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On Path-Lifting Mechanisms and Unwinding in Quaternion-based Attitude Control*

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***Abstract*—The unit quaternion is a pervasive representation of rigid-body attitude used for the design and analysis of feedback control laws. Because the space of unit quaternions constitutes a double cover of the rigid-body attitude space, quaternion-based control laws are often—by design—inconsistent, i.e., they do not have a unique value for each rigid-body attitude. Inconsistent quaternion-based control laws require an additional mechanism that uniquely convert an attitude estimate into its quaternion representation; however, conversion mechanisms that are memoryless—e.g., selecting the quaternion having positive scalar component—have a limited domain where they remain injective and, when used globally, introduce discontinuities into the closed-loop system. We show—through an explicit construction and Lyapunov analysis—that such discontinuities can be hijacked by arbitrarily small measurement disturbances to stabilize attitudes far from the desired attitude. To remedy this limitation, we propose a hybrid-dynamic algorithm for smoothly lifting an attitude path to the unit-quaternion space. We show that this hybrid-dynamic mechanism allows us to directly translate quaternion-based controllers and their asymptotic stability properties (obtained in the unit-quaternion space) to the actual rigid-body-attitude space. We also show that when quaternion-based controllers are not designed to account for the double covering of the rigid-body-attitude space by a unit-quaternion parameterization, they can give rise to the unwinding phenomenon, which we characterize in terms of the projection of asymptotically stable sets. Finally, we employ the main results to show that certain hybrid feedbacks can globally asymptotically stabilize the attitude of a rigid body.**

I. Introduction

Controlling the attitude of a rigid body is one of the canonical nonlinear control problems, with applications in aerospace and publications spanning many decades [1]–[5]. A fundamental characteristic of attitude control that imparts a fascinating difficulty is the topological complexity of the underlying state space of rotation matrices, $\mathsf{SO}(3)$: a boundaryless compact manifold that is not diffeomorphic to any vector space. This property of $\mathsf{SO}(3)$ precludes the existence of a continuous time-invariant state-feedback control law that globally asymptotically stabilizes a particular attitude [6], [7]. For the same reason, no periodic or discontinuous feedback can *robustly* globally asymptotically stabilize a particular attitude [8].

Often, unit quaternions are used to parametrize $\mathsf{SO}(3)$. While this parametrization yields the minimal globally nonsingular[1] parametrization of rigid-body attitude [9], its state space, $\mathbb{S}^3$ (the set of unit-magnitude vectors in $\mathbb{R}^4$) is a double cover of $\mathsf{SO}(3)$. That is, there are two (antipodal) unit quaternions corresponding to every rigid-body attitude. This creates the need to stabilize a disconnected set in the covering space [5], which has its own topological obstructions [10]. As discussed in [6], these topological subtleties can cause confusion and sometimes, lead to dubious claims regarding the globality of asymptotic stability (see e.g. [1], [11]). Nevertheless, unit quaternions are still used by many authors (including the authors of this paper) today to design feedback control algorithms for attitude control.

A feedback controller designed using a quaternion representation of attitude may not be *consistent* with a control law defined on $\mathsf{SO}(3)$. That is, for every rigid-body attitude, the quaternion-based feedback may take on one of two possible values. When this is the case, analysis for quaternion-based feedback is often carried out in $\mathbb{S}^3$ with a lifted dynamic equation. However, such analyses are not directly related to a feedback system defined on $\mathsf{SO}(3)$. This obviously begs the following questions. How is a unit quaternion representation obtained from available measurements? On what state-space is an inconsistent quaternion-based feedback defined? How is stability analysis done in the covering space related to a stability result for the actual system?

Given an estimated attitude, it is a fairly simple operation to compute the corresponding set of unit quaternions (see e.g. [12], [13]); however, the process of selecting which quaternion to use for feedback is a less obvious operation. As noted in [4], it is often the case that the quaternion with positive "scalar" component is used for feedback. This operation is non-global and discontinuous. As we show in this work, the act of paring such a discontinuous quaternion-selection scheme with a widely used inconsistent quaternion-based feedback opens the door for an undesirable chattering effect. In fact, we construct an explicit disturbance—defined on $\mathsf{SO}(3)$—that exploits the discontinuity to stabilize a region about the manifold of $180°$ rotations with zero angular velocity.

To remedy this behavior, we propose a hybrid-dynamic algorithm for smoothly lifting path from $\mathsf{SO}(3)$ onto $\mathbb{S}^3$. Our approach allows us to make an equivalence between any

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[1]The term "globally nonsingular" here means that the covering map from $\mathbb{S}^3$ to $\mathsf{SO}(3)$ is everywhere a local diffeomorphism.

asymptotic (in)stability result for a closed-loop system in the covering space and a corresponding (in)stability result for the actual plant. This justifies carrying out stability analysis in a unit-quaternion setting; however, when a quaternion-based feedback does not respect the two-to-one covering of $\mathsf{SO}(3)$, this translated stability result may not be desirable.

Often, quaternion-based feedbacks are designed to stabilize only one of two quaternions corresponding to the desired attitude. When these inconsistent feedbacks are paired with a path-lifting algorithm, they cause the so-called "unwinding phenomenon," where the feedback can unnecessarily rotate the rigid body through a full rotation. This behavior was discussed at length in [6] in terms of lifts of paths and vector fields from $\mathsf{SO}(3)$ to $\mathbb{S}^3$. In this paper, we characterize unwinding in terms of asymptotically stable sets in an extended state space projected onto the plant state space.

In practice, an explicit measurement of attitude is not available. Instead, the attitude must be reconstructed from measurements of known inertial-frame vectors expressed in body-frame coordinates [14]. With measurements of at least two such linearly independent vectors, the attitude can be algebraically reconstructed by in various ways, such as solving a least-squares problem (often called "Wahba's problem" [15]) [16], [17]. When using only a static attitude-reconstruction algorithm, a path-lifting mechanism (like the one herein proposed) is necessary to choose the quaternion consistently if an inconsistent feedback is used. Alternatively, dynamical filters can be used to estimate the attitude from vector observations (or IMU measurements) or from the results of static attitude attitude-reconstruction algorithms [18], [19]. Regardless of the process that ultimately forms an estimate of attitude, the message of this work is clear: when an inconsistent quaternion-based feedback is used, a dynamic mechanism is needed to resolve the ambiguity in which quaternion is used for feedback. Furthermore, regardless of the mechanism that fills this role (e.g. the hybrid algorithm proposed herein or a dynamic filter as in [18], [19]), the additional state(s) of the mechanism should be considered to correctly assess the stability properties of the closed-loop system and to rule out any possibility of unwinding.

This paper is organized as follows. Section II provides the background material for attitude control and hybrid systems used in this paper. Section III reconstructs the "select-the-quaternion-with-positive-scalar-component" mechanism in terms of a static map that selects a quaternion according to a metric. In Section IV we show by Lyapunov analysis that, when composed with a widely used inconsistent feedback, the aforementioned quaternion-selection scheme makes the closed-loop system susceptible to arbitrarily small measurement disturbances that can act to stabilize attitudes far from the desired attitude. Section V constructs a hybrid-dynamic system that smoothly lifts paths from $\mathsf{SO}(3)$ to $\mathbb{S}^3$. We couple this system with a quaternion-based feedback in Section VI and establish an equivalence of stability between two closed systems: one is defined in the unit-quaternion space and the other one is defined in the rigid-body-attitude space extended by a unit-quaternion memory state. Section VII discusses the unwinding phenomenon in terms of the projection of asymptotically stable sets and suggests how to avoid the behavior. Finally, we present conclusions in Section VIII.

II. Preliminaries

A. Notation

In this paper, $\mathbb{R}$ ($\mathbb{R}_{\geq 0}$) denotes the (nonnegative) real numbers, $\mathbb{R}^n$ denotes n-dimensional Euclidean space, and $\mathbb{R}^{m\times n}$ denotes the vector space of $m\times n$ real matrices. Given vectors $x, y \in \mathbb{R}^n$ and matrices $A, B \in \mathbb{R}^{m\times n}$, their inner products are defined as $\langle x, y\rangle := x^\top y$ and $\langle A, B\rangle := \mathrm{trace}(A^\top B)$, respectively. The 2-norm of a vector $y \in \mathbb{R}^n$ is $|y| = \sqrt{\langle y, y\rangle}$ and the Frobenius norm of a matrix $A \in \mathbb{R}^{n\times m}$ is $\|A\|_F = \sqrt{\langle A, A\rangle}$. The n-dimensional unit sphere embedded in $\mathbb{R}^{n+1}$ is denoted as $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$, the closed unit ball in $\mathbb{R}^n$ is $\mathbb{B} = \{x \in \mathbb{R}^n : |x| \leq 1\}$, and the closed unit ball in $\mathbb{R}^{m\times n}$ is $\mathbb{B} = \{A \in \mathbb{R}^{m\times n} : \|A\|_F \leq 1\}$. A set-valued map is denoted as $\rightrightarrows$. That is, $F : X \rightrightarrows Y$ indicates that for each $x \in X$, $F(x) \subset Y$.

Given differentiable functions $h : \mathbb{R}^n \to \mathbb{R}$, and $k : \mathbb{R}^{m\times n} \to \mathbb{R}$, we denote their gradients as $\nabla h : \mathbb{R}^n \to \mathbb{R}^n$ and $\nabla k : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$. That is,

$$\nabla h(x) = \begin{bmatrix} \dfrac{\partial h(x)}{\partial x_1} \\ \vdots \\ \dfrac{\partial h(x)}{\partial x_n} \end{bmatrix} \quad \nabla k(x) = \begin{bmatrix} \dfrac{\partial k(x)}{\partial x_{11}} & \cdots & \dfrac{\partial k(x)}{\partial x_{1n}} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial k(x)}{\partial x_{m1}} & \cdots & \dfrac{\partial k(x)}{\partial x_{mn}} \end{bmatrix}. \tag{1}$$

Let $y : \mathbb{R} \to \mathbb{R}^n$ and $z : \mathbb{R} \to \mathbb{R}^{m\times n}$ be differentiable functions and define $\alpha = h \circ y$ and $\beta = k \circ z$. Then, the matrix calculus by vectorization [20] yields the consistent notation

$$\begin{aligned} \dot{\alpha}(t) &= \nabla h(y(t))^\top \dot{y}(t) = \langle \nabla h(y(t)), \dot{y}(t)\rangle \\ \dot{\beta}(t) &= \mathrm{trace}(\nabla k(z(t))^\top \dot{z}(t)) = \langle \nabla k(z(t)), \dot{z}(t)\rangle . \end{aligned}$$

B. Attitude kinematics, dynamics, and representation by unit quaternions

The attitude of a rigid body is defined as the relative rotation of a body-fixed frame to an inertial frame and is represented by a 3×3 orthogonal matrix with unitary determinant: an element of the special orthogonal group of order three,

$$\mathsf{SO}(3) = \left\{R \in \mathbb{R}^{3\times 3} : R^\top R = I,\ \det R = 1\right\}.$$

The kinematic and dynamic equations of a rigid body are

$$\dot{R} = R\,[\omega]_\times \tag{2a}$$

$$J\dot{\omega} = [J\omega]_\times\,\omega + \tau, \tag{2b}$$

respectively, where $R \in \mathsf{SO}(3)$ is the attitude, $\omega \in \mathbb{R}^3$ is the the angular velocity given in the body-fixed frame, $J = J^\top > 0$ is the inertia matrix, $\tau \in \mathbb{R}^3$ is an external torque, and the cross product between vectors $y, z \in \mathbb{R}^3$, is defined by a matrix multiplication: $y \times z = [y]_\times\, z$, where

$$[y]_\times = \begin{bmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{bmatrix}.$$

Members of $\mathsf{SO}(3)$ are often parametrized in terms of a rotation $\theta \in \mathbb{R}$ about a fixed axis $u \in \mathbb{S}^2$ by the so-called Rodrigues formula: the map $\mathcal{U} : \mathbb{R} \times \mathbb{S}^2 \to \mathsf{SO}(3)$ defined as

$$\mathcal{U}(\theta, u) = I + \sin(\theta) [u]_\times + (1 - \cos(\theta)) [u]_\times^2 . \tag{3}$$

The unit-quaternion parametrization of $\mathsf{SO}(3)$ associates every element of $\mathsf{SO}(3)$ with two elements of $\mathbb{S}^3$. In the sense of (3), a unit quaternion q is defined as

$$q = \begin{bmatrix} \eta & \epsilon^\top \end{bmatrix}^\top = \pm \begin{bmatrix} \cos(\theta/2) & \sin(\theta/2)u^\top \end{bmatrix}^\top \in \mathbb{S}^3 \tag{4}$$

and represents an element of $\mathsf{SO}(3)$ through the map $\mathcal{R} : \mathbb{S}^3 \to \mathsf{SO}(3)$ defined as

$$\mathcal{R}(q) = I + 2\eta [\epsilon]_\times + 2 [\epsilon]_\times^2 . \tag{5}$$

Note the important property that for $q_1 \neq q_2 \in \mathbb{S}^3$, $\mathcal{R}(q_1) = \mathcal{R}(q_2)$ if and only if $q_1 = -q_2$. We denote the double-valued inverse map $\mathcal{Q} : \mathsf{SO}(3) \rightrightarrows \mathbb{S}^3$ as

$$\mathcal{Q}(R) = \{q \in \mathbb{S}^3 : \mathcal{R}(q) = R\}. \tag{6}$$

Conveniently, we will often write a quaternion as a pair $q = (\eta, \epsilon)$, rather than as a vector.

