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## Quasiconformal extensions, Loewner chains, and the $\lambda$ -Lemma

*in memory of Alexander Vasil'ev*

Pavel Gumenyuk · István Prause

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**Abstract** In 1972, J. Becker [J. Reine Angew. Math. **255**] discovered a sufficient condition for quasiconformal extendibility of Loewner chains. Many known conditions for quasiconformal extendibility of holomorphic functions in the unit disk can be deduced from his result. We give a new proof of (a generalization of) Becker's result based on Slodkowski's Extended  $\lambda$ -Lemma. Moreover, we characterize all quasiconformal extensions produced by Becker's (classical) construction and use that to obtain examples in which Becker's extension is extremal (i.e. optimal in the sense of maximal dilatation) or, on the contrary, fails to be extremal.

**Keywords** Quasiconformal extension · Loewner chain · Becker extension · evolution family · Loewner – Kufarev equation · Loewner range

**Mathematics Subject Classification (2000)** Primary: 30C62; Secondary: 30C35 · 30D05

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## 1 Introduction

The study of conformal mappings of the unit disk  $\mathbb{D} := \{z: |z| < 1\}$  admitting quasiconformal extensions is a classical topic in Geometric Function Theory closely related with the Teichmüller Theory, see, e.g. [4, 23, 28, 39]. One of the main results of Loewner Theory states that the class  $\mathcal{S}$  of all conformal mappings  $f: \mathbb{D} \rightarrow \mathbb{C}$  normalized by  $f(0) = 0$ ,  $f'(0) = 1$ , coincides with the range of the map that takes each function  $p(z, t)$ ,  $z \in \mathbb{D}$ ,  $t \geq 0$ , measurable in  $t$ , holomorphic in  $z$  with  $\operatorname{Re} p > 0$ , and normalized by  $p(0, t) = 1$ , to  $f(z) := \lim_{t \rightarrow +\infty} e^t w(z, t)$ , where  $w$  is the unique solution to the Loewner–Kufarev ODE  $dw/dt = -wp(w, t)$ ,  $t \geq 0$ ,  $w(z, 0) = z \in \mathbb{D}$ . At a first glance, this representation of  $\mathcal{S}$  seems to be too complicated. Nevertheless, it proved to be a very efficient tool in many problems of Complex Analysis, e.g., in extremal problems for conformal mappings, see [9] and references therein.

Given a subclass  $\tilde{\mathcal{S}} \subset \mathcal{S}$ , a natural problem arise: find a set of functions  $p$  that generates  $\tilde{\mathcal{S}}$ . For the subclass  $\mathcal{S}_k$ ,  $k \in (0, 1)$ , formed by all  $f \in \mathcal{S}$  admitting  $k$ -q.c. extensions  $F: \mathbb{C} \rightarrow \mathbb{C}$  with  $F(\infty) = \infty$ , see Sect. 2 for precise definitions, a partial answer was given in 1972 by Becker [10] who discovered a quite explicit construction of a  $k$ -q.c. extension of functions  $f \in \mathcal{S}$  which are generated by  $p$  satisfying  $p(\mathbb{D}, t) \subset U_k := \{\zeta: |\zeta - 1| \leq k|\zeta + 1|\}$  for a.e.  $t \geq 0$ . There are many indications that the class  $\mathcal{S}_k^B$  generated by such  $p$ 's does not coincide with  $\mathcal{S}_k$ . In Sect. 4, we characterize all q.c.-extensions arising from Becker's construction, see Theorem 2. As a corollary, we are able to prove rigorously that  $\mathcal{S}_k^B \neq \mathcal{S}_k$ , see Theorem 3 in Sect. 5. Although Becker's condition is only sufficient for  $k$ -q.c. extendibility, it seems to be worth for thorough investigation. Many well-known explicit sufficient conditions can be deduced from Becker's result, see, e.g. [12, §§5.3-5.4], [20, 21]. In fact, for certain  $f \in \mathcal{S}_k$ , Becker's extension has smallest maximal dilatation among all q.c.-extensions of  $f$ , see, e.g. Examples 1 and 2 in Sect. 5. Moreover, in [17], Becker's construction has been extended to the so-called chordal analogue of the Loewner–Kufarev equation [8, §8]. A bit surprisingly, but Becker's condition appears to be sufficient for the  $k$ -q.c. extendibility in a much more general version of Loewner Theory developed in [8, 7, 13, 5]. The proof of that fact has been recently obtained in [22]. In Sect. 3, we give a shorter proof based on the Extended  $\lambda$ -Lemma due Slodkowski [34].

In the next section, we give necessary preliminaries from quasiconformal mappings and Loewner Theory. Our main results are stated and proved in Sect. 3–5. The paper is concluded with a brief discussion in Sect. 6 on an auxiliary question concerning the Loewner range, which constitutes also some independent interest for Loewner Theory.

## 2 Preliminaries

There are several ways to define quasiconformality. One of the equivalent definitions is as follows.

**Definition 1** Let  $k \in [0, 1)$ . By a  $k$ -quasiconformal map (in short,  $k$ -q.c. map, or simply q.c.-map if we do not have to specify  $k$ ) of a domain  $U \subset \overline{\mathbb{C}}$  we mean a homeomorphism  $F : U \rightarrow \overline{\mathbb{C}}$  in the Sobolev class  $W_{\text{loc}}^{1,2}$  such that  $|\bar{\partial}F| \leq k|\partial F|$  for a.e.  $z \in U$ , where as usual  $\bar{\partial} := \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$  and  $\partial := \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$ .

Every  $k$ -q.c. map  $F$  is a solution to the Beltrami equation

$$\bar{\partial}F(z) = \mu_F(z) \partial F(z) \quad \text{for a.e. } z, \quad (2.1)$$

where  $\mu_F$  is a complex-valued measurable function satisfying  $\text{ess sup } |\mu_F| \leq k$ . The map  $F$  is conformal if  $\mu_F$  vanishes almost everywhere.

**Definition 2** The function  $\mu_F$  in (2.1) is called the *Beltrami coefficient* or *complex dilatation* of  $F$ ; and  $\text{ess sup } |\mu_F|$  is called the *maximal dilatation* of  $F$ .

Note that in this paper we use the “ $k$ -small notation”. Often instead of  $k$ , the deviation from conformality is measured by the parameter  $K := (1+k)/(1-k)$ . For further details on quasiconformal mappings, see, e.g. [2, 6, 27, 28].

Criteria for q.c.-extendibility of univalent maps  $f : \mathbb{D} \rightarrow \mathbb{C}$  is a classical topic in Geometric Function Theory, see, e.g. [4, 23, 28, 39].

