategory, $\mathsf{Thick}_{\mathsf{T}} X$ is the smallest thick subcategory of T containing X. See [7, Section 2] for an inductive construction of $\mathsf{Thick}_{\mathsf{T}} X$ and a discussion of the related concept of *levels*. If Y is an object of $\mathsf{Thick}_{\mathsf{T}}\, X$, then we say that X *finitely builds* Y.

2.3.3. Fix triangulated categories (T, Σ) and (T', Σ'). An exact functor $\mathsf{T} \to \mathsf{T}'$ is a pair (F, η) where $F : \mathsf{T} \to \mathsf{T}'$ is a functor and $\eta : F\Sigma \to \Sigma' F$ is a natural isomorphism that sends each exact triangle in T to an exact triangle in T'. Explicitly, given an exact triangle $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma}$ in T we get an exact triangle

$$F(X) \xrightarrow{F(\alpha)} F(Y) \xrightarrow{F(\beta)} F(Z) \xrightarrow{\eta_X F(\gamma)} .$$

We will often write $F : \mathsf{T} \to \mathsf{T}'$ for an exact functor and suppress η unless we need to specifically reference it.

Let A be a DG Q-algebra.

2.3.4. Let $\mathsf{D}(A)$ denote the derived category of A (see [20] for an explicit construction). Loosely speaking, $\mathsf{D}(A)$ is obtained by formally inverting quasi-isomorphisms between DG A-modules. Recall that $\mathsf{D}(A)$, equipped with Σ, is a triangulated category. Define $\mathsf{D}^f(A)$ to be the full subcategory of $\mathsf{D}(A)$ consisting of all M such that $\mathrm{H}(M)$ is a finitely generated graded module over $\mathrm{H}(A)$. We use $\simeq$ to denote isomorphisms in $\mathsf{D}(A)$ and reserve $\cong$ for isomorphisms of DG A-modules.

2.3.5. Let $\alpha : M \to N$ be a morphism of DG A-modules. We let $\mathrm{cone}(\alpha)$ denote the mapping cone of α and

$$0 \to N \xrightarrow{\iota} \mathrm{cone}(\alpha) \xrightarrow{\pi} \Sigma M \to 0$$

be the mapping cone exact sequence. Then

$$M \xrightarrow{\alpha} N \xrightarrow{\iota} \mathrm{cone}(\alpha) \xrightarrow{\pi}$$

is an exact triangle in $\mathsf{D}(A)$. Moreover, α is a quasi-isomorphism if and only if $\operatorname{cone}(\alpha) \simeq 0$ in $\mathsf{D}(A)$.

Example 2.3.6. Let R be a commutative ring. A complex of R-modules M is *perfect* if it is quasi-isomorphic to a bounded complex of finitely generated projective R-modules. By [12, 3.7], the objects of $\mathsf{Thick}_{\mathsf{D}(R)} R$ are exactly the perfect complexes. Moreover, $\mathsf{Thick}_{\mathsf{D}(R)} R$ contains the *compact objects*, or *small objects*, of $\mathsf{D}(R)$. That is, each object M of $\mathsf{Thick}_{\mathsf{D}(R)} R$ satisfies that $\operatorname{Hom}_{\mathsf{D}(R)}(M, -)$ preserves arbitrary direct sums.

We sketch the argument here that $\mathsf{Thick}_{\mathsf{D}(R)} R$ consists exactly of the perfect complexes for the convenience of the reader. Moreover, this gives a nice illustration of some techniques used while proving things about thick subcategories.

Let T denote the full subcategory of $\mathsf{D}(R)$ consisting of perfect complexes. It is left to the reader to check that T is a thick subcategory of $\mathsf{D}(R)$. As R is an object of T and $\mathsf{Thick}_{\mathsf{D}(R)} R$ is the smallest thick subcategory of $\mathsf{D}(R)$ containing R (see 2.3.2), $\mathsf{Thick}_{\mathsf{D}(R)} R$ is a subcategory of T.

Conversely, an object of T is isomorphic in $\mathsf{D}(R)$ to a complex

$$P = 0 \to P_s \to \ldots \to P_i \to 0$$

where each P_j is a finitely generated projective R-module. We will show every such P is an object of $\mathsf{Thick}_{\mathsf{D}(R)} R$ by inducting on the number of degrees such a P is concentrated in. If $s - i = 0$, then P is a finitely generated projective R-module concentrated in a single degree. Hence, P is an object of $\mathsf{Thick}_{\mathsf{D}(R)} R$. For $s - i > 0$, consider the exact triangle

$$P' \to P \to \Sigma^s P_s \to$$

where $P' = P_{<s}$, i.e., the truncation of P below degree s. It is clear that P' and $\Sigma^s P_s$ are bounded complexes of finitely generated projective R-modules concentrated in a fewer number of homological degrees. Hence, by induction P' and $\Sigma^s P_s$ are objects of $\mathsf{Thick}_{\mathsf{D}(R)} R$ and so P is an object of $\mathsf{Thick}_{\mathsf{D}(R)} R$ (see 2.3.1(2)). Therefore, every bounded complex of finitely generated projective R-modules is an object of $\mathsf{Thick}_{\mathsf{D}(R)} R$. Since $\mathsf{Thick}_{\mathsf{D}(R)} R$ is closed under isomorphisms in $\mathsf{D}(R)$ it follows that every object of T is an object of $\mathsf{Thick}_{\mathsf{D}(R)} R$, finishing the sketch.

2.3.7. Let $\varphi : A \to A'$ be a morphism of DG Q-algebras. Suppose M and N are objects of $\mathsf{D}(A')$. Via restriction of scalars each DG A'-module is a DG A-module and hence, we can regard M and N as objects of $\mathsf{D}(A)$ by restricting scalars along φ. This defines an exact functor $\varphi^* : \mathsf{D}(A') \to \mathsf{D}(A)$. If M is in $\mathsf{Thick}_{\mathsf{D}(A')} N$, then M is an object of $\mathsf{Thick}_{\mathsf{D}(A)} N$ when M and N are viewed as DG A-modules via restriction of scalars along φ. More precisely, M is an object of $\mathsf{Thick}_{\mathsf{D}(A')} N$ implies that $\varphi^*(M)$ is an object of $\mathsf{Thick}_{\mathsf{D}(A)} \varphi^*(N)$.

Chapter 3

Support for Complexes and Graded Modules

3.1 Support of a Complex of Modules

Let R be a commutative noetherian ring and let $\operatorname{Spec} R$ denote the set of prime ideals of R. Recall that $\operatorname{Spec} R$ is a topological space with the Zariski topology.

3.1.1. Let S be a flat R-algebra. As the functor $- \otimes_R S$ is exact we get a well-defined exact functor of derived categories $- \otimes_R S : \mathsf{D}(R) \to \mathsf{D}(S)$.

3.1.2. For each $\mathfrak{p} \in \operatorname{Spec} R$, localization at $\mathfrak{p}$ defines an exact functor $\mathsf{D}(R) \to \mathsf{D}(R_\mathfrak{p})$ as

$$(-)_\mathfrak{p} \cong - \otimes_R R_\mathfrak{p}.$$

In particular, each exact triangle

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma}$$

in $\mathsf{D}(R)$ is sent to an exact triangle

$$L_\mathfrak{p} \xrightarrow{\alpha_\mathfrak{p}} M_\mathfrak{p} \xrightarrow{\beta_\mathfrak{p}} N_\mathfrak{p} \xrightarrow{\gamma_\mathfrak{p}}$$

in $\mathsf{D}(R_\mathfrak{p})$. Moreover, it is straightforward to see the localization functor is essentially surjective.

3.1.3. For a complex of R-modules M, define the *support of* M to be

$$\operatorname{Supp}_R M := \{\mathfrak{p} \in \operatorname{Spec} R : M_{\mathfrak{p}} \not\simeq 0 \text{ in } \mathsf{D}(R_{\mathfrak{p}})\}.$$

As localization is exact,

$$\operatorname{Supp}_R M = \bigcup_{i \in \mathbb{Z}} \operatorname{Supp}_R \mathrm{H}_i(M)$$

where the the supports on the right-hand side of the equation above are the classical supports defined for R-modules. In particular, $\operatorname{Supp}_R M = \emptyset$ if and only if $M \simeq 0$.

