The Effects of Making a Subsidy Inversely Related to the Product Price under Cournot Competition

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Abstract

This paper complements the analysis by Kiso (2019), which is based on Bertrand competition with product differentiation. With a model of Cournot competition with product differentiation, I investigate the effect of making a subsidy inversely related to the product price. I show that the Cournot framework overall leads to similar outcomes to the Bertrand framework: relative to the specific or ad valorem subsidy, the IPR subsidy induces the same sales with less government outlay and allows the regulator to flexibly adjust the incidence on producers.

Keywords: subsidy design; subsidy efficiency; subsidy incidence; imperfect competition; Cournot oligopoly

JEL Classification: D43, H21, H22, L13, Q58

1 Subsidy Policies and Corresponding Equilibria

Consider a market with n firms, each of which produces a symmetrically differentiated product at a constant marginal cost c. The inverse demand function for firm i's product is represented by a twice continuously differentiable function $P(q_i, \lambda_i)$, where q_i is the output of product i and $\lambda_i \equiv \lambda(\mathbf{q}_{-i})$ is an aggregator of the effects of the outputs of the other $n-1$ products ($q_j\ \forall j \neq i$). We assume that inverse demand is decreasing in q_i: $P_q(q_i, \lambda_i) \equiv \frac{\partial P(q_i,\lambda_i)}{\partial q_i} < 0$. Also, products are substitutes in the sense that $\frac{\partial P(q_i,\lambda_i)}{\partial q_j} = \frac{\partial P(q_i,\lambda_i)}{\partial \lambda_i}\frac{\partial \lambda_i}{\partial q_j} < 0$ for $j \neq i$. Without loss of generality, $\lambda_i = \lambda(\mathbf{q}_{-i})$ is defined in such a way that $\frac{\partial \lambda_i}{\partial q_j} > 0\ \forall j \neq i$, so that $P_\lambda(q_i, \lambda_i) \equiv \frac{\partial P(q_i,\lambda_i)}{\partial \lambda_i} < 0$. Due to symmetry, permutating the order of the other $n-1$ products does not affect λ_i.

[Policy A: No Subsidy]

First, we look at the baseline case of no subsidy. The firms engage in Cournot competition. Firm i sets q_i to maximize its profits $\pi_A(q_i, \lambda_i) = [P(q_i, \lambda_i) - c]q_i$. Assuming an interior

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solution, the best response to λ_i, $q_i = q_A^b(\lambda_i)$, is determined by the first order condition (FOC):

$$P_q(q_i, \lambda_i)q_i + P(q_i, \lambda_i) = c. \tag{1}$$

[Policy B: Specific Subsidy]

Suppose that the government offers consumers a specific subsidy of z per unit of the good purchased, where $z \in (0, c)$. The subsidy is provided as a rebate or tax credit, for example. Importantly, it makes no difference whether the direct recipients of the subsidy are consumers or producers (physical neutrality). The paper mainly thinks in terms of consumption subsidies (offered directly to consumers), but its results are valid for production subsidies (offered directly to producers) as well.

We interpret $P(q_i, \lambda_i)$ as the consumer price, or the the effective price that consumers pay out of pocket after accounting for the subsidy. Demand depends on this effective price. The producer price p_p (i.e., the price received by a firm) equals $P(q_i, \lambda_i) + z$. Firm i sets q_i to maximize its profits $\pi_B(q_i, \lambda_i) = [P(q_i, \lambda_i) + z - c]q_i$, so the best response to λ_i, $q_i = q_B^b(\lambda_i)$, is determined by the FOC

$$P_q(q_i, \lambda_i)q_i + P(q_i, \lambda_i) = c - z. \tag{2}$$

[Policy C: *Ad Valorem* Subsidy]

Suppose that the government offers consumers an *ad valorem* subsidy of $vP(q_i, \lambda_i)$ per unit of the good purchased ($v > 0$). As under Policy B, the consumer price is $P(q_i, \lambda_i)$, and the producer price is $(1 + v)P(q_i, \lambda_i)$. Firm i sets q_i to maximize its profits $\pi_C(q_i, \lambda_i) = [(1 + v)P(q_i, \lambda_i) - c]q_i$, so the best response to λ_i, $q_i = q_C^b(\lambda_i)$, is determined by the FOC

$$P_q(q_i, \lambda_i)q_i + P(q_i, \lambda_i) = \frac{c}{1 + v}. \tag{3}$$

In each of the three cases above, the implicit function theorem gives

$$\frac{\partial q_X^b}{\partial q_j} = \frac{dq_X^b}{d\lambda_i}\frac{\partial \lambda_i}{\partial q_j} = -\frac{P_\lambda(q_i, \lambda_i) + q_i P_{q\lambda}(q_i, \lambda_i)}{2P_q(q_i, \lambda_i) + q_i P_{qq}(q_i, \lambda_i)}\bigg|_{q_i = q_X^b(\lambda_i)} \cdot \frac{\partial \lambda_i}{\partial q_j}, \tag{4}$$

where $X \in \{A, B, C\}$ and $j \neq i$. It is assumed that $\frac{\partial q_X^b}{\partial q_j} = -\frac{P_\lambda(q_i, \lambda_i) + q_i P_{q\lambda}(q_i, \lambda_i)}{2P_q(q_i, \lambda_i) + q_i P_{qq}(q_i, \lambda_i)}\big|_{q_i = q_X^b(\lambda_i)} \cdot \frac{\partial \lambda_i}{\partial q_j} \in (-1, 0)$, so that products are strategic substitutes but firm i's reaction to Δq_j is smaller in magnitude than Δq_j (or, equivalently, a marginal change in q_i has a larger (in magnitude) effect on firm i's marginal revenue than a marginal change in q_j does. This implies that there exists a symmetric and unique Nash equilibrium under each of Policies A–C, where the output per firm q_X^* is such that the corresponding FOC is satisfied with $q_i = q_X^*$ $\forall i$.[1] Denote these equilibria by E_A, E_B, and E_C, respectively.

[1]Suppose that an asymmetric Nash equilibrium exists under Policy X. Then, there are (at least) two firms (denoted by 1 and 2) such that $q_{1X}^* \neq q_{2X}^*$, where q_{iX}^* is firm i's output in this equilibrium. Without loss of generality, assume $q_{1X}^* < q_{2X}^*$. By definition, $q_{1X}^* = q_X^b(\lambda(q_{2X}^*, q_{3X}^*, \cdots, q_{nX}^*))$ and $q_{2X}^* = q_X^b(\lambda(q_{1X}^*, q_{3X}^*, \cdots, q_{nX}^*))$.