**Definition 3** Let  $k \in [0, 1)$ . A holomorphic map  $f : \mathbb{D} \rightarrow \mathbb{C}$  is said to be  $k$ -q.c. extendible to  $\mathbb{C}$ , or to  $\overline{\mathbb{C}}$ , if there exists a  $k$ -q.c. map  $F : \mathbb{C} \rightarrow \mathbb{C}$ , or respectively,  $F : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$  such that  $F|_{\mathbb{D}} = f$ .

Thanks to the removability property, see, e.g. [27, Chapter I, §8.1],  $f$  is  $k$ -q.c. extendible to  $\mathbb{C}$  if and only if it admits a  $k$ -q.c. extension  $F : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$  with  $F(\infty) = \infty$ . We will denote by  $\mathcal{S}_k$  the class of all  $k$ -q.c. extendible to  $\mathbb{C}$  holomorphic functions  $f : \mathbb{D} \rightarrow \mathbb{C}$  normalized by  $f(0) = f'(0) - 1 = 0$ .

Below we collect necessary basics of Loewner Theory following [8, 7, 13, 5]. In 1972, Becker [10] obtained an explicit construction of q.c.-extensions based on so-called *Loewner chains*.

**Definition 4** ([13, Definition 1.2]) A *Loewner chain* in the unit disk  $\mathbb{D}$  is a family  $(f_t)_{t \geq 0} \subset \text{Hol}(\mathbb{D}, \mathbb{C})$  satisfying the following conditions:

- LC1.  $f_t$  is univalent in  $\mathbb{D}$  for any  $t \geq 0$ ,
- LC2.  $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$  whenever  $t \geq s \geq 0$ ,
- LC3. for any compact  $K \subset \mathbb{D}$  there exists a non-negative locally integrable function  $k_K$  on  $[0, +\infty)$  such that

$$\max_K |f_t - f_s| \leq \int_s^t k_K(\xi) d\xi \quad \text{whenever } t \geq s \geq 0.$$

Any Loewner chain  $(f_t)$  solves, in the sense of [15, Definition 2.1], the Loewner–Kufarev PDE

$$\frac{\partial f_t}{\partial t} = -f'_t(z)G(z, t), \quad z \in \mathbb{D}, \quad t \geq 0, \quad (2.2)$$

where  $G$  is some *Herglotz vector field*, defined by  $(f_t)$  uniquely up to a null-set on the  $t$ -axis. According to [8, Theorem 4.8], one of the two equivalent definitions of Herglotz vector fields is as follows.

**Definition 5** A *Herglotz function* in  $\mathbb{D}$  is a map  $p : \mathbb{D} \times [0, +\infty) \rightarrow \mathbb{C}$  satisfying the following conditions:

- HF1. for each  $z \in \mathbb{D}$ ,  $p(z, \cdot)$  is locally integrable on  $[0, +\infty)$ , and  
 HF2. for a.e.  $t \geq 0$ ,  $p(\cdot, t)$  is holomorphic in  $\mathbb{D}$  and  $\operatorname{Re} p(\cdot, t) \geq 0$ .

A *Herglotz vector field* in  $\mathbb{D}$  is a map  $G : \mathbb{D} \times [0, +\infty) \rightarrow \mathbb{C}$  of the form

$$G(z, t) = (\tau(t) - z)(1 - \overline{\tau(t)}z)p(z, t) \quad \text{for all } z \in \mathbb{D} \text{ and a.e. } t \geq 0, \quad (2.3)$$

where  $\tau : [0, +\infty) \rightarrow \overline{\mathbb{D}}$  is a measurable function and  $p$  is a Herglotz function.

*Remark 1* It is known [8, Theorem 4.8] that the Herglotz function  $p$  in representation (2.3) is uniquely defined by  $G$  up to a null-set on the  $t$ -axis.

Furthermore, it is known that the so-called *evolution family*  $(\varphi_{s,t})_{t \geq s \geq 0}$ ,  $\varphi_{s,t} := f_t^{-1} \circ f_s$ , associated with  $(f_t)$  is the unique solution to the following initial values problem for the (generalized) Loewner–Kufarev ODE:

$$\frac{d\varphi_{s,t}(z)}{dt} = G(\varphi_{s,t}(z), t), \quad t \geq s \geq 0, \quad z \in \mathbb{D}; \quad \varphi_{s,s}(z) = z. \quad (2.4)$$

*Remark 2* In general, the right-hand side of (2.4) is discontinuous in  $t$ . The equation is to be understood as Carathéodory’s first order ODE; see, e.g. [14, §2] and references therein for basic theory of such ODEs.

Evolution families can be defined independently, with no relation to Loewner chains.

**Definition 6** ([8, Definition 3.1]) An *evolution family* in  $\mathbb{D}$  is a two-parameter family  $(\varphi_{s,t})_{t \geq s \geq 0}$  satisfying the following conditions:

- EF1.  $\varphi_{s,s} = \operatorname{id}_{\mathbb{D}}$  for any  $s \geq 0$ ,  
 EF2.  $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$  whenever  $t \geq u \geq s \geq 0$ ,  
 EF3. for each  $z \in \mathbb{D}$  there exists a non-negative locally integrable function  $k_z$  on  $[0, +\infty)$  such that

$$|\varphi_{s,u}(z) - \varphi_{s,t}(z)| \leq \int_u^t k_z(\xi) d\xi \quad \text{whenever } t \geq u \geq s \geq 0.$$

Equation (2.4) establishes a one-to-one correspondence between evolution families  $(\varphi_{s,t})$  and Herglotz vector fields  $G$ , see [8, Theorem 1.1]. Moreover, given  $(\varphi_{s,t})$  or  $G$ , one can reconstruct the corresponding Loewner chain  $(f_t)$ , which turns out to be unique up to the post-composition with a conformal map, see [13, Theorems 1.3, 1.6] and [5].

**Definition 7** By a *radial Loewner chain* we mean a Loewner chain  $(f_t)$  satisfying  $f_t(0) = f_0(0)$  for all  $t \geq 0$ .

*Remark 3* Clearly, a Loewner chain  $(f_t)$  is radial if and only if  $G(0, t) = 0$  for a.e.  $t \geq 0$ , i.e. if  $\tau(t) = 0$  and hence  $G(z, t) = -zp(z, t)$  for a.e.  $t \geq 0$ .

### 3 Becker's condition and q.c.-extensions of evolution families and general Loewner chains

The classical Loewner Theory developed in [29,24,30,18], see also [31, §6.1], deals with radial Loewner chains  $(f_t)$  whose Herglotz functions  $p$  are normalized by  $p(0, t) = 1$  for a.e.  $t \geq 0$ . This normalization implies that the Loewner range  $\cup_{t \geq 0} f_t(\mathbb{D})$  coincides with  $\mathbb{C}$  and hence the Loewner chain  $(f_t)$  corresponding to a given Herglotz function  $p$  is defined uniquely up to linear transformations. Therefore, many properties of  $(f_t)$ , such as quasiconformal extendibility, are determined by the properties of  $p$ .