With the identity element $\mathbf{i} = (1, 0) \in \mathbb{S}^3$, each unit quaternion $q \in \mathbb{S}^3$ has an inverse $q^{-1} = (\eta, -\epsilon)$ under the quaternion multiplication rule

$$q_1 \odot q_2 = \begin{bmatrix} \eta_1\eta_2 - \epsilon_1^\top \epsilon_2 & (\eta_1\epsilon_2 + \eta_2\epsilon_1 + [\epsilon_1]_\times \epsilon_2)^\top \end{bmatrix}^\top ,$$

where $q_i = (\eta_i, \epsilon_i) \in \mathbb{R}^4$ and $i \in \{1, 2\}$. Then, the map $\mathcal{R}$ is a group homomorphism satisfying

$$\mathcal{R}(q_1)\mathcal{R}(q_2) = \mathcal{R}(q_1 \odot q_2). \tag{7}$$

The manifold $\mathbb{S}^3$ is a *covering space* for $\mathsf{SO}(3)$ and $\mathcal{R} : \mathbb{S}^3 \to \mathsf{SO}(3)$ is the *covering map*. Precisely, for every $R \in \mathsf{SO}(3)$, there exists an open neighborhood $U \subset \mathsf{SO}(3)$ of R such that $\mathcal{Q}(U) = \mathcal{O}_1 \cup \mathcal{O}_2$, where $\mathcal{O}_1, \mathcal{O}_2 \subset \mathbb{S}^3$ are open, $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$, and for each $k \in \{1, 2\}$, the restriction of $\mathcal{R}$ to $\mathcal{O}_k$ is a diffeomorphism. In particular, $\mathcal{R}$ is everywhere a local diffeomorphism.

A fundamental property of a covering space is that a continuous path in the base space can be uniquely "lifted" to a continuous path in the covering space once a base point is specified. In terms of $\mathsf{SO}(3)$ and $\mathbb{S}^3$, for every continuous path $R : [0, 1] \to \mathsf{SO}(3)$ and for every $p \in \mathcal{Q}(R(0))$, there exists a unique continuous path $q_p : [0, 1] \to \mathbb{S}^3$ satisfying $q_p(0) = p$ and $\mathcal{R}(q_p(t)) = R(t)$ for every $t \in [0, 1]$ [21, Theorem 54.1]. We call any such path q_p a lift of R over $\mathcal{R}$. We refer the reader to see [21], [22] for general information about covering spaces.

In addition to paths, vector fields defined on $\mathsf{SO}(3)$ can be lifted onto $\mathbb{S}^3$ as well [6]. In this direction, given a Lebesgue-measurable function $\omega : [0, 1] \to \mathbb{R}^3$ and an absolutely continuous path $R : [0, 1] \to \mathsf{SO}(3)$ satisfying (2a) for almost all $t \in [0, 1]$, any $q : [0, 1] \to \mathbb{S}^3$ that is a lift of R over $\mathcal{R}$ satisfies the *quaternion kinematic equation*

$$\dot{q} = \begin{bmatrix} \dot{\eta} \\ \dot{\epsilon} \end{bmatrix} = \frac{1}{2} q \odot \nu(\omega) = \frac{1}{2}\Lambda(q)\omega, \tag{8}$$

for almost all $t \in [0, 1]$, where the maps $\nu : \mathbb{R}^3 \to \mathbb{R}^4$ and $\Lambda : \mathbb{S}^3 \to \mathbb{R}^{4\times3}$ are defined as

$$\nu(\omega) = \begin{bmatrix} 0 \\ \omega \end{bmatrix} \qquad \Lambda(q) = \begin{bmatrix} -\epsilon^\top \\ \eta I + [\epsilon]_\times \end{bmatrix}.$$

C. Hybrid systems framework

In this work, we appeal to the hybrid systems framework [23], [24]. This is in part due to the fact that the authors have developed quaternion-based hybrid feedback controllers that achieve global asymptotic stabilization of rigid-body attitude in [5], [25], [26] and also because the path-lifting algorithm presented here is hybrid. A hybrid system allows for both continuous and discrete evolution of the state. A hybrid system $\mathcal{H}$ with state $x \in \mathbb{R}^n$ is defined by four objects: a *flow map*, $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, governing continuous evolution of the state by a differential inclusion, a *jump map*, $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, governing discrete evolution of the state by a difference inclusion, a *flow set*, $C \subset \mathbb{R}^n$, dictating where continuous state evolution is allowed, and a *jump set*, $D \subset \mathbb{R}^n$, dictating where discrete state evolution is allowed. We write a hybrid system in the compact form,

$$\mathcal{H} \begin{cases} \dot{x} \in F(x) & x \in C \\ x^+ \in G(x) & x \in D. \end{cases}$$

Often, we will refer to a hybrid system by its data as $\mathcal{H} = (F, G, C, D)$.

Solutions to hybrid systems are defined on *hybrid time domains* and are parametrized by t, the amount of time spent flowing and j, the number of jumps that have occurred. A *compact hybrid time domain* is a set $E \subset \mathbb{R}_{\geq0} \times \mathbb{Z}_{\geq0}$ of the form

$$E = \bigcup_{j=0}^{J} ([t_j, t_{j+1}], j), \tag{9}$$

where J is a nonnegative integer, $0 = t_0 \leq t_1 \leq \cdots \leq t_{J+1}$. We say that E is a *hybrid time domain* if, for each $(T, J) \in E$, the set $E \cap ([0, T] \times \{0, 1, \ldots, J\})$ is a compact hybrid time domain. On every hybrid time domain, points are naturally ordered as $(t, j) \preceq (s', k')$ if $t + j \leq s + k$ and $(t, j) \prec (s, k)$ if $t + j < s + k$.

A *hybrid arc* is a function $x : \operatorname{dom} x \to \mathbb{R}^n$, where $\operatorname{dom} x$ is a hybrid time domain and, for each fixed j, the map $t \to x(t, j)$ is a locally absolutely continuous function on the interval

$$\mathcal{I}_j = \{t : (t, j) \in \operatorname{dom} x\}. \tag{10}$$

When a hybrid arc has several components, we adopt the economical notation

$$(x_1(t, j), \ldots, x_k(t, j)) = (x_1, \ldots, x_k)|_{(t,j)}.$$

A hybrid arc x is a *solution to the hybrid system* $\mathcal{H} = (F, G, C, D)$ if $x(0, 0) \in C \cup D$ and

1) for each $j \in \mathbb{Z}_{\geq0}$ such that $\mathcal{I}_j$ has nonempty interior, $\dot{x}(t, j) \in F(x(t, j))$ for almost all $t \in \mathcal{I}_j$ and $x(t, j) \in C$ for all $t \in [\min \mathcal{I}_j, \sup \mathcal{I}_j)$,
2) for each $(t, j) \in \operatorname{dom} x$ such that $(t, j + 1) \in \operatorname{dom} x$, $x(t, j + 1) \in G(x(t, j))$ and $x(t, j) \in D$.

Solutions are not unique if G is multi-valued for some $x \in D$, there is more than one flowing solution from some $x \in C$, or it is possible to flow from some point $x \in C \cap D$.

A solution x to $\mathcal{H}$ is *maximal* if it is not a truncation of another solution and it is *complete* if $\operatorname{dom} x$ is unbounded. Given a hybrid arc x, let $\bar{T}(x) = \sup\{t : \exists j \in \mathbb{Z}_{\geq 0}\ (t,j) \in \operatorname{dom} x\}$ and let $\bar{J}(t) = \max\{j : (t,j) \in \operatorname{dom} x\}$. Then, the *time projection* of x is the function $x\downarrow_t : [0, \bar{T}(x)) \to \mathbb{R}^n$ defined as

$$x\downarrow_t(t) = x(t, \bar{J}(t)). \tag{11}$$

In this work, we assume that the hybrid system $\mathcal{H}$ satisfies the *hybrid basic conditions*:

1) C and D are closed sets in $\mathbb{R}^n$.
2) $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is an outer semicontinuous[2] set-valued mapping, locally bounded on C, and such that $F(x)$ is nonempty and convex for each $x \in C$.
3) $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is an outer semicontinuous set-valued mapping, locally bounded on D, and such that $G(x)$ is nonempty for each $x \in D$.

These properties ensure, among other things, that asymptotic stability is nominally robust [24].

A compact set $\mathcal{A} \subset \mathbb{R}^n$ is *stable* for $\mathcal{H}$ if for each open set $U_\epsilon \supset \mathcal{A}$, there exists an open set $U_\delta \supset \mathcal{A}$ such that for each solution $x : \operatorname{dom} x \to \mathbb{R}^n$ to $\mathcal{H}$ satisfying $x(0,0) \in U_\delta$, it follows that $x(t,j) \in U_\epsilon$ for all $(t,j) \in \operatorname{dom} x$. A compact set $\mathcal{A}$ is *unstable* if it is not stable. A set $\mathcal{A}$ is attractive from a set $\mathcal{B}$ if each solution with initial condition in $\mathcal{B}$ converges to $\mathcal{A}$, i.e., for each solution $x : \operatorname{dom} x \to \mathbb{R}^n$ with $x(0,0) \in \mathcal{B}$ and each open set $U_\epsilon \supset \mathcal{A}$, there exists $T > 0$ such that $x(t,j) \in U_\epsilon$ for all $(t,j) \in \operatorname{dom} x$ satisfying $t + j \geq T$. The set of points in $\mathbb{R}^n$ from which each solution is complete, bounded, and converges to $\mathcal{A}$ is called the *basin of attraction* of $\mathcal{A}$. Note that each point in $\mathbb{R}^n \setminus (C \cup D)$ belongs to the basin of attraction of any set $\mathcal{A}$, since no solutions exist from these points. A compact set $\mathcal{A}$ is *asymptotically stable* if it is stable and attractive from an open neighborhood of $\mathcal{A}$ and is *globally asymptotically stable* if its basin of attraction is $\mathbb{R}^n$.

Finally, we remark that while the above definitions are written in terms of $\mathbb{R}^n$, they equally apply to manifolds embedded in $\mathbb{R}^n$. In particular, they apply to the state spaces that we will be using in this paper: $\mathbb{S}^3$, $\mathsf{SO}(3)$, and discrete sets of logic variables.

III. Inconsistent Quaternion-based Feedback and Memoryless Path Lifting

It is quite commonplace to design an attitude control law based upon a quaternion representation. That is, the control designer creates a continuous function $\kappa : \mathbb{S}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ and closes a feedback loop around (2) by setting $\tau(t) = \kappa(q(t), \omega(t))$, where $q(t)$ is selected to satisfy $\mathcal{R}(q(t)) = R(t)$, for each $t \in \mathbb{R}_{\geq 0}$. When the feedback κ satisfies

$$\kappa(q,\omega) = \kappa(-q,\omega) \qquad \forall (q,\omega) \in \mathbb{S}^3 \times \mathbb{R}^3, \tag{12}$$

[2] A set-valued map $F : X \rightrightarrows Y$ is outer semicontinuous if the set $\{(x,y) \in X \times Y : y \in F(x)\}$ is closed. It is locally bounded on C if for each compact K, $F(K)$ is bounded.

we say that κ is *consistent*. Smooth and consistent feedback control algorithms are investigated in [27] for adaptive attitude control without angular velocity measurements and recently in [28] for attitude synchronization of a formation of spacecraft. In such cases, there is little need for a quaternion representation for analysis, as κ could be defined in terms of $R \in \mathsf{SO}(3)$.

When a quaternion-based feedback is *inconsistent*, that is,

$$\exists (q,\omega) \in \mathbb{S}^3 \times \mathbb{R}^3 \qquad \kappa(q,\omega) \neq \kappa(-q,\omega), \tag{13}$$

the resulting feedback *does not define a unique vector field on* $\mathsf{SO}(3) \times \mathbb{R}^3$ because for $R \in \mathsf{SO}(3)$ satisfying $\mathcal{Q}(R) = \{-q, q\}$, the feedback $\kappa(\mathcal{Q}(R), \omega)$ is a two-element set [6]. At this point, the control designer must, for every $t \in \mathbb{R}_{\geq 0}$, choose which $q(t) \in \mathcal{Q}(R(t))$ to use for feedback. In this direction, we provide a quote from the seminal paper [4]:

> "In many quaternion extraction algorithms, the sign of [the 'scalar' part of the quaternion] is arbitrarily chosen positive. This approach is not used here, instead, the sign ambiguity is resolved by choosing the one that satisfies the associated kinematic differential equation. In implementation, this would probably imply keeping some immediate past values of the quaternion."

There is much to be gleaned from this quotation. In particular, it suggests that inconsistent quaternion-based control laws require an extra memory state to lift a trajectory from $\mathsf{SO}(3)$ to a trajectory in $\mathbb{S}^3$. In what follows, we reconstruct the discontinuous quaternion "extraction" algorithm mentioned in the quotation above in terms of a metric and use the ensuing discussion to motivate a hybrid algorithm for on-line lifting of an attitude trajectory from $\mathsf{SO}(3)$ to $\mathbb{S}^3$.

We define a metric $d : \mathbb{S}^3 \times \mathbb{S}^3 \to [0,2]$ and an associated distance function from $q \in \mathbb{S}^3$ to a set $Q \subset \mathbb{S}^3$ as

$$d(q,p) = 1 - q^\top p, \quad \operatorname{dist}(q,Q) = \inf\{d(q,p) : p \in Q\}. \tag{14}$$

From a geometric viewpoint, $d(q,p)$ is the height of $p \in \mathbb{S}^3$ "above" the plane orthogonal to the vector $q \in \mathbb{S}^3$ at q. When the set Q in (14) takes the form of $\mathcal{Q}(R)$ for some $R \in \mathsf{SO}(3)$, the distance function also takes a special form. In particular, let $\mathcal{Q}(R) = \{p, -p\}$. Then, $\operatorname{dist}(q, \mathcal{Q}(R)) = 1 - |q^\top p|$.

One possible method to lift a path from $\mathsf{SO}(3)$ to $\mathbb{S}^3$ is to simply pick the quaternion representation of R that is closest to a specific quaternion in terms of the metric d. In particular, let us define the map $\Phi : \mathbb{S}^3 \times \mathsf{SO}(3) \rightrightarrows \mathbb{S}^3$ as

$$\Phi(q,R) = \operatorname*{argmin}_{p \in \mathcal{Q}(R)} d(q,p) = \operatorname*{argmax}_{p \in \mathcal{Q}(R)} q^\top p. \tag{15}$$

The map Φ has some useful properties, which we summarize in the following lemmas.

Lemma 1. *Let $q \in \mathbb{S}^3$ and $R \in \mathsf{SO}(3)$. The following are equivalent:*

1) $\Phi(q,R)$ *is single-valued and* $q^\top \Phi(q,R) > 0$
2) $0 \leq \operatorname{dist}(q, \mathcal{Q}(R)) < 1$
3) $q^\top p \neq 0$ *for all* $p \in \mathcal{Q}(R)$
4) $R \neq \mathcal{U}(\pi, u)\mathcal{R}(q)$ *for any* $u \in \mathbb{S}^2$, *where the map* $\mathcal{U} : \mathbb{R} \times \mathbb{S}^3 \to \mathsf{SO}(3)$ *was defined in* (3).

