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ON THE MOMENT-ANGLE MANIFOLD CONSTRUCTED BY FAN, CHEN, MA AND WANG

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Abstract

Fan, Chen, Ma and Wang [5] constructed a moment-angle manifold whose cohomology ring is isomorphic to that of the connected sum of sphere products consisting of one product of three spheres. In this paper, we show that these are in fact diffeomorphic.

1. Introduction

The topology of moment-angle manifolds has been studied by many authors [9, 4, 2, 6], and it is now known that this can be rather complicated. A connected sum of sphere products gives a typical example of such manifolds.

Theorem 1.1 (McGavran [9] and Bosio-Meersseman [2])**.** *Let K be the triangulation of a sphere that is dual to the simple polytope obtained from the k-simplex by cutting off $\ell > 0$ vertices, i.e., the boundary of a stacked polytope. Then, the moment-angle manifold associated to K is diffeomorphic to a connected sum of sphere products $Z_K \cong \#_{j=1}^{\ell}\left(S^{j+2} \times S^{2k+\ell-j-1}\right)^{\# j\binom{\ell+1}{j+1}}$.*

Here, $X^{\# j}$ denotes the connected sum of j-copies of a manifold X without boundary. Moreover, for $k = 2, 3$ Bosio and Meersseman characterized precisely the spherical triangulation that gives rise to a connected sum of sphere products as a moment-angle manifold. See Proposition 11.6 of [2]. In addition, see the paper [6]. In these observations, only a product of two spheres appears.

Fan, Chen, Ma, and Wang [5] found that the cohomology ring of the moment-angle manifold $Z_{\partial P^8_{28}}$ is isomorphic to that of the connected sum of sphere products

$$M = (S^3 \times S^3 \times S^6)\#(S^5 \times S^7)^{\#8}\#(S^6 \times S^6)^{\#8}, \tag{1.1}$$

where ∂P^8_{28} is the boundary of P^8_{28} (a simplicial 4-polytope with 8 vertices) described in [7]. The combinatorial structure of ∂P^8_{28} is described in §2 in terms of missing faces and facets.

In this paper, we show that the moment-angle manifold $Z_{\partial P^8_{28}}$ is in fact diffeomorphic to the connected sum of sphere products given above, and we affirm the conjecture of [5].

Theorem 1.2. *The moment-angle manifold $Z_{\partial P^8_{28}}$ is diffeomorphic to M as defined by* (1.1).

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2. Moment-angle manifold

To prove that $Z_{\partial P^8_{28}}$ is diffeomorphic to M, we follow the standard method, i.e., the use of h-cobordism theory. The following theorem is a simple modification of Theorem A1 of Gitler and López [6].

Theorem 2.1. *Let Q be a compact smooth manifold of dimension $d+1 \geq 6$ with boundary ∂Q, satisfying the following:*

(1) *Q is simply connected with a simply connected boundary.*
(2) *There is a finite collection $\{X_j\}$ of disjointly embedded closed smooth manifolds inside Q with trivial normal bundles.*
(3) *The embedding $\coprod_j X_j \to Q$ induces isomorphisms of integral homology groups of positive dimensions, and $H_i(Q) = 0$ for $i \geq d-1$.*

Then, Q is diffeomorphic to the boundary connected sum of $\coprod_j X_j \times D^{d+1-\dim X_j}$, and therefore ∂Q is diffeormophic to $\#_j(X_j \times S^{d-\dim X_j})$.

To construct a manifold Q with boundary $Z_{\partial P^8_{28}}$, we employ the polyhedral product.