Following Becker [11], [12, §5.1], we replace the normalization  $p(0, t) = 1$  by a weaker condition  $\int_0^{+\infty} \operatorname{Re} p(0, t) dt = +\infty$ , which still guarantees that  $\cup_{t \geq 0} f_t(\mathbb{D}) = \mathbb{C}$ . Becker discovered the following remarkable result.

**Theorem A ([10,11])** *Let  $k \in [0, 1)$  and let  $(f_t)$  be a radial Loewner chain whose Herglotz function  $p$  satisfies*

$$p(\mathbb{D}, t) \subset U(k) := \left\{ w \in \mathbb{C} : \left| \frac{w-1}{w+1} \right| \leq k \right\} \quad \text{for a.e. } t \geq 0. \quad (3.1)$$

*Then for every  $t \geq 0$ , the function  $f_t$  admits a  $k$ -q.c. extension to  $\overline{\mathbb{C}}$  that fixes  $\infty$ . In particular, such an extension for  $f_0$  is given by*

$$F(\rho e^{i\theta}) := \begin{cases} f_0(\rho e^{i\theta}), & \text{if } 0 \leq \rho < 1, \\ f_{\log \rho}(e^{i\theta}), & \text{if } \rho \geq 1. \end{cases} \quad (3.2)$$

Many sufficient conditions for q.c.-extendibility can be deduced from the above theorem, see e.g. [12, §§5.3-5.4] and [20,21]. A similar result for chordal Loewner chains, i.e. the Loewner chains associated with Herglotz vector fields of the form  $G(z, t) = (1-z)^2 p(z, t)$ , was obtained in [17]. More generally, Hotta [22] showed that Becker's condition (3.1) is sufficient for q.c.-extendibility also in case of Herglotz vector fields (2.3) with arbitrary measurable function  $\tau : [0, +\infty) \rightarrow \overline{\mathbb{D}}$ . Below we give a simpler proof of this fact using the Extended  $\lambda$ -Lemma due to Ślodkowski [34].

**Theorem 1** *Let  $k \in [0, 1)$  and let  $(f_t)$  be a Loewner chain associated with a Herglotz vector field  $G$  such that the Herglotz function  $p$  in representation (2.3) satisfies Becker's condition (3.1). Then:*

- (i)  $\cup_{t \geq 0} f_t(\mathbb{D}) = \mathbb{C}$ ;
- (ii) *all the elements of the Loewner chain  $(f_t)$  and of the evolution family  $(\varphi_{s,t})$  associated with  $G$  admit  $k$ -q.c. extensions to  $\overline{\mathbb{C}}$ .*

*Remark 4* By [13, Theorem 1.6], for any Herglotz vector field  $G$  there exists a *unique* associated Loewner chain  $(f_t)$  such that  $f_0(0) = f'_0(0) - 1 = 0$  and  $\cup_{t \geq 0} f_t(\mathbb{D})$  is the whole complex plane or a disk centered at the origin. A Loewner chain  $(f_t)$  satisfying these conditions is called *standard*.

**Proof of Theorem 1** First we will replace (3.1) by a weaker condition and prove (ii) for the standard Loewner chain  $(f_t)$  associated with  $G$ . Namely, assume that there exists a locally integrable function  $a : [0, +\infty) \rightarrow \mathbb{H} \cup i\mathbb{R}$ , where  $\mathbb{H} := \{z : \operatorname{Re} z > 0\}$ , such that

$$p(\mathbb{D}, t) \subset D_t \quad \text{for a.e. } t \geq 0, \quad (3.3)$$

where  $D_t$  is the closed hyperbolic disk in  $\mathbb{H}$  of radius  $\frac{1}{2} \log \frac{1+k}{1-k}$  centered at  $a(t)$  when  $a(t) \in \mathbb{H}$ , and  $D_t := \{a(t)\}$  when  $a(t) \in i\mathbb{R}$ . If  $a(t) \equiv 1$ , then (3.3) becomes Becker's condition (3.1).

For all  $t \in \mathcal{Q} := \{t \geq 0 : \operatorname{Re} a(t) > 0\}$  and  $\lambda \in \mathbb{D}$ , set  $p_\lambda(\cdot, t) := H_t \circ \phi_\lambda(\cdot, t)$ , where

$$\phi_\lambda(\cdot, t) := \frac{\lambda}{k} H_t^{-1} \circ p(\cdot, t) \quad \text{and} \quad H_t(z) := \frac{1+z}{1-z} \operatorname{Re} a(t) + i \operatorname{Im} a(t).$$

For  $t \in [0, +\infty) \setminus \mathcal{Q}$ ,  $p(\cdot, t)$  is a constant belonging to  $i\mathbb{R}$  and we set  $p_\lambda(\cdot, t) := p(\cdot, t)$  for all such  $t$  and all  $\lambda \in \mathbb{D}$ . Since  $H_t^{-1}(D_t) = \{z : |z| \leq k\}$  for all  $t \in \mathcal{Q}$ , condition (3.3) implies that  $|\phi_\lambda(z, t)| \leq 1$  for a.e.  $t \in \mathcal{Q}$  and all  $z, \lambda \in \mathbb{D}$ . It follows that  $p_\lambda$  is a Herglotz function for any  $\lambda \in \mathbb{D}$ . For each  $\lambda \in \mathbb{D}$ , let  $(\varphi_{s,t}^\lambda)$  stand for the evolution family associated with the Herglotz vector field  $G_\lambda(z, t) := (\tau(t) - z)(1 - \overline{\tau(t)}z) p_\lambda(z, t)$ . Note that  $p_k = p$  and hence  $(\varphi_{s,t}^k) = (\varphi_{s,t})$ .

By the very construction,  $\mathbb{D} \ni \lambda \mapsto p_\lambda(z, t)$  is holomorphic for all  $z \in \mathbb{D}$  and a.e.  $t \geq 0$ . Moreover, for any compact sets  $K_1, K_2 \subset \mathbb{D}$ ,

$$t \mapsto \max_{(\lambda, z) \in K_1 \times K_2} |p_\lambda(z, t)|$$

is locally integrable on  $[0, +\infty)$ . Following the standard arguments used in [14, §2], it is easy to conclude that  $\mathbb{D} \ni \lambda \mapsto \varphi_{s,t}^\lambda(z)$  is holomorphic whenever  $t \geq s \geq 0$  and  $z \in \mathbb{D}$ .