Proof: For the remainder of this proof, we let $\mathcal{Q}(R) = \{p, -p\}$. By the definition of Φ in (15), we see that $\Phi(q, R)$ is single-valued if and only if $d(q, p) \neq d(q, -p) \Leftrightarrow 1 - q^\top p \neq 1 + q^\top p \Leftrightarrow q^\top p \neq -q^\top p \Leftrightarrow q^\top p \neq 0 \Leftrightarrow 0 \leq \text{dist}(q, \mathcal{Q}(R)) < 1$. This provides an equivalence between 1), 2), and 3), above.

Now, let $\theta \in \mathbb{R}$ and $u \in \mathbb{S}^2$ be such that $R = \mathcal{U}(\theta, u)\mathcal{R}(q)$. Since $R = \mathcal{R}(\pm p)$, the fact that $\mathcal{R}$ satisfies (7) provides the following equivalent series of expressions: $\mathcal{R}(p) = \mathcal{U}(\theta, u)\mathcal{R}(q) \Leftrightarrow \mathcal{R}(p)\mathcal{R}(q)^{-1} = \mathcal{U}(\theta, u) \Leftrightarrow \mathcal{R}(p \odot q^{-1}) = \mathcal{U}(\theta, u)$.

Now, since $p \odot q^{-1} = (p^\top q, *)$, the form of $\mathcal{R} : \mathbb{S}^3 \to \mathsf{SO}(3)$ in (5) guarantees that $\mathcal{R}(p \odot q^{-1}) = \mathcal{R}(p \odot q^{-1})^\top \neq I$ if and only if $p^\top q = 0$. But $\mathcal{U}(\theta, u) = \mathcal{U}(\theta, u)^\top \neq I$ if and only if $\sin\theta = 0$ and $\cos\theta = -1$, which is satisfied for $\theta = \pi$. ■

Lemma 2. *For every $\hat{q} \in \mathbb{S}^3$, every continuous $R : [0, 1] \to \mathsf{SO}(3)$, and every continuous $q : [0, 1] \to \mathbb{S}^3$ satisfying $d(\hat{q}, q(0)) < 1$ and for all $t \in [0, 1]$ $\mathcal{R}(q(t)) = R(t)$ and $\text{dist}(\hat{q}, \mathcal{Q}(R(t))) < 1$, it follows that $\Phi(\hat{q}, R(t)) = q(t)$ for all $t \in [0, 1]$.*

Proof: Under the assumptions of the lemma, suppose further that for some $t' \in [0, 1]$, $\Phi(\hat{q}, R(t')) = -q(t')$. This implies that $d(\hat{q}, -q(t')) < d(\hat{q}, q(t'))$ and that $d(\hat{q}, q(t')) > 1$. But since $q(t)$ is continuous and $d(\hat{q}, q(0)) < 1$, it follows that $d(\hat{q}, q(t))$ is continuous and from the intermediate value theorem, there exists $t^* \in [0, t']$ such that $d(\hat{q}, q(t^*)) = d(\hat{q}, -q(t^*)) = \text{dist}(\hat{q}, \mathcal{Q}(R(t^*))) = 1$. This is a contradiction. ■

Lemma 3. *For all $\hat{q} \in \mathbb{S}^3$ and $R \in \mathsf{SO}(3)$ satisfying $\text{dist}(\hat{q}, \mathcal{Q}(R)) < 1$, it follows that*

$$\Phi(\Phi(\hat{q}, R), R) = \Phi(\hat{q}, R). \tag{16}$$

Proof: Without loss of generality, let $\mathcal{Q}(R) = \{q, -q\}$ and $\text{dist}(\hat{q}, q) < 1$. Then,

$$\Phi(\Phi(\hat{q}, R), R) = \underset{q' \in \{q, -q\}}{\text{argmin}} \ \text{dist}(q, q') = q,$$

so that $\Phi(\Phi(\hat{q}, R), R) = \Phi(\hat{q}, R)$. ■

Since a goal of attitude control is to regulate R to I (or, in general, an error attitude to I), one might choose $\mathbf{i}$ as a point of reference (since $\mathcal{R}(\mathbf{i}) = I$) and use the map $\Phi_\mathbf{i} : \mathsf{SO}(3) \rightrightarrows \mathbb{S}^3$ defined as

$$\Phi_\mathbf{i}(R) = \Phi(\mathbf{i}, R) \qquad \forall R \in \mathsf{SO}(3). \tag{17}$$

Now, following 3) from Lemma 1 we see that $\mathbf{i}^\top \Phi_\mathbf{i}(R) > 0$, that is, $\Phi_\mathbf{i}$ always chooses the quaternion with positive scalar component, so long as it is single-valued. Further, Lemma 2 allows one to lift curves with $\Phi_\mathbf{i}$ so long as R does not cross the manifold of $180°$ rotations where $\Phi_\mathbf{i}$ is multi-valued, or else $\Phi_\mathbf{i}$ will produce a quaternion trajectory that is discontinuous. As we now show, this leads to an undesirable chattering effect when $\Phi_\mathbf{i}$ is composed with an inconsistent feedback.

IV. Non-Robustness

Let $c > 0$ and let $\Psi : \mathbb{R}^3 \to \mathbb{R}^3$ be a continuous function satisfying

$$\gamma(|\omega|) \leq \omega^\top \Psi(\omega), \tag{18}$$

where $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a strictly increasing continuous function satisfying $\gamma(0) = 0$.

$$\mathcal{E}(q) = \mathcal{E}(\eta, \epsilon) = \Lambda(q)^\top \mathbf{i} = \epsilon \tag{19}$$

and consider the inconsistent feedback

$$\kappa^*(q, \omega) = -c\mathcal{E}(q) - \Psi(\omega), \tag{20}$$

where $c > 0$. While this control law makes the set $\{(\mathbf{i}, 0), (-\mathbf{i}, 0)\}$ globally attractive for the lifted closed-loop system defined by (8), (2b), and setting $\tau = \kappa^*(q, \omega)$, it renders $(\mathbf{i}, 0)$ stable and $(-\mathbf{i}, 0)$ unstable equilibrium. When composed with $\Phi_\mathbf{i}$, one might expect that the resulting feedback globally asymptotically stabilizes the identity element of $\mathsf{SO}(3)$; however, we show that any such expected global attractivity property is not robust to arbitrarily small disturbances.

Define the function $\sigma : \mathbb{R} \to \{-1, 0, 1\}$ as

$$\sigma(s) = \begin{cases} s/|s| & s \neq 0 \\ 0 & s = 0. \end{cases} \tag{21}$$

Then, for $0 \leq \delta < \pi$, consider the function $\mu : \mathsf{SO}(3) \times \mathbb{R}^3 \to \mathcal{U}(\delta, \mathbb{S}^2)$ defined implicitly in terms of the Rodrigues formula as, for every $R \in \mathsf{SO}(3)$ and every $(\theta, u) \in \mathbb{R} \times \mathbb{S}^2$ satisfying $\mathcal{U}(\theta, u) = R$,

$$\mu(\mathcal{U}(\theta, u), \omega) = \begin{cases} \mathcal{U}(-\delta\sigma(\omega^\top u), u) & \cos\theta < \cos(\pi + \delta) \\ I & \text{otherwise.} \end{cases} \tag{22}$$

For any $(R, \omega) \in \mathsf{SO}(3) \times \mathbb{R}^3$, the rotation matrix $\mu(R, \omega)R$ constitutes an angular perturbation of R about the eigenaxis $u \in \mathbb{S}^2$. The parameter δ controls the size of the disturbance. We note that (22) is well defined on $\mathsf{SO}(3)$.

Lemma 4. *For every $\delta \in [0, \pi)$ and $(R, \omega) \in \mathsf{SO}(3) \times \mathbb{R}^3$, $\mu(R, \omega)$ is uniquely defined.*

Proof: Suppose that $R = \mathcal{U}(\theta, u)$ for some $\theta \in \mathbb{R}$ and $u \in \mathbb{S}^2$. Clearly, $\mu(R, \omega)$ is uniquely defined when $\omega = 0$ or $\cos\theta \geq \cos(\pi \pm \delta)$, since it does not depend on R or ω in this case.

Suppose that $\cos\theta < \cos(\pi \pm \delta)$ and $\omega \neq 0$. This implies that $R \neq I$, since $0 < \delta < \pi$. Then, it follows from the Rodrigues formula that for any $v \in \mathbb{S}^2$ and ϕ such that $R = \mathcal{U}(\phi, v)$, it must be the case that $u = v$ or $u = -v$ (only when $R \neq I$). Moreover, since $\mathcal{U}(-\theta, -u) = \mathcal{U}(\theta, u)$, it follows that

$$\mu(\mathcal{U}(\phi, v), \omega) = \mathcal{U}(-\delta\sigma(\omega^\top v), v) = \mathcal{U}(-\delta\sigma(\omega^\top u), u).$$

Then, we have shown that the value of μ is independent of the angle-axis representation of R, hence, it is uniquely defined on $\mathsf{SO}(3) \times \mathbb{R}^3$. ■

Let $\phi_\mathbf{i} : \mathsf{SO}(3) \to \mathbb{S}^3$ be any single-valued selection of $\Phi_\mathbf{i}$, that is, $\phi_\mathbf{i}(R) = \Phi_\mathbf{i}(R)$ for all $R \neq \mathcal{U}(\pi, u)$ and $\phi_\mathbf{i}(R) \in \Phi_\mathbf{i}$ otherwise. Now, we apply the disturbance μ to measurements of attitude before being converted to a quaternion for use with the inconsistent feedback (20) and analyze the resulting closed-loop system. That is, we replace q with $\phi_\mathbf{i}(\mu(R, \omega)R)$ in the control law κ^* defined in (20).

Because $\phi_\mathbf{i}$ and μ are discontinuous, we use the notion of Krasovskii solutions for discontinuous systems [29]. We note that the following definition is equally valid for product spaces such as $\mathbb{R}^{m\times n}\times\mathbb{R}^p$, once $\mathbb{R}^{m\times n}$ is isometrically identified with $\mathbb{R}^{mn}$ by vectorization.

Definition 5. *Let $f:\mathbb{R}^n\to\mathbb{R}^n$. The* Krasovskii regularization *of f is the set-valued mapping*

$$\mathbf{K}\,f(x)=\bigcap_{\epsilon>0}\overline{\mathrm{conv}}\,f(x+\epsilon\mathbb{B}) \tag{23}$$

where $\overline{\mathrm{conv}}B$ denotes the closed convex hull of the set $B\subset\mathbb{R}^n$. Then, given a function $f:\mathbb{R}^n\to\mathbb{R}^n$, a Krasovskii solution *to $\dot{x}=f(x)$ on an interval $\mathcal{I}\subset\mathbb{R}_{\geq 0}$ is an absolutely continuous function satisfying*

$$\dot{x}(t)\in\mathbf{K}\,f(x(t)) \tag{24}$$

for almost all $t\in\mathcal{I}$.

We now state the main result of this section: the discontinuity created by pairing an inconsistent quaternion-based feedback with a discontinuous quaternion selection scheme makes the closed-loop system susceptible to *arbitrarily small* measurement disturbances that can exploit how feedback term $c\mathcal{E}(\phi_{\mathbf{i}}(R))$ opposes itself about the discontinuity of $\phi_{\mathbf{i}}$.

Theorem 6. *Let $a>0$, $c>0$, and $\delta>0$ satisfy*

$$0<\delta<\frac{1}{2}\left(-\frac{a}{c}+\sqrt{\left(\frac{a}{c}\right)^2+8}\right) \tag{25}$$

and define

$$\mathscr{B}=\{(\mathcal{U}(\theta,u),\omega):\cos\theta+(1/a)\omega^\top J\omega\leq\cos(\pi+\delta)\}.$$

Then, the set $\{\mathcal{U}(\pi,\mathbb{S}^2)\}\times\{0\}$ is stable and $\mathscr{B}$ is invariant for the closed-loop system

$$\begin{aligned}\dot{R}&=R\,[\omega]_\times\\ J\dot{\omega}&=[J\omega]_\times\,\omega+\kappa^*(\phi_{\mathbf{i}}(\mu(R,\omega)R),\omega)\end{aligned} \tag{26}$$

Proof: See Appendix A. ∎

The various failures of $\Phi_{\mathbf{i}}$ have led several authors (e.g. [30]) to derive sufficient conditions on the initial conditions of (2) to ensure that these 180° attitudes are never approached, thus obviating the use of a globally nonsingular representation of attitude like unit quaternions. However, the issues with using $\Phi_{\mathbf{i}}$ as a path-lifting algorithm are not a problem with the quaternion representation—they arise because $\Phi_{\mathbf{i}}$ is a memoryless map from $\mathsf{SO}(3)$ to $\mathbb{S}^3$. In particular, $\Phi_{\mathbf{i}}$ always chooses the closest quaternion to $\mathbf{i}$ and in general, when one compares $\mathcal{Q}(R)$ with q for some $R\in\mathsf{SO}(3)$ and $q\in\mathbb{S}^3$, $\Phi(p,R)$ is multi-valued on the 2-D manifold $\{p\in\mathbb{S}^3:p^\top q=0\}$. However, when the reference point for choosing the closest quaternion is allowed to change, it is then possible to create a dynamic algorithm for smoothly lifting a trajectory from $\mathsf{SO}(3)$ to $\mathbb{S}^3$. We now explore such an algorithm that is *hybrid* in nature.

V. A Hybrid Algorithm for Dynamic Path Lifting

In this section, we present a simple dynamic algorithm for lifting a path from $\mathsf{SO}(3)$ to $\mathbb{S}^3$. The main feature of the algorithm is a memory state $\hat{q}\in\mathbb{S}^3$ that provides a reference point for choosing the closest quaternion with respect to d. This memory state usually remains constant, but is updated when necessary to ensure that $\mathrm{dist}(\hat{q},\mathcal{Q}(R))<1$. The basic logic behind the algorithm is pictured in Fig. 1 as a flow chart.