Let K be a simplicial complex on the vertex set $[m] = \{1, 2, \cdots, m\}$. A moment-angle complex Z_K associated to K is defined by

$$Z_K = \bigcup_{\sigma \in K} (D^2)^\sigma \times (S^1)^{[m]\setminus\sigma} \subset (D^2)^m.$$

This construction has been generalized to the polyhedral product. See [1, 3]. Let $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i\in[m]}$ be a set of pairs of spaces (X_i, A_i). For a subset $I \subset [m]$, we define

$$(\underline{X}, \underline{A})^I = \{(x_1, \cdots, x_m) \in \prod_{i=1}^m X_i \mid x_j \in A_j \text{ if } j \notin I\}.$$

Then, the polyhedral product $Z_K(\underline{X}, \underline{A})$ is defined as the union of $(\underline{X}, \underline{A})^\sigma$ for all faces σ of K:

$$Z_K(\underline{X}, \underline{A}) = \bigcup_{\sigma \in K} (\underline{X}, \underline{A})^\sigma.$$

If K is an n-dimensional triangulated sphere with m vertices, then Z_K has the structure of an $(n+m+1)$-dimensional manifold. See Theorem 4.1.4 of [3]. Moreover, if K is the boundary of a simplicial polytope, then Z_K is a smooth manifold. See Theorem 6.2.4 of [3]. Therefore, Z_K is called a moment-angle manifold if K is a triangulated sphere.

Now, we recall the simplicial complex ∂P^8_{28} in the manner described by Fan, Chen, Ma, and Wang [5].

Definition 2.2. For a simplicial complex K on $[m]$, a subset $I \subset [m]$ is called a *missing face* or *minimal non-face* of K if I is not a simplex of K, but all of its proper subsets are faces of K. The set of missing faces of K is denoted by $\mathrm{MF}(K)$.

For a sequence of integers $i_1 < i_2 < \cdots < i_k$, $i_1 i_2 \cdots i_k$ denotes the set $\{i_1, i_2, \cdots, i_k\}$. ∂P^8_{28}

is a simplicial complex with the vertex set [8], and is characterized by its missing faces

$$\mathrm{MF}(\partial P^8_{28}) = \{56,\ 78,\ 123,\ 134,\ 235,\ 346,\ 147,\ 467,\ 128,\ 258\},$$

and ∂P^8_{28} has 18 facets (maximal faces):

1245, 1246, 1257, 1267, 1357, 1367, 2347, 2367, 2457,
3457, 1458, 1468, 1358, 1368, 2348, 2368, 2468, 3458.

$K = \partial P^8_{28}$ is decomposed as $K = K_{[6]} \cup \mathrm{link}_K(7) * 7 \cup \mathrm{link}_K(8) * 8$, where $\mathrm{link}_K(v)$ denotes the link of v in K. For reader's convenience, $K_{[6]}$, $\mathrm{link}_K(7)$ and $\mathrm{link}_K(8)$ are described in Figure 1. As noted in [5], $K_{[6]}$ can be viewed as a thick 2-sphere with two 3-simplices 1245 and 1246.