For any  $s \geq 0$  and  $\lambda \in \mathbb{D}$ , the Schwarzian  $S\varphi_{s,t}^\lambda$  of  $\varphi_{s,t}^\lambda$  satisfies

$$\frac{d}{dt} S\varphi_{s,t}^\lambda(z) = (d\varphi_{s,t}^\lambda(z)/dz)^2 G_\lambda'''(\varphi_{s,t}^\lambda(z), t) \quad \text{a.e. } t \geq s.$$

In particular,  $G_0''' \equiv 0$  and hence  $\varphi_{s,t}^0$ 's are linear-fractional maps.

Now, fix some  $s \geq 0$  and  $t \geq s$ . Note that  $\varphi_{s,t}^\lambda : \mathbb{D} \rightarrow \mathbb{D}$  is univalent for any  $\lambda \in \mathbb{D}$ . Therefore,  $(\lambda, z) \mapsto \psi_\lambda(z) := (\varphi_{s,t}^0)^{-1} \circ \varphi_{s,t}^\lambda$  is a holomorphic motion of  $\mathbb{D}$ . By the Extended  $\lambda$ -Lemma, see, e.g. [6, Theorem 12.3.2 on p. 298], it extends to a holomorphic motion  $\mathbb{D} \times \overline{\mathbb{C}} \ni (\lambda, z) \mapsto \Psi_\lambda(z) \in \overline{\mathbb{C}}$  of  $\overline{\mathbb{C}}$  and moreover, for any  $\lambda \in \mathbb{D}$ , the map  $\Phi_{s,t}^\lambda$  is a  $|\lambda|$ -q.c. automorphism of  $\overline{\mathbb{C}}$ . In particular, for  $\lambda := k$  we obtain a  $k$ -q.c. extension of  $\varphi_{s,t} = \varphi_{s,t}^k$  defined by the formula  $\Phi_\lambda := \varphi_{s,t}^0 \circ \Psi_\lambda$ . To prove  $k$ -q.c. extendibility of  $f_t$ 's, it remains to use the explicit construction of the associated standard Loewner chain given in the proof of [13, Theorem 1.6] and apply [35, Theorem 14.1 on p. 148].

Now let us assume that  $p$  satisfies (3.1). Then (i) holds by Theorem 4, which we will prove in Sect. 6. As a consequence, according to [13, Theorem 1.6], any two Loewner chains associated with  $G$  differ by a linear map. Therefore, (ii) holds for any Loewner chain  $(f_t)$  associated with  $G$ .  $\square$

*Remark 5* In contrast to Becker's classical result, the q.c.-extensions of the evolution family and of the standard Loewner chain  $(f_t)$  in Theorem 1 do not have to fix  $\infty$ . For the special case of constant  $\tau$ , the linear-fractional maps  $\varphi_{s,t}^0$  have a fixed point at  $\tau^* := 1/\bar{\tau}$  and hence before extending to  $\bar{\mathbb{C}}$ , we may define the holomorphic motion  $(\lambda, z) \mapsto \psi_\lambda(z)$  also at the point  $\tau^*$  by setting  $\psi_\lambda(\tau^*) := \tau^*$  for all  $\lambda \in \mathbb{D}$ . As a result, in this case,  $\varphi_{s,t}$ 's admit  $k$ -q.c. extensions to  $\bar{\mathbb{C}}$  with a fixed point at  $\tau^*$ .

*Remark 6* Another way to apply the Extended  $\lambda$ -Lemma to the problem of q.c.-extendibility (not related to Loewner chains) was found by Sugawa [38]. The Loewner–Kufarev equation for q.c.-extendible functions as an evolution in the universal Teichmüller space was studied by Vasil'ev [40, 41].

#### 4 Characterization of Becker's Extensions

The q.c.-extensions produced by Becker's construction form a proper subclass in the class of all q.c.-maps of  $\bar{\mathbb{C}}$  that are conformal in  $\mathbb{D}$  and have a fixed point at  $\infty$ . Below we give a comparatively simple characterization of Becker's extensions. To be precise, we start by introducing the following definition.

**Definition 8** A *Becker extension* of a function  $f \in \mathcal{S}$  is a q.c.-map  $F : \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$  with  $F|_{\mathbb{D}} = f$ ,  $F(\infty) = \infty$  and such that there exists a radial Loewner chain  $(f_t)$  satisfying the following conditions:

- (i)  $f_0 = f$ ;
- (ii) for any  $t \geq 0$ , the function  $f_t$  extends continuously to  $\partial\mathbb{D}$ , with  $f_t(\zeta) = F(e^t\zeta)$  for all  $\zeta \in \partial\mathbb{D}$ .

If exists, this radial Loewner chain  $(f_t)$  is clearly unique; in what follows it will be referred to as the Loewner chain *associated* with the Becker extension  $F$ .

Denote by  $\mathcal{H}^\infty(\mathbb{D})$  the Hardy class of all bounded holomorphic functions in  $\mathbb{D}$ . We say that a complex-valued function  $\nu \in L^\infty(\partial\mathbb{D})$  *represents the boundary values* of a function  $\varphi \in \mathcal{H}^\infty(\mathbb{D})$ , if  $\lim_{r \rightarrow 1^-} \varphi(r\zeta) = \nu(\zeta)$  for a.e.  $\zeta \in \partial\mathbb{D}$ .

**Theorem 2** Let  $k \in [0, 1)$  and let  $F : \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$ ,  $F(\infty) = \infty$ , be a  $k$ -q.c. extension of some function  $f \in \mathcal{S}$ . The following assertions hold:

- (I)  $F$  is a Becker extension of  $f$  if and only if the complex dilatation  $\mu_F$  of  $F$  obeys the following property: for a.e.  $\rho > 1$  the map  $\partial\mathbb{D} \ni \zeta \mapsto \mu_F(\rho\zeta)$  represents boundary values of some  $\varphi_\rho \in \mathcal{H}^\infty(\mathbb{D})$  with  $\varphi_\rho(0) = \varphi'_\rho(0) = 0$ .
- (II) If  $F$  is a Becker extension of  $f$ , then the Loewner chain  $(f_t)$  associated with  $F$  satisfies the Loewner–Kufarev equation

$$\frac{\partial f_t(z)}{\partial t} = z f'_t(z) p(z, t), \quad t \geq 0, \quad z \in \mathbb{D},$$

where  $p(z, t) := (1 + \varphi_{e^t}(z)/z^2)/(1 - \varphi_{e^t}(z)/z^2)$  for all  $z \in \mathbb{D}$  and a.e.  $t \geq 0$ . In particular,  $p$  satisfies Becker's condition (3.1).



