HELSINGIN YLIOPISTO — HELSINGFORS UNIVERSITET — UNIVERSITY OF HELSINKI

Tiedekunta/Osasto — Fakultet/Sektion — Faculty	Laitos — Institution — Department
Faculty of Science	Department of Mathematics and Statistics

Tekijä — Författare — Author

Yasir Mahmood

Työn nimi — Arbetets titel — Title

Locality: A useful notion for proving inexpressibility in Finite Model Theory

Oppiaine — Läroämne — Subject

Mathematical Logic

Työn laji — Arbetets art — Level	Aika — Datum — Month and year	Sivumäärä — Sidoantal — Number of pages
Master's Thesis	June 2018	46

Tiivistelmä — Referat — Abstract

This thesis discusses the notion of locality used in finite model theory to obtain results about the expressive power of first order logic. It turns out that the most commonly used Ehrenfeucht-Fraïssé games are also applicable over finite structures. However, we analyze with an example the need for simpler tools for finite structures due to the complex combinatorial arguments required while using EF-games. We argue that locality is such a tool, although the gap between games and locality is quite narrow as the latter is in fact based on the former. Intuitively speaking: locality of FO implies that in order to check the satisfiability of a FO-formula over a finite structure, it is enough to look at a small portion of the universe (which will be called the neighborhood of a point). We discuss two commonly known notions of locality given by William Hanf and Haim Gaifman. We provide the original results of the authors and then their modified versions suitable for finite structures. We then show that first order logic over any relational vocabulary has both of these locality properties. In order to grasp the idea of locality we also include examples wherever required. Towards the end of the thesis we also discuss deficiencies and limitations of the two types of locality and possible solutions to overcome them. In the last section we also discuss locality of order-invariant first order formulas.

Avainsanat — Nyckelord — Keywords

Finite model theory, Expressibility, Ehrenfeucht-Fraïssé games, Locality,

Säilytyspaikka — Förvaringsställe — Where deposited

Kumpula Science Library

Muita tietoja — Övriga uppgifter — Additional information

Locality:
A useful notion for proving inexpressibility in Finite Model Theory

Yasir Mahmood

May 24, 2018

0.1 Acknowledgment

I want to pay special thanks to the people who have helped me in any way while writing this thesis. All the credit goes, first and foremost to my parents who have always supported me morally. My father has always inspired me and motivated me throughout this master degree by saying: you must do it even if you pass it with 33 marks (the lowest passing marks in my country).

Mr. Juha Kontinen has been a great supervisor. He was always just one email away. I remember walking his room for the first time with no idea at all of what a "Master's Thesis" is and here I am today, completing it in a very short time. His detailed comments helped me to complete this thesis as well as gaining knowledge of academic writing.

I am also thankful to my friend Mohsin Ali, a PhD researcher in China for all the technical support regarding thesis writing. Moreover, my fellows in Finland who have always supported me and helped me in any need, food for example, Asad Sohail, Moeed Ahmad Syed and Farhan Muzaffer to mention a few and most importantly my little niece Faira Rehman for always making me laugh and literally forget every stress.

YES ! I DID IT ...

Yasir Mahmood

May 24, 2018

0.2 Abstract

Finite model theory is the branch of mathematical logic where we deal with finite structures. Often the questions addressed are related to the expressive power of a logic. That is, what can we express and what we cannot, in a given logic $\mathcal{L}$.

This thesis discusses the notion of locality which can be used in finite model theory to obtain results about expressibility and inexpressibility in first order logic. It turns out that the most commonly used Ehrenfeucht-Fraïssé games are also applicable over finite structures. However, we analyze with an example the need for simpler tools for finite structures due to the complex combinatorial arguments required while using EF-games. We argue that locality is more effective if we want to avoid complex arguments. Although the gap between games and locality is quite narrow the latter is in fact based on the former. Intuitively speaking: locality of FO says that in order to check the satisfiability of a FO-formula over a finite structure, it is enough to look at a small portion of the universe (which will be called the neighborhood of a point).

We discuss the inexpressibility of queries over finite structures rather than the general first order formulas and sentences. We include two commonly known notions of locality given by William Hanf and Haim Gaifman. We provide the original results of the authors and then their modified versions that suit the theme of this writing. In order to grasp the idea of locality we also include examples wherever required.

Towards the end of this thesis we also present deficiencies and limitations of the two types of locality with possible solutions to overcome them. This thesis concludes with the results that FO over any relational vocabulary has both of these locality properties. This allows one to check the queries over small

neighborhoods of parameters. Moreover, at the end we give an introduction to the notion of locality in first order logic with the built in linear order.

Contents

1 Preliminaries

1.1 Structure of the thesis

In Section 1.2 we introduce finite model theory and look at the differences between the classical and the finite case. Then we present some general terminology which we will often use. We define most of the concepts and notions at the beginning of each section where they are introduced.

In Section 2 we look at Ehrenfeucht-Fraïssé games and an example illustrating the need for new tools for showing inexpressibility over finite structures.

In Section 3 we introduce the notion of locality followed by the two main locality criteria given by Hanf and Gaifman. This section includes results for first order logic and their applications for showing inexpressibility. In Section 4 we look at the shortcomings and difficulties in generalizing this tool for the extensions of FO. At the end we introduce the locality of FO in the presence of built-in linear order and the numerical predicates induced by this order.

1.2 General introduction

Finite model theory has its essential applications in computer science. Specifically, the context of database theory and complexity of problems interests us in finite model theory where logic and model theory meet computers that we see and work with. The same complexity theory leads to verification and model checking problems.

Measuring computational complexity of a problem has been greatly discussed in computer science which studies the amount of resources such as time and space which a Turing machine requires to solve a problem. The other type of complexity is descriptive complexity which refers to describing a problem in

some logical formalism. The connection between these two kind of complexities allows us to use the knowledge of logical methods of a particular logic to infer results about computational complexity of a certain class of problems. One such result by Fagin states that a problem is in NP (that is, computable by a non-deterministic Turing machine in polynomial time) if and only if its formal counterpart is expressible in existential second order logic (the existential quantifier is allowed to range over the subsets/relations of the structures)
The expressive power of a logic is an important topic studied in mathematical logic. The question addressed is: Given a class of sentences Ψ and a class of structures $\mathcal{K}$, determine whether a property P about given structures is expressible by a sentence from Ψ. For example, let $\mathcal{K}$ be the class of all finite graphs. We want to check if the property 'there is an edge between any two points' is expressible by a FO sentence ψ. This statement is true for a graph it it is complete and false other wise. This example is quite easy to work with since the result is positive. In general if the given property is indeed expressible then it is enough to come up with a sentence ψ that expresses P in $\mathcal{K}$. However, proving a negative result requires sometimes much work since it is necessary to show that no sentence of the logic can express the given property. Usually the class of sentences in the given vocabulary is very large (mostly infinite) which makes the task of showing inexpressibility or negative results a difficult job. For this reason we need tools for showing inexpressibility.
In classical model theory (where universe of structures is infinite) we may use results such as the compactness theorem. That is, we can make extra assumptions such as, 'take a countable collection of new and distinct constant symbols' where the new constants are interpreted by the distinct elements of the structure, guaranteed by the infiniteness of our structure. Hence we can prove the desired results in new vocabularies. This is however not the case

when we talk about finite structures since compactness fails over finite models as do most of the results and tools from classical model theory

The reason why the results such as compactness fail over finite structures is the result below:

Theorem 1. *Let σ be a vocabulary and $\mathcal{A}$ be a finite σ-structure. Then there is a FO sentence Φ that completely determines $\mathcal{A}$ up to isomorphism. That is, for any σ-structure $\mathcal{B}$ it is the case that $\mathcal{B} \models \Phi$ iff $\mathcal{A} \cong \mathcal{B}$.*

□

In finite model theory we study the expressive power of a logic over finite models. We are especially interested in showing inexpressibility. That is, a certain property cannot be expressed in a given logic. The reason for this interest is that it gives us limitations of the expressive power of a logic. As an example, we will see shortly that we cannot express in first order whether the domain of a structure $\mathcal{A}$ has even cardinality. This thesis discusses such tools to show such negative results in first order logic.

1.3 Terminology

Basics from Mathematical Logic: Throughout this thesis we talk about purely relational vocabularies. However, we can allow constant symbols since each constant symbol can be thought of as a unary relation symbol with its interpretation as a singleton in the structure. A vocabulary or a language is set of relation symbols $\{P_1, \ldots, P_l\}$, where each relation symbol P_i has an associated arity denoted by $\#P_i$. From now on our vocabularies as well as structures will be finite. If $\sigma = \{P_1, \ldots, P_l\}$ is a vocabulary then a σ-structure is determined by a set $\mathbf{A}$, called the universe of the structure, together with a mapping that associates each relation symbol $P_i \in \sigma$ with a relation $R_i \subseteq \mathbf{A}^{\#P_i}$. We use the standard notations. So, the capital Greek letters $\mathcal{A}, \mathcal{B}$ etc. represent structures (now finite) and their underlying universes are denoted by the same letters in Roman $\mathbf{A}, \mathbf{B}$ etc.

We use the usual Tarski-Vaught's truth definition. For a formula $\phi(x)$ in a vocabulary σ, $\phi(\mathcal{A})$ denotes the subset $\{a \in \mathbf{A} : \mathcal{A} \models \phi(a)\}$ of $\mathbf{A}$. There should be no confusion whether we write $a \in \mathcal{A}$ or $a \in \mathbf{A}$. Similarly when the context is clear, rather than writing $\mathbf{a} = (a_1, \dots a_k) \in \mathbf{A}^k$ for some k-tuple $\mathbf{a}$ we can also write boldface $\mathbf{a}$ without specifying k. For example, $\mathcal{A} \models \phi(\mathbf{a})$ should be unambiguous when it is clear from the context what is the length of $\mathbf{a}$.

In order to distinguish formulas from sentences, $\phi(\mathbf{x})$ will denote a formula with free variables among $\mathbf{x} = (x_1, \dots, x_k)$ while sentences will be denoted without free variables such as ψ.

Definition 1. *Let σ be a vocabulary and $\mathcal{A}, \mathcal{B}$ be two σ-structures with tuples $\boldsymbol{a} = (a_1, \dots, a_n)$ $\boldsymbol{b} = (b_1, \dots, b_n)$ from $\mathcal{A}$ and $\mathcal{B}$ respectively. We say that $\boldsymbol{a} \mapsto \boldsymbol{b}$ defines a partial isomorphism if:*

- *For every $i, j \leq n, a_i = a_j \iff b_i = b_j$*
- *For every k-ary $P \in \sigma$ and $i_1, \dots, i_k \in \{1, \dots, n\}$ it holds that:* $(a_{i_1}, \dots, a_{i_k}) \in P^{\mathcal{A}} \iff (b_{i_1}, \dots, b_{i_k}) \in P^{\mathcal{B}}$.

In other words, $p : \boldsymbol{a} \mapsto \boldsymbol{b}$ is a partial isomorphism if p is an isomorphism of the substructures of $\mathcal{A}$ and $\mathcal{B}$ generated by $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively. We denote the set of all partial isomorphisms between $\mathcal{A}$ and $\mathcal{B}$ by $Part(\mathcal{A}, \mathcal{B})$.

Definition 2. *For any formula $\phi \in FO$ the quantifier rank of ϕ is denoted by $qr(\phi)$ which is defined as,*

- *If ϕ is an atomic formula then $qr(\phi) = 0$.*
- *For ϕ_1, ϕ_2, $qr(\phi_1 \wedge \phi_2) = qr(\phi_1 \vee \phi_2) = max\{qr(\phi_1), qr(\phi_2)\}$.*
- $qr(\neg\phi) = qr(\phi)$.
- $qr(\forall x\phi) = qr(\exists x\phi) = qr(\phi) + 1$.

By $FO_k[\sigma]$ we denote the collection of all the $FO[\sigma]$-formulas and sentences of quantifier rank at most k.

Definition 3. *Let σ be a vocabulary and $\mathcal{A}$,$\mathcal{B}$ be σ-structures. Then $\mathcal{A}$ is elementary equivalent to $\mathcal{B}$, denoted as $\mathcal{A} \equiv \mathcal{B}$ if and only if for every σ-sentence ψ, $\mathcal{A} \models \psi \iff \mathcal{B} \models \psi$.*
We use the notation $\mathcal{A} \equiv_k \mathcal{B}$ if $\mathcal{A} \models \psi \iff \mathcal{B} \models \psi$ for every sentence $\psi \in FO_k[\sigma]$. Furthermore, we use the notation $\mathcal{A} \equiv_\Phi \mathcal{B}$ when it is the case that: $\mathcal{A} \models \phi \iff \mathcal{B} \models \phi$ for all $\phi \in \Phi$.