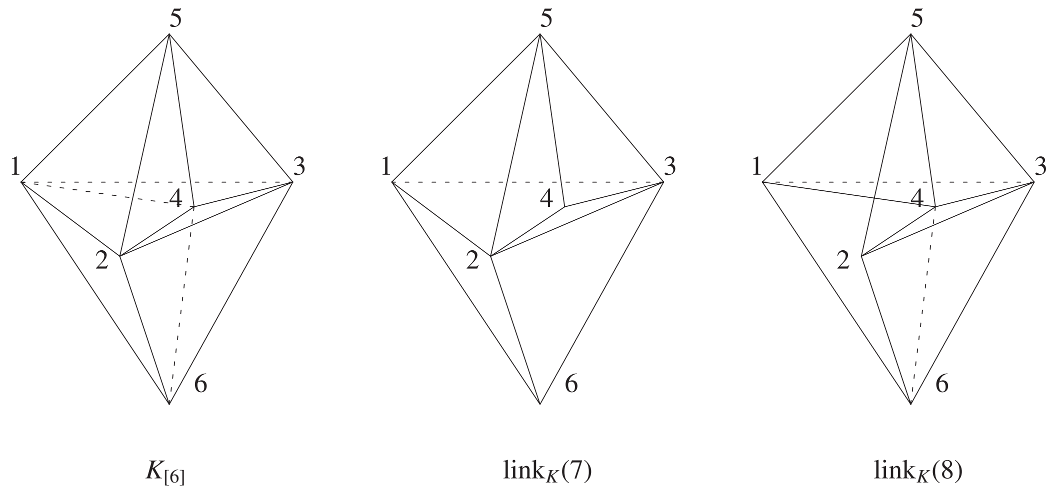


Fig. 1. $K = \partial P^8_{28}$

Let $D^3_+ = \{(x,z) \in \mathbf{R} \times \mathbf{C} \mid x^2 + |z|^2 \le 1,\ x \ge 0\}$ and $S^2_+ = \{(x,z) \in D^3_+ \mid x^2 + |z|^2 = 1\}$. Set $Q = Z_{\partial P^8_{28}}((D^3_+, S^2_+), (D^2, S^1), \cdots, (D^2, S^1))$. Then, Q is a 13-dimensional smooth manifold with boundary $Z_{\partial P^8_{28}}$. Because (D^3_+, S^2_+) is contractible as a pair of spaces, Q is homotopy equivalent to $Z_{\partial P^8_{28}}((*,*), (D^2, S^1), \cdots, (D^2, S^1))$, which is identified with $Z_{\partial P^8_{28}-1}$, where $\partial P^8_{28} - 1 = \{\sigma \subset \{2, \cdots, 8\} \mid \sigma \in \partial P^8_{28}\}$. To apply Theorem 2.1, we need to know the homotopy type of $Z_{\partial P^8_{28}-1}$. For a subset $I \subset [8]$, we denote the full subcomplex of ∂P^8_{28} on I by $(\partial P^8_{28})_I$.

Theorem 2.3. *The following homotopy equivalence holds:*

$$Q \simeq Z_{\partial P^8_{28}-1} \simeq (S^3_{56} \times S^3_{78}) \vee S^5_{235} \vee S^5_{346} \vee S^5_{467} \vee S^5_{258}$$
$$\vee S^6_{4678} \vee S^6_{4567} \vee S^6_{3467} \vee S^6_{3456} \vee S^6_{2578} \vee S^6_{2568} \vee S^6_{2358} \vee S^6_{2356} \vee S^7_{45678} \vee S^7_{25678} \vee S^7_{23568} \vee S^7_{34567},$$

where S^k_I denotes the sphere S^k, which appears for the first time in $Z_{(\partial P^8_{28})_I}$ as a wedge summand.

3. Fat wedge filtration

To prove Theorem 2.3, we make use of the fat wedge filtration introduced by Kishimoto and the author in [8].

Let K be a simplicial complex on $[m]$. For each subset $I \subset [m]$, K_I denotes the full subcomplex on I. That is, $K_I = \{\sigma \in K \mid \sigma \subset I\}$. We regard Z_{K_I} as a subspace of Z_K, by identifying it with $Z_{K_I} \times \{-1\}^{I^c}$. On the other hand, the projection $(D^2)^m \to (D^2)^I$ induces the projection $Z_K \to Z_{K_I}$. In particular, Z_{K_I} is a retract of Z_K.

Now, we recall the fat wedge filtration of Z_K. For $0 \leq i \leq m$, the fat wedge filtration of Z_K is given by

$$Z_K^i = \{(x_1, \cdots, x_m) \in Z_K \mid \text{at least } m-i \text{ of } x_j \text{ are } -1\},$$

which induces the following filtration of Z_K:

$$Z_K^0 = \{*\} \subset Z_K^1 \subset \cdots \subset Z_K^i \subset \cdots \subset Z_K^m = Z_K,$$

where $* = (-1, \cdots, -1)$. Then, it is easy to see that

$$Z_K^i = \bigcup_{I \subset [m],\ |I| \leq i} Z_{K_I}.$$

The key property of the fat wedge filtration of the moment-angle complex is the following.

Theorem 3.1 (Theorem 5.1 of [8])**.** *For $i = 1, \ldots, m$, Z_K^i is obtained from Z_K^{i-1} by attaching a cone to the composition of maps $\varphi_{K_I} : \Sigma^i|K_I| \to Z_{K_I}^{i-1} \xrightarrow{\text{incl}} Z_K^{i-1}$, for each $I \subset [m]$ with $|I| = i$.*

There exist some classes of simplicial complexes whose attaching maps for Z_K^i are trivial. One such complex is the fillable complex.

Definition 3.2. A simplicial complex K is *fillable* if there are missing faces $L_1, \cdots, L_r$ of K such that $|K \cup \{L_1, \cdots, L_r\}|$ is contractible.