Some additional regularity conditions are assumed on inverse demand $P(q_i, \lambda_i)$ so that a subsidy payment will have reasonable effects on firm behavior and equilibrium outcomes. First, given λ_i, marginal revenue (the LHS of (1)) is decreasing in q_i (i.e., $2P_q(q_i, \lambda_i) + q_i P_{qq}(q_i, \lambda_i) < 0$). Thus, conditional on λ_i, the best response $q^b_X(\lambda_i)$ is a singleton, and an increase in the subsidy rate (z or v) or reduction in the production cost (c) leads to a larger output, so that $q^b_A(\lambda_i) < q^b_B(\lambda_i)$ and $q^b_A(\lambda_i) < q^b_C(\lambda_i)$. It is also assumed that, when $q_i = q\ \forall i$, $2P_q(q, \lambda(q, \cdots, q)) + qP_{qq}(q, \lambda(q, \cdots, q)) + qP_{q\lambda}(q, \lambda(q, \cdots, q))\frac{d\lambda(q,\cdots,q)}{dq} < 0$, where $\frac{d\lambda(q,\cdots,q)}{dq} \equiv (n-1)\frac{\partial\lambda(q,\cdots,q)}{\partial q_j}$. Together with the FOCs (1), (2), and (3), this means that a higher subsidy rate or lower production cost raises equilibrium outputs and profits under Policies A–C.[2] Hence, $q^*_A < q^*_B$ and $\pi_A(q^*_A, \lambda(q^*_A, \cdots, q^*_A)) < \pi_B(q^*_B, \lambda(q^*_B, \cdots, q^*_B))$, and $q^*_A < q^*_C$ and $\pi_A(q^*_A, \lambda(q^*_A, \cdots, q^*_A)) < \pi_C(q^*_C, \lambda(q^*_C, \cdots, q^*_C))$.

[Policy D: IPR Subsidy]

The government conditionally offers consumers a subsidy that is inversely related to the product price $p_i = P(q_i, \lambda_i)$. No subsidy is provided if the price of a good is greater than or equal to a government-set threshold $\overline{p}$ (i.e., if $p_i \geq \overline{p}$), where we assume $r\overline{p} < c < \overline{p}$.[3] If the price is below $\overline{p}$, the subsidy per unit of the good increases linearly as the price *decreases*. Specifically, if the consumer price $p_i < \overline{p}$, a subsidy of $r[\overline{p} - p_i]$ is provided per unit of the good, where $0 < r < 1$. Thus, the producer price $p^p_i = (1-r)p_i + r\overline{p}$.[4] The assumption $r < 1$ ensures $\frac{dp^p_i}{dp_i} = 1 - r > 0$, so that the subsidy is not so generous that the producer price can be raised by lowering the consumer price p_i.

This leads to a contradiction:

$$-1 = \frac{q^*_{1X} - q^*_{2X}}{q^*_{2X} - q^*_{1X}} = \frac{q^b_X(\lambda(q^*_{2X}, q^*_{3X}, \cdots, q^*_{nX})) - q^b_X(\lambda(q^*_{1X}, q^*_{3X}, \cdots, q^*_{nX}))}{q^*_{2X} - q^*_{1X}} > -1, \tag{5}$$

where the inequality holds because $\frac{\partial q^b_X}{\partial q_j} > -1$. A similar argument shows the uniqueness of a symmetric Nash equilibrium provided that $\frac{\partial q^b_X}{\partial q_j} < \frac{1}{n-1}$.

[2]In a symmetric equilibrium with $q_i = q\ \forall i$, each firm's equilibrium marginal revenue decreases with q because $d[P_q(q, \lambda(q, \cdots, q))q + P(q, \lambda(q, \cdots, q))]/dq = 2P_q(q, \lambda(q, \cdots, q)) + qP_{qq}(q, \lambda(q, \cdots, q)) + [P_\lambda(q, \lambda(q, \cdots, q)) + qP_{\lambda q}(q, \lambda(q, \cdots, q))]\frac{d\lambda(q,\cdots,q)}{dq} < 0$. This implies that a higher subsidy rate or lower production cost increases the equilibrium output, so that $q^*_A < q^*_B$ and $q^*_A < q^*_C$. By substituting the FOCs, the profits at E_A, E_B, and E_C are $-P_q(q^*_A, \lambda^*_A) \cdot q^{*2}_A$, $-P_q(q^*_B, \lambda^*_B) \cdot q^{*2}_B$, and $-(1+v)P_q(q^*_C, \lambda^*_C) \cdot q^{*2}_C$, respectively, where $\lambda^*_X = \lambda(q^*_X, \cdots, q^*_X)$ for $X \in \{A, B, C\}$. Since $d[-P_q(q, \lambda(q, \cdots, q)) \cdot q^2]/dq = -q[2P_q(q, \lambda(q, \cdots, q)) + qP_{qq}(q, \lambda(q, \cdots, q)) + qP_{q\lambda}(q, \lambda(q, \cdots, q))\frac{d\lambda(q,\cdots,q)}{dq}] > 0$, a higher subsidy rate or lower production cost, which raises the equilibrium output q, leads to higher equilibrium profits under each of Policies A–C.

[3]This assumption sets the range on the generosity of the subsidy: $c < \overline{p}$ means that the subsidy is generous enough to give positive profits for marginal cost pricing ($p_i = c$), while $r\overline{p} < c$ means that it is not so generous that even the price of zero does not result in losses.

[4]Subsidy payment $r[\overline{p} - p_i](= p^p_i - p_i)$ is defined in terms of the consumer price p_i. Alternatively, it can be expressed with the producer price p^p_i as $r^p[\overline{p} - p^p_i]$, where the parameter r^p differs from r, while $\overline{p}$ is, by construction of the policy, identical to the one in the consumer price-based definition above. Equating the values from the two definitions gives $p^p_i - p_i = r[\overline{p} - p_i] = r^p[\overline{p} - p^p_i]$. Rearranging this, we obtain $r^p = r/(1-r)$. Since the function $g : (0,1) \to (0, \infty)$ defined as $g(r) = r/(1-r)$ is bijective (one-to-one and onto), it does not matter whether the subsidy is defined in terms of the consumer or producer price.