Definition 4. *An m-ary query over a class of structures is a mapping $\mathcal{Q}$ that associates $\mathcal{A}$ with an m-ary relation $\mathcal{Q}(\mathcal{A}) \subseteq \mathbf{A}^m$ on its universe. Queries are preserved under isomorphism, that is: if $h : \mathcal{A} \longrightarrow \mathcal{B}$ is an isomorphism of structures then $(a_1, \ldots, a_m) \in \mathcal{Q}(\mathcal{A})$ if and only if $(h(a_1), \ldots, h(a_m)) \in \mathcal{Q}(\mathcal{B})$. If $\mathcal{Q}$ is 0-ary it is called a Boolean query which assigns each class of structures $\mathcal{C}$ with its subclass $\{\mathcal{A} \in \mathcal{C} : \mathcal{Q} \text{ is true of } \mathcal{A}\}$.*

Example 1. *Let $\sigma = \{E\}$ and the class $\mathcal{K}$ of σ-structures be finite graphs. The binary query ($\mathcal{TC}$) defined below is called Transitive Closure with respect to E.*

$$\mathcal{TC}(\mathcal{G}) = \{(a, b) \in \boldsymbol{G}^2 \ : \ \text{there is path from } a \text{ to } b\}$$

for every $\mathcal{G} \in \mathcal{K}$.

The output of this query is all those pairs (a,b) such that there is a path from 'a' to 'b'.

Example 2. *Let σ and $\mathcal{K}$ be as in Example 1. The query $\mathcal{C}$ defined below is a Boolean Query.*

$$\mathcal{C}(\mathcal{G}) = \{\mathcal{G} \in \mathcal{K} : \mathcal{G} \models \forall x \forall y E(x, y)\}.$$

This query checks if a graph is complete.

Definition 5. *An m-ary query $\mathcal{Q}$ over σ-structures is expressible in a logic $\mathcal{L}$ if there is an $\mathcal{L}$-formula $\phi(\boldsymbol{x})$ such that $\mathcal{Q}(\mathcal{A}) = \{\boldsymbol{a} : \mathcal{A} \models \phi(\boldsymbol{a})\}$ for every σ-structure $\mathcal{A}$, in other words $\mathcal{Q}(\mathcal{A}) = \phi(\mathcal{A})$. Similarly a Boolean query is expressible in $\mathcal{L}$ if there is a sentence ψ such that, $\mathcal{Q} = \{\mathcal{A} : \mathcal{A} \models \psi\}$.*

Example 3. *The query $\mathcal{TC}$ of Example 1 above is not expressible in FO as we will see in Section 3. However $\mathcal{C}$ of Example 2 is FO-expressible by the sentence: $\forall x \forall y E(x, y)$.*

2 Ehrenfeucht-Fraïssé Games and Inexpressibility

As we discussed earlier, our main goal is to look for tools that might help us in showing inexpressibility of certain properties and hence the limitations of a logic $\mathcal{L}$. The only tool that survives the transition from classical to finite model theory is Ehrenfeucht-Fraïssé games. EF-games work for both finite and infinite models even though these are not as widely studied and popular in classical model theory as other tools, such as compactness.

The idea of the EF-game is that it can show that two structures are *similar* by using quantifier rank of formulas as a parameter. The game goes like this: there are two players called the duplicator whose task is to show that the two structures are alike and the spoiler whose task is as the name indicates, to stop the duplicator from doing his task. In fact the spoiler plays first in each round. Given two structures $\mathcal{A}$ and $\mathcal{B}$ as the game board, in each round the spoiler tries to show that $\mathcal{A}$ and $\mathcal{B}$ are *different* by picking an element from one of the structures. Then the duplicator's task is to show that they are similar by responding with an element from the other structure. We formalize below what is meant by the structures being *similar* and *different*.

2.1 EF-Games

Definition 6. *The k-round Ehrenfeucht-Fraïssé game is denoted by $EF_k(\mathcal{A}, \boldsymbol{a}; \mathcal{B}, \boldsymbol{b})$ where $(\mathcal{A}, \boldsymbol{a}; \mathcal{B}, \boldsymbol{b})$ is called an initial configuration. The game is played as follows: For $i = 0$, the spoiler wins even before the game starts if the map $\boldsymbol{a} \mapsto \boldsymbol{b}$ is not a partial isomorphism. In the following round the spoiler first picks an element from either $\mathcal{A}$ or $\mathcal{B}$ and then the duplicator responds by selecting an* *element from $\mathcal{B}$ or $\mathcal{A}$ respectively. Let c_1 be the point selected in $\mathcal{A}$ and d_1 in $\mathcal{B}$ then the new configuration becomes $(\mathcal{A}, \boldsymbol{a}c_1\ ;\ \mathcal{B}, \boldsymbol{b}d_1)$. The correspondence $\boldsymbol{a} \mapsto \boldsymbol{b}$ extends to $\boldsymbol{a}c_1 \mapsto \boldsymbol{b}d_1$. At the end of the k-th round the final configuration (or a play) is: $(\mathcal{A}, \boldsymbol{ac}; \mathcal{B}, \boldsymbol{bd})$, where $\boldsymbol{c} = (c_1, \ldots, c_k)$ are the points selected from $\mathcal{A}$ and $\boldsymbol{d} = (d_1, \ldots, d_k)$ from $\mathcal{B}$.*

Definition 7. *We say that the spoiler wins a play $(\mathcal{A}, \boldsymbol{ac}; \mathcal{B}, \boldsymbol{bd})$ of $EF_k(\mathcal{A}, \boldsymbol{a}; \mathcal{B}, \boldsymbol{b})$ if the mapping $\boldsymbol{ac} \mapsto \boldsymbol{bd}$ is not a partial isomorphism. Otherwise the duplicator wins if he can maintain the correspondence of configurations through all the rounds until k. That is, the mapping: $c_i \mapsto d_i$ $i = 1, 2, \ldots, k$ extended by the map $\boldsymbol{a} \mapsto \boldsymbol{b}$ is a partial isomorphism.*

If the duplicator wins then we talk about the extensions of the isomorphic configurations $\mathbf{a} \mapsto \mathbf{b} \in Part(\mathcal{A}, \mathcal{B})$ of the game. We say that either player has a winning strategy for k-rounds if he can play in a way that guarantees his success in any k-rounds play of $EF_k(\mathcal{A}, \mathbf{a}; \mathcal{B}, \mathbf{b})$. If the duplicator has a winning strategy for k rounds we denote this by $\mathcal{A} \cong_k \mathcal{B}$. Since the number of rounds is finite exactly one player has a winning strategy in an EF-game. We denote the above partial isomorphism by $(\mathbf{c},\mathbf{d})$ or $\mathbf{c} \mapsto \mathbf{d}$.

Definition 8. *Let σ be a relational vocabulary, $\mathcal{A}$ and $\mathcal{B}$ be σ-structures with $\boldsymbol{a} \in \mathcal{A}$, $\boldsymbol{b} \in \mathcal{B}$.*

1. *Let $I \subseteq Part(\mathcal{A}, \mathcal{B})$ and $p := \boldsymbol{a} \mapsto \boldsymbol{b} \in Part(\mathcal{A}, \mathcal{B})$. We say that p has back-and-forth extensions in I if*

$$forth : \forall c \in \mathcal{A} \exists d \in \mathcal{B} : \boldsymbol{a}c \mapsto \boldsymbol{b}d \in I$$

$$back : \forall d \in \mathcal{B} \exists c \in \mathcal{A} : \boldsymbol{a}c \mapsto \boldsymbol{b}d \in I$$

2. *Let* $I_i \subseteq Part(\mathcal{A}, \mathcal{B})$ *for each* $i \leq k$. *Then we say that* $(I_i)_{0 \leq i \leq k}$ *is a back-and-forth system for* $EF_k(\mathcal{A}, \boldsymbol{a}; \mathcal{B}, \boldsymbol{b})$ *if*

   - $\boldsymbol{a} \mapsto \boldsymbol{b} \in I_0$
   - *For* $0 \leq j < k$, *every* $p \in I_j$ *has back-and-forth extension in* I_{j+1}.

   We can also put a restriction on the size of each partial isomorphism. That is, for every $p \in I_i$, $|dom(p)| = |\boldsymbol{a}| + i$.

3. *If* $(I_i)_{0 \leq i \leq k}$ *is a back-and-forth system for* $EF_k(\mathcal{A}, \boldsymbol{a}; \mathcal{B}, \boldsymbol{b})$ *then we say that* $(\mathcal{A}, \boldsymbol{a})$ *and* $(\mathcal{B}, \boldsymbol{b})$ *are* k*-isomorphic and write* $(\mathcal{A}, \boldsymbol{a}) \simeq_k (\mathcal{B}, \boldsymbol{b})$.

The following theorem by Ehrenfeucht-Fraïssé characterizes the relations k-isomorphism and k-equivalence with the duplicator's winning strategy for k rounds.

Theorem 2. *Let* σ *be a relational vocabulary and* $\mathcal{A}, \mathcal{B}$ *be* σ*-structures with* $\boldsymbol{a} \in \mathcal{A}$, $\boldsymbol{b} \in \mathcal{B}$. *Then the following are equivalent.*

1. $(\mathcal{A}, \boldsymbol{a}) \equiv_k (\mathcal{B}, \boldsymbol{b})$
2. $(\mathcal{A}, \boldsymbol{a}) \cong_k (\mathcal{B}, \boldsymbol{b})$
3. $(\mathcal{A}, \boldsymbol{a}) \simeq_k (\mathcal{B}, \boldsymbol{b})$

□

Since we want to show the inexpressibility of queries in FO, the following version of the above theorem is often used.

Theorem 3. *Let σ be a vocabulary. An m-ary query $\mathcal{Q}$ on σ-structures is FO-expressible by a σ-formula $\phi(\boldsymbol{x})$ with quantifier rank k if and only if $\mathcal{Q}$ is closed under the relation $\simeq_k$. That is, for every $\mathcal{A}_k$, $\mathcal{B}_k$ and two m-tuples $\boldsymbol{a}$ and $\boldsymbol{b}$ the following holds.*

$$(\mathcal{A}_k, \boldsymbol{a}) \simeq_k (\mathcal{B}_k, \boldsymbol{b}) \implies (\boldsymbol{a} \in \mathcal{Q}(\mathcal{A}_k) \iff \boldsymbol{b} \in \mathcal{Q}(\mathcal{B}_k))$$

□

Hence, in order to show that an m-ary query $\mathcal{Q}$ is not FO-expressible, we need to find two structures $\mathcal{A}_k, \mathcal{B}_k$ for each k and two m-tuples $\mathbf{a}, \mathbf{b}$ in them such that: $(\mathcal{A}_k, \mathbf{a}) \simeq_k (\mathcal{B}_k, \mathbf{b})$ but $\mathbf{a} \in \mathcal{Q}(\mathcal{A}_k) \iff \mathbf{b} \notin \mathcal{Q}(\mathcal{B}_k)$. Then Theorem 3 implies the inexpressibility of $\mathcal{Q}$ by any FO-formula.
The question we study in the following examples (in fact in whole thesis) is: How to find those two structures? What is k? How do we ensure the k-isomorphism of the two structures?

2.2 Inexpressibility using EF-Game

We present two examples as applications to Theorem 3.

Corollary 1. *Let σ be empty vocabulary. Then the following Boolean query which is true if size of the universe is even is not FO-expressible.*

$$\mathcal{Q}_{even} = \{\mathcal{A} \ : \ |\mathbf{A}| = 0 \ (\mathrm{mod}\ 2)\}$$

Proof. In the empty vocabulary structures are simply sets of elements. For any k, let $\mathcal{S}_1$ and $\mathcal{S}_2$ be sets of elements of sizes at least k. Let $I_i \subseteq Part(\mathcal{S}_1, \mathcal{S}_2), i \leq k$ be such that,

$$I_i = \{\mathbf{a} \mapsto \mathbf{b} \ : \ a_r = a_s \iff b_r = b_s\}$$

for every $\mathbf{a} = (a_1, \ldots, a_i), \mathbf{b} = (b_1, \ldots, b_i), i \leq k$.

Then $(I_i)_{0 \leq i \leq k}$ is a back-and-forth system for $EF_k(\mathcal{S}_1, \mathcal{S}_2)$. Since every $p \in I_i$ is simply a bijection and hence have a back-and-forth extension in I_{i+1}. That is, for every $\mathbf{a} \mapsto \mathbf{b} \in I_i$, if $a_{i+1} \in \mathcal{S}_1 - \{a_1, \ldots, a_i\}$, $\mathbf{a}a_{i+1} \mapsto \mathbf{b}b_{i+1} \in I_{i+1}$ where $b_{i+1} \in \mathcal{S}_2 - \{b_1, \ldots, b_i\}$ and vice versa.

If $\mathcal{Q}_{even}$ were FO-expressible by a sentence ψ with $qr(\psi) = k$. Then we can find two sets $\mathcal{S}_1$ and $\mathcal{S}_2$ with cardinalities k and $k+1$ respectively such that $\mathcal{S}_1 \simeq_k \mathcal{S}_2$. Now, $\mathcal{Q}_{even}$ must agree on $\mathcal{S}_1$ and $\mathcal{S}_2$ but it does not. Hence $\mathcal{Q}_{even}$ is not FO-expressible.

□

Note that forming a back-and-forth system in this example was quite simple. This simplicity vanishes and we need a deeper argument when our vocabulary contains relation symbols of various arities. Below is an example with a single relational symbol.

Corollary 2. *The query $\mathcal{Q}_{even}$ is not FO-expressible in the vocabulary $\sigma = \{<\}$ where the intended class of structures is finite linear orders.*

Proof. Suppose to the contrary that $\mathcal{Q}$ is expressible by a sentence of quantifier rank k. In order to get a contradiction we need to find two linear orders $\mathcal{L}_1$, $\mathcal{L}_2$ such that $\mathcal{L}_1 \simeq_k \mathcal{L}_2$ and still $\mathcal{Q}(\mathcal{L}_1) \iff \neg\mathcal{Q}(\mathcal{L}_2)$.