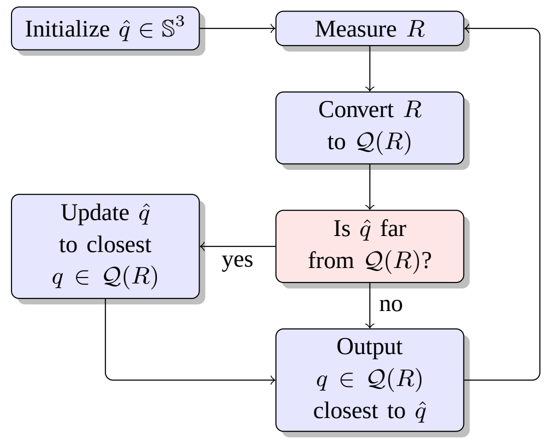

Fig. 1. Flow chart for dynamic path lifting from $\mathsf{SO}(3)$ to $\mathbb{S}^3$.

Given a distance threshold $\alpha\in(0,1)$, we define the sets $C_\ell,D_\ell\subset\mathbb{S}^3\times\mathsf{SO}(3)\times$ as

$$\begin{aligned}C_\ell&=\{(\hat{q},R)\in\mathbb{S}^3\times\mathsf{SO}(3):\mathrm{dist}(\hat{q},\mathcal{Q}(R))\leq\alpha\}\\ D_\ell&=\{(\hat{q},R)\in\mathbb{S}^3\times\mathsf{SO}(3):\mathrm{dist}(\hat{q},\mathcal{Q}(R))\geq\alpha\}.\end{aligned} \tag{27a}$$

Then, we propose the hybrid path-lifting algorithm as the system

$$\mathcal{H}_\ell=\begin{cases}\dot{\hat{q}}=0 & (\hat{q},R)\in C_\ell\\ \hat{q}^+\in\Phi(\hat{q},R) & (\hat{q},R)\in D_\ell,\end{cases} \tag{27b}$$

with continuous input $R:\mathbb{R}_{\geq 0}\to\mathsf{SO}(3)$ and output

$$q=\begin{cases}\Phi(\hat{q},R) & (\hat{q},R)\in C_\ell\\ \emptyset & \text{otherwise.}\end{cases} \tag{27c}$$

We analyze the properties of the hybrid path-lifting algorithm by analyzing the solutions of an autonomous system that generates a wide class of useful trajectories in $\mathsf{SO}(3)$ as input to $\mathcal{H}_\ell$.

Theorem 7. *Let $\alpha\in(0,1)$ and $M>0$. The hybrid system*

$$\underbrace{\begin{aligned}\dot{\hat{q}}&=0\\ \dot{R}&\in R\,[M\mathbb{B}]_\times\end{aligned}}_{(\hat{q},R)\in C_\ell}\qquad\underbrace{\begin{aligned}\hat{q}^+&\in\Phi(\hat{q},R)\\ R^+&=R\end{aligned}}_{(\hat{q},R)\in D_\ell} \tag{28}$$

and its output q defined in (27c) *have the following properties:*

1) *Closed loop system* (28) *satisfies the hybrid basic conditions.*
2) *For each $(\hat{q},R)\in\mathbb{S}^3\times\mathsf{SO}(3)\supset D_\ell$ and each $p\in\Phi(\hat{q},R)$, it follows that $(p,R)\in C_\ell\setminus D_\ell$.*
3) *The flow set C_ℓ is invariant.*
4) *For any solution $(\hat{q},R)$ to* (28),
$$\{(t,j):\mathrm{dist}(\hat{q}(t,j),\mathcal{Q}(R(t,j)))>\alpha\}\subset\{(0,0)\}.$$
5) *All maximal solutions are complete.*
6) *The time between jumps is bounded below by $2\alpha/M$.*

7) *The function* $q{\downarrow_t} : [0,\infty) \to \mathbb{S}^3$ *is continuous and satisfies* $\mathcal{R}(q{\downarrow_t}(t)) = R{\downarrow_t}(t)$.

Proof: See Appendix B. ■

From Theorem 7 and its proof, one could append dynamical equations for the output $q(t,j) = \Phi(\hat{q}(t,j), R(t,j))$ to (28) as $\dot{q} = \frac{1}{2} q \odot \nu(\omega)$ and $q^+ = q$, where $\omega \in M\mathbb{B}$ and $\dot{R} = R\,[\omega]_\times$. In practice, one should choose $\alpha \in (0,1)$ such that for each $(\hat{q}, \mathcal{Q}(R)) \in C_\ell$ and each expected measurement disturbance $R_d \in \mathsf{SO}(3)$, it follows that $\mathrm{dist}(\hat{q}, \mathcal{Q}(R_d R)) < 1$. That is, α should be selected so that no measurement disturbance can make the choice of quaternion ambiguous.

VI. Quaternion Feedback with Dynamic Lifting

With a hybrid algorithm for path lifting in place, we consider the feedback interconnection of (2) with the hybrid path-lifting system and the quaternion-based hybrid controller $\mathcal{H}_c$, that takes a measurement $y \in \mathbb{S}^3 \times \mathbb{R}^3$ as input, has a state $\xi \in \mathcal{X} \subset \mathbb{R}^n$, has dynamics

$$\mathcal{H}_c \begin{cases} \dot{\xi} \in F_c(y,\xi) & (y,\xi) \in C_c \\ \xi^+ \in G_c(y,\xi) & (y,\xi) \in D_c, \end{cases} \tag{29}$$

and produces a continuous torque $\kappa : \mathbb{S}^3 \times \mathbb{R}^3 \times \mathcal{X} \to \mathbb{R}^3$.

Often, quaternion-based controllers are analyzed using the lifted attitude dynamics, defined by equations (8) and (2b), thus neglecting any auxiliary lifting system. The next theorem essentially justifies this approach by relating solutions of the whole closed-loop system (including the hybrid path-lifting system) to a reduced system that has the quaternion-based hybrid controller in feedback with the lifted system defined by (8) and (2b).

Before stating the theorem, we define two closed-loop systems. The first closed-loop system is the feedback interconnection of (2) with the series interconnection of $\mathcal{H}_\ell$ and $\mathcal{H}_c$. This yields the system $\mathcal{H}_1$ with state $(R,\omega,\hat{q},\xi) \in \mathsf{SO}(3) \times \mathbb{R}^3 \times \mathbb{S}^3 \times \mathcal{X}$ defined as

$$\underbrace{\begin{aligned} \dot{R} &= R\,[\omega]_\times \\ J\dot{\omega} &= [J\omega]_\times \omega + \kappa(\Phi(\hat{q},R),\omega,\xi) \\ \dot{\hat{q}} &= 0 \\ \dot{\xi} &\in F_c(\Phi(\hat{q},R),\omega,\xi) \end{aligned}}_{(\hat{q},R) \in C_\ell,\ (\Phi(\hat{q},R),\omega,\xi) \in C_c} \tag{30}$$

$$\underbrace{\begin{aligned} R^+ &= R \\ \omega^+ &= \omega \\ \hat{q}^+ &\in \Phi(\hat{q},R) \\ \xi^+ &= \xi \end{aligned}}_{(\hat{q},R) \in D_\ell,} \qquad \underbrace{\begin{aligned} R^+ &= R \\ \omega^+ &= \omega \\ \hat{q}^+ &= \hat{q} \\ \xi^+ &\in G_c(\Phi(\hat{q},R),\omega,\xi) \end{aligned}}_{(\hat{q},R) \in C_\ell,\ (\Phi(\hat{q},R),\omega,\xi) \in D_c.}$$

In (30), we mean that flows can occur when flows can occur for *both* the controller and lifting subsystems. Jumps can occur when *either* the controller or lifting subsystems can jump. It may be possible that both $(\hat{q},R) \in D_\ell$ and $(\Phi(\hat{q},R),\omega,\xi) \in D_c$ are satisfied at the same "time," i.e., $D_\ell \cap D_c \neq \emptyset$, in which case, *either* jump is possible. That is, either $\hat{q}^+ \in \Phi(\hat{q},R)$ *or* $\xi^+ \in G_c(\Phi(\hat{q},R),\omega,\xi)$ (the other states do not change). This is necessary to ensure that the closed-loop system satisfies the hybrid basic conditions.

Now, we define the feedback interconnection of the lifted attitude system and the hybrid controller $\mathcal{H}_c$. This yields the reduced system $\mathcal{H}_2$ with state $(q,\omega,\xi) \in \mathbb{S}^3 \times \mathbb{R}^3 \times \mathcal{X}$ defined as

$$\underbrace{\begin{aligned} \dot{q} &= \tfrac{1}{2} q \odot \nu(\omega) \\ J\dot{\omega} &= [J\omega]_\times \omega + \kappa(q,\omega,\xi) \\ \dot{\xi} &\in F_c(q,\omega,\xi) \end{aligned}}_{(q,\omega,\xi) \in C_c} \qquad \underbrace{\begin{aligned} q^+ &= q \\ \omega^+ &= \omega \\ \xi^+ &\in G_c(q,\omega,\xi) \end{aligned}}_{(q,\omega,\xi) \in D_c.} \tag{31}$$

Lemma 8. *For every solution* $(R_1,\omega_1,\hat{q}_1,\xi_1) : E_1 \to \mathsf{SO}(3) \times \mathbb{R}^3 \times \mathbb{S}^3 \times \mathcal{X}$ *to* $\mathcal{H}_1$ *of* (30) *such that* $\mathrm{dist}(\hat{q}_1, \mathcal{Q}(R_1))|_{(0,0)} < 1$, *there exists a solution* $(q_2,\omega_2,\xi_2) : E_2 \to \mathbb{S}^3 \times \mathbb{R}^3 \times \mathcal{X}$ *to* $\mathcal{H}_2$ *of* (31) *such that for every* $(t,j) \in E_1$, *there exists* $j' \le j$ *such that* $(t,j') \in E_2$ *and*

$$(R_1, \Phi(\hat{q}_1,R_1), \omega_1, \xi_1)|_{(t,j)} = (\mathcal{R}(q_2), q_2, \omega_2, \xi_2)|_{(t,j')}. \tag{32}$$

Conversely, for every solution $(q_2,\omega_2,\xi_2) : E_2 \to \mathbb{S}^3 \times \mathbb{R}^3 \times \mathcal{X}$ *to* (31), *there exists a solution* $(R_1,\omega_1,\hat{q}_1,\xi_1) : E_1 \to \mathsf{SO}(3) \times \mathbb{R}^3 \times \mathbb{S}^3 \times \mathcal{X}$ *to* (30) *such that for every* $(t,j') \in E_2$, *there exists* $j \ge j'$ *such that* $(t,j) \in E_1$ *and* (32) *is satisfied.*

Proof: See Appendix C. ■

Now, we state one of our main results. The following theorem is a "separation principle" that allows one to design a feedback for the lifted system defined by (8), (2b) and then expect the results to translate directly to the actual system when the hybrid-dynamic path-lifting system $\mathcal{H}_\ell$ is used to lift the trajectory in $\mathsf{SO}(3)$ to $\mathbb{S}^3$.

Theorem 9. *Let* $\alpha \in (0,1)$. *A compact set* $\mathcal{A}_\ell \subset \mathbb{S}^3 \times \mathbb{R}^3 \times \mathcal{X}$ *is stable (unstable) for the system* $\mathcal{H}_2$ *of* (31) *if and only if the compact set*

$$\mathcal{A} = \{(R,\omega,\hat{q},\xi) : (\Phi(\hat{q},R),\omega,\xi) \in \mathcal{A}_\ell,\ \mathrm{dist}(\hat{q},\mathcal{Q}(R)) \le \alpha\} \tag{33}$$

is stable (unstable) for the system $\mathcal{H}_1$ *of* (30). *Moreover,* $\mathcal{A}_\ell$ *is attractive from* $\mathscr{B}_\ell \subset \mathbb{S}^3 \times \mathbb{R}^3 \times \mathcal{X}$ *for the system* $\mathcal{H}_2$ (31) *if and only if* $\mathcal{A}$ *is attractive from*

$$\mathscr{B} = \{(R,\omega,\hat{q},\xi) : (\Phi(\hat{q},R),\omega,\xi) \in \mathscr{B}_\ell,\ \mathrm{dist}(\hat{q},\mathcal{Q}(R)) < 1\} \tag{34}$$

for the system $\mathcal{H}_1$ *of* (30).

Proof: See Appendix D. ■

Interestingly, the result of Theorem 9 is not always desired! When the set $\mathcal{A}$ above is not designed correctly, the resulting closed-loop system can exhibit the symptom of *unwinding*.

VII. The Unwinding Phenomenon

In Theorem 6, we showed how a particular class of inconsistent control laws (20) can be hijacked by small measurement disturbances when $\Phi_{\mathbf{i}}$ defined in (17) is used to lift paths from $\mathsf{SO}(3)$ to $\mathbb{S}^3$. In light of Section V and Theorem 9, one might ask how the control law (20) behaves in feedback with the hybrid path lifting system $\mathcal{H}_\ell$. The answer is that it induces "unwinding."

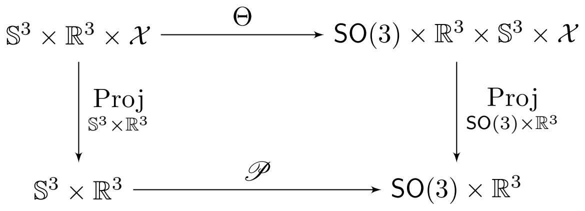

Fig. 2. Commutative diagram of set projections.

Though the behavior has been documented for decades (see e.g. [3]), the term unwinding was perhaps first coined by [6] to describe a symptom of controllers that are designed for systems evolving on topologically complex manifolds using local coordinates in a covering space. In particular, the ambiguity arising from the quaternion representation can cause inconsistent quaternion-based controllers to unnecessarily rotate the rigid body through a full rotation. This behavior can be induced by inconsistent control laws like (20) that are designed to stabilize a *single* point in $\mathbb{S}^3$ while leaving the antipodal point *unstable*, despite the fact that they both correspond to the same physical orientation. This behavior was elegantly described in [6] in terms of the lifts of paths and vector fields. We now provide a characterization in terms of projections of asymptotically stable sets onto the plant state space.