Under this policy, firm i's profits are expressed as

$$\pi_D(q_i,\lambda_i) = \begin{cases} \pi_A(q_i,\lambda_i) & \text{if} \quad P(q_i,\lambda_i) \geq \overline{p}, \\ \pi_{D_0}(q_i,\lambda_i) & \text{if} \quad P(q_i,\lambda_i) < \overline{p}, \end{cases} \tag{6}$$

where $\pi_A(q_i,\lambda_i)$ is the firm's profits under Policy A, as defined above, and $\pi_{D_0}(q_i,\lambda_i) \equiv \{P(q_i,\lambda_i) + r[\overline{p} - P(q_i,\lambda_i)] - c\}q_i$. Note that, by definition, $\pi_D(q_i,\lambda_i)$ is continuous, including at q_i such that $P(q_i,\lambda_i) = \overline{p}$.

We first consider maximizing the function $\pi_{D_0}(q_i,\lambda_i)$ with respect to q_i, conditional on λ_i, with ignoring the eligibility condition $P(q_i,\lambda_i) < \overline{p}$ in (6) for the moment. The maximizer $q_i = q^b_{D_0}(\lambda_i)$ satisfies the FOC

$$P_q(q_i,\lambda_i)q_i + P(q_i,\lambda_i) = \frac{c - r\overline{p}}{1-r} \; (< c). \tag{7}$$

As under Policies A–C, the slope of $q_i = q^b_{D_0}(\lambda_i)$ is given by (4) with $X = D_0$ and assumed to be $\in (-1,0)$, implying that the map $\Pi^n_{i=1} q^b_{D_0}(\lambda_i)$, where $\lambda_i = \lambda(\mathbf{q}_{-i})$, has a symmetric and unique fixed point E_{D_0} with $q_i = q^*_{D_0}$ $\forall i$. Also, $q^b_A(\lambda_i) < q^b_{D_0}(\lambda_i)$, and $q^*_A < q^*_{D_0}$ (but it may *not* be the case that $\pi_A(q^*_A, \lambda(q^*_A, \cdots, q^*_A)) < \pi_{D_0}(q^*_{D_0}, \lambda(q^*_{D_0}, \cdots, q^*_{D_0}))$, as will be analyzed later in the paper).

Next, define $G(\lambda_i)$ as the difference between $\max_{q_i} \pi_A(q_i,\lambda_i) = \pi_A(q^b_A(\lambda_i),\lambda_i)$ and $\max_{q_i} \pi_{D_0}(q_i,\lambda_i) = \pi_{D_0}(q^b_{D_0}(\lambda_i),\lambda_i)$, where maximization is unconditional on the (in)eligibility conditions given in (6), and $q^b_A(\lambda_i)$ and $q^b_{D_0}(\lambda_i)$ are defined by (1) and (7), respectively:

$$\begin{aligned} G(\lambda_i) &\equiv \pi_A(q^b_A(\lambda_i),\lambda_i) - \pi_{D_0}(q^b_{D_0}(\lambda_i),\lambda_i) \\ &= [P(q^b_A(\lambda_i),\lambda_i) - c] \cdot q^b_A(\lambda_i) - [(1-r)P(q^b_{D_0}(\lambda_i),\lambda_i) + r\overline{p} - c] \cdot q^b_{D_0}(\lambda_i). \end{aligned} \tag{8}$$

It is assumed that $G(\lambda_i) = 0$ implies $G'(\lambda_i) < 0$. This can be interpreted as follows. Suppose that, given the subsidy policy (r and $\overline{p}$) and other firms' actions represented by λ_i, firm i is indifferent between opting in and out. Under these circumstances, a marginal increase in the aggressiveness of the other firms' aggregate behavior (i.e., a marginal increase in λ_i) and the resulting downward shift of the residual demand curve for product i (because $P_\lambda < 0$) makes opting in and receiving government support the (strictly) preferred choice for firm i.[5] A sufficient condition for this to be the case is $P_\lambda(q_i,\lambda_i) - \frac{P_\lambda(q_i,\lambda_i) + q_i P_{q\lambda}(q_i,\lambda_i)}{2P_q(q_i,\lambda_i) + q_i P_{qq}(q_i,\lambda_i)} \cdot P_q(q_i,\lambda_i) < 0$, which implies that $P(q_i,\lambda_i)$ goes down if λ_i is increased and i's optimal choice of q_i under each policy is adjusted accordingly.[6]

Proposition 1. *If $G(\lambda(0,\cdots,0)) \geq 0$ and $G(M) \leq 0$ for some M ($\geq \lambda(0,\cdots,0)$), there exists a unique $\tilde{\lambda} \in [\lambda(0,\cdots,0), M]$ such that $G(\tilde{\lambda}) = 0$. In this case, the best response correspondence under*

[5]As shown by (19) in the proof of the proposition, $G < 0$ means opting in is strictly preferred to opting out.

[6]Recall that $\frac{\partial q^b_X(\lambda_i)}{\partial \lambda_i} = -\frac{P_\lambda(q_i,\lambda_i) + q_i P_{q\lambda}(q_i,\lambda_i)}{2P_q(q_i,\lambda_i) + q_i P_{qq}(q_i,\lambda_i)}\Big|_{q_i = q^b_X(\lambda_i)}$.

Policy D is

$$q_D^b(\lambda_i) = \begin{cases} q_A^b(\lambda_i) & \text{if} \quad \lambda_i \leq \tilde{\lambda}, \\ q_{D_0}^b(\lambda_i) & \text{if} \quad \lambda_i \geq \tilde{\lambda}. \end{cases} \tag{9}$$

If $G(\lambda(0, \cdots, 0)) < 0$, the best response correspondence under Policy D is

$$q_D^b(\lambda_i) = q_{D_0}^b(\lambda_i) \;\; \forall \lambda_i. \tag{10}$$

If $G(\lambda_i) > 0$ for all λ_i, the best response correspondence under Policy D is

$$q_D^b(\lambda_i) = q_A^b(\lambda_i) \;\; \forall \lambda_i. \tag{11}$$

Proof. See the Appendix. ∎

In the latter two cases, (10) and (11), the subsidy scheme is either so generous or parsimonious that the choice of opting in or out does not depend on other firms' actions, resulting in the Nash equilibrium with $q_i = q_{D_0}^* \; \forall i \; (E_{D_0})$ or with $q_i = q_A^* \; \forall i \; (E_A)$. These extreme cases are of little interest and not considered in the following, so there exists a unique $\tilde{\lambda}$ such that $G(\tilde{\lambda}) = 0$.