In the case of the empty vocabulary we were able to do this by simply taking the universes large enough to cover k rounds. But due to the order, showing k-isomorphism becomes a difficult task. Take for example when $\mathcal{L}_1$ is $\langle 0, 1, 2\rangle$ and $\mathcal{L}_2$ is $\langle 0, 1\rangle$ with natural order. Then there is no winning strategy for the duplicator for $EF_2(\mathcal{L}_1, \mathcal{L}_2)$. The point to be noted is that the arguments for showing k-isomorphism and hence for the duplicator's winning strategy gets complicated and so does the conditions required to obtain the structures for showing inexpressibility. We present those arguments in the lemma below.

Lemma 1. *Let $\sigma = \{\prec\}$ and the class $\mathcal{K}$ of σ-structures be linear orders. Let $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{K}$ then the duplicator has a winning strategy for $EF_k(\mathcal{L}_1, \mathcal{L}_2)$ if the sizes of both orders are at least 2^k.*

Proof. Let $\mathcal{L}_1 = \langle [m], \prec \rangle$ and $\mathcal{L}_2 = \langle [n], \prec \rangle$ be two initial segments of $(\mathbf{N}, \prec)$ with $m, n > 2^k$. We prove that there is a back-and-forth system $(I_i)_{i \leq k}$ such that any map in I_i which preserves a certain distance between two points has a back-and-forth extension in $I_{i+1}, i < k$. In order to formalize the idea of preserving distance we denote the distance between two points by $d(x, y)$ which is simply $|x - y|$.

Now let $\mathbf{a} = (a_1, \ldots, a_i), \mathbf{b} = (b_1, \ldots, b_i)$ be tuples of points from $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. Let us define:

$$d_i(a_r, a_s) = \begin{cases} |a_r - a_s| & \text{if } |a_r - a_s| < 2^{k-i} \\ \infty & \text{otherwise} \end{cases}$$

for $r, s \leq i$.

Then our claim is that $(I_i)_{i \leq k}$ witnesses that $\mathcal{L}_1 \simeq_k \mathcal{L}_2$. Where $I_i \subseteq Part(\mathcal{L}_1, \mathcal{L}_2)$ for each i is defined as

$$I_0 = \{\emptyset\}$$

$$I_i = \{\mathbf{a} \mapsto \mathbf{b} \ : \ \mathbf{a} \in \mathcal{L}_1, \mathbf{b} \in \mathcal{L}_2 \text{ and satisfy } (\mathbf{??})\}.$$

Where $\mathbf{a} = (a_1, \ldots, a_i)$, $\mathbf{b} = (b_1, \ldots, b_i)$ and for $r, s \leq i$,

$$\left.\begin{aligned} d_i(1, a_r) &= d_i(1, b_r) \\ d_i(a_r, m) &= d_i(b_r, n) \\ d_i(a_r, a_s) &= d_i(b_r, b_s) \\ a_r \leq a_s &\iff b_r \leq b_s. \end{aligned}\right\} \tag{1}$$

Now we prove that every $p \in I_i$ has back-and-forth extension in I_{i+1}. Let $p : \mathbf{a} \mapsto \mathbf{b}$.

Forth: $\forall a_{i+1} \in \mathcal{L}_1, \exists b_{i+1} \in \mathcal{L}_2$ such that $p \cup \{(a_{i+1}, b_{i+1})\} \in I_{i+1}$ (to prove).

If $a_{i+1} \in dom(p)$ then $b_{i+1} = p(a_{i+1})$. Otherwise, $a_{i+1} \in (a_j, a_l)$ or $a_{i+1} \in (1, a_l)$ or $a_{i+1} \in (a_j, m)$ for $j, l < i+1$ such that there is no $a' \in dom(p)$ in the same interval. Similarly the interval (b_j, b_l) (or any of $(1, b_l)$ or (b_j, n) respectively) contains no point b' with $b' \in range(p)$. Furthermore, $d_i(a_j, a_l) = d_i(b_j, b_l)$. Let $b_{i+1} \in \mathcal{L}_2$ be such that $d_i(a_j, a_{i+1}) = d_i(b_j, b_{i+1})$ and $d_i(a_{i+1}, a_l) = d_i(b_{i+1}, b_l)$. Then $d_i(1, b_{i+1}) = d_i(1, a_{i+1})$ and $d_i(b_{i+1}, n) = d_i(a_{i+1}, m)$. Now, $(a_1, \ldots a_{i+1})$ and $(b_1, \ldots b_{i+1})$ also satisfy (**??**). Therefore, $p \cup \{(a_{i+1}, b_{i+1})\} \in I_{i+1}$.

Similarly by interchanging the roles of a_{i+1} and b_{i+1} we get the desired 'back' condition. Hence our claim is true and $\mathcal{L}_1 \simeq_k \mathcal{L}_2$.

□

(*Proof cont.*) Now our result for the inexpressibility of the query $\mathcal{Q}$ follows as: Suppose $\mathcal{Q}$ is expressible by a sentence of quantifier rank k, Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be of sizes 2^k and $2^k + 1$ respectively. Since $\mathcal{L}_1 \simeq_k \mathcal{L}_2$ the query must agree on $\mathcal{L}_1$ and $\mathcal{L}_2$ but clearly it does not.

□

One purpose of including this example was to have a look at the complex combinatorial arguments involved in the applications of the EF-game when our vocabulary is non-trivial. Moreover, note that none of the arguments used in the examples above work in general scenarios and we must present the arguments in each case separately. Even though Theorem 2 is useful while showing inexpressibility, it is a fact that this is the only result we can apply while using EF-games argument. This motivates us to come up with some generic versions of the games or similar criteria that could be used for showing inexpressibility in general settings. Obviously it is a good idea to have some general conditions at hand which can guarantee the existence of back-and-forth systems or the winning strategy for the duplicator. Two such criteria are presented in the following chapter, namely the Hanf's condition and the Gaifman's theorem.

3 Locality, A Tool in FMT

In the previous chapter we discussed the complexity of the EF-game and the need for further tools for showing inexpressibility. One such tool that simplifies the proofs (at least for FO) is locality.

Locality is a tool useful for many logics. The idea is that in logics having this locality property it is enough to look at a *small portion* of the universe of a structure in order to decide the truth value of a formula in the structure. In other words, the truth value of a formula can be determined locally by looking at small neighborhoods of the free variables of a formula along with the substructures generated by these neighborhoods up to isomorphism.

The problem we addressed at the end of the last chapter can be answered by the connection between EF-game arguments and locality. That is, the latter helps in guaranteeing a winning strategy for the duplicator. In this way his strategy is not based on some combinatorial arguments but rather on the assumptions which follow from the property of a given logic $\mathcal{L}$, either being local or not. Thus, instead of coming up with a winning strategy for a game we can make use of the winning strategy provided by the result.

3.1 Preliminary Concepts

We start by formalizing the concepts that we will need.

Definition 9. *Let σ be a vocabulary and $\mathcal{A}$ be a σ-structure. The Gaifman graph $\mathcal{G}(\mathcal{A})$ of $\mathcal{A}$ is defined as the structure $(\mathbf{A}, E)$ where the set of nodes is $\mathbf{A}$, the universe of $\mathcal{A}$ and $(a_1, a_2) \in E^{\mathcal{G}(\mathcal{A})}$ iff $a_1 = a_2$ or there is a relation $R \in \sigma$ such that for some tuple $\mathbf{t} \in R^{\mathcal{A}}$, both a_1 and a_2 occur in $\mathbf{t}$.*

If $\mathcal{A}$ is itself a graph then it's Gaifman graph is the structure along with the *loops* (a, a) *for every* $a \in \mathbf{A}$.

The degree of a node $a \in \mathbf{A}$ *is the size of the set* $\{b : (a, b) \in E^{\mathcal{G}(\mathcal{A})}\}$.

Definition 10. *Let* $\mathcal{A}$ *be a* σ*-structure and* $\mathcal{G}(\mathcal{A})$ *be it's Gaifman graph. For any* $a, b \in \mathbf{A}$*, the distance* $d_{\mathcal{A}}(a, b)$ *is the length of the shortest path between* a *and* b *in* $\mathcal{G}(\mathcal{A})$. *If there is no such path then* $d_{\mathcal{A}}(a, b) = \infty$. *Moreover,* $d(a, a) = 0$.

For $\boldsymbol{a} = (a_1, \ldots, a_n)$ *and* $\boldsymbol{b} = (b_1, \ldots, b_m)$ *and a point* $c \in \mathbf{A}$*, we define:*

$$d_{\mathcal{A}}(\boldsymbol{a}, c) = min\{d(a_i, c) : 1 \leq i \leq n\}$$

and

$$d_{\mathcal{A}}(\boldsymbol{a}, \boldsymbol{b}) = min\{d(a_i, b_j) : 1 \leq i \leq n, 1 \leq j \leq m\}.$$

Definition 11. *Let* $\mathcal{A}$ *be a* σ*-structure and* $a \in \mathbf{A}$. *The* r*–ball around* a *is the set* $\mathrm{B}_r^{\mathcal{A}}(a)$ *defined as*

$$\mathrm{B}_r^{\mathcal{A}}(a) \quad = \quad \{b \in \mathcal{A} : d_{\mathcal{A}}(a, b) \leq r\}.$$

For a tuple $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbf{A}^n$,

$$\mathrm{B}_r^{\mathcal{A}}(\boldsymbol{a}) \quad = \quad \{b \in \mathcal{A} : d_{\mathcal{A}}(\boldsymbol{a}, b) \leq r\} \quad = \quad \bigcup_{i=1}^{n} B_r^{\mathcal{A}}(a_i).$$

The r*–neighborhood of* $\boldsymbol{a}$ *is the substructure* $\mathcal{N}_r^{\mathcal{A}}(\boldsymbol{a})$ *of* $\mathcal{A}$ *in the vocabulary* $\sigma \cup \{c_i : i \leq n\}$ *(henceforth this vocabulary will be denoted by* σ_n*). The universe of* $\mathcal{N}_r^{\mathcal{A}}(\boldsymbol{a})$ *is* $\mathrm{B}_r^{\mathcal{A}}(\boldsymbol{a})$. *Each* k*-ary relation* $R \in \sigma$ *is interpreted as* $R^{\mathcal{A}} \cap (\mathrm{B}_r^{\mathcal{A}}(\boldsymbol{a}))^k$ *and the new constants* $\boldsymbol{c}$ *denote the center of ball with* $\boldsymbol{a}$ *as its interpretation.*

If two neighborhoods $\mathcal{N}_r^{\mathcal{A}}(\mathbf{a})$ and $\mathcal{N}_r^{\mathcal{A}}(\mathbf{b})$ are isomorphic via an isomorphism h

then it must be the case that $h(a_i) = h(b_i)$ for every $i \leq n$. An equivalence class of this isomorphic relation on the class of all σ_n-structures (of neighborhoods) is called an isomorphism type of σ_n-structures. Rather than saying that a structure $\mathcal{N}_r^{\mathcal{A}}(\mathbf{a})$ belongs to an isomorphism type τ we say that this structure is of type τ. Furthermore, in this case we say that $\mathbf{a}$ r–realizes τ.

Definition 12. *Let $\mathcal{A}$ and $\mathcal{B}$ be σ-structures and $\boldsymbol{a} \in \mathbf{A}^n$, $\boldsymbol{b} \in \mathbf{B}^n$. We write $(\mathcal{A}, \boldsymbol{a}) \leftrightarrows_r (\mathcal{B}, \boldsymbol{b})$ if there exists a bijection $f : \mathbf{A} \to \mathbf{B}$ such that*

$$\mathcal{N}_r^{\mathcal{A}}(\boldsymbol{a}c) \cong \mathcal{N}_r^{\mathcal{B}}(\boldsymbol{b}f(c))$$

for every $c \in \mathbf{A}$ where $\boldsymbol{a}c$ simply stands for the tuple $(a_1, \ldots, a_n, c)$.
For n=0 $\mathcal{A} \leftrightarrows_r \mathcal{B}$ means that there is a bijection $f : \mathbf{A} \to \mathbf{B}$ such that

$$\mathcal{N}_r^{\mathcal{A}}(c) \cong \mathcal{N}_r^{\mathcal{B}}(f(c))$$

for every $c \in \mathbf{A}$. In other words, the relation $\mathcal{A} \leftrightarrows_r \mathcal{B}$ says that locally $\mathcal{A}$ and $\mathcal{B}$ look alike with respect to a bijection f.
We will use the notation $\boldsymbol{a} \approx_r^{\mathcal{A},\mathcal{B}} \boldsymbol{b}$ as a shorthand for $\mathcal{N}_r^{\mathcal{A}}(\boldsymbol{a}) \cong \mathcal{N}_r^{\mathcal{B}}(\boldsymbol{b})$.