3.1.4. Let M be a nontrivial object of $\mathsf{D}^f(R)$. As the (classical) support of a finitely generated R-module is a closed subset of $\operatorname{Spec} R$, it follows that $\operatorname{Supp}_R M$ is a finite union of closed subsets of $\operatorname{Spec} R$. Thus, $\operatorname{Supp}_R M$ is a closed subset of $\operatorname{Spec} R$. In particular, there exists a maximal ideal in $\operatorname{Supp}_R M$.

3.1.5. Let M be an object of $\mathsf{D}(R)$. It is clear that

$$\operatorname{Supp}_R M = \operatorname{Supp}_R \mathsf{\Sigma}^i M$$

for any $i \in \mathbb{Z}$. Moreover, for any exact triangle $M(1) \to M(2) \to M(3) \to$ in $\mathsf{D}(R)$,

$$\operatorname{Supp}_R M(i) \subseteq \operatorname{Supp}_R M(j) \cup \operatorname{Supp}_R M(\ell)$$

where $\{i, j, \ell\} = \{1, 2, 3\}$.

3.1.6. Let M be in $\mathsf{D}^f(R)$ and let $\boldsymbol{x} = x_1, \ldots, x_n$ generate an ideal I of R. It follows from Nakayama's lemma that

$$\operatorname{Supp}_R(M \otimes_R \operatorname{Kos}^R(\boldsymbol{x})) = \operatorname{Supp}_R M \cap \operatorname{Supp}_R(R/I).$$

In particular, if $\boldsymbol{x}$ generates a maximal ideal $\mathfrak{m}$ of R with $\mathfrak{m} \in \operatorname{Supp}_R M$, then

$$\operatorname{Supp}_R(M \otimes_R \operatorname{Kos}^R(\boldsymbol{x})) = \{\mathfrak{m}\}.$$

Example 3.1.7. Let R be a commutative ring and let $\mathfrak{m}$ be a maximal ideal of R. By [12, 3.10], $\mathsf{Thick}_{\mathsf{D}(R)}(R/\mathfrak{m})$ consists of all objects M of $\mathsf{D}^f(R)$ such that $\operatorname{Supp}_R M = \{\mathfrak{m}\}$. In particular, if $\boldsymbol{x}$ is a generating set of $\mathfrak{m}$, then $\operatorname{Kos}^R(\boldsymbol{x})$ is an object of $\mathsf{Thick}_{\mathsf{D}(R)}(R/\mathfrak{m})$.

Lemma 3.1.8. *Let n be a nonzero integer and let M be in $\mathsf{D}^f(R)$. If $\alpha : M \to \Sigma^n M$ is a morphism in $\mathsf{D}(R)$, then*

$$\operatorname{Supp}_R M = \operatorname{Supp}_R(\operatorname{cone}(\alpha)).$$

Proof. Let $C := \operatorname{cone}(\alpha)$. We have an exact triangle

$$M \xrightarrow{\alpha} \Sigma^n M \to C \to \tag{3.1}$$

in $\mathsf{D}(R)$ (c.f. 2.3.5). Now using 3.1.5, it follows that $\operatorname{Supp}_R C \subseteq \operatorname{Supp}_R M$.

Conversely, suppose $\mathfrak{p} \notin \operatorname{Supp}_R C$. Localizing (3.1), we obtain an exact triangle

$$M_\mathfrak{p} \to \Sigma^n M_\mathfrak{p} \to C_\mathfrak{p} \to$$

in $\mathsf{D}(R_{\mathfrak{p}})$. By assumption, $C_{\mathfrak{p}} \simeq 0$ and since

$$\operatorname{cone}(\alpha_{\mathfrak{p}}) \simeq C_{\mathfrak{p}}$$

it follows that $\operatorname{cone}(\alpha_{\mathfrak{p}}) \simeq 0$. By 2.3.5, we conclude that

$$M_{\mathfrak{p}} \simeq \Sigma^n M_{\mathfrak{p}} \tag{3.2}$$

in $\mathsf{D}(R_{\mathfrak{p}})$. However, since $M_{\mathfrak{p}}$ is in $\mathsf{D}^f(R_{\mathfrak{p}})$ and $n \neq 0$, (3.2) implies that $M_{\mathfrak{p}} \simeq 0$. Thus, $\mathfrak{p} \notin \operatorname{Supp}_R M$. □

3.2 Cohomological Support for Graded Modules

Let $\mathcal{A} = \{\mathcal{A}^i\}_{i \geq 0}$ be a cohomologically graded, commutative noetherian ring. Define $\operatorname{Proj} \mathcal{A}$ to be the set of homogeneous prime ideals of $\mathcal{A}$ not containing the irrelevant homogeneous maximal ideal $\mathcal{A}^{>0} := \{\mathcal{A}^i\}_{i>0}$.

3.2.1. Let X be a graded $\mathcal{A}$-module. For each $\mathsf{p} \in \operatorname{Proj} \mathcal{A}$ we let X_{p} denote the homogeneous localization of X at p. Define the *homogeneous support of* X *over* $\mathcal{A}$ to be

$$\mathsf{Supp}^+_{\mathcal{A}}\mathsf{X} := \{\mathsf{p} \in \operatorname{Proj} \mathcal{A} : \mathsf{X}_{\mathsf{p}} \neq 0\}.$$

3.2.2. Fix homogeneous elements $a_1, \dots, a_m \in \mathcal{A}$. Define

$$\mathcal{V}(a_1, \dots, a_m) = \{\mathsf{p} \in \operatorname{Proj} \mathcal{A} : a_i \in \mathsf{p} \text{ for each } i\}.$$

It is straightforward to check that

$$\mathcal{V}(a_1, \dots, a_m) = \mathsf{Supp}^+_{\mathcal{A}}(\mathcal{A}/(a_1, \dots, a_m)).$$

The following properties of (cohomologically) graded $\mathcal{A}$-modules follow easily from the definition of homogeneous support; see [8, 2.2].

Proposition 3.2.3. *Let* $\mathcal{A} = \{\mathcal{A}^i\}_{i\geq 0}$ *be a cohomologically graded, commutative noetherian ring.*

1. *Let* X *be a graded* $\mathcal{A}$*-module and* $n \in \mathbb{Z}$*. Then* $\mathsf{Supp}^+_{\mathcal{A}}\mathsf{X} = \mathsf{Supp}^+_{\mathcal{A}}(\Sigma^n\mathsf{X})$.

2. *Given an exact sequence of graded* $\mathcal{A}$*-modules* $0 \to \mathsf{X}' \to \mathsf{X} \to \mathsf{X}'' \to 0$ *then*

$$\mathsf{Supp}^+_{\mathcal{A}}\mathsf{X} = \mathsf{Supp}^+_{\mathcal{A}}\mathsf{X}' \cup \mathsf{Supp}^+_{\mathcal{A}}\mathsf{X}''.$$

3. *If* X *is a finitely generated graded* $\mathcal{A}$*-module, then* $\mathsf{Supp}^+_{\mathcal{A}}\mathsf{X} = \emptyset$ *if and only if* $\mathsf{X}^{\gg 0} = 0$.

Chapter 4

Cohomology Operators and Support Varieties

4.1 Koszul Complexes

Fix a commutative noetherian ring Q. Let $\boldsymbol{f} = f_1, \ldots, f_n$ be a list of elements in Q. Define $\mathrm{Kos}^Q(\boldsymbol{f})$ to be the DG Q-algebra with $\mathrm{Kos}^Q(\boldsymbol{f})^\natural$ the exterior algebra on a free Q-module with basis $\xi_1, \ldots, \xi_n$ of homological degree 1, and differential $\partial\xi_i = f_i$. One can realize $\mathrm{Kos}^Q(\boldsymbol{f})$ as the DG Q-algebra obtained by adjoining homological variables of degree one, $\xi_1, \ldots, \xi_n$, to kill the cycles $\boldsymbol{f}$ (see [3, Section 6], [17, Chapter 1], or [26]). Hence, we write

$$\mathrm{Kos}^Q(\boldsymbol{f}) = Q\langle \xi_1, \ldots, \xi_n | \partial\xi_i = f_i \rangle.$$

4.1.1. Let $\boldsymbol{f}' = f_1', \ldots, f_m'$ be in Q and set $E' := Q\langle \xi_1', \ldots, \xi_m' | \partial\xi_i' = f_i' \rangle$. Assume that there exists $a_{ij} \in Q$ such that

$$f_i = \sum_{j=1}^{m} a_{ij} f_j'.$$

Then there exists a morphism of DG Q-algebras $\mathrm{Kos}^Q(\boldsymbol{f}) \to E'$ which is uniquely determined by

$$\xi_i \mapsto \sum_{j=1}^{m} a_{ij} \xi_j'.$$

Therefore, E' is a DG $\mathrm{Kos}^Q(\boldsymbol{f})$-module where the action is determined by

$$\xi_i \cdot e' = \sum_{j=1}^{m} a_{ij}\xi'_j e'$$

for all $e' \in E'$.