Proposition 2. *There are at least one, and at most two Nash equilibria under Policy D. All Nash equilibria are symmetric. In one possible equilibrium, all firms follow $q_A^b(\lambda_i)$, realizing E_A. In the other possible equilibrium, all firms follow $q_{D_0}^b(\lambda_i)$, realizing E_{D_0}. Given a unique $\tilde{\lambda}$ such that $G(\tilde{\lambda}) = 0$, consider a unique $\tilde{q}$ such that $\tilde{\lambda} = \lambda(\tilde{q}, \cdots, \tilde{q})$.*

1. *If $\tilde{q} < q_A^*$, the only equilibrium under Policy D is E_{D_0}.*
2. *If $q_A^* \leq \tilde{q} \leq q_{D_0}^*$, the two equilibria under Policy D are E_A and E_{D_0};*
3. *If $q_{D_0}^* < \tilde{q}$, the only equilibrium under Policy D is E_A;*

Proof. See the Appendix. ∎

2 Comparing the Outcomes of Different Policies

Section 1 has considered firm behavior and market equilibria under exogenous subsidy policies. Suppose now that the government aims to increase social and consumer surplus by setting Policies B, C, or D (z, v, or r and $\overline{p}$) to induce the firms to raise the output (per-firm) from the no-subsidy level q_A^* to a common target level $\hat{q}$ $(= q_B^* = q_C^* = q_{D_0}^*)$.[7] This section compares equilibrium outcomes of Policies A–D when the specific, *ad valorem*, and IPR subsidies are all designed to achieve the same target.

[7] If no externality is associated with the consumption/production of the good, social surplus (consumer surplus + producer surplus − government expenditure) is maximized with $\hat{q}$ such that $P(\hat{q}, \lambda(\hat{q}, \cdots, \hat{q})) = c$. With positive externalities, which are often the very reason for subsidization but not considered explicitly in this paper, socially optimal $\hat{q}$ will be greater than the level that equates the (consumer) price with c. Note that the following analysis is not about setting $\hat{q}$ optimally, and thus is not restricted to socially optimal $\hat{q}$.

[Policies B and C]

Under Policies B and C, substituting $q_i = \hat{q}\ \forall i$ into the FOCs (2) and (3) shows that the government target $\hat{q}$ ($> q_A^*$) is induced by setting $z = -P_q(\hat{q},\hat{\lambda})\hat{q} - P(\hat{q},\hat{\lambda}) + c$ and $v = \frac{c}{P_q(\hat{q},\hat{\lambda})\hat{q}+P(\hat{q},\hat{\lambda})} - 1$, respectively, where $\hat{\lambda} \equiv \lambda(\hat{q},\cdots,\hat{q})$. With these values, the subsidy payment per unit under each policy is

$$\begin{aligned} \sigma_B(\hat{q}) &\equiv -P_q(\hat{q},\hat{\lambda})\hat{q} - P(\hat{q},\hat{\lambda}) + c, \\ \sigma_C(\hat{q}) &\equiv \big(\frac{c}{P_q(\hat{q},\hat{\lambda})\hat{q} + P(\hat{q},\hat{\lambda})} - 1\big)P(\hat{q},\hat{\lambda}). \end{aligned} \tag{12}$$

Using the FOCs, equilibrium profits per firm are expressed as $\pi_B(\hat{q},\hat{\lambda}) = -P_q(\hat{q},\hat{\lambda})\hat{q}^2$ and $\pi_C(\hat{q},\hat{\lambda}) = -\frac{cP_q(\hat{q},\hat{\lambda})q^2}{P_q(\hat{q},\hat{\lambda})q+P(\hat{q},\hat{\lambda})}$.

[Policy D]

The FOC (7) implies that pairs of r and $\overline{p}$ that satisfy the following equation result in $q_{D_0}^* = \hat{q}$:

$$(1-r)[P_q(\hat{q},\hat{\lambda}) \cdot \hat{q} + P(\hat{q},\hat{\lambda})] + r\overline{p} - c = 0. \tag{13}$$

From (13), $\overline{p}$ is given as a function of r and $\hat{q}$, and $\frac{\partial \overline{p}}{\partial r} < 0$.[8] Hereafter, $\overline{p}$ is eliminated by using (13), which means that when we consider below the effect of changing r conditional on $\hat{q}$, $\overline{p}$ is also implicitly changed to satisfy (13), and we observe the total effect of the changes in both r and $\overline{p}$.

First, we investigate the conditions on r (and $\overline{p}$) under which Policy D can induce a Nash equilibrium with $q_{D_0}^* = \hat{q}$. Let r_2 and r_3 be determined by $G(\lambda_A^*; r_2, \overline{p}(r_2,\hat{q})) = 0$ and $G(\hat{\lambda}; r_3, \overline{p}(r_3,\hat{q})) = 0$, where $\lambda_A^* \equiv \lambda(q_A^*,\cdots,q_A^*)$ and $\hat{\lambda} \equiv \lambda(\hat{q},\cdots,\hat{q})$. In words, given the government target $\hat{q}$ and other firms' choices $q_j = q_A^*\ \forall j \neq i$, r_2 makes firm i indifferent between opting in and out (i.e., between $q_i = q_{D_0}^b(\lambda_A^*; r_2, \overline{p}(r_2,\hat{q}))$ and $q_i = q_A^b(\lambda_A^*)$, and r_3 is described analogously. It can be shown that $r_2 \in (0,1)$ and $r_3 \in (0,1)$.[9] The following proposition rephrases Proposition 2 of Section 1 in terms of policy variables (r, $\hat{q}$, and implicitly $\overline{p}$), and is useful from a policy-making perspective.

Proposition 3. *Given $\hat{q}$ ($> q_A^*$), $r_2 < r_3$. Moreover,*

1. *if $r < r_2$, the only equilibrium under Policy D is E_{D_0};*
2. *if $r_2 \leq r \leq r_3$, the two equilibria under Policy D are E_A and E_{D_0};*
3. *if $r_3 < r$, the only equilibrium under Policy D is E_A.*

Proof. See the Appendix. ■

[8] $\overline{p} = [c - P_q(\hat{q},\hat{\lambda}) \cdot \hat{q} - P(\hat{q},\hat{\lambda})]/r + P_q(\hat{q},\hat{\lambda}) \cdot \hat{q} + P(\hat{q},\hat{\lambda})$, and $\partial\overline{p}/\partial r = [P_q(\hat{q},\hat{\lambda}) \cdot \hat{q} + P(\hat{q},\hat{\lambda}) - c]/r^2 < 0$.