Lemma 2. *Let σ be a vocabulary and $\mathcal{A}$ and $\mathcal{B}$ be σ-structures with $\boldsymbol{a} \in \mathbf{A}^n$, $\boldsymbol{b} \in \mathbf{B}^n$. Then the following holds.*

1. *If $h : \mathcal{N}_r^{\mathcal{A}}(\boldsymbol{a}) \mapsto \mathcal{N}_r^{\mathcal{B}}(\boldsymbol{b})$ is an isomorphism and $r' < r$ then $h' : \mathcal{N}_{r'}^{\mathcal{A}}(\boldsymbol{a}) \to \mathcal{N}_{r'}^{\mathcal{B}}(\boldsymbol{b})$ is also an isomorphism. Where $h' = h \restriction B_{r'}^{\mathcal{A}}(\boldsymbol{a})$.*

2. *Let $h : \mathcal{N}_r^{\mathcal{A}}(\boldsymbol{a}) \mapsto \mathcal{N}_r^{\mathcal{B}}(\boldsymbol{b})$ be an isomorphism. Suppose $r_1 + r_2 \leq r$ and $\boldsymbol{x} \in B_{r_1}^{\mathcal{A}}$ then $h(B_{r_2}^{\mathcal{A}}(\boldsymbol{x})) = B_{r_2}^{\mathcal{B}}(h(\boldsymbol{x}))$ and the restriction $h' : \mathcal{N}_{r_2}^{\mathcal{A}}(\boldsymbol{x}) \to \mathcal{N}_{r_2}^{\mathcal{B}}(h(\boldsymbol{x}))$ is also an isomorphism.*

3. Let $\boldsymbol{a}_1 \in \mathcal{A}^m, \boldsymbol{b}_1 \in \mathcal{B}^m$ *and* $\boldsymbol{a}_2 \in \mathcal{A}^n, \boldsymbol{b}_2 \in \mathcal{B}^n$ *for* $m, n \geq 1$. *If* $d_{\mathcal{A}}(\boldsymbol{a}_1, \boldsymbol{a}_2), d_{\mathcal{B}}(\boldsymbol{b}_1, \boldsymbol{b}_2) > 2r + 1$. *Then*

$$\boldsymbol{a}_1 \approx_r^{\mathcal{A},\mathcal{B}} \boldsymbol{b}_1 \text{ and } \boldsymbol{a}_2 \approx_r^{\mathcal{A},\mathcal{B}} \boldsymbol{b}_2 \implies \boldsymbol{a}_1\boldsymbol{a}_2 \approx_r^{\mathcal{A},\mathcal{B}} \boldsymbol{b}_1\boldsymbol{b}_2$$

4. $(\mathcal{A}, \boldsymbol{a}) \leftrightarrows_r (\mathcal{B}, \boldsymbol{b}) \implies (\mathcal{A}, \boldsymbol{a}) \leftrightarrows_{r'} (\mathcal{B}, \boldsymbol{b})$ *for every* $r' < r$.

□

Now we present two well known notions of locality namely Hanf and Gaifman locality.

3.2 Hanf locality

Definition 13. *An m-ary query* $\mathcal{Q}$ *on a class* $\mathcal{K}$ *of* σ*-structures is Hanf-local iff there is* $r \in \mathbb{N}$ *such that for every* $\mathcal{A}, \mathcal{B} \in \mathcal{K}$ *and every* $\boldsymbol{a} \in \mathbf{A}^m$, $\boldsymbol{b} \in \mathbf{B}^m$,

$$(\mathcal{A}, \boldsymbol{a}) \leftrightarrows_r (\mathcal{B}, \boldsymbol{b}) \text{ implies } \boldsymbol{a} \in \mathcal{Q}(\mathcal{A}) \iff \boldsymbol{b} \in \mathcal{Q}(\mathcal{B}).$$

The smallest such r is called the Hanf-locality rank of $\mathcal{Q}$ *and denoted by* $hlr(\mathcal{Q})$.

Example 4. *Let* $\sigma = \{\subseteq\}$ *and the class* $\mathcal{K}$ *of* σ*-structures be Boolean algebras. Then every query over* $\mathcal{K}$ *is Hanf-local with locality rank 2. Since* $\emptyset$ *is the subset of every other set so the radius 2-ball of each point is the whole structure.*

Example 5. *Let* $\sigma = \{E\}$ *and the intended class* $\mathcal{K}$ *of* σ*-structures be graphs. Then the query Graph Acyclicity is not Hanf-local (Example 9). This Boolean query checks whether there are cycles in the graph. That is,*

$$\mathcal{Q} = \{\mathcal{A} \in \mathcal{K} \ : \ \mathcal{A} \; has \; no \; cycle \; of \; length \; n \; for \; every \; n \in \mathbb{N}\}$$

This criteria for locality is derived from Hanf's technique for proving the elementary equivalence of two structures as stated below. Even though Hanf's original work was for classical model theory, most of his work was later extended to finite structures.

Theorem 4. *Let σ be a vocabulary and $\mathcal{A}$, $\mathcal{B}$ be σ-structures. Assume that the neighborhood of every point in $\mathcal{A}$ and $\mathcal{B}$ contains finitely many points. If $\mathcal{A}$ and $\mathcal{B}$ have the same number of points with r–type τ for every type τ of neighborhoods then $\mathcal{A} \equiv \mathcal{B}$.*

□

We modify the above statement to suit the writing of this thesis. So Hanf's theorem states that if $\mathcal{A} \leftrightarrows_r \mathcal{B}$ then any FO-expressible Boolean query agrees on the two structures provided that every r–neighborhood in both structures is finite. Hence the above result does not apply if the Gaifman graph of a structure has infinite degree which cannot happen in finite structures.

The following lemma provides an equivalent formulation of Hanf-locality in terms of the number of points realizing same types.

Lemma 3. *A Boolean query is Hanf-local if and only if for each isomorphism type τ of σ_1-structures, if the number of points of $\mathcal{A}$ and $\mathcal{B}$ realizing τ are same then $\mathcal{A}$ and $\mathcal{B}$ agree on the given query.*

□

We will present the main result for this section showing that every FO-definable query is Hanf-local. The technique for showing inexpressibility then follows the same pattern as the EF-games. We construct two structures $\mathcal{A}$ and $\mathcal{B}$ and tuples $\mathbf{a} \in \mathcal{A}, \mathbf{b} \in \mathcal{B}$ by using Hanf-locality rank (*hlr*) as a parameter such that $\mathcal{A}$ and $\mathcal{B}$ disagree on the given query. The only difference now is that the Hanf's result gives a sufficient condition for the duplicator to have a winning strategy in the EF-game.

Hanf-locality of FO

As mentioned earlier Hanf was not working in finite model theory and his results were later extended to the finite case in in (Fagin, Stockmeyer, & Vardi, 1995). The original criterion derived from Hanf's technique is for more general setting and has been described in Section 4.1. However, we use the following result for now giving a sufficient condition for the existence of a winning strategy for the duplicator in terms of local equivalence.

Theorem 5. *Let σ be a vocabulary, $\mathcal{A}$ and $\mathcal{B}$ be σ-structures. Let $k \geq 1$ then there is a positive integer r depending only on k such that $\mathcal{A} \leftrightarrows_r \mathcal{B}$ implies $\mathcal{A} \simeq_k \mathcal{B}$. Moreover, this r can be taken to be 3^{k-1}.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be two σ-structures such that $\mathcal{A} \leftrightarrows_r \mathcal{B}$. We now form a back-and-forth system for the game $EF_k(\mathcal{A}, \mathcal{B})$ where $r = 3^{k-1}$. Assume without loss of generality that the universes of $\mathcal{A}$ and $\mathcal{B}$ are disjoint.
We argue that $(I_i)_{i \leq k}$ is the required system of partial isomorphisms where

$$I_i = \{\mathbf{a} \mapsto \mathbf{b} : \mathcal{N}^{\mathcal{A}}_{3^{k-i}}(\mathbf{a}) \cong \mathcal{N}^{\mathcal{A}}_{3^{k-i}}(\mathbf{b})\} \tag{2}$$

Where $\mathbf{a} = (a_1, \ldots, , a_i), \mathbf{b} = (b_1, \ldots, b_i)$.
By induction on i we show that every $p \in I_i$ has a back-and-forth extension in $I_{i+1}, i < k$. In fact it is enough to prove the forth condition.
For $i = 0$, Suppose $a_1 \in \mathcal{A}$ and let τ be the r–type of a_1. Since $\mathcal{A} \leftrightarrows_r \mathcal{B}$, there is a $b_1 \in \mathcal{B}$ with r–type τ. That is, $\mathcal{N}^{\mathcal{A}}_r(a_1) \cong \mathcal{N}^{\mathcal{A}}_r(b_1)$ where $r = 3^{k-1}$. So

$a_1 \mapsto b_1 \in I_1$.

For the induction step let $p = \mathbf{a} \mapsto \mathbf{b} \in I_i$, where $\mathbf{a} = (a_1, \ldots, a_i), \mathbf{b} = (b_1, \ldots, b_i)$. Let $a_{i+1} \in \mathcal{A}$ then we argue that there is a $b_{i+1} \in \mathcal{B}$ so that (**??**) holds for $i+1$. For the sake of simplicity let us write $3^{k-i} = r_i$ and $3^{k-(i+1)} = r_{i+1}$. There are two cases:

Case 1. If $a_{i+1} \in B_{2r_{i+1}}(\mathbf{a}) := X^{\mathcal{A}}_{i+1}$. Then $B_{r_{i+1}}(a_{i+1}) \subseteq B_{r_i}(\mathbf{a})$. Since $\mathcal{N}^{\mathcal{A}}_{r_i}(\mathbf{a}) \cong \mathcal{N}^{\mathcal{A}}_{r_i}(\mathbf{b})$, the required b_{i+1} can be taken to be $f(a_{i+1}) \in B_{r_i}(\mathbf{a})$ where f is the isomorphism of neighborhoods.

Case 2. If Case 1 fails. Let τ be the r_{i+1}–type of a_{i+1}. Let m be the number of points in $X^{\mathcal{A}}_{i+1}$ with r_{i+1}–type τ. Then, for every $c \in X^{\mathcal{A}}_{i+1}$ we have:

$$B_{r_{i+1}}(c) \subseteq B_{r_i}(\mathbf{a})$$

Hence, for every $c \in X^{\mathcal{A}}_{i+1}$ the r_{i+1}–type of c in $\mathcal{A}$ is same as its r_{i+1}–type in the substructure $\mathcal{N}^{\mathcal{A}}_{r_i}(\mathbf{a})$. Therefore, m also equals the number of points in $X^{\mathcal{A}}_{i+1}$ whose r_{i+1}–type in $\mathcal{N}^{\mathcal{A}}_{r_i}(\mathbf{a})$ is τ.

Now let $X^{\mathcal{B}}_{i+1} := B_{2r_{i+1}}(\mathbf{b})$. Then by (**??**) m is equal to the number of points in $X^{\mathcal{B}}_{i+1}$ whose r_{i+1}–type in $\mathcal{N}^{\mathcal{B}}_{r_i}(\mathbf{b})$ is τ. By a similar argument as in $\mathcal{A}$ and $\mathcal{N}^{\mathcal{A}}_{r_i}(\mathbf{a})$ it follows that m also equals the number of points in $X^{\mathcal{B}}_{i+1}$ with r_{i+1}–type τ in $\mathcal{B}$.

Now there are at least $m+1$ points in $\mathcal{A}$ whose r_{i+1}–type is τ (m points in $X^{\mathcal{A}}_{i+1}$ and a_{i+1}). Since $\mathcal{A} \leftrightarrows_r \mathcal{B}$ and $r_{i+1} < r$, this implies $\mathcal{A} \leftrightarrows_{r_{i+1}} \mathcal{B}$. Hence there are at least $m+1$ points in $\mathcal{B}$ with r_{i+1}–type τ. In particular there is a point $x \notin X^{\mathcal{B}}_{i+1}$, let this x be the witness b_{i+1}. Since Case 1 fails, $\mathcal{N}^{\mathcal{A}}_{r_{i+1}}(\mathbf{a}) \cup \mathcal{N}^{\mathcal{A}}_{r_{i+1}}(a_{i+1}) \cong \mathcal{N}^{\mathcal{A}}_{r_{i+1}}(\mathbf{a}a_{i+1})$. Similarly, $\mathcal{N}^{\mathcal{B}}_{r_{i+1}}(\mathbf{b}) \cup \mathcal{N}^{\mathcal{B}}_{r_{i+1}}(b_{i+1}) \cong \mathcal{N}^{\mathcal{B}}_{r_{i+1}}(\mathbf{b}b_{i+1})$. Hence $\mathcal{N}^{\mathcal{A}}_{r_{i+1}}(\mathbf{a}a_{i+1}) \cong \mathcal{N}^{\mathcal{B}}_{r_{i+1}}(\mathbf{b}b_{i+1})$. This completes the induction step.

This proves that $(I_i)_{1\leq i\leq k}$ is a back-and-forth system for $EF_k(\mathcal{A},\mathcal{B})$. In particular

$$I_k = \{(a_1,\ldots,a_k)\mapsto(b_1,\ldots,b_k) : \mathbf{a}\cong\mathbf{b}\}$$

Which implies that the substructure of $\mathcal{A}$ generated by $\{a_1,\ldots,a_k\}$ is isomorphic to that generated by $\{b_1,\ldots,b_k\}$ under the map $f : a_i \mapsto b_i, i\leq k$. This completes the proof that $\mathcal{A}\simeq_k\mathcal{B}$.

□

Locality of FO is an immediate Corollary of Theorem 6. Before moving to the main result of this section let us make an observation that will help in the transition from the Boolean to the non-Boolean case.