4.1.2. Assume that $(Q, \mathfrak{n}, k)$ is a commutative noetherian local ring. Define K^Q to be the Koszul complex on some minimal generating set for $\mathfrak{n}$. Then K^Q is unique up to DG Q-algebra isomorphism.

4.1.3. Let $E := \mathrm{Kos}^Q(\boldsymbol{f})$ and $R := Q/(\boldsymbol{f})$. Since $\mathrm{H}(E)$ is a finitely generated graded Q-module, it follows that M is an object of $\mathsf{D}^f(E)$ if and only if $\mathrm{H}(M)$ is a finitely generated graded Q-module (equivalently, $\mathrm{H}(M)$ is a finitely generated graded R-module as $(\boldsymbol{f})\,\mathrm{H}(M) = 0$). Moreover, since $\mathrm{H}_0(E) = R$ and using 2.1.11, $E \to R$ is a morphism of DG Q-algebras. Thus, every complex of R-modules (i.e., a DG R-module) can be regarded as a DG E-module via restriction of scalars (see 2.3.7). Finally, each object of $\mathsf{D}^f(R)$ can be regarded as an object of $\mathsf{D}^f(E)$ restricting along the augmentation map $E \to R$.

4.2 Map on Ext

This section is devoted to the following technical theorem; its proof uses the theory of DG Γ-algebras. See [3, Section 6] or [17, Chapter 1] as a reference for definitions and notation employed in the proof of Theorem 4.2.1.

Theorem 4.2.1. *Assume $(Q, \mathfrak{n}, k)$ is a regular local ring. Let $R = Q/I$ where I is minimally generated by $\boldsymbol{f} = f_1, \ldots, f_n \in \mathfrak{n}^2$. Let E be the Koszul complex on $\boldsymbol{f}$ over*

Q. Let $\varphi : E \to R$ denote the augmentation map. The canonical map

$$\operatorname{Ext}^*_\varphi(k,k) : \operatorname{Ext}^*_R(k,k) \to \operatorname{Ext}^*_E(k,k)$$

is surjective.

Proof. Write $E = Q\langle \xi_1, \dots, \xi_n | \partial \xi_i = f_i \rangle$. For an element $a \in Q$, let $\overline{a}$ denote the image of a in R. Let $s_1, \dots, s_e$ be a minimal generating set for $\mathfrak{n}$. Let $X = \{x_1, \dots, x_e\}$ be a set of exterior variables of homological degree 1 and $Y = \{y_1, \dots, y_n\}$ a set of divided power variables of homological degree 2. By [3, 7.2.10], the morphism of DG Γ-algebras $\varphi : E \to R$ extends to a morphism of DG Γ-algebras

$$\varphi\langle X \rangle : E\langle X | \partial x_i = s_i \rangle \to R\langle X | \partial x_i = \overline{s}_i \rangle$$

such that $\varphi\langle X \rangle(x_i) = x_i$ for each $1 \le i \le e$.

Since $f_i \in \mathfrak{n}^2$, there exists $a_{ij} \in \mathfrak{n}$ such that

$$f_i = \sum_{j=1}^{e} a_{ij} s_j.$$

For each $1 \le i \le n$, we have degree 1 cycles

$$z_i := \sum_{j=1}^{n} a_{ij} x_j - \xi_i \quad \text{and} \quad \overline{z}_i := \sum_{j=1}^{n} \overline{a}_{ij} x_j$$

in $E\langle X \rangle$ and $R\langle X \rangle$, respectively. Moreover, for each $1 \le i \le n$

$$\varphi\langle X \rangle(z_i) = \overline{z_i}.$$

Applying [3, 7.2.10] yields a morphism of DG Γ-algebras

$$\varphi\langle X,Y\rangle : E\langle X\rangle\langle Y|\partial y_i = z_i\rangle \to R\langle X\rangle\langle Y|\partial y_i = \overline{z_i}\rangle$$

extending $\varphi\langle X\rangle$ such that $\varphi\langle X,Y\rangle(y_i) = y_i$ for each $1 \leq i \leq n$.

By [3, 6.3.2], $E\langle X,Y\rangle$ is an acyclic closure of k over E. In particular, $E\langle X,Y\rangle$ is a semiprojective resolution of k over E. Next, $\overline{s}_1,\dots,\overline{s}_e$ is a minimal generating set for the maximal ideal of R. Also, since $f_1,\dots,f_n$ minimally generates I, it follows that $[\overline{z}_1],\dots,[\overline{z}_n]$ is a minimal generating set for $H_1(R\langle X\rangle)$ (see [26, Theorem 4] or [17, 1.5.4]). Thus, $R\langle X,Y\rangle$ is the second step in forming an acyclic closure of k over R. Let $\iota : R\langle X,Y\rangle \hookrightarrow R\langle X,Y,V\rangle$ denote the inclusion of DG Γ-algebras where $R\langle X,Y,V\rangle$ is an acyclic closure of k over R and V consists of Γ-variables of homological degree at least 3. Define $\alpha : E\langle X,Y\rangle \to R\langle X,Y,V\rangle$ to be the morphism of DG Γ-algebras that is the the following composition of DG Γ-algebra

$$E\langle X,Y\rangle \xrightarrow{\varphi\langle X,Y\rangle} R\langle X,Y\rangle \xrightarrow{\iota} R\langle X,Y,V\rangle.$$

The following is a commutative diagram of Γ-algebras

$$\begin{array}{ccc} E\langle X,Y\rangle\otimes_E k & \xrightarrow{\alpha\otimes k} & R\langle X,Y,V\rangle\otimes_R k \\ \cong\downarrow & & \downarrow\cong \\ k\langle X,Y\rangle & \xrightarrow{\subseteq} & k\langle X,Y,V\rangle \end{array}$$

Therefore, $\alpha\otimes k$ is an injective morphism of Γ-algebras. In particular, $\alpha\otimes k$ is injective as a map of graded k-vector spaces. Also, the following is a commutative diagram of graded k-vector spaces

$$\begin{array}{ccc} \operatorname{Hom}_k(R\langle X,Y,V\rangle\otimes_R k,k) & \xrightarrow{\operatorname{Hom}_k(\alpha\otimes k,k)} & \operatorname{Hom}_k(E\langle X,Y\rangle\otimes_E k,k) \\ \cong\downarrow & & \downarrow\cong \\ \operatorname{Hom}_R(R\langle X,Y,V\rangle,k) & \xrightarrow{\operatorname{Hom}_\varphi(\alpha,k)} & \operatorname{Hom}_E(E\langle X,Y\rangle,k) \end{array}$$

Since $\alpha \otimes k$ is injective, $\operatorname{Hom}_k(\alpha \otimes k, k)$ is surjective. Thus, $\operatorname{Hom}_\varphi(\alpha, k)$ is surjective. Moreover, $\operatorname{Hom}_E(E\langle X, Y\rangle, k)$ and $\operatorname{Hom}_R(R\langle X, Y, V\rangle, k)$ have trivial differential (see [3, 6.3.4]). Thus, $\operatorname{Ext}^*_\varphi(k, k) = \operatorname{Hom}_\varphi(\alpha, k)$, and so $\operatorname{Ext}^*_\varphi(k, k)$ is surjective. □

4.3 Cohomology Operators and Support Varieties

Notation 4.3.1. Throughout this section, and the next, we fix the following notation. Let Q be a commutative noetherian ring. When Q is local, we will let $\mathfrak{n}$ denote its maximal ideal and k its residue field.

Let I be an ideal of Q and fix a generating set $\boldsymbol{f} = f_1, \dots, f_n$ for I. Set $R := Q/I$ and $E := Q\langle \xi_1, \dots, \xi_n | \partial \xi_i = f_i \rangle$. The augmentation map $E \to R$ is a map of DG Q-algebras. Hence, we consider each complex of R-modules as DG E-modules via restriction of scalars along $E \to R$ (c.f. 4.1.3).