*Remark 7* Statement (I) of the above theorem can be rewritten as follows:  $F$  is a Becker extension if and only if the Fourier coefficients  $a_n(\rho) := \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} \mu_F(\rho e^{i\theta}) d\theta$  vanish for a.e.  $\rho > 1$  and all integer  $n \leq 1$ .

In the proof, we will need the following lemma.

**Lemma 1** *Let  $F$  be a Becker extension of some  $f \in \mathcal{S}$ . Then for a.e.  $t \geq 0$ , the Loewner chain  $(f_t)$  associated with  $F$  satisfies:*

$$\lim_{r \rightarrow 1^-} i e^{i\theta} f'_t(r e^{i\theta}) = \frac{\partial F(e^{t+i\theta})}{\partial \theta} \quad \text{for a.e. } \theta \in [0, 2\pi], \quad (4.1)$$

$$\lim_{r \rightarrow 1^-} \frac{\partial f_t(r e^{i\theta})}{\partial t} = \frac{\partial F(e^{t+i\theta})}{\partial t} \quad \text{for a.e. } \theta \in [0, 2\pi]. \quad (4.2)$$

**Proof** For a.e.  $t \geq 0$  the map  $\mathbb{R} \ni \theta \rightarrow F(e^{t+i\theta})$  is absolutely continuous and hence by a theorem of F. Riesz, see, e.g. [16, Theorem 1 in §IX.5], equality (4.1) takes place. To prove (4.2) fix for a moment some  $t \geq 0$  and write

$$\frac{f_{t+h}(z) - f_t(z)}{h} = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{P}(e^{-i\theta} z) \frac{F(e^{t+h+i\theta}) - F(e^{t+i\theta})}{h} d\theta \quad (4.3)$$

for all  $z \in \mathbb{D}$  and all  $h \geq -t$ , where  $\mathcal{P}(z) := \operatorname{Re}((1+z)/(1-z))$  is the Poisson kernel for  $\mathbb{D}$ . According to [6, Theorem 3.5.3 on p.66] and [6, Corollary 3.4.7 on p.62], for any  $\varepsilon > 0$  the function

$$L_\varepsilon(z) := \sup \{ |F(w) - F(z)| / |w - z| : 0 < |w - z| < \varepsilon \}$$

is locally integrable in  $\mathbb{C}$ . It follows that  $\theta \mapsto \sup\{(F(e^{t+h+i\theta}) - F(e^{t+i\theta}))/h : h \in \mathbb{R}, 0 < |h| < \varepsilon\}$  is integrable on  $[0, 2\pi]$  for a.e.  $t \geq 0$ . Hence for all such  $t$ 's we can apply Lebesgue's dominated convergence theorem to pass in (4.3) to the limit as  $h \rightarrow 0$ . As a result we get

$$\frac{\partial f_t(z)}{\partial t} = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{P}(e^{-i\theta} z) \frac{\partial F(e^{t+i\theta})}{\partial t} d\theta$$

for all  $z \in \mathbb{D}$  and a.e.  $t \geq 0$ . Equality (4.2) follows now by properties of the Poisson integral, see, e.g. [16, Corollary 1 in §IX.1].  $\square$

**Proof of Theorem 2** Suppose that  $F$  is a Becker extension of some  $f \in \mathcal{S}$  and let  $(f_t)$  be the associated Loewner chain. Furthermore, let  $p(z, t) := (\frac{\partial}{\partial t} f_t(z)) / (z f'_t(z))$  be the Herglotz function of  $(f_t)$ . Since  $\operatorname{Re} p(\cdot, t) \geq 0$ ,

$$\varphi_{e^t}(z) := \frac{p(z, t) - 1}{p(z, t) + 1} z^2 = \frac{\partial f_t(z) / \partial t + i(iz f'_t(z))}{\partial f_t(z) / \partial t - i(iz f'_t(z))} z^2$$

is a holomorphic bounded function in  $\mathbb{D}$  for a.e.  $t \geq 0$ , with zero of the second order at  $z = 0$ . The Jacobian of  $F$ ,  $J_F(z) = |\partial F|^2 - |\bar{\partial} F|^2$  is positive for a.e.  $z$ ,

see, e.g. [6, Corollary 3.7.6 on p. 75]. Therefore,  $(\frac{\partial}{\partial t} - i\frac{\partial}{\partial \theta})F(e^{t+i\theta}) \neq 0$  for a.e.  $(t, \theta) \in \mathbb{R} \times [0, 2\pi]$ . Taking this into account, we immediately deduce from Lemma 1 that for a.e.  $t \geq 0$  and a.e.  $\theta \in [0, 2\pi]$ ,  $\lim_{r \rightarrow 1^-} \varphi_{e^t}(re^{i\theta}) = \mu_F(e^{t+i\theta})$ . Since every bounded harmonic function can be recovered from its radial limits on  $\partial\mathbb{D}$  by means of the Poisson integral, see, e.g. [19, p. 38], we conclude, in particular, that  $\varphi_{e^t}(\mathbb{D}) \subset k\mathbb{D}$  for a.e.  $t \geq 0$  and hence  $p$  satisfies Becker's condition (3.1).

The above argument proves (II) and the necessity part of (I). To complete the proof of the theorem suppose now that for a.e.  $\rho > 1$ , the function  $\partial\mathbb{D} \ni \zeta \mapsto \mu_F(\rho\zeta)$  represents the boundary values of some  $\varphi_\rho \in \mathcal{H}^\infty(\mathbb{D})$  with  $\varphi_\rho(0) = \varphi'_\rho(0) = 0$ . We have to show that  $F$  is a Becker extension. Recall that  $\varphi_\rho$  can be recovered from its radial limits on  $\partial\mathbb{D}$  by means of the Poisson integral. Therefore,  $\sup_{z \in \mathbb{D}} |\varphi_\rho(z)| \leq k$  for a.e.  $\rho > 1$  and  $\rho \mapsto \varphi_\rho(z)$  is measurable for each  $z \in \mathbb{D}$  by Fubini's Theorem applied to  $(\rho, \theta) \mapsto \mu_F(\rho e^{i\theta})(1 + e^{-i\theta}z)/(1 - e^{-i\theta}z)$ . It follows that the formula  $p(z, t) := (1 + \varphi_{e^t}(z)/z^2)/(1 - \varphi_{e^t}(z)/z^2)$  defines a Herglotz function satisfying (3.1). By Becker's Theorem A, the radial Loewner chain  $(f_t)$  associated with  $p$  generates a  $k$ -q.c. extension  $\Phi : \mathbb{C} \rightarrow \mathbb{C}$  of  $f_0$  with  $\Phi(e^{t+i\theta}) = f_t(e^{i\theta})$ ,  $t \geq 0$ ,  $\theta \in \mathbb{R}$ , and  $\Phi(\infty) = \infty$ . By definition,  $\Phi$  is a Becker extension. Hence to complete the proof it remains to check that  $F = \Phi$ . Fix for a moment some  $r \in (0, 1)$ . Following Becker's proof, we consider the  $k$ -q.c. map  $\Phi_r$  defined by equalities  $\Phi_r(z) := f_0(rz)/r$  for all  $z \in \mathbb{D}$  and  $\Phi_r(e^{t+i\theta}) := f_t(re^{i\theta})/r$  for all  $t \geq 0$  and all  $\theta \in [0, 2\pi]$ . Then  $\mu_{\Phi_r}|_{\mathbb{D}} = 0$  and for all  $\theta \in [0, 2\pi]$  and a.e.  $t \geq 0$  we have