Theorem 3.3 (Theorem 7.2 of [8])**.** *If K is fillable, then the attaching map φ_K is null homotopic.*

4. Proof of Theorem 2.3

From this point on, K denotes ∂P^8_{28}. Here, we remark that $(K-1)_I = K_I$ for a subset $I \subset [8] - 1$. Because $K - 1$ does not have ghost vertices, it is easy to see that $Z^1_{K-1} \simeq *$.

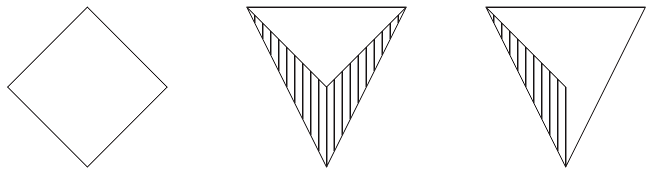
Step I): Z^2_{K-1}. Because by Theorem 3.1 we have a cofiber sequence

$$\bigvee_{I \subset [8]-1,\ |I|=2} \Sigma^2|K_I| \to Z^1_{K-1} \simeq * \to Z^2_{K-1}$$

and $|K_I|$ is not contractible for $I \subset [8] - 1$ with $|I| = 2$ if and only if $I = 56$ or $I = 78$, we have a homotopy equivalence $Z^2_{K-1} \simeq S^3_{56} \vee S^3_{78}$.

Step II): Z^3_{K-1}. $|K_I|$ is not contractible for $I \subset [8] - 1$ with $|I| = 3$ if and only if I is one of the four missing faces with three vertices in $\mathrm{MF}(K)$, it does not contain the vertex 1, and in each case $K_I \cong \partial\Delta^2$. Clearly, $\partial\Delta^2$ is fillable, and the attaching maps φ_{K_I} are all trivial by Theorem 3.3. Thus, we have that $Z^3_{K-1} \simeq S^3_{56} \vee S^3_{78} \vee S^5_{235} \vee S^5_{346} \vee S^5_{467} \vee S^5_{258}$.

Step III): Z^4_{K-1}. In [5], the authors classified non-contractible full subcomplexes of K with four vertices. There are three possible types described in Figure 2.



A B C

Fig. 2

If K_I is of type A, then $I = 5678$ and $Z_{K_I} = S^3 \times S^3$. If K_I is of type B or type C, then it is a fillable complex, and the attaching map is trivial. Thus, we have the homotopy equivalence

$$Z^4_{K-1} \simeq (S^3_{56} \times S^3_{78}) \vee S^5_{235} \vee S^5_{346} \vee S^5_{467} \vee S^5_{258} \\ \vee S^6_{4678} \vee S^6_{4567} \vee S^6_{3467} \vee S^6_{3456} \vee S^6_{2578} \vee S^6_{2568} \vee S^6_{2358} \vee S^6_{2356}.$$

Step IV): Z^5_{K-1}. $|K_I|$ is not contractible for $I \subset [8] - 1$ with $|I| = 5$ if and only if I is the complement of one of the four missing faces with three vertices in MF(K) that contains the vertex 1. In all cases, it is easy to see that K_I is fillable, and the attaching map is trivial.

$$Z^5_{K-1} \simeq (S^3_{56} \times S^3_{78}) \vee S^5_{235} \vee S^5_{346} \vee S^5_{467} \vee S^5_{258} \vee S^6_{4678} \vee S^6_{4567} \vee S^6_{3467} \vee S^6_{3456} \\ \vee S^6_{2578} \vee S^6_{2568} \vee S^6_{2358} \vee S^6_{2356} \vee S^7_{45678} \vee S^7_{25678} \vee S^7_{23568} \vee S^7_{34567}.$$

Step V). Z^6_{K-1} and Z_{K-1}. For $|I| = 6$, $|K_I|$ is not contractible if and only if I is the complement of one of the two missing faces with two vertices in MF(K). That is, $I = 123456$ or $I = 123478$. In both cases I contains the vertex 1, and therefore $|(K-1)_I|$ is contractible for all $I \subset [8] - 1$ with $|I| = 6$.