Let $\mathcal{S} := Q[\chi_1, \dots, \chi_n]$ be a graded polynomial ring where each χ_i has cohomological degree 2. When Q is local, set

$$\mathcal{A} := \mathcal{S} \otimes_Q k = k[\chi_1, \dots, \chi_n].$$

Define Γ to be the graded Q-linear dual of $\mathcal{S}$, i.e., Γ is the graded Q-module with

$$\Gamma_i := \operatorname{Hom}_Q(\mathcal{S}^i, Q).$$

Let $\{y^{(H)}\}_{H \in \mathbb{N}^n}$ be the Q-basis of Γ dual to $\{\chi^H := \chi_1^{h_1} \dots \chi_n^{h_n}\}_{H \in \mathbb{N}^n}$ the standard Q-basis of $\mathcal{S}$. Then Γ is a graded $\mathcal{S}$-module via the action

$$\chi_i \cdot y^{(H)} := \begin{cases} y^{(h_1, \dots, h_{i-1}, h_i - 1, h_{i+1}, \dots, h_n)} & h_i \geq 1 \\ 0 & h_i = 0 \end{cases}$$

4.3.2. Let M be a DG E-module. A *Koszul resolution of M* is a surjective quasi-isomorphism of DG E-modules $\epsilon : P \xrightarrow{\simeq} M$ such that $\epsilon : P \to M$ is a semiprojective resolution of M over Q where we view each DG E-module as DG Q-module via restriction of scalars along the structure map $Q \to E$. A semiprojective resolution of M over E is a Koszul resolution of M, and hence Koszul resolutions exist (c.f. 2.2.2). By [5, 2.1], when M is perfect over Q and Q is local, there exists a Koszul resolution $P \xrightarrow{\simeq} M$ where

$$P^{\natural} \cong \coprod_{i=0}^{t} \Sigma^i Q^{\beta_i}$$

for some nonnegative integers t and β_i.

Construction 4.3.3. Let $\epsilon : P \xrightarrow{\simeq} M$ be a Koszul resolution of M. Define $U_E(P)$ to be the DG E-module with

$$U_E(P)^{\natural} \cong (E \otimes_Q \Gamma \otimes_Q P)^{\natural}$$

and differential given by the formula

$$\partial = \partial^E \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \partial^P + \sum_{i=1}^{n} (1 \otimes \chi_i \otimes \lambda_i - \lambda_i \otimes \chi_i \otimes 1)$$

where λ_i denotes left multiplication by ξ_i. By [5, 2.4], $U_E(P) \to M$ is a semiprojective resolution over E where the augmentation map is given by

$$a \otimes y^{(H)} \otimes x \mapsto \begin{cases} a\epsilon(x) & |H| = 0 \\ 0 & |H| > 1 \end{cases}$$

Notice that $U_E(P)$ has a DG $\mathcal{S}$-module structure where $\mathcal{S}$ acts on $U_E(P)$ via its action on Γ. For a DG E-module N, $\operatorname{Hom}_E(U_E(P), N)$ is a DG $\mathcal{S}$-module and hence,

$$\operatorname{Ext}_E^*(M, N) \cong \operatorname{H}(\operatorname{Hom}_E(U_E(P), N)) \tag{4.1}$$

is a graded module over $\mathcal{S}$.

Remark 4.3.4. Let M and M' be DG E-modules and assume that $\alpha : M \to M'$ is a morphism of DG E-modules. Let F and F' be semiprojective resolutions of M and M' over E, respectively. Since F is semiprojective over E, there exists a morphism of DG E-modules $\tilde{\alpha} : F \to F'$ lifting α that is unique up to homotopy. Moreover, $\tilde{\alpha}$ induces a morphism of DG E-modules $1 \otimes 1 \otimes \tilde{\alpha} : U_E(F) \to U_E(F')$ that is $\mathcal{S}$-linear and unique up to homotopy.

In particular, if F and F' are both semiprojective resolutions of a DG E-module M, then there exists a DG E-module homotopy equivalence $U_E(F) \to U_E(F')$ that is $\mathcal{S}$-linear and unique up to homotopy. Thus, the $\mathcal{S}$-module structures of

$$\operatorname{H}(\operatorname{Hom}_E(U_E(F), N)) \text{ and } \operatorname{H}(\operatorname{Hom}_E(U_E(F'), N))$$

coincide when F and F' are both semiprojective resolutions of M over E.

Proposition 4.3.5. *Let M and N be in $\mathsf{D}(E)$. Then the $\mathcal{S}$-module structure on $\operatorname{Ext}_E^*(M, N)$ given by (4.1) is independent of choice of Koszul resolution for M. Moreover, the $\mathcal{S}$-module action on $\operatorname{Ext}_E^*(M, N)$ is functorial in M and given an exact triangle*

$$M' \to M \to M'' \to$$

in $\mathsf{D}(E)$ the canonical maps, induced by applying $\operatorname{Ext}_E^*(-, N)$, form an exact sequence *of graded $\mathcal{S}$-modules*

$$\Sigma^{-1}\operatorname{Ext}_E^*(M',N) \to \operatorname{Ext}_E^*(M'',N) \to \operatorname{Ext}_E^*(M,N) \to \operatorname{Ext}_E^*(M',N).$$

Proof. Let P be a Koszul resolution of M and F a semiprojective resolution of M over E. There exists a morphism of DG E-modules $\tilde{\alpha} : F \to P$ lifting the identity on M which is unique up to homotopy. This induces a DG E-module homotopy equivalence $1 \otimes 1 \otimes \tilde{\alpha} : U_E(F) \to U_E(P)$ that is $\mathcal{S}$-linear and unique up to homotopy. Thus, F and P determine the same $\mathcal{S}$-module structure on $\operatorname{Ext}_E^*(M,N)$. From Remark 4.3.4, it follows that the $\mathcal{S}$-module structure on $\operatorname{Ext}_E^*(M,N)$ is independent of choice of Koszul resolution for M.

Moreover, by Remark 4.3.4 the $\mathcal{S}$-module structure on $\operatorname{Ext}_E^*(M,N)$ is functorial in M. Thus, $\operatorname{Ext}_E^*(-,N)$ sends exact triangles in $\mathsf{D}(E)$ to exact sequences of graded $\mathcal{S}$-modules. □

4.3.6. Assume that $(Q,\mathfrak{n},k)$ is a local ring and recall that $\mathcal{A} = \mathcal{S} \otimes_Q k$. Let M be in $\mathsf{D}(E)$. The $\mathcal{S}$-action on $\operatorname{Ext}_E^*(M,k)$ factors through $\mathcal{S} \to \mathcal{A}$, and hence, $\operatorname{Ext}_E^*(M,k)$ is a graded $\mathcal{A}$-module. Therefore, by Proposition 4.3.5, for any exact triangle $M' \to M \to M'' \to$ in $\mathsf{D}(E)$, we get an exact sequence of graded $\mathcal{A}$-modules

$$\Sigma^{-1}\operatorname{Ext}_E^*(M',k) \to \operatorname{Ext}_E^*(M'',k) \to \operatorname{Ext}_E^*(M,k) \to \operatorname{Ext}_E^*(M',k).$$

Lemma 4.3.7. *Assume that $(Q,\mathfrak{n},k)$ is a local ring and M is in $\mathsf{D}(E)$. For any $x \in \mathfrak{n}$, there exists an exact sequence of graded $\mathcal{A}$-modules*

$$0 \to \Sigma^{-1}\operatorname{Ext}_E^*(M,k) \to \operatorname{Ext}_E^*(M \otimes_Q \operatorname{Kos}^Q(x),k) \to \operatorname{Ext}_E^*(M,k) \to 0.$$

Proof. By 4.3.6, applying $\operatorname{Ext}_E^*(-,k)$ to the exact triangle

$$M \to M \to M \otimes_Q \operatorname{Kos}^Q(x) \to$$

in $\mathsf{D}(E)$ gives us an exact sequences of graded $\mathcal{A}$-modules

$$\Sigma^{-1}\operatorname{Ext}_E^*(M,k) \to \operatorname{Ext}_E^*(M \otimes_Q \operatorname{Kos}^Q(x), k) \to \operatorname{Ext}_E^*(M,k) \xrightarrow{x\cdot} \operatorname{Ext}_E^*(M,k).$$

Since x is in $\mathfrak{n}$, we obtain the desired result. □

Proposition 4.3.8. *Assume that $(Q,\mathfrak{n},k)$ is a regular local ring. For each M in $\mathsf{D}^f(E)$, $\operatorname{Ext}_E^*(M,k)$ is a finitely generated graded $\mathcal{A}$-module.*