Recall that for some set $Z \subset X \times Y$, its projection onto X is defined as

$$\operatorname*{Proj}_X Z = \{x \in X : \exists y \in Y \ (x, y) \in Z\}. \tag{35}$$

Now, we characterize how a set of interest in the covering space (including extra dynamic states of the controller) appears when projected to the actual plant state space $\mathsf{SO}(3) \times \mathbb{R}^3$.

In this direction, we define the operator $\Theta : \mathbb{S}^3 \times \mathbb{R}^3 \times \mathcal{X} \rightrightarrows \mathsf{SO}(3) \times \mathbb{R}^3 \times \mathbb{S}^3 \times \mathcal{X}$ as

$$\Theta(q, \omega, \xi) = \{(R, \omega, \hat{q}, \xi) : q = \Phi(\hat{q}, R),\ \operatorname{dist}(\hat{q}, \mathcal{Q}(R)) \leq \alpha\}. \tag{36}$$

Further, we define the covering projection $\mathscr{P} : \mathbb{S}^3 \times \mathbb{R}^3 \to \mathsf{SO}(3) \times \mathbb{R}^3$ as

$$\mathscr{P}(q, \omega) = (\mathcal{R}(q), \omega). \tag{37}$$

Lemma 10. *The maps* $\mathscr{P}$ *and* Θ *satisfy*

$$\mathscr{P} \circ \operatorname*{Proj}_{\mathbb{S}^3 \times \mathbb{R}^3} = \operatorname*{Proj}_{\mathsf{SO}(3) \times \mathbb{R}^3} \circ \Theta, \tag{38}$$

that is, the diagram Fig. 2 commutes.

Proof: Let $(q, \omega, \xi) \in \mathbb{S}^3 \times \mathbb{R}^3 \times \mathcal{X}$ and let $R = \mathcal{R}(q)$. It is easy to see that $\mathscr{P}\left(\operatorname{Proj}_{\mathbb{S}^3 \times \mathbb{R}^3}(q, \omega, \xi)\right) = \mathscr{P}(q, \omega) = (\mathcal{R}(q), \omega) = (R, \omega)$. Similarly, for every $(R, \omega, \hat{q}, \xi) \in \Theta(q, \omega, \xi)$, it follows that $R = \mathcal{R}(q)$. Thus, $\operatorname{Proj}_{\mathsf{SO}(3) \times \mathbb{R}^3} \Theta(q, \omega, \xi) = (R, \omega)$, and so, (38) is satisfied. ∎

Let

$$\Pi = \mathscr{P} \circ \operatorname*{Proj}_{\mathbb{S}^3 \times \mathbb{R}^3} = \operatorname*{Proj}_{\mathsf{SO}(3) \times \mathbb{R}^3} \circ \Theta. \tag{39}$$

Lemma 10 clarifies the purpose of controllers designed in the covering space. Suppose it is desired to asymptotically stabilize some set $\mathcal{A}_p \subset \mathsf{SO}(3) \times \mathbb{R}^3$ (in the sense that $\mathcal{A}_p$ is the projection of an asymptotically stable set in the extended state space including controller states). If the dynamic controller (29) is designed to stabilize $\mathcal{A}_\ell \subset \mathbb{S}^3 \times \mathbb{R}^3 \times \mathcal{X}$ in the extended covering state space (as in Lemma 10), one would obviously desire that $\Pi(\mathcal{A}_\ell) = \mathcal{A}_p$, but this should not be the only requirement. In fact, one should design $\mathcal{A}_\ell$ to satisfy

$$\operatorname*{Proj}_{\mathbb{S}^3 \times \mathbb{R}^3} \mathcal{A}_\ell = \mathscr{P}^{-1}(\Pi(\mathcal{A}_\ell)), \tag{40}$$

in which case, we say that $\mathcal{A}_\ell$ is *consistent*. That is, the controller should stabilize *all* points in the lifted state space whose projections under $\mathscr{P}$ map to a point in $\mathcal{A}_p$. As the following lemma states, when (40) is not satisfied, there may be points in the plant state space whose stability relies on the controller's quaternion representation of attitude, which is hardly a desired quality.

Lemma 11. *Let* $\mathcal{A}_\ell \subset \mathbb{S}^3 \times \mathbb{R}^3 \times \mathcal{X}$. *If* $\mathcal{A}_\ell$ *is not consistent, that is, it does not satisfy* (40), *then there exists* $(R, \omega) \in \Pi(\mathcal{A}_\ell)$ *and* $q \in \mathcal{Q}(R)$ *such that for every* $\hat{q} \in \mathbb{S}^3$ *satisfying* $d(q, \hat{q}) \leq \alpha$ *and every* $\xi \in \mathcal{X}$, $(R, \omega, \hat{q}, \xi) \notin \Theta(\mathcal{A}_\ell)$.

Proof: If $\mathcal{A}_\ell$ does not satisfy (40), then, clearly, there exists $(R, \omega) \in \mathsf{SO}(3) \times \mathbb{R}^3$ and $q \in \mathcal{Q}(R)$ such that $(q, \omega) \notin \operatorname{Proj}_{\mathbb{S}^3 \times \mathbb{R}^3} \mathcal{A}_\ell$. Then, by definition of the Proj operator, for every $\xi \in \mathcal{X}$, $(q, \omega, \xi) \notin \mathcal{A}_\ell$. Finally, Lemma 1 asserts that $\Phi(\hat{q}, R) = q$ whenever $\operatorname{dist}(\hat{q}, q) \leq 1$ and by definition of Θ, it follows that for every $\hat{q} \in \mathbb{S}^3$ satisfying $d(\hat{q}, q) \leq \alpha < 1$, that $(R, \omega, \hat{q}, \xi) \notin \Theta(\mathcal{A}_\ell)$. ∎

Unfortunately, many designs proposed in the literature (see, e.g., [1], [3], [4], [11], [27], [28], [30]–[36]) do not satisfy (40). Instead, many designs, like the inconsistent feedback (20) (having $\mathcal{X} = \emptyset$), render the point $(\mathbf{i}, 0) \in \mathbb{S}^3 \times \mathbb{R}^3$ a stable equilibrium, while rendering $(-\mathbf{i}, 0) \in \mathbb{S}^3$ an unstable equilibrium. In this situation, $\Pi((\mathbf{i}, 0)) = \Pi((-\mathbf{i}, 0)) = (I, 0)$. When seen through the map Θ, this creates two distinct, disconnected equilibrium sets in the extended state space, $\mathsf{SO}(3) \times \mathbb{R}^3 \times \mathbb{S}^3$ with one set asymptotically stable and the other, unstable. However, both equilibrium sets project to $(I, 0)$. As the next result shows, the desired attitude can be stable, or unstable, depending on the controller's knowledge of the quaternion representation of the attitude.

Corollary 12. *Let* $\alpha \in (0, 1)$. *Then,* $(\mathbf{i}, 0)$ *is asymptotically stable and* $(-\mathbf{i}, 0)$ *is unstable for the system*

$$\left.\begin{aligned} \dot{q} &= \frac{1}{2} q \odot \nu(\omega) \\ J\dot{\omega} &= [J\omega]_\times \omega + \kappa^*(q, \omega) \end{aligned}\right\} \qquad (q, \omega) \in \mathbb{S}^3 \times \mathbb{R}^3, \tag{41}$$

where κ^* *was defined in* (20). *Similarly, the compact set* $\mathcal{A}_s = \{(I, 0, \hat{q}) : 1 - \hat{q}^\top \mathbf{i} \leq \alpha\}$ *is asymptotically stable and the compact set* $\mathcal{A}_u = \{(I, 0, \hat{q}) : 1 + \hat{q}^\top \mathbf{i} \leq \alpha\}$ *is unstable for the hybrid system*

$$\underbrace{\begin{aligned} \dot{R} &= R\,[\omega]_\times \\ J\dot{\omega} &= [J\omega]_\times \omega + \kappa^*(\Phi(\hat{q}, R), \omega) \\ \dot{\hat{q}} &= 0 \end{aligned}}_{(\hat{q}, R) \in C_\ell} \qquad \underbrace{\begin{aligned} R^+ &= R \\ \omega^+ &= \omega \\ \hat{q}^+ &\in \Phi(\hat{q}, R) \end{aligned}}_{(\hat{q}, R) \in D_\ell.} \tag{42}$$

Proof: We note that the stability and instability of $(\mathbf{i},0)$ for (41) is easily obtained by a simple Lyapunov analysis using the proper and positive definite function $V:\mathbb{S}^3\times\mathbb{R}^3\to\mathbb{R}$ defined as $V(q,\omega)=2c(1-\eta)+\omega^\top J\omega$. Instability of $(-\mathbf{i},0)$ can be shown in numerous ways. To show that $\mathcal{A}_s$ is asymptotically stable for the hybrid system (42), we apply Theorem 9. From (33) and (34), we obtain that $\mathcal{B}_s=\{(R,\omega,\hat{q}):(\Phi(\hat{q},R),\omega)=(\mathbf{i},0),\ \mathrm{dist}(\hat{q},\mathcal{Q}(R))\leq\alpha\}$ is asymptotically stable for (42). By the properties of the maps Φ, $\mathcal{Q}$, $\mathcal{R}$, and dist, it follows that $\mathcal{B}_s=\mathcal{A}_s$. Theorem 9 implies, in a similar fashion, that $\mathcal{A}_u$ is unstable for (42). ■

Finally, we note that in recent works, the authors have presented a hybrid strategy for achieving a global result that is robust to measurement disturbances in [5]. The results in [5] satisfy (40) and can be applied to 6-DOF rigid bodies [25] and synchronization of a network of rigid bodies [26]. Several works also suggest the use of a memoryless (i.e., $\mathcal{X}=\emptyset$) discontinuous quaternion-based feedback using the term $-\sigma(\eta)\epsilon$. Such methods have been suggested in [3], [31], [37]–[40] and indeed avoid the unwinding phenomenon; however, these control laws are susceptible to measurement disturbances like the result in Theorem 6.

Corollary 13. *Let $\alpha,\delta\in(0,1)$, $\mathcal{S}=\mathbb{S}^3\times\mathbb{R}^3\times\{-1,1\}$ and define $C_c\subset\mathcal{S}$, and $D_c\subset\mathcal{S}$ as*

$$C_c=\{(q,\omega,\xi):\eta\xi\geq-\delta\}\quad D_c=\{(q,\omega,\xi):\eta\xi\leq-\delta\}$$

Then, $\mathcal{A}_\ell=\{(q,0,\xi):q=\pm\mathbf{i},\ \xi=\pm1\}$ is globally asymptotically stable for

$$\underbrace{\begin{aligned}\dot{q}&=\frac{1}{2}q\odot\nu(\omega)\\ J\dot{\omega}&=[J\omega]_\times\omega+\kappa^*(\xi q,\omega)\\ \dot{\xi}&=0\end{aligned}}_{(q,\omega,\xi)\in C_c}\qquad\underbrace{\begin{aligned}q^+&=q\\ \omega^+&=\omega\\ \xi^+&=-\xi\end{aligned}}_{(q,\omega,\xi)\in D_c.}\tag{43}$$

Similarly, the compact set $\mathcal{A}=\{(I,0,\hat{q},\xi):1-|\hat{q}^\top\mathbf{i}|<\alpha,\ \xi=\pm1\}$ is globally asymptotically stable for

$$\underbrace{\begin{aligned}\dot{R}&=R\,[\omega]_\times\\ J\dot{\omega}&=[J\omega]_\times\omega+\kappa^*(\xi\Phi(\hat{q},R),\omega)\\ \dot{\hat{q}}&=0\\ \dot{\xi}&=0\end{aligned}}_{(\hat{q},R)\in C_\ell,\ (\Phi(\hat{q},R),\omega,\xi)\in C_c}$$

$$\underbrace{\begin{aligned}R^+&=R\\ \omega^+&=\omega\\ \hat{q}^+&\in\Phi(\hat{q},R)\\ \xi^+&=\xi\end{aligned}}_{(\hat{q},R)\in D_\ell,}\qquad\underbrace{\begin{aligned}R^+&=R\\ \omega^+&=\omega\\ \hat{q}^+&=\hat{q}\\ \xi^+&=-\xi\end{aligned}}_{(\hat{q},R)\in C_\ell,\ (\Phi(\hat{q},R),\omega,\xi)\in D_c.}\tag{44}$$

Proof: Global asymptotic stability of $\mathcal{A}_\ell$ for (43) is obtained by using Lyapunov and invariance analysis [41] with the function $V:\mathbb{S}^3\times\mathbb{R}^3\to\mathbb{R}$ defined as $V(q,\omega,\xi)=2c(1-\xi\eta)+\omega^\top J\omega$. See [5], for example. To show that $\mathcal{A}$ is asymptotically stable for the hybrid system (44), we note that Theorem 7 implies that $\{(R,\omega,\hat{q},\xi):(\hat{q},R)\in C_\ell\}$ is globally attractive and then we apply Theorem 9. ■

VIII. Conclusion

Obtaining global asymptotic stability of rigid-body attitude is a fundamentally difficult task. Often, feedback controllers are designed and analyzed on a state space that is topologically simpler than $\mathsf{SO}(3)$; however, it is not always clear how the analysis of such algorithms can be translated to $\mathsf{SO}(3)$. When unit quaternions are used to parametrize rigid-body attitude and design feedback control laws, their actual implementation relies on an algorithm to translate measurements from $\mathsf{SO}(3)$ to $\mathbb{S}^3$. When a memoryless map is used for this task, the resulting quaternion trajectory may be discontinuous, creating an extreme measurement-disturbance sensitivity for a widely used class of quaternion-based feedback control laws. An alternative is to dynamically lift the paths using a hybrid mechanism. Such a hybrid algorithm allows one to translate stability results obtained in the covering space directly to the actual plant; however, such a feedback system can induce an undesirable unwinding response when the quaternion-based feedback is not designed to stabilize *all* quaternion representations of the desired attitude. Finally, when hybrid path-lifting mechanisms are used in conjunction with the hybrid quaternion-based feedbacks proposed in [5], [25], [26], the result is global asymptotic stabilization of the identity element of $\mathsf{SO}(3)$.