Theorem 6. *If a logic $\mathcal{L}$ is closed under first order operations then Hanf-locality of sentences implies the Hanf-locality of formulas.*

□

The proof of this theorem is based on the well-known technique from classical model theory. The idea is to extend the vocabulary by adding constants and after proving the desired results we restrict our attention back to the original vocabulary.
Let σ be a vocabulary and let $\sigma_m = \sigma\cup\{c_1,\cdots,c_m\}$. Then every σ-formula $\phi(x_1,\ldots,x_m)$ corresponds to a σ_m-sentence Φ obtained by replacing each variable x_i by the constant c_i, for $i\leq m$. For any σ-structure $\mathcal{A}$ and $\mathbf{a}\in\mathcal{A}$, let $\mathcal{A}[\mathbf{a}]$ be the corresponding σ_m-structure with $\mathbf{a}$ being the interpretation of new constants symbols. Then it holds that $\mathcal{A}\models\phi(\mathbf{a}) \iff \mathcal{A}[\mathbf{a}]\models\Phi$.

Lemma 4. *Let $\mathcal{L}$ be a logic closed under first order operations. If every $\mathcal{L}$-definable Boolean query is Hanf-local then every non-Boolean query is also Hanf-local.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be σ-structures and $\mathcal{Q}$ be an m-ary query on the class of σ-structures. Consider the vocabulary σ_m.
Suppose that $(\mathcal{A}, \mathbf{a}) \leftrightarrows_r (\mathcal{B}, \mathbf{b})$. Then this is also true for structures in the vocabulary σ_m. That is, $\mathcal{A}[\mathbf{a}] \leftrightarrows_r \mathcal{B}[\mathbf{b}]$. Define a Boolean query $\mathcal{Q}'$ over σ_m structures as,

$$\mathcal{Q}' = \{\mathcal{A}[\mathbf{a}] : \mathcal{A} \text{ is } \sigma - \text{structure, } \mathbf{a} \in \mathcal{A} \text{ and } \mathbf{a} \in \mathcal{Q}(\mathcal{A})\}.$$

Then for any m-tuples $\mathbf{a}$ and $\mathbf{b}$ in $\mathcal{A}$, the following is true.

$$\mathbf{a} \in \mathcal{Q}(\mathcal{A}) \iff \mathcal{A}[\mathbf{a}] \in \mathcal{Q}' \iff \mathcal{B}[\mathbf{b}] \in \mathcal{Q}' \iff \mathbf{b} \in \mathcal{Q}(\mathcal{B}).$$

Where the equivalence for $\mathcal{Q}'$ is guaranteed by Hanf's theorem and the fact that $\mathcal{A}[\mathbf{a}] \leftrightarrows_r \mathcal{B}[\mathbf{b}]$. Hence, the m-ary query $\mathcal{Q}$ is Hanf-local provided that the Boolean query $\mathcal{Q}'$ is Hanf-local.

□

Corollary 3. *Every FO definable query $\mathcal{Q}$ is Hanf-local with locality rank 3^{k-1} where k is the quantifier rank of the formula expressing $\mathcal{Q}$.*

Proof. Let $\mathcal{Q}$ be FO-expressible by a sentence ψ and let k be the quantifier rank of ψ. Now suppose $\mathcal{A} \leftrightarrows_r \mathcal{B}$ where $r = 3^{k-1}$. Then Theorem 6 implies that $\mathcal{A} \equiv_k \mathcal{B}$, which in turn implies that $\mathcal{A} \models \psi \iff \mathcal{B} \models \psi$ and hence $\mathcal{Q}$ must agree on $\mathcal{A}$ and $\mathcal{B}$ as required.
The case for non-Boolean queries follows from Lemma 4.

□

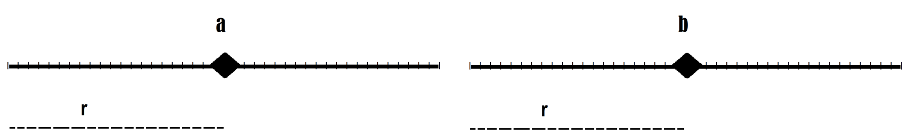
Applications of Hanf-Locality

One way of looking at locality is that a local logic can only define local properties of structures. In order to show that a certain property is not expressible it suffices to show that this property is not local.

Corollary 4. *Let* $\sigma = \{E\}$ *and the class* $\mathcal{K}$ *of* σ*-structures be finite graphs. Then the query* $\mathcal{Q} = \{\mathcal{A} : \mathcal{A} \text{ is connected}\}$ *is not FO-expressible.*

Proof. Suppose to the contrary that $\mathcal{Q}$ is expressible in FO. Since every FO-definable query is Hanf-local, let $hlr(\mathcal{Q}) = r$.
Take two graphs with $2(2r+1)$ nodes each such that $\mathcal{G}_1$ is a single cycle and $\mathcal{G}_2$ is union of two disjoint cycles with $2r+1$ nodes each. Now the r–type of every point in $\mathcal{G}_1$ and $\mathcal{G}_2$ is a chain of length $2r+1$. Moreover, $\mathcal{G}_1$ and $\mathcal{G}_2$ have equal number of points all with the same r–types. Thus it holds that $\mathcal{G}_1 \leftrightarrows_r \mathcal{G}_2$. Hence $\mathcal{Q}(\mathcal{G}_1) \iff \mathcal{Q}(\mathcal{G}_2)$, which is a contradiction.

Neighborhoods of two points, $a \in \mathcal{G}_1,\ b \in \mathcal{G}_2$



□

Corollary 5. *The query Graph Acyclicity is not Hanf-local (and hence not FO-expressible). That is,*

$$\mathcal{Q} = \{\mathcal{A} : \mathcal{A}\ has\ no\ cycle\ of\ length\ n\ for\ every\ n \in \mathbb{N}\}$$

Proof. The same arguments as in the above example work except that we take $\mathcal{G}_1$ to be a chain of $4r + 2$ points and $\mathcal{G}_2$ to be a disjoint union of a chain of length $2r + 1$ and a cycle of length $2r + 1$. Then $\mathcal{G}_1 \in \mathcal{Q}$ but $\mathcal{G}_2 \notin \mathcal{Q}$.

□

3.3 Gaifman Locality

Definition 14. *An m-ary query $\mathcal{Q}$ on a class $\mathcal{K}$ of σ-structures is Gaifman-local iff there is an $r \in \mathbb{N}$ such that for every σ-structure $\mathcal{A} \in \mathcal{K}$ and m-tuples $\boldsymbol{a}_1, \boldsymbol{a}_2 \in \mathbf{A}^m$:*

$$\boldsymbol{a}_1 \approx_r^{\mathcal{A}} \boldsymbol{a}_2 \quad \textit{implies} \quad \boldsymbol{a}_1 \in \mathcal{Q}(\mathcal{A}) \iff \boldsymbol{a}_2 \in \mathcal{Q}(\mathcal{A}).$$

As with Hanf-locality, the Gaifman-locality rank of a query $\mathcal{Q}$ is denoted by lr($\mathcal{Q}$).

The difference between the two kinds of locality is that Hanf-locality relates two different structures and Gaifman-locality is useful when we can show inexpressibility using one structure. For the same reason Hanf-locality is useful when we want to show inexpressibility of Boolean queries since we can compare two structures for whether a Boolean query agrees on them or not. However, the method for showing inexpressibility remains same.

Example 6. *Let $\sigma = \{\prec\}$ and the class $\mathcal{K}$ of σ-structures be linear orders. Then every query over $\mathcal{K}$ is trivially Gaifman-local with locality rank 1. Since for every a, b, connectedness implies that either $a \prec b$ or $a \succeq b$. This implies that the 1–ball of each point is the whole structure.*

Example 7. *Let $\sigma = \{E\}$ and the class $\mathcal{K}$ of σ-structures be graphs. The* *binary query Same Generation is not Gaifman-local (Example 12).*

$$\mathcal{SG}(\boldsymbol{G}) = \{(a,b) \in \boldsymbol{G}^2 \ : \ \exists c, d(a,c) = d(b,c)\}$$

Where $d(x,y)$ is the length of the shortest path between x and y in $\boldsymbol{G}$. This query tests whether two nodes a and b are in the same generation by checking if both have a common ancestor c equidistant from both a and b.

Facts and Observations

We defined the Gaifman graph, distance and neighborhoods before giving the locality criteria. Now we formalize these concepts in first order logic.

Fact 1. *Let σ be a vocabulary having m as the maximum arity of any relation symbol such that $m \geq 2$. For any σ-structure $\mathcal{A}$, its Gaifman graph $\mathcal{G}(\mathcal{A})$ is definable in $\mathcal{A}$ by a FO-formula of quantifier rank $m-2$.*
For example, let $\sigma = \{P, R\}$ where P is a binary and R a ternary relation. Then $\mathcal{G}(\mathcal{A})$ is defined by: $(a,b) \in E^{\mathcal{G}(\mathcal{A})} \iff \mathcal{A} \models \phi(a,b)$ where, $\phi(x,y) := (x = y) \vee (xPy) \quad \bigvee \quad \exists z \langle R(x,y,z) \vee R(x,z,y) \vee R(y,x,z) \vee R(y,z,x) \vee R(z,x,y) \vee R(z,y,x) \rangle$.

Fact 2. *We can define distance in $\mathcal{G}(\mathcal{A})$ by a FO-formula as, $d(a,b) \leq 1 \iff d^{\leq 1}(a,b)$ where $d^{\leq 1}(x,y) := (x = y \vee xEy)$. Now, $d(a,b) \leq r$ can be defined by $d^{\leq r}(x,y) := \exists z_0 \ldots \exists z_r (x = z_0 \wedge y = z_r \wedge \bigwedge_{n<r} d^{\leq 1}(z_n, z_{n+1})$. Similarly, we can define the FO formulas $d^{=r}(\boldsymbol{x}, y)$ and $d^{\geq r}(\boldsymbol{x}, y)$ which are true in $\mathcal{A}$ for $\boldsymbol{a}$ and b exactly when $d_{\mathcal{A}}(\boldsymbol{a}, b) = r$ and $d_{\mathcal{A}}(\boldsymbol{a}, b) \geq r$ respectively. The r–ball around some tuple $\boldsymbol{a} \in \mathcal{G}(\mathcal{A})$ is also FO-definable as $B_r^{\mathcal{A}}(\boldsymbol{a}) = d^{\leq r}(\boldsymbol{a}, \mathcal{A})$.*

Definition 15. *Let $\phi(\boldsymbol{x}) \in FO(\sigma)$ and $\boldsymbol{y}$ a tuple of variables not bounded in ϕ. We denote by $\phi^{(r)}(\boldsymbol{x}, \boldsymbol{y})$ the formula that relativizes $\phi(\boldsymbol{x})$ to $\mathcal{N}_r(\boldsymbol{y})$. That* *is, a formula which has quantification of a local nature. This can be defined by induction on ϕ.*

- *For atomic $\phi(\boldsymbol{x})$, $\phi^{(r)}(\boldsymbol{x}, \boldsymbol{y}) = \phi(\boldsymbol{x})$.*
- *For $\phi = \psi_1 \wedge \psi_2$, $\phi^{(r)} = \psi_1^{(r)} \wedge \psi_2^{(r)}$ and similarly for disjunction.*
- *For $\phi = \neg\psi$, $\phi^{(r)} = \neg\psi^{(r)}$.*
- *For $\phi(\boldsymbol{x}) = \exists z \psi(z, \boldsymbol{x})$, $\phi^{(r)}(\boldsymbol{x}, \boldsymbol{y}) = \exists z \left(d^{\leq r}(\boldsymbol{y}, z) \wedge \psi^{(r)}(\boldsymbol{x}, \boldsymbol{y})\right)$.*
- *For $\phi(\boldsymbol{x}) = \forall z \psi(z, \boldsymbol{x})$, $\phi^{(r)}(\boldsymbol{x}, \boldsymbol{y}) = \forall z \left(d^{\leq r}(\boldsymbol{y}, z) \rightarrow \psi^{(r)}(\boldsymbol{x}, \boldsymbol{y})\right)$.*

The essential model theoretic property of this relativization is that, for $\mathbf{a}, \mathbf{b} \in \mathcal{A}$ with $\mathbf{a} \in \mathbf{B}_r(\mathbf{b})$:

$$\mathcal{A} \models \phi^{(r)}(\mathbf{a}, \mathbf{b}) \iff \mathcal{N}_r^{\mathcal{A}}(\mathbf{b}) \models \phi(\mathbf{a}).$$

Definition 16. *Let σ be a vocabulary, $\mathcal{A}$ and $\mathcal{B}$ be σ-structures.*

1. *A σ-formula $\phi(\boldsymbol{x})$ is r–local around $\boldsymbol{x}$ if and only if it has the local quantification around $\boldsymbol{x}$. That is, $\phi(\boldsymbol{x}) := \psi^{(r)}(\boldsymbol{x})$ for some $\psi \in FO[\sigma]$. If the quantifier rank of ψ is k then the corresponding local formula $\psi^{(r)}$ is said to have the Gaifman rank (r, k).*

2. *A sentence of the form below is called a basic local sentence (also called existentially local),*

$$\exists x_1 \ldots \exists x_s \left(\bigwedge_{i,j<s} d^{>2r}(x_i, x_j) \wedge \bigwedge_i \psi^{(r)}(x_i) \right)$$

   If k is the quantifier rank of the r–local formula ψ then the Gaifman rank of this local sentence is (r, k, s).