Proof. As $\mathrm{H}(M)$ is finitely generated over Q and Q is regular, there exists a Koszul resolution $P \xrightarrow{\simeq} M$ such that P is a bounded complex of finitely generated free Q-modules (see 4.3.2). Also, we have an isomorphism of graded $\mathcal{A}$-modules

$$\operatorname{Hom}_E(U_E(P),k)^\natural \cong \mathcal{A} \otimes_k \operatorname{Hom}_Q(P,k)^\natural.$$

Thus, $\operatorname{Hom}_E(U_E(P),k)$ is a noetherian graded $\mathcal{A}$-module. As $\mathcal{A}$ is a noetherian graded ring and $\operatorname{Ext}_E^*(M,k)$ is a graded subquotient of $\operatorname{Hom}_E(U_E(P),k)$, it follows that $\operatorname{Ext}_E^*(M,k)$ is a noetherian graded $\mathcal{A}$-module. □

Remark 4.3.9. Suppose the local ring $(Q,\mathfrak{n},k)$ is regular. By 4.1.1, K^Q is a DG E-module. Assume that $I \subseteq \mathfrak{n}^2$. Left multiplication by ξ_i on K^Q is zero modulo $\mathfrak{n}$. Thus, we have an isomorphism of DG $\mathcal{A}$-modules

$$\operatorname{Hom}_E(U_E(K^Q),k) \cong \mathcal{A} \otimes_k \operatorname{Hom}_Q(K^Q,k),$$

where both DG $\mathcal{A}$-modules have trivial differential (see 4.1.1). Therefore, there is an isomorphism of graded $\mathcal{A}$-modules

$$\operatorname{Ext}_E^*(k,k) \cong \mathcal{A} \otimes_k \operatorname{Hom}_Q(K^Q, k).$$

In particular,

$$\mathsf{Supp}_{\mathcal{A}}^+(\operatorname{Ext}_E^*(k,k)) = \operatorname{Proj} \mathcal{A}.$$

Remark 4.3.10. We import the assumptions from 4.3.9. We give an alternative proof of Proposition 4.3.8 in this case that uses thick subcategories.

Let T be the full subcategory of $\mathsf{D}(E)$ consisting of objects M such that $\operatorname{Ext}_E^*(M,k)$ is a finitely generated graded $\mathcal{A}$-module. Then T is a thick subcategory.

Let $M \in \mathsf{D}^f(E)$. By [7, 3.10], $M \otimes_Q K^Q$ is an object of $\mathsf{Thick}_{\mathsf{D}(E)}\, k$. Since k is an object of T and T is a thick subcategory of $\mathsf{D}(E)$, we conclude that $M \otimes_Q K^Q$ is an object of T. That is, $\operatorname{Ext}_E^*(M \otimes_Q K^Q, k)$ is finitely generated as an $\mathcal{A}$-module. By applying Lemma 4.3.7, $\operatorname{Ext}_E^*(M,k)$ is a finitely generated graded $\mathcal{A}$-module.

4.4 Support Varieties

We import the notation set in Notation 4.3.1. Further assume that $(Q, \mathfrak{n}, k)$ is a regular local ring, $\boldsymbol{f}$ *minimally* generates I, and $I \subseteq \mathfrak{n}$.

By Proposition 4.3.8, $\operatorname{Ext}_E^*(M,k)$ is a finitely generated graded $\mathcal{A}$-module for each M in $\mathsf{D}^f(E)$. This leads to the following definition which recovers the support varieties of Avramov in [2] in the case that $\boldsymbol{f}$ is a Q-regular sequence. The varieties, defined below, are investigated and further developed in [25].

Definition 4.4.1. Let M be in $\mathsf{D}^f(E)$. Define the *support variety of* M *over* E to be

$$\mathsf{V}_E(M) := \mathsf{Supp}^+_{\mathcal{A}}\left(\operatorname{Ext}^*_E(M,k)\right).$$

Theorem 4.4.2. *With the assumptions above, the following hold.*

1. *Let M and N be in $\mathsf{D}^f(E)$. If N is in $\mathsf{Thick}_{\mathsf{D}(E)} M$, then $\mathsf{V}_E(N) \subseteq \mathsf{V}_E(M)$.*

2. *For any M in $\mathsf{D}^f(E)$, $\mathsf{V}_E(M) = \mathsf{V}_E(M \otimes_Q K^Q)$.*

3. *$\boldsymbol{f}$ is a regular Q-sequence if and only if $\mathsf{V}_E(R) = \emptyset$.*

Proof. Using 4.3.6 and Proposition 3.2.3, it follows that the full subcategory of $\mathsf{D}^f(E)$ consisting of objects L such that $\mathsf{V}_E(L) \subseteq \mathsf{V}_E(M)$ is a thick subcategory of $\mathsf{D}^f(E)$. Therefore, (1) holds.

Iteratively applying Lemma 4.3.7 and Proposition 3.2.3(2), establishes (2).

For (3), first assume that $\boldsymbol{f}$ is a Q-regular sequence. Hence, the augmentation map $E \to R$ is a quasi-isomorphism. Therefore, 2.2.5 yields an isomorphism

$$\operatorname{Ext}^*_E(R,k) \cong \operatorname{Ext}^*_R(R,k) = k.$$

Thus, by Proposition 3.2.3(3) we conclude that

$$\mathsf{V}_E(R) = \mathsf{Supp}^+_{\mathcal{A}} k = \emptyset.$$

Conversely, assume that $\mathsf{V}_E(R) = \emptyset$. Hence, by Proposition 4.3.8 and Proposition 3.2.3(3),

$$\operatorname{Ext}^{\gg 0}_E(R,k) = 0. \tag{4.2}$$

Next, let $\boldsymbol{g} = g_1, \ldots, g_n$ be a minimal generating set for I such that $\boldsymbol{g}' = g_1, \ldots, g_c$ is a maximal Q-regular sequence in I for some $c \leq g$. Set $\overline{Q} := Q/(\boldsymbol{g}')$, $\overline{\boldsymbol{g}}$ to be the image of $g_{c+1}, \dots, g_n$ in $\overline{Q}$, and $\overline{E} := \mathrm{Kos}^{\overline{Q}}(\overline{\boldsymbol{g}})$. Since $\boldsymbol{g}'$ is a Q-regular sequence, we have a quasi-isomorphism of DG Q-algebras $E \xrightarrow{\simeq} \overline{E}$. Hence, 2.2.5 yields an isomorphism of graded k-vector spaces

$$\mathrm{Ext}^*_{\overline{E}}(R,k) \cong \mathrm{Ext}^*_E(R,k).$$

In particular, $\mathrm{Ext}^{\geqslant 0}_{\overline{E}}(R,k) = 0$ by (4.2). Hence, R has a semiprojective resolution P over $\overline{E}$ where

$$P^\natural \cong \coprod_{j=0}^{t} \Sigma^j (\overline{E}^{\beta_j})^\natural$$

(c.f. [9, B.9]). Therefore, R has finite projective dimension over $\overline{Q}$.

As $R = \overline{Q}/I\overline{Q}$ where $I\overline{Q}$ contains no $\overline{Q}$-regular element, it follows that $I\overline{Q} = 0$ (see [11, 1.4.7]). Thus, $\boldsymbol{g} = \boldsymbol{g}'$, that is, I is generated by a Q-regular sequence. Therefore, by [11, 1.6.19], $\boldsymbol{f}$ is Q-regular sequence. □

Remark 4.4.3. In [25], a different argument is used to establish Theorem 4.4.2(3). In fact, the following is shown: *$\boldsymbol{f}$ is a Q-regular sequence if and only if $\mathsf{V}_E(M) = \emptyset$ for some nonzero finitely generated R-module M.* The proof in [25] is simpler but uses the *amplitude inequality* established by Jørgensen in [19, 4.1]. One of the key lemmas Jørgensen proved to establish [19, 4.1] depends on the *new intersection theorem*, which was proved in full generality by P. Roberts.