Appendix A
Proof of Theorem 6

In what follows, we denote $x=(R,\omega)\in\mathsf{SO}(3)\times\mathbb{R}^3$ and define, for pairs $(A_1,a_1),(A_2,a_2)\in\mathbb{R}^{m\times n}\times\mathbb{R}^p$, $\langle(A_1,a_1),(A_2,a_2)\rangle=\langle A_1,A_2\rangle+\langle a_1,a_2\rangle$. Finally, in accordance with (1), for some function $V:\mathbb{R}^{m\times n}\times\mathbb{R}^p\to\mathbb{R}$, we denote $\nabla V(x)=(\nabla_R V(x),\nabla_\omega V(x))\in\mathbb{R}^{m\times n}\times\mathbb{R}^p$, where $\nabla_R V(x)$ denotes the matrix of partial derivatives V with respect to $R\in\mathbb{R}^{m\times n}$ and $\nabla_\omega V(x)$ denotes the vector of partial derivatives of V with respect to $\omega\in\mathbb{R}^p$.

Define the function $f_\omega:\mathbb{R}^{3\times3}\times\mathbb{R}^3\to\mathbb{R}^3$ and the vector field $F:\mathbb{R}^{3\times3}\times\mathbb{R}^3\to\mathbb{R}^{3\times3}\times\mathbb{R}^3$ as

$$\begin{aligned}f_\omega(R,\omega)&=J^{-1}\left([J\omega]_\times\omega-c\mathcal{E}(\phi_{\mathbf{i}}(\mu(R,\omega)R))-\Psi(\omega)\right)\\ F(R,\omega)&=\left(R\,[\omega]_\times,f_\omega(R,\omega)\right).\end{aligned}\tag{45}$$

By some abuse of notation, the Krasovskii regularization of F is $\mathbf{K}\,F$, where the arguments of F are perturbed as in (23) with respect to the norms defined on each space. That is,

$$\mathbf{K}\,F(R,\omega)=\bigcap_{\epsilon>0}\overline{\mathrm{conv}}F(R+\epsilon\mathbb{B},\omega+\epsilon\mathbb{B}).\tag{46}$$

Let $\mathcal{W}(R,\omega)=\phi_{\mathbf{i}}(\mu(R,\omega)R)$. From the definition (23), one can show that

$$\begin{aligned}\mathbf{K}\,f_\omega(R,\omega)&=\big\{J^{-1}\left([J\omega]_\times\omega-c\mathcal{E}(q)-\Psi(\omega)\right):\\ &\qquad q\in\mathbf{K}\,\mathcal{W}(R,\omega)\big\}\\ \mathbf{K}\,F(R,\omega)&=\left\{(R\,[\omega]_\times,\tau):\tau\in\mathbf{K}\,f_\omega(R,\omega)\right\}.\end{aligned}\tag{47}$$

Since we are studying Krasovskii solutions to (26), we might normally need to evaluate $\mathbf{K}\,\mathcal{W}$; however, the analysis in this proof obviates the need for calculating the Krasovskii regularization for regions where the calculation is nontrivial. By definition of μ and $\phi_{\mathbf{i}}$, the map $\mathcal{W}$ is continuous on the set $\mathcal{O} = \{(\mathcal{U}(\theta,u),\omega) : \cos\theta < \cos(\pi+\delta),\ \omega \neq 0\}$, so $\mathbf{K}\,\mathcal{W}(x) = \mathcal{W}(x)$ for all $x \in \mathcal{O}$.

Consider the Lyapunov function

$$V(R,\omega) = a(1 - \operatorname{trace}(I-R)/4) + \frac{1}{2}\omega^\top J\omega. \tag{48}$$

Expressed in terms of rotation angle, we have equivalently,

$$V(\mathcal{U}(\theta,u),\omega) = \frac{a}{2}(1+\cos\theta) + \frac{1}{2}\omega^\top J\omega.$$

Since $\operatorname{trace}(I - \mathcal{U}(\theta,u)) = 2(1-\cos\theta)$, it follows that $V(R,\omega) \geq 0$ for all $(R,\omega) \in \mathsf{SO}(3)\times\mathbb{R}^3$ and $V(R,\omega) = 0$ if and only if $R = \mathcal{U}(\pi,v)$ and $\omega = 0$. Furthermore, the sub-level sets of V are compact.

Define the function $\psi : \mathbb{R}^{3\times 3} \to \mathbb{R}^3$ as

$$\psi(A) = \frac{1}{2}\begin{bmatrix} A_{32}-A_{23} & A_{13}-A_{31} & A_{21}-A_{12}\end{bmatrix}^\top. \tag{49}$$

Then, ψ satisfies $\operatorname{trace}(A\,[\omega]_\times) = -2\omega^\top\psi(A)$ and $\psi(\mathcal{U}(\theta,u)) = u\sin\theta$. Employing (47), we calculate

$$\begin{aligned}\max_{z\in\mathbf{K}\,F(x)} &\langle\nabla V(\mathcal{U}(\theta,u),\omega), z\rangle = \\ &-\omega^\top\Psi(\omega) - \frac{a}{2}\omega^\top\psi(R) - \min_{y\in\mathbf{K}\,\mathcal{W}(x)} c\omega^\top\mathcal{E}(y),\end{aligned} \tag{50}$$

where we have used the fact that $\omega^\top[J\omega]_\times\omega = 0$. Note that $\max_{z\in\mathbf{K}\,\mathcal{W}(x)}\langle\nabla V(R,0), z\rangle = 0$ no matter what values the Krasovskii regularization may take.

Now, we let $R = \mathcal{U}(\theta,u)$ and henceforth constrain our analysis to the case where $\cos\theta < \cos(\pi+\delta)$ and $\omega \neq 0$, so that $\mu(R,\omega)R = \mathcal{U}(\theta - \delta\sigma(\omega^\top u), u)$ and $\phi_{\mathbf{i}}(\mu(R,\omega)R)$ is single-valued. Also, in this region, the Krasovskii regularization of (26) is identical to (26). Recalling that $\phi_{\mathbf{i}}$ selects the quaternion with positive scalar component and noting that $\mathcal{U}(\phi,u)\mathcal{U}(\theta,u) = \mathcal{U}(\theta+\phi,u)$, we can now write

$$\begin{aligned}&\phi_{\mathbf{i}}(\mu(R,\omega)R) = \\ &\sigma\left(\cos\left((\theta-\delta\sigma(\omega^\top u))/2\right)\right)\begin{bmatrix}\cos\left((\theta-\delta\sigma(\omega^\top u))/2\right)\\ \sin\left((\theta-\delta\sigma(\omega^\top u))/2\right)u\end{bmatrix},\end{aligned}$$

and in particular,

$$\begin{aligned}&\mathcal{E}(\phi_{\mathbf{i}}(\mu(R,\omega)R)) = \\ &\sigma\left(\cos\left((\theta-\delta\sigma(\omega^\top u))/2\right)\right)\sin\left((\theta-\delta\sigma(\omega^\top u))/2\right)u.\end{aligned} \tag{51}$$

Applying (51) and (18) to (50),

$$\begin{aligned}\max_{z\in\mathbf{K}\,F(x)}\langle\nabla V(\mathcal{U}(\theta,u),\omega), z\rangle &\leq -\gamma(\|\omega\|_2) - \omega^\top u\frac{a}{2}\sin\theta \\ &- \omega^\top u\left(c\sigma\left(\cos\left((\theta-\delta\sigma(\omega^\top u))/2\right)\right)\right. \\ &\left.*\sin\left((\theta-\delta\sigma(\omega^\top u))/2\right)\right).\end{aligned} \tag{52}$$

Note that when $\omega^\top u = 0$, it follows that $\max_{z\in\mathbf{K}\,F(x)}\langle\nabla V(x), z\rangle \leq 0$, so we further constrain our analysis from this point to the case when $\omega^\top u \neq 0$. Now, without loss of generality, we assume that $\pi - \delta < \theta < \pi + \delta$, where

$$\sigma\left(\cos\left((\theta-\delta\sigma(\omega^\top u))/2\right)\right) = \sigma\left(\pi - (\theta - \delta\sigma(\omega^\top u))\right). \tag{53}$$

Now, since $\sigma(\omega^\top u)^2 = 1$ and $s\sigma(s) = |s|$, we factor this term to arrive at

$$\begin{aligned}\dot V(\mathcal{U}(\theta,u),\omega) &\leq -\gamma(\|\omega\|_2) - |\omega^\top u|\frac{a}{2}\sigma(\omega^\top u)\sin\theta \\ &- |\omega^\top u|c\sigma(\omega^\top u)\sigma\left(\pi - (\theta-\delta\sigma(\omega^\top u))\right) \\ &*\sin\left((\theta-\delta\sigma(\omega^\top u))/2\right).\end{aligned} \tag{54}$$

Moreover, for any $r,s \in \mathbb{R}$, it follows that $\sigma(s)\sigma(r) = \sigma(r\sigma(s))$. Applying this relation to (54), we have

$$\begin{aligned}\max_{z\in\mathbf{K}\,F(x)}&\langle\nabla V(\mathcal{U}(\theta,u),\omega), z\rangle \leq \\ &-\gamma(\|\omega\|_2) - |\omega^\top u|\frac{a}{2}\sigma(\omega^\top u)\sin\theta \\ &- |\omega^\top u|c\sigma\left((\pi-\theta)\sigma(\omega^\top u)+\delta\right) \\ &*\sin\left((\theta-\delta\sigma(\omega^\top u))/2\right).\end{aligned} \tag{55}$$

It follows that $\max_{z\in\mathbf{K}\,F(x)}\langle\nabla V(x), z\rangle < 0$ whenever

$$\begin{aligned}c\sigma\left((\pi-\theta)\sigma(\omega^\top u)+\delta\right)&\sin\left((\theta-\delta\sigma(\omega^\top u))/2\right) \\ &+ \frac{a}{2}\sigma(\omega^\top u)\sin\theta > 0.\end{aligned} \tag{56}$$

Since we have assumed that $\delta > |\pi - \theta|$, it follows that $\sigma\left((\theta-\pi)\sigma(\omega^\top u)+\delta\right) = 1$. Moreover, since $|\sin\theta| \leq |\theta - \pi|$ and $1-\cos\theta \leq \frac{1}{2}\theta^2$, we can use the properties of $\sin$ and $\cos$ to deduce that $\sin(\theta/2) \geq 1 - \frac{1}{8}(\theta-\pi)^2$. Hence, (56) holds when

$$c\left(1 - \frac{1}{8}\left(\theta - \pi - \delta\sigma(\omega^\top u)\right)^2\right) > \frac{a}{2}|\theta-\pi|. \tag{57}$$

Again employing the assumption that $\delta > |\pi - \theta|$, we have $1 - \frac{1}{8}(\theta-\pi-\delta\sigma(\omega^\top u))^2 \geq 1 - \frac{1}{2}\delta^2$ and that (56) holds when

$$c\left(1-\delta^2/2\right) > a\delta/2 \iff 0 > \delta^2 + (a/c)\delta - 2. \tag{58}$$

Since $\delta \geq 0$, we have at least for small δ that $0 > \delta^2 + a\delta/c - 2$, so we can bound δ by the positive root of $\lambda(x) = x^2 + (a/c)x - 2$ located at $x = (-(a/c) \pm \sqrt{(a/c)^2+8})/2$. Hence, we have that $\max_{z\in\mathbf{K}\,F(x)}\langle\nabla V(x), z\rangle \leq 0$ on the set $W = \{(R,\omega) : \cos\theta < \cos(\pi+\delta) \text{ or } \omega = 0\} \supset \{\mathcal{U}(\pi,\mathbb{S}^2)\}\times\{0\}$, where $0 < \delta < \left(-(a/c)+\sqrt{(a/c)^2+8}\right)/2$. This implies that $\{\mathcal{U}(\pi,\mathbb{S}^2)\}\times\{0\}$ is stable.

To estimate an invariant set using V, we find a sub-level set of V contained in the set W. In fact, the set $\mathscr{B}$ is a sub-level set of V corresponding to the set $\{(\mathcal{U}(\theta,u),\omega) : V(\mathcal{U}(\theta,u),\omega) \leq \frac{a}{2}(1+\cos(\pi+\delta))\}$. Moreover, $\mathscr{B} \subset W$ and so it is invariant.

Appendix B
Proof of Theorem 7

First, note that (28) satisfies the hybrid basic conditions. The map $R \mapsto R\,[M\mathbb{B}]_\times$ is nonempty, locally bounded, outer semicontinuous, and convex-valued. Moreover, $(\hat q, R) \mapsto \Phi(\hat q, R)$ is outer semicontinuous, locally bounded, and nonempty. To see this, note that $\Phi(\hat q, R)$ is continuous on the set

$\{(\hat{q}, R) : \operatorname{dist}(\hat{q}, \mathcal{Q}(R)) < 1\}$ and that $\Phi(\hat{q}, R) = \mathcal{Q}(R)$ when $\operatorname{dist}(\hat{q}, \mathcal{Q}(R)) = 1$.

Let $(\hat{q}, R) \in \mathbb{S}^3 \times \mathsf{SO}(3)$. By definition of Φ in (15), it follows that for all $p \in \Phi(\hat{q}, R)$, $\operatorname{dist}(p, \mathcal{Q}(R)) = 0$. Thus, $(p, R) \in C_\ell \setminus D_\ell$. Now, let $\mathsf{T}_x M$ denote the tangent space of a manifold M at $x \in M$. Since $R[\omega]_\times \in \mathsf{T}_R\mathsf{SO}(3)$ for all $(R, \omega) \in \mathsf{SO}(3) \times \mathbb{R}^3$ and $0 \in \mathsf{T}_q\mathbb{S}^3$, the set $\mathbb{S}^3 \times \mathsf{SO}(3)$ is viable under the flow (i.e., there exists a nontrivial solution from any initial condition in C_ℓ). This combined with the previous fact that any jump maps the state to $C_\ell \setminus D_\ell$ makes C_ℓ invariant and implies that for any solution $(\hat{q}, R)$ of (28), $\{(t,j) \in \operatorname{dom}\hat{q} : (\hat{q}, R) \in D_\ell \setminus C_\ell\} = \{(0,0)\}$. Finally, since $C_\ell \cup D_\ell = \mathbb{S}^3 \times \mathsf{SO}(3)$ and $\mathsf{SO}(3) \times \mathbb{S}^3$ is compact, no finite escape of solutions is possible. It follows from [24, Proposition 2.4] that every maximal solution is complete. This proves 2)—5).