3. *We say that two σ-structures $\mathcal{A}$ and $\mathcal{B}$ with tuples $\boldsymbol{a}, \boldsymbol{b}$ are (r, k, s)-Gaifman equivalent, denoted as $(\mathcal{A}, \boldsymbol{a}) \equiv^r_{k,s} (\mathcal{B}, \boldsymbol{b})$ if,*

   (a) *For every r–local formula $\phi^{(r)}(\boldsymbol{x})$ with $qr(\phi) \leq k$, $\mathcal{A} \models \phi^{(r)}(\boldsymbol{a}) \iff \mathcal{B} \models \phi^{(r)}(\boldsymbol{b})$.*

   (b) *For every existentially local sentence Φ with Gaifman rank $(r', k', s'), r' < r, k' < k, s' < s$, $\mathcal{A} \models \Phi \iff \mathcal{B} \models \Phi$.*

If σ is a finite vocabulary then up to logical equivalence the set of sentences with Gaifman ranks at most (r, k, s) is also finite. Hence the relation $\equiv^r_{k,s}$ has finite index over the class of finite σ-structures.

Gaifman-locality of FO

Theorem 7 (**Gaifman**). *Let σ be a vocabulary. Then every first order σ-formula $\phi(\boldsymbol{x})$ with $\boldsymbol{x} = \{x_1, \ldots, x_n\}$ is logically equivalent to a Boolean combination of the following:*
(I) The basic local sentences, $\exists z_1 \ldots \exists z_s \left(\bigwedge_{i,j \leq s} d^{>2r}(z_i, z_j) \wedge \bigwedge_{i \leq s} \psi^{(r)}(z_i) \right)$
(II) t-local formulas $\alpha^{(t)}(\boldsymbol{u})$ where $\boldsymbol{u} \subset \{x_1, \ldots, x_n\}$.
Similarly, every FO sentence ϕ is logically equivalent to a Boolean combination of sentences of the form (I). Furthermore, if the quantifier rank of $\phi(\boldsymbol{x})$ is k then upper bounds for the numbers r, s and t are given as:

$$r \leq 7^k, \qquad s \leq n + k \quad \textit{and} \quad t \leq \frac{7^k - 1}{2}.$$

Before proving the theorem we need the following definition of Hintikka formula and some facts concerning the equivalence of structures in terms of this new definition.

Definition 17. *Let* $\boldsymbol{x} = x_1, \ldots, x_n$*, define;*

$$\mathcal{H}^0_{\mathcal{A},\boldsymbol{a}}(\boldsymbol{x}) = \bigwedge \{\phi(\boldsymbol{x}) \ : \ \phi \text{ is atomic or negated atomic with } \mathcal{A} \models \phi(\boldsymbol{a})\}$$

$$\mathcal{H}^{k+1}_{\mathcal{A},\boldsymbol{a}}(\boldsymbol{x}) = \bigwedge_{c \in \mathcal{A}} \exists x_{s+1} \mathcal{H}^k_{\mathcal{A},\boldsymbol{a}c}(\boldsymbol{x}, x_{s+1}) \wedge \forall x_{s+1} \bigvee_{c \in \mathcal{A}} \mathcal{H}^k_{\mathcal{A},\boldsymbol{a}c}(\boldsymbol{x}, x_{s+1}).$$

That is, $\mathcal{H}^0_{\mathcal{A},\boldsymbol{a}}(\boldsymbol{x})$ *describes the isomorphism type of the substructure of* $\mathcal{A}$ *induced by* $\boldsymbol{a}$ *while* $\mathcal{H}^k_{\mathcal{A},\boldsymbol{a}}(\boldsymbol{x})$ *describes the isomorphism types to which the tuple* $\boldsymbol{a}$ *can be extended in k-steps. The formula* $\mathcal{H}^k_{\mathcal{A},\boldsymbol{a}}(\boldsymbol{x})$ *is a so-called k-Hintikka formula and it characterizes the substructure of* $\mathcal{A}$ *induced by* $\boldsymbol{a}$ *upto* k*-equivalence.*

When there is no confusion among the structures we can write $\mathcal{H}^k_{\mathbf{a}}(\boldsymbol{x})$ instead of $\mathcal{H}^k_{\mathcal{A},\mathbf{a}}(\boldsymbol{x})$. Note that $qr(\mathcal{H}^k_{\mathcal{A},\mathbf{a}}(\boldsymbol{x})) = k$.

Theorem 8. *Let* σ *be a vocabulary and* $\mathcal{A}, \mathcal{B}$ *be* σ*-structures with* $\boldsymbol{a} \in \mathcal{A}$, $\boldsymbol{b} \in \mathcal{B}$ *then the following are equivalent.*

- $(\mathcal{A}, \boldsymbol{a}) \equiv_k (\mathcal{B}, \boldsymbol{b})$
- $(\mathcal{B}, \boldsymbol{b}) \models \mathcal{H}^k_{\mathcal{A},\boldsymbol{a}}(\boldsymbol{x})$

□

We use the following lemma to prove Gaifman's theorem.

Lemma 5. *Let* σ *be a vocabulary and* Φ *be a finite set of* σ*-sentences up to logical equivalence such that* Φ *is closed under Boolean combinations. Let* ψ *be a* σ*-sentence then the following are equivalent.*

1. *For every* σ*-structures* $\mathcal{A}$ *and* $\mathcal{B}$, $\mathcal{A} \equiv_\Phi \mathcal{B}$ *implies* $\mathcal{A} \models \psi \iff \mathcal{B} \models \psi$.

2. ψ is logically equivalent to some $\phi \in \Phi$.

$\square$

Proof: [Gaifman's Theorem]

We prove the theorem for sentences only since the version for formulas with free variables uses the similar arguments. Let Φ be a finite set of basic local sentences. The Gaifman rank of these sentences will emerge below. It suffices to prove that for every k (quantifier rank of a formula ϕ) there are r, f and s depending only on k, such that: $\mathcal{A} \equiv^{r}_{f,s} \mathcal{B} \implies \mathcal{A} \simeq_k \mathcal{B}$. Then Lemma 5 implies that ϕ is equivalent to a Boolean combination of basic local sentences of Gaifman ranks up to (r, f, s).

We take $r = 7^k$ while the values of f and s will emerge during the proof. The idea is that for every $i \leq k$, $f(i)$-equivalence in each round i guarantees that $\mathcal{A}$ and $\mathcal{B}$ satisfy a particular sentence (Γ) however s will ensure that the local formula appearing in basic local sentence is satisfied by exactly s elements of $\mathcal{A}$ and $\mathcal{B}$. Our task is to show that $\mathcal{A} \simeq_k \mathcal{B}$. Let

$$I_i = \{\mathbf{a} \mapsto \mathbf{b} / \mathcal{N}^{\mathcal{A}}_{7^{k-i}}(\mathbf{a}) \equiv_{f(i)} \mathcal{N}^{\mathcal{B}}_{7^{k-i}}(\mathbf{b})\}$$

where $\mathbf{a} = (a_1, \ldots, a_i) \in \mathcal{A}, \mathbf{b} = (b_1, \ldots, b_i) \in \mathcal{B}$. Then our claim is that $(I_i)_{i \leq k} : \mathcal{A} \simeq_k \mathcal{B}$. Let us write, $r_i := 7^{k-i}$ for the sake of simplicity. We use induction and only prove the forth property.

For $i = 0$. Let $a_1 \in \mathcal{A}$. Then $\mathcal{H}^{f(1)}_{a_1}(x)$ characterizes the substructure of $\mathcal{A}$ generated by a_1 upto $f(1)$-equivalence, that is, $\mathcal{N}^{\mathcal{A}}_{r_1}(a_1)$. Let $\hat{\mathcal{H}}^1_{a_1}(x)$ denotes the relativised version of $\mathcal{H}^{f(1)}_{a_1}(x)$ to $\mathcal{N}^{\mathcal{A}}_{r_1}(a_1)$. Now, $\exists x \hat{\mathcal{H}}^1_{a_1}(x)$ is an existentially local sentence and $\mathcal{A} \models \exists x \hat{\mathcal{H}}^1_{a_1}(x)$. By our assumption $\mathcal{B} \models \exists x \hat{\mathcal{H}}^1_{a_1}(x)$. That is, there is a $b_1 \in \mathcal{B}$ such that $\mathcal{B} \models \hat{\mathcal{H}}^1_{a_1}(b_1)$. Moreover, since $\mathcal{N}^{\mathcal{B}}_{r_1}(b_1) \models \hat{\mathcal{H}}^1_{a_1}(b_1)$, this implies $\mathcal{N}^{\mathcal{A}}_{r_1}(a_1) \equiv_{f(1)} \mathcal{N}^{\mathcal{B}}_{r_1}(b_1)$ by Theorem 8. Hence, $a_1 \mapsto b_1 \in I_1$.

For $i+1$, let $\mathbf{a} \mapsto \mathbf{b} \in I_i$ and $a_{i+1} \in \mathcal{A}$. We need to ensure there is a $b_{i+1} \in \mathcal{B}$ such that the $(i+1)$-Hintikka types of $\mathbf{a}a_{i+1}$ and $\mathbf{b}b_{i+1}$ are the same.

Now, there are three cases to consider.

Case 1. $d(\mathbf{a}, a_{i+1}) \leq 2r_{i+1}$:

This implies $B^{\mathcal{A}}_{r_{i+1}}(a_{i+1}) \subseteq B^{\mathcal{A}}_{r_i}(\mathbf{a})$. Let

$$\Gamma(\boldsymbol{x}) := \exists x_{i+1} \left(d^{\leq 2r_{i+1}}(\boldsymbol{x}, x_{i+1}) \wedge \hat{\mathcal{H}}^{i+1}_{\mathbf{a},a_{i+1}}(\boldsymbol{x}, x_{i+1}) \right).$$

Then $\mathcal{N}^{\mathcal{A}}_{r_i}(\mathbf{a}) \models \Gamma(\mathbf{a})$. Now, by taking $f(i) \geq qr(\Gamma)$ the condition $\mathcal{N}^{\mathcal{A}}_{r_i}(\mathbf{a}) \equiv_{f(i)} \mathcal{N}^{\mathcal{B}}_{r_i}(\mathbf{b})$ implies $\mathcal{N}^{\mathcal{B}}_{r_i}(\mathbf{b}) \models \Gamma(\mathbf{b})$. That is, there is a $b_{i+1} \in \mathcal{B}$ such that,

$$\mathcal{N}^{\mathcal{B}}_{r_i}(\mathbf{b}) \models \left(d^{\leq 2r_{i+1}}(\mathbf{b}, b_{i+1}) \wedge \hat{\mathcal{H}}^{i+1}_{\mathbf{a},a_{i+1}}(\mathbf{b}, b_{i+1}) \right).$$

Hence, $\mathcal{N}^{\mathcal{B}}_{r_{i+1}}(\mathbf{b}, b_{i+1}) \models \hat{\mathcal{H}}^{i+1}_{\mathbf{a},a_{i+1}}(\mathbf{b}, b_{i+1})$.

This implies

$$\mathcal{N}^{\mathcal{A}}_{r_{i+1}}(\mathbf{a}a_{i+1}) \equiv_{f(i+1)} \mathcal{N}^{\mathcal{B}}_{r_{i+1}}(\mathbf{b}b_{i+1})$$

and hence $\mathbf{a}a_{i+1} \mapsto \mathbf{b}b_{i+1} \in I_{i+1}$.

Case 2. $2r_{i+1} < d(\mathbf{a}, a_{i+1}) \leq 6r_{i+1}$:

Let

$$\Gamma(\mathbf{x}) := \exists x_{i+1} \left(d^{>2r_{i+1}}(\mathbf{x}, x_{i+1}) \wedge d^{\leq 6r_{i+1}}(\mathbf{x}, x_{i+1}) \wedge \hat{\mathcal{H}}^{i+1}_{a_{i+1}}(x_{i+1}) \right)$$

Again by letting $f(i) \geq qr(\Gamma)$ it holds that $\mathcal{B} \models \Gamma(\mathbf{b})$. Furthermore, there is a $b_{i+1} \notin \mathcal{N}^{\mathcal{B}}_{2r_{i+1}}(\mathbf{b})$ with same $f(i+1)$-Hintikka type as a_{i+1}.