Theorem 4.4.4. *Assume $(Q, \mathfrak{n}, k)$ is a regular local ring. Let $R = Q/I$ where I is minimally generated by $\boldsymbol{f} = f_1, \dots, f_n \in \mathfrak{n}^2$. Let E be the Koszul complex on $\boldsymbol{f}$ over Q and set $\mathcal{A} = k[\chi_1, \dots, \chi_n]$. For each homogeneous element $g \in \mathcal{A}$, there exists a complex of R-modules $C(g)$ in $\mathsf{Thick}_{\mathsf{D}(R)}\, k$ such that*

$$\mathsf{V}_E(C(g)) = \mathcal{V}(g).$$

Proof. As Q is regular, the Koszul complex K^Q is a free resolution of k over Q. Moreover, 4.1.1 says that K^Q is a Koszul resolution of k. By Construction 4.3.3, $\epsilon : U_E(K^Q) \xrightarrow{\simeq} k$ is a semiprojective resolution of k over E. Set $U := U_E(K^Q)$ and let d denote the degree of g. Define

$$\tilde{C}(g) := \operatorname{cone}(U \xrightarrow{g\cdot} \Sigma^d U).$$

The same proof[1] given in [8, 3.10] and Remark 4.3.9 yield

$$\mathsf{V}_E(\tilde{C}(g)) = \mathcal{V}(g). \tag{4.3}$$

Fix a projective resolution $\delta : P \xrightarrow{\simeq} k$ over R. Since U is a semiprojective DG E-module there exists a morphism of DG E-modules $\alpha : U \to P$ such that $\delta\alpha = \epsilon$. Note that α is a quasi-isomorphism.

By Theorem 4.2.1 and 2.2.3, there exists a morphism of complexes of R-modules $\gamma : P \to \Sigma^d k$ such that

$$\begin{array}{ccc} U & \xrightarrow{g\cdot} & \Sigma^d U \\ {\scriptstyle \alpha}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\downarrow{\scriptstyle \Sigma^d \epsilon} \\ P & \xrightarrow[\gamma]{} & \Sigma^d k \end{array} \tag{4.4}$$

is a diagram of DG E-modules that commutes up to homotopy. Define

$$C(g) := \operatorname{cone}(\gamma).$$

Since $P \simeq k$ and γ is a morphism of complexes of R-modules, it follows that $C(g)$ is in $\mathsf{Thick}_{\mathsf{D}(R)}\, k$. Also, as α are $\Sigma^d \epsilon$ quasi-isomorphisms and (4.4) commutes up to

[1]A proof of Equation (4.3) follows the proof of Theorem 4.4.4.

homotopy, we get an isomorphism

$$C(g) \simeq \tilde{C}(g)$$

in $\mathsf{D}(E)$. Therefore, Equation (4.3) yields

$$\mathsf{V}_E(C(g)) = \mathsf{V}_E(\tilde{C}(g)) = \mathcal{V}(g). \qquad \square$$

We sketch the argument of (4.3) for the convenience of the reader.

Proof. We import the notation from the proof of Theorem 4.4.4. Set $\mathcal{E} := \operatorname{Ext}^*_E(k,k)$ and $\mathcal{E}' := \operatorname{Ext}^*_E(\tilde{C}(g),k)$. Applying $\operatorname{Ext}^*_E(-,k)$ to the exact triangle

$$U \xrightarrow{g\cdot} \Sigma^d U \to \tilde{C}(g) \to$$

gives us an exact sequence of graded $\mathcal{A}$-modules

$$\Sigma^{-d-1}\mathcal{E} \xrightarrow{g\cdot} \Sigma^{-1}\mathcal{E} \to \mathcal{E}' \to \Sigma^{-d}\mathcal{E} \xrightarrow{g\cdot} \mathcal{E}$$

(see 4.3.6). Thus, we obtain the exact sequence

$$0 \to \Sigma^{-1}\mathcal{E}/g\mathcal{E} \to \mathcal{E}' \to \Sigma^{-d}(0 :_{\mathcal{E}} g) \to 0$$

of graded $\mathcal{A}$-modules where $(0 :_{\mathcal{E}} g)$ is graded $\mathcal{A}$-submodule of $\mathcal{E}$ consisting of elements of $\mathcal{E}$ that are annihilated by g. The exact sequence of graded $\mathcal{A}$-modules yields

$$\mathsf{Supp}^+_{\mathcal{A}}\mathcal{E}' = \mathsf{Supp}^+_{\mathcal{A}}\mathcal{E}/g\mathcal{E} \cup \mathsf{Supp}^+_{\mathcal{A}}(0 :_{\mathcal{E}} g) \qquad (4.5)$$

(see Proposition 3.2.3(2)). As $\mathcal{E}$ is a finitely generated $\mathcal{A}$-module,

$$\mathsf{Supp}^+_{\mathcal{A}}\mathcal{E}/g\mathcal{E} = \mathsf{Supp}^+_{\mathcal{A}}\mathcal{E} \cap \mathsf{Supp}^+_{\mathcal{A}}\mathcal{A}/(g)$$

and hence,

$$\mathsf{Supp}^+_{\mathcal{A}}\mathcal{E}/g\mathcal{E} = \mathsf{Supp}^+_{\mathcal{A}}\mathcal{E} \cap \mathcal{V}(g).$$

Furthermore, since $(0 :_{\mathcal{E}} g)$ is a submodule of $\mathcal{E}$ that is annihilated by g,

$$\mathsf{Supp}^+_{\mathcal{A}}(0 :_{\mathcal{E}} g) \subseteq \mathsf{Supp}^+_{\mathcal{A}}\mathcal{E} \cap \mathcal{V}(g).$$

Combining these equalities with Equation (4.5), we conclude that

$$\mathsf{Supp}^+_{\mathcal{A}}\mathcal{E}' = \mathsf{Supp}^+_{\mathcal{A}}\mathcal{E} \cap \mathcal{V}(g).$$

Finally, using Remark 4.3.9 we obtain the desired result.

□

Chapter 5

Virtually Small Complexes and Complete Intersections

5.1 Thick subcategories revisited

5.1.1. Let $F : \mathsf{T} \to \mathsf{T}'$ be an exact functor between triangulated categories with right adjoint exact functor G. Let $\epsilon : FG \to \mathrm{id}_{\mathsf{T}'}$ and $\eta : \mathrm{id}_{\mathsf{T}} \to GF$ be the co-unit and unit transformations.

The full subcategory of T consisting of all objects X such that the natural map $\eta_X : X \to GF(X)$ is an isomorphism is a thick subcategory of T. For each X in T, the composition

$$F(X) \xrightarrow{F(\eta_X)} FGF(X) \xrightarrow{\epsilon_{F(X)}} F(X)$$

is an isomorphism. Therefore, if η_X is an isomorphism in T then $\epsilon_{F(X)}$ is an isomorphism in T' and F induces an equivalence of categories

$$\mathsf{Thick}_{\mathsf{T}} X \xrightarrow{\cong} \mathsf{Thick}_{\mathsf{T}'} F(X).$$

Lemma 5.1.2. *Let $\varphi : R \to S$ be flat morphism of commutative rings. Suppose M is in $\mathsf{D}(R)$ and the natural map $M \to M \otimes_R S$ is an isomorphism in $\mathsf{D}(R)$. Then the* *functor* $-\otimes_R S : \mathsf{D}(R) \to \mathsf{D}(S)$ *induces an equivalence of categories*

$$\mathsf{Thick}_{\mathsf{D}(R)} M \xrightarrow{\cong} \mathsf{Thick}_{\mathsf{D}(S)}(M \otimes_R S).$$

In particular, for each N *in* $\mathsf{Thick}_{\mathsf{D}(R)} M$ *the natural map* $N \to N \otimes_R S$ *is an isomorphism in* $\mathsf{D}(R)$*.*

Proof. The restriction of scalar functor $\varphi^* : \mathsf{D}(S) \to \mathsf{D}(R)$ is right adjoint to $-\otimes_R S : \mathsf{D}(R) \to \mathsf{D}(S)$. By assumption, the natural map

$$M \to \varphi^*(M \otimes_R S)$$

is an isomorphism in $\mathsf{D}(R)$. Hence, 5.1.1 completes the proof. □

Let R be a commutative noetherian ring. Recall that a complex of R-modules is perfect provided that it is an object of $\mathsf{Thick}_{\mathsf{D}(R)} R$ (see Example 2.3.6).

5.1.3*.* The following theorem is a remarkable and beautiful result due to M. Hopkins [18] and Neeman [22]: *For objects* M *and* N *of* $\mathsf{Thick}_{\mathsf{D}(R)} R$*,* M *is an object of* $\mathsf{Thick}_{\mathsf{D}(R)} N$ *if and only if* $\operatorname{Supp}_R M \subseteq \operatorname{Supp}_R N$.

5.2 Virtualy Small Complexes

Let R be a commutative noetherian ring. A complex of R-modules M is *virtually small* if $M \simeq 0$ or there exists a nontrivial object P in

$$\mathsf{Thick}_{\mathsf{D}(R)} M \cap \mathsf{Thick}_{\mathsf{D}(R)} R.$$

If in addition P can be chosen with $\operatorname{Supp}_R M = \operatorname{Supp}_R P$, we say M is *proxy small.* These notions were introduced by Dwyer, Greenlees, and Iyengar in [12] and [13], where the authors apply methods from commutative algebra to homotopy theory and vice versa.