Now, we prove 6). Suppose that $\mathcal{I}_j \times \{j\} \subset \operatorname{dom}\hat{q}$ has a nonempty interior. Then, for all $(t,j) \in \mathcal{I}_j \times \{j\}$, it follows that $(\hat{q}(t,j), R(t,j)) \in C_\ell$ so that $\operatorname{dist}(\hat{q}(t,j), \mathcal{Q}(R(t,j))) \leq \alpha < 1$. This fact combined with Lemma 2 implies that $t \mapsto q(t,j)$ is absolutely continuous on $\mathcal{I}_j$, $\mathcal{R}(q(t,j)) = R(t,j)$, and $\dot{q}(t,j) = \frac{1}{2}q(t,j) \otimes \nu(\omega(t))$ for some Lebesgue measurable $\omega : \mathcal{I}_j \to M\mathbb{B}$ and for almost all $(t,j) \in \mathcal{I}_j \times \{j\}$.

Recalling Lemma 1, we have $\operatorname{dist}(\hat{q}(t,j), \mathcal{Q}(R(t,j))) = 1 - \hat{q}(t,j)^\top q(t,j)$, which implies $\frac{d}{dt}\operatorname{dist}(\hat{q}, \mathcal{Q}(R)) = \frac{d}{dt}\left(1 - \hat{q}^\top q\right) = -\hat{q}^\top \dot{q} \leq \|\hat{q}\|_2\|\dot{q}\|_2 = \|\dot{q}\|_2$. It follows that $\frac{d}{dt}\operatorname{dist}(\hat{q}, \mathcal{Q}(R)) \leq \|\dot{q}\|_2 = \frac{1}{2}|\Lambda(q)\omega|^2 = \frac{1}{2}\|\omega\|_2 \leq M/2$. Since $\operatorname{dist}(\hat{q}(t,j), \mathcal{Q}(R(t,j))) = 0$ whenever $(t, j-1) \in \operatorname{dom}\hat{q}$, the time between jumps must be at least $2\alpha/M$.

To show 7), we recall our previous conclusion that for every $\mathcal{I}_j \times \{j\} \subset \operatorname{dom}\hat{q}$, $t \mapsto q(t,j)$ is absolutely continuous on $\mathcal{I}_j$ and satisfies $\mathcal{R}(q(t,j)) = R(t,j)$. Recalling the definition of the time projection, we now need only show that the value of q does not change over jumps (note also that jumps occurring at $t = 0$ are ignored). Now, suppose that $\{(t,j), (t,j+1)\} \subset \operatorname{dom}\hat{q}$. Then, $\Phi(\hat{q}(t,j+1), R(t,j+1)) = \Phi(\hat{q}(t,j+1), R(t,j)) = \Phi(\Phi(\hat{q}(t,j), R(t,j)), R(t,j))$, and by Lemma 3, it follows that $\Phi(\hat{q}(t,j+1), R(t,j+1)) = \Phi(\hat{q}(t,j), R(t,j))$. By the definition of q in (27c), it follows that $q(t,j) = q(t,j+1)$ so that $\mathcal{R}(q{\downarrow_t}(t)) = R{\downarrow_t}(t)$.

Appendix C
Proof of Lemma 8

First, we assume the existence of a solution $(R_1, \omega_1, \hat{q}_1, \xi_1) : E_1 \to \mathsf{SO}(3) \times \mathbb{R}^3 \times \mathbb{S}^3 \times \mathcal{X}$ to system $\mathcal{H}_1$ of (30) such that $\operatorname{dist}(\hat{q}_1, \mathcal{Q}(R_1))|_{(0,0)} < 1$. Now, we will recursively define the solution (q_2, ω_2, ξ_2) and its associated hybrid time domain in terms of $(R_1, \omega_1, \hat{q}_1, \xi_1)$ and E_1. In this direction, we define $E_2^{-1} = \emptyset$, $T^0 = 0$, and $J_1^0 = 0$. Now, for each $k \in \mathbb{Z}_{\geq 0}$ we define

$$\begin{aligned}(T^{k+1}, J_1^{k+1}) = \\ \min\big\{&(t,j) \in E_1 : (t, j-1) \in E_1,\ j > J_1^k \\ &\hat{q}(t,j) = \hat{q}(t,j-1), \\ &\xi_1(t,j) \in G_c(\Phi(\hat{q}_1, R_1), \omega_1, \xi_1)|_{(t,j-1)}\big\},\end{aligned} \tag{59}$$

where $\min$ is taken with respect to the natural ordering on E_1. That is, $(T^{k+1}, J_1^{k+1}) \in E_1$ is the time immediately after the first jump due to the controller after $j = J_1^k$.

There can be two cases. If $(T^{k+1}, J_1^{k+1}) \neq \emptyset$, then a controller jump occurs and we define

$$E_2^{k+1} = E_2^k \cup ([T^k, T^{k+1}], k) \tag{60}$$

and for every $t \in [T^k, T^{k+1}]$ and $J_1^k \leq j \leq J_1^{k+1} - 1$ such that $(t,j) \in E_1$, we define the solution

$$(q_2, \omega_2, \xi_2)|_{(t,k)} = (\Phi(\hat{q}_1, R_1), \omega_1, \xi_1)|_{(t,j)} \tag{61}$$

We now verify that this is indeed a solution to (31). First, we ensure that the jump dynamics are satisfied. Note that $\Phi(\hat{q}_1(t,j), R_1(t,j))$ is single-valued for every $(t,j) \in E_1$, since $\operatorname{dist}(\hat{q}_1(0,0), \mathcal{Q}(R_1(0,0))) < 1$ by assumption and then Theorem 7 provides that $\operatorname{dist}(\hat{q}_1(t,j), \mathcal{Q}(R_1(t,j))) \leq \alpha < 1$ for each $(t,j) \succeq (0,0)$. When $k \neq 0$, we consider the jump from $(T^k, k-1)$ to (T^k, k). By (61), we have that $(q_2, \omega_2, \xi_2)|_{(T^k, k-1)} = (\Phi(\hat{q}_1, R_1), \omega_1, \xi_1)|_{(T^k, J_1^k - 1)}$ and $(q_2, \omega_2, \xi_2)|_{(T^k, k)} = (\Phi(\hat{q}_1, R_1), \omega_1, \xi_1)|_{(T^k, J_1^k)}$. From the definition of (T^k, J_1^k) in (59), it follows that

$$\begin{aligned}&(R_1, \omega_1, \hat{q}_1, \xi_1)|_{(T^k, J_1^k)} \in \\ &\qquad (R_1, \omega_1, \hat{q}_1, G_c(\Phi(\hat{q}_1, R_1), \omega_1, \xi_1))|_{(T^k, J_1^k - 1)},\end{aligned}$$

which implies

$$\begin{aligned}&(\Phi(\hat{q}_1, R_1), \omega_1, \xi_1)|_{(T^k, J_1^k)} \in \\ &\qquad (\Phi(\hat{q}_1, R_1), \omega_1, G_c(\Phi(\hat{q}_1, R_1), \omega_1, \xi_1))|_{(T^k, J_1^k - 1)},\end{aligned}$$

and so, $(q_2, \omega_2, \xi_2)|_{(T^k, k)} \in (q_2, \omega_2, G_c(q_2, \omega_2, \xi_2))|_{(T^k, k-1)}$. Thus, (q_2, ω_2, ξ_2) satisfies the jump dynamics of (31) for each pair $\{(t,j), (t,j+1)\} \subset E_2^k$ when $j + 1 \leq k$.

Now, we verify that (61) is a solution to (31) along flows for all $t \in [T^k, T^{k+1}]$. First note that along solutions of (30), if $J_1^k < J_1^{k+1} - 1$, there are jumps due to the lifting system. That is, $\hat{q}_1(t, j+1) \in \Phi(\hat{q}_1(t,j), \mathcal{Q}(R_1(t,j)))$ for some $t \in [T^k, T^{k+1}]$ and $J_1^k \leq j < J_1^{k+1}$, while other states are unchanged. Theorem 7 then implies that over any such jumps of the lifting system, $\Phi(\hat{q}_1, R_1)|_{(t,j+1)} = \Phi(\hat{q}_1, R_1)|_{(t,j)}$. Then, the definition of $(q_2, \omega_2, \xi_2)|_{(t,k)}$ is well-defined in the sense that there is no ambiguity in the definition due to possible jumps of the lifting system.

Furthermore, over the interval $t \in [T^k, T^{k+1}]$ and $J_1^k \leq j \leq J_1^{k+1} - 1$ such that $(t,j) \in E_1$, Theorem 7 provides that $t \mapsto \Phi(\hat{q}_1, R_1)|_{(t,j)}$ (where j is taken implicitly from t) is a continuous trajectory satisfying $\mathcal{R}(\Phi(\hat{q}_1, R_1)|_{(t,j)}) = R_1(t,j)$ and so it also satisfies (8). Since ω_1 and ξ_1 do not change over jumps due to the lifting system and obey the same differential inclusions for (30) and (31), this implies that (61) is a solution to (31) on E_2^k for all k such that $(T^k, J_1^k) \neq \emptyset$.

We now handle the second case. If there is no such $k \in \mathbb{Z}_{\geq 0}$ such that $(T^k, J_1^k) = \emptyset$, we let

$$E_2 = \bigcup_{k=0}^{\infty} ([T^k, T^{k+1}], k).$$

Then, (q_2, ω_2, ξ_2) is a solution to (31) on E_2. Moreover, since jumps from the lifting system are not counted in solutions to

(31), we have that for every $(t,j) \in E_1$ there exists $j' \leq j$ such that $(t,j') \in E_2$ and $(R_1, \Phi(\hat{q}_1, R_1), \omega_1, \xi_1)|_{(t,j)} = (\mathcal{R}(q_2), q_2, \omega_2, \xi_2)|_{(t,j')}$.

Now suppose that for some $k^* \in \mathbb{Z}_{\geq 0}$, $(T^{k^*+1}, J_1^{k^*+1}) = \emptyset$. That is, after T^{k^*}, there are no further jumps due to the controller. In this case, we let

$$E_2 = E_2^{k^*} \cup ([T^{k^*}, \bar{T}), k^*) = \left(\bigcup_{k=0}^{k^*-1} ([T^k, T^{k+1}], k) \right) \cup ([T^{k^*}, \bar{T}), k^*),$$

where $\bar{T} = \sup\{t : \exists j \in \mathbb{Z}_{\geq 0}(t,j) \in E_1\}$ and we allow $\bar{T} = \infty$ when E_1 is unbounded in the t direction. Then, for all $t \in [T^{k^*}, \bar{T})$, we define the solution

$$(q_2, \omega_2, \xi_2)|_{(t,k^*)} = (\Phi(\hat{q}_1, R_1), \omega_1, \xi_1)\downarrow_t(t). \tag{62}$$

Similar to previous arguments, Theorem 7 assures that $\Phi(\hat{q}_1, R_1)\downarrow_t(t)$ is a continuous trajectory satisfying $\mathcal{R}(\Phi(\hat{q}_1, R_1))\downarrow_t(t) = R_1\downarrow_t(t)$ for every $t \in [0, \bar{T})$ and so also satisfies (8). Since the ω and ξ solution-components of (30) do not exhibit changes over jumps due to the lifting system and otherwise have identical dynamics to solutions of (31) when there are no controller jumps, it follows that the hybrid arc defined in (61) and (62) is a solution to (31) on E_2.

In particular, it follows that (q_2, ω_2, ξ_2) is a solution to (31) on E_2 and for every $(t,j) \in E_1$ there exists $j' \leq j$ such that $(t,j') \in E_2$ and $(R_1, \Phi(\hat{q}_1, R_1), \omega_1, \xi_1)|_{(t,j)} = (\mathcal{R}(q_2), q_2, \omega_2, \xi_2)|_{(t,j')}$. This concludes the first part of the lemma. A converse follows similarly by adding in jumps due to the lifting system.

APPENDIX D
PROOF OF THEOREM 9

First, we note that since $\mathcal{R} : \mathbb{S}^3 \to \mathsf{SO}(3)$ is a covering map and in particular, is everywhere a local diffeomorphism, we can easily write open neighborhoods of $\mathcal{A}$ in terms of open neighborhoods of $\mathcal{A}_\ell$. In particular, an open neighborhood $U^\epsilon \supset \mathcal{A}$ can be written as

$$U^\epsilon = \{(R, \omega, \hat{q}, \xi) : . \quad (\Phi(\hat{q}, R), \omega, \xi) \in U_\ell^\epsilon,\ \operatorname{dist}(\hat{q}, \mathcal{Q}(R)) < \alpha + \epsilon\},$$

where U_ℓ^ϵ is an open neighborhood of $\mathcal{A}_\ell$, when every $(R, \omega, \hat{q}, \xi) \in U^\epsilon$ satisfies $\operatorname{dist}(\hat{q}, \mathcal{Q}(R)) < 1$. Since $\alpha < 1$, this holds true for small open neighborhoods of $\mathcal{A}$, where $\epsilon < 1 - \alpha$.

Suppose that $\mathcal{A}_\ell$ is stable for (31). Let U_ℓ^ϵ be an open neighborhood of $\mathcal{A}_\ell$. then, there exists an open set $U_\ell^\delta \subset U_\ell^\epsilon$ such that for any solution $(q_2, \omega_2, \xi_2) : E_2 \to \mathbb{S}^3 \times \mathbb{R}^3 \times \mathcal{X}$ satisfying $(q_2, \omega_2, \xi_2)|_{(0,0)} \in U_\ell^\delta$, it follows that $(q_2, \omega_2, \xi_2)|_{(t,j)} \in U_\ell^\epsilon$ for all $(t,j) \in E_2$. Without loss of generality, suppose that $(R_1, \omega_1, \hat{q}_1, \xi_1) : E_1 \to \mathsf{SO}(3) \times \mathbb{R}^3 \times \mathbb{S}^3 \times \mathcal{X}$ is a solution to (31) satisfying $\operatorname{dist}(\hat{q}_1(0,0), \mathcal{Q}(R_1(0,0))) < \alpha + \epsilon < 1$ for some $\epsilon > 0$ and $(\Phi(\hat{q}, R_1), \omega_1, \xi_1)|_{(0,0)} \in U_\ell^\delta$. Then, Lemma 8 guarantees the existence of a solution $(q_2, \omega_2, \xi_2) : E_2 \to \mathbb{S}^3 \times \mathbb{R}^3 \times \mathcal{X}$ such that for every $(t,j') \in E_2$, there exists $j \geq j'$ such that $(t,j) \in E_1$ and $(R_1, \Phi(\hat{q}_1, R_1), \omega_1, \xi_1)|_{(t,j)} = (\mathcal{U}(q_2), q_2, \omega_2, \xi_2)|_{(t,j')}$.