[9] By the definition of G (with the FOCs substituted), $1 - r_2 = \frac{P_q(q_A^*,\lambda_A^*)\cdot q_A^{*2}}{P_q(q_{D_0}^b(\lambda_A^*;r_2,\overline{p}(r_2,\hat{q})),\lambda_A^*)\cdot[q_{D_0}^b(\lambda_A^*;r_2,\overline{p}(r_2,\hat{q}))]^2}$ and $1 - r_3 = \frac{P_q(q_A^b(\hat{\lambda},\hat{\lambda})\cdot[q_A^b(\hat{\lambda}]^2}{P_q(\hat{q},\hat{\lambda})\cdot\hat{q}^2}$. Since $\partial[P_q(q,\lambda)\cdot q^2]/\partial q = q\big[2P_q(q,\lambda) + qP_{qq}(q,\lambda)\big] < 0$ and $P_q < 0$, $1 - r_2 \in (0,1)$ and $1 - r_3 \in (0,1)$.

By Proposition 3, if $r \leq r_3$, E_{D_0} is a Nash equilibrium and the subsidy payment per unit (i.e., $r(\overline{p} - \hat{p})$ with (13) satisfied) in this equilibrium is

$$\sigma_{D_0}(r, \hat{p}) \equiv -[P(\hat{q}, \hat{\lambda}) - c] - (1 - r)P_q(\hat{q}, \hat{\lambda}) \cdot \hat{q}. \tag{14}$$

Using the FOCs, equilibrium profits per firm are expressed as $\pi_{D_0}(\hat{q}, \hat{\lambda}) = -(1 - r)P_q(\hat{q}, \hat{\lambda}) \cdot \hat{q}^2$. The next proposition is the main result of the paper, showing the relative efficiency of the different subsidy schemes in achieving a given government target ($\hat{q}$).

Proposition 4. *Suppose Policies B, C, and D (with $r \leq r_3$) are all designed to attain $\hat{q}$ ($> q_A^*$). Then, as to the subsidy payment per unit under these policies, $\sigma_{D_0}(r, \hat{q}) < \sigma_B(\hat{q}) < \sigma_C(\hat{q})$. Specifically, the differences in subsidy payment (or, equivalently, in profits) per unit of a good are as follows:*

$$\begin{aligned} &\sigma_B(\hat{q}) - \sigma_{D_0}(r, \hat{q}) = -rP_q(\hat{q}, \hat{\lambda}) \cdot \hat{q} > 0, \\ &\sigma_C(\hat{q}) - \sigma_B(\hat{q}) = \Big(\frac{c}{P_q(\hat{q}, \hat{\lambda})\hat{q} + P(\hat{q}, \hat{\lambda})} - 1\Big)\Big(-P_q(\hat{q}, \hat{\lambda}) \cdot \hat{q}\Big) > 0. \end{aligned} \tag{15}$$

Proof. The results follow from (12) and (14). ■

Next, I compare a firm's profits at the two potential equilibria E_A and E_{D_S} that can be realized under Policy D. Using the FOCs (1) and (7), $\pi_A(q_A^*, \lambda_A^*) = -P_q(q_A^*, \lambda_A^*) \cdot q_A^{*2}$, and $\pi_{D_0}(\hat{q}, \hat{\lambda}) = -(1 - r)P_q(\hat{q}, \hat{\lambda}) \cdot \hat{q}^2$. Therefore, with $r_1 \equiv 1 - \frac{P_q(q_A^*, \lambda_A^*) \cdot q_A^{*2}}{P_q(\hat{q}, \hat{\lambda}) \cdot \hat{q}^2} \in (0, 1)$,[10]

$$\pi_A(q_A^*, \lambda_A^*) \begin{cases} < \pi_{D_0}(\hat{q}, \hat{\lambda}) & \text{if} \quad r < r_1, \\ = \pi_{D_0}(\hat{q}, \hat{\lambda}) & \text{if} \quad r = r_1, \\ > \pi_{D_0}(\hat{q}, \hat{\lambda}) & \text{if} \quad r > r_1. \end{cases} \tag{16}$$

Proposition 5. *Given $\hat{q}$ ($> q_A^*$), $r_1 < r_2 (< r_3)$. Therefore, Policy D can make a firm's profits at E_{D_0} higher than, equal to, and lower than at E_A by setting r to satisfy $0 < r < r_1$, $r = r_1$, and $r_1 < r \leq r_3$, respectively, and $\overline{p}$ by (13). In particular, if $r_1 < r < r_2$, the unique Nash equilibrium under Policy D (E_{D_0}) results in lower profits than E_A.*

Proof. See the Appendix. ■

References

Kiso, Takahiko. 2019. "A Subsidy Inversely Related to the Product Price."

[10]Note that $0 < r_1 < 1$ because $d[P_q(q, \lambda(q, \cdots, q)) \cdot q^2]/dq = q\big[2P_q(q, \lambda(q, \cdots, q)) + qP_{qq}(q, \lambda(q, \cdots, q)) + qP_{q\lambda}(q, \lambda(q, \cdots, q))\frac{d\lambda(q, \cdots, q)}{dq}\big] < 0$, $P_q < 0$, and $q_A^* < \hat{q}$.

A Proof of Proposition 1

Proof. By the definition of $G(\lambda_i)$,

$$G(\lambda_i) \geq 0 \iff \pi_A(q_A^b(\lambda_i), \lambda_i) = \max_{q_i}\{\max\{\pi_A(q_i, \lambda_i),\ \pi_{D_0}(q_i, \lambda_i)\}\}. \tag{17}$$

Thus, given λ_i, if $G(\lambda_i) \geq 0$ and additionally if the eligibility requirement is not satisfied at $q_i = q_A^b(\lambda_i)$ (that is, $P(q_A^b(\lambda_i), \lambda_i) \geq \overline{p}$), then (6) and (17) imply that $q_{iA}^b(\lambda_i)$ is firm i's best response. Similarly, conditional on λ_i,

$$G(\lambda_i) \leq 0 \iff \pi_{D_0}(q_{D_0}^b(\lambda_i), \lambda_i) = \max_{q_i}\{\max\{\pi_A(q_i, \lambda_i),\ \pi_{D_0}(q_i, \lambda_i)\}\}. \tag{18}$$

Thus, given λ_i, if $G_i(\lambda_i) \leq 0$ and additionally if the eligibility requirement is satisfied at $q_i = q_{D_0}^b(\lambda_i)$ (that is, $P(q_{D_0}^b(\lambda_i), \lambda_i) < \overline{p}$), then (6) and (18) imply that $q_{D_0}^b(\lambda_i)$ is firm i's best response. The following lemma implies that the (in)eligibility conditions in (6) can be ignored in deriving the optimal response because they are implied by the conditions on $G(\lambda_i)$.