Case 3. $d(\mathbf{a}, a_{i+1}) > 6r_{i+1}$:

Let n be such that

$$\mathcal{N}^{\mathcal{A}}_{2r_{i+1}}(\mathbf{a}) \models \exists x_1 \dots \exists x_n \left(\bigwedge_{j,l \leq n} d^{>4r_{i+1}}(x_j, x_l) \wedge \bigwedge_{j \leq n} \hat{\mathcal{H}}^{i+1}_{a_{i+1}}(x_j) \right)$$

and

$$\mathcal{N}^{\mathcal{A}}_{2r_{i+1}}(\mathbf{a}) \not\models \exists x_1 \dots \exists x_{n+1} \left(\bigwedge_{j,l} d^{>4r_{i+1}}(x_j, x_l) \wedge \bigwedge_{j} \hat{\mathcal{H}}^{i+1}_{a_{i+1}}(x_j) \right)$$

That is, n is the maximal such that there are n elements in $\mathcal{N}^{\mathcal{A}}_{2r_{i+1}}(\mathbf{a})$ pairwise $4r_{i+1}$ distance apart, each with the same $(i+1)$-Hintikka type as a_{i+1}

Recall that $\hat{\mathcal{H}}^{i+1}_{a_{i+1}}(x_j)$ is the local version of $\mathcal{H}^{f(i+1)}_{a_{i+1}}(x_j)$. Let $f(i)$ be such that

$$\mathcal{N}^{\mathcal{B}}_{2r_{i+1}}(\mathbf{b}) \models \exists x_1 \dots \exists x_n \left(\bigwedge_{j,l} d^{>4r_{i+1}}(x_j, x_l) \wedge \bigwedge_{j} \hat{\mathcal{H}}^{i+1}_{a_{i+1}}(x_j) \right)$$

and

$$\mathcal{N}^{\mathcal{B}}_{2r_{i+1}}(\mathbf{b}) \not\models \exists x_1 \dots \exists x_{n+1} \left(\bigwedge_{j,l} d^{>4r_{i+1}}(x_j, x_l) \wedge \bigwedge_{j} \hat{\mathcal{H}}^{i+1}_{a_{i+1}}(x_j) \right).$$

Now,

$$\mathcal{A} \models \exists x_1 \dots \exists x_{n+1} \left(\bigwedge_{j,l} d^{>4r_{i+1}}(x_j, x_l) \wedge \bigwedge_{j} \hat{\mathcal{H}}^{i+1}_{a_{i+1}}(x_j) \right).$$

Moreover, since the above sentence is an existentially local sentence. By our assumption and the choice of $f(i)$ and n, the following holds.

$$\mathcal{B} \models \exists x_1 \dots \exists x_{n+1} \left(\bigwedge_{j,l} d^{>4r_{i+1}}(x_j, x_l) \wedge \bigwedge_{j} \hat{\mathcal{H}}^{i+1}_{a_{i+1}}(x_j) \right).$$

That is, there are $(n+1)$ elements in $\mathcal{B}$ with same $f(i+1)$-Hitikka type as a_{i+1}. In particular there is a y such that $d(\mathbf{b}, y) > 6r_{i+1}$, call this y to be b_{i+1}. Then $\mathcal{N}^{\mathcal{B}}_{r_{i+1}}(b_{i+1}) \models \hat{\mathcal{H}}^{i+1}_{a_{i+1}}(b_{i+1})$.

Since $d(\mathbf{a}, a_{i+1}) > 6r_{i+1}$. This implies $\mathcal{N}^{\mathcal{A}}_{r_{i+1}}(\mathbf{a}) \cup \mathcal{N}^{\mathcal{A}}_{r_{i+1}}(a_{i+1}) \cong \mathcal{N}^{\mathcal{A}}_{r_{i+1}}(\mathbf{a}a_{i+1})$. Hence $\mathcal{N}^{\mathcal{A}}_{r_{i+1}}(\mathbf{a}a_{i+1}) \equiv_{f(i+1)} \mathcal{N}^{\mathcal{B}}_{r_{i+1}}(\mathbf{b}b_{i+1})$ which completes the induction step and hence the proof.

□

Corollary 6. *Every m-ary FO-definable query $\mathcal{Q}$ is Gaifman-local with locality rank $\frac{1}{2}(7^k - 1)$, where k is the quantifier rank of formula expressing the query.*

Proof. Suppose $\mathcal{Q}$ is definable by a FO-formula $\phi(\boldsymbol{x})$ where $\boldsymbol{x} = \{x_1, \ldots, x_m\}$, and quantifier rank of $\phi(\boldsymbol{x})$ is k. By Gaifman's Theorem $\phi(\boldsymbol{x})$ is equivalent to a Boolean combination of local formulas $\alpha_i^{t_i}(\boldsymbol{x})$, where every $\alpha_i(\boldsymbol{x})$ is t_i–local, and basic local sentences Θ_j. Let $t = max\{t_i\}$. Then our claim is that $\phi(\boldsymbol{x})$ (and hence $\mathcal{Q}$) is Gaifman t–local where $t \leq \frac{1}{2}(7^k - 1)$.
Let $\mathbf{a} \approx_t^{\mathcal{A}} \mathbf{b}$ for some $\mathbf{a}, \mathbf{b} \in \mathcal{A}$. Then

$$\mathcal{A} \models \alpha_i(\mathbf{a}) \iff \mathcal{A} \models \alpha_i(\mathbf{b})$$

since every α_i is local around $\boldsymbol{x}$. Furthermore, every Θ_j is a basic local sentence. We have:

$$\mathcal{A} \models \phi(\mathbf{a}) \iff \mathcal{A} \models \phi(\mathbf{b})$$

That is, $\mathbf{a} \in \mathcal{Q}(\mathcal{A}) \iff \mathbf{b} \in \mathcal{Q}(\mathcal{A})$. Hence $\mathcal{Q}$ is Gaifman local.

□

Note that Hanf-locality of queries is a stronger condition that also implies Gaifman-locality. We could have used this fact to prove the Gaifman-locality of FO but as discussed earlier we need generally applicable tools rather than mere implications so that in the cases where Hanf-Locality does not hold we can still check for Gaifman-locality.

Theorem 9. *Any Hanf local non-Boolean query is also Gaifman local with the following locality rank:* $lr(\mathcal{Q}) \leq 3hlr(\mathcal{Q}) + 1$.

□

Applications of Gaifman Locality

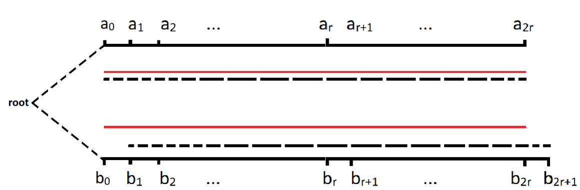
Corollary 7. *The query* $\mathcal{SG}(\boldsymbol{G})$*:*

$$\mathcal{SG} = \{(a, b) \in \boldsymbol{G}^2 \ : \ \exists c \ d(a, c) = d(b, c)\}$$

Testing whether a *and* b *are equidistant from their common ancestor* c*, is not expressible in FO.*

This query on graphs tests whether two nodes a and b are in same generation by checking if they both have a common ancestor equidistant from both a and b. In terms of trees $\mathcal{SG}(\mathbf{T})$ determines whether a and b are equidistant from the root.

Proof. Assume that $\mathcal{SG}$ is FO-definable and let r be its Gaifman-locality rank. Consider a tree $\mathbf{T}$ with two branches of length $2r + 1$ and $2r + 2$ where every a_{i+1} is the successor of a_i.



$$\mathcal{N}_r(a_r, b_r) \cong \mathcal{N}_r(a_r, b_{r+1})$$

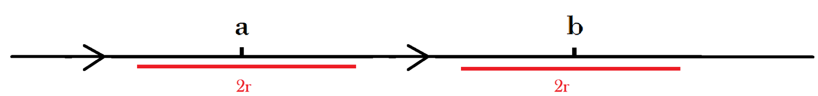
Now, the r–neighborhoods of both (a_r, b_r) and (a_r, b_{r+1}) are isomorphic since each of these two is a disjoint union of two chains (represented as red continuous and black dashed lines above). Clearly $(a_r, b_r) \in \mathcal{SG}(\mathbf{T})$ but $(a_r, b_{r+1}) \notin \mathcal{SG}(\mathbf{T})$. Which proves that $\mathcal{SG}$ is not Gaifman-local and hence not FO-expressible.

□

Corollary 8. *The query Transitive Closure $\mathcal{TC}$ is not expressible in FO. That is,*

$$\mathcal{TC}(\mathcal{G}) = \{(a, b) \in \boldsymbol{G}^2 \ : \ \text{there is a finite path from a to b}\}$$

Proof. If both a and b are at distance $2r + 1$ from each other and from both endpoints then $\mathcal{N}_r(a, b) \cong \mathcal{N}_r(b, a)$ each being the disjoint union $\mathcal{N}_r(a) \cup \mathcal{N}_r(b)$. However there is path *from* a to b but not the other way.



□

We conclude this section here with the main results being that first order logic is both Hanf and Gaifman-local. An obvious question is now to look at the strengths and weaknesses of these two notions and study the variations of locality. Next section discuses this topic.

4 Variations in the notion of Locality

As discussed in the previous section the locality has turned out to be an effective method for inexpressibility results. Before extending its scope to the logics beyond FO, let us examine its limitations and make alterations where possible so that locality indeed turns out to be an effective and efficient tool in logics beyond FO as well.

4.1 Limitations of locality notion

As we noted not very many assumptions were required to show inexpressibility for first order logic. It turns out that both notions of locality have some limitations which we must address before studying locality of logics (for example, FO with built-in predicates). We state below three such problems with possible solutions.

1. *Smaller locality ranks does not make models simpler.*

For the structures with $d(x, y) \leq r$ for every pair $x, y \in \mathcal{A}$, the r–neighborhood of each point is the whole structure. Hence if this number r is very small then locality does not make inexpressibility proofs easier.
For example, let $\sigma = \{+, -, *, ^{-1}, 0, 1\}$ with the intended class of structures as fields (or rings in general). Due to the closure property under $+$ and $*$, the notion of locality becomes trivial. Since for every $a, b \in \mathcal{F}, a + b, a * b \in \mathcal{F}$ there is always an edge in the Gaifman graph. Hence distance between any two points is always 1. For this reason, locality is useful only for structures having locality ranks (r) neither too small (like above) nor too big in order to assure that r–neighborhoods are relatively simpler than the whole structures.

One solution to this deficiency was suggested by Gaifman (Gaifman, 1982) which uses the following idea. Recall that we defined the neighborhoods in terms of distance in the Gaifman graph. In some cases we can alter the definition of distance measure in order to assure that the new distance (based on a different collection of basic relations) is non-trivial. For example, in the case of linear orders where the distance between any two points is always 1. We can still study the locality of those formulas and sentences whose truth value does not depend on the particular order of a structure. Thus, in a sense we are *neglecting* the linear order. This leads to a well known notion of 'order-invariance' which has been largely studied and still found to be Gaifman local.

Definition 18. *Let σ be a vocabulary and $\sigma' = \{\prec\}$ with $\mathcal{K}$ as the class of linear orders. Let $\mathcal{A}$ be a σ-structure and $\boldsymbol{a} \in \mathbf{A}$.*

- *A formula $\phi(\boldsymbol{x})$ in the language $\sigma \cup \{\prec\}$ is called $\mathcal{K}$-invariant (or $\prec$-invariant) on $\mathcal{A}$ if for any two linear orderings $\prec_1, \prec_2 \in \mathcal{K}$ of $\boldsymbol{A}$ the following holds.*

$$(\mathcal{A}, \prec_1) \models \phi(\boldsymbol{a}) \iff (\mathcal{A}, \prec_2) \models \phi(\boldsymbol{a})$$

where $(\mathcal{A}, \prec_i)$ for $i = 1, 2$ is the structure with universe $\boldsymbol{A}$, the relations in σ are interpreted as in $\mathcal{A}$ and $\prec_i$ is inherited from the linear order $\langle \mathbf{A}, \prec_i \rangle$.

- *A formula $\phi(\boldsymbol{x})$ is $\prec$-invariant if it is $\prec$-invariant on every σ-structure.*

We denote the the class of all $\prec$-invariant queries by $(FO+ \prec)_{inv}$. (There might be different ways to express this notation, this one has been taken from (Libkin, 2013). The following theorem from (Grohe, & Schwentick, 2000) characterizes the expressive power of order-invariant first order logic.

Theorem 10. *Let σ be a vocabulary. Then every m-ary query in $(FO+ \prec)_{inv}$ for $m \geq 1$ is Gaifman local.*

$\square$

2. *Weaker logics cannot show isomorphism of neighborhoods*

Both of the locality notions are defined using isomorphism of neighborhoods. This assumption is rather strong since 'having local neighborhoods to be isomorphic' is not expressible in FO. For example, Hanf's criterion requires that some structures $\mathcal{A}$ and $\mathcal{B}$ realize same isomorphism types of radius r for some r. We cannot show in FO whether this requirement is met or not. A possible solution to this is to lower the isomorphims condition to k-equivalence ($\equiv_k$) of neighborhoods.

The resulting new notion of locality is called the game-based locality (corresponding to usual isomosphism-based locality and referring to the difference between the two). The intuitive idea of game-based locality is that we shifted from EF-games to the locality in order to avoid the complexity of arguments but the locality requires too strong assumptions. So the solution is that we combine game arguments with the locality which simplifies the problems from both of these tools. We play EF-games at the small neighborhood structures. Hence we lower the assumption of neighborhoods being isomorphic to the neighborhoods being indistinguishable. The following result from (Arenas, Barceló, & Libkin, 2008) justifies this relaxation at least in case of FO.