$$\mu_{\Phi_r}(e^{t+i\theta}) = \frac{\partial f_t(re^{i\theta})/\partial t - z f'_t(re^{i\theta})}{\partial f_t(re^{i\theta})/\partial t + z f'_t(re^{i\theta})} e^{2i\theta} = \frac{p(re^{i\theta}, t) - 1}{p(re^{i\theta}, t) + 1} e^{2i\theta} = \frac{\varphi_{e^t}(re^{i\theta})}{r^2}.$$

The radial limit of the r.h.s. exists for a.e.  $\theta \in [0, 2\pi]$  and equals to  $\mu_F(e^{t+i\theta})$ . It follows that  $\mu_{\Phi_r} \rightarrow \mu_F$  as  $r \rightarrow 1^-$  a.e. in  $\mathbb{C}$ . Note also that  $\Phi_r$  satisfies the same normalization as  $F$ , i.e.  $\Phi_r(0) = 0$ ,  $\Phi'_r(0) = 1$ , and  $\Phi_r(\infty) = \infty$ . Therefore, according to [6, Lemma 5.3.5 on p. 171],  $\Phi_r \rightarrow F$  in  $\overline{\mathbb{C}}$  as  $r \rightarrow 1^-$ . On the other hand, it follows easily from the construction that  $\Phi_r \rightarrow \Phi$  as  $r \rightarrow 1^-$ . This completes the proof.  $\square$

*Remark 8* It is interesting to compare Becker's explicit construction with the machinery used in the proof of Theorem 1 when applied to radial Loewner chains. Suppose that the Herglotz function  $p$  associated with  $(f_t)$  satisfies Becker's condition (3.1) and let  $F$  be the Becker extension of  $f_0$  defined by equality (3.2) in Theorem A. Consider the holomorphic motion  $(\lambda, z) \mapsto F_\lambda(z)$ , where  $F_\lambda$  is the unique  $|\lambda|$ -q.c. automorphism of  $\mathbb{C}$  satisfying the Beltrami equation  $\partial F_\lambda(z) = k^{-1}\lambda\mu_F(z)\partial F_\lambda(z)$  with the normalization  $F_\lambda(\infty) = \infty$ ,  $F_\lambda(0) = 0$ ,  $F'_\lambda(0) = 1$ . Then according to Theorem 2, for any  $\lambda \in \mathbb{D}$ ,  $F_\lambda$  is a Becker extension of some  $f^\lambda \in \mathcal{S}$ . Let  $(f_t^\lambda)$  be its associated classical Loewner chain. It is easy to see from the proof of Theorem 2 that the Herglotz function of  $(f_t^\lambda)$  coincides with  $p_\lambda$  defined in the proof of Theorem 1. Therefore, in the classical setting, Becker's explicit q.c.-extension is one of the q.c.-extensions that may be obtained via our implicit construction based on the  $\lambda$ -Lemma.

## 5 Examples and remarks

For  $k \in [0, 1)$ , denote by  $\mathcal{S}_k$  the set of all  $f \in \mathcal{S}$  admitting a  $k$ -q.c. extension  $F: \mathbb{C} \rightarrow \mathbb{C}$  with  $F(\infty) = \infty$ . We would like to get some idea about relation between the classes  $\mathcal{S}_k$  and the classes  $\mathcal{S}_k^B$  formed by all  $f \in \mathcal{S}$  admitting a  $k$ -q.c. Becker extension. It is difficult to believe that  $\mathcal{S}_k^B$  coincides with  $\mathcal{S}_k$  or constitutes a “large” part of it. However,

$$\mathcal{S}_k \subset \mathcal{S}_{3k}^B \quad \text{for any } k \in [0, \frac{1}{3}).$$

The above inclusion follows immediately from the following two facts. On the one hand, for any  $f \in \mathcal{S}_k$ , the Schwarzian  $S_f$  of  $f$  satisfies the inequality  $(1 - |z|^2)^2 |S_f(z)| \leq 6k$  for all  $z \in \mathbb{D}$ , see [25, Satz 3\*] or [26, Corollary 2]. On the other hand, the condition  $(1 - |z|^2)^2 |S_f(z)| \leq 2k'$  for all  $z \in \mathbb{D}$  is sufficient for  $k'$ -q.c. extendibility [1, 3] and, moreover, implies the existence of a  $k'$ -q.c. Becker extension [10, Satz 4.2], [12, p. 62–68].

Moreover, in certain examples, a Becker extension is the best possible in the sense of the maximal dilatation.

*Example 1* Fix  $k \in [0, 1)$ . Let  $f_1(z) := z/(1 - kz)^2$  and  $f_2(z) := z/(1 - kz^2)$ . It follows readily from [25, Satz 3\*], and from similar results in [26, Corollaries 1 and 3], that  $f_j$ ,  $j = 1, 2$ , have unique  $k$ -q.c. extensions  $F_j: \mathbb{C} \rightarrow \mathbb{C}$  with  $F_j(\infty) = \infty$ . In fact,  $F_j$ 's are Becker extensions associated with the Loewner chains  $f_t^j := e^t f_j$ ,  $t \geq 0$ ,  $j = 1, 2$ .