Because $|K - 1|$ is contractible, $Z^7_{K-1} \simeq Z^6_{K-1} \simeq Z^5_{K-1}$, and we have completed the proof of Theorem 2.3.

5. Proof of Theorem 1.2

We call $S^3_{56} \times S^3_{78}$ or S^k_I a wedge summand when they appear as a wedge summand of Q. To construct an embedding for a wedge summand to Q, we make use of the smooth embedding $Z_{\mathrm{link}_K(\sigma)} \to Z_K$, where $\mathrm{link}_K(\sigma) = \{\tau \in \sigma^c \,|\tau \cup \sigma \in K\}$ is the link of σ in K. To embed $S^3_{56} \times S^3_{78}$, we use $\mathrm{link}_K(13) = \{\emptyset, 5, 6\} * \{\emptyset, 7, 8\}$. Then, $Z_{\mathrm{link}_K(13)} = S^3 \times S^3$ and the embedding $Z_{\mathrm{link}_K(13)} = S^3 \times S^3 \to Z_K \subset Q$ represents the wedge summand $S^3_{56} \times S^3_{78}$.

To embed the other wedge summands, we use $\mathrm{link}_K(2)$ and $\mathrm{link}_K(4)$.

The facets of $\mathrm{link}_K(2)$ are 145, 146, 157, 167, 347, 367, 457, 348, 368, 468, and therefore it is the boundary of a stacked polytope as $\mathrm{link}_K(2)$ is described below.

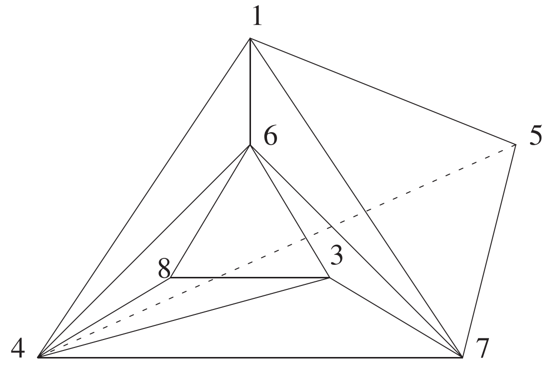


Fig. 3. $\mathrm{link}_K(2)$

Thus $\mathrm{link}_K(2)$ is dual to the simple polytope obtained from the 3-simplex by cutting off 3 vertices. By Theorem 1.1, we see that $Z_{\mathrm{link}_K(2)} \cong (S^3 \times S^7)^{\#6} \# (S^4 \times S^6)^{\#8} \# (S^5 \times S^5)^{\#3}$. It is easy to see where a sphere factor of $Z_{\mathrm{link}_K(2)}$ occurs, and we see that

$$\begin{aligned} Z_{\mathrm{link}_K(2)} \cong\ & (S^3_{13} \times S^7_{45678}) \# (S^3_{18} \times S^7_{34567}) \# (S^3_{35} \times S^7_{14678}) \\ & \# (S^3_{56} \times S^7_{13478}) \# (S^3_{58} \times S^7_{13467}) \# (S^3_{78} \times S^7_{13456}) \\ & \# (S^4_{135} \times S^6_{4678}) \# (S^4_{138} \times S^6_{4567}) \# (S^4_{158} \times S^6_{3467}) \# (S^4_{178} \times S^6_{3456}) \\ & \# (S^4_{356} \times S^6_{1478}) \# (S^4_{358} \times S^6_{1467}) \# (S^4_{568} \times S^6_{1347}) \# (S^4_{578} \times S^6_{1346}) \\ & \# (S^5_{147} \times S^5_{3568}) \# (S^5_{346} \times S^5_{1578}) \# (S^5_{467} \times S^5_{1358}). \end{aligned}$$