Lemma 1. *$G(\lambda_i) \geq 0$ implies $P(q_A^b(\lambda_i), \lambda_i) \geq \overline{p}$, and $G(\lambda_i) \leq 0$ implies $P(q_{D_0}^b(\lambda_i), \lambda_i) < \overline{p}$. Equivalently, $P(q_A^b(\lambda_i), \lambda_i) < \overline{p}$ implies $G(\lambda_i) < 0$, and $P(q_{D_0}^b(\lambda_i), \lambda_i) \geq \overline{p}$ implies $G(\lambda_i) > 0$.*

Proof. Suppose $P(q_A^b(\lambda_i), \lambda_i) < \overline{p}$. Then,

$$\begin{aligned}
\pi_{D_0}(q_{D_0}^b(\lambda_i), \lambda_i) &> \pi_{D_0}(q_A^b(\lambda_i), \lambda_i) \\
&= [(1-r)P(q_A^b(\lambda_i), \lambda_i) + r\overline{p} - c] \cdot q_A^b(\lambda_i) \\
&= [P(q_A^b(\lambda_i), \lambda_i) - c] \cdot q_A^b(\lambda_i) + r[\overline{p} - P(q_A^b(\lambda_i), \lambda_i)] \cdot q_A^b(\lambda_i) \\
&= \pi_A(q_A^b(\lambda_i), \lambda_i) + r[\overline{p} - P(q_A^b(\lambda_i), \lambda_i)] \cdot q_A^b(\lambda_i) \\
&> \pi_A(q_A^b(\lambda_i), \lambda_i).
\end{aligned}$$

That is, $G(\lambda_i) < 0$. In other words, if $G(\lambda_i) \geq 0$, then $P(q_A^b(\lambda_i), \lambda_i) \geq \overline{p}$.

Suppose $P(q_{D_0}^b(\lambda_i), \lambda_i) \geq \overline{p}$. Then,

$$\begin{aligned}
\pi_A(q_A^b(\lambda_i), \lambda_i) &> \pi_A(q_{D_0}^b(\lambda_i), \lambda_i) \\
&= [P(q_{D_0}^b(\lambda_i), \lambda_i) - c] \cdot q_{D_0}^b(\lambda_i) \\
&\geq [P(q_{D_0}^b(\lambda_i), \lambda_i) - c] \cdot q_{D_0}^b(\lambda_i) + r[\overline{p} - P(q_{iD_0}^b(\lambda_i), \lambda_i)] \cdot q_{D_0}^b(\lambda_i) \\
&= [(1-r)P(q_{D_0}^b(\lambda_i), \lambda_i) + r\overline{p} - c] \cdot q_{D_0}^b(\lambda_i) \\
&= \pi_{D_0}(q_{D_0}^b(\lambda_i), \lambda_i).
\end{aligned}$$

That is, $G(\lambda_i) > 0$. In other words, if $G(\lambda_i) \leq 0$, then $P(q_{D_0}^b(\lambda_i), \lambda_i) < \overline{p}$. ■

Lemma 1 and the argument preceding it suggest that the best response correspondence under

Policy D conditional on λ_i, denoted by $q_D^b(\lambda_i)$, is as follows:

$$q_D^b(\lambda_i) = \begin{cases} q_A^b(\lambda_i) & \text{if} \quad G(\lambda_i) \geq 0, \\ q_{D_0}^b(\lambda_i) & \text{if} \quad G(\lambda_i) \leq 0. \end{cases} \tag{19}$$

Next, the conditions in (19) are simplified. By the assumption that $G(\lambda_i) = 0$ implies $G'(\lambda_i) < 0$, as λ_i increases, the curve $G(\lambda_i)$ can cross the λ_i axis ($G = 0$) only from above and at most once. Together with the intermediate value theorem, this implies that if $G(\lambda(0, \cdots, 0)) \geq 0$ and $G(M) \leq 0$ for some M ($\geq \lambda(0, \cdots, 0)$), then there exists a unique $\tilde{\lambda} \in [\lambda(0, \cdots, 0), M]$ such that $G(\tilde{\lambda}) = 0$; $G(\lambda_i) > 0$ for all $\lambda_i < \tilde{\lambda}$; and $G(\lambda_i) < 0$ for all $\lambda_i > \tilde{\lambda}$.

Lemma 2. *1. If there exists $\check{\lambda}$ such that $G(\check{\lambda}) \geq 0$, then $G(\lambda_i) > 0$ for all $\lambda_i < \check{\lambda}$.*

2. If there exists $\check{\lambda}$ such that $G(\check{\lambda}) \leq 0$, then $G(\lambda_i) < 0$ for all $\lambda_i > \check{\lambda}$.

3. There is at most one $\tilde{\lambda}$ such that $G(\tilde{\lambda}) = 0$.

Proof. By the envelope theorem, $\frac{d\pi_A(q_A^b(\lambda_i),\lambda_i)}{d\lambda_i} = P_\lambda(q_A^b(\lambda_i), \lambda_i) \cdot q_A^b(\lambda_i)$ and $\frac{d\pi_{D_0}(q_{D_0}^b(\lambda_i),\lambda_i)}{d\lambda_i} = (1-r)P_\lambda(q_{D_0}^b(\lambda_i), \lambda_i) \cdot q_{D_0}^b(\lambda_i)$. Thus,

$$\begin{aligned} G'(\lambda_i) &= P_\lambda(q_A^b(\lambda_i), \lambda_i) \cdot q_A^b(\lambda_i) - (1-r)P_\lambda(q_{D_0}^b(\lambda_i), \lambda_i) \cdot q_{D_0}^b(\lambda_i) \\ &= P_q(q_A^b(\lambda_i), \lambda_i) \cdot [q_A^b(\lambda_i)]^2 \cdot \frac{P_\lambda(q_A^b(\lambda_i), \lambda_i)}{P_q(q_A^b(\lambda_i), \lambda_i) \cdot q_A^b(\lambda_i)} \\ &\qquad - (1-r)P_q(q_{D_0}^b(\lambda_i), \lambda_i) \cdot [q_{D_0}^b(\lambda_i)]^2 \cdot \frac{P_\lambda(q_{D_0}^b(\lambda_i), \lambda_i)}{P_q(q_{D_0}^b(\lambda_i), \lambda_i) \cdot q_{D_0}^b(\lambda_i)} \\ &< \left\{ P_q(q_A^b(\lambda_i), \lambda_i) \cdot [q_A^b(\lambda_i)]^2 - (1-r)P_q(q_{D_0}^b(\lambda_i), \lambda_i) \cdot [q_{D_0}^b(\lambda_i)]^2 \right\} \frac{P_\lambda(q_A^b(\lambda_i), \lambda_i)}{P_q(q_A^b(\lambda_i), \lambda_i) \cdot q_A^b(\lambda_i)} \\ &= -G(\lambda_i) \frac{P_\lambda(q_A^b(\lambda_i), \lambda_i)}{P_q(q_A^b(\lambda_i), \lambda_i) \cdot q_A^b(\lambda_i)}. \end{aligned} \tag{20}$$