*Example 2* Let  $\sigma \in (0, 2)$  and consider  $f_\sigma \in \mathcal{S}$  given by  $f_\sigma := \sigma^{-1} H^{-1} \circ g_\sigma \circ H$ , where  $g_\sigma(\zeta) := \zeta^\sigma$ ,  $\operatorname{Re} \zeta > 0$ ,  $g_\sigma(1) = 1$ , and  $H(z) := (1+z)/(1-z)$ . Let  $F$  be a q.c.-extension of  $f_\sigma$  to  $\mathbb{C}$  with  $F(\infty) = \infty$ . Define  $\psi := (2-\sigma)^{-1} h^{-1} \circ (\sigma F) \circ h$ , where  $h(w) := H^{-1}(-e^w)$  is a conformal mapping of  $\Pi := \{w: |\operatorname{Im} w| < \pi/2\}$  onto  $\mathbb{C} \setminus \mathbb{D}$ . Then  $\psi$  is a q.c.-automorphism of  $\Pi$  continuously extendible to  $\partial\Pi$  with

$$\psi\left(x \pm \frac{i\pi}{2}\right) = \frac{\sigma x}{2 - \sigma} \pm \frac{i\pi}{2} \quad \text{for all } x \in \mathbb{R}. \quad (5.1)$$

Conversely, any q.c.-automorphism of  $\Pi$  satisfying (5.1) defines a q.c.-extension  $F$  of  $f$  to  $\mathbb{C}$ , with  $\mu_F(z) = \mu_\psi(h^{-1}(z))(z^2 - 1)/(\bar{z}^2 - 1)$ ,  $|z| > 1$ . Let  $k := \operatorname{ess\,sup} |\mu_\psi|$  be the maximal dilatation of  $\psi$ . To estimate  $k$  from below, consider the  $k$ -q.c. automorphism of  $\mathbb{D}$  defined by  $\varphi := H^{-1} \circ \exp \circ \psi \circ \log \circ H$ , where  $\log$  stands for the branch of the logarithm that maps the right half-plane onto  $\Pi$ . Note that  $\varphi$  and  $\varphi^{-1}$ , as well as their homeomorphic extensions to  $\overline{\mathbb{D}}$ , must be  $(1-k)/(1+k)$ -Hölder continuous; see, e.g. [2, p. 30]. It follows that

$$\inf_{a \in \mathbb{R}} \sup_{w \in P_a} \frac{\operatorname{Re} \psi(w)}{\operatorname{Re} w} \leq K, \quad \sup_{a \in \mathbb{R}} \inf_{w \in P_a} \frac{\operatorname{Re} \psi(w)}{\operatorname{Re} w} \geq \frac{1}{K},$$

$$\text{where } P_a := \left\{w: \operatorname{Re} w \geq a, |\operatorname{Im} w| \leq \frac{\pi}{2}\right\}.$$

Using (5.1), we get  $k \geq |\sigma - 1|$ . Equality occurs when  $\psi = \psi_0$ ,  $\psi_0(x + iy) = \sigma x/(2 - \sigma) + iy$  for all  $x \in \mathbb{R}$ ,  $y \in (-\pi/2, \pi/2)$ . Moreover,  $\psi_0$  is the only extremal q.c.-map in this case; see, e.g. [32, Example 1.4.2 on p. 85].

Let  $F_0$  be the q.c.-extension of  $f_\sigma$  corresponding to the automorphism  $\psi_0$ . Then  $F_0$  is the unique  $|\sigma - 1|$ -q.c. extension of  $f_\sigma$  to  $\overline{\mathbb{C}}$ . A simple computation shows that  $F_0(\infty) = \infty$  and that  $\mu_{F_0}(z) = (\sigma - 1)(z^2 - 1)/(\overline{z}^2 - 1)$  for all  $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$ . By Theorem 2,  $F_0$  is a Becker extension of  $f_\sigma$ .

*Remark 9* The Becker extensions  $F_0$  and  $F_2$  in the above two examples are uniquely extremal Teichmüller mappings of  $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$  with infinite and finite norm, respectively; see, e.g. [33] or [32], for the terminology and related results. The Becker extension  $F_1$  is a Teichmüller mapping of  $\mathbb{C} \setminus \overline{\mathbb{D}}$ , but not of  $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ , and it is not extremal without the condition  $F(\infty) = \infty$ .

On the one hand, Examples 1 and 2 along with Remark 9 indicate that  $\mathcal{S}_k^B$  should be an ‘‘important’’ part of  $\mathcal{S}$ . In particular, one can construct a series of similar examples, e.g., by considering  $f_n(z) := (f_1(z^n))^{1/n} = z/(1 - kz^n)^{2/n}$ ,  $n = 2, 3, \dots$ . The Loewner chain  $f_t^n := e^t f_n$ ,  $t \geq 0$ , defines a  $k$ -q.c. Becker extension  $F_n$  of the function  $f_n$ , with  $\mu_{F_n}(z) = k\varphi(z)/|\varphi(z)|$  for all  $z \in \mathbb{C} \setminus \overline{\mathbb{D}}$ , where  $\varphi(z) := -1/z^{n+2}$ . Therefore,  $F_n$  is a Teichmüller map of  $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$  with finite norm. It follows [36] that  $F_n$  is the unique q.c.-extension of  $f_n$  for which the maximal dilatation has the least possible value. On the other hand, the same idea allows us to construct, in an implicit way, many functions  $f \in \mathcal{S}_k$  not belonging to  $f \in \mathcal{S}_k^B$ .

**Theorem 3** *For any  $k \in (0, 1)$ ,  $\mathcal{S}_k^B \neq \mathcal{S}_k$ . In particular, if  $\varphi$  is a holomorphic function in  $\mathbb{C} \setminus \overline{\mathbb{D}}$  with finite norm  $\|\varphi\|_1 := \iint_{\mathbb{C} \setminus \overline{\mathbb{D}}} |\varphi(z)| \, dx dy$ , and if there exists  $\rho > 1$  such that  $\partial\mathbb{D} \ni \zeta \mapsto \zeta^{-2} \overline{\varphi(\rho\zeta)}/|\varphi(\rho\zeta)|$  does not admit a holomorphic extension to  $\mathbb{D}$ , then  $F|_{\mathbb{D}} \in \mathcal{S}_k \setminus \mathcal{S}_k^B$ , where  $F$  stands for the unique solution to the Beltrami equation  $\partial F = \mu \partial F$ ,  $\mu := k\overline{\varphi}/|\varphi|$  in  $\mathbb{C} \setminus \overline{\mathbb{D}}$ ,  $\mu \equiv 0$  in  $\mathbb{D}$ , normalized by  $F(0) = 0$ ,  $F'(0) = 1$ ,  $F(\infty) = \infty$ .*

**Proof** By [37, Theorem 4],  $F$  is the unique  $k$ -q.c. extension of  $f := F|_{\mathbb{D}}$ , but it is not a Becker extension by Theorem 2.  $\square$

At the end of this section, let us recall that Theorem 1 states also q.c.-extendibility of the evolution family. Fix  $k \in [0, 1)$  and denote by  $\mathcal{U}_k$  the union of all evolution families generated by Herglotz vector fields given by (2.3) with  $p$  satisfying condition (3.3).