Here, we remark that in the formula above, factors of S^k_I arise from $Z_{(\mathrm{link}_K(2))_I}$. The composite of maps $Z_{\mathrm{link}_K(2)} \to Z_K \subset Q \xrightarrow{\simeq} Z_{K-1}$ maps a summand S^k_I in $Z_{\mathrm{link}_K(2)}$ to S^k_I in Z_{K-1} in a homotopy equivalent manner for $I = 346,\ 467,\ 4678,\ 4567,\ 3467,\ 3456,\ 45678,\ 34567$. To see this, it is sufficient to check that $|(\mathrm{link}_K(2))_I| \to |K_I|$ is homotopy equivalent for those I's by Theorem 3.1. For $I = 346,\ 467,\ 4678,\ 4567$ or 3467, it is a routine work to check that $(\mathrm{link}_K(2))_I = K_I$. For $I = 3456$, $|(\mathrm{link}_K(2))_{346}| = |K_{346}|$ is a deformation retract both of $|(\mathrm{link}_K(2))_I|$ and $|K_I|$. For $I = 45678$, $|(\mathrm{link}_K(2))_{467}| = |K_{467}|$ is a deformation retract both of $|(\mathrm{link}_K(2))_I|$ and $|K_I|$. For $I = 34567$, $|(\mathrm{link}_K(2))_{346}| = |K_{346}|$ is a deformation retract both of $|(\mathrm{link}_K(2))_I|$ and $|K_I|$. Thus in all cases the inclusion map $|(\mathrm{link}_K(2))_I| \to |K_I|$ induces a homotopy equivalence.

Similarly, the facets of $\mathrm{link}_K(4)$ are 126, 125, 168, 158, 238, 358, 268, 237, 357, 257. These are just those of $\mathrm{link}_K(2)$, except with 4 replaced by 2 and $7 \leftrightarrow 8$ exchanged with $5 \leftrightarrow 6$. Therefore, we see that

$$\begin{aligned} Z_{\mathrm{link}_K(4)} \cong\ & (S^3_{13} \times S^7_{25678}) \# (S^3_{17} \times S^7_{23568}) \# (S^3_{36} \times S^7_{12578}) \\ & \# (S^3_{56} \times S^7_{12378}) \# (S^3_{67} \times S^7_{12358}) \# (S^3_{78} \times S^7_{12356}) \\ & \# (S^4_{136} \times S^6_{2578}) \# (S^4_{137} \times S^6_{2568}) \# (S^4_{167} \times S^6_{2358}) \# (S^4_{178} \times S^6_{2356}) \\ & \# (S^4_{356} \times S^6_{1278}) \# (S^4_{367} \times S^6_{1258}) \# (S^4_{567} \times S^6_{1238}) \# (S^4_{678} \times S^6_{1235}) \\ & \# (S^5_{128} \times S^5_{3567}) \# (S^5_{235} \times S^5_{1678}) \# (S^5_{258} \times S^5_{1367}). \end{aligned}$$

Thus, we have obtained a necessary embedding for each wedge summand. Because all wedge summands are embedded in the boundary of Q, the embeddings can be made mutually disjoint and lying in a collar neighborhood of the boundary Z_K.

Finally, we see that the normal bundles of the embedded manifolds are trivial. The normal bundle of S^5 (resp. S^7) embedded in Z_K has been classified by the homotopy group $[S^5, BO(7)] \cong [S^4, O(7)] \cong 0$ (resp. $[S^7, BO(5)] \cong [S^6, O(5)] \cong 0$) in [11] and [10]. Thus, these are also trivial in Q. The normal bundle of S^6 embedded in Q is classified by the homotopy group $[S^6, BO(7)] \cong [S^5, O(7)] \cong 0$. The normal bundle of $S^3 \times S^3$ embedded in Q is classified by the homotopy set $[S^3 \times S^3, BO(7)] = *$. To see this, we consider the exact sequence associated with the cofiber sequence $S^3 \vee S^3 \to S^3 \times S^3 \to S^6$:

$$0 \cong [S^6, BO(7)] \to [S^3 \times S^3, BO(7)] \to [S^3 \vee S^3, BO(7)].$$

Because $[S^3 \vee S^3, BO(7)] \cong [S^3, BO(7)] \times [S^3, BO(7)] \cong [S^2, O(7)] \times [S^2, O(7)] \cong 0$, we see that $[S^3 \times S^3, BO(7)] = *$. Thus, all of the normal bundles are trivial, and we have proved Theorem 1.2 by applying Theorem 2.1.

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