The inequality in (20) follows because $q_A^b(\lambda_i) < q_{D_0}^b(\lambda_i)$ and

$$\frac{\partial \frac{P_\lambda(q,\lambda)}{qP_q(q,\lambda)}}{\partial q} = -\frac{2P_q + qP_{qq}}{(qP_q)^2} \Big[P_q(-\frac{P_\lambda + qP_{q\lambda}}{2P_q + qP_{qq}}) + P_\lambda \Big] < 0 \tag{21}$$

imply $\frac{P_\lambda(q_A^b(\lambda_i),\lambda_i)}{P_q(q_A^b(\lambda_i),\lambda_i) \cdot q_A^b(\lambda_i)} > \frac{P_\lambda(q_{D_0}^b(\lambda_i),\lambda_i)}{P_q(q_{D_0}^b(\lambda_i),\lambda_i) \cdot q_{D_0}^b(\lambda_i)}$. The last equality in (20) results from substituting the FOCs (1) and (7) into (8). Thus, $G(\lambda_i) = 0$ implies $G'(\lambda_i) < 0$. Therefore, as λ_i increases, the curve $G(\lambda_i)$ can cross the λ_i axis ($G = 0$) only from above and at most once, so the three statements of Lemma 2 follow. ∎

Thus, based on (19), the best response correspondence is given by (9) if $G(\lambda(0, \cdots, 0)) \geq 0$ and $G(M) \leq 0$ for some M ($\geq \lambda(0, \cdots, 0)$), by (10) if $G(\lambda(0, \cdots, 0)) < 0$, and by (11) if $G(\lambda_i) > 0$ for all λ_i.

∎

B Proof of Proposition 2

Proof. First, it is proved by contradiction that no asymmetric Nash equilibrium exists under Policy D. If there is one, there are (at least) two firms (denoted by 1 and 2) such that $q^*_{1D} \neq q^*_{2D}$, where q^*_{iD} is firm i's output in this equilibrium. Without loss of generality, assume $q^*_{1D} < q^*_{2D}$. By definition, $q^*_{1D} = q^b_D(\lambda(q^*_{2D}, q^*_{3D}, \cdots, q^*_{nD}))$ and $q^*_{2D} = q^b_D(\lambda(q^*_{1D}, q^*_{3D}, \cdots, q^*_{nD}))$.

Recall that $q^b_D(\cdot) = q^b_A(\cdot)$ or $q^b_D(\cdot) = q^b_{D_0}(\cdot)$. Suppose that at this equilibrium $q^b_D(\cdot) = q^b_A(\cdot)$ for both firms or $q^b_D(\cdot) = q^b_{D_0}(\cdot)$ for both firms. This leads to a contradiction as in footnote 1. Next, suppose that $q^*_{1D} = q^b_D(\lambda(q^*_{2D}, q^*_{3D}, \cdots, q^*_{nD})) = q^b_{D_0}(\lambda(q^*_{2D}, q^*_{3D}, \cdots, q^*_{nD}))$ and $q^*_{2D} = q^b_D(\lambda(q^*_{1D}, q^*_{3D}, \cdots, q^*_{nD})) = q^b_A(\lambda(q^*_{1D}, q^*_{3D}, \cdots, q^*_{nD}))$, then

$$\begin{aligned} -1 = \frac{q^*_{1D} - q^*_{2D}}{q^*_{2D} - q^*_{1D}} &= \frac{q^b_{D_0}(\lambda(q^*_{2D}, q^*_{3D}, \cdots, q^*_{nD})) - q^b_A(\lambda(q^*_{1D}, q^*_{3D}, \cdots, q^*_{nD}))}{q^*_{2D} - q^*_{1D}} \\ &> \frac{q^b_A(\lambda(q^*_{2D}, q^*_{3D}, \cdots, q^*_{nD})) - q^b_A(\lambda(q^*_{1D}, q^*_{3D}, \cdots, q^*_{nD}))}{q^*_{2D} - q^*_{1D}} > -1, \end{aligned} \tag{22}$$

which is a contradiction. Lastly, suppose that $q^*_{1D} = q^b_D(\lambda(q^*_{2D}, q^*_{3D}, \cdots, q^*_{nD})) = q^b_A(\lambda(q^*_{2D}, q^*_{3D}, \cdots, q^*_{nD}))$ and $q^*_{2D} = q^b_D(\lambda(q^*_{1D}, q^*_{3D}, \cdots, q^*_{nD})) = q^b_{D_0}(\lambda(q^*_{1D}, q^*_{3D}, \cdots, q^*_{nD}))$. Then, (9) implies that $\lambda(q^*_{2D}, q^*_{3D}, \cdots, q^*_{nD}) \leq \tilde{\lambda} \leq \lambda(q^*_{1D}, q^*_{3D}, \cdots, q^*_{nD})$, but $q^*_{1D} < q^*_{2D}$ means $\lambda(q^*_{1D}, q^*_{3D}, \cdots, q^*_{nD}) < \lambda(q^*_{2D}, q^*_{3D}, \cdots, q^*_{nD})$, which is a contradiction.

Next, we look into symmetric equilibria. Proposition 1 implies that there are two potential cases to occur at a symmetric equilibrium: [1] all firms follow $q^b_A(\cdot)$, or [2] all firms follow $q^b_{D_0}(\cdot)$. In case [1], the potential equilibrium is nothing other than E_A, the symmetric and unique equilibrium under Policy A. According to Proposition 1, $q^b_D(\lambda_i) = q^b_A(\lambda_i)$ if and only if $\lambda_i \leq \tilde{\lambda}$ (note that the trivial cases (10) and (11) have been disregarded). Therefore, q^*_A is a fixed point of the map $q^b_D(\lambda(q, \cdots, q))$ (that is, $q_i = q^*_A = q^b_D(\lambda(q^*_A, \cdots, q^*_A))$ for all i, and E_A is a Nash equilibrium under Policy D) if and only if $\lambda(q^*_A, \cdots, q^*_A) \leq \tilde{\lambda}$, or $q^*_A \leq \tilde{q}$.