Remark 5.2.1. In [12] and [13], the objects of $\mathsf{Thick}_{\mathsf{D}(R)} R$ are called the *small objects of* $\mathsf{D}(R)$. With this terminology, the nontrivial virtually small objects of $\mathsf{D}(R)$ are the complexes that finitely build a nontrivial small object.

Example 5.2.2. A trivial class of virtually small complexes are the perfect complexes.

Example 5.2.3. Let $\mathfrak{m}$ be a maximal ideal of R and suppose it is generated by $\boldsymbol{x}$. By 3.1.7, $\operatorname{Kos}^R(\boldsymbol{x})$ is an object of $\mathsf{Thick}_{\mathsf{D}(R)}(R/\mathfrak{m})$. Thus, $R/\mathfrak{m}$ is virtually small. Notice that $R/\mathfrak{m}$ is a perfect complex if and only if $R_\mathfrak{m}$ is a regular local ring. Indeed, if $R/\mathfrak{m}$ is a perfect complex then $(R/\mathfrak{m})_\mathfrak{m}$ has finite projective dimension over the local ring $R_\mathfrak{m}$. Moreover, $R_\mathfrak{m}$ is a local ring with residue field

$$R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m} \cong (R/\mathfrak{m})_\mathfrak{m}$$

and hence, by the Auslander-Buchsbaum and Serre theorem we conclude that $R_\mathfrak{m}$ is a regular local ring.

Conversely, assume that $R_\mathfrak{m}$ is a regular local ring. Since $R_\mathfrak{m}$ is regular, $(R/\mathfrak{m})_\mathfrak{m}$ has finite projective dimension over $R_\mathfrak{m}$. Moreover, $(R/\mathfrak{m})_\mathfrak{p} = 0$ for all $\mathfrak{p} \neq \mathfrak{m}$. Thus, $R/\mathfrak{m}$ has finite projective dimension over R. That is, $R/\mathfrak{m}$ is a perfect complex.

In summary, whenever R is not a regular ring then there is some maximal ideal $\mathfrak{m}$ of R such that $R/\mathfrak{m}$ is virtually small but not a perfect complex.

5.2.4. A nontrivial object M of $\mathsf{D}^f(R)$ is virtually small if and only if there exists a maximal ideal $\mathfrak{m} = (\boldsymbol{x})$ of R such that $\operatorname{Kos}^R(\boldsymbol{x})$ is in $\mathsf{Thick}_{\mathsf{D}(R)} M$. In particular, if

R is local, a nontrivial complex M in $\mathsf{D}^f(R)$ is virtually small if and only if K^R is in $\mathsf{Thick}_{\mathsf{D}(R)} M$. This was observed in [12, 4.5], and is a consequence 5.1.3.

As a matter of notation, let $\mathsf{VS}(R)$ to be the full subcategory of $\mathsf{D}^f(R)$ consisting of all virtually small complexes. In the following lemma, the argument for "(1) implies (2)" is abstracted from the proof of [12, 9.4].

Lemma 5.2.5. *Let R be a commutative noetherian ring. The following are equivalent:*

1. $\mathsf{Thick}_{\mathsf{D}(R)}(R/\mathfrak{m})$ *is a subcategory of* $\mathsf{VS}(R)$ *for each maximal ideal* $\mathfrak{m}$ *of* R.
2. $\mathsf{D}^f(R) = \mathsf{VS}(R)$.
3. $\mathsf{VS}(R)$ *is a thick subcategory of* $\mathsf{D}(R)$.

Proof. (1) $\implies$ (2): Let M be a nontrivial object of $\mathsf{D}^f(R)$. Since M is nontrivial, there exists a maximal ideal $\mathfrak{m}$ in $\operatorname{Supp}_R M$. Let $\boldsymbol{x}$ generate $\mathfrak{m}$ and set

$$N := M \otimes_R \operatorname{Kos}^R(\boldsymbol{x}).$$

By 3.1.6, $\operatorname{Supp}_R N = \{\mathfrak{m}\}$ and hence, N is in $\mathsf{Thick}_{\mathsf{D}(R)}(R/\mathfrak{m})$ (see 3.1.7). By assumption, there exists a nontrivial object P in $\mathsf{Thick}_{\mathsf{D}(R)} N \cap \mathsf{Thick}_{\mathsf{D}(R)} R$. Finally, since N is in $\mathsf{Thick}_{\mathsf{D}(R)} M$, $\mathsf{Thick}_{\mathsf{D}(R)} N$ is a subcategory of $\mathsf{Thick}_{\mathsf{D}(R)} M$. Thus, P is in $\mathsf{Thick}_{\mathsf{D}(R)} M$. That is, M is virtually small.

(2) $\implies$ (3): Whenever R is noetherian, $\mathsf{D}^f(R)$ is a thick subcategory of $\mathsf{D}(R)$.

(3) $\implies$ (1): Let $\mathfrak{m}$ be a maximal ideal of R and suppose $\boldsymbol{x}$ generates $\mathfrak{m}$. By 3.1.7, $\operatorname{Kos}^R(\boldsymbol{x})$ is in $\mathsf{Thick}_{\mathsf{D}(R)}(R/\mathfrak{m})$. Thus, $R/\mathfrak{m}$ is in $\mathsf{VS}(R)$. Since $\mathsf{VS}(R)$ is a thick subcategory of $\mathsf{D}(R)$, it follows that $\mathsf{Thick}_{\mathsf{D}(R)}(R/\mathfrak{m})$ is contained in $\mathsf{VS}(R)$. $\square$

Lemma 5.2.6. *Let $\varphi : R \to S$ be a flat morphism of commutative noetherian rings. Suppose $\mathfrak{m}$ is a maximal ideal of R such that $\mathfrak{m}S$ is a maximal ideal of S and the* *canonical map* $R/\mathfrak{m} \to S/\mathfrak{m}S$ *is an isomorphism. Then* $\mathsf{Thick}_{\mathsf{D}(R)}(R/\mathfrak{m})$ *is a subcategory of* $\mathsf{VS}(R)$ *if and only if* $\mathsf{Thick}_{\mathsf{D}(S)}(S/\mathfrak{m}S)$ *is a subcategory of* $\mathsf{VS}(S)$.

Proof. Set $K := \mathrm{Kos}^R(\boldsymbol{x})$ where $\boldsymbol{x}$ generates $\mathfrak{m}$. Let $\boldsymbol{x}'$ denote the image of $\boldsymbol{x}$ under φ and set $K' := \mathrm{Kos}^S(\boldsymbol{x}')$. Hence, we have an isomorphism of DG S-algebras $K' \cong K \otimes_R S$.

Assume $\mathsf{Thick}_{\mathsf{D}(R)}(R/\mathfrak{m})$ is a subcategory of $\mathsf{VS}(R)$. Let N be a nontrivial object of $\mathsf{Thick}_{\mathsf{D}(S)}(S/\mathfrak{m}S)$. By Lemma 5.1.2, there exists a nontrivial complex M in $\mathsf{Thick}_{\mathsf{D}(R)}(R/\mathfrak{m})$ such that $M \otimes_R S \simeq N$ in $\mathsf{D}(S)$. By assumption and 5.2.4, K is in $\mathsf{Thick}_{\mathsf{D}(R)} M$. Hence, $K \otimes_R S$ is in $\mathsf{Thick}_{\mathsf{D}(S)}(M \otimes_R S)$. Since $K' \cong K \otimes_R S$ and $N \simeq M \otimes_R S$, we conclude that K' is in $\mathsf{Thick}_{\mathsf{D}(S)} N$. Thus, N is in $\mathsf{VS}(S)$.

Let M be a nontrivial object of $\mathsf{Thick}_{\mathsf{D}(R)}(R/\mathfrak{m})$. Thus, $M \otimes_R S$ is a nontrivial object of $\mathsf{Thick}_{\mathsf{D}(S)}(S/\mathfrak{m}S)$. By assumption and 5.2.4, K' is in $\mathsf{Thick}_{\mathsf{D}(S)}(M \otimes_R S)$. Therefore,

$$K' \in \mathsf{Thick}_{\mathsf{D}(R)}(M \otimes_R S). \tag{5.1}$$

Since the natural map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism and K and M are in $\mathsf{Thick}_{\mathsf{D}(R)}(R/\mathfrak{m})$, by applying Lemma 5.1.2 we obtain the following isomorphisms in $\mathsf{D}(R)$

$$K \xrightarrow{\simeq} K \otimes_R S \cong K' \text{ and } M \xrightarrow{\simeq} M \otimes_R S.$$

These isomorphisms and (5.1) imply that K is in $\mathsf{Thick}_{\mathsf{D}(R)} M$. That is, M is in $\mathsf{VS}(R)$. $\square$

Proposition 5.2.7. *Let* R *be a commutative noetherian ring.*

1. *Then* $\mathsf{D}^f(R) = \mathsf{VS}(R)$ *if and only if* $\mathsf{D}^f(R_\mathfrak{m}) = \mathsf{VS}(R_\mathfrak{m})$ *for every maximal ideal* $\mathfrak{m}$ *of* R.