Theorem 11. *Let σ be a vocabulary. Then for each σ-formula $\phi(\boldsymbol{x})$ of quantifier rank k, there exist r and l depending on k, such that if $\mathcal{N}_r(\boldsymbol{a}) \equiv_l \mathcal{N}_r(\boldsymbol{b})$ then $\mathcal{A} \models \phi(\boldsymbol{a}) \iff \mathcal{A} \models \phi(\boldsymbol{b})$.*

$\square$

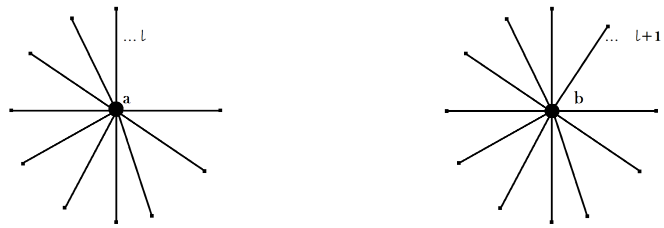
The above theorem is out of the scope of this thesis. However the purpose of adding it was to give a solution to the deficiency we discussed. Now we give an application of Theorem 11 to sum up the deficiency addressed above with the the solution just mentioned.

Example 8. *Let* $\sigma = \{U, E\}$ *with* U *a unary and* E *a binary relation symbol. Consider the query*

$$\mathcal{Q}(\mathcal{A}) = \{a \in \mathcal{A} \ : \ a \in U \textit{ and } |\{c : (a, c) \in E\}| = 0 \bmod 2\}.$$

Where $\mathcal{Q}$ *returns those elements of the universe which are in* U *and have an even out degree with respect to* E*. Then* $\mathcal{Q}$ *is not FO-expressible.*

Proof: First we examine why we cannot apply any locality notion in this example. Being isomorphic the size of both $\mathcal{N}_1(a)$ and $\mathcal{N}_1(b)$ would be same. Hence, the condition $\mathcal{N}_1(a) \cong \mathcal{N}_1(b)$ implies that $\mathcal{Q}$ cannot differentiate between a and b. Applying game-based locality uses the fact that if $\mathcal{Q}$ were definable by a formula of quantifier rank k then there would be an r and l such that, for every $\mathcal{A}$ and $a, b \in \mathcal{A}$, if $\mathcal{N}_r(a) \equiv_l \mathcal{N}_r(b)$ then $a \in \mathcal{Q}(\mathcal{A}) \iff b \in \mathcal{Q}(\mathcal{A})$. Take $\mathcal{A}$ with a and b so that both $a, b \in U$ and have out degrees with respect to E to be l and $l + 1$ respectively. clearly $\mathcal{N}_1(a) \equiv_l \mathcal{N}_1(b)$ but still $a \in \mathcal{Q}(\mathcal{A})$ and $b \notin \mathcal{Q}(\mathcal{A})$.



3. *Structures of different sizes.*

Notice a kind of limitation in the definition of Hanf-locality. The notation $\mathcal{A} \leftrightarrows_r \mathcal{B}$ requires the existence of a bijection between the universes of $\mathcal{A}$ and $\mathcal{B}$. Hence Hanf-locality is inapplicable if we have structures of different sizes. For this reason we extend the notion of r–local equivalence to (r, m)–equivalence or the so called threshold-equivalence.

Definition 19. *Two structures $\mathcal{A}$ and $\mathcal{B}$ are (r,m)–equivalent if for every type τ of isomorphic neighborhoods, either $\mathcal{A}$ and $\mathcal{B}$ have the same number of points realizing τ or they both have at least m points realizing this type. This is denoted by: $\mathcal{A} \leftrightarrows_{r,m}^{thr} \mathcal{B}$.*

The usual definition of r–local equivalence is the special case of above (r, m)–equivalence with m allowed to be ∞. The following theorem is the refinement of Theorem 5 and often called the original Hanf criterion.

Theorem 12. *Let σ be a vocabulary and $\mathcal{A}, \mathcal{B}$ be σ-structures. For every positive k and d, with d being the maximum degree of any point in $\mathcal{A}$ and $\mathcal{B}$, there are positive integers r and m such that $\mathcal{A} \leftrightarrows_{r,m}^{thr} \mathcal{B}$ implies $\mathcal{A} \equiv_k \mathcal{B}$ where r depends only on k. Moreover, we may choose $r = 3^{k-1}$ and $m = kd^{r-1}$.*

□

Example 9. *We show that the Boolean query* $\mathcal{Q}_l$ *defined on the class* $\mathcal{C}$ *of simple cyclic graphs with loops on some nodes is not FO-expressible. That is, for any* $\mathcal{A} = \langle\{a_1, \dots, a_n\}, E\rangle$ *with* $(a_i, a_{i+1}) \in E^{\mathcal{A}}, i < n$, $(a_n, a_1) \in E^{\mathcal{A}}$ *as well as* $(a_i, a_i) \in E^{\mathcal{A}}$ *for some* $i < n$

$$\mathcal{Q}_l = \{\mathcal{A} : |(a_i, a_i) \in E^{\mathcal{A}}| = 0 \; mod \; 2\}$$

$\mathcal{Q}_l$ *checks whether there are an even number of loops in the graph.*

First we look at why Hanf-locality cannot be used. For any $\mathcal{A}$ and $\mathcal{B}$, the requirement $\mathcal{A} \leftrightarrows_r \mathcal{B}$ implies that the number of different r–neighborhood types are realized the same number of times in $\mathcal{A}$ and $\mathcal{B}$. This implies that the number of nodes with loops are also equal. However we can apply Theorem 12 to get two structures of different sizes. Let $\mathcal{Q}_l$ be definable by a sentence of quantifier rank k. In cyclic graphs the maximum degree of any point is 2. Let r and $m \geq 0$ be given by Theorem 12. Let $\mathcal{G}_{r,m}$ be a cycle with m loops such that the distance between two consecutive loops is $2(r+1)$. Then $\mathcal{G}_{r,m+1} \leftrightarrows^{thr}_{r,m} \mathcal{G}_{r,m+2}$. Since both structures realize the same types except for the number of points realizing those types (upto threshold). Now $\mathcal{Q}_l$ cannot be FO-expressible since it distinguishes $\mathcal{G}_{r,m+1}$ and $\mathcal{G}_{r,m+2}$.

4.2 Locality of FO with built-in predicates

Locality of first order logic with built-in predicate symbols has been largely studied, for example (Schweikardt, 2011, 2013). Prominently when only the linear order $\{\prec\}$ or the order and induced addition symbol $\{\prec, +\}$ are allowed with the condition that formulas and sentences we consider are $\{\prec, +\}$-invariant.

There is an obvious reason for allowing the extra symbol $\prec$. Since, a stored database on a drive generates an order (hidden let's say) and a query can use this order with the restriction that the output should not be affected by what the order stands for. The queries like $\mathcal{Q}_{even}$ can use this order and obviously it does not matter in which order is the information stored, the answer remains unchanged.

This independence of interpretation also allows us to talk about locality of logic with order since otherwise a linear order poses a problem while using locality as we discussed earlier. As suggested by Gaifman, using variations of those predicates which trivialize distance function.

First order logic with built-in order (or all numerical predicates induced by this order) is a whole different area to study and hence it cannot be discussed in detail in this thesis. Recall we defined the notion of order-invariant FO earlier. This linear order allows the interpretation of every numerical predicate from $\mathbb{N}$ to the structure $\mathcal{A}$.

Definition 20. *Let σ be a vocabulary and σ_{arb} be another vocabulary containing $\prec$ and a relation symbol R for each numerical predicate (that is, $R \subseteq \mathbb{N}^k$ for some k. Let $\mathcal{A}$ be a σ-structure.*

- *An arb-expansion of $\mathcal{A}$ is a $(\sigma \cup \sigma_{arb})$-structure $\mathcal{A}'$ such that dom$(\mathcal{A}') = \mathbf{A}$, the relations in σ are interpreted as in $\mathcal{A}$, $\prec$ is interpreted as linear order over $\mathbf{A}$. This interpretation induces a bijection $i : \mathbf{A} \rightarrow [n]$ where $n = |\mathbf{A}|$. Then every relation symbol $R \in \sigma_{arb}$ is interpreted over $\mathbf{A}$ via i and $R^{\mathbb{N}}$.*

- *A formula $\phi(\boldsymbol{x})$ is arb-invariant on $\mathcal{A}$ if for any tuple $\boldsymbol{a} \in \mathbf{A}$ and any* *two arb-expansions $\mathcal{A}_1, \mathcal{A}_2$ of $\mathcal{A}$, we have*

$$(\mathcal{A}, \mathcal{A}_1) \models \phi(\boldsymbol{a}) \iff (\mathcal{A}, \mathcal{A}_2) \models \phi(\boldsymbol{a}).$$

- *A formula $\phi(\boldsymbol{x})$ is arb-invariant if it is arb-invariant over every σ-structure $\mathcal{A}$.*

We denote the class of all arb-invariant FO-formulas by $(FO + Arb)_{inv}$. We sum up this thesis with a result from (Schweikardt, 2011) about arb-invariant first order logic.

Theorem 13. *Let σ be a vocabulary. Then every m-ary query in $(FO+Arb)_{inv}$ is Gaifman-local. Moreover the locality rank of any query $\mathcal{Q} \in (FO + Arb)_{inv}$ no longer depends only on $\mathcal{Q}$ but also on the sizes of the structures.*

□

5 Conclusion

We started this thesis with the need for tools in order to show inexpressibility of certain properties in finite model theory. The driving force being that compactness fails over finite structures. We also presented the fact that FO over finite structures is too strong that it can define a whole structure upto isomorphism by a single sentence and at the same time too weak that it cannot express some simple properties such as even cardinality of the domain (the query $\mathcal{Q}_{even}$).

Ehrenfeucht-Fraïssé games, even though worked in case of finite structures turned out to be sort of complex since every time we used this result a winning strategy must to be presented.

The burden was lowered by the use of Hanf's criterion for guaranteeing winning strategy for the duplicator with the assumption that the structure look alike at the local level. Our main result is that FO is Hanf-local. Hence, rather than forming the duplicator's winning strategy for k-rounds, we only need to prove that the 3^{k-1}–neighborhoods in two structures are isomorphic. This property allowed us to infer that FO can only define local properties of finite structures. Thus in order to show that a certain property is not FO-expressible we simply prove that it is a global property.

We also described another locality criterion given by Gaifman in which we consider one structure and look at neighborhoods of two points. If these neighborhoods are isomorphic a FO-formula cannot distinguish the two points. It turned out that first order logic enjoys Gaifman-locality as well which appeared to be a nice tool for showing inexpressibility of non-Boolean queries.

Finally we discussed three limitations of these locality criteria; isomorphism of neighborhoods being too strong condition, the major one. Even though this condition was relaxed to neighborhoods being k-equivalent it is found in (Arenas, Barceló, & Libkin, 2008) that the Hanf-locality is then lost. Moreover the implication Hanf-locality $\implies$ Gaifman-locality also fails.

Threshold equivalence turned out to an efficient refinement of Hanf-locality since structures of different sizes were also added in to the discussion.

As a final word, the writer believes that Theorem 12 can be further strengthened by diverting the attention from isomorphism-based locality to the game-based. That is, Theorem 11 together with Theorem 12 implies that the condition of having the maximum degree to be some fixed d can be eliminated. So that if some elements have larger degrees we can still talk about k-rounds of EF-game for some k and thereby lowering the condition of isomorphism to k-equivalence.

References

Arenas, M., Barceló, P., and Libkin, L. (2008). Game-based notions of locality over finite models. *Annals of Pure and Applied Logic*, 152(1-3):3–30.

Dawar, A. (2008). Finite model theory tutorial. In *Lecture slides 2*. Modnet summer school, Manchester.

Fagin, R. (1997). Easier ways to win logical games. *Descriptive complexity and finite models*, 31:1–32.

Fagin, R., Stockmeyer, L. J., and Vardi, M. Y. (1995). On monadic np vs monadic co-np. *Information and Computation*, 120(1):78–92.

Gaifman, H. (1982). On local and non-local properties. In *Studies in Logic and the Foundations of Mathematics*, volume 107, pages 105–135. Elsevier.

Grohe, M. and Schwentick, T. (2000). Locality of order-invariant first-order formulas. *ACM Transactions on Computational Logic (TOCL)*, 1(1):112–130.

Gurevich, Y. (1984). Toward logic tailored for computational complexity. In *Computation and proof theory*, pages 175–216. Springer.

Hella, L., Libkin, L., and Nurmonen, J. (1999). Notions of locality and their logical characterizations over finite models. *The Journal of Symbolic Logic*, 64(4):1751–1773.

Libkin, L. (2001). Logics capturing local properties. *ACM Transactions on Computational Logic (TOCL)*, 2(1):135–153.

Libkin, L. (2006). Locality of queries and transformations. *Electronic Notes in Theoretical Computer Science*, 143:115–127.

Libkin, L. (2013). *Elements of finite model theory*. Springer Science & Business Media.

Montanari, A. and Vitacolonna, N. (2015). Ehrenfeucht-fraïssé games: Applications and complexity. In *Lecture slides*. Department of Mathematics and Computer Science University of Udine, Italy.

Niemistö, H. et al. (2007). Locality and order-invariant logics.

Otto, M. (2005-06). Finite model theory. In *Winter term course*.

Pichler, R. (2017). Ehrenfeucht-fraïssé games. In *Lecture slides on Database theory*. Institut für Informationssysteme Arbeitsbereich DBAI Technische Universität Wien.

Schweikardt, N. (2012). On the expressive power of logics with invariant uses of arithmetic predicates. In *International Workshop on Logic, Language, Information, and Computation*, pages 85–87. Springer.