Now, since $(\Phi(\hat{q}_1, R_1), \omega_1, \hat{q}_1, \xi_1)|_{(0,0)} \in U_\ell^\delta$, this implies that $(q_2, \omega_2, \xi_2)|_{(0,0)} \in U_\ell^\delta$ and so, $(q_2, \omega_2, \xi_2)|_{(t,j')} \in U_\ell^\epsilon$ for all $(t,j') \in E_2$. But then, this implies that $(\Phi(\hat{q}_1, R_1), \omega_1, \xi_1)|_{(t,j)} \in U_\ell^\epsilon$ for all $(t,j) \in E_1$. Finally, by Theorem 7, $\operatorname{dist}(\hat{q}_1(0,0), \mathcal{Q}(R_1(0,0))) < 1$ implies that $\operatorname{dist}(\hat{q}_1(t,j), \mathcal{Q}(R_1(t,j))) \leq \alpha$, and so, $\mathcal{A}$ is stable.

Proceeding, we suppose that $\mathcal{A}$ is stable. Let

$$U^\epsilon = \{(R, \omega, \hat{q}, \xi) : \quad (\Phi(\hat{q}, R), \omega, \xi) \in U_\ell^\epsilon,\ \operatorname{dist}(\hat{q}, \mathcal{Q}(R)) < \alpha + \epsilon\}$$

be an open neighborhood of $\mathcal{A}$, where $\epsilon + \alpha < 1$. Now, there exists $0 < \delta < \epsilon < 1 - \alpha$ and an open set $U^\delta \subset U^\epsilon$ written as

$$U^\delta = \{(R, \omega, \hat{q}, \xi) : \quad (\Phi(\hat{q}, R), \omega, \xi) \in U_\ell^\delta,\ \operatorname{dist}(\hat{q}, \mathcal{Q}(R)) < \alpha + \delta\}$$

where $U_\ell^\delta \subset U_\ell^\epsilon$ such that for any solution $(R_1, \omega_1, \hat{q}_1, \xi_1) : E_1 \to \mathsf{SO}(3) \times \mathbb{R}^3 \times \mathbb{S}^3 \times \mathcal{X}$ satisfying $(R_1, \omega_1, \hat{q}_1, \xi_1)|_{(0,0)} \in U^\delta$, it follows that $(R_1, \omega_1, \hat{q}_1, \xi_1)|_{(t,j)} \in U^\epsilon$ for all $(t,j) \in E_1$. Or, equivalently, $(\Phi(\hat{q}_1, R_1), \omega_1, \xi_1)|_{(0,0)} \in U_\ell^\delta$ and $\operatorname{dist}(\hat{q}_1(0,0), \mathcal{Q}(R_1(0,0))) < \alpha + \delta$ implies that $(\Phi(\hat{q}_1, R_1), \omega_1, \xi_1)|_{(t,j)} \in U_\ell^\epsilon$ and $\operatorname{dist}(\hat{q}_1(0,0), \mathcal{Q}(R_1(0,0))) < \alpha + \epsilon$ for all $(t,j) \in E_1$.

Now, suppose $(q_2, \omega_2, \xi_2) : E_2 \to \mathbb{S}^3 \times \mathbb{R}^3 \times \mathcal{X}$ is a solution to (31) satisfying $(q_2, \omega_2, \xi_2)|_{(0,0)} \in U_\ell^\delta$. Then, Lemma 8 guarantees the existence of a solution $(R_1, \omega_1, \hat{q}_1, \xi_1) : E_1 \to \mathsf{SO}(3) \times \mathbb{R}^3 \times \mathbb{S}^3 \times \xi_1$ such that for every $(t,j) \in E_1$, there exists $j' \leq j$ such that $(t,j) \in E_2$ and $(R_1, \Phi(\hat{q}_1, R_1), \omega_1, \xi_1)|_{(t,j)} = (\mathcal{R}(q_2), q_2, \omega_2, \xi_2)|_{(t,j')}$. But then, such a solution would satisfy $(\Phi(\hat{q}_1, R_1), \omega_1, \xi_1)|_{(0,0)} \in U_\ell^\delta$ and $\operatorname{dist}(\hat{q}_1(0,0), \mathcal{Q}(R_1(0,0))) < \alpha + \delta$, which implies that $(\Phi(\hat{q}_1, R_1), \omega_1, \xi_1)|_{(t,j)} \in U_\ell^\epsilon$ and $\operatorname{dist}(\hat{q}_1(0,0), \mathcal{Q}(R_1(0,0))) < \alpha + \epsilon$ for all $(t,j) \in E_1$. Finally, this implies that $(q_2, \omega_2, \xi_2)|_{(t,j)} \in U_\ell^\epsilon$ for all $(t,j) \in E_2$ and that $\mathcal{A}_\ell$ is stable.

From the arguments above, the proofs of instability follow similarly. While we do not prove attractivity here, we emphasize that the proofs are largely the same in character and ultimately rely on comparing solutions of (31) with (30) through Lemma 8.

REFERENCES

[1] R. E. Mortensen, “A globally stable linear attitude regulator,” *International Journal of Control*, vol. 8, no. 3, pp. 297–302, 1968.

[2] G. Meyer, “Design and global analysis of spacecraft attitude control systems,” NASA, Tech. Rep. R-361, Mar. 1971.

[3] B. Wie and P. M. Barba, “Quaternion feedback for spacecraft large angle maneuvers,” *Journal of Guidance, Control, and Dynamics*, vol. 8, no. 3, pp. 360–365, 1985.

[4] J. T.-Y. Wen and K. Kreutz-Delgado, “The attitude control problem,” *IEEE Transactions on Automatic Control*, vol. 36, no. 10, pp. 1148–1162, Oct. 1991.

[5] C. G. Mayhew, R. G. Sanfelice, and A. R. Teel, “Quaternion-based hybrid control for robust global attitude tracking,” *IEEE Transactions on Automatic Control*, vol. 56, no. 11, pp. 2555–2566, Nov. 2011.

[6] S. P. Bhat and D. S. Bernstein, “A topological obstruction to continuous global stabilization of rotational motion and the unwinding phenomenon,” *Systems & Control Letters*, vol. 39, no. 1, pp. 63–70, Jan. 2000.

[7] E. Sontag, *Mathematical Control Theory*. Springer, 1998.
[8] C. G. Mayhew and A. R. Teel, “On the topological structure of attraction basins for differential inclusions,” *Systems & Control Letters*, vol. 60, no. 12, pp. 1045–1050, Dec. 2011.
[9] J. Stuelpnagel, “On the parametrization of the three-dimensional rotation group,” *SIAM Review*, vol. 6, no. 4, pp. 422–430, Oct. 1964.
[10] R. G. Sanfelice, M. J. Messina, S. E. Tuna, and A. R. Teel, “Robust hybrid controllers for continuous-time systems with applications to obstacle avoidance and regulation to disconnected set of points,” in *Proceedings of the American Control Conference*, 2006, pp. 3352–3357.
[11] S. M. Joshi, A. G. Kelkar, and J. T.-Y. Wen, “Robust attitude stabilization of spacecraft using nonlinear quaternion feedback,” *IEEE Transactions on Automatic Control*, vol. 40, no. 10, pp. 1800–1803, 1995.
[12] C. Grubin, “Derivation of the quaternion scheme via the Euler axis and angle,” *Journal of Spacecraft and Rockets*, vol. 7, no. 10, pp. 1261–1263, Oct. 1970.
[13] A. Klumpp, “Singularity-free extraction of a quaternion from a direction-cosine matrix,” *Journal of Spacecraft and Rockets*, vol. 13, no. 12, pp. 754–755, 1976.
[14] J. R. Wertz, Ed., *Spacecraft Attitude Determiniation and Control*. Kluwer Academic Publishers, 1978.
[15] G. Wahba, “Problem 65-1,” *SIAM Review*, vol. 7, p. 409, 1965.
[16] M. D. Shuster, “Three-axis attitude determination from vector observations,” *Journal of Guidance, Control, and Dynamics*, vol. 4, no. 1, pp. 70–77, Jan.–Feb. 1981.
[17] F. Markley, “Attitude determination using vector observations and the singular value decomposition,” *The Journal of the Astronomical Sciences*, vol. 36, no. 3, pp. 245–258, 1988.
[18] R. Mahony, T. Hamel, and J.-M. Pflimlin, “Nonlinear complementary filters on the special orthogonal group,” *IEEE Transactions on Automatic Control*, vol. 53, no. 5, pp. 1203–1218, Jun. 2008.
[19] A. Tayebi, A. Roberts, and A. Benallegue, “Inertial measurements based dynamic attitude estimation and velocity-free stabilization,” in *Proceedings of the American Control Conference*, 2011, pp. 1027–1032.
[20] J. R. Magnus and H. Neudecker, “Matrix differential calculus with applications to simple, Hadamard, and Kronecker products,” *Journal of Mathematical Psychology*, vol. 29, no. 4, pp. 474–492, Dec. 1985.
[21] J. R. Munkres, *Topology*. Prentice Hall, 2000.
[22] M. A. Armstrong, *Basic Topology*. Springer, 1983.
[23] R. Goebel, R. G. Sanfelice, and A. R. Teel, “Hybrid dynamical systems,” *IEEE Control Systems*, vol. 29, no. 2, pp. 28–93, Apr. 2009.
[24] R. Goebel and A. R. Teel, “Solutions to hybrid inclusions via set and graphical convergence with stability theory applications,” *Automatica*, vol. 42, no. 4, pp. 573–587, Apr. 2006.
[25] C. G. Mayhew, R. G. Sanfelice, and A. R. Teel, “Robust global asymptotic stabilization of a 6-DOF rigid body by quaternion-based hybrid feedback,” in *Proceedings of the 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference*, 2009, pp. 1094–1099.
[26] C. G. Mayhew, R. G. Sanfelice, M. Arcak, and A. R. Teel, “Robust global asymptotic attitude synchronization by hybrid control,” in *Proceedings of the American Control Conference*, 2010, pp. 3666–3671.
[27] O. Egeland and J.-M. Godhavn, “Passivity-based adaptive attitude control of a rigid spacecraft,” *IEEE Transactions on Automatic Control*, vol. 39, no. 4, pp. 842–846, Apr. 1994.
[28] J. Erdong and S. Zhaowei, “Robust attitude synchronisation controllers design for spacecraft formation,” *IET Control Theory & Applications*, vol. 3, no. 3, pp. 325–339, Mar. 2009.
[29] O. Hájek, “Discontinuous differential equations, I,” *Journal of Differential Equations*, vol. 32, no. 2, pp. 149–170, May 1979.
[30] J. R. Lawton and R. W. Beard, “Synchronized multiple spacecraft rotations,” *Automatica*, vol. 38, no. 8, pp. 1359–1364, Aug. 2002.
[31] B. Wie, H. Weiss, and A. Arapostahis, “Quaternion feedback regulator for spacecraft eigenaxis rotations,” *Journal of Guidance, Control, and Dynamics*, vol. 12, no. 3, pp. 375–380, 1989.
[32] F. Lizarralde and J. T. Wen, “Attitude control without angular velocity measurement: a passivity approach,” *IEEE Transactions on Automatic Control*, vol. 41, no. 3, pp. 468–472, Mar. 1996.
[33] J. Ahmed, V. Coppola, and D. Bernstein, “Adaptive asymptotic tracking of spacecraft attitude motion with inertia matrix identification,” *Journal of Guidance, Control, and Dynamics*, vol. 21, no. 5, pp. 684–691, 1998.
[34] F. Caccavale and L. Villani, “Output feedback control for attitude tracking,” *Systems & Control Letters*, vol. 38, no. 2, pp. 91–98, Oct. 1999.
[35] A. Tayebi, “Unit quaternion-based output feedback for the attitude tracking problem,” *IEEE Transactions on Automatic Control*, vol. 53, no. 6, pp. 1516–1520, Jul. 2008.
[36] H. Bai, M. Arcak, and J. T. Wen, “Rigid body attitude coordination without inertial frame information,” *Automatica*, vol. 44, no. 12, pp. 3170–3175, Dec. 2008.
[37] O. E. Fjellstad and T. I. Fossen, “Singularity-free tracking of unmanned underwater vehicles in 6 DOF,” in *Proceedings of the 33rd IEEE Conference on Decision and Control*, vol. 2, 1994, pp. 1128–1133.
[38] J. Thienel and R. M. Sanner, “A coupled nonlinear spacecraft attitude controller and observer with an unknown constant gyro bias and gyro noise,” *IEEE Transactions on Automatic Control*, vol. 48, no. 11, pp. 2011–2015, Nov. 2003.
[39] D. Fragopoulos and M. Innocenti, “Stability considerations in quaternion attitude control using discontinuous Lyapunov functions,” *IEE Proceedings - Control Theory and Applications*, vol. 151, no. 3, pp. 253–258, May 2004.
[40] R. Kristiansen, P. J. Nicklasson, and J. T. Gravdahl, “Satellite attitude control by quaternion-based backstepping,” *IEEE Transactions on Control Systems Technology*, vol. 17, no. 1, pp. 227–232, Jan. 2009.
[41] R. G. Sanfelice, R. Goebel, and A. R. Teel, “Invariance principles for hybrid systems with connections to detectability and asymptotic stability,” *IEEE Transactions on Automatic Control*, vol. 52, no. 12, pp. 2282–2297, Dec. 2007.

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