2. *In addition, assume $(R, \mathfrak{m}, k)$ is local and let $\widehat{R}$ denote its $\mathfrak{m}$-adic completion. Then $\mathsf{D}^f(R) = \mathsf{VS}(R)$ if and only if $\mathsf{D}^f(\widehat{R}) = \mathsf{VS}(\widehat{R})$.*

Proof. By Lemma 5.2.5, $\mathsf{D}^f(R) = \mathsf{VS}(R)$ if and only if $\mathsf{Thick}_{\mathsf{D}(R)}(R/\mathfrak{m})$ is a subcategory of $\mathsf{VS}(R)$ for each maximal ideal of $\mathfrak{m}$ of R. By Lemma 5.2.6, the latter holds if and only if $\mathsf{Thick}_{\mathsf{D}(R_\mathfrak{m})}(\kappa(\mathfrak{m}))$ is a subcategory of $\mathsf{VS}(R_\mathfrak{m})$ for each maximal ideal $\mathfrak{m}$ of R where $\kappa(\mathfrak{m}) = R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}$. Equivalently, $\mathsf{D}^f(R_\mathfrak{m}) = \mathsf{VS}(R_\mathfrak{m})$ for each maximal ideal $\mathfrak{m}$ of R by Lemma 5.2.5. Thus, (1) holds.

Next, Lemma 5.2.6 yields that $\mathsf{Thick}_{\mathsf{D}(R)}\, k$ is a subcategory of $\mathsf{VS}(R)$ if and only if $\mathsf{Thick}_{\mathsf{D}(\widehat{R})}\, k$ is a subcategory of $\mathsf{VS}(\widehat{R})$. Applying Lemma 5.2.5, finishes the proof of (2). □

5.3 The Main Results

Let $(R, \mathfrak{m})$ be a commutative noetherian local ring and let $\widehat{R}$ denote its $\mathfrak{m}$-adic completion. The local ring R is said to be a *complete intersection* provided

$$\widehat{R} \cong Q/(f_1, \ldots, f_c)$$

where Q is a regular local ring and $f_1, \ldots, f_c$ is a Q-regular sequence. In [12, 9.4], the following was established: *if R is a complete intersection every object of $\mathsf{D}^f(R)$ virtually small. If in addition R is a quotient of a regular local ring, every object of $\mathsf{D}^f(R)$ is proxy small.* Moreover, the authors posed the following question:

Question 5.3.1. *[12, 9.4] If every object of $\mathsf{D}^f(R)$ is virtually small, is R a complete intersection?*

Theorem 5.3.2, below, answers Question 5.3.1 in the affirmative. Much of the work in establishing "(1) implies (3)" is done in the proof of a theorem of Bergh [10, 3.2]. The theory of support varieties developed in Section 4.4 is the key ingredient used to prove "(2) implies (1)."

Theorem 5.3.2. *Let R be a commutative noetherian local ring. The following are equivalent.*

1. *R is a complete intersection.*
2. *Every object of $\mathsf{D}^f(R)$ is virtually small.*
3. *Every object of $\mathsf{D}^f(R)$ is proxy small.*

Proof. (1) $\implies$ (3): Let M be in $\mathsf{D}^f(R)$. In the proof of [10, 3.2], it is shown there exist positive integers $n_1, \ldots, n_t$ and exact triangles in $\mathsf{D}(R)$

$$\begin{aligned} &M \to \Sigma^{n_1} M \to M(1) \to \\ &M(1) \to \Sigma^{n_2} M(1) \to M(2) \to \\ &\ \vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\quad \vdots \\ &M(t-1) \to \Sigma^{n_t} M(t-1) \to M(t) \to \end{aligned}$$

such that $M(t)$ is in $\mathsf{Thick}_{\mathsf{D}(R)} R$. Also, it is clear that $M(t)$ is in $\mathsf{Thick}_{\mathsf{D}(R)} M$. Since each $n_i \neq 0$, Lemma 3.1.8 yields

$$\operatorname{Supp}_R M = \operatorname{Supp}_R(M(1)) = \ldots = \operatorname{Supp}_R(M(t)).$$

Thus, M is proxy small.

(3) $\implies$ (2): Clear from the definitions.

(2) $\implies$ (1): By Proposition 5.2.7(2), we may assume that R is complete. Write $R = Q/I$ where $(Q, \mathfrak{n}, k)$ is a regular local ring. Assume I is minimally generated by $\boldsymbol{f} = f_1, \ldots, f_n \in \mathfrak{n}^2$ and let E be the Koszul complex on $\boldsymbol{f}$.

Fix $1 \leq i \leq n$. By Theorem 4.4.4, there exists $C(i)$ in $\mathsf{Thick}_{\mathsf{D}(R)}\, k$ with

$$\mathsf{V}_E(C(i)) = \mathcal{V}(\chi_i).$$

Since each $C(i)$ is an object of $\mathsf{Thick}_{\mathsf{D}(R)}\, k$, it follows that each $C(i)$ is an object of $\mathsf{D}^f(R)$. Hence, by assumption each $C(i)$ is virtually small. Therefore, 5.2.4 implies that K^R is in $\mathsf{Thick}_{\mathsf{D}(R)}\, C(i)$. Hence,

$$\mathsf{V}_E(K^R) \subseteq \mathsf{V}_E\,(C(i)) = \mathcal{V}(\chi_i)$$

by Theorem 4.4.2(1). Applying Theorem 4.4.2(2) with $M = R$ yields

$$\mathsf{V}_E(R) = \mathsf{V}_E(K^R),$$

and hence, $\mathsf{V}_E(R) \subseteq \mathcal{V}(\chi_i)$.

Therefore,

$$\mathsf{V}_E(R) \subseteq \mathcal{V}(\chi_1) \cap \ldots \cap \mathcal{V}(\chi_n).$$

That is, $\mathsf{V}_E(R) = \emptyset$ and so by Theorem 4.4.2(3), $\boldsymbol{f}$ is a Q-regular sequence. Thus, R is a complete intersection. □

This structural characterization of a complete intersection's derived category yields the following corollary which was first established by Avramov in [4].

Corollary 5.3.3. *Assume a commutative noetherian local ring R is a complete intersection. For any $\mathfrak{p} \in \operatorname{Spec} R$, $R_{\mathfrak{p}}$ is is a complete intersection.*

Proof. For any $\mathfrak{p} \in \operatorname{Spec} R$, the functor $- \otimes_R R_{\mathfrak{p}} : \mathsf{D}^f(R) \to \mathsf{D}^f(R_{\mathfrak{p}})$ is essentially surjective. Also, the property of proxy smallness localizes. These observations and

Theorem 5.3.2 complete the proof. □

Let R be a commutative noetherian ring. We say that R is *locally a complete intersection* if $R_{\mathfrak{p}}$ is a complete intersection for each $\mathfrak{p} \in \operatorname{Spec} R$. By Corollary 5.3.3, R is locally a complete intersection if and only if $R_{\mathfrak{m}}$ is a complete intersection for every maximal ideal $\mathfrak{m}$ of R. We obtain the following homotopical characterization of rings that are locally complete intersections.

Theorem 5.3.4. *A commutative noetherian ring R is locally a complete intersection if and only if every object of $\mathsf{D}^f(R)$ is virtually small.*

Proof. As remarked above, R is locally a complete intersection if and only if $R_{\mathfrak{m}}$ is a complete intersection for each maximal ideal $\mathfrak{m}$ of R. By Theorem 5.3.2, the latter holds if and only if $\mathsf{D}^f(R_{\mathfrak{m}}) = \mathsf{VS}(R_{\mathfrak{m}})$ for each maximal ideal $\mathfrak{m}$ of R. Equivalently, $\mathsf{D}^f(R) = \mathsf{VS}(R)$ by Proposition 5.2.7(1). □

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