*Remark 10* An interesting fact about  $\mathcal{U}_k$  is that on the one hand, it is closed w.r.t. taking compositions, i.e.  $\varphi \circ \psi \in \mathcal{U}_k$  for any  $\varphi, \psi \in \mathcal{U}_k$ , but on the other hand, each  $\varphi \in \mathcal{U}_k$  admits a q.c.-extension  $\Phi$  to  $\overline{\mathbb{C}}$  with the same bound for the maximal dilatation:  $\text{ess sup } |\mu_\Phi| \leq k$ , although the composition of  $k$ -q.c. extensions of  $\varphi$  and  $\psi$ , clearly, does not need to be a  $k$ -q.c. map.

## 6 A question concerning the Loewner range

In order to complete the proof of Theorem 1, we have to show that under its hypothesis, the Loewner range  $L[(f_t)] := \cup_{t \geq 0} f_t(\mathbb{D})$  is the whole complex plane.

**Theorem 4** *Let  $p$  be a Herglotz function such that  $p(\mathbb{D}, t) \subset K$  for a.e.  $t \geq 0$  and some compact set  $K \subset \mathbb{H}$ . Then for any measurable  $\tau: [0, +\infty) \rightarrow \overline{\mathbb{D}}$ , the Loewner chains  $(f_t)$  associated with the Herglotz vector field  $G(z, t) := (\tau(t) - z)(1 - \tau(t)z)p(z, t)$  satisfy  $L[(f_t)] = \mathbb{C}$ .*

Before giving the proof of above theorem, let us place some remarks. If  $\tau$  is a constant function with the value in  $\mathbb{D}$ , then it is a simple exercise to show that  $L[(f_t)] = \mathbb{C}$  if and only if  $\int_0^\infty \operatorname{Re} p(0, t) dt = +\infty$ . In case of constant  $\tau$  with the value on  $\partial\mathbb{D}$ , a sufficient condition is that

$$C_1 < \operatorname{Re} p(z, t) < C_2 \quad \text{for all } z \in \mathbb{D} \text{ and a.e. } t \geq 0 \quad (6.1)$$

with some positive constants  $C_1$  and  $C_2$ , see [17, Proposition 3.7]. However, Example 3 given after the proof of Theorem 4 shows that for arbitrary measurable functions  $\tau$ , condition (6.1) does not imply that  $L[(f_t)] = \mathbb{C}$ .

**Proof of Theorem 4** Denote by  $(\varphi_{s,t})$  the evolution family associated with the Herglotz vector field  $G$ . Let  $\psi_{s,t} := h_t \circ \varphi_{s,t} \circ h_s^{-1}$ , where

$$h_t(w) := \frac{w - a(t)}{1 - \overline{a(t)}w}, \quad a(t) := \varphi_{0,t}(0); \quad w \in \mathbb{D}, \quad t \geq s \geq 0.$$

By [13, Lemma 2.8],  $(\psi_{s,t})$  is an evolution family. Thanks to [13, Theorem 1.6], it is sufficient to show that  $|\psi'_{0,t}(0)| \rightarrow 0$  as  $t \rightarrow +\infty$ . Denote by  $G_0$  the Herglotz vector field of  $(\psi_{s,t})$ . Since  $\psi_{s,t}(0) = 0$  whenever  $0 \leq s \leq t$ , we have  $G_0(z, t) = -zq(z, t)$  for all  $z \in \mathbb{D}$  and a.e.  $t \geq 0$ , where  $q$  is a Herglotz function. From (2.4) we find that  $\operatorname{Re} q(0, t) =$

$$= -\operatorname{Re} G'_0(0, t) = \frac{1 - |a(t)|^2}{|1 + \overline{a(t)}\kappa(t)|^2} \operatorname{Re} \left[ (1 + |\kappa(t)|^2)p_1(0, t) - \kappa(t)p'_1(0, t) \right],$$

for a.e.  $t \geq 0$ , where  $p_1(z, t) := p(h_t^{-1}(z), t)$  and  $\kappa(t) := h_t(\tau(t))$ . Using the fact that holomorphic maps are non-expansive w.r.t. the hyperbolic metric, in the same way as in the proof of [17, Proposition 3.7] we see that  $|p'_1(0, t)| \leq 2\nu \operatorname{Re} p_1(0, t)$  for a.e.  $t \geq 0$  and some constant  $\nu \in (0, 1)$  depending only on the compact set  $K$ . Therefore, for a.e.  $t \geq 0$ ,

$$\operatorname{Re} \left[ (1 + |\kappa(t)|^2)p_1(0, t) - \kappa(t)p'_1(0, t) \right] \geq (1 - \nu)(1 + |\kappa(t)|^2) \operatorname{Re} p_1(0, t).$$

To show that  $\lim_{t \rightarrow +\infty} \log |\psi'_{0,t}(0)| = -\int_0^{+\infty} \operatorname{Re} q(0, t) dt = -\infty$ , it remains to notice that

$$\begin{aligned} \frac{d|a(t)|^2/dt}{2} &= \operatorname{Re} \left[ \overline{a(t)} G(a(t), t) \right] \leq x |p(a(t), t)| (1 - |a(t)|^2) - x^2 \operatorname{Re} p(a(t), t) \\ &\leq \frac{|p(a(t), t)|^2}{4 \operatorname{Re} p(a(t), t)} (1 - |a(t)|^2)^2 \quad \text{for a.e. } t \geq 0, \end{aligned}$$

where  $x := |1 - \overline{\tau(t)}a(t)|$ , and hence  $\int_0^{+\infty} (1 - |a(t)|^2) dt = +\infty$ .  $\square$

*Example 3* Let  $\rho : [0, +\infty) \rightarrow [0, 1)$  be a locally absolutely continuous function with  $\rho(t) = 0$  if and only if  $t = 0$  and such that  $\int_0^1 dt/\rho(t) < +\infty$ . Consider the evolution family  $(\varphi_{s,t})$  associated with the Herglotz vector field  $G$  given by (2.3) with

$$p(z, t) := 1 - i\rho'(t) \frac{(1 + \rho(t)^2)}{(1 - \rho(t)^2)^2} \quad \text{and} \quad \tau(t) := ie^{i\theta(t)} \frac{(1 - i\rho(t))^2}{1 + \rho(t)^2},$$

$$\text{where } \theta(t) := \int_0^t \frac{(1 - \rho(s)^2)^2}{1 + \rho(s)^2} \frac{ds}{\rho(s)},$$

for all  $z \in \mathbb{D}$  and  $t \geq 0$ . It is easy to check that  $a(t) := \varphi_{0,t}(0) = \rho(t)e^{i\theta(t)}$ .

Calculations in the proof of Theorem 4 show that  $\psi'_{0,t}(0) \rightarrow 0$  as  $t \rightarrow +\infty$  if and only if  $\int_0^{+\infty} (1 - \rho(t)^2) dt = +\infty$ . It follows that the hypothesis of Theorem 4 cannot be replaced by the weaker condition (6.1).

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