Analogously, in case [2], the potential equilibrium is nothing but E_{D_0}, the symmetric and unique fixed point of the map $\Pi^n_{i=1} q^b_{D_0}(\lambda_i)$, where $\lambda_i = \lambda(\mathbf{q}_{-i})$. Proposition 1 shows that $q^b_D(\lambda_i) = q^b_{D_0}(\lambda_i)$ if and only if $\lambda_i \leq \tilde{\lambda}$. Therefore, $q^*_{D_0}$ is a fixed point of the map $q^b_D(\lambda(q, \cdots, q))$ (that is, $q_i = q^*_{D_0} = q^b_D(\lambda(q^*_{D_0}, \cdots, q^*_{D_0}))$ for all i, and E_{D_0} is a Nash equilibrium under Policy D) if and only if $\lambda(q^*_{D_0}, \cdots, q^*_{D_0}) \geq \tilde{\lambda}$, or $q^*_{D_0} \geq \tilde{q}$.

Since $q^*_A < q^*_{D_0}$ as discussed below (7), statements 1–3 of the proposition follow.

∎

C Proof of Proposition 3

Proof. By substituting (13),

$$\pi_{D_0}(q^b_{D_0}(\lambda_i),\lambda_i;r,\overline{p}(r,\hat{q})) = (1-r)[P(q^b_{D_0}(\lambda_i),\lambda_i) - P_q(\hat{q},\hat{\lambda})\cdot\hat{q} - P(\hat{q},\hat{\lambda})]\cdot q^b_{D_0}(\lambda_i). \tag{23}$$

The FOC to be satisfied is

$$[P(q^b_{D_0}(\lambda_i),\lambda_i) - P_q(\hat{q},\hat{\lambda})\cdot\hat{q} - P(\hat{q},\hat{\lambda})] + P_q(q^b_{D_0}(\lambda_i),\lambda_i)\cdot q^b_{D_0}(\lambda_i) = 0. \tag{24}$$

By the envelope theorem,

$$\begin{aligned}\frac{dG(\lambda_i;r,\overline{p}(r,\hat{q}))}{dr} &= -\frac{d\pi_{D_0}(q^b_{D_0}(\lambda_i),\lambda_i;r,\overline{p}(r,\hat{q}))}{dr}\\ &= [P(q^b_{D_0}(\lambda_i),\lambda_i) - P_q(\hat{q},\hat{\lambda})\cdot\hat{q} - P(\hat{q},\hat{\lambda})]\cdot q^b_{D_0}(\lambda_i)\\ &= -P_q(q^b_{D_0}(\lambda_i),\lambda_i)\cdot[q^b_{D_0}(\lambda_i)]^2\\ &> 0.\end{aligned} \tag{25}$$

By the maintained assumption that $G(\lambda_i;r,\overline{p}) = 0$ implies $G'(\lambda_i;r,\overline{p}) < 0$, $G(\lambda^*_A;r_2,\overline{p}(r_2,\hat{q})) = 0$ and $\hat{\lambda} > \lambda^*_A$ imply $G(\hat{\lambda};r_2,\overline{p}(r_2,\hat{q})) < 0$. By (25), $G(\hat{\lambda};r_2,\overline{p}(r_2,\hat{q})) < 0$ and $G(\hat{\lambda};r_3,\overline{p}(r_3,\hat{q})) = 0$ mean $r_2 < r_3$.

Also, $G(\lambda^*_A;r_2,\overline{p}(r_2,\hat{q})) = 0$ and (25) imply that $G(\lambda^*_A;r,\overline{p}(r,\hat{q})) < (>)\ 0$ if $r < (>)\ r_2$. Thus, according to (19), if $r < r_2$, then $q^b_D(\lambda^*_A;r,\overline{p}(r,\hat{q})) = q^b_{D_0}(\lambda^*_A;r,\overline{p}(r,\hat{q})) > q^*_A$, so that E_A is not a Nash equilibrium under Policy D. If $r \geq r_2$, then $q^b_D(\lambda^*_A;r,\overline{p}(r,\hat{q})) = q^b_A(\lambda^*_A) = q^*_A$, so that E_A is a Nash equilibrium under Policy D.

Similarly, $G(\hat{\lambda};r_3,\overline{p}(r_3,\hat{q})) = 0$ and (25) imply that $G(\hat{\lambda};r,\overline{p}(r,\hat{q})) < (>)\ 0$ if $r < (>)\ r_3$. Thus, according to (19), if $r \leq r_3$, then $q^b_D(\hat{\lambda};r,\overline{p}(r,\hat{q})) = q^b_{D_0}(\hat{\lambda};r,\overline{p}(r,\hat{q})) = \hat{q}$, so that E_{D_0} is a Nash equilibrium under Policy D. If $r > r_3$, then $q^b_D(\hat{\lambda};r,\overline{p}(r,\hat{q})) = q^b_A(\hat{\lambda}) < \hat{q}$, so that E_{D_0} is a Nash equilibrium under Policy D.

Hence, the three statements of the proposition follow. ■

D Proof of Proposition 5

Proof. By the definitions of r_1 and r_2, $\tilde{\pi}_A(q^*_A,\lambda^*_A) = \tilde{\pi}_{D_0}(\hat{q},\hat{\lambda};r_1,\overline{p}(r_1;\hat{q})) = \tilde{\pi}_{D_0}(q^b_{D_0}(\lambda^*_A),\lambda^*_A;r_2,\overline{p}(r_2;\hat{q}))$. By the envelope theorem, $\frac{d\tilde{\pi}_{D_0}(q^b_{D_0}(\lambda_i),\lambda_i;r,\overline{p})}{d\lambda_i} = (1-r)P_\lambda(q^b_{D_0}(\lambda_i),\lambda_i)\cdot q^b_{D_0}(\lambda_i) < 0$. Then, because $\lambda^*_A < \hat{\lambda}$, $\tilde{\pi}_{D_0}(q^b_{D_0}(\lambda^*_A),\lambda^*_A;r_2,\overline{p}(r_2;\hat{q})) > \tilde{\pi}_{D_0}(q^b_{D_0}(\hat{\lambda}),\hat{\lambda};r_2,\overline{p}(r_2;\hat{q})) = \tilde{\pi}_{D_0}(\hat{q},\hat{\lambda};r_2,\overline{p}(r_2;\hat{q}))$. Thus, $\tilde{\pi}_{D_0}(\hat{q},\hat{\lambda};r_2,\overline{p}(r_2;\hat{q})) < \tilde{\pi}_{D_0}(\hat{q},\hat{\lambda};r_1,\overline{p}(r_1;\hat{q}))$ by the first sentence of this proof. This implies $r_1 < r_2$ because $\frac{d\pi_{D_0}(q^b_{D_0}(\lambda_i),\lambda_i;r,\overline{p}(r,\hat{q}))}{dr} < 0$ by (25). The remaining statements follow from Proposition 3 and